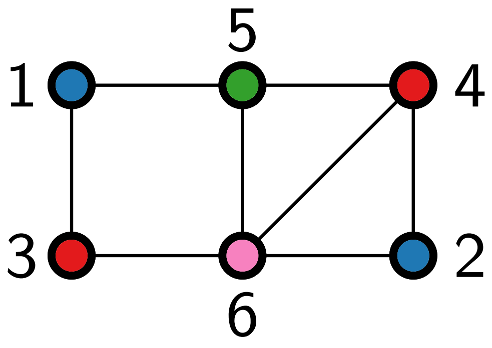
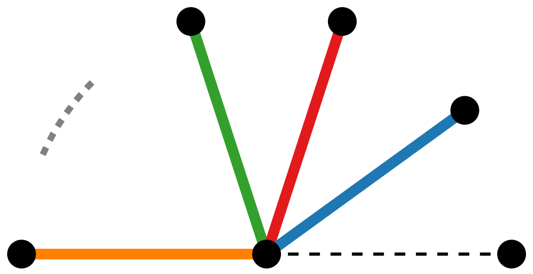
Advanced Algorithms

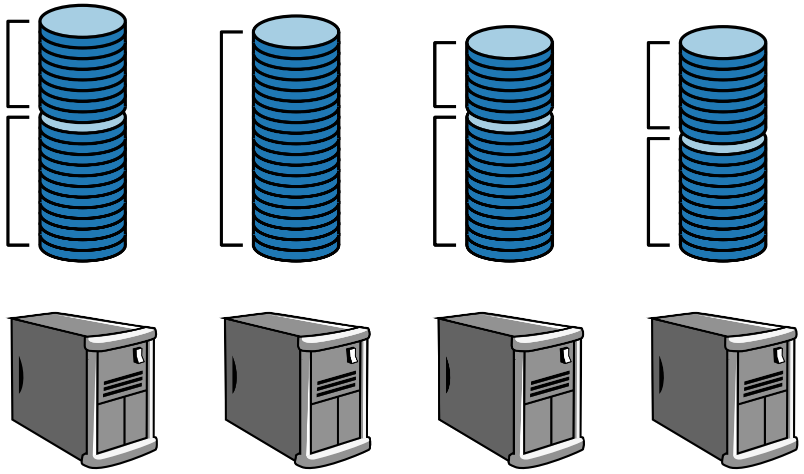
Approximation Algorithms

Coloring and Scheduling Problems

Alexander Wolff · WS22



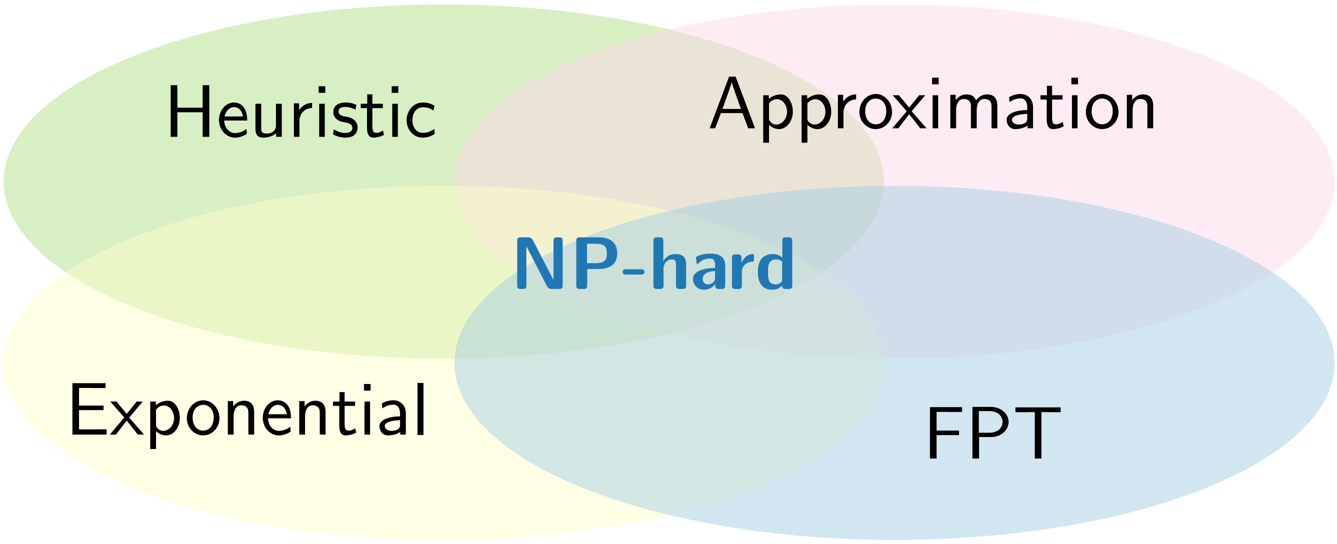




Dealing with NP-Hard Optimization Problems

What should we do?

- Sacrifice optimality for speed
  - Heuristics
  - Approximation algorithms
- Optimal solutions
  - Exact exponential-time algorithms
  - Fine-grained analysis – parameterized algorithms



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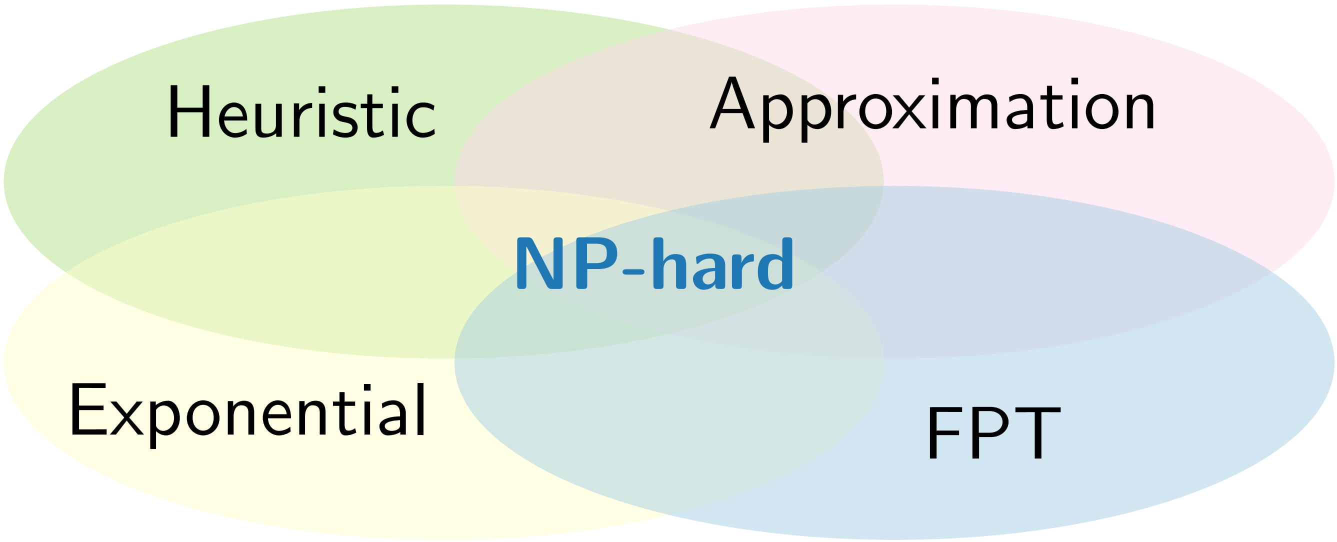

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- Approximation algorithms that compute solutions with/that are
  - additive guarantee,
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PTAS (*polynomial-time approximation scheme*)

Approximation with Additive Guarantee

Definition.
Let Π be an optimization problem,
let $\mathcal{A}$ be a polynomial-time algorithm for Π,
let I be an instance of Π, and
let $\mathsf{ALG}(I)$ be the value of the objective function of the solution that $\mathcal{A}$ computes given I.

Then $\mathcal{A}$ is called an **approximation algorithm with additive guarantee** δ (which can depend on I) if

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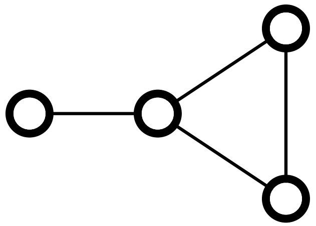
- Most problems that we know do not admit an approximation algorithm with additive guarantee.

Minimum Vertex Coloring

Input. A graph $G = (V, E)$. Let Δ be the maximum degree of G.



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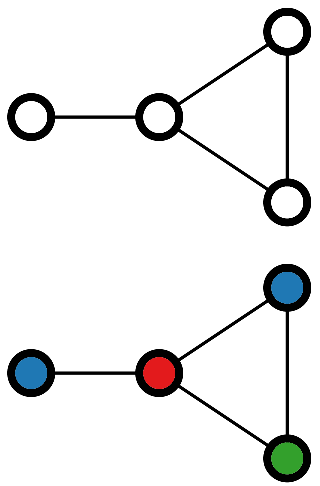

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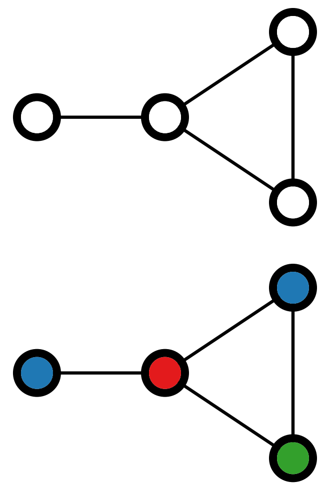

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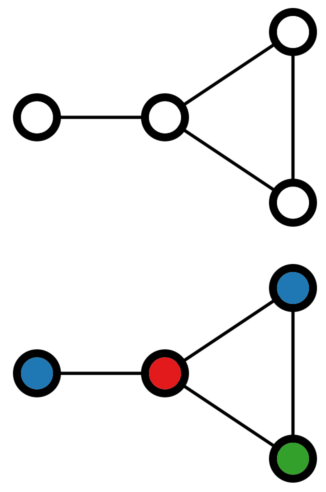

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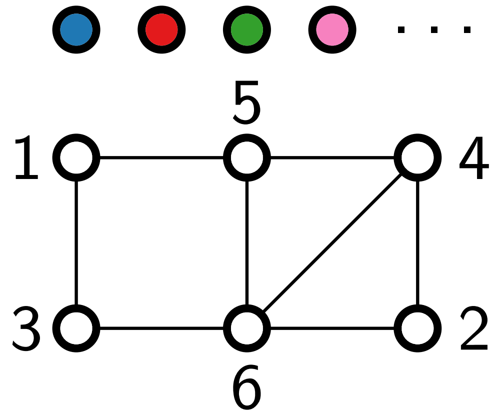

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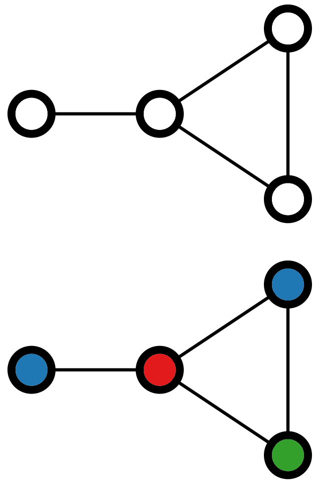

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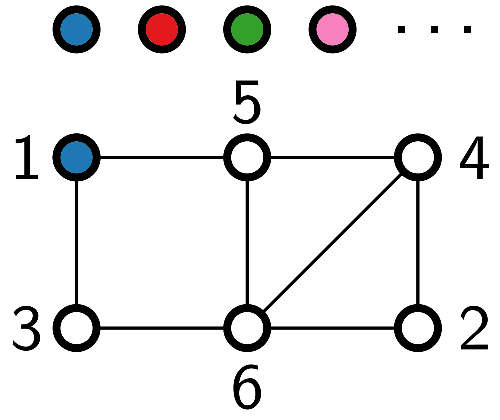

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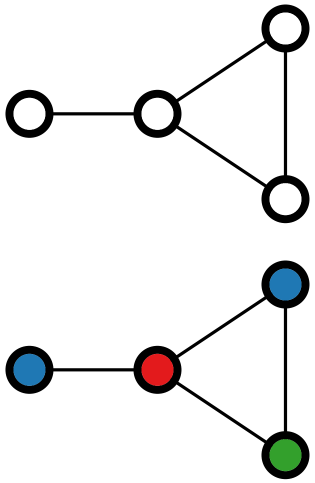

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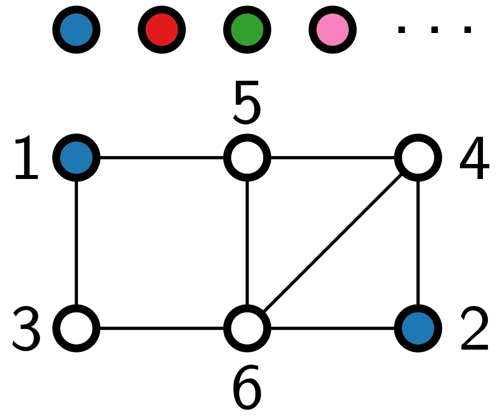

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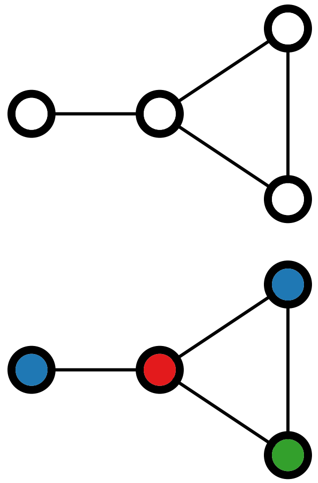

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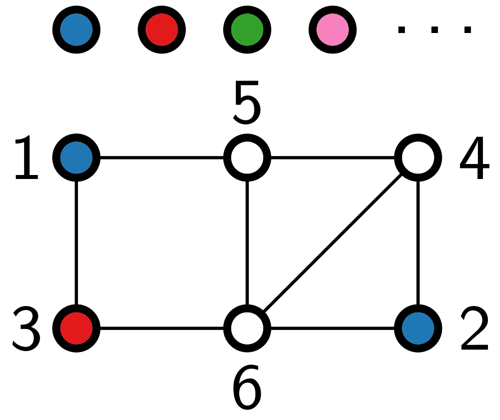

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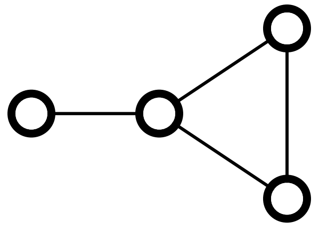

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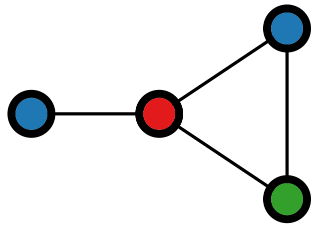

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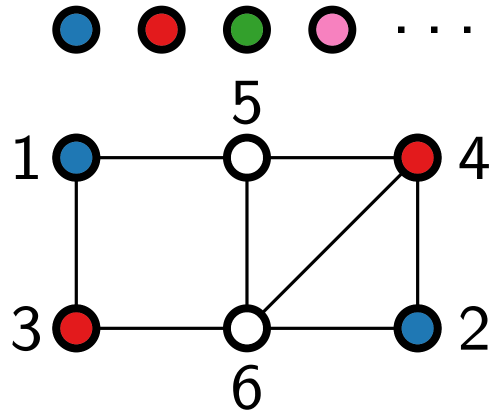

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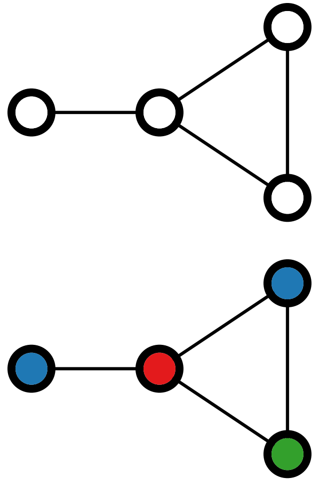

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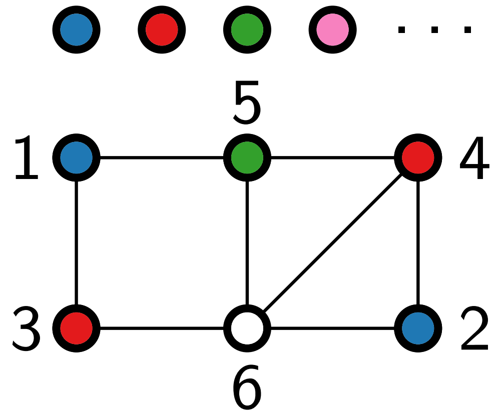

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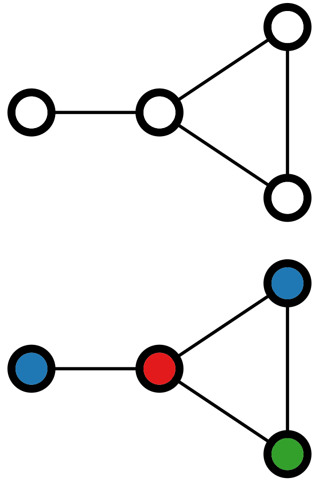

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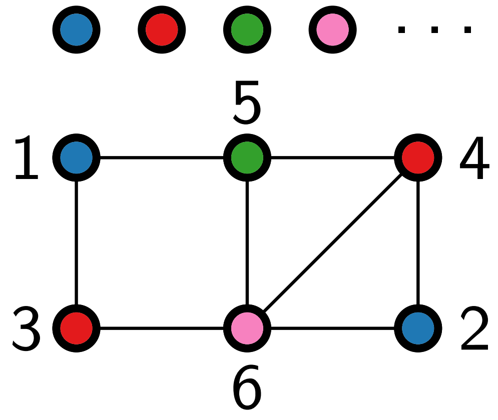

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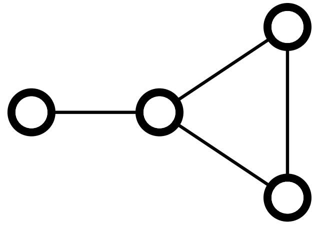

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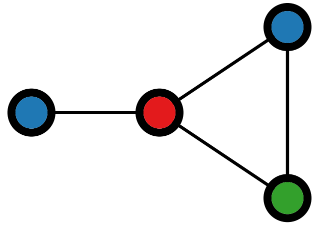

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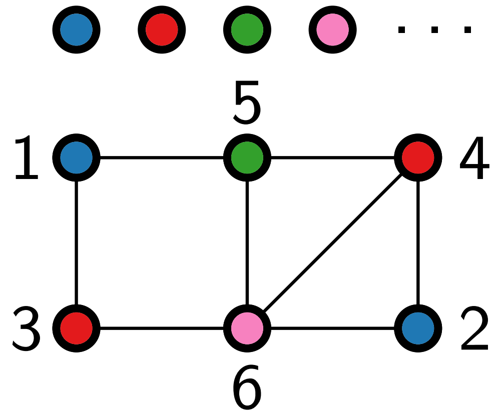

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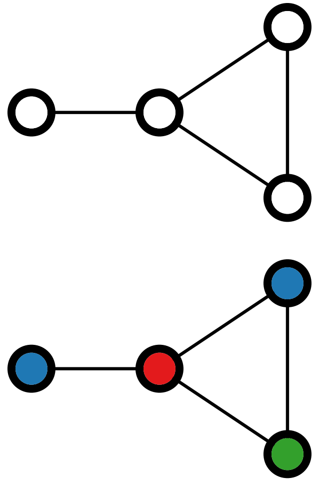

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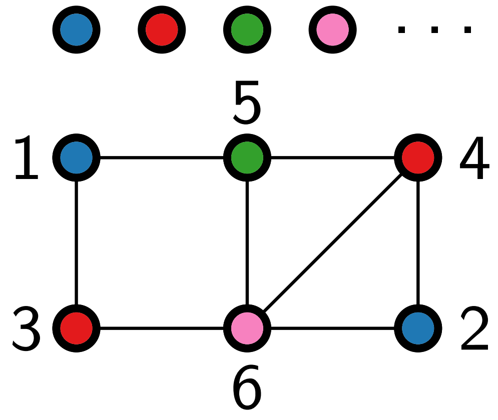

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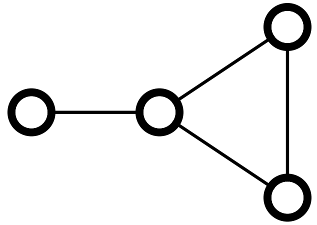

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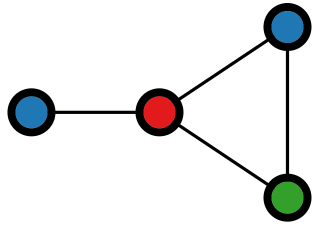

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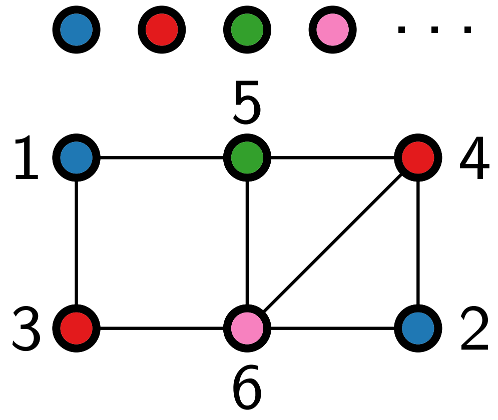

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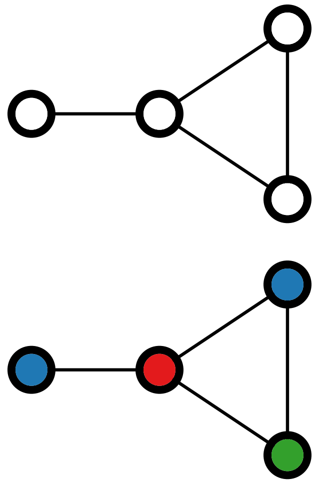

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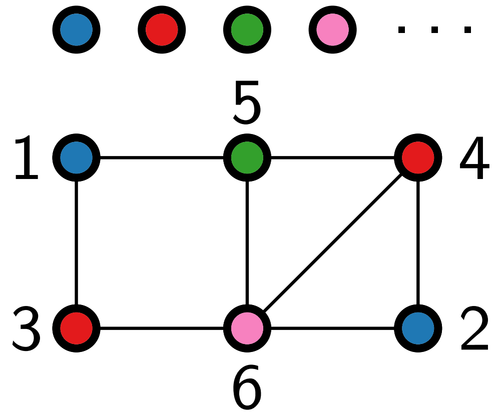

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We can get $\Delta - 2$ if we return a 2-coloring whenever G is bipartite.

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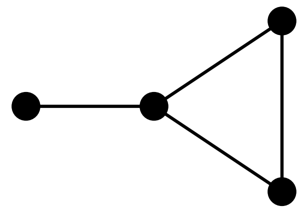

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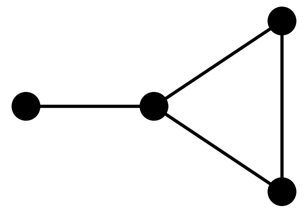

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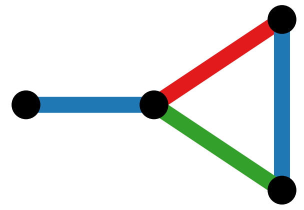

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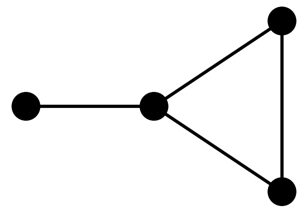

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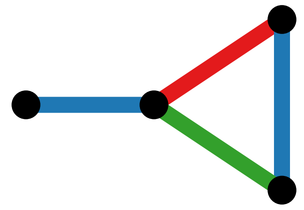

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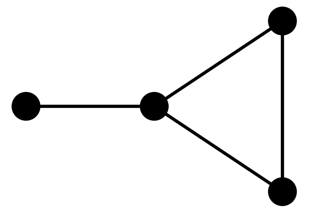

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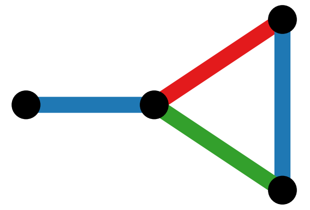

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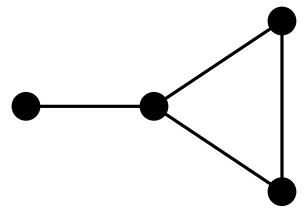

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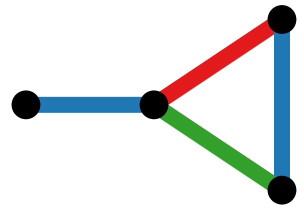

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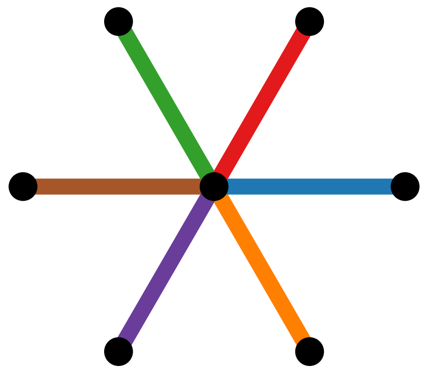

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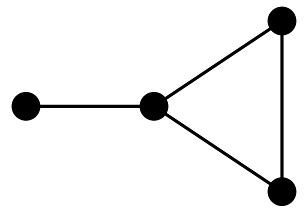

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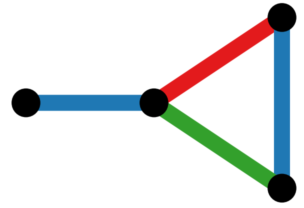

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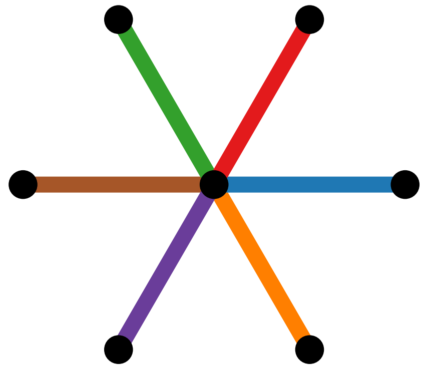

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- $\chi'(G)$ is lowerbounded by Δ.
- We show that $\chi'(G) \leq \Delta + 1$.

Minimum Edge Coloring – Upper Bound

Vizing's Theorem.
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(Kiew 1937 – 2017 Odessa)

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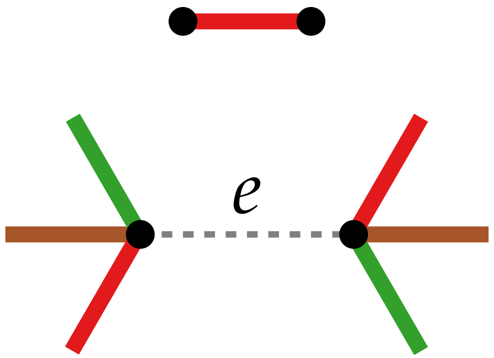

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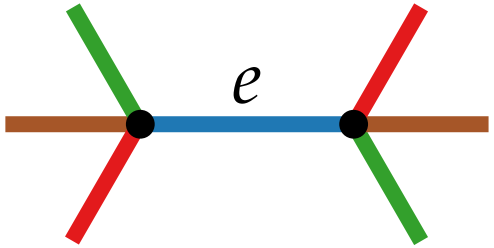

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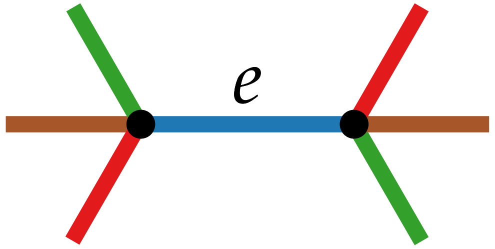

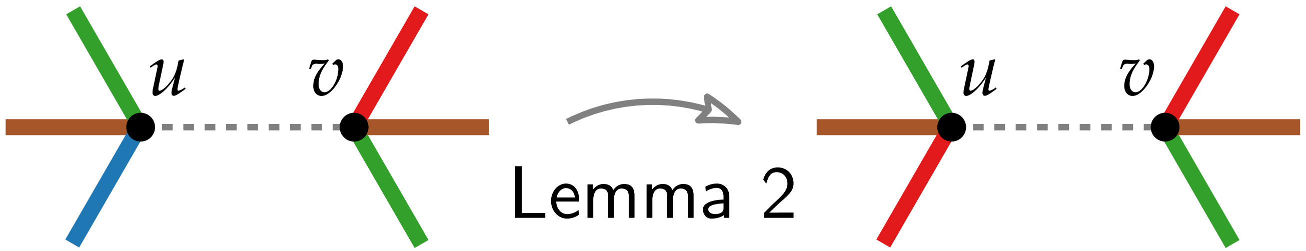

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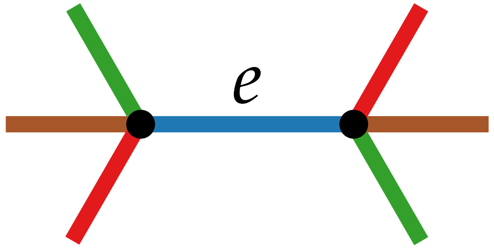

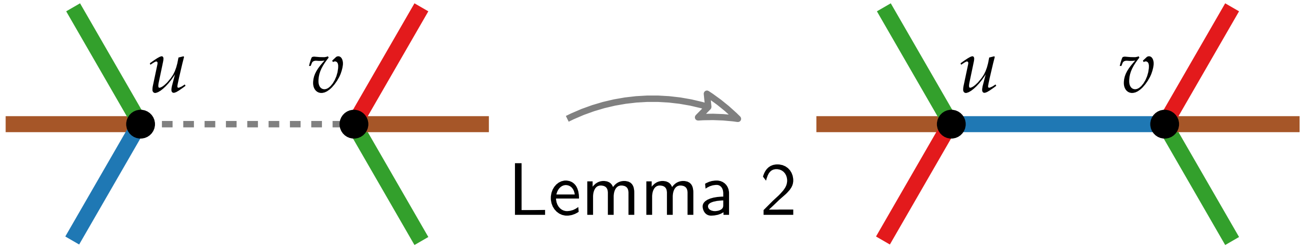

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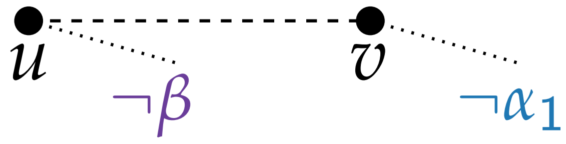

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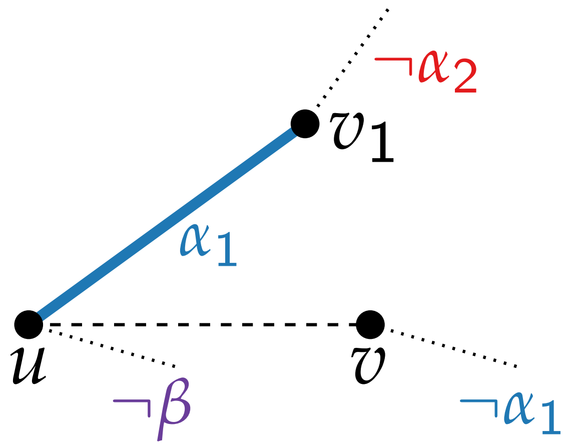

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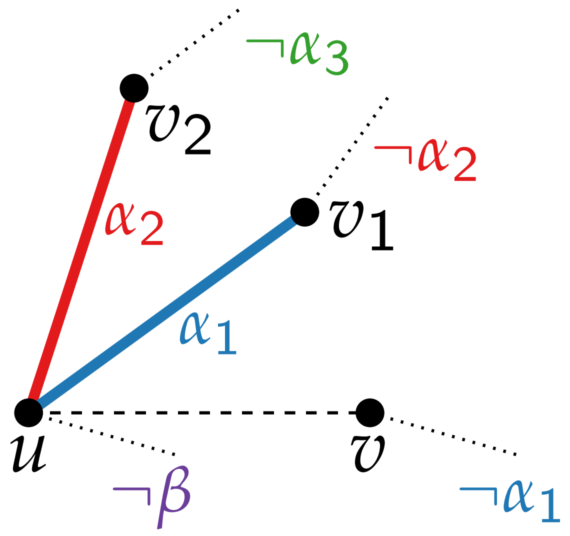

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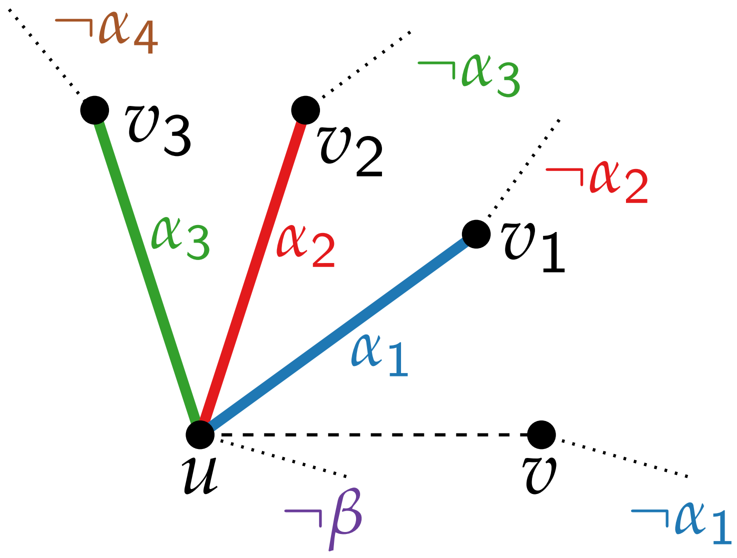

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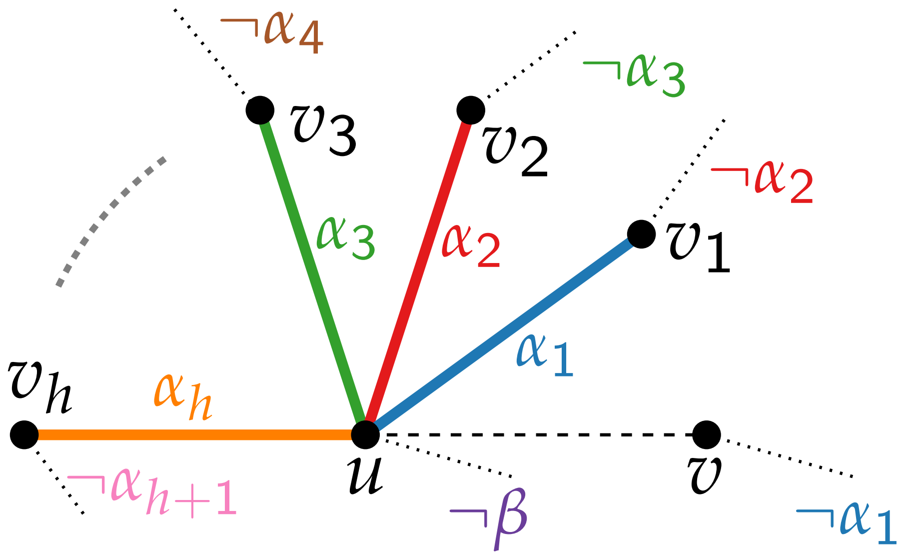

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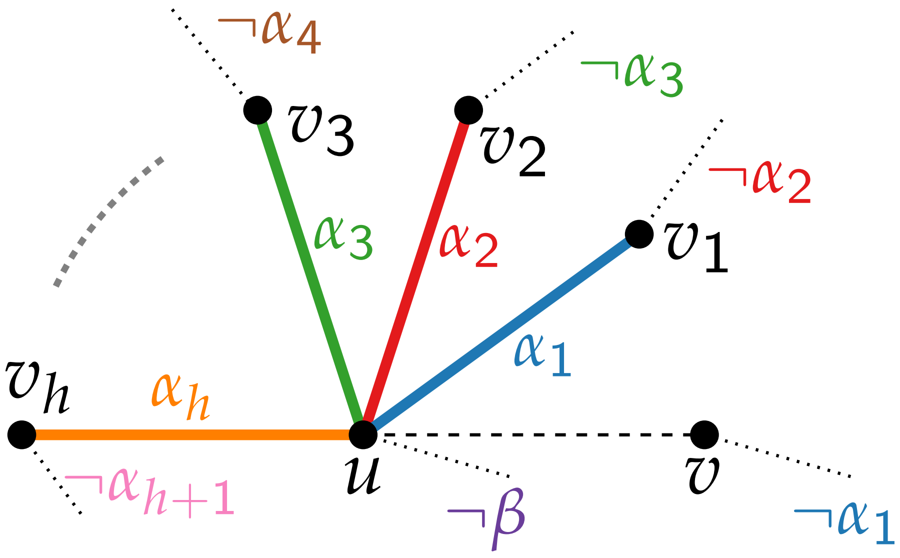

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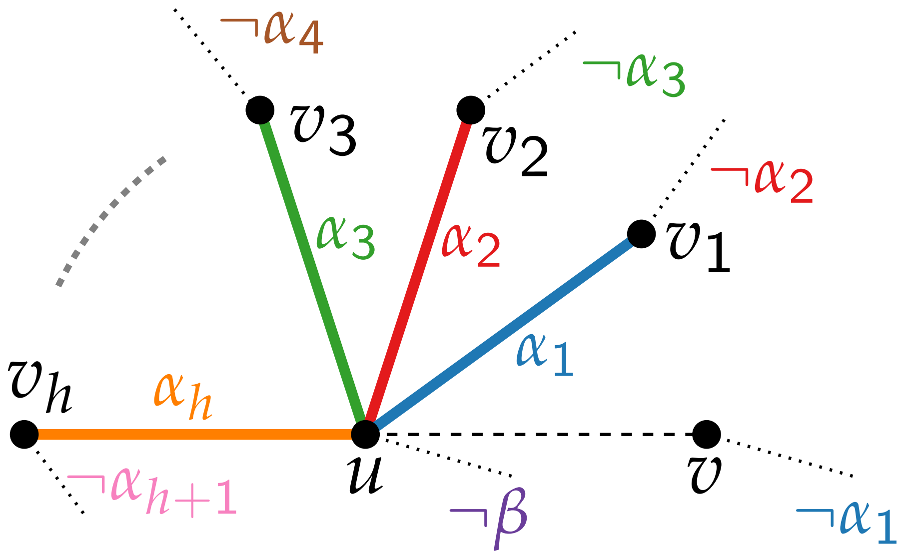

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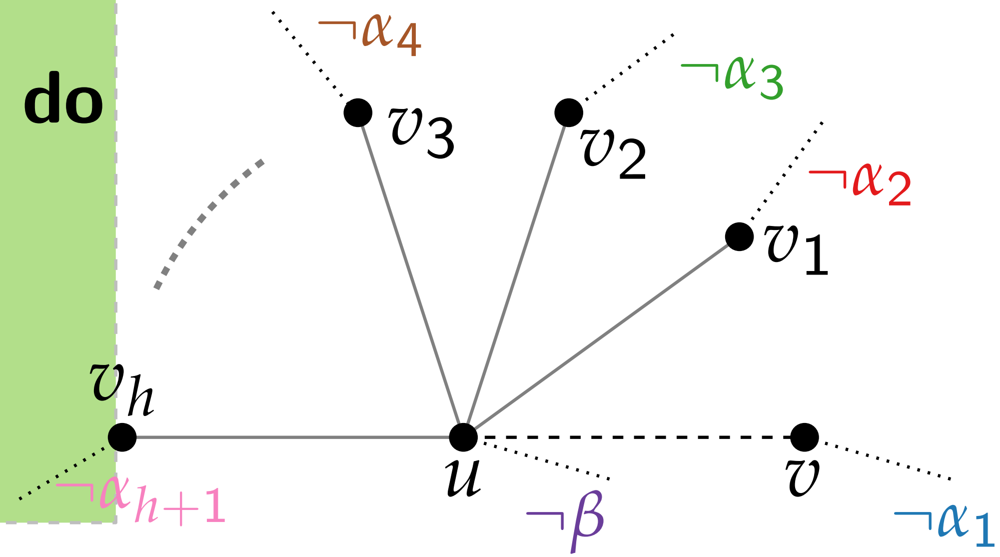

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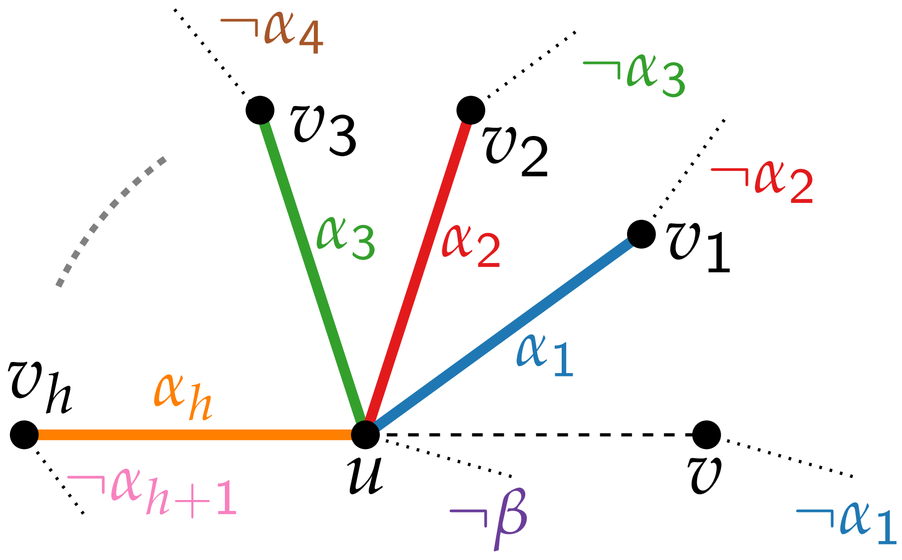

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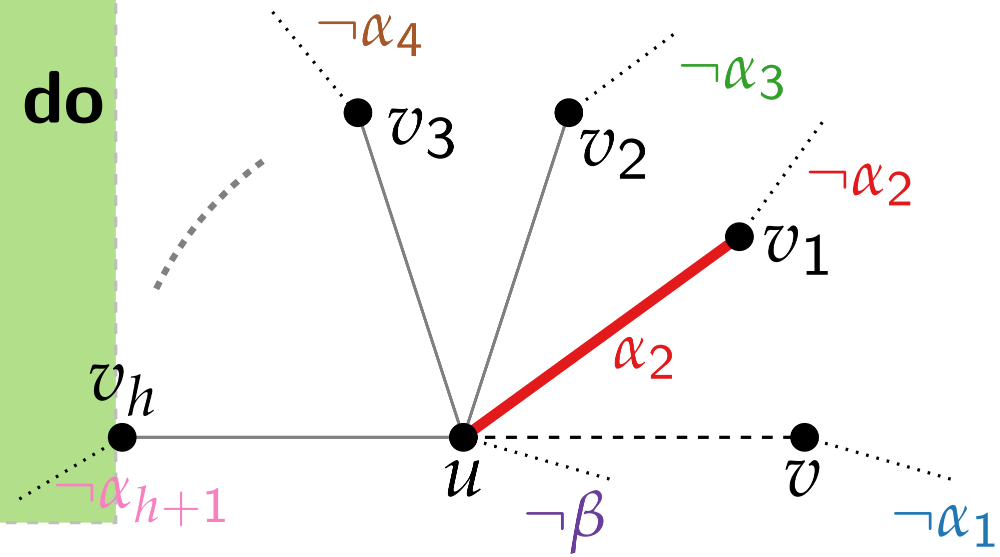

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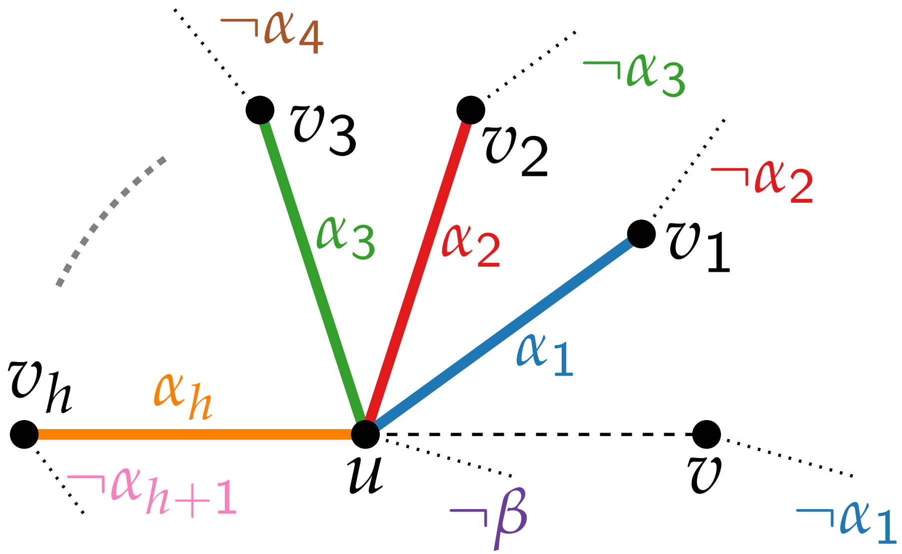

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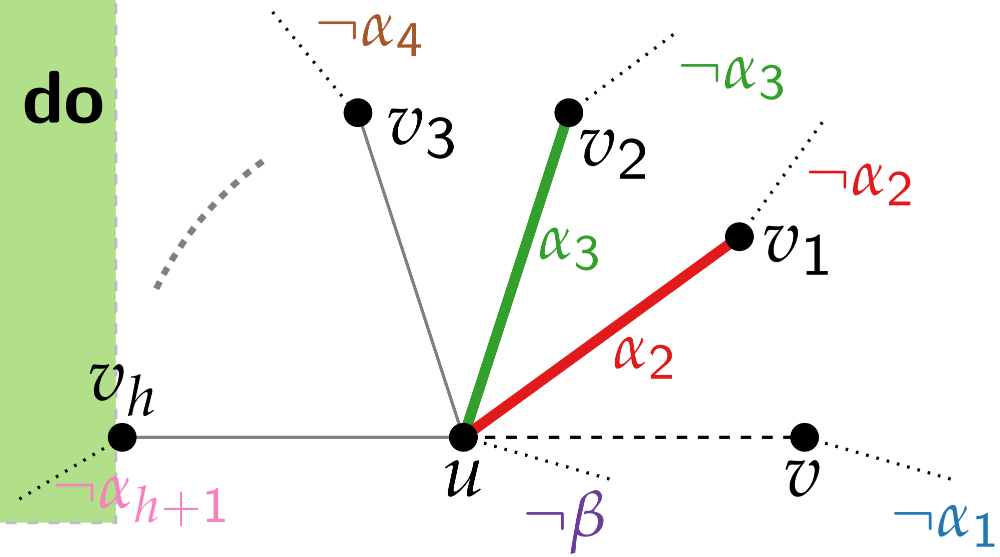

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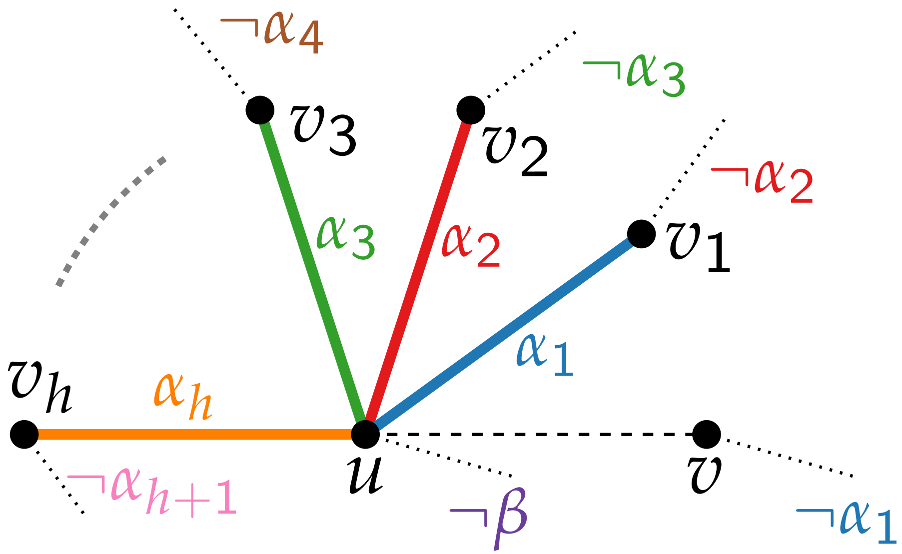

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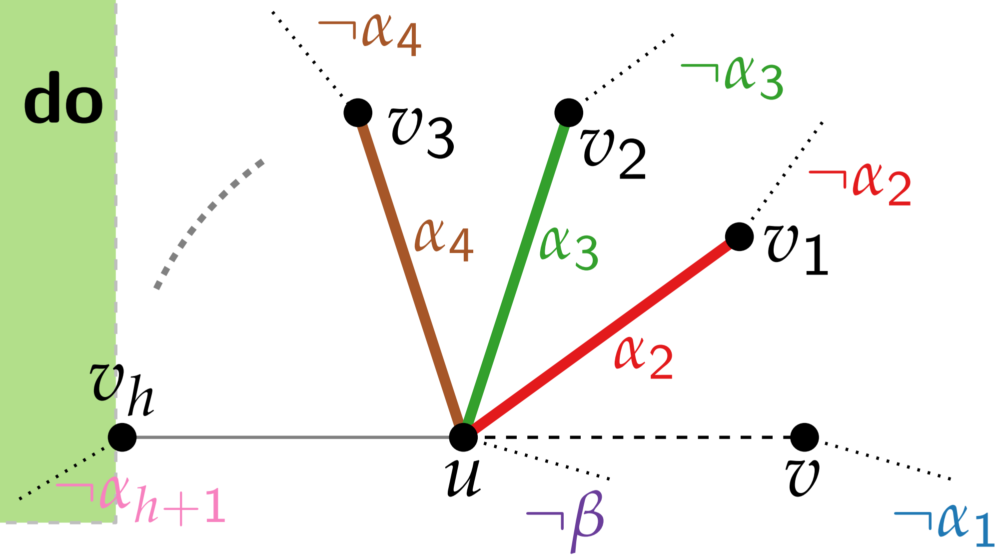

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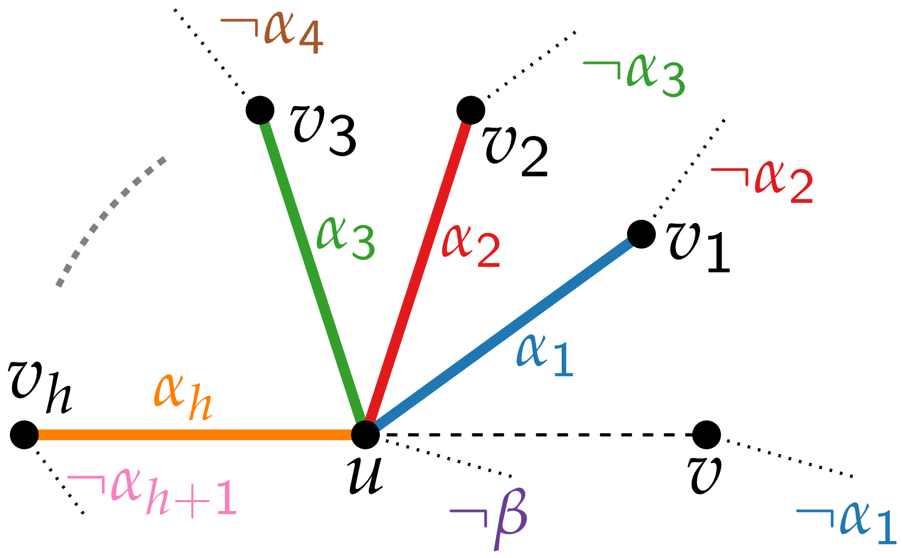

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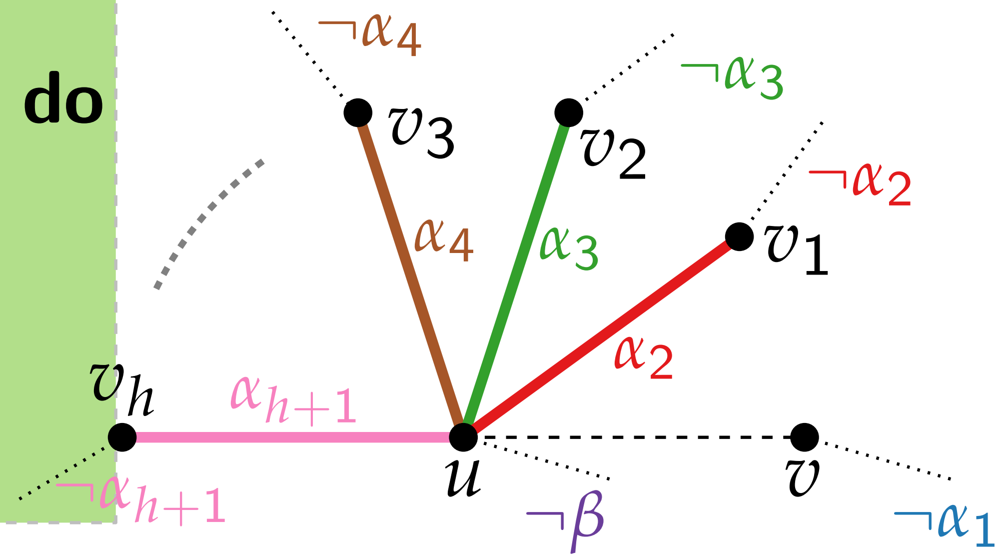

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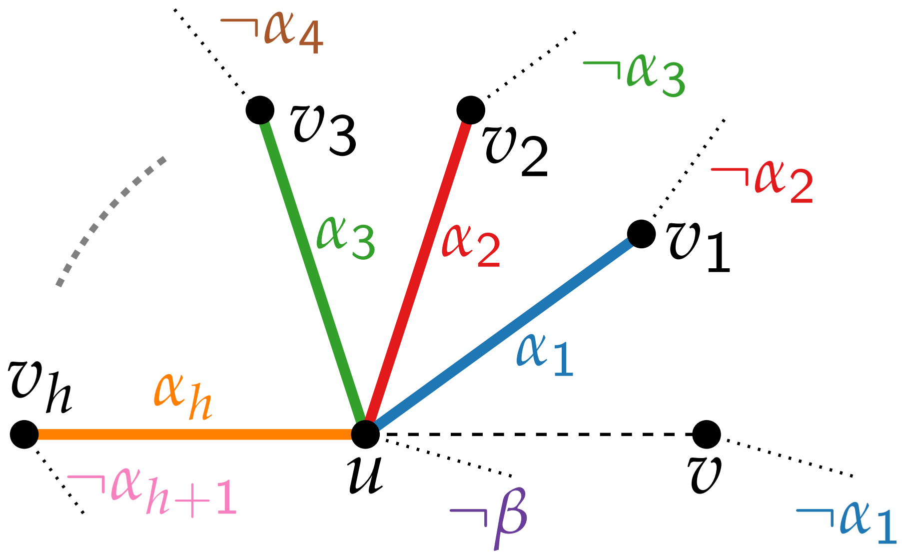

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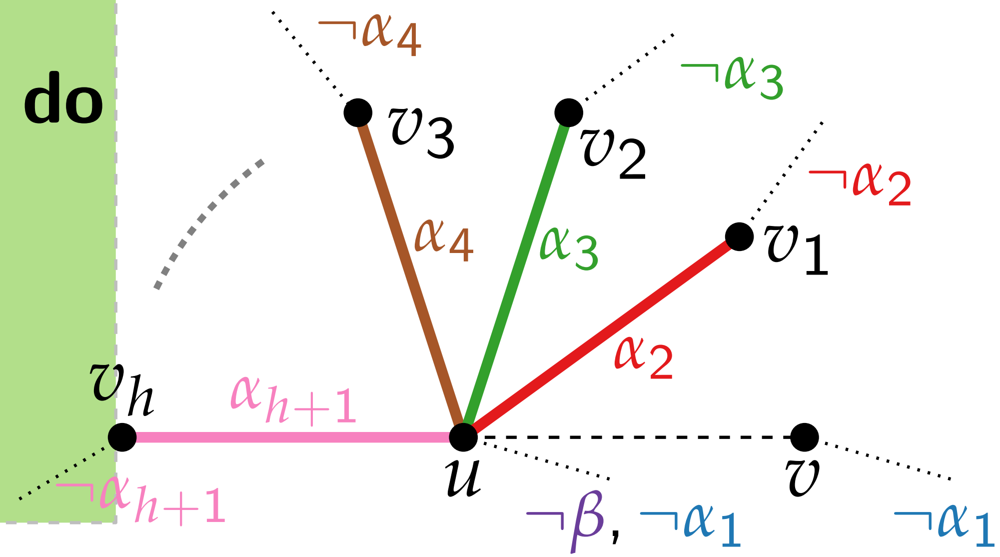

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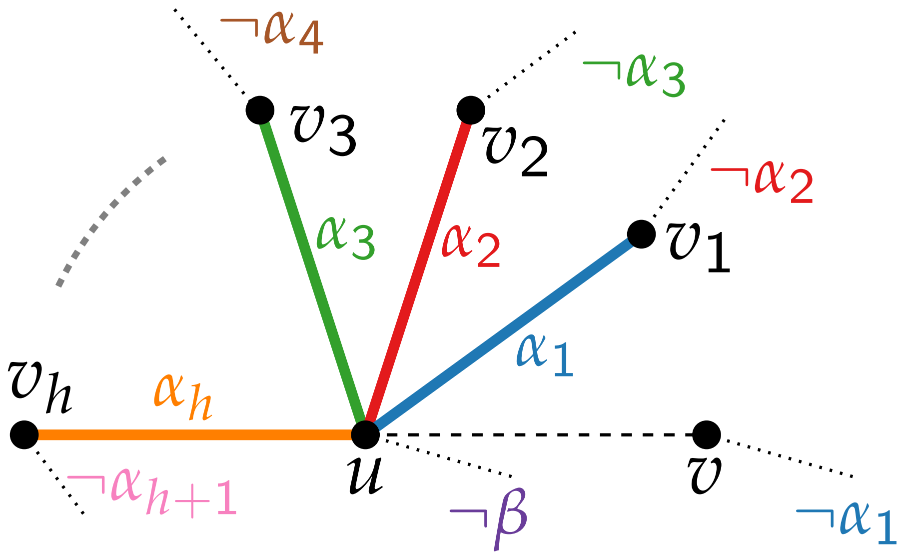

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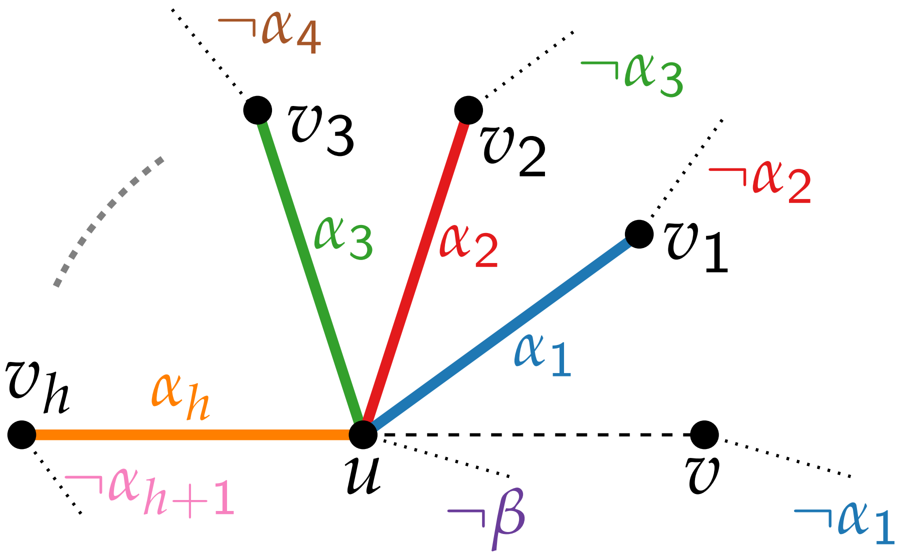

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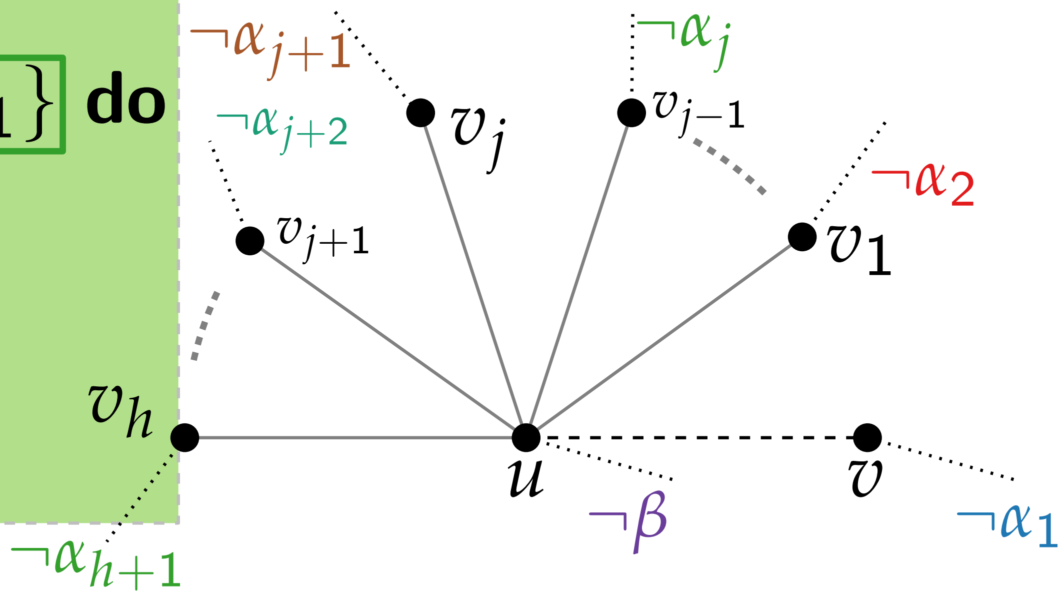

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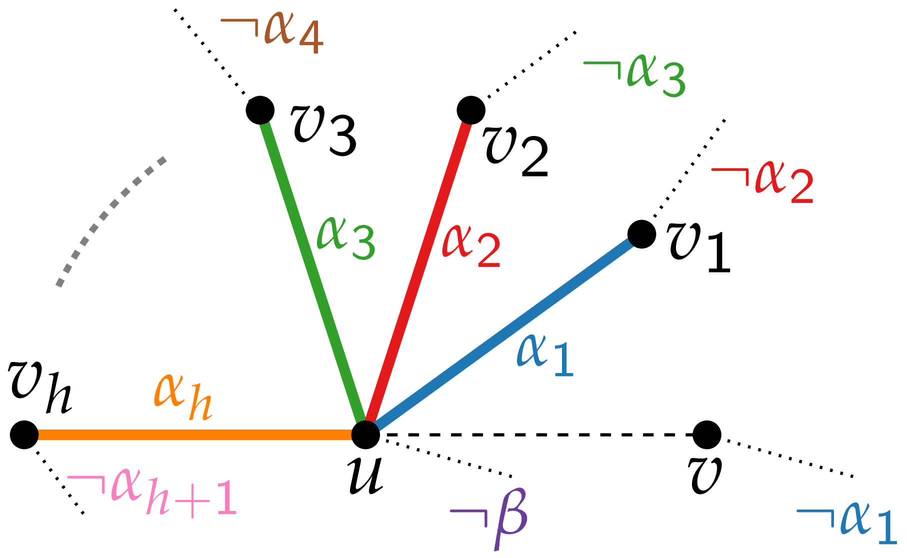

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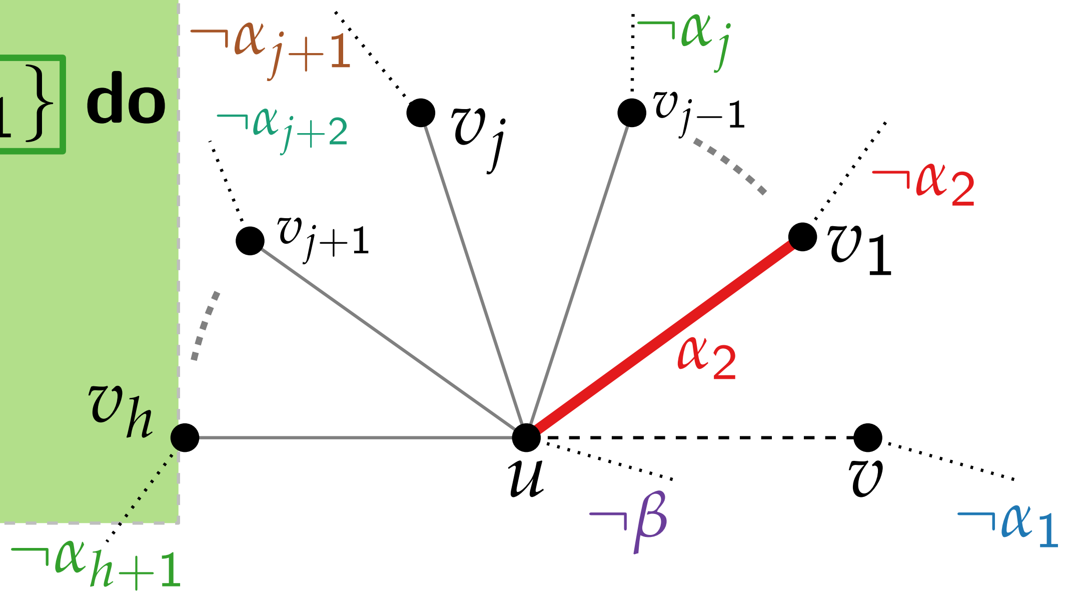
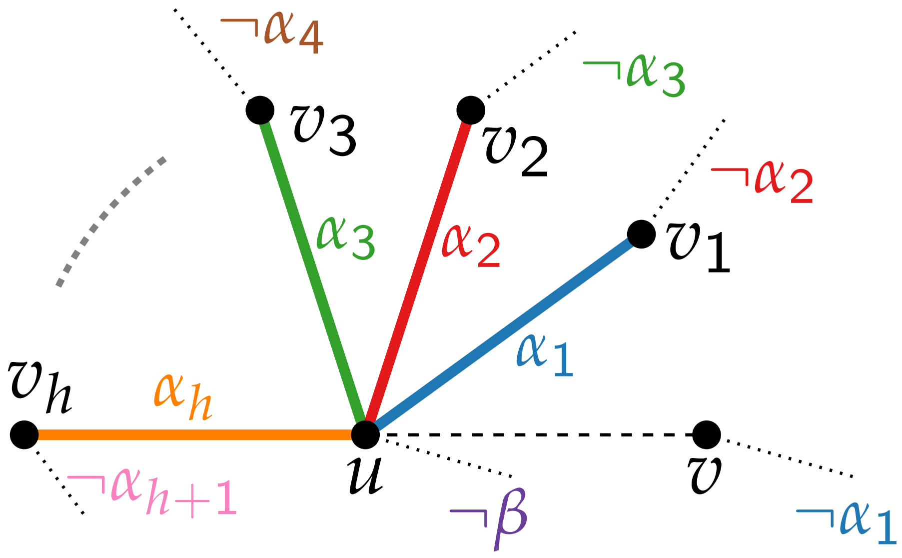


Minimum Edge Coloring – Recoloring

Lemma 2.
Let G be a graph with a $(\Delta+1)$-edge coloring c,
let u, v be non-adjacent vertices with $\deg(u), \deg(v) < \Delta$.
Then c can be changed s.t. u and v miss the same color.



Proof. Note that every vertex is **missing** a color.
Let u miss β and v miss α_1; apply the following algorithm:

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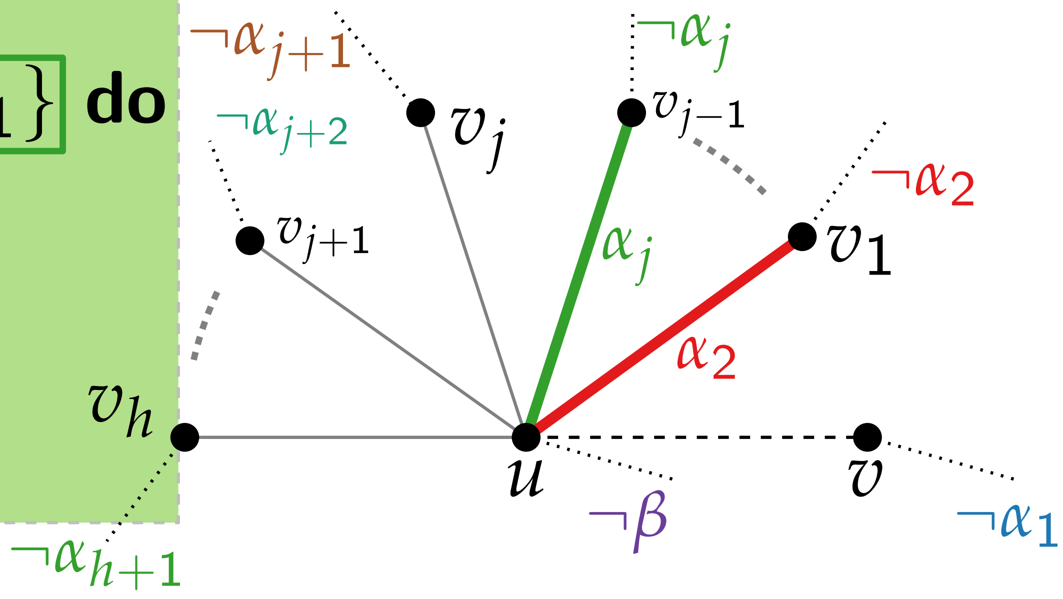

Minimum Edge Coloring – Recoloring

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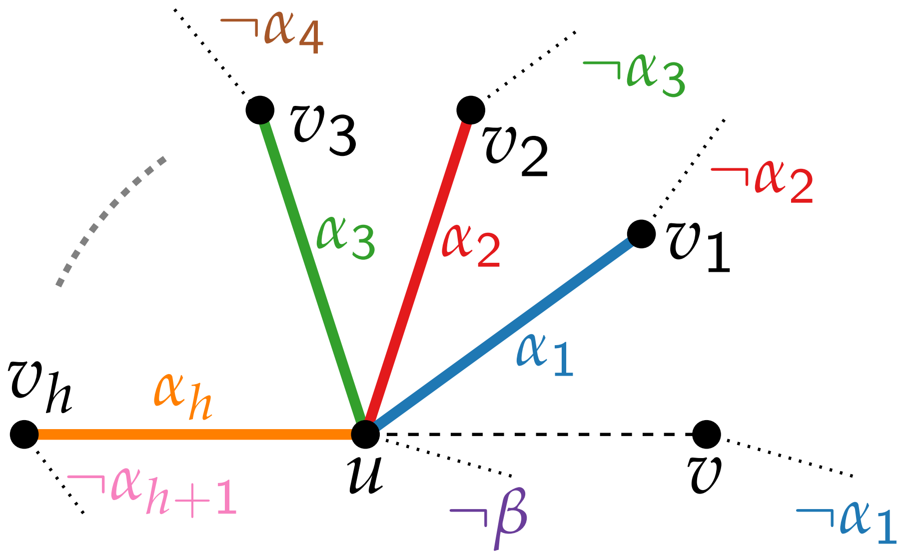

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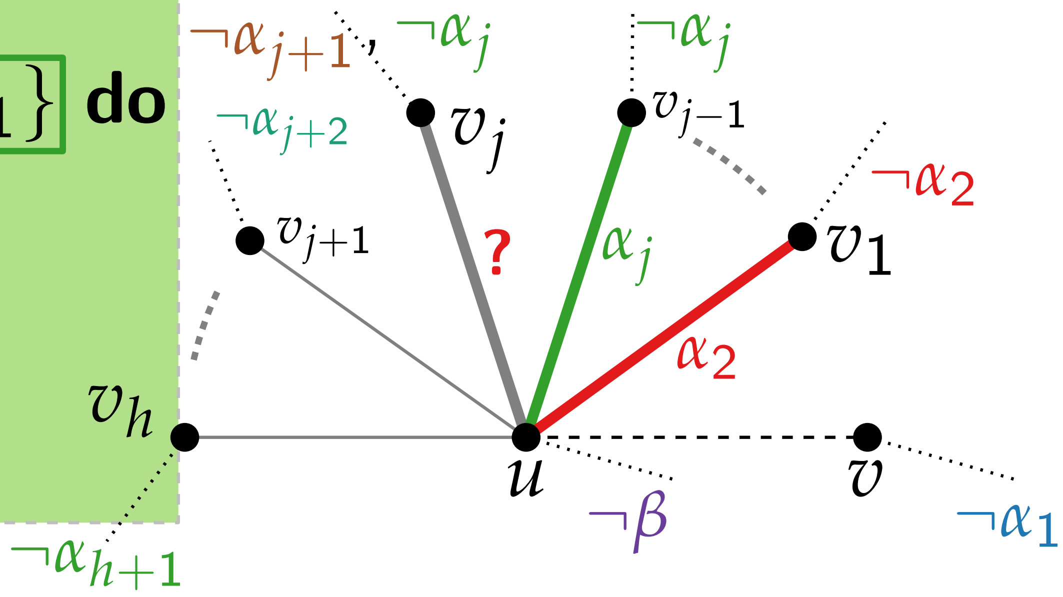

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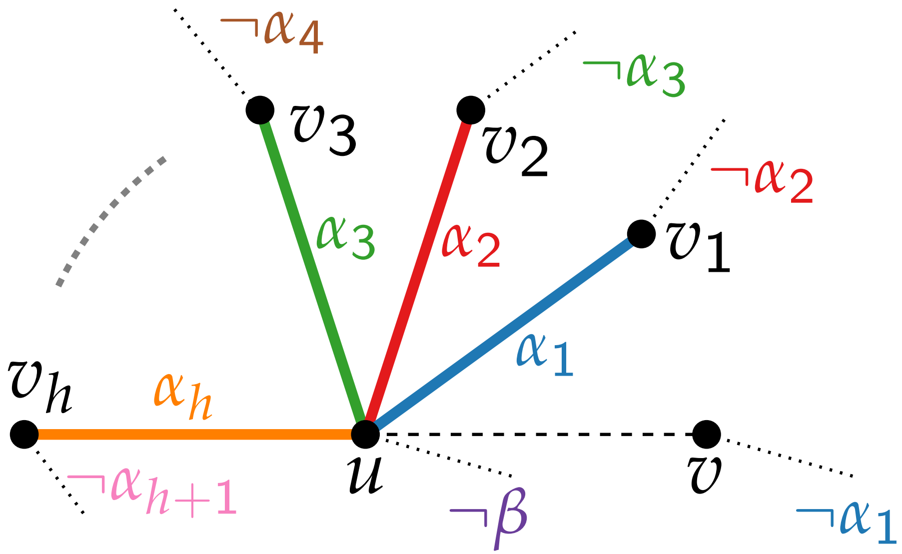

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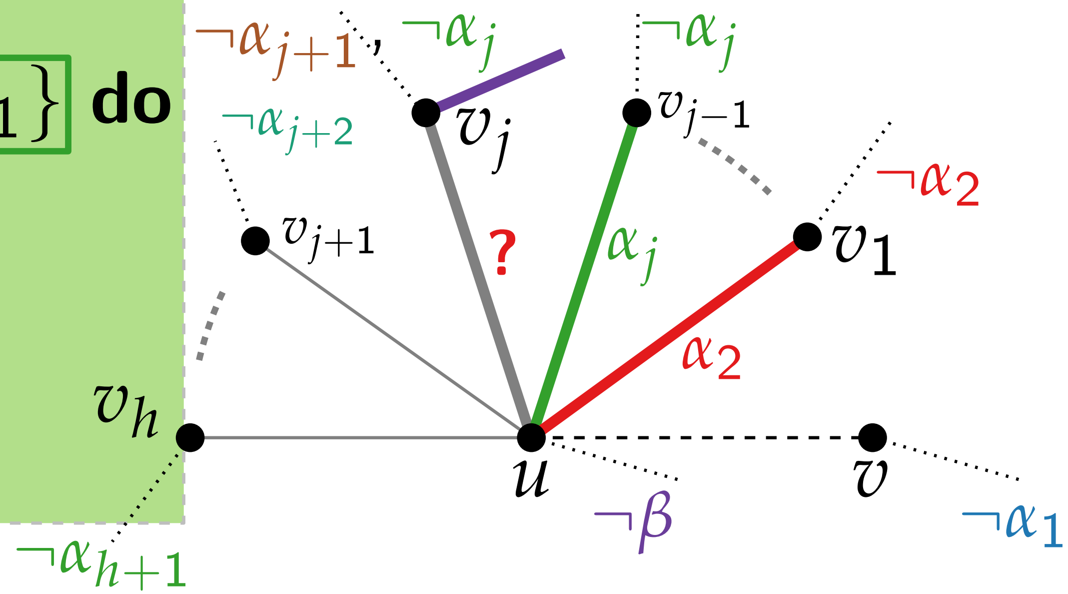

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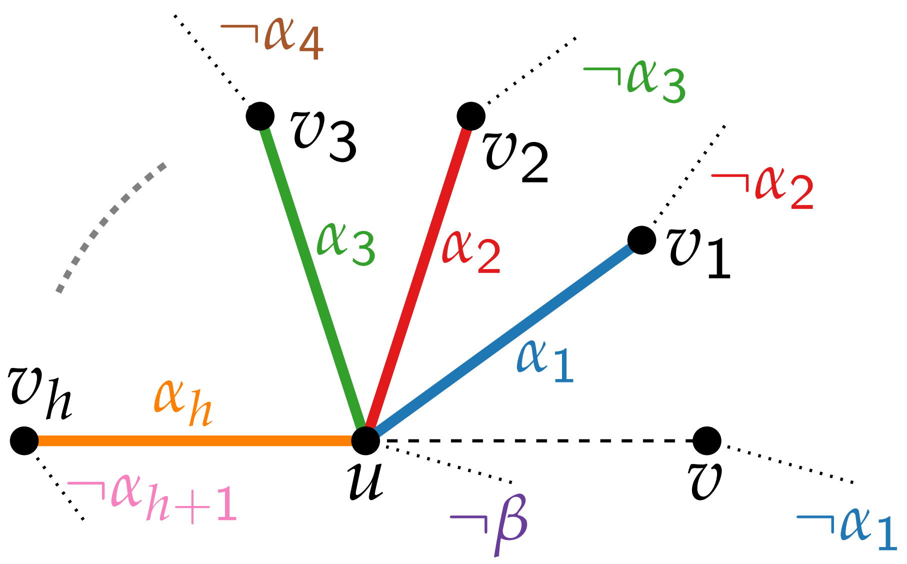

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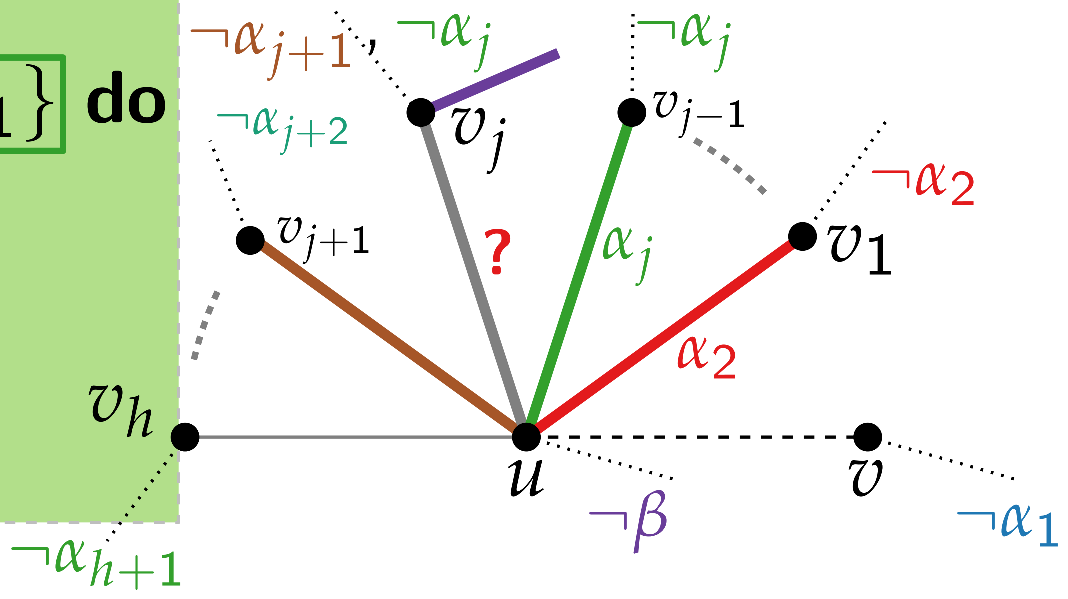

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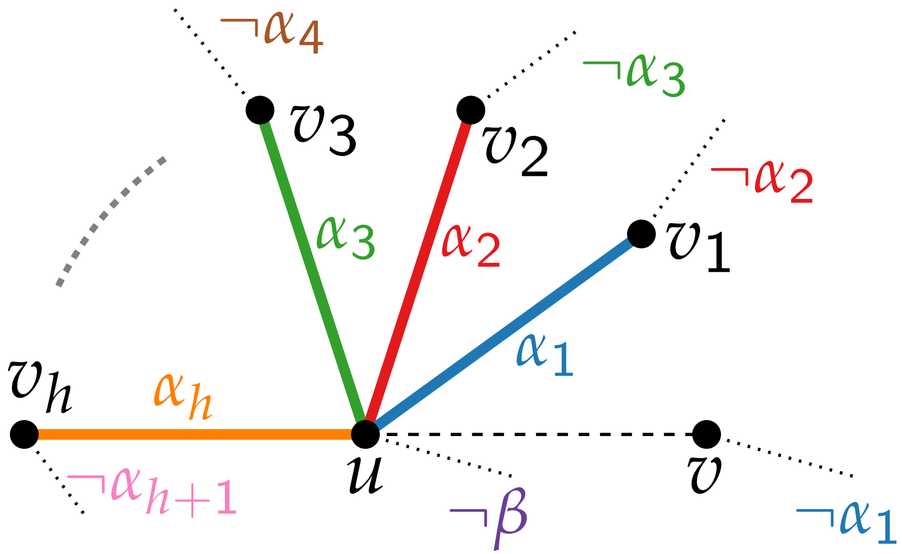

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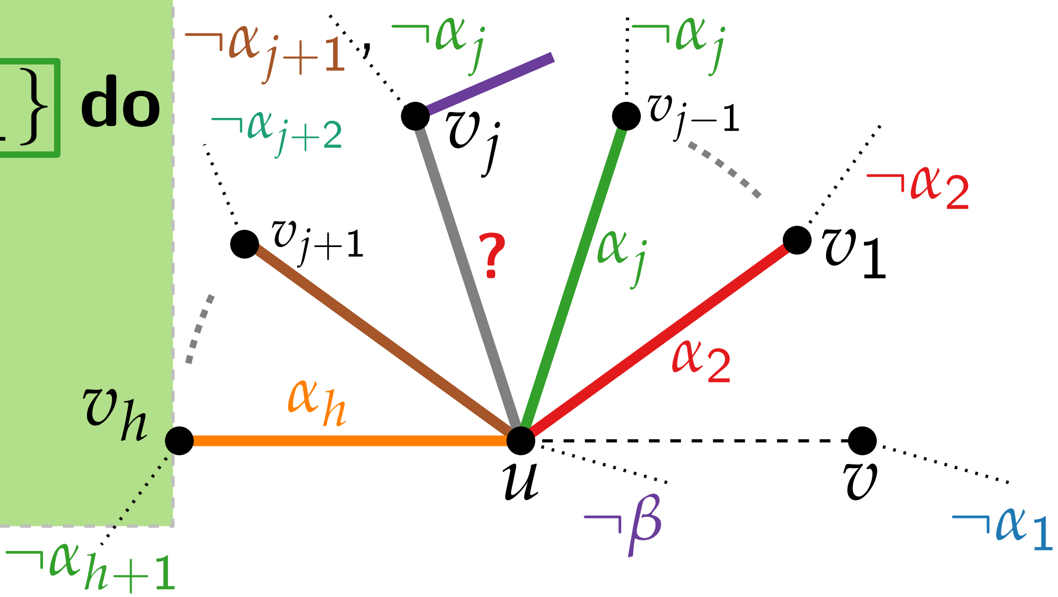

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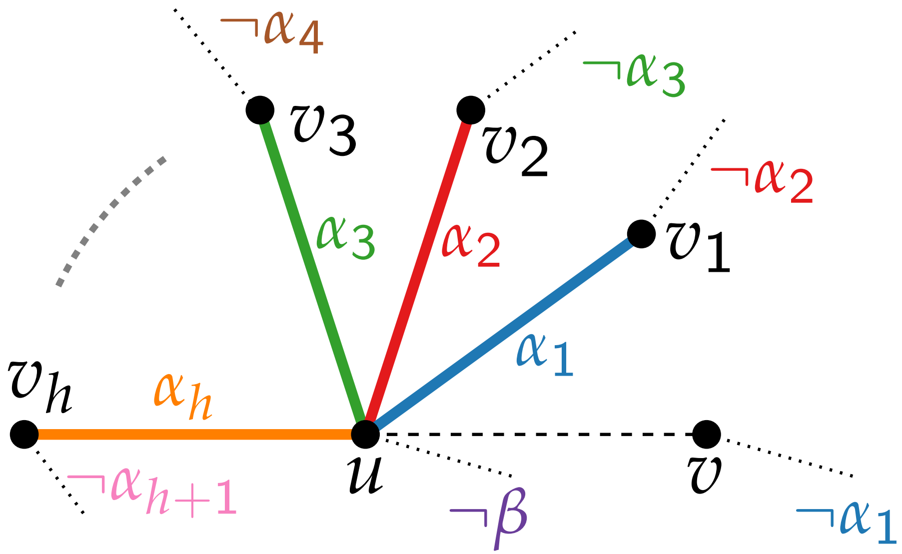

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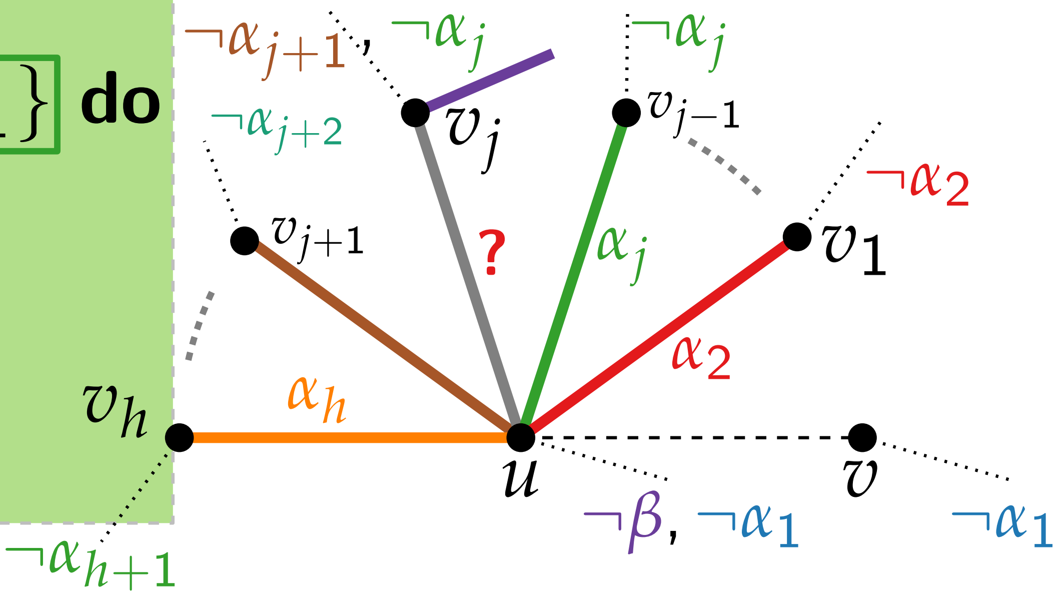

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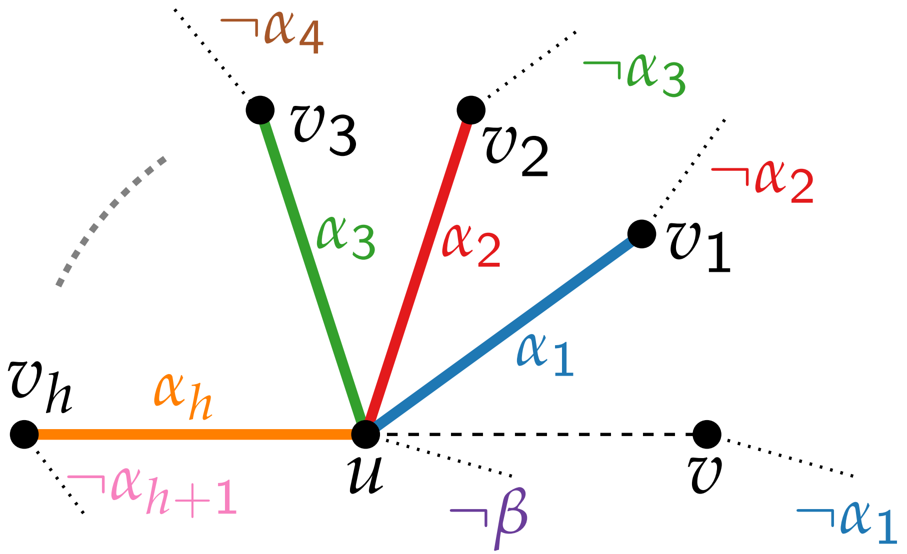

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Need color for edge uv_j!

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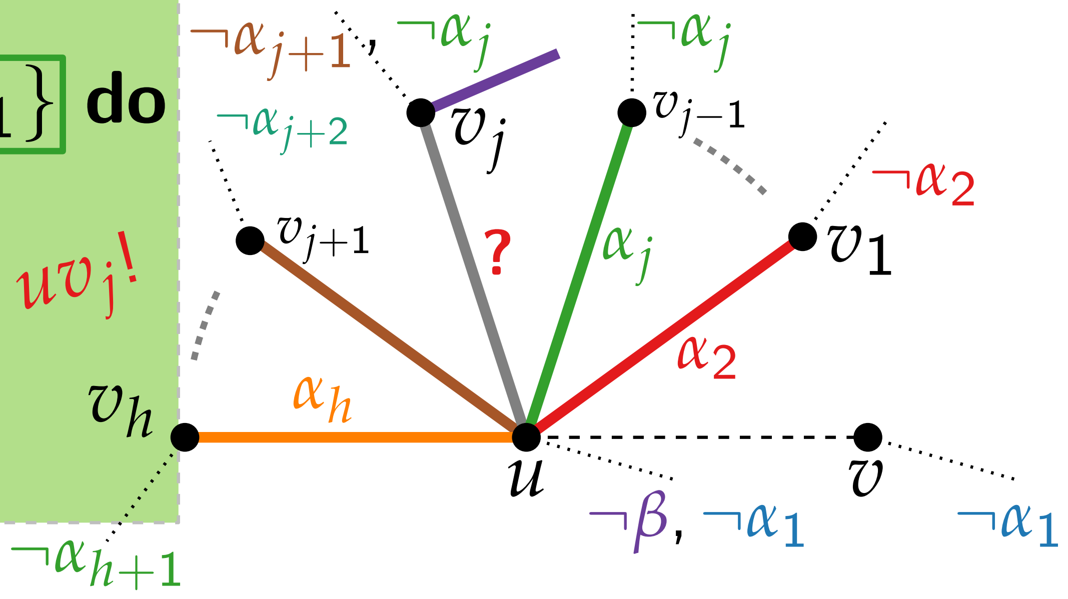

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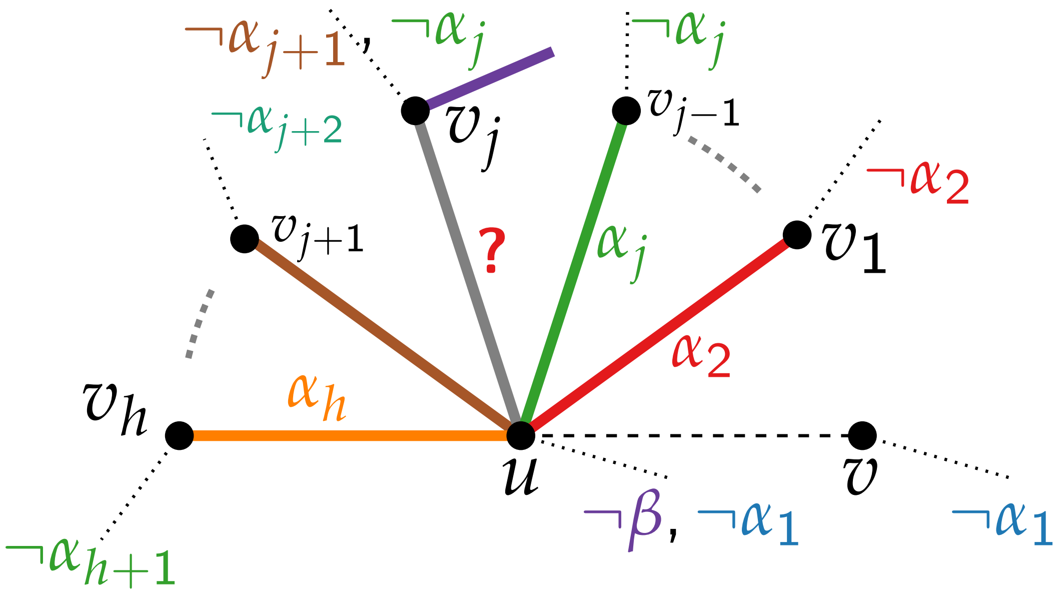

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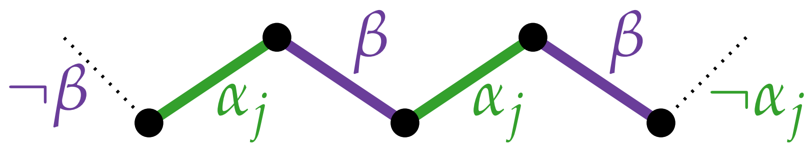

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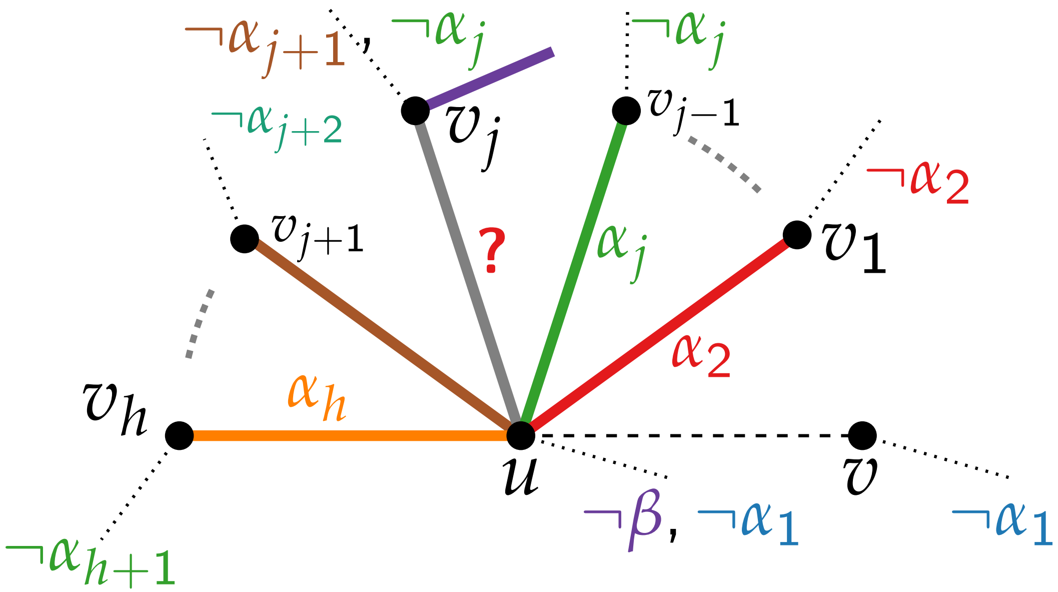

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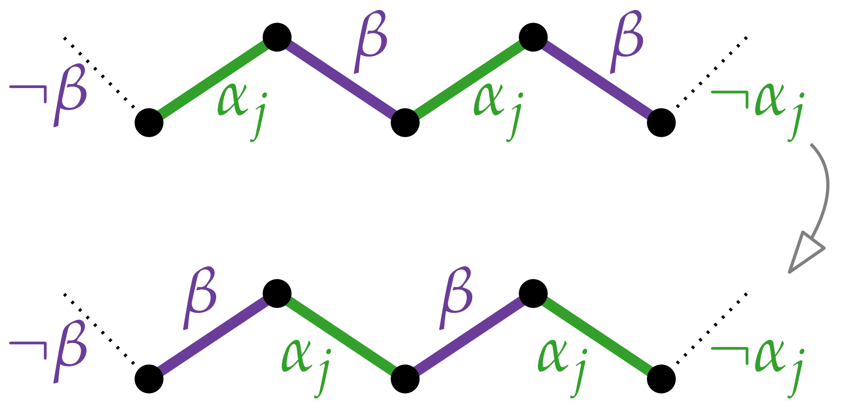

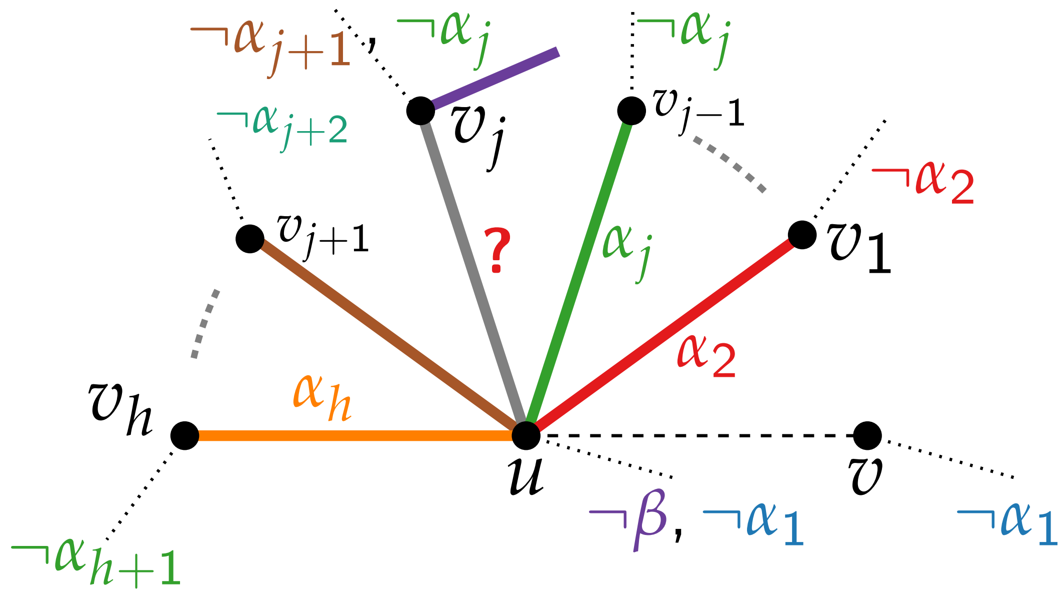

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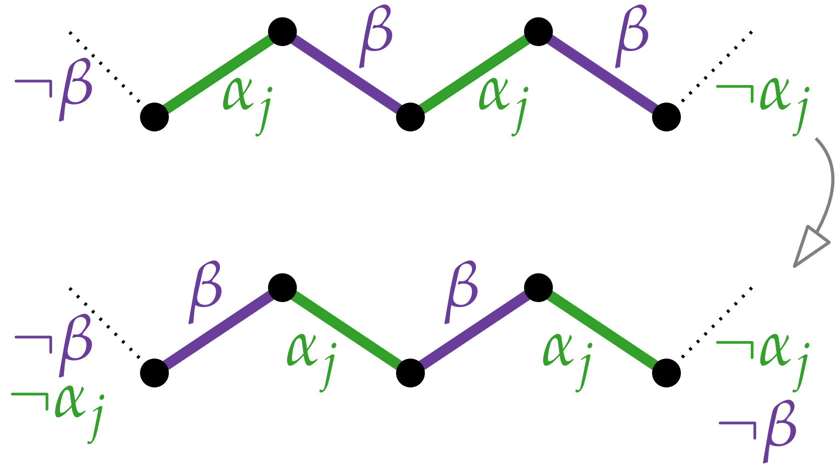

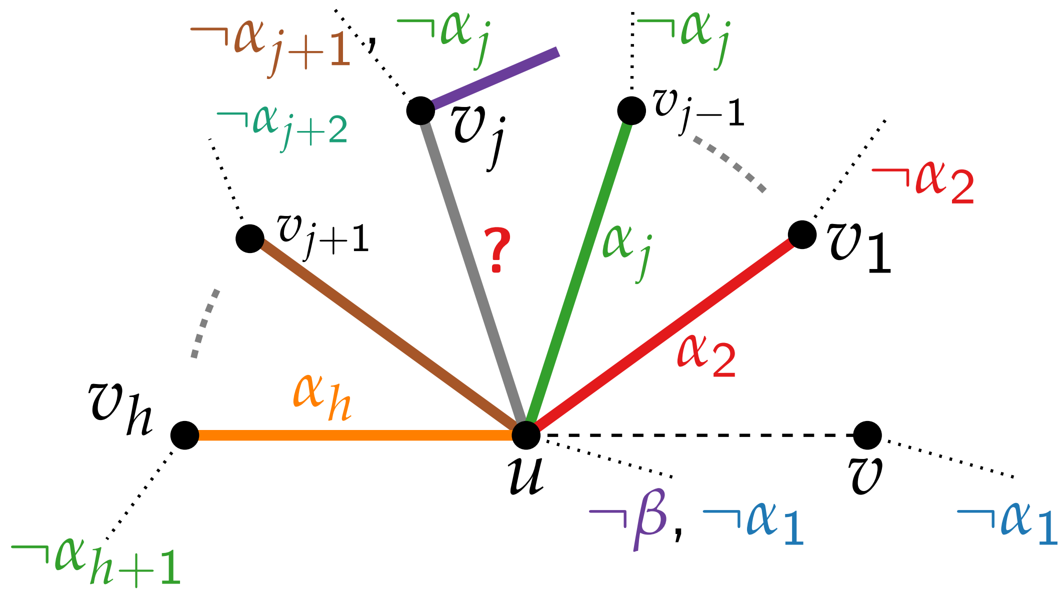

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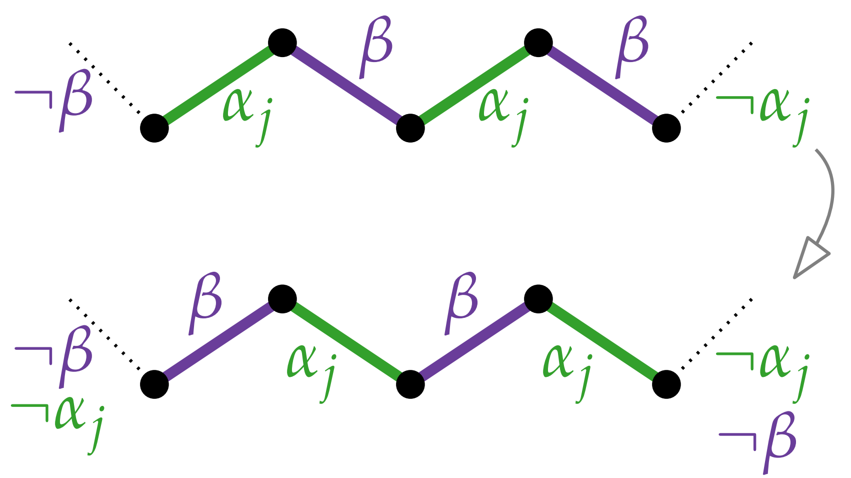

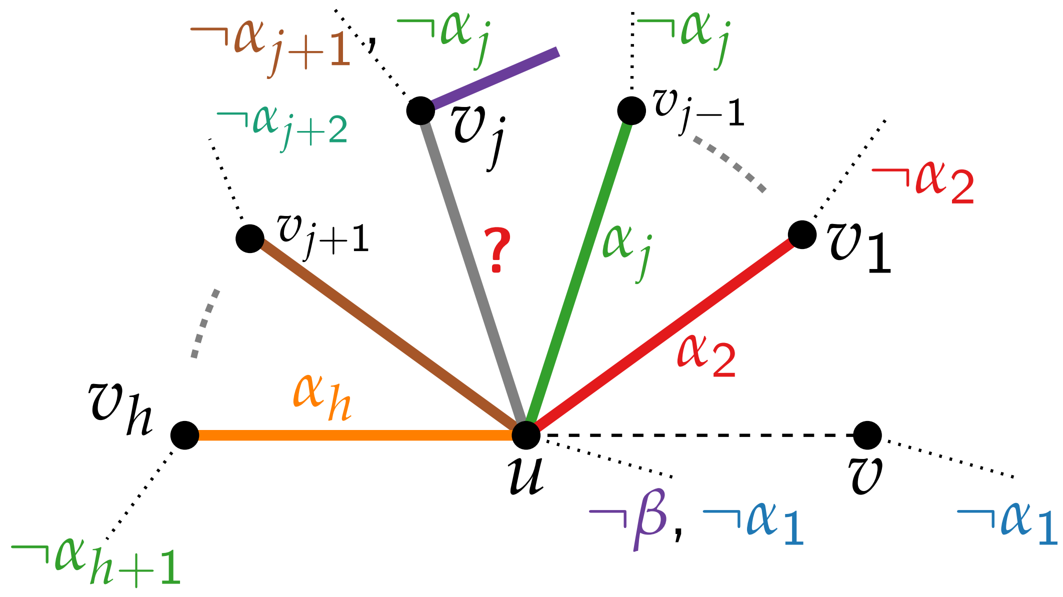

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- If u and v_j are not in the same component:
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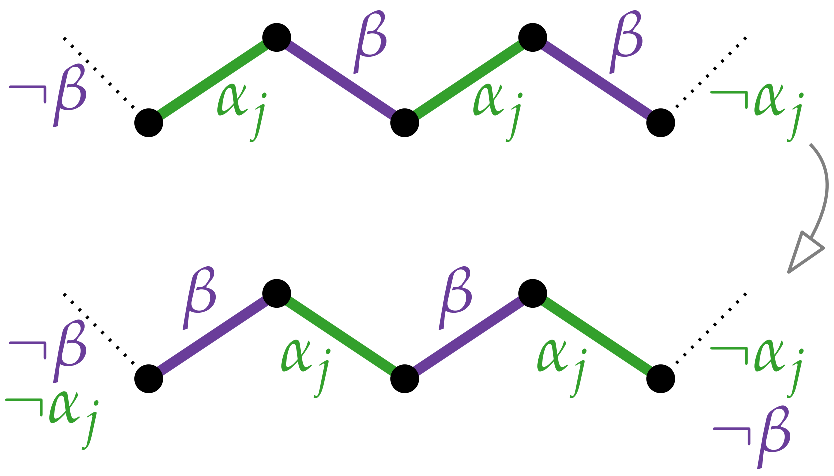

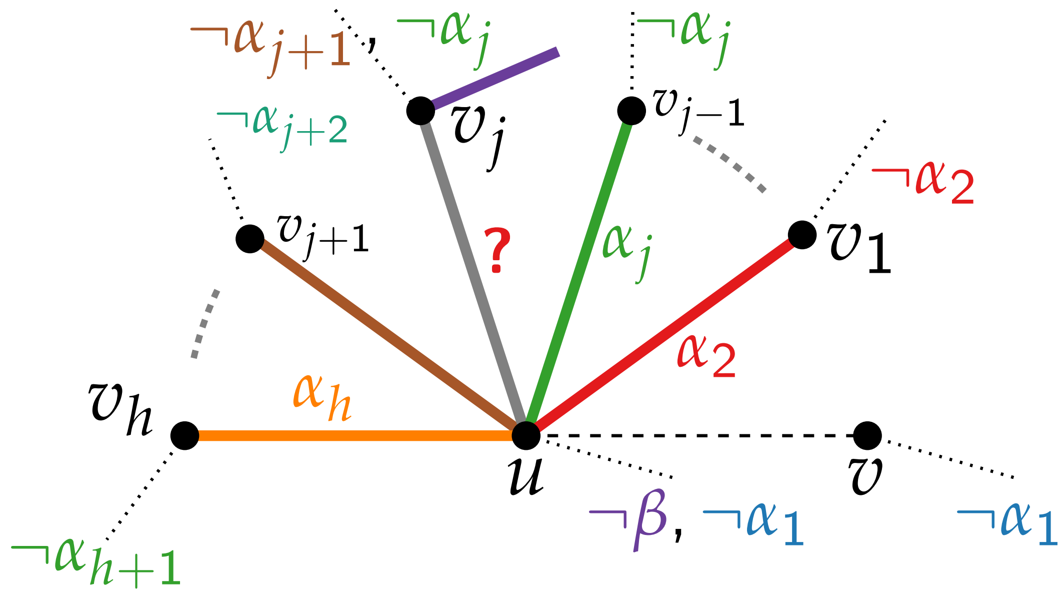

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- If u and v_j are not in the same component:
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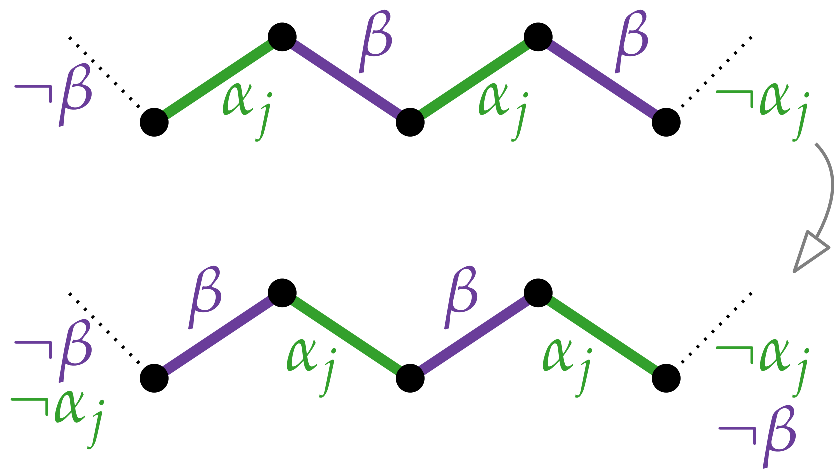

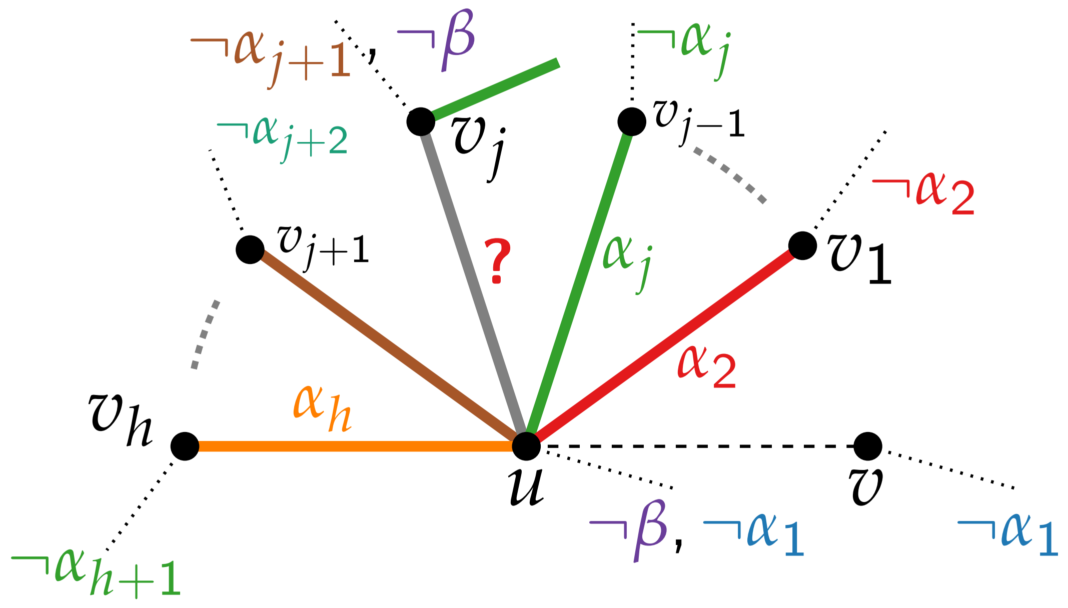

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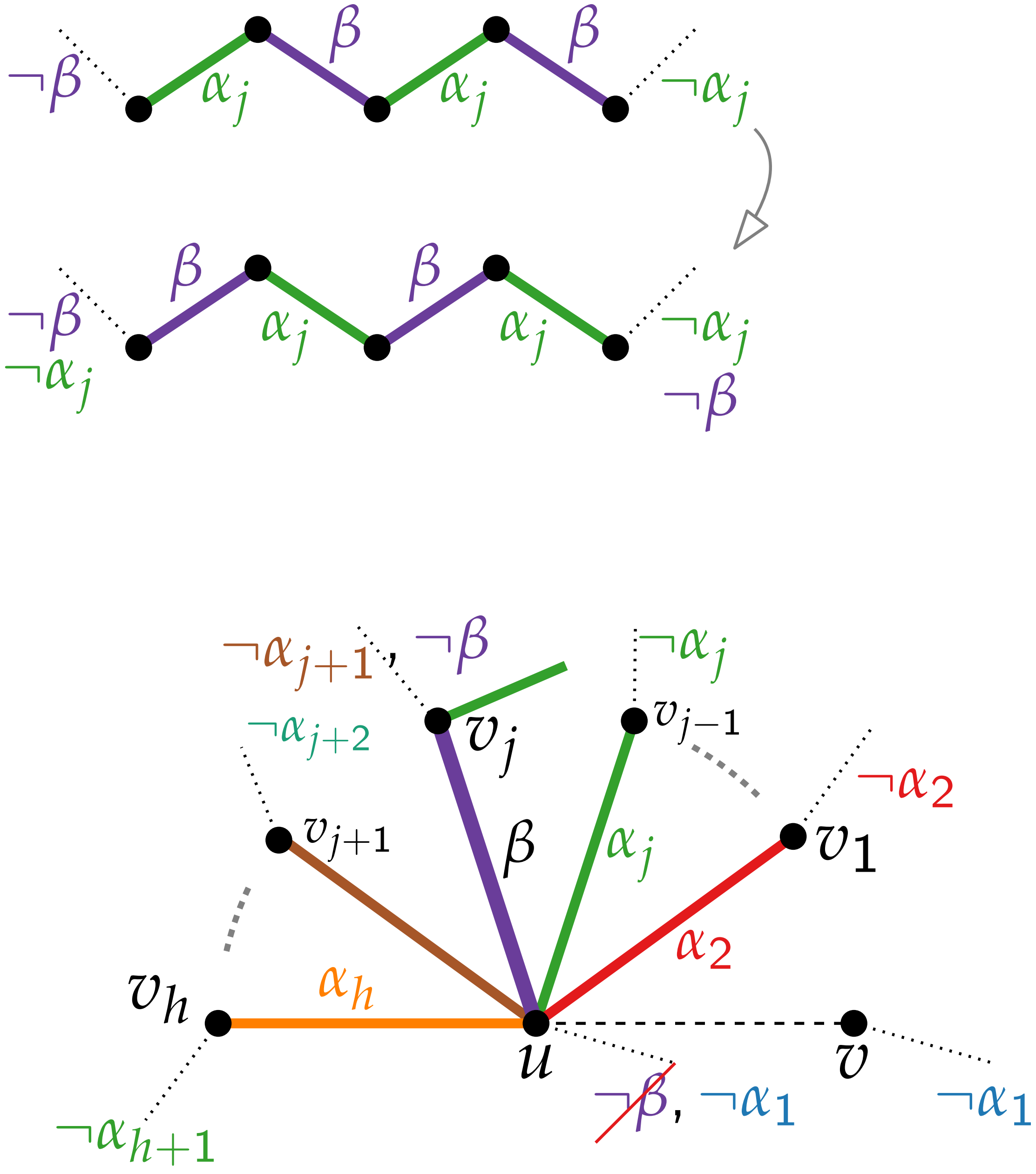

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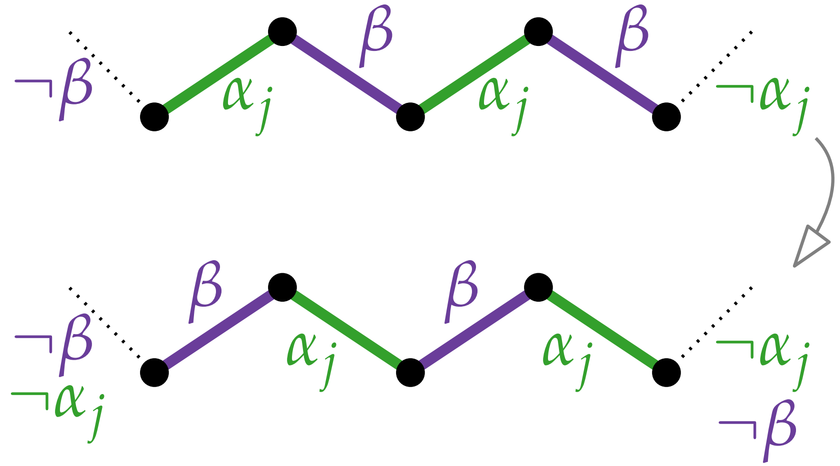

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- If u and v_j are not in the same component:
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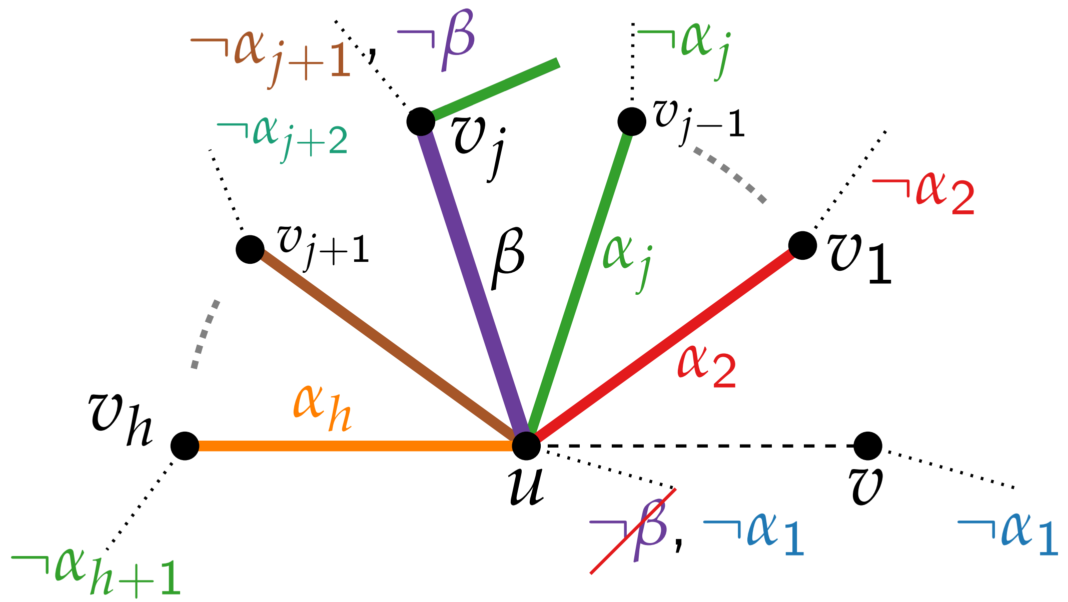
- What if u and v_j are in the same component?



Minimum Edge Coloring – Algorithm

`VizingEdgeColoring`(graph G, coloring $c \equiv 0$)

if $E(G) \neq \emptyset$ **then**
- Let $e = uv$ be an arbitrary edge of G.
- $G_e \leftarrow G - e$
- `VizingEdgeColoring`(G_e, c)
- **if** $\Delta(G_e) < \Delta(G)$ **then**
  - Color e with lowest free color.
- **else**
  - Recolor G_e as in Lemma 2.
  - Color e with color now missing at u and v.

Minimum Edge Coloring – Algorithm

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VizingEdgeColoring(graph G, coloring c ≡ 0)
  if E(G) ≠ ∅ then
      Let e = uv be an arbitrary edge of G.
      G_e ← G − e
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      if Δ(G_e) < Δ(G) then
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Theorem 4.
VizingEdgeColoring is an approximation algorithm with additive approximation guarantee $\mathrm{ALG}(G) - \mathrm{OPT}(G) \le 1$.

Approximation with Relative Factor

- An additive approximation guarantee can rarely be achieved; but sometimes, there is a multiplicative approximation!

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Let Π be a minimization problem, and let $\alpha \in \mathbb{Q}^+$.
A **factor-α approximation algorithm** for Π is a polynomial-time algorithm $\mathcal{A}$ that computes, for every instance I of Π, a solution of value $\mathsf{ALG}(I)$ such that

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We call α the **approximation factor** of $\mathcal{A}$.

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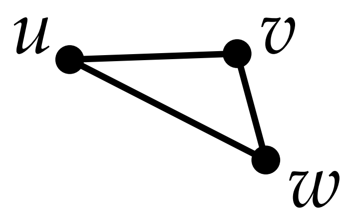
Input. Complete graph $G = (V, E)$ and a distance function $d \colon E \to \mathbb{R}_{\geq 0}$ that satisfies the triangle inequality, i.e., $\forall u, v, w \in V \colon d(u, w) \leq d(u, v) + d(v, w)$.



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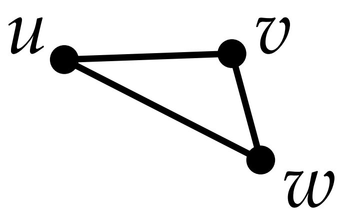

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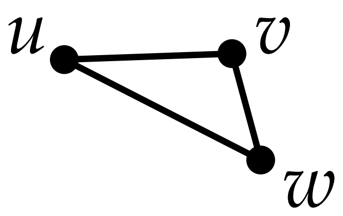

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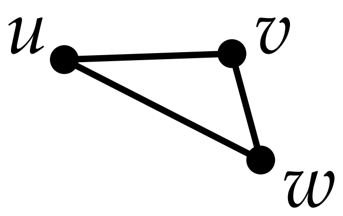

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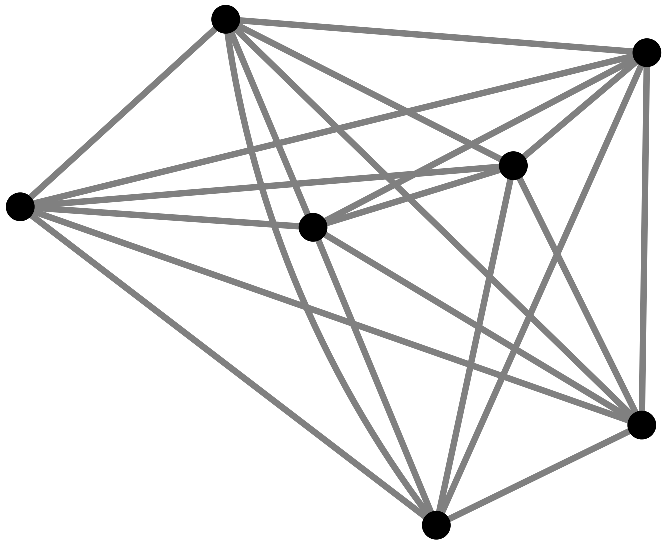

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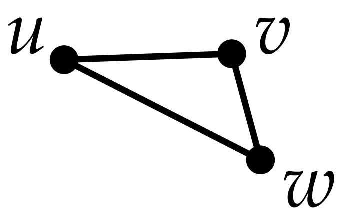

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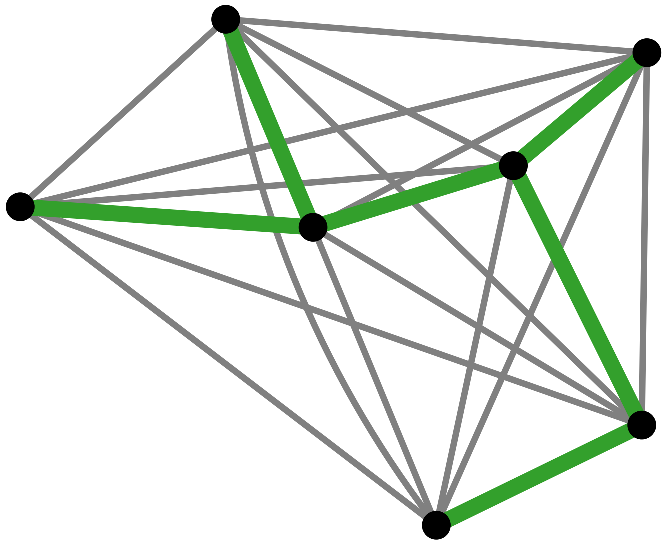
- Compute MST.



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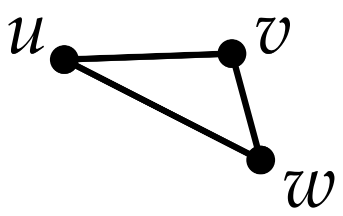

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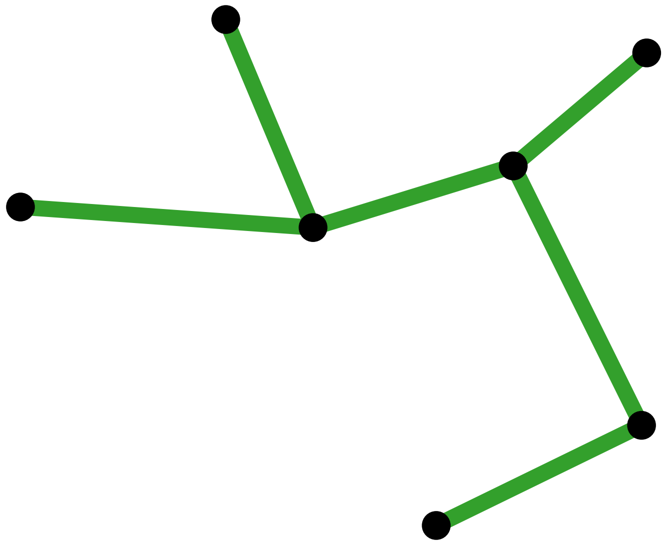

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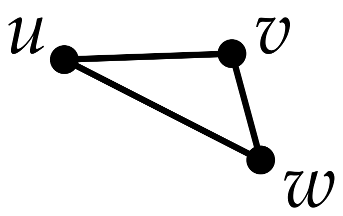

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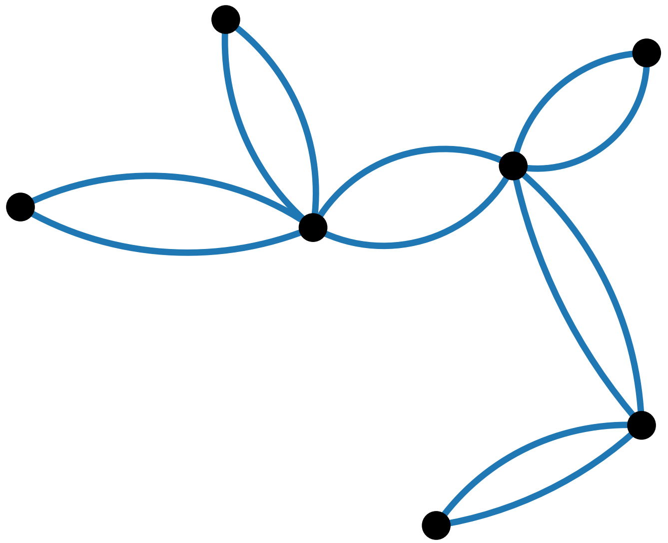

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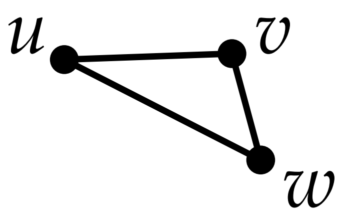

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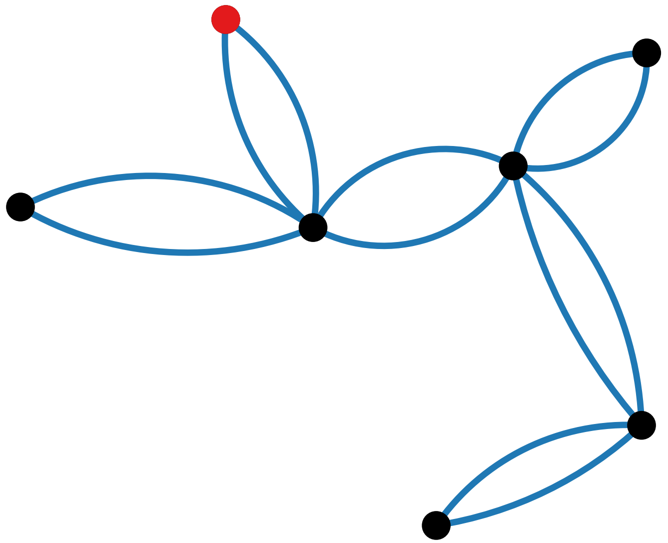

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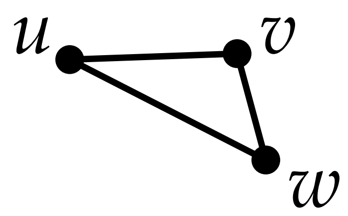

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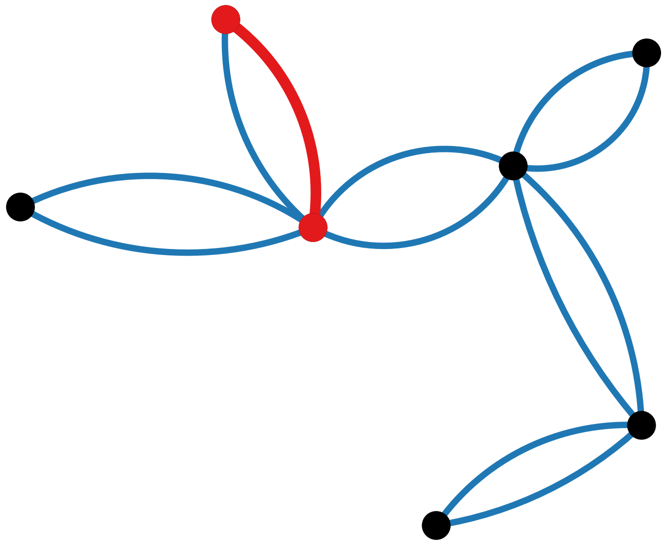

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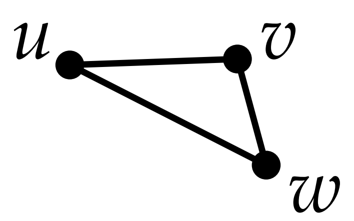

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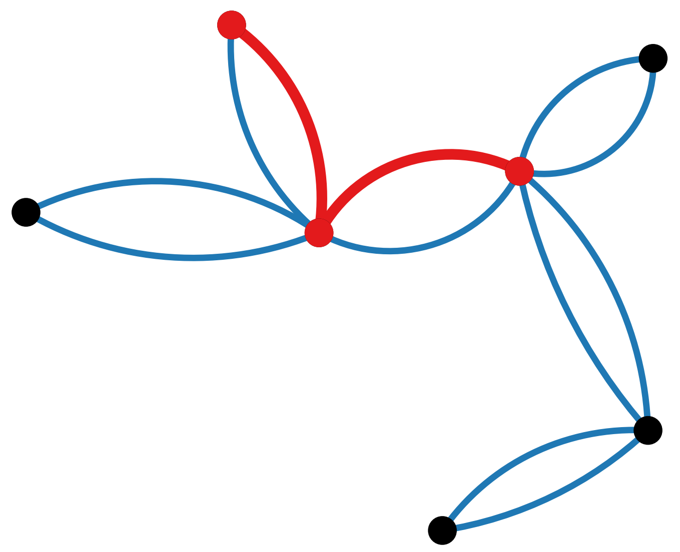

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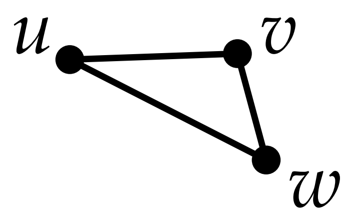

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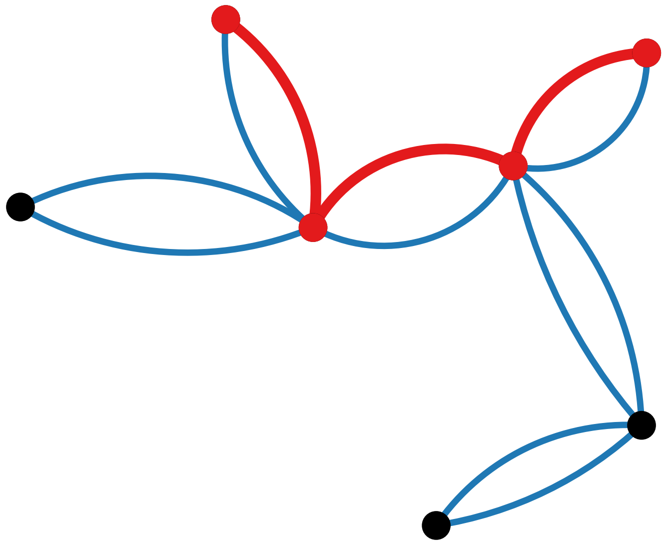

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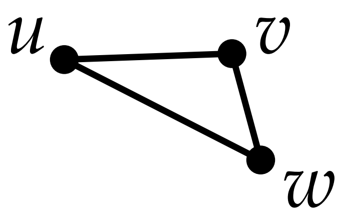

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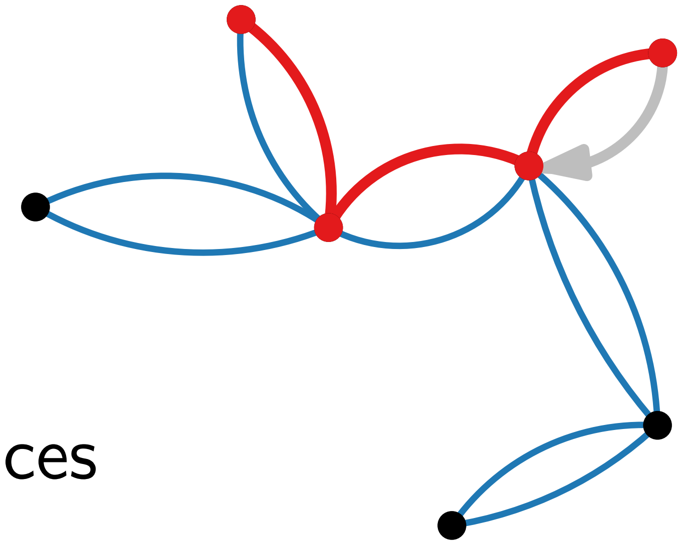

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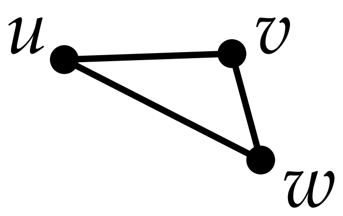

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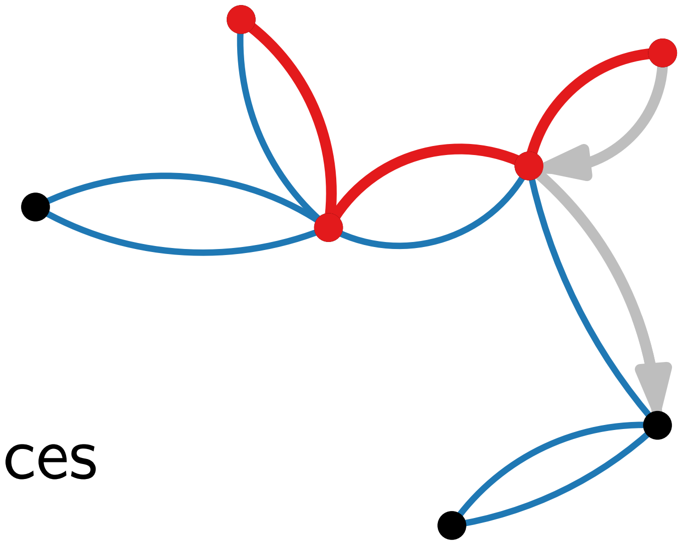

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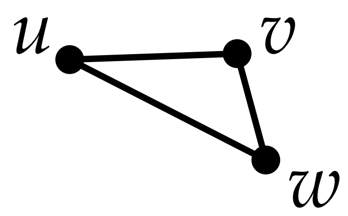

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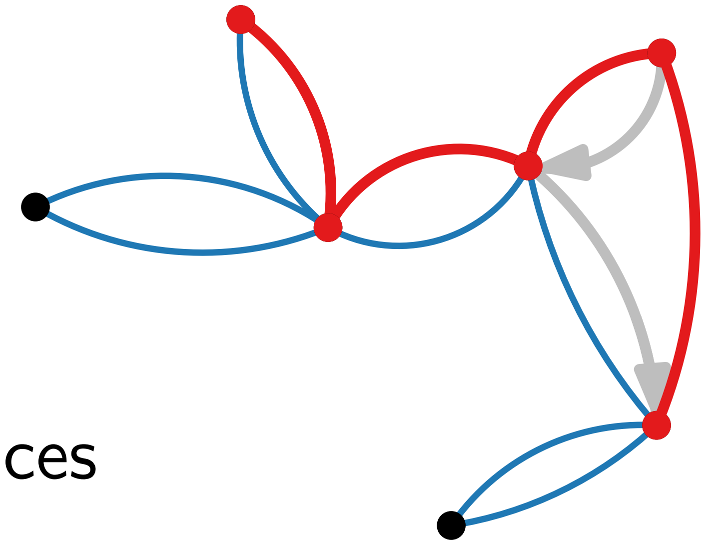

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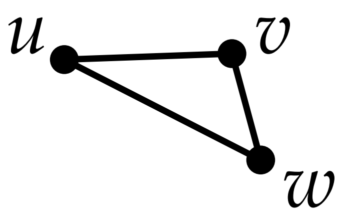

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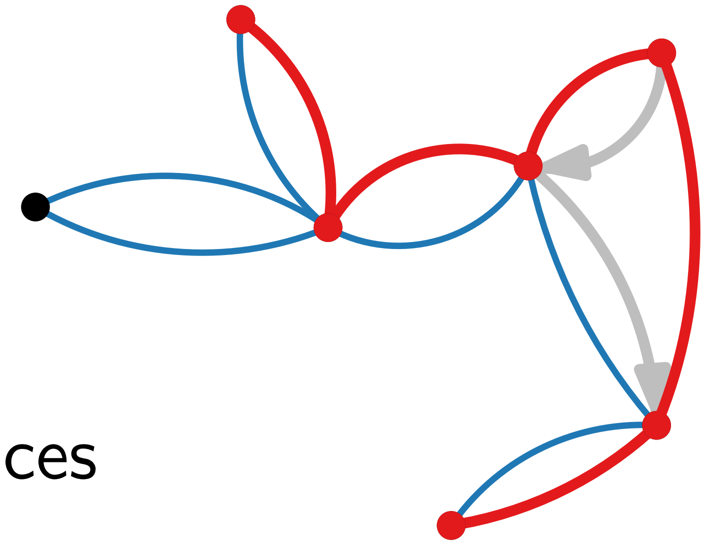

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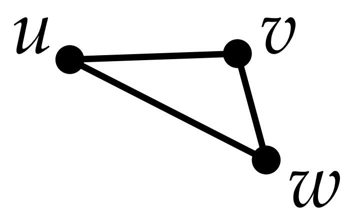

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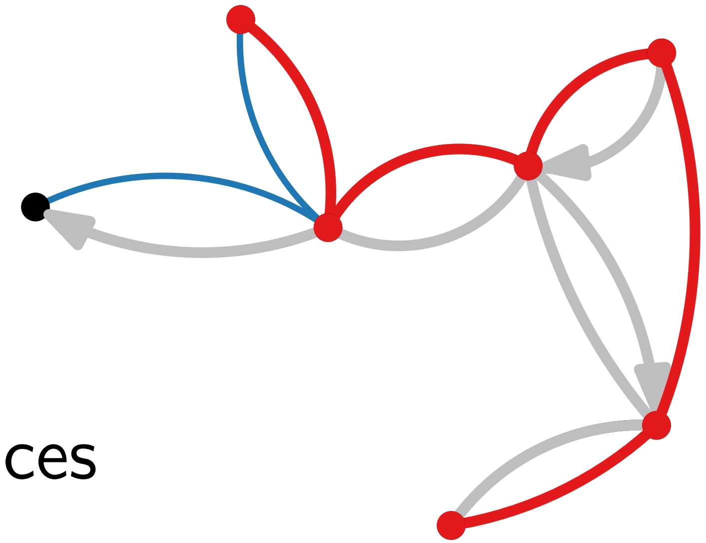

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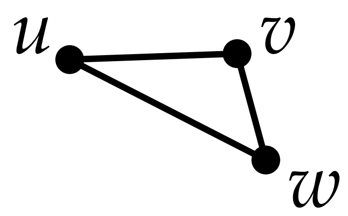

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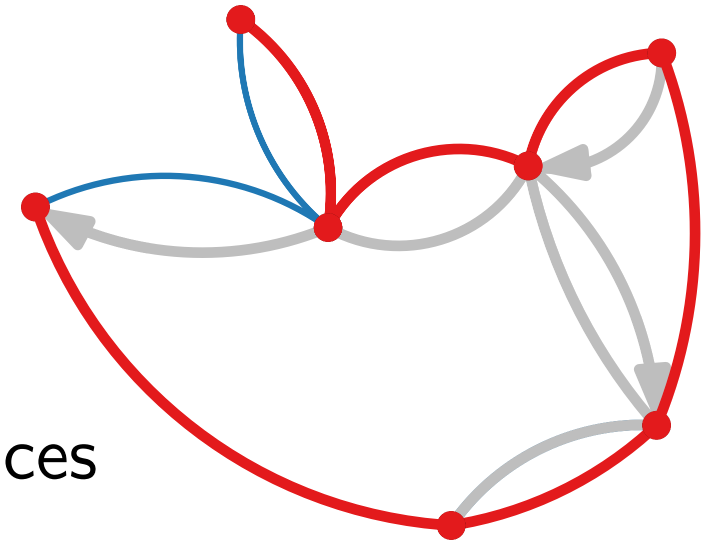

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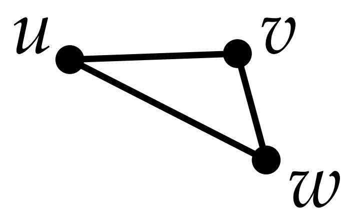

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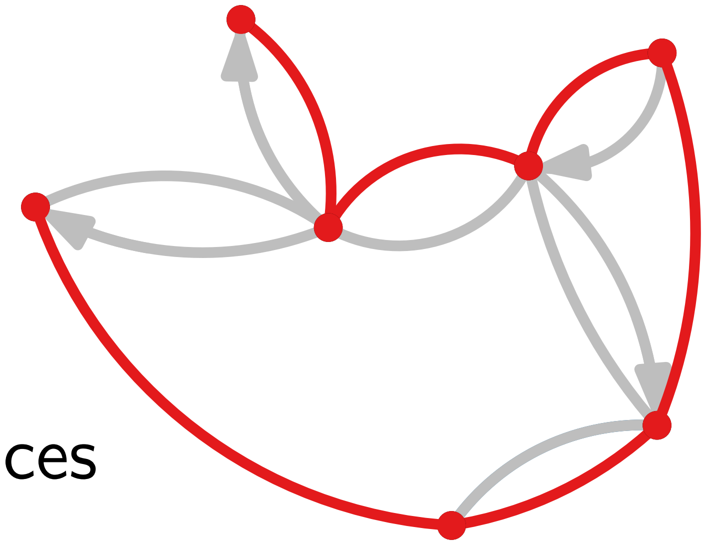

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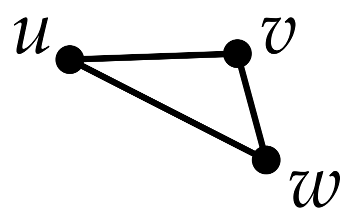

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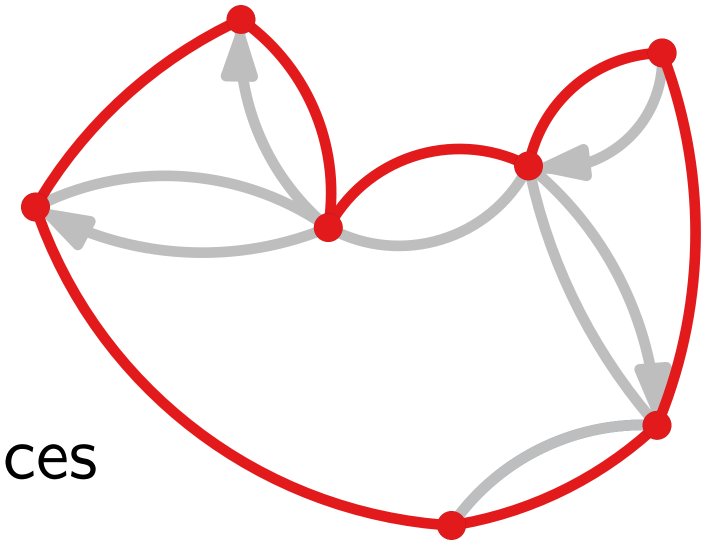

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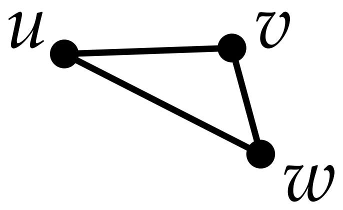

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The MST edge doubling algorithm is a 2-approximation algorithm for metric TSP.

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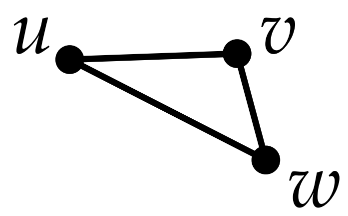

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Proof.
$\text{ALG} \leq d(\text{cycle}) = 2d(\text{MST}) \leq 2\text{OPT}$.

Nearest Addition Algorithm for Metric TSP

`NearestAdditionAlgorithm`$(G = (V, E), d)$

Find closest pair, say i and k.
Set tour T to go from i to k to i (clockwise).
while $T \subsetneq V$ **do**
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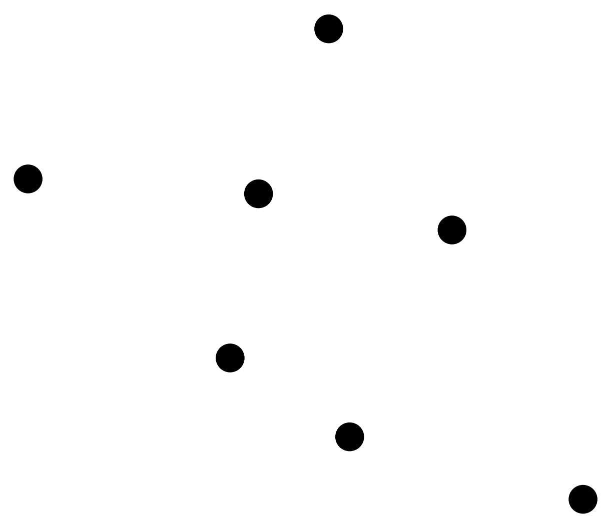

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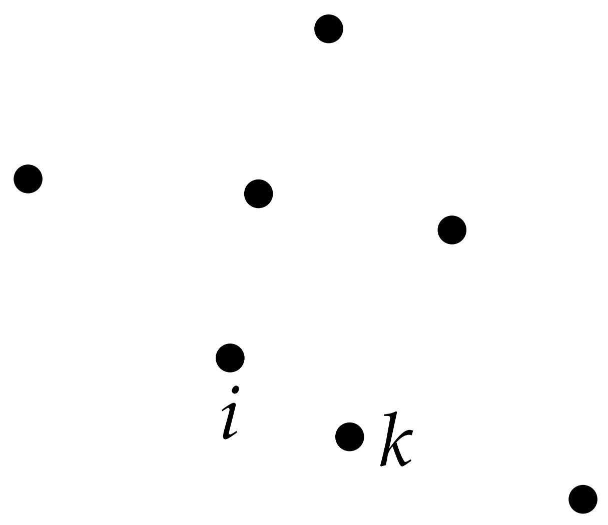

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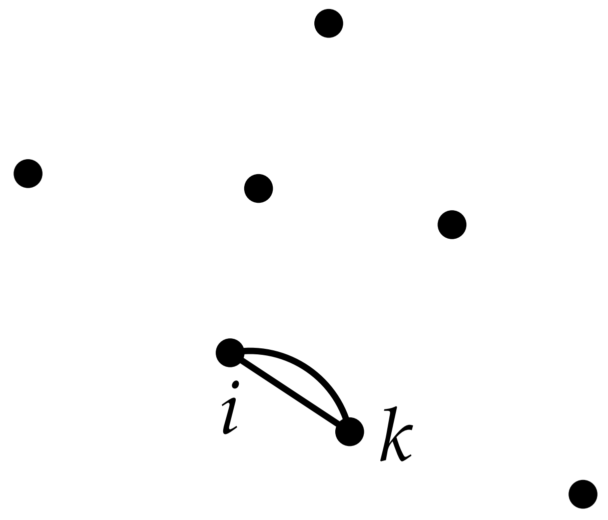

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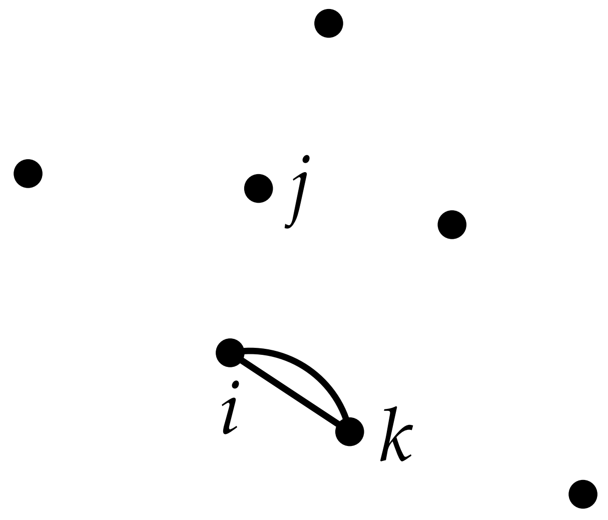

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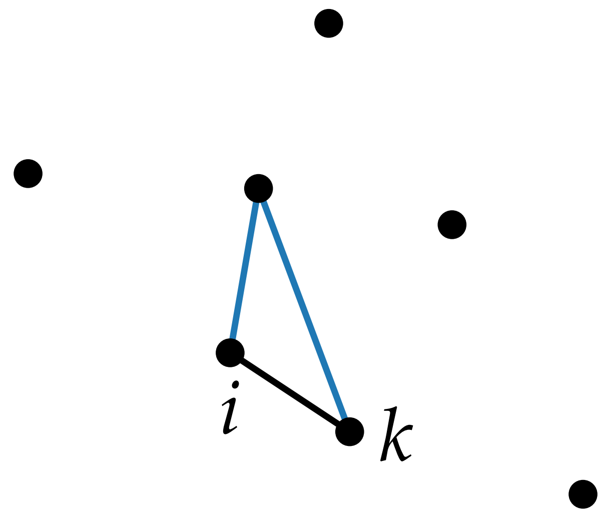

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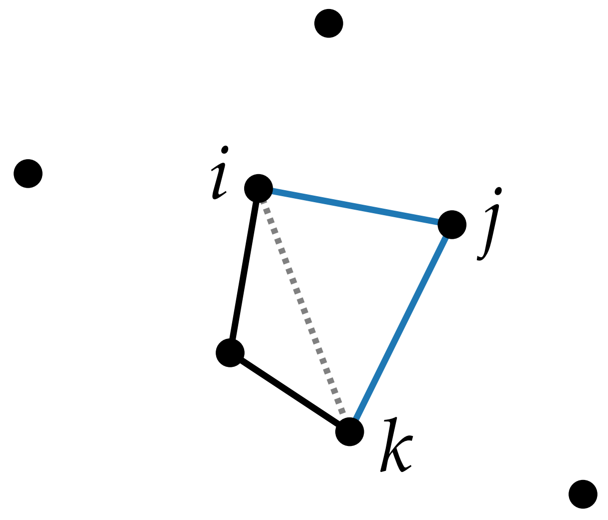

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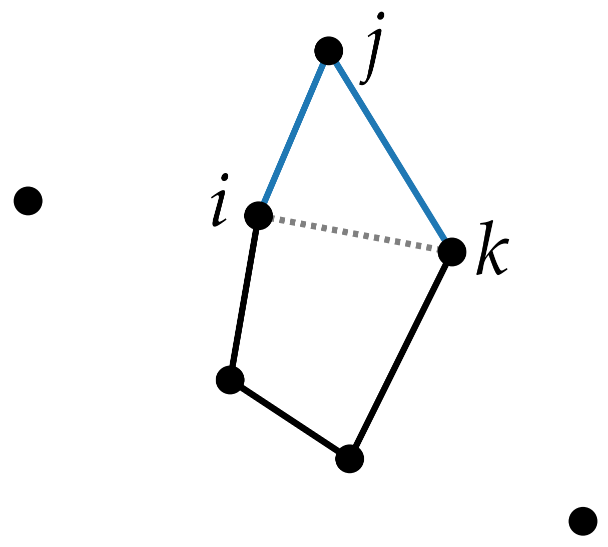

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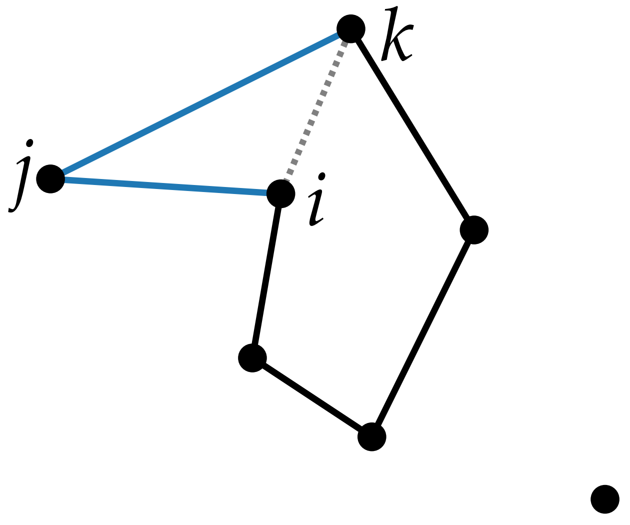

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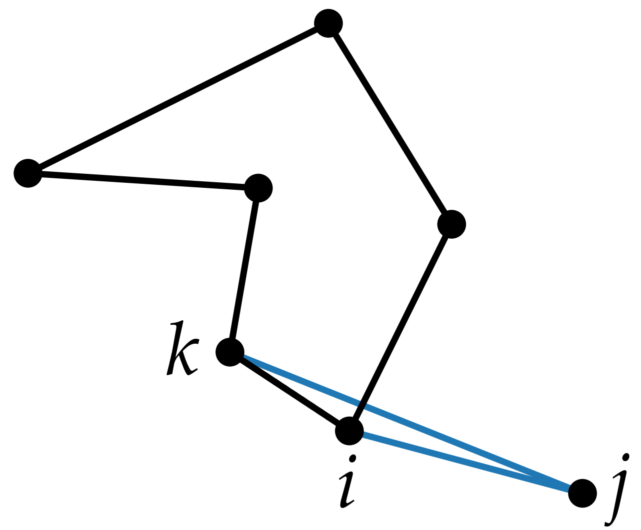

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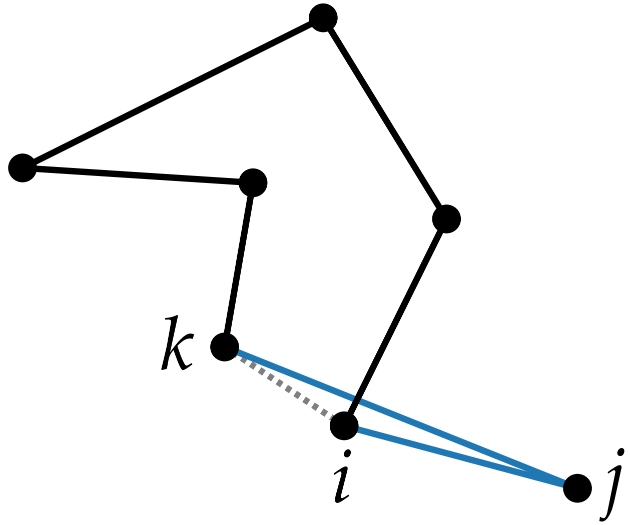

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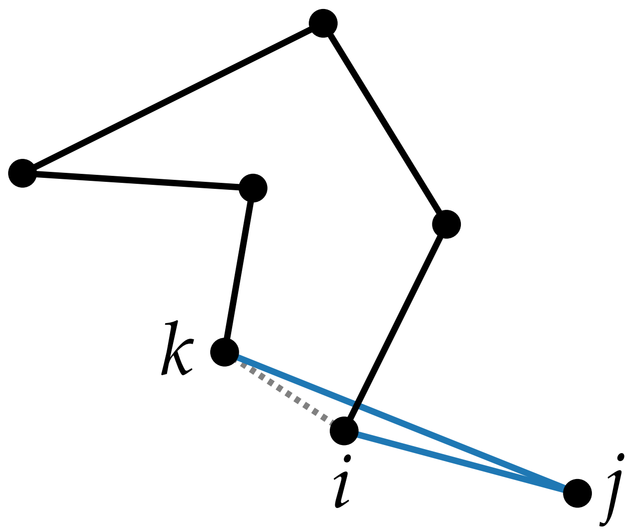

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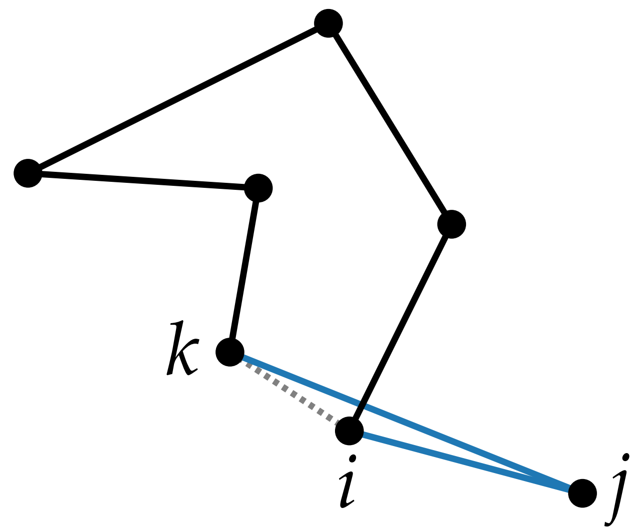

Theorem 6.
`NearestAdditionAlgorithm` is a 2-approximation algorithm for metric TSP.

Proof.

- Exercise.
- *Hints:* MST and Prim's algorithm.

Approximation Schemes

- In some cases, we can get arbitrarily good approximations.

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Let Π be a minimization problem. An algorithm $\mathcal{A}$ is called a **polynomial-time approximation scheme (PTAS)** if $\mathcal{A}$ computes, for every input (I, ε) (consisting of an instance I of Π and a real $\varepsilon > 0$), a value $\mathrm{ALG}(I)$ such that:

- $\mathrm{ALG}(I) \leq (1+\varepsilon) \cdot \mathrm{OPT}(I)$, and
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$\mathcal{A}$ is called a **fully polynomial-time approximation scheme (FPTAS)** if it runs in time polynomial in $|I|$ and $1/\varepsilon$.

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Examples.

- $\mathcal{O}\left(n^2 + n^{\frac{1}{\varepsilon}}\right) \Rightarrow$
- $\mathcal{O}\left(n^2 \cdot 3^{\frac{1}{\varepsilon}}\right) \Rightarrow$
- $\mathcal{O}\left(n^4 \cdot \left(\frac{1}{\varepsilon}\right)^2\right) \Rightarrow$

Approximation Schemes

- In some cases, we can get arbitrarily good approximations.

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Let Π be a minimization problem. An algorithm $\mathcal{A}$ is called a **polynomial-time approximation scheme (PTAS)** if $\mathcal{A}$ computes, for every input (I, ε) (consisting of an instance I of Π and a real $\varepsilon > 0$), a value $\text{ALG}(I)$ such that:

$\geq (1-\varepsilon)$

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Examples.

- $\mathcal{O}\left(n^2 + n^{\frac{1}{\varepsilon}}\right) \Rightarrow$ PTAS but not FPTAS
- $\mathcal{O}\left(n^2 \cdot 3^{\frac{1}{\varepsilon}}\right) \Rightarrow$ PTAS but not FPTAS
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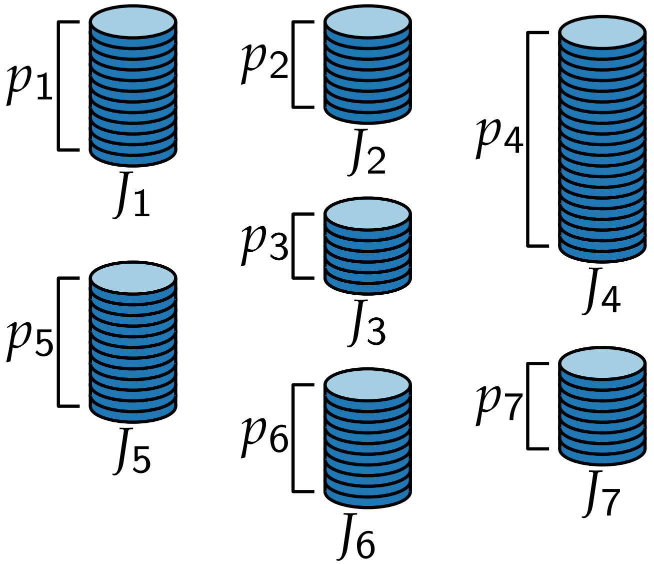
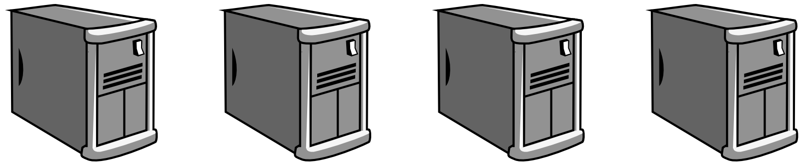
Multiprocessor Scheduling

Input.

- n jobs $J_1, \dots, J_n$ with durations $p_1, \dots, p_n$.



- m identical machines ($m < n$)



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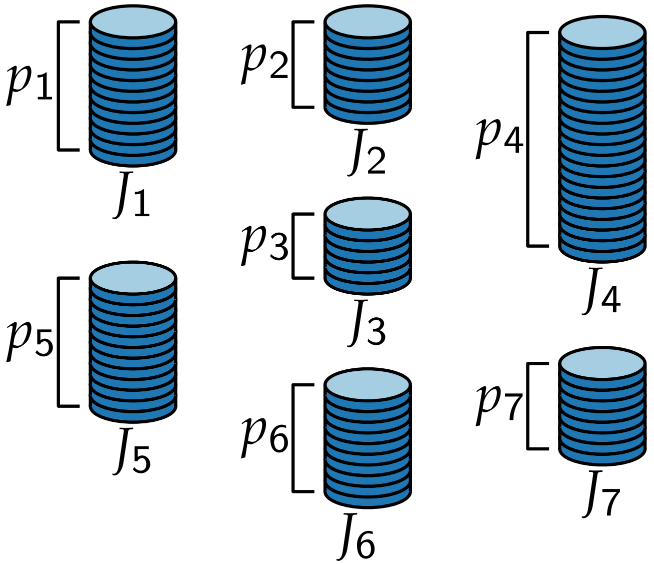

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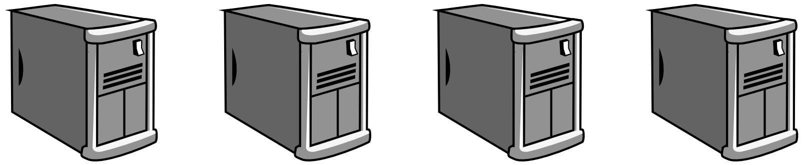

Output. Assignment of jobs to machines such that the time when all jobs have been processed is minimum.
This is called the **makespan** of the assignment.

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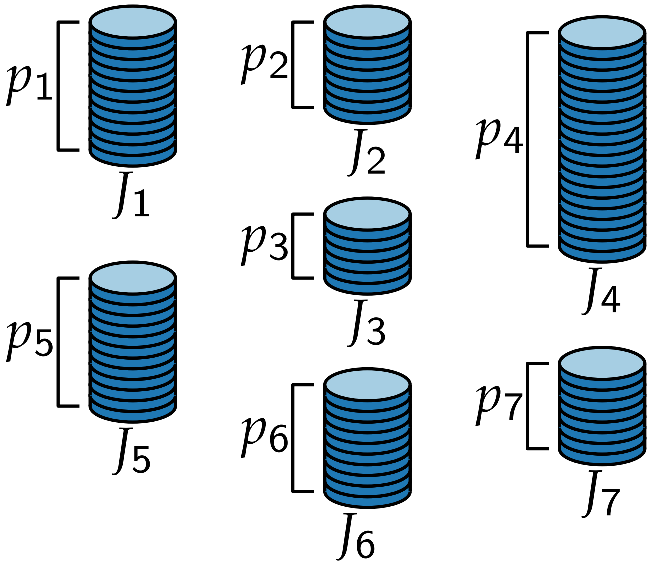

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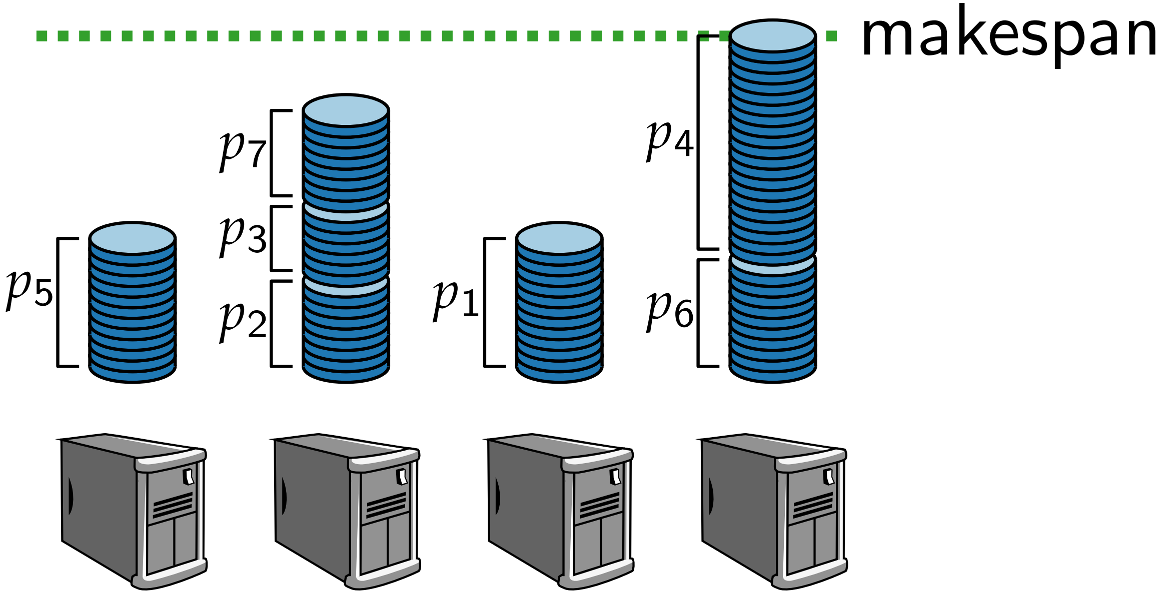

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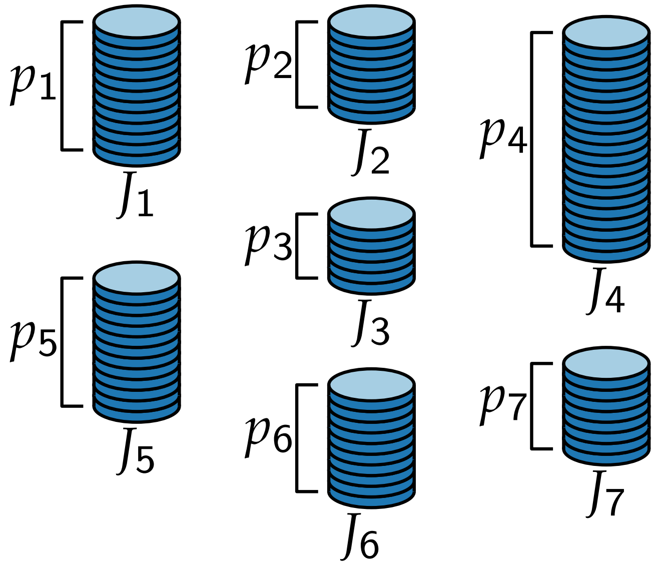

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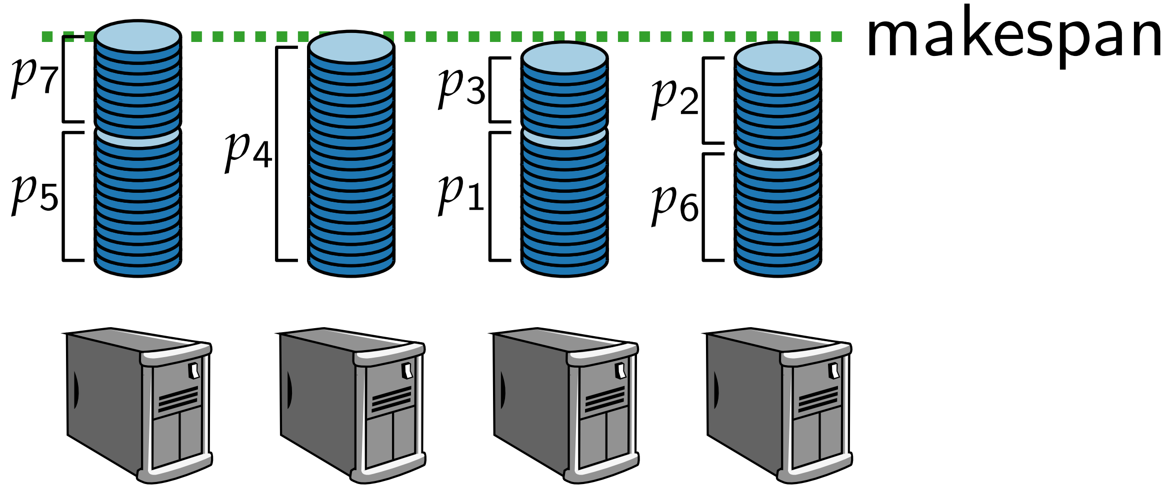

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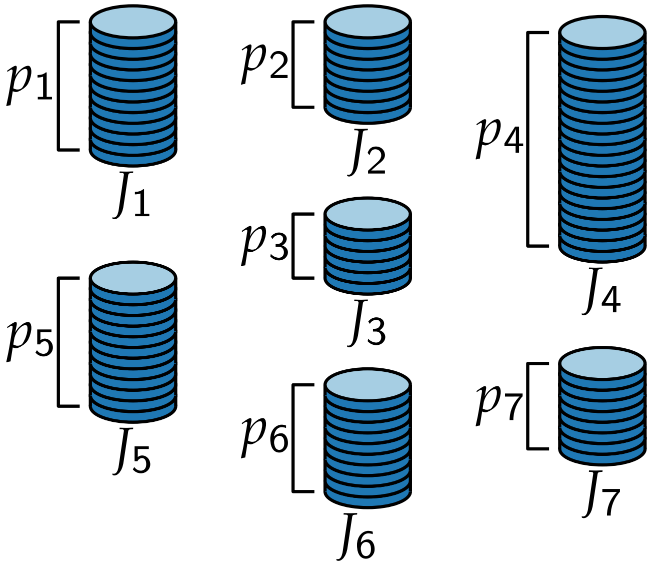

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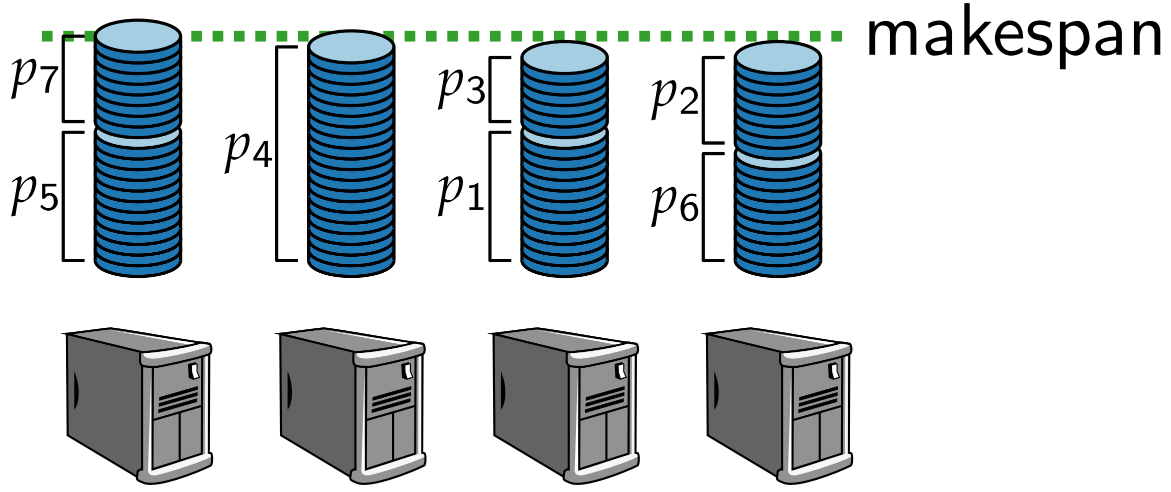


Output. Assignment of jobs to machines such that the time when all jobs have been processed is minimum.
This is called the **makespan** of the assignment.

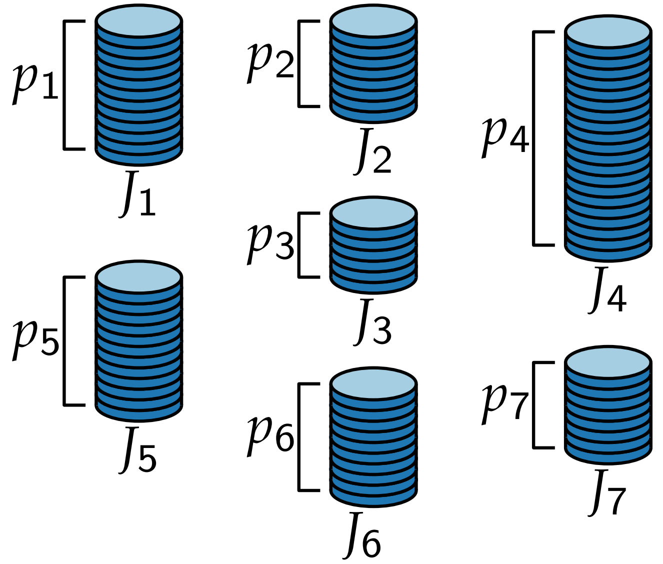
- Multiprocessor scheduling is NP-hard.

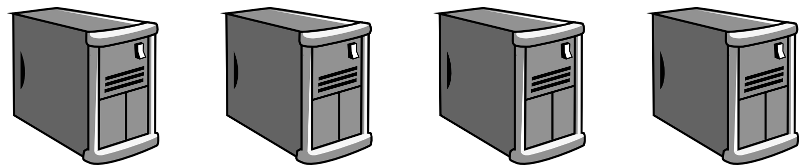
Multiprocessor Scheduling – List Scheduling

ListScheduling($J_1, \ldots, J_n, m$)

Put the first m jobs on the m machines.
Put the next job on the first free machine.

Example.





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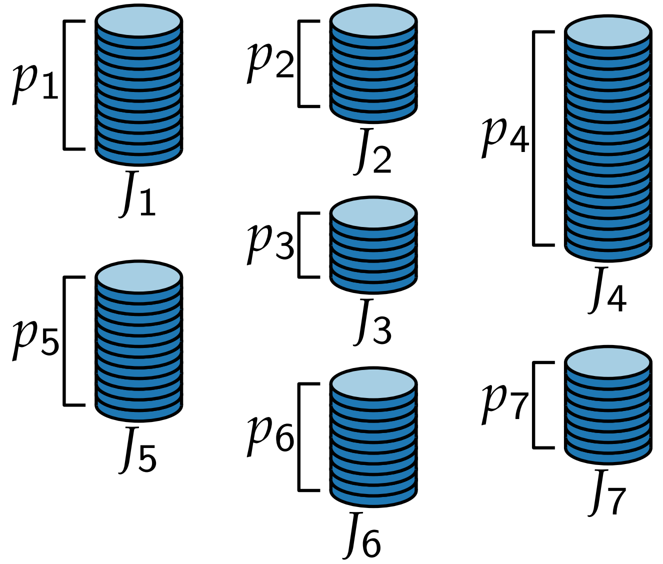

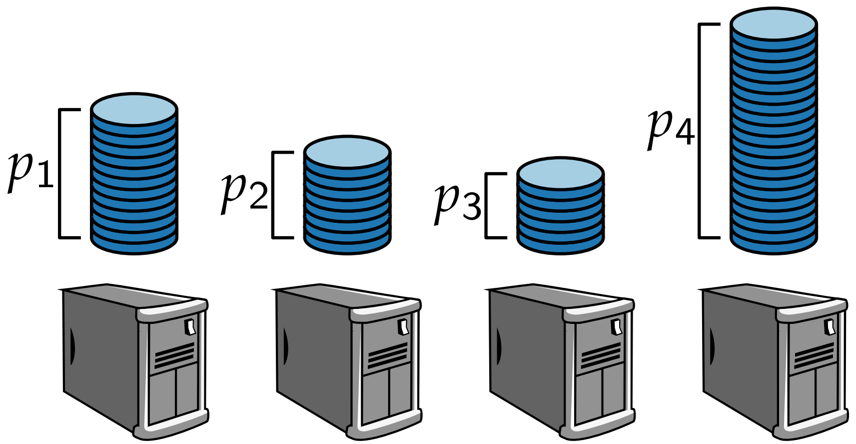

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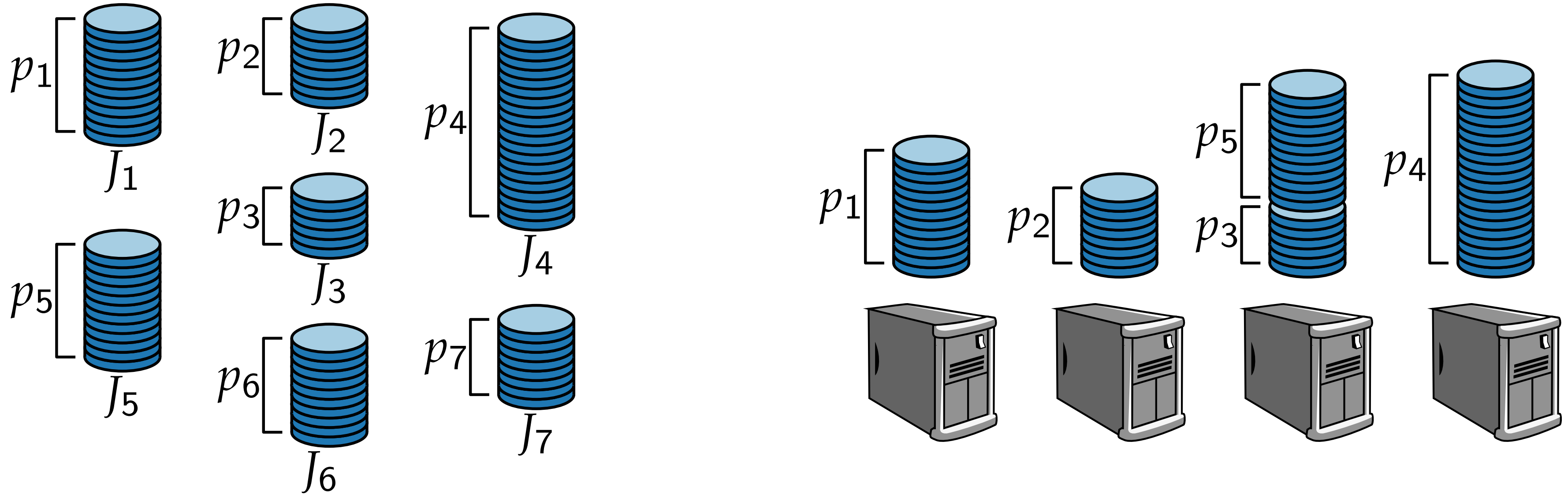

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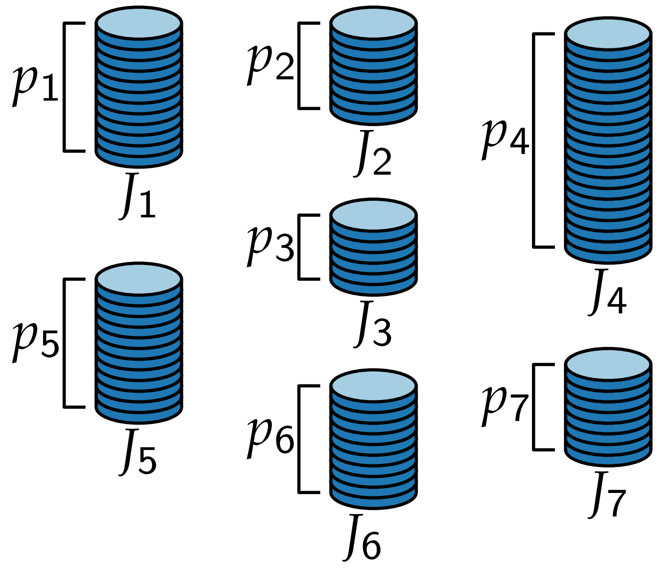

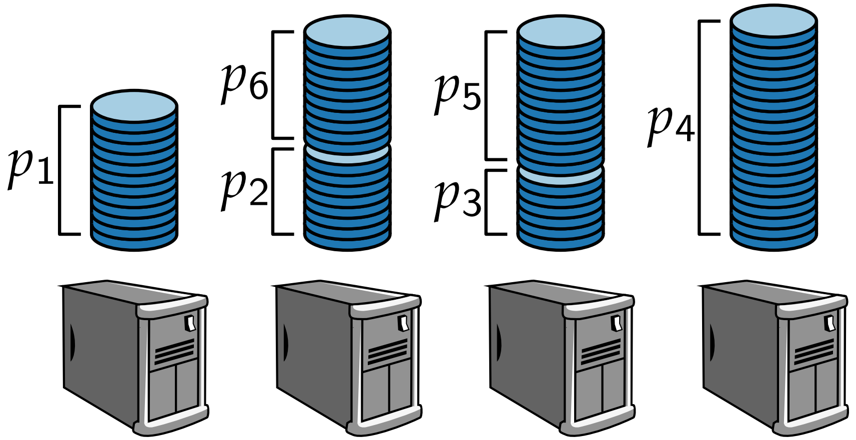

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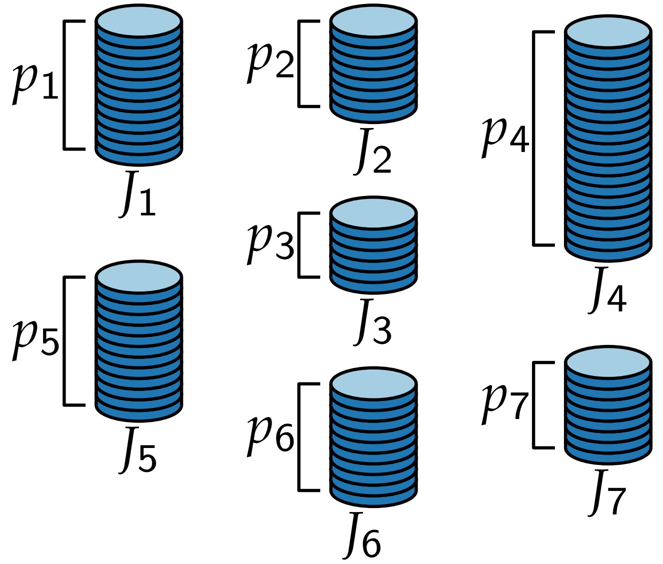

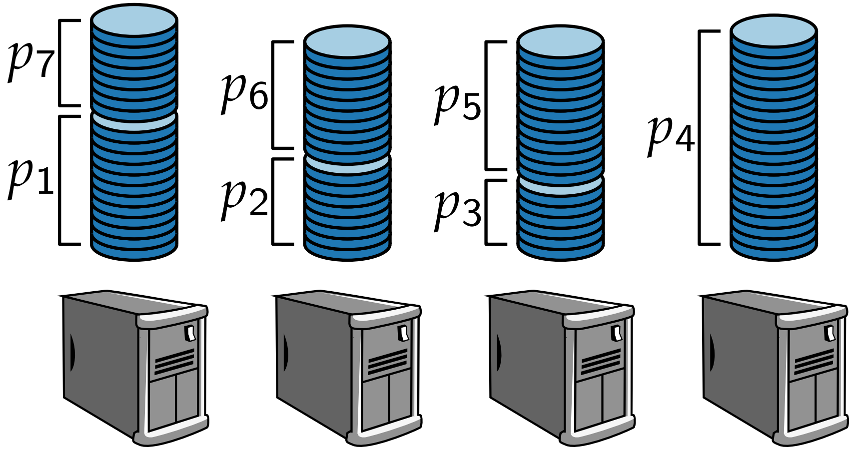

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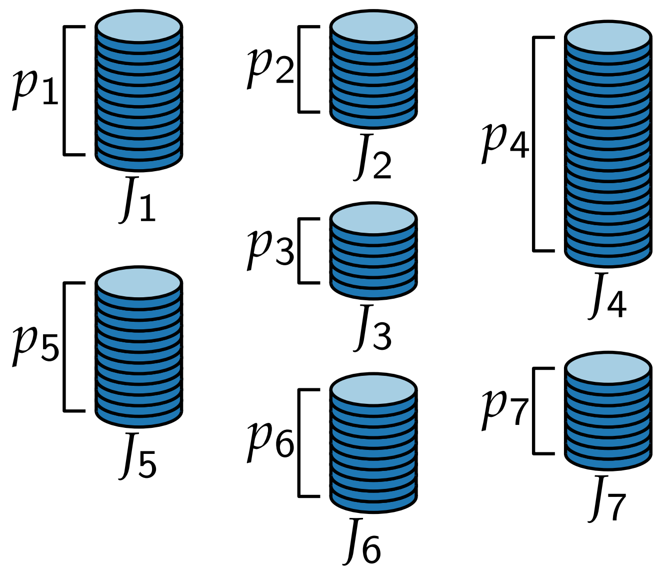

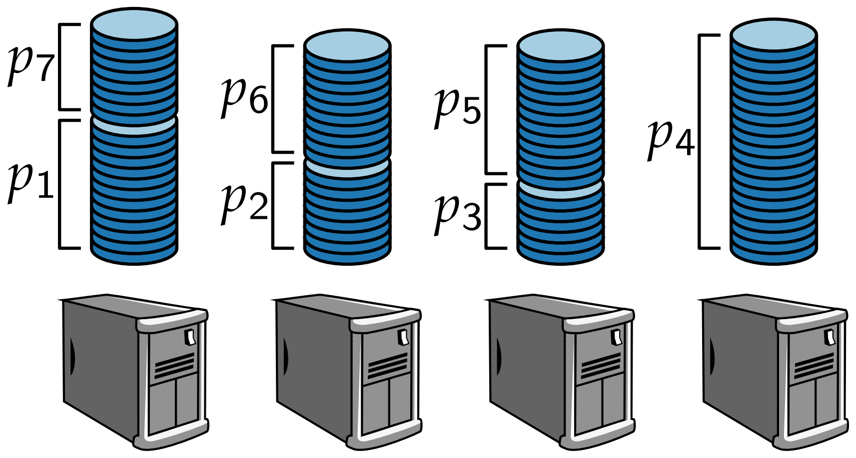

- **ListScheduling** runs in

Multiprocessor Scheduling – List Scheduling

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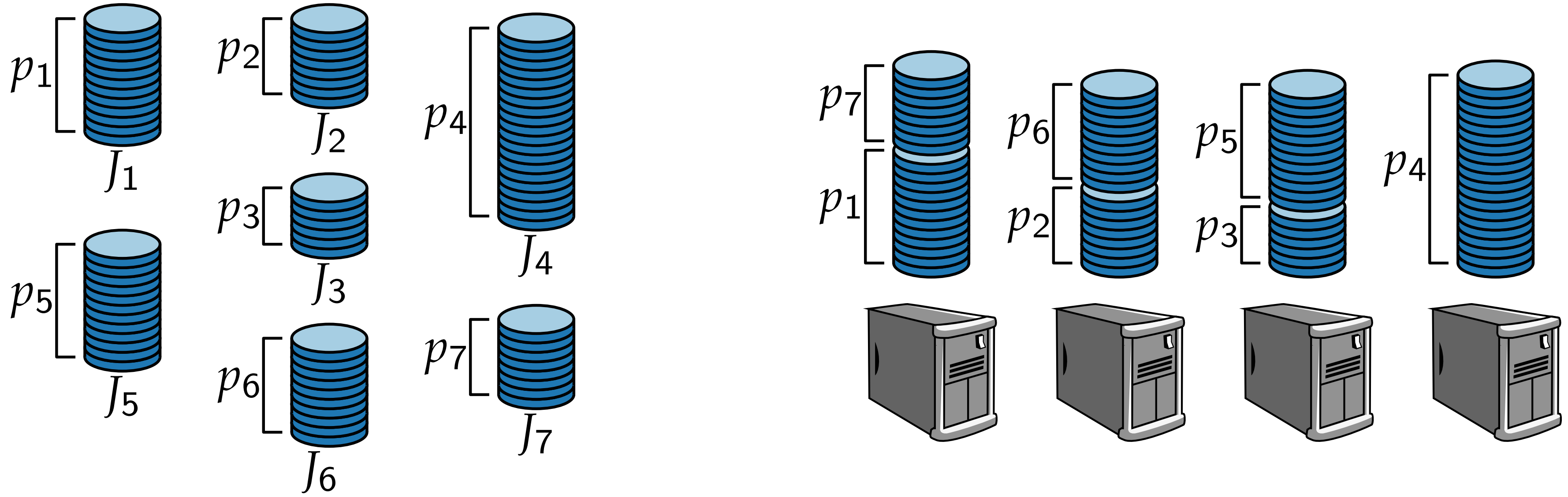

- ListScheduling runs in $\mathcal{O}(n)$ time.

Multiprocessor Scheduling – List Scheduling

ListScheduling($J_1, \ldots, J_n, m$)

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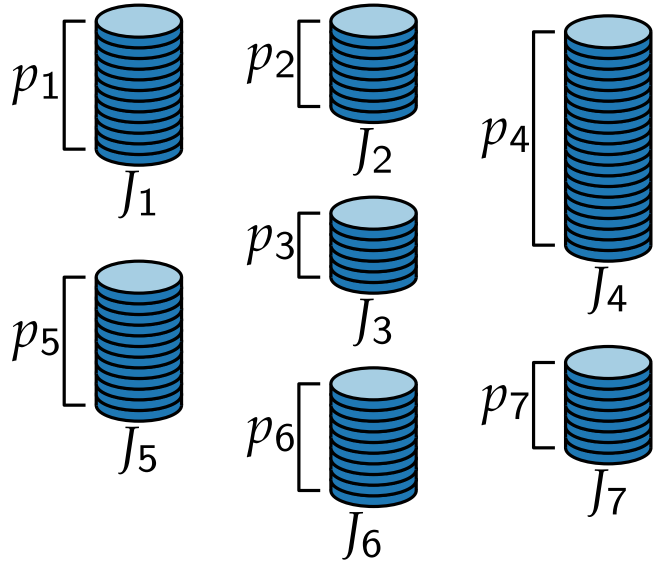
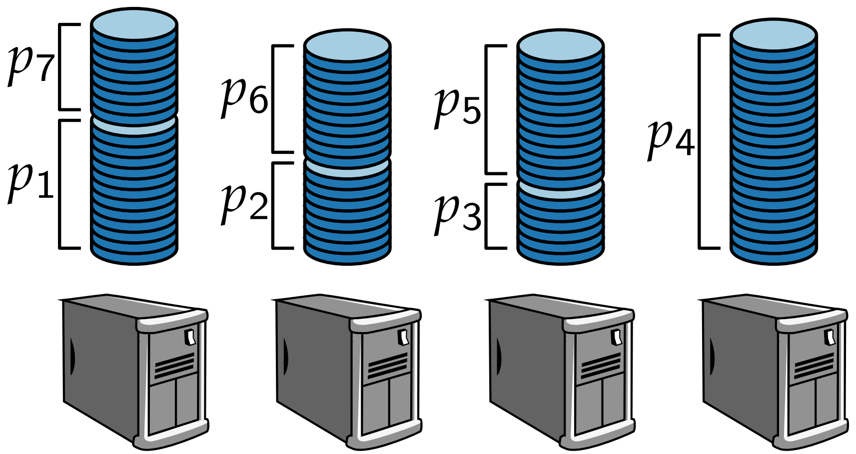
Theorem 7.
ListScheduling is a factor-approximation algorithm.

Example.





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Multiprocessor Scheduling – List Scheduling

ListScheduling$(J_1, \dots, J_n, m)$

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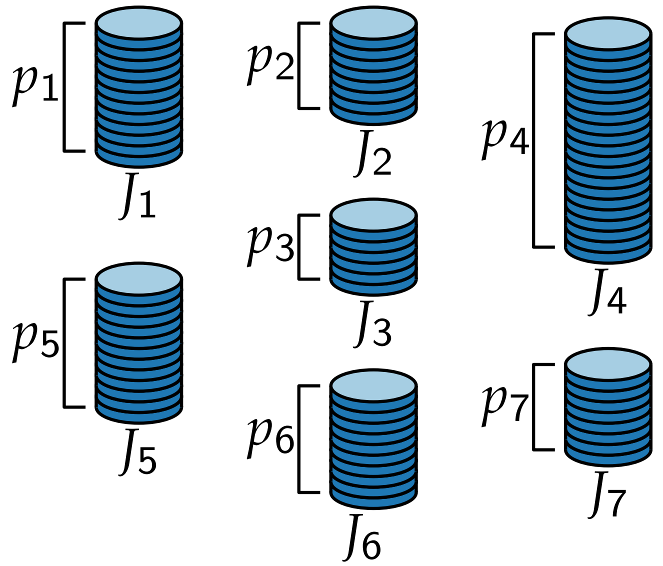
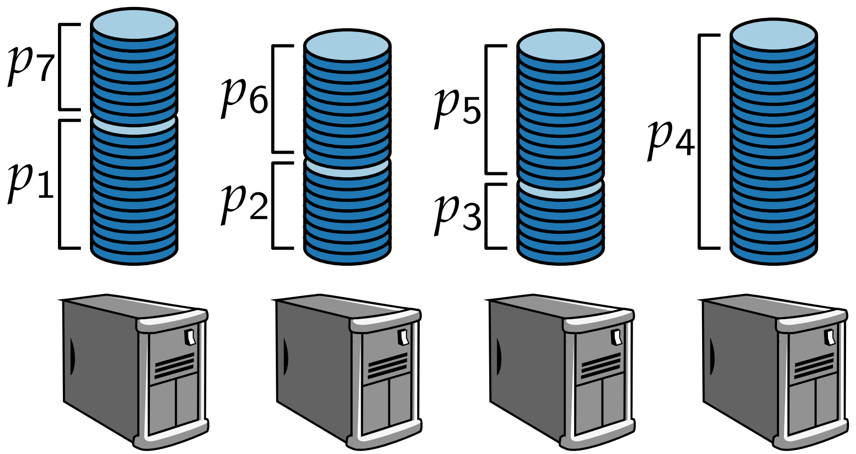
Theorem 7.
ListScheduling is a factor-$\left(2 - \frac{1}{m}\right)$ approximation algorithm.

Example.





- ListScheduling runs in $\mathcal{O}(n)$ time.

Multiprocessor Scheduling – List Scheduling (Proof)

ListScheduling($J_1, \dots, J_n, m$)

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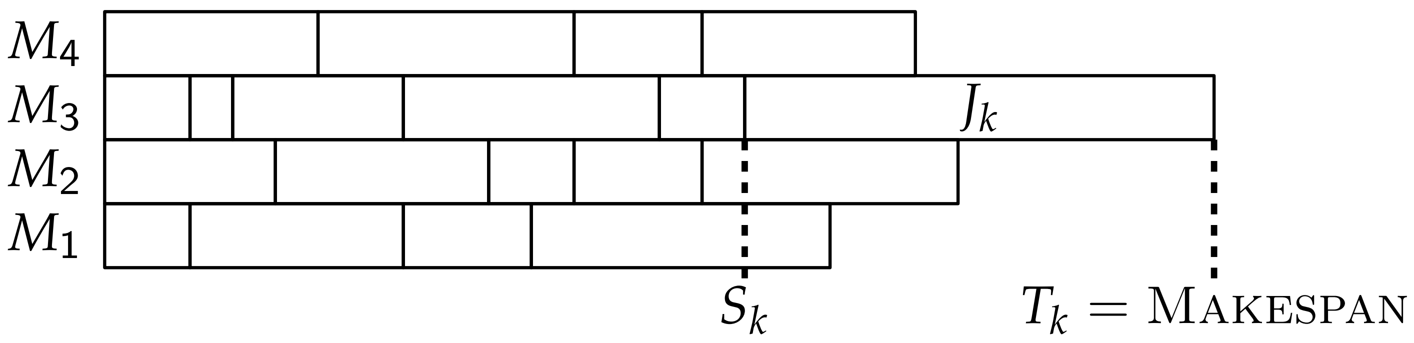
Theorem 7.
ListScheduling is a $\left(2 - \frac{1}{m}\right)$-approximation alg.

Proof. Let $J_k = (S_k, T_k)$ be the last job, that is, T_k determines the makespan.



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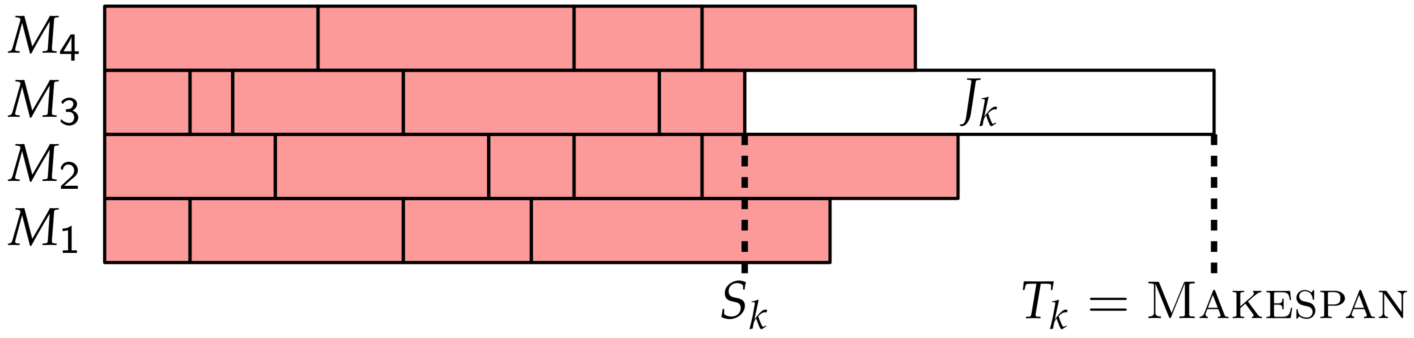
- No machine idles at time S_k.

$$S_k \leq \frac{1}{m} \sum_{i \neq k} p_i$$

weight of all jobs but J_k
evenly distributed on m machines



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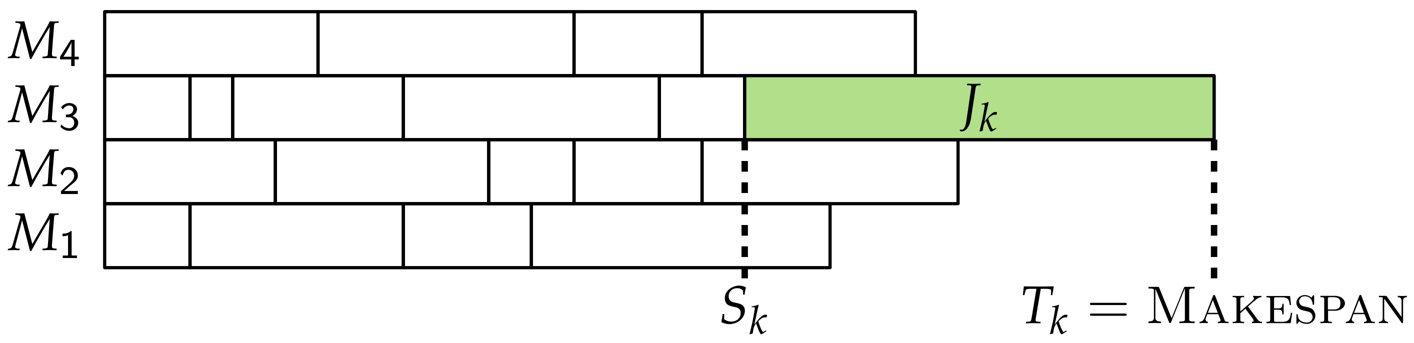
- $T_{\text{OPT}} \ge p_k$



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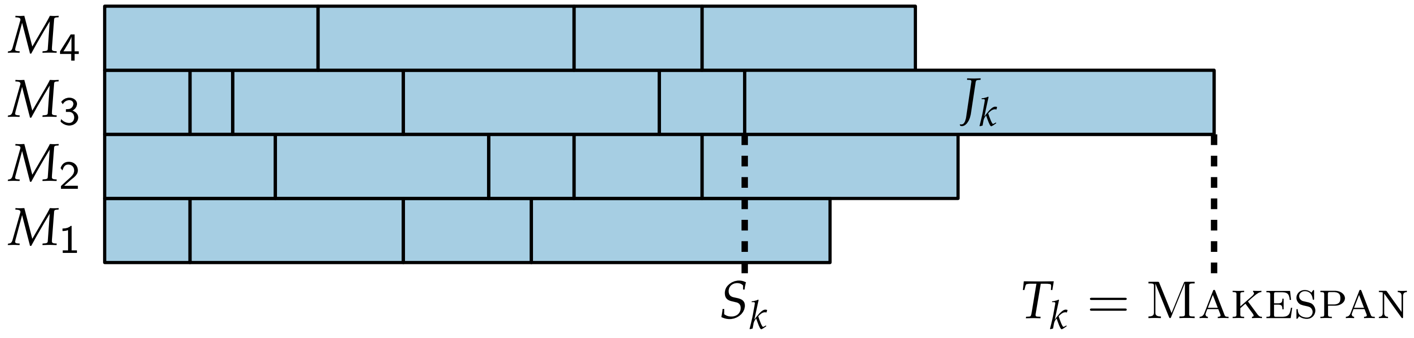
- $T_{\mathrm{OPT}} \ge \frac{1}{m} \sum_{i=1}^{n} p_i$ weight of all jobs evenly distributed



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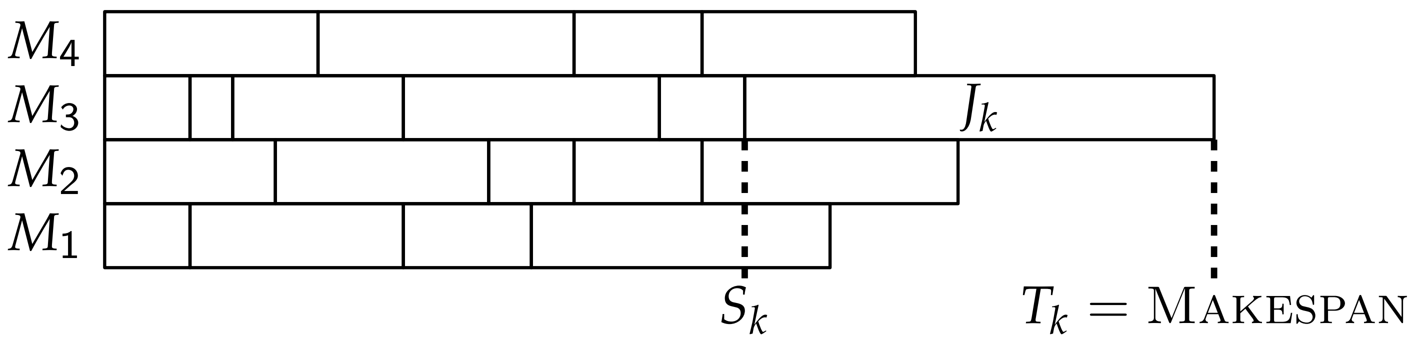
- Hence:

$$T_k = S_k + p_k$$



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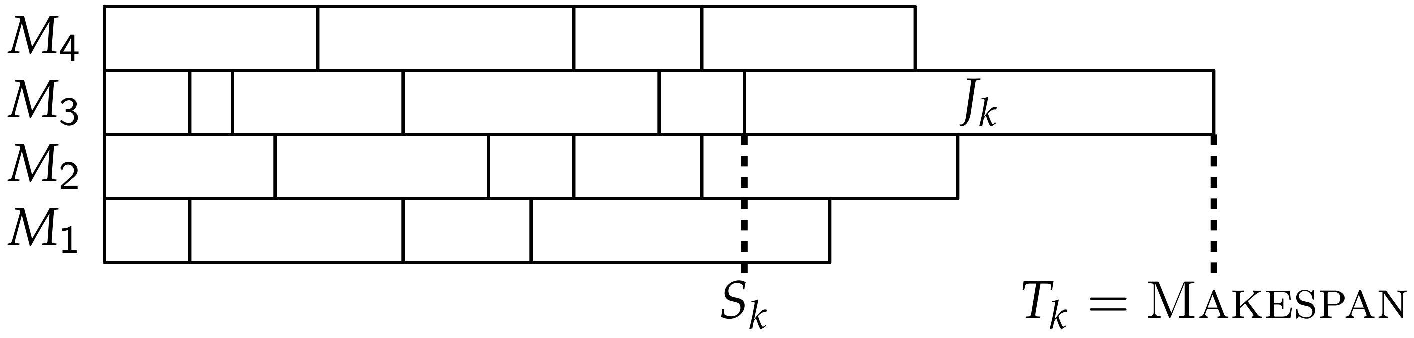
- Hence:

$$T_k = S_k + p_k \leq \frac{1}{m} \cdot \sum_{i \neq k} p_i + p_k$$



Multiprocessor Scheduling – List Scheduling (Proof)

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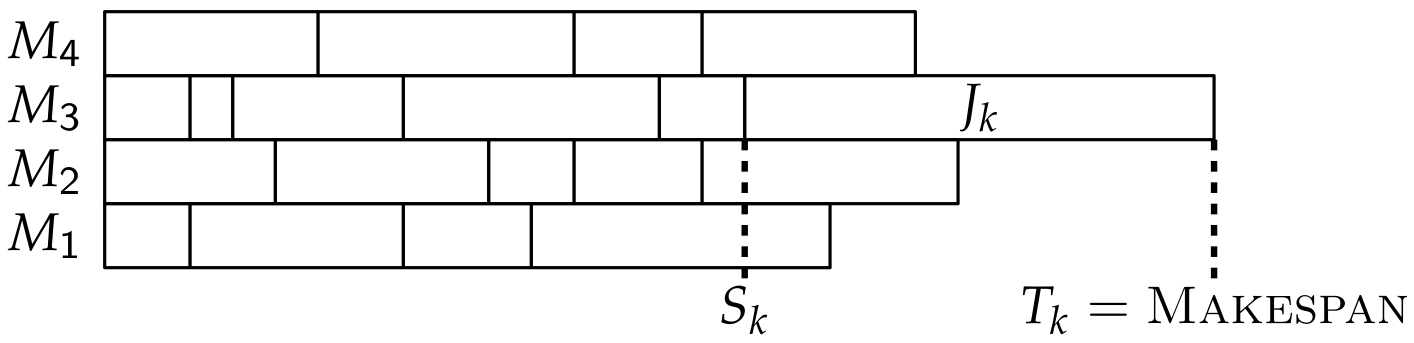
- Hence:

$$T_k = S_k + p_k$$
$$\leq \frac{1}{m} \cdot \sum_{i \neq k} p_i + p_k$$
$$= \frac{1}{m} \cdot \sum_{i=1}^{n} p_i + \left(1 - \frac{1}{m}\right) \cdot p_k$$



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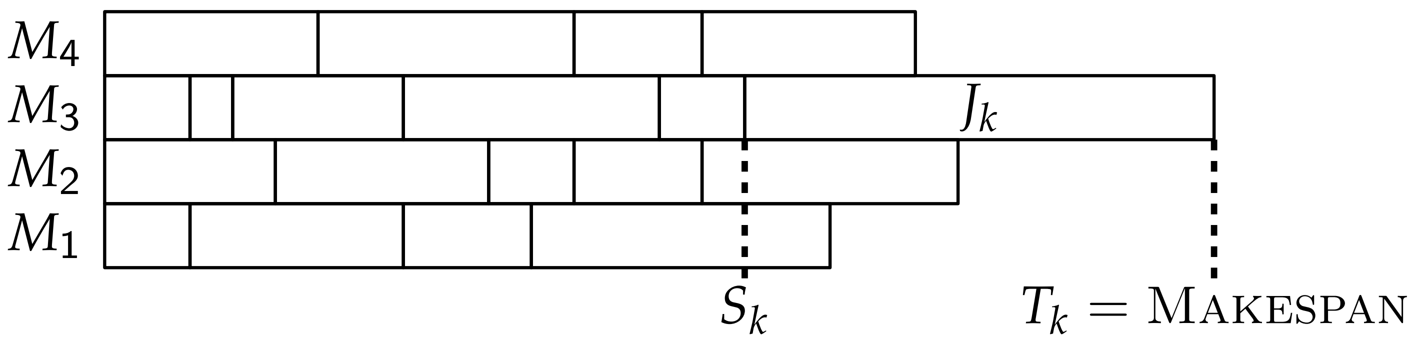
$$\begin{aligned} T_k &= S_k + p_k \\ &\le \frac{1}{m} \cdot \sum_{i \neq k} p_i + p_k \\ &= \frac{1}{m} \cdot \sum_{i=1}^{n} p_i + \left(1 - \frac{1}{m}\right) \cdot p_k \\ &\le T_{\mathrm{OPT}} + \left(1 - \frac{1}{m}\right) \cdot T_{\mathrm{OPT}} \end{aligned}$$



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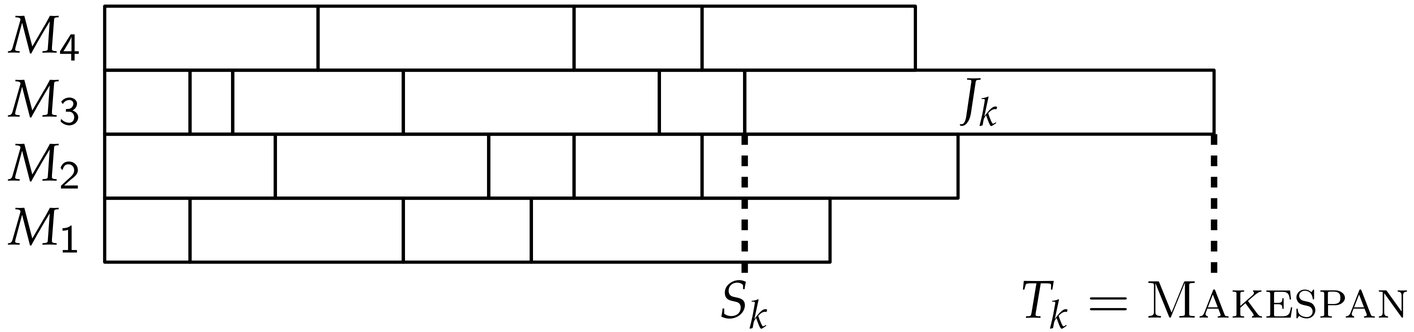
  - $T_{\mathrm{OPT}} \geq \frac{1}{m} \sum_{i=1}^{n} p_i$ weight of all jobs evenly distributed



- Hence:

$$\begin{aligned}
T_k &= S_k + p_k \\
&\leq \frac{1}{m} \cdot \sum_{i \neq k} p_i + p_k \\
&= \frac{1}{m} \cdot \sum_{i=1}^{n} p_i + \left(1 - \frac{1}{m}\right) \cdot p_k \\
&\leq T_{\mathrm{OPT}} + \left(1 - \frac{1}{m}\right) \cdot T_{\mathrm{OPT}} \\
&= \left(2 - \frac{1}{m}\right) \cdot T_{\mathrm{OPT}}
\end{aligned}$$

Multiprocessor Scheduling – PTAS

For a constant ℓ $(1 \leq \ell \leq n)$ define the algorithm $\mathcal{A}_\ell$ as follows.

$\mathcal{A}_\ell(J_1, \ldots, J_n, m)$

Sort jobs in descending order of runtime.

Schedule the ℓ longest jobs $J_1, \ldots, J_\ell$ optimally.

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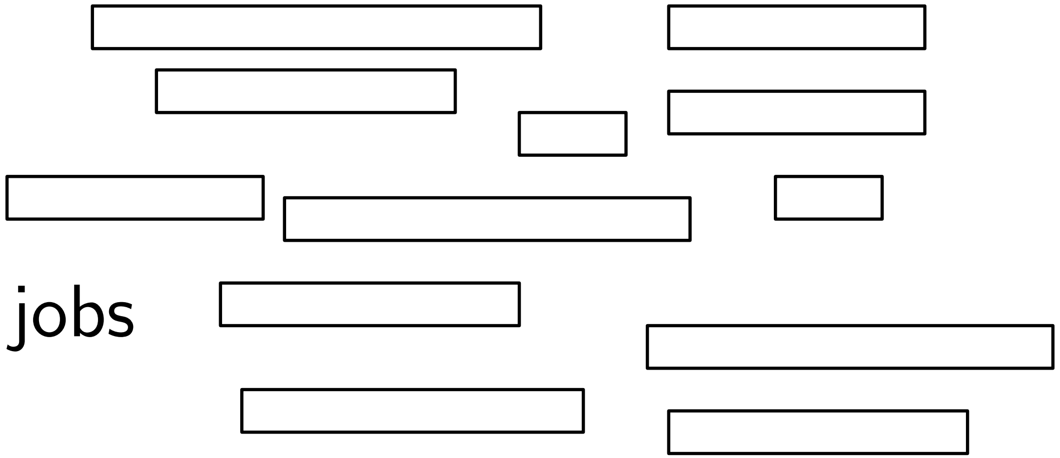
Example.

$\ell = 6$



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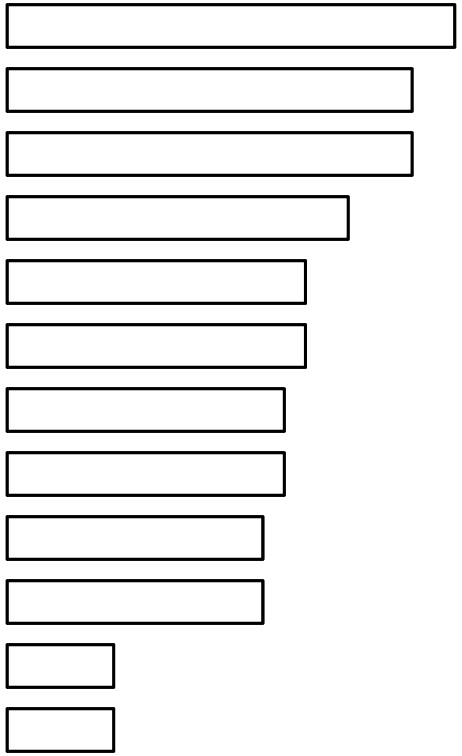

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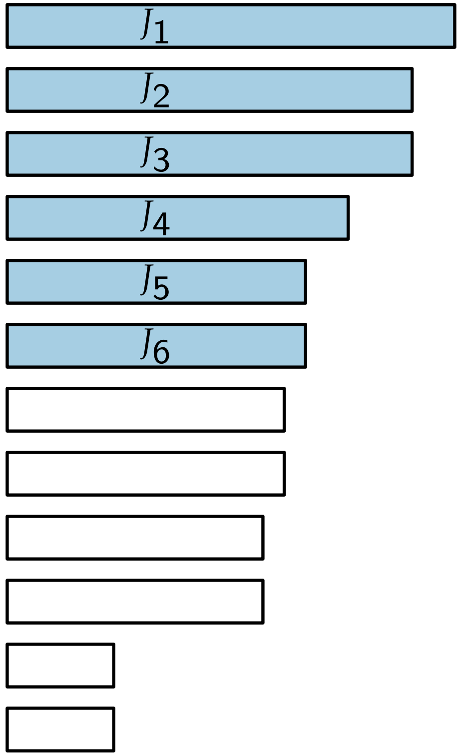

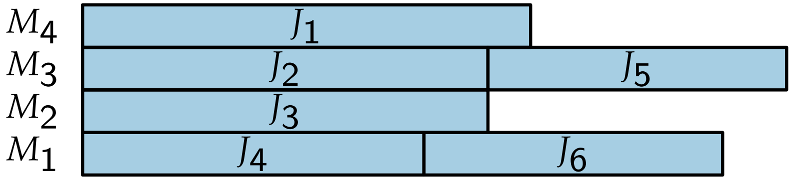

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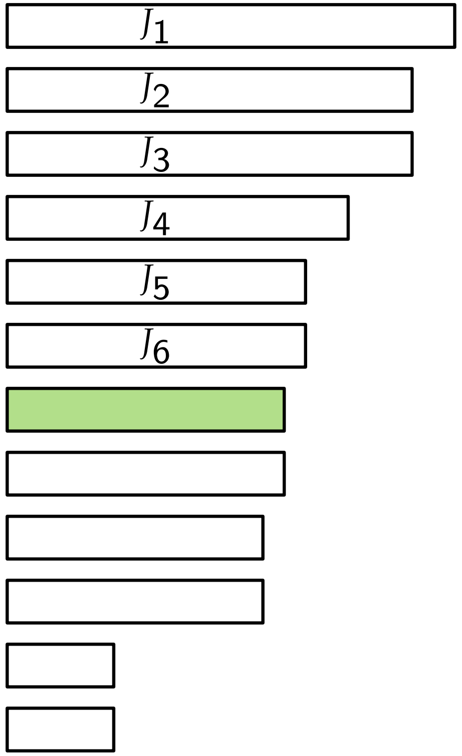

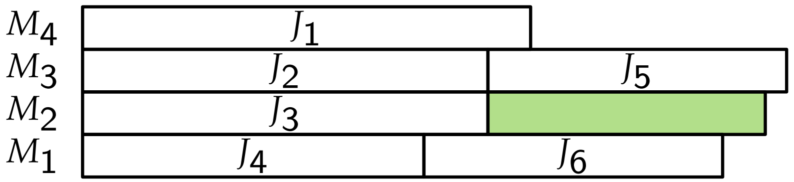

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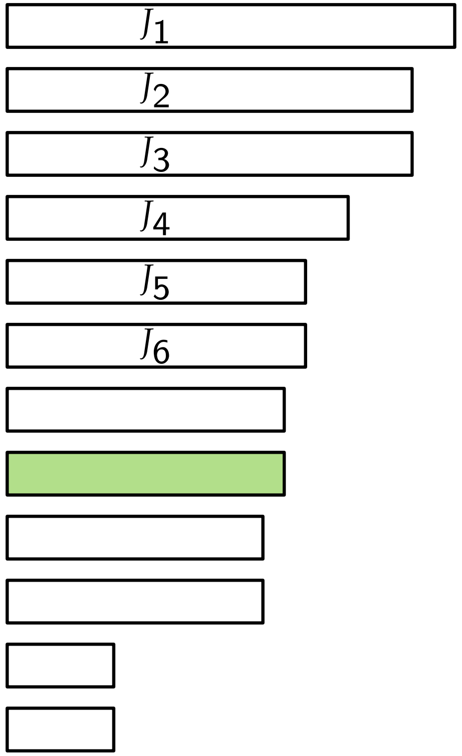

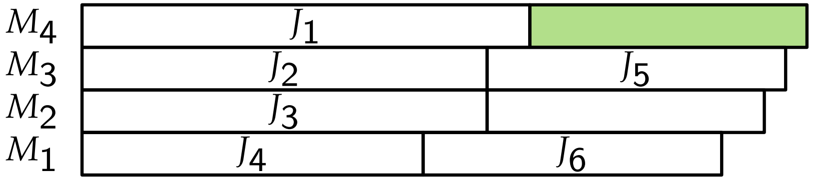

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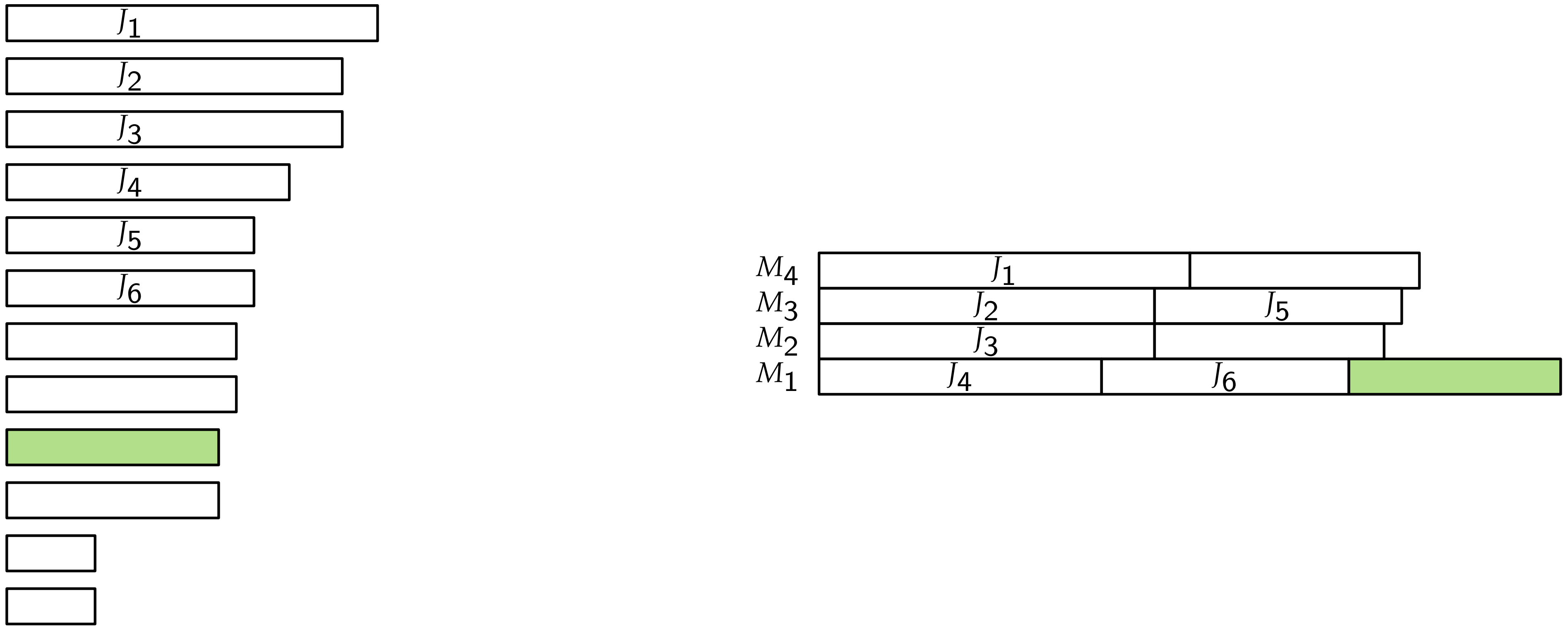

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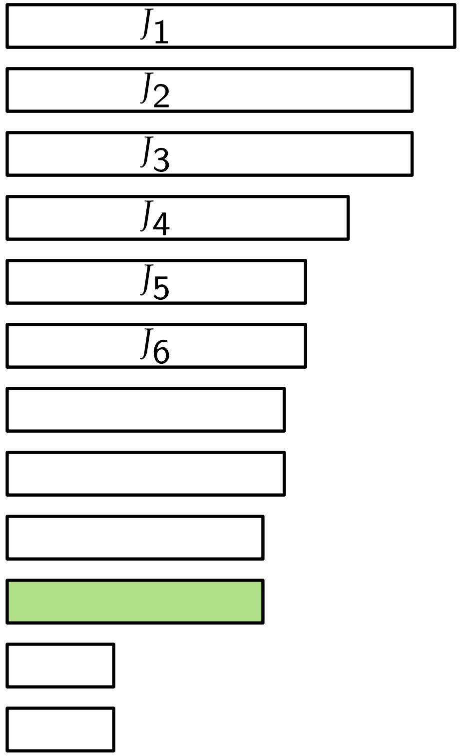

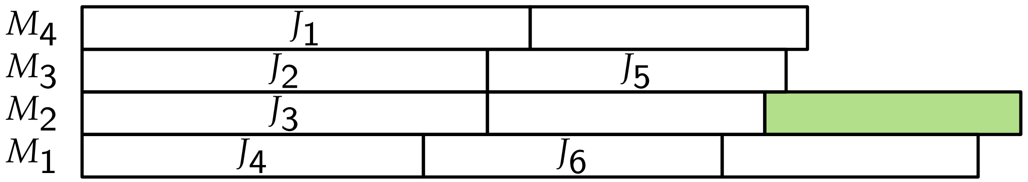

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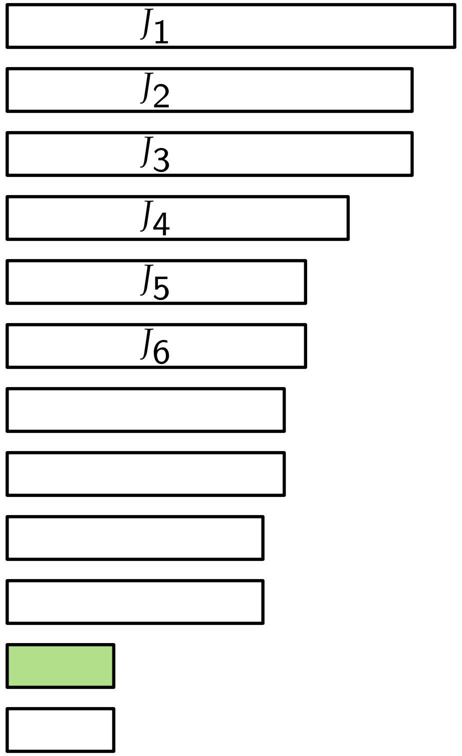

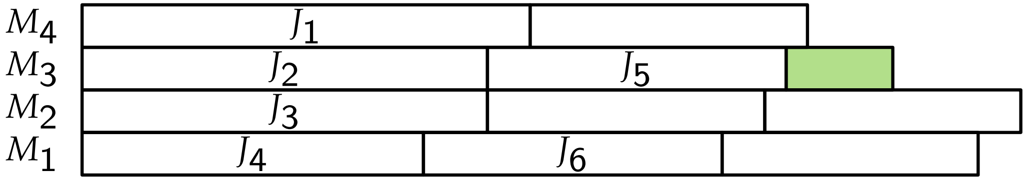

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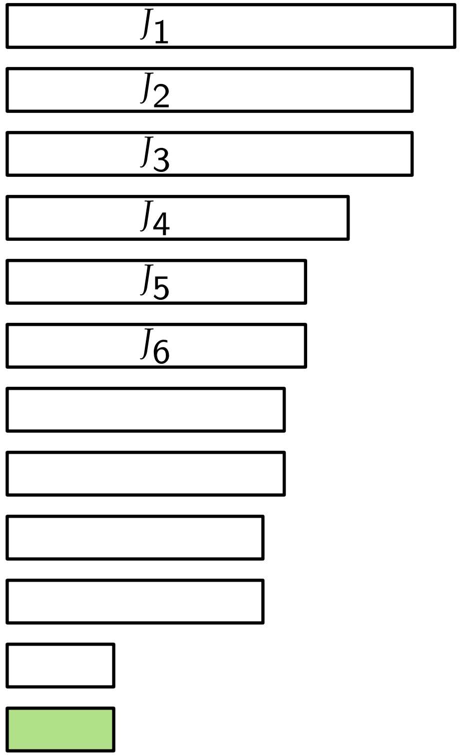

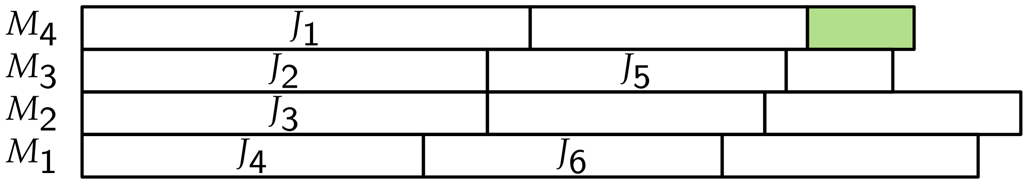

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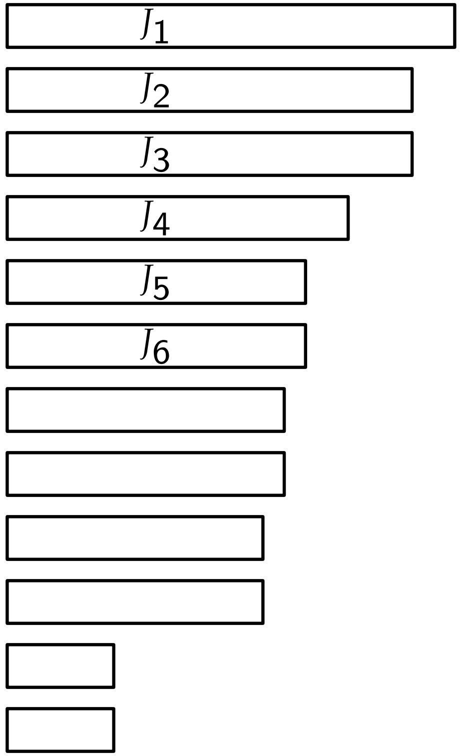

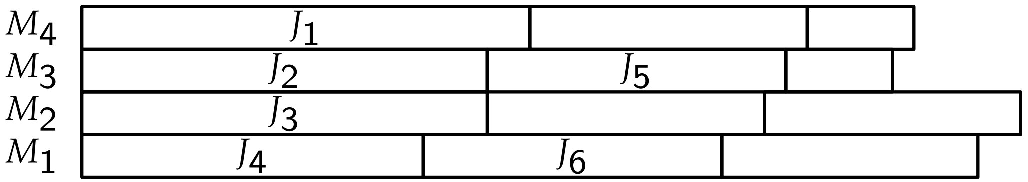

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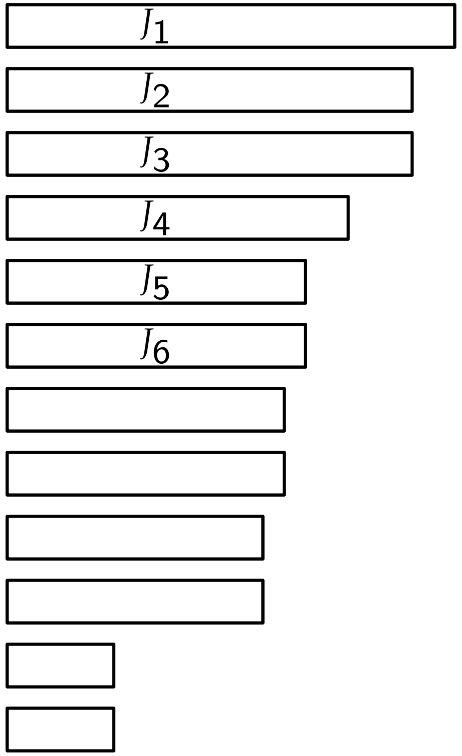

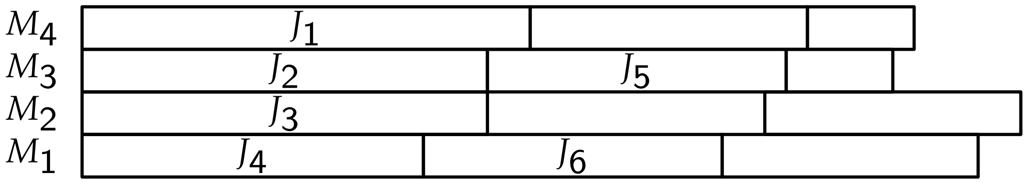

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For constant $1 \leq \ell \leq n$, the algorithm $\mathcal{A}_\ell$ is a $1 + \frac{1-\frac{1}{m}}{1+\lfloor \frac{\ell}{m} \rfloor}$-approximation algorithm.

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Corollary 9.
For a constant number of machines, $\{\mathcal{A}_\varepsilon \mid \varepsilon > 0\}$ is a PTAS.

Multiprocessor Scheduling – PTAS

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- $\{\mathcal{A}_\varepsilon \mid \varepsilon > 0\}$ is not an FPTAS since the running time is not polynomial in $\frac{1}{\varepsilon}$.

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Multiprocessor Scheduling – PTAS (Proof)

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Proof. Let $J_k = (S_k, T_k)$ be the last job, that is, T_k determines the makespan.

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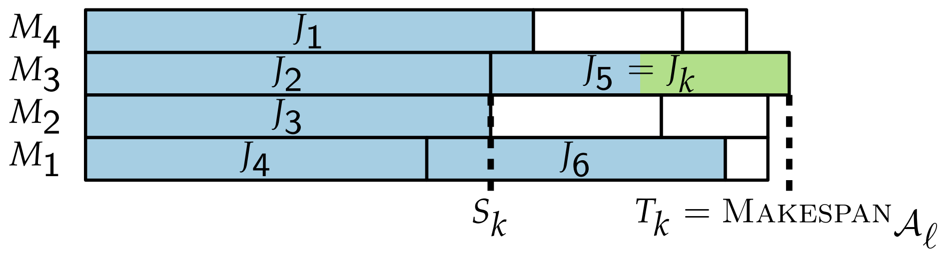

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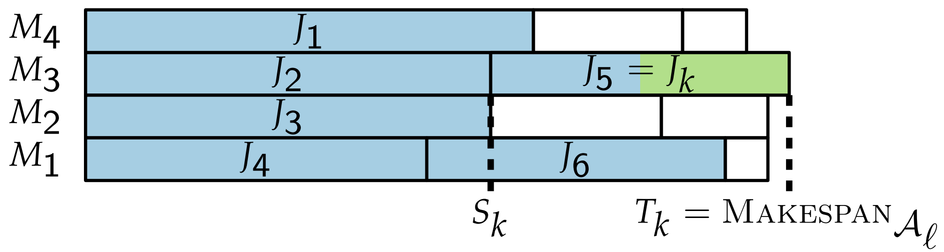

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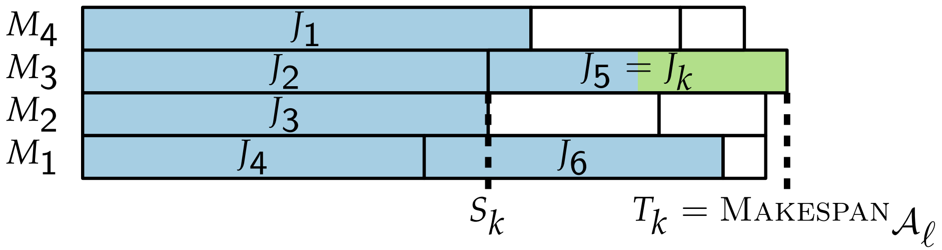

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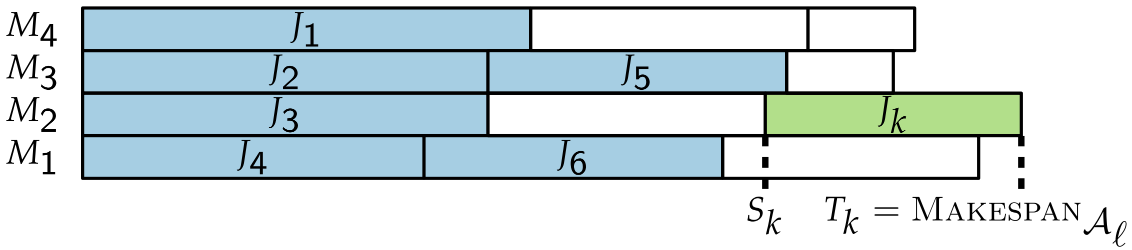

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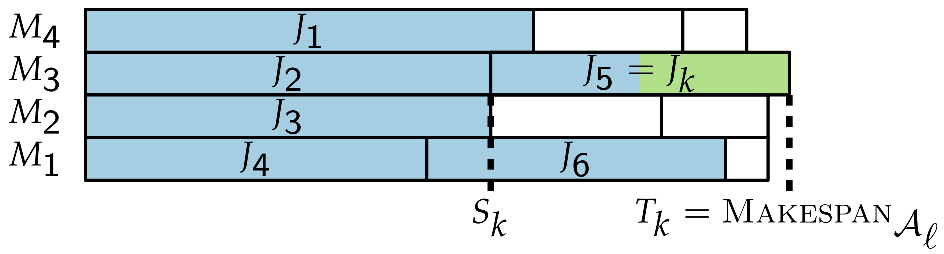

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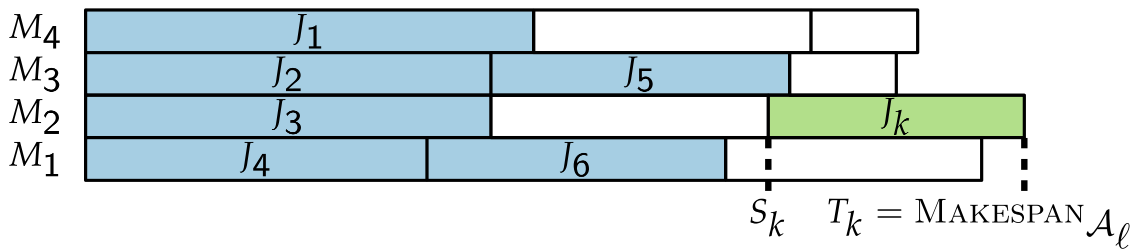
- Similar analysis to ListScheduling
- Use that there are $\ell + 1$ jobs that are at least as long as J_k (including J_k).



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Proof of **Case 2**.

- $S_k \leq \frac{1}{m} \sum_{i \neq k} p_i$
- $T_{\mathrm{OPT}} \geq \frac{1}{m} \sum_{i=1}^{n} p_i$
- $T_{\mathrm{OPT}} \geq p_k$

$$T_k = S_k + p_k$$

Multiprocessor Scheduling – PTAS (Proof)

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&\leq T_{\mathrm{OPT}} + \left(1 - \frac{1}{m}\right) \cdot T_{\mathrm{OPT}}
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Multiprocessor Scheduling – PTAS (Proof)

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For constant $1 \le \ell \le n$, the algorithm $\mathcal{A}_\ell$ is a $1 + \frac{1-\frac{1}{m}}{1+\lfloor \frac{\ell}{m} \rfloor}$-approximation algorithm.

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- $S_k \le \frac{1}{m} \sum_{i \ne k} p_i$
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$$\begin{aligned} T_k &= S_k + p_k \\ &\le \frac{1}{m} \cdot \sum_{i \ne k} p_i + p_k \\ &= \frac{1}{m} \cdot \sum_{i=1}^{m} p_i + \left(1 - \frac{1}{m}\right) \cdot p_k \\ &\le T_{\mathrm{OPT}} + \left(1 - \frac{1}{m}\right) \cdot T_{\mathrm{OPT}} \end{aligned}$$

can we do better?

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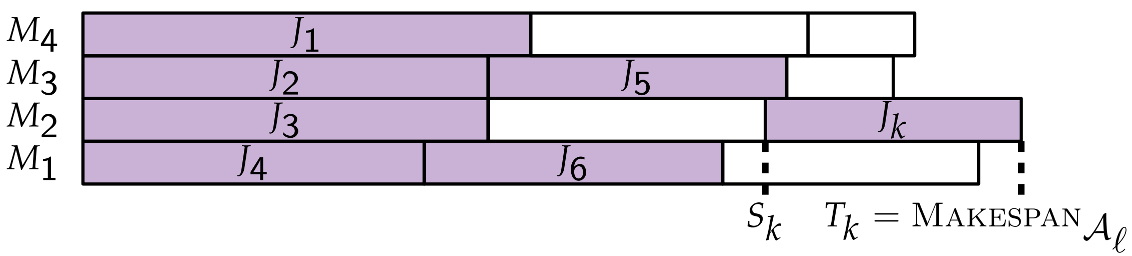

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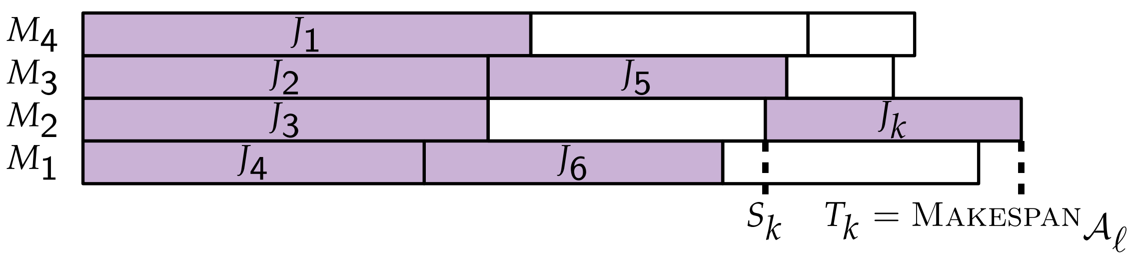

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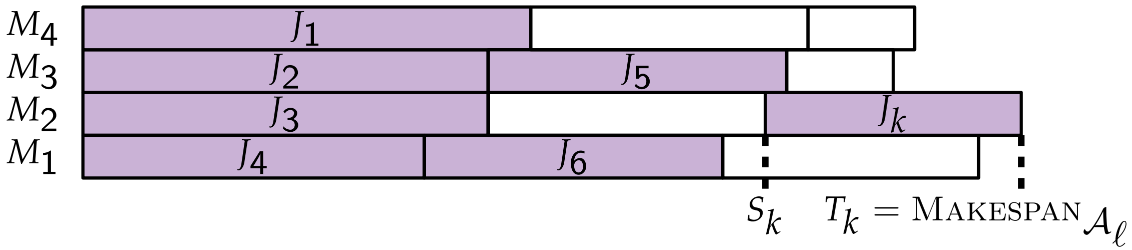

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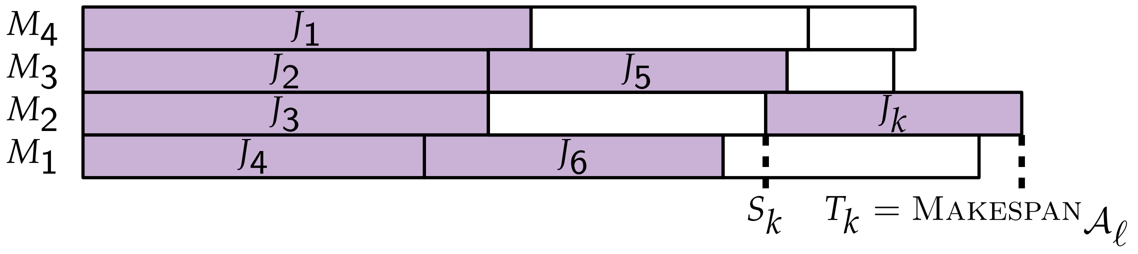

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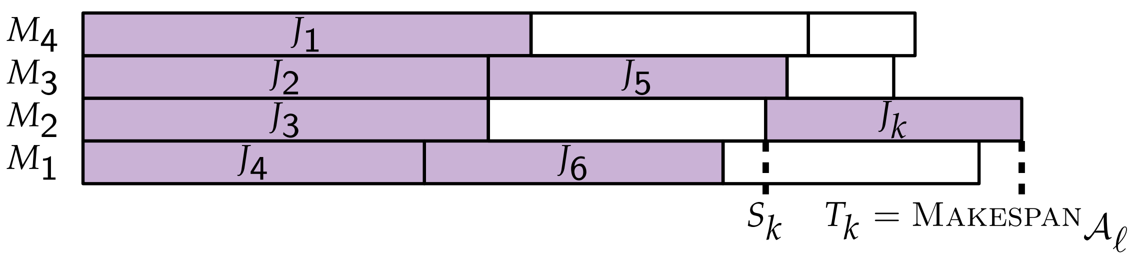

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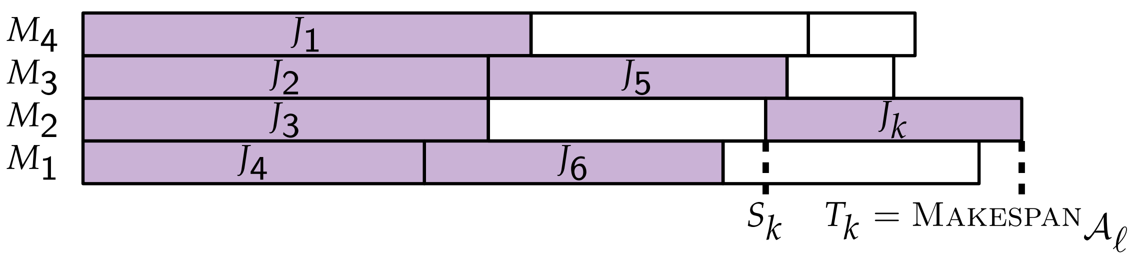

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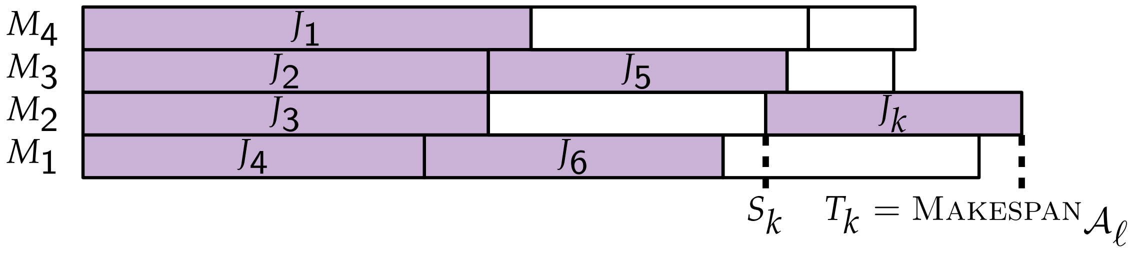

Discussion

- Only “easy” NP-hard problems admit FPTAS (PTAS).
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  greedy, local search, geometric, DP, . . .
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- Minimum Vertex Coloring on planar graphs can be approximated
  with an additive approximation guarantee of 2.
- Christofides' approximation algorithm for Metric TSP has approximation factor 1.5.

Literature

Main references

- [Jansen & Margraf, 2008: Ch3]
  "Approximative Algorithmen und Nichtapproximierbarkeit"
- [Williamson & Shmoys, 2011: Ch3]
  "The Design of Approximation Algorithms"

Another book recommendation:

- [Vazirani, 2013] "Approximation Algorithms"