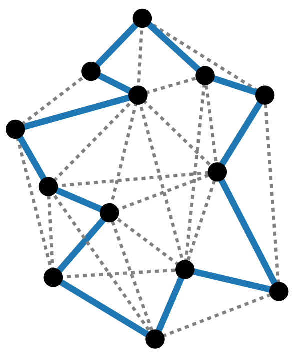
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

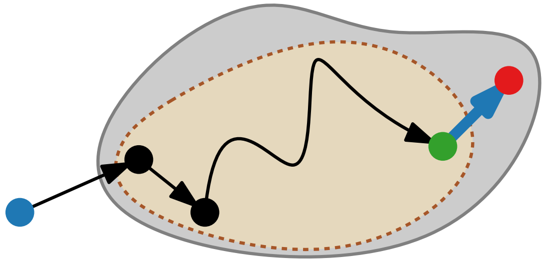
Advanced Algorithms

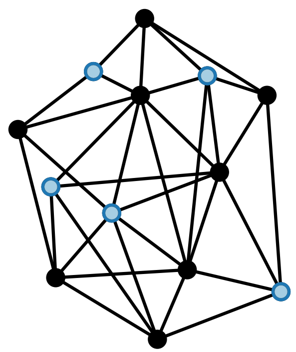
Exact Algorithms for NP-hard Problems

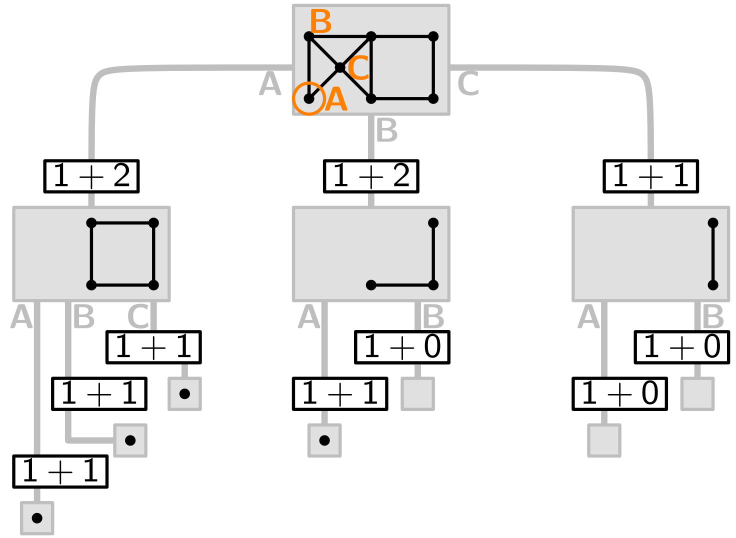
Traveling Salesman Problem and Maximal Independent Set

Diana Sieper · WS22









Examples of NP-hard Problems

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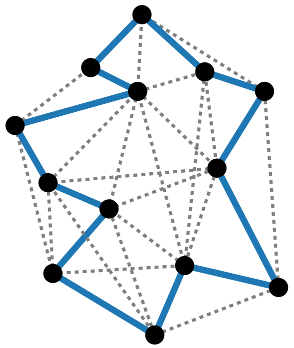
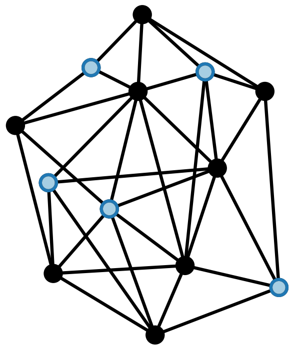
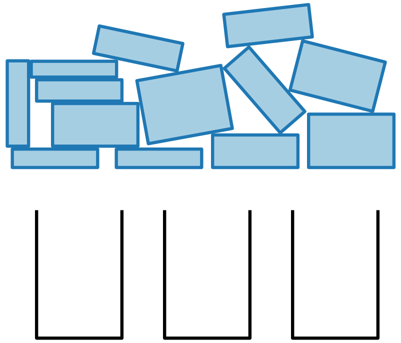


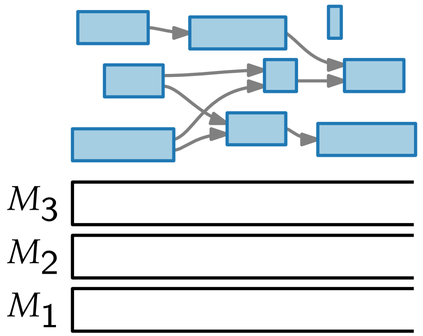
TSP



MIS



Bin Packing



Scheduling

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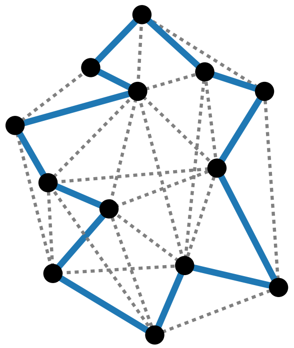

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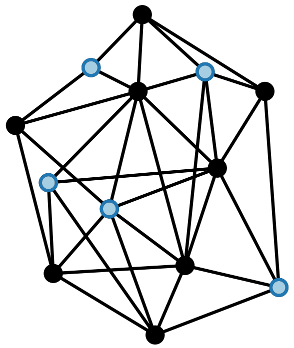

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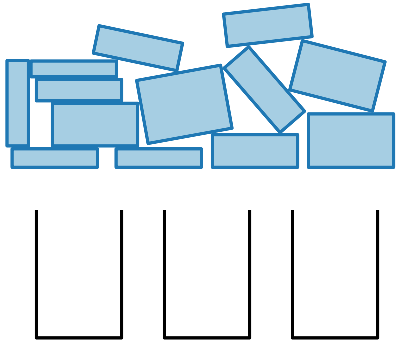

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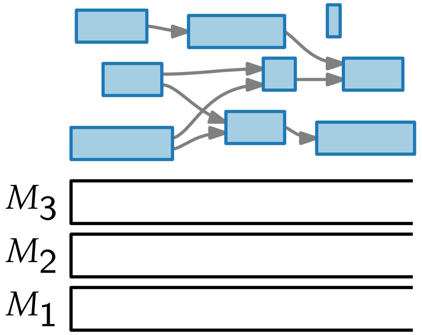
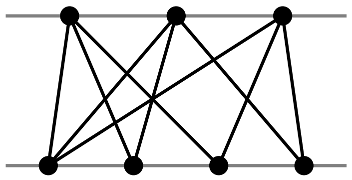


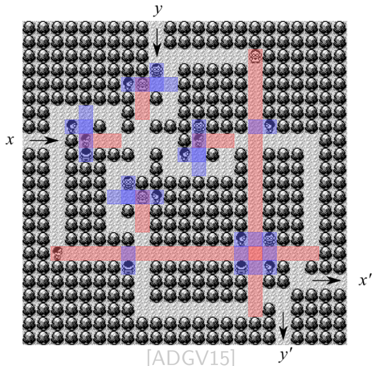
Scheduling

$$(x_1 \vee x_2 \vee \neg x_4) \wedge$$
$$(\neg x_2 \vee x_3 \vee \neg x_4) \wedge$$
$$(x_3 \vee x_7 \vee \neg x_8) \wedge$$
$$\ldots$$

SAT



Graph Drawing



[ADGV15]

Games

...

Formal View on NP-Hardness

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- or: There is a polynomial-time many-one reduction from an NP-hard problem L to H.
- If P $\neq$ NP, then NP-hard problems cannot be solved in polynomial time.

Misconceptions about NP-Hardness

Common misconceptions [Mann '17]

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- Problems that are hard to solve in practice by an engineer are NP-hard.
- NP-hard problems cannot be solved optimally.
- NP-hard problems cannot be solved more efficiently than by exhaustive search.
- For solving NP-hard problems, the only practical possibility is the use of heuristics.

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  - Heuristics (Simulated Annealing, Tabu-Search)
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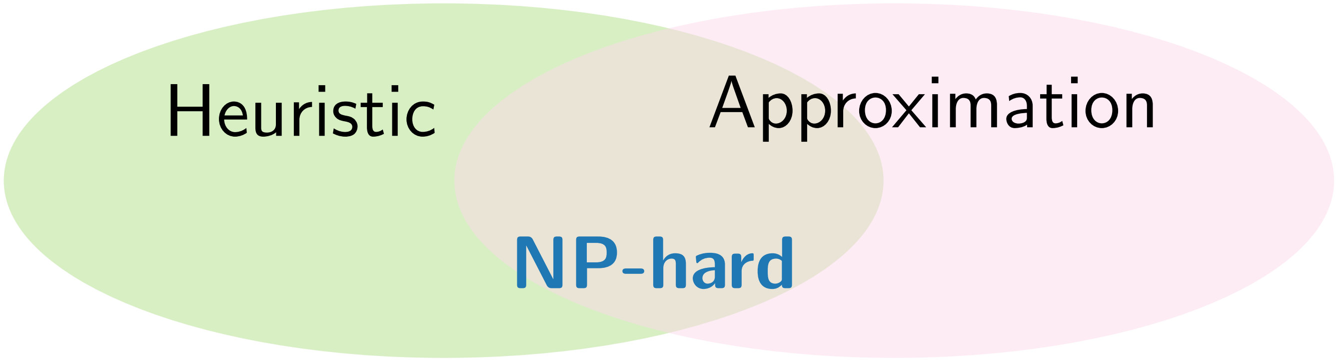

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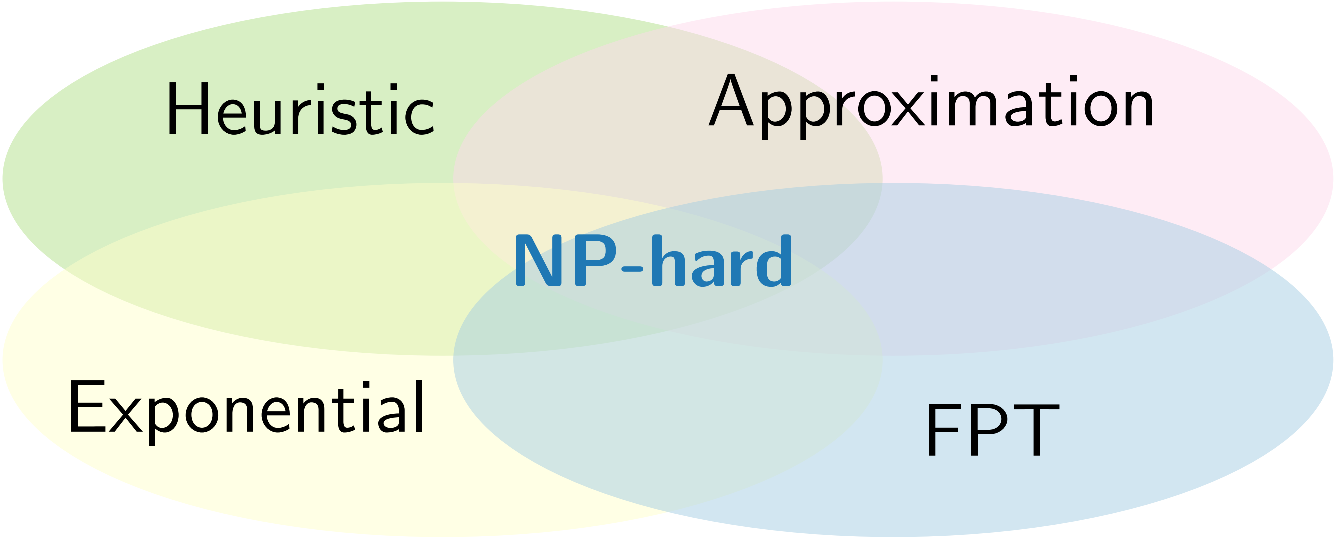

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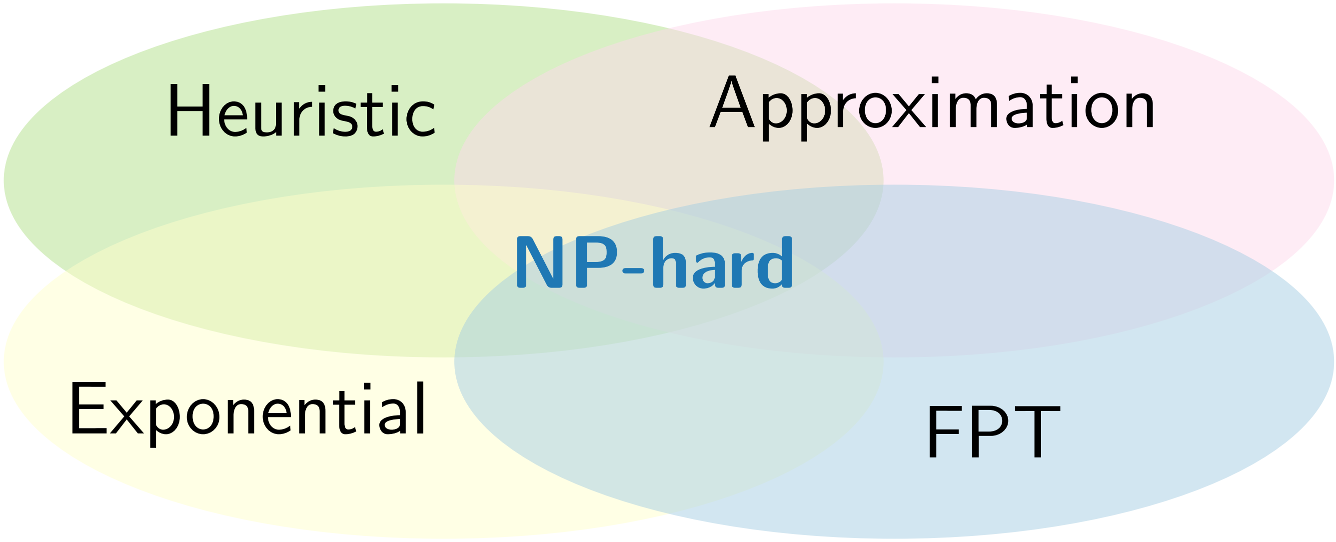

this lecture

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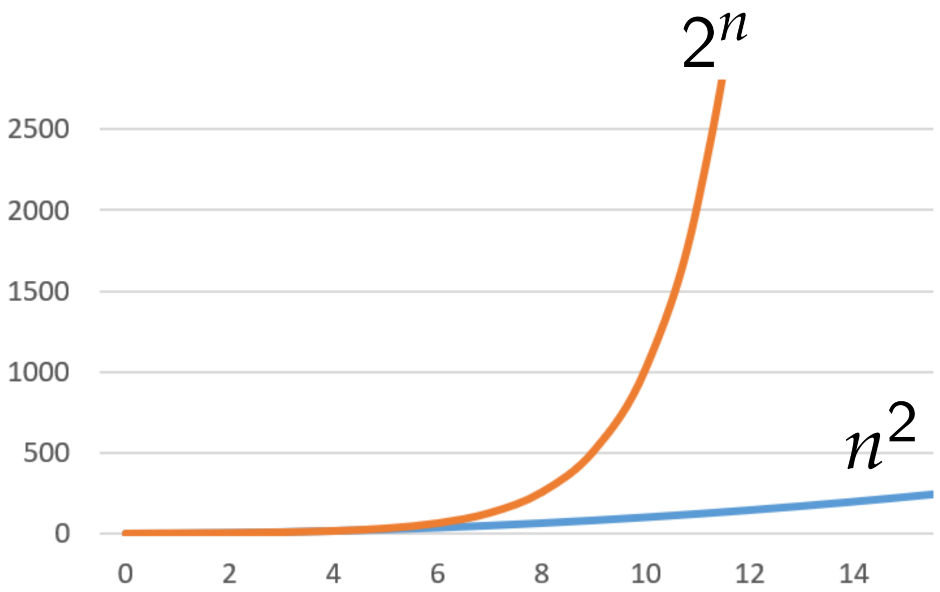

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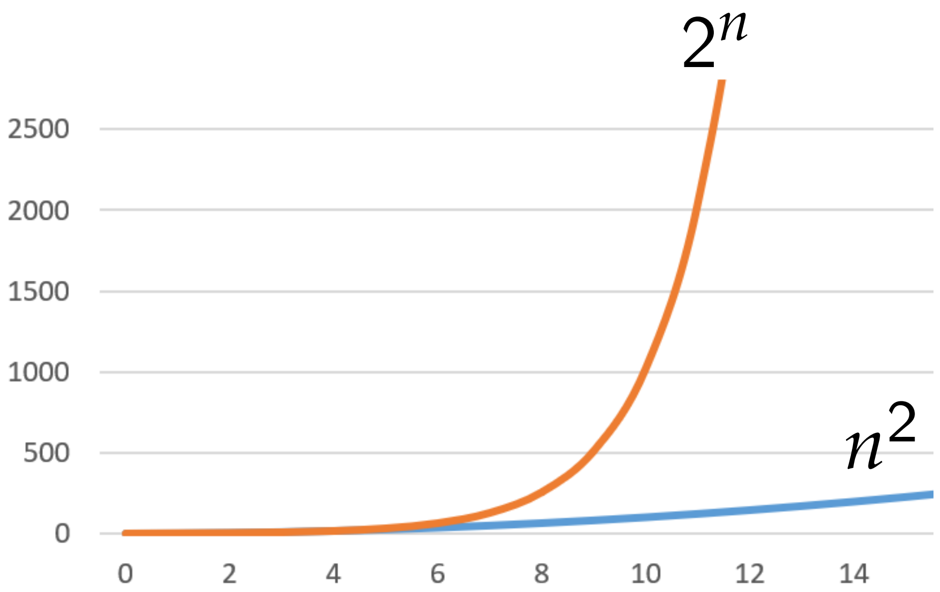

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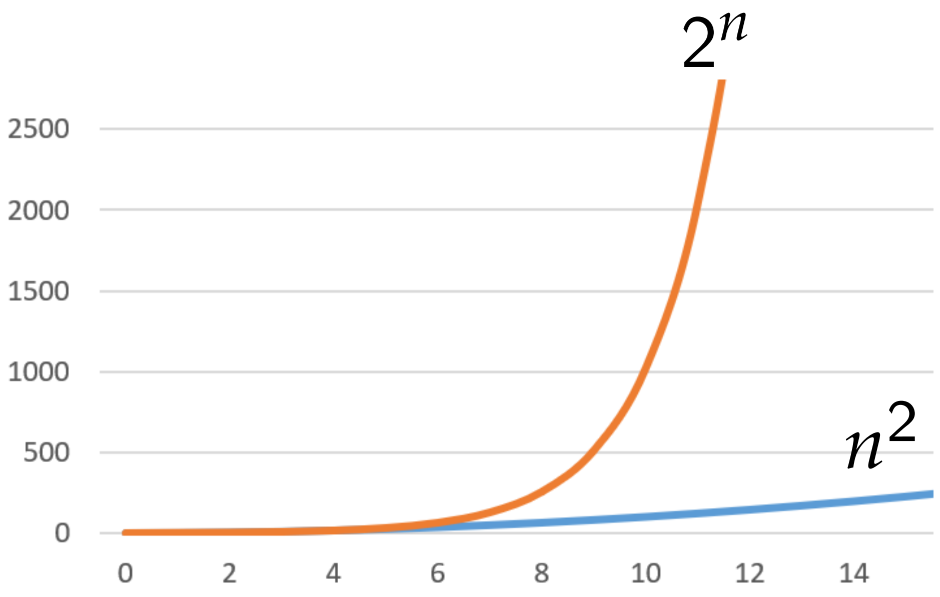

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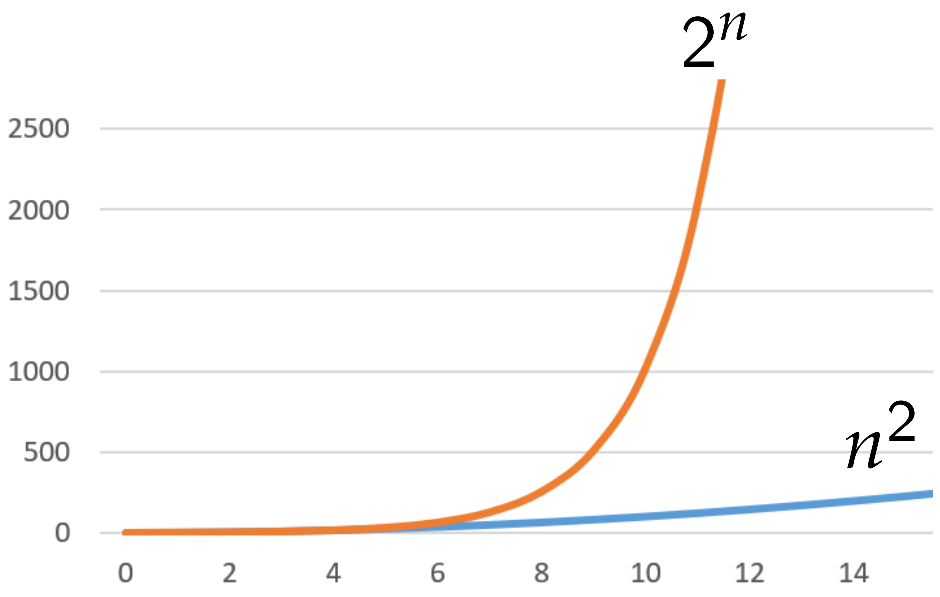

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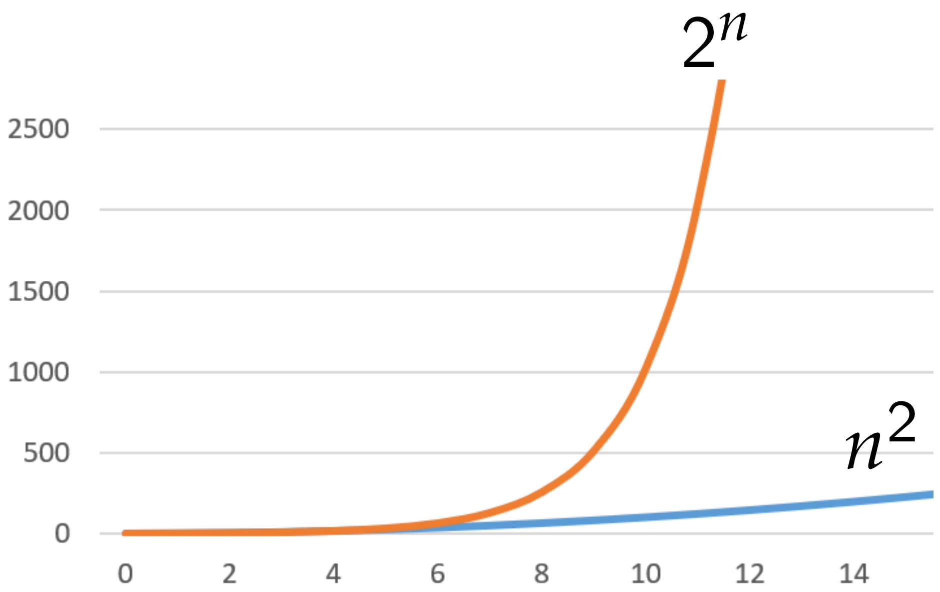

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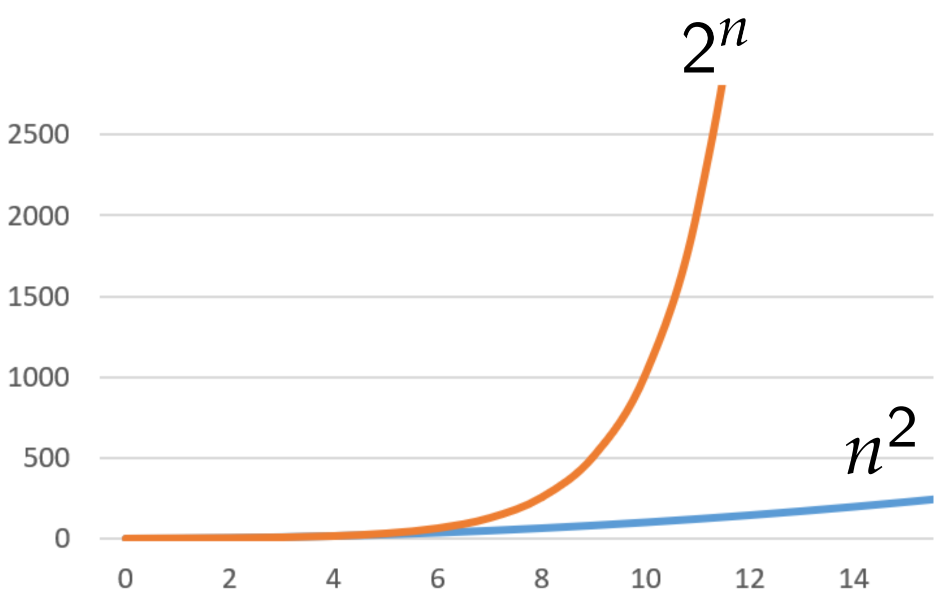

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  - TSP solvable exactly for $n \leq 2000$ and specialized instances with $n \leq 85900$

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- Reducing the base of the runtime to $b < a$ results in a *multiplicative* increase:

$$b^{n_0'} = a^{n_0} \rightsquigarrow n_0' = n_0 \cdot \log_b a$$

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- SAT: No better algorithm than trivial brute-force search known.

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- typical result

Approach	Runtime in $\mathcal{O}$-Notation	$\mathcal{O}^*$-Notation
Brute-Force	$\mathcal{O}(2^n)$	$\mathcal{O}^*(2^n)$
Algorithm A	$\mathcal{O}(1.5^n \cdot n)$	$\mathcal{O}^*(1.5^n)$
Algorithm B	$\mathcal{O}(1.4^n \cdot n^2)$	$\mathcal{O}^*(1.4^n)$

Traveling Salesperson Problem (TSP)

Input. Distinct cities $\{v_1, v_2, \ldots, v_n\}$ with distances $d(c_i, c_j) \in Q_{\geq 0}$;
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$$\sum_{i=1}^{n-1} d(v_{\pi(i)}, v_{\pi(i+1)}) + d(v_{\pi(n)}, v_{\pi(1)})$$

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Brute-force.

- Try all permutations and pick the one with smallest weight.
- Runtime: $\Theta(n! \cdot n) = n \cdot 2^{\Theta(n \log n)}$

TSP – Dynamic Programming

Bellman-Held-Karp Algorithm

Idea.

- Reuse optimal substructures with dynamic programming.

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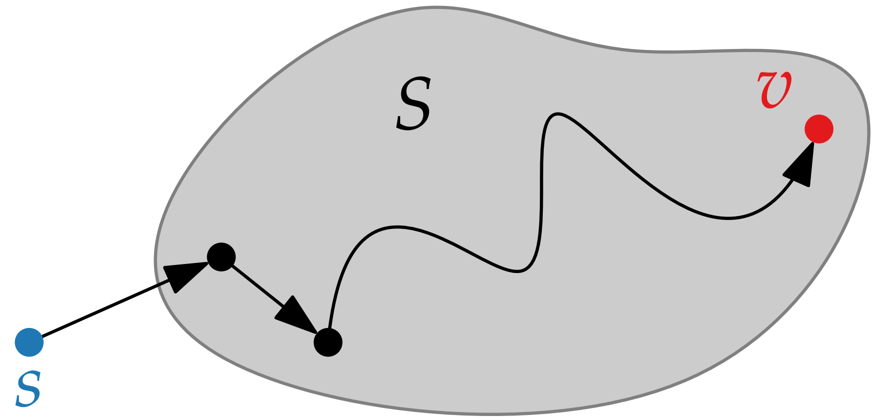

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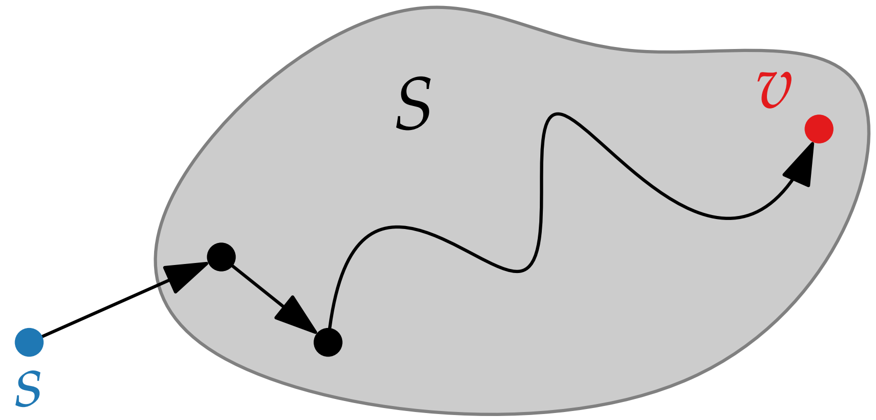

- Use $\mathrm{OPT}[S - v, u]$ to compute $\mathrm{OPT}[S, v]$.

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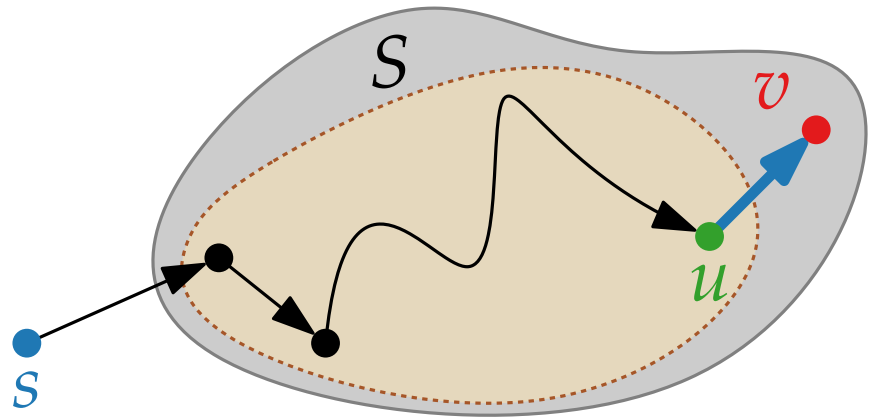

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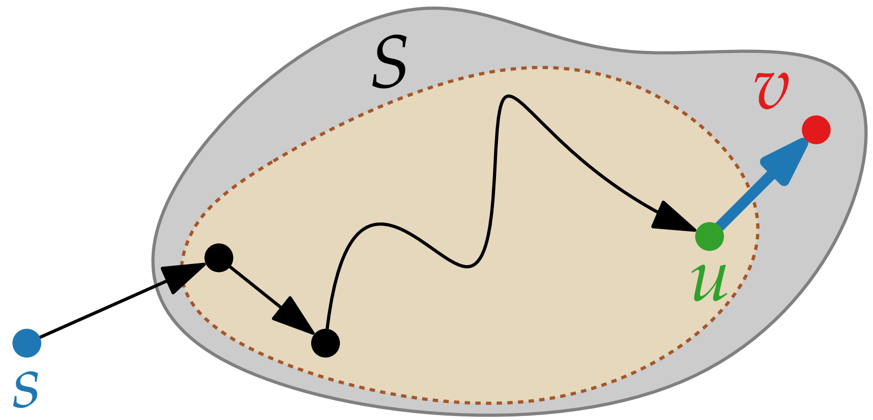

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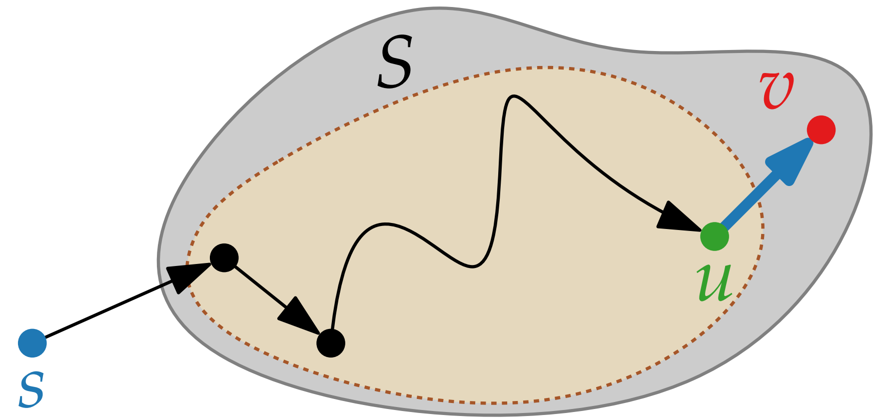

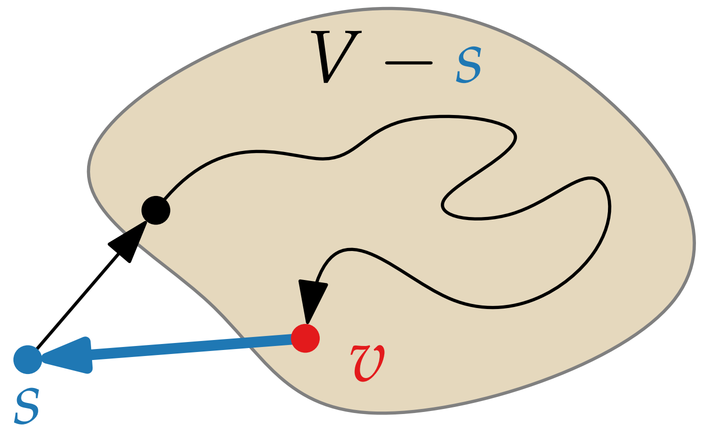

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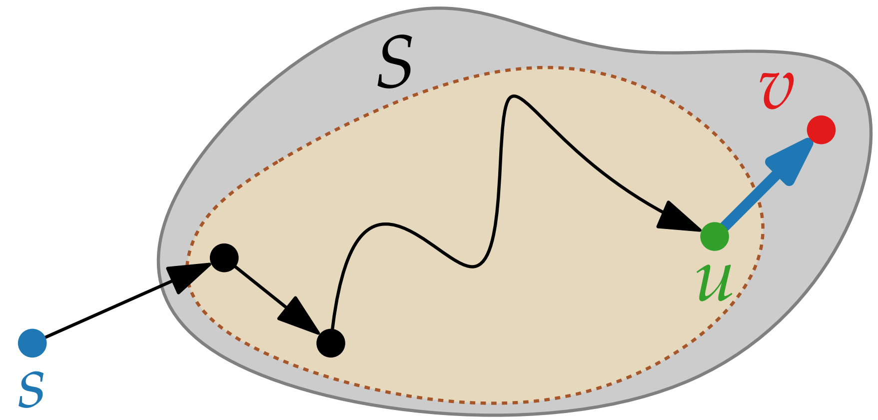

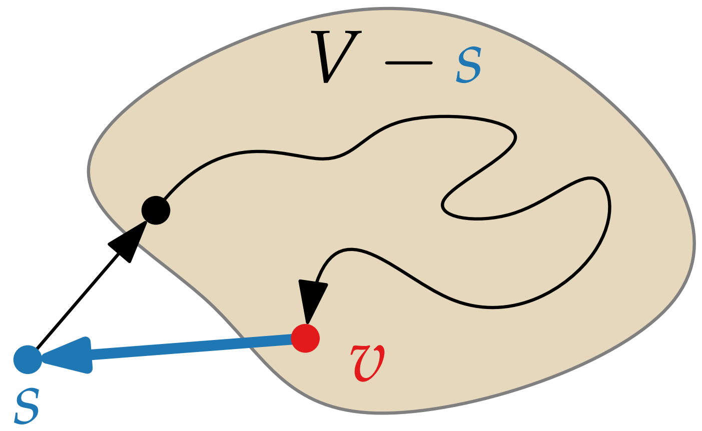

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Pseudocode.

Algorithm Bellmann-Held-Karp(G, c)

```
foreach v ∈ V − s do
    OPT[{v}, v] = c(s, v)
for j ← 2 to n − 1 do
    foreach S ⊆ V − s with |S| = j do
        foreach v ∈ S do
            OPT[S, v] ← min{ OPT[S − v, u]
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TSP – Dynamic Programming

Pseudocode.

Algorithm Bellmann-Held-Karp(G, c)

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foreach v ∈ V − s do
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- Or actually better? What table values do we need to store?

TSP – Discussion

- DP algorithm that runs in $\mathcal{O}^*(2^n)$ time and $\mathcal{O}(2^n \cdot \overset{?}{n})$ space
- Brute-force runs in $2^{\mathcal{O}(n \log n)}$ time
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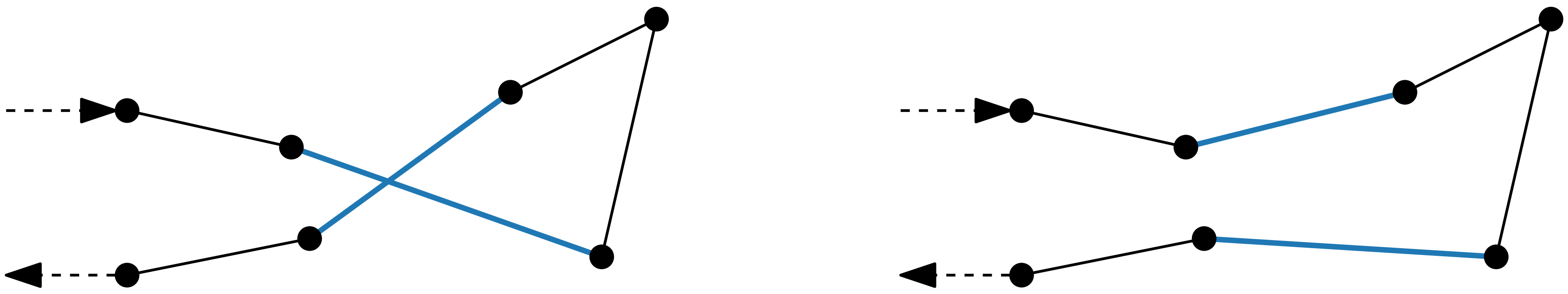
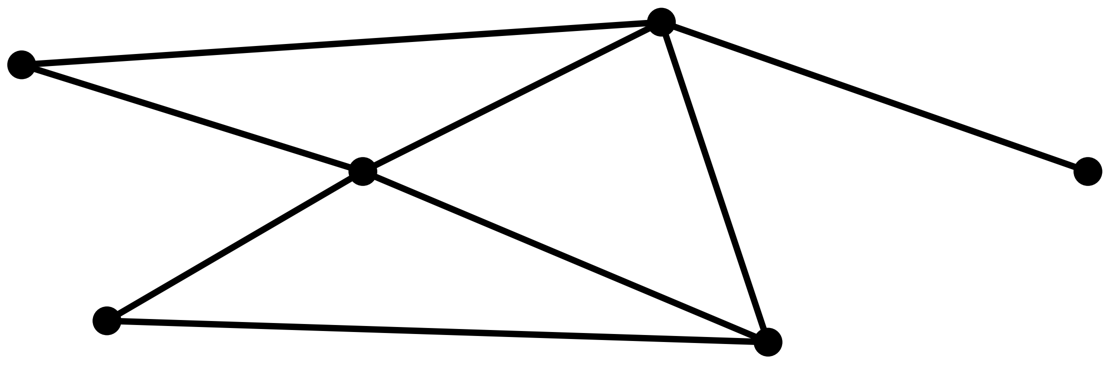
- In practice, one successful approach is to start with a greedily computed Hamiltonian cycle and then use 2-OPT and 3-OPT swaps to improve it.



Maximum Independent Set (MIS)

Input. Graph $G = (V, E)$ with n vertices.



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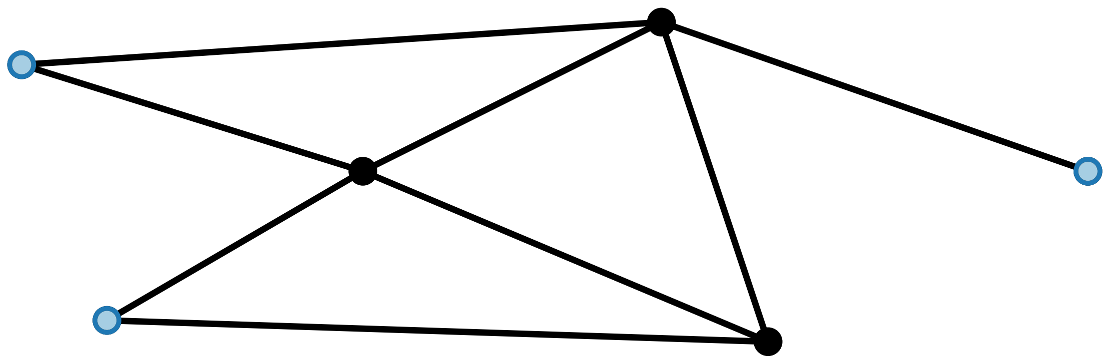

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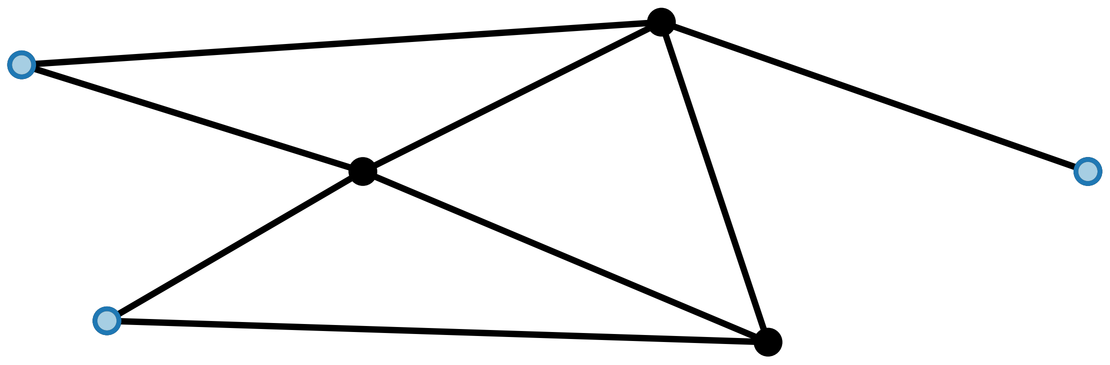

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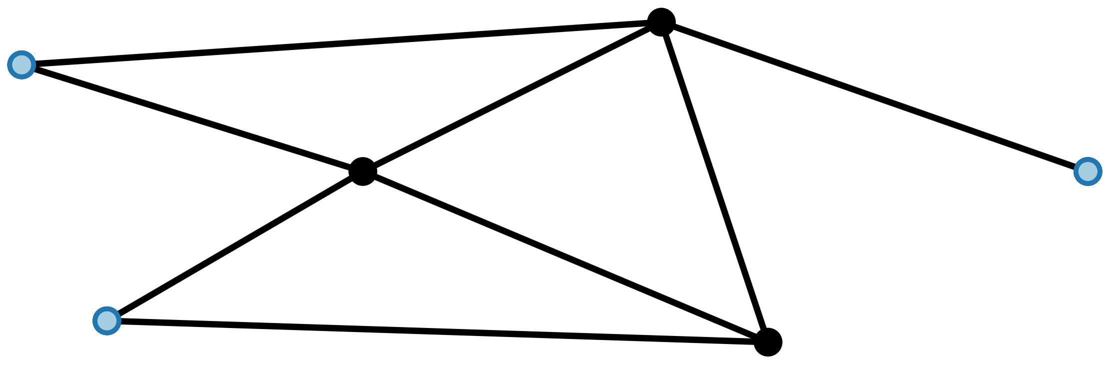

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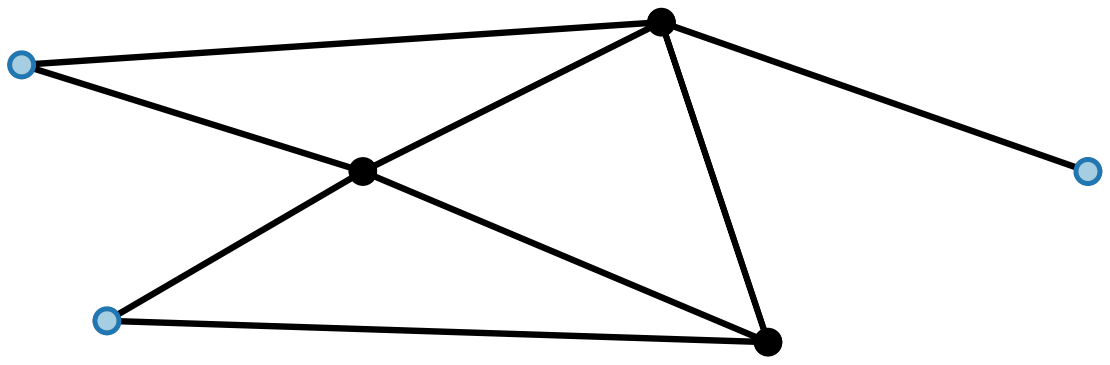

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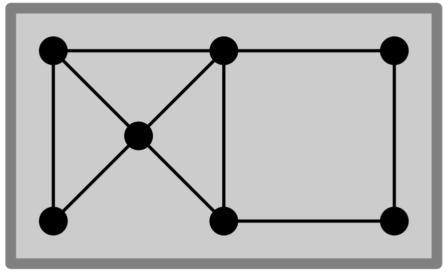
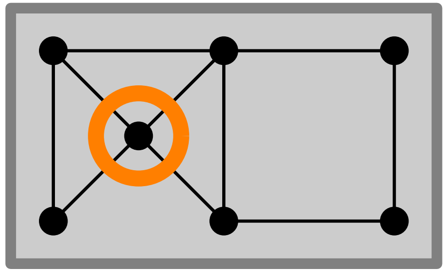
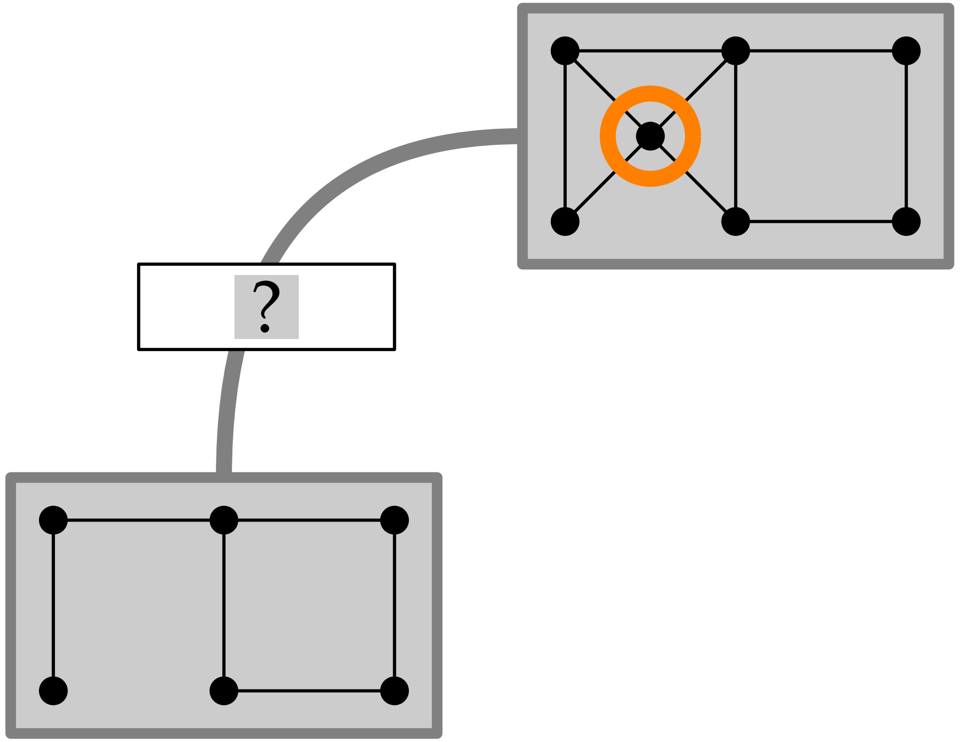
Algorithm NaiveMIS(G)

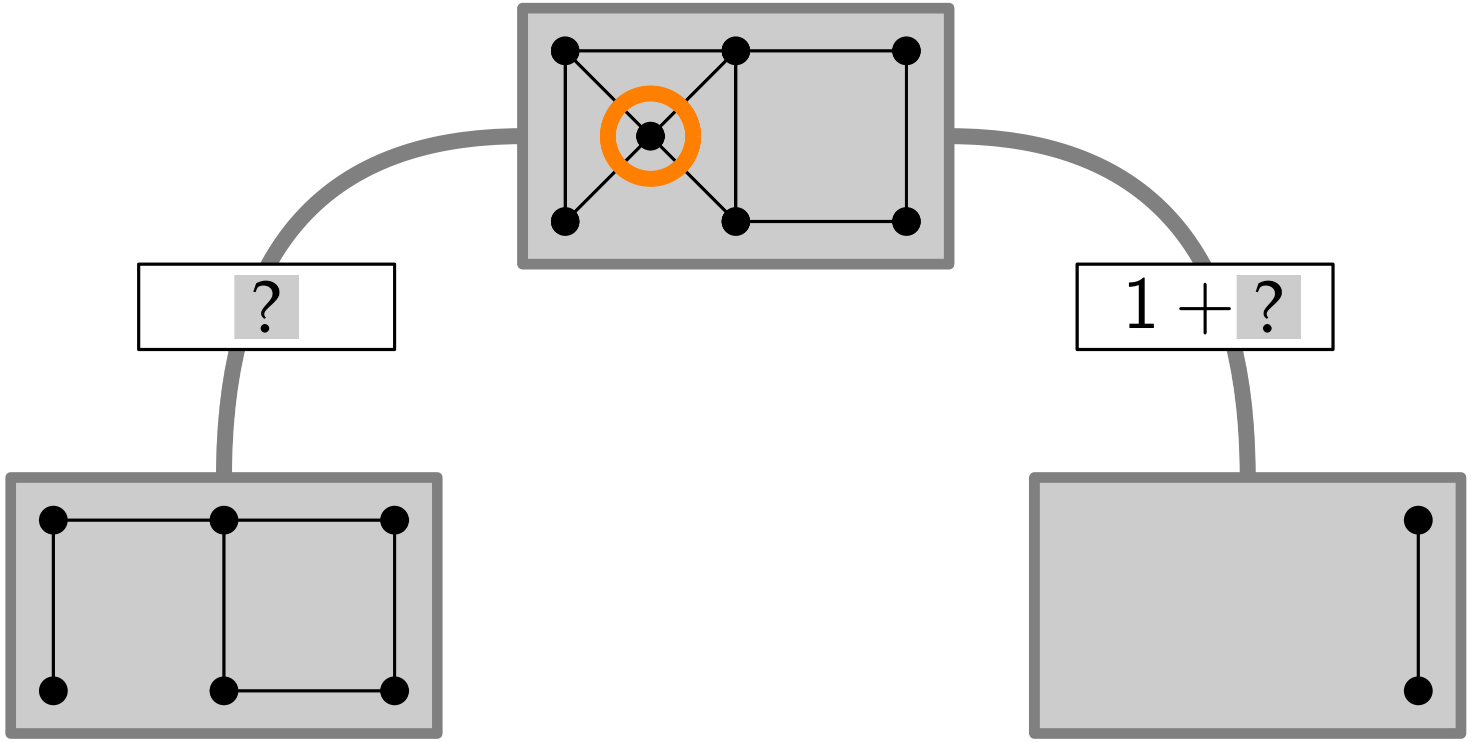
if $V = \emptyset$ **then**
  return 0

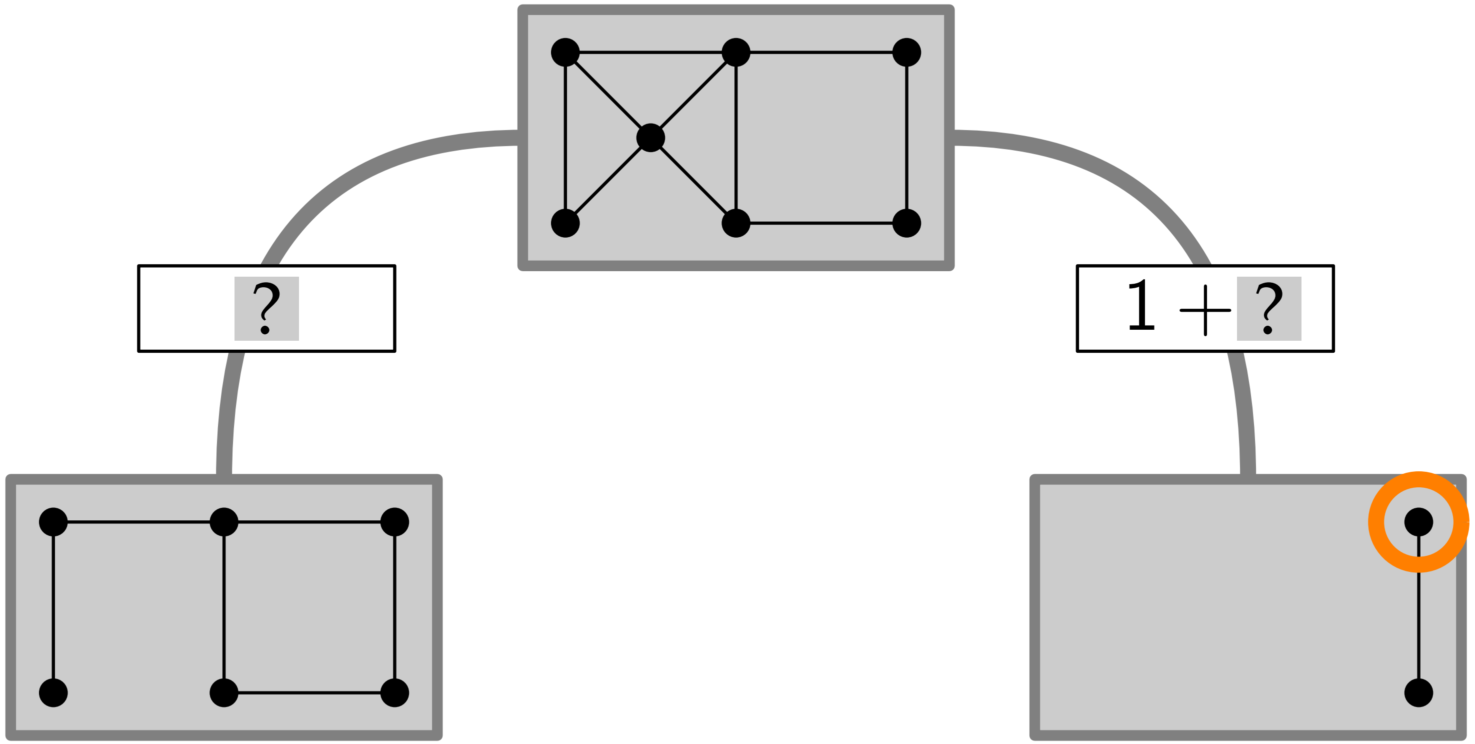
$v \leftarrow$ arbitrary vertex in $V(G)$
return max$\{1+$ NaiveMIS$(G - N(v) - \{v\})$, NaiveMIS$(G - \{v\})\}$

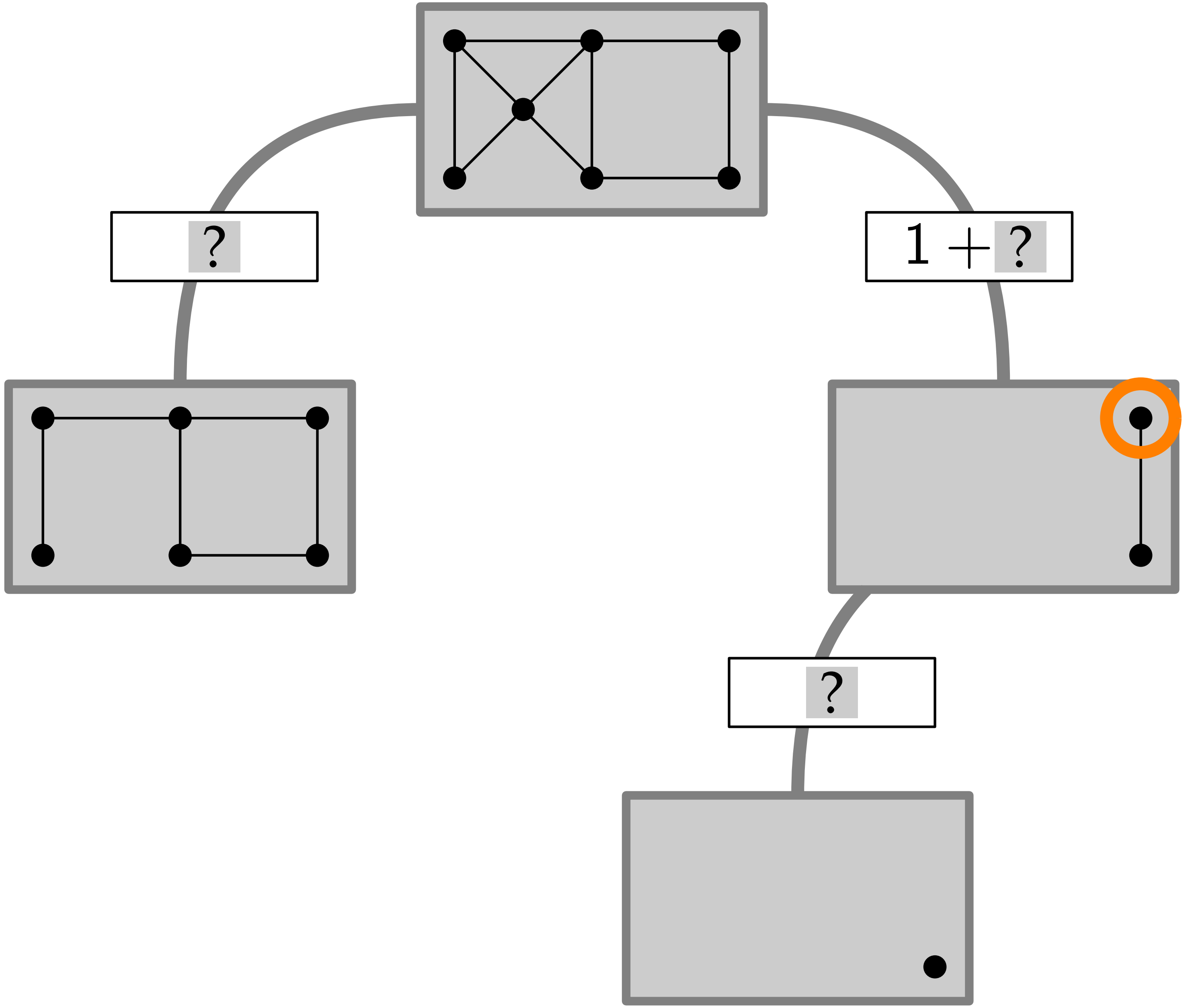







?
1 + ?


?
1 + ?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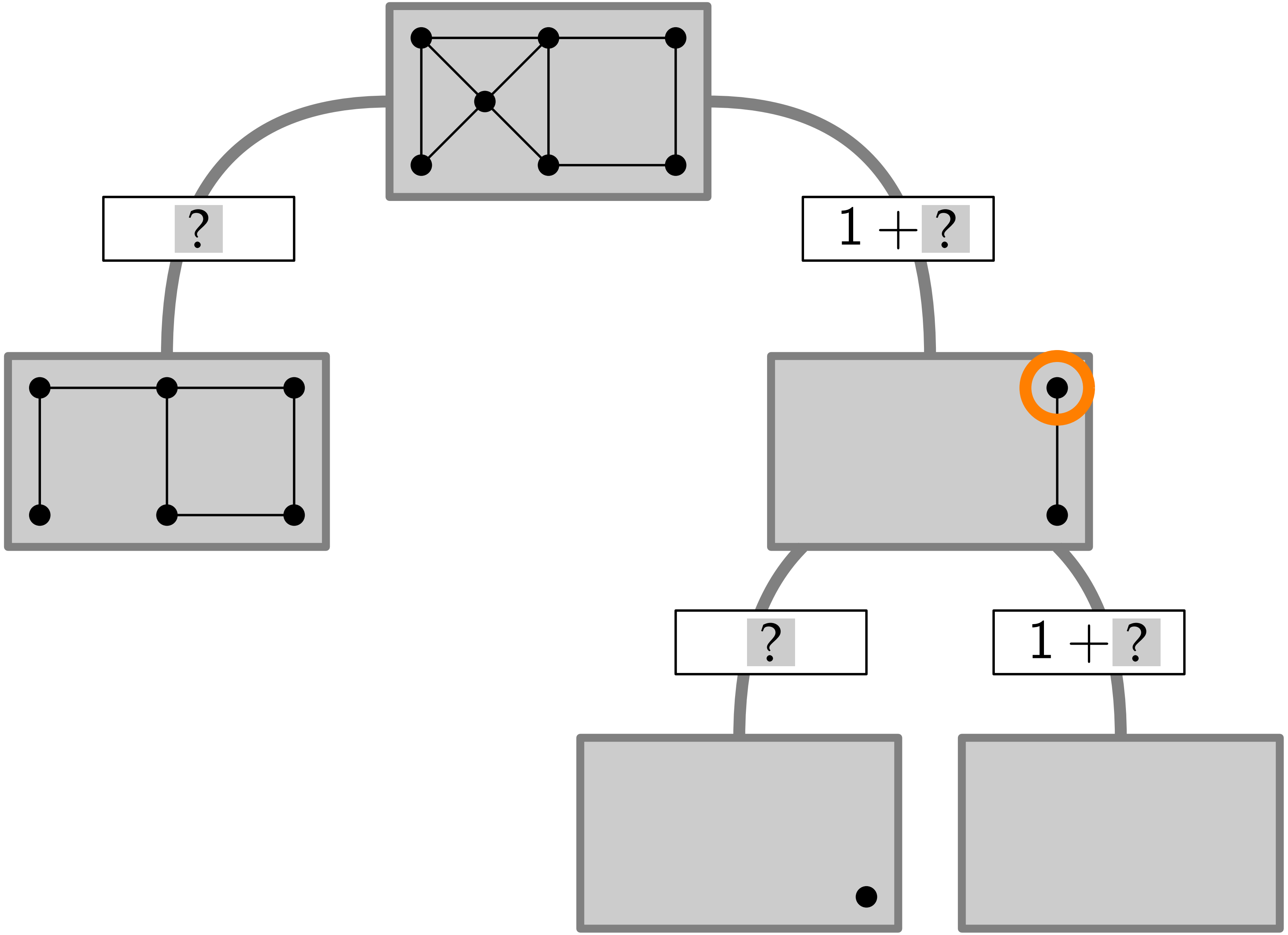
1 + ?
?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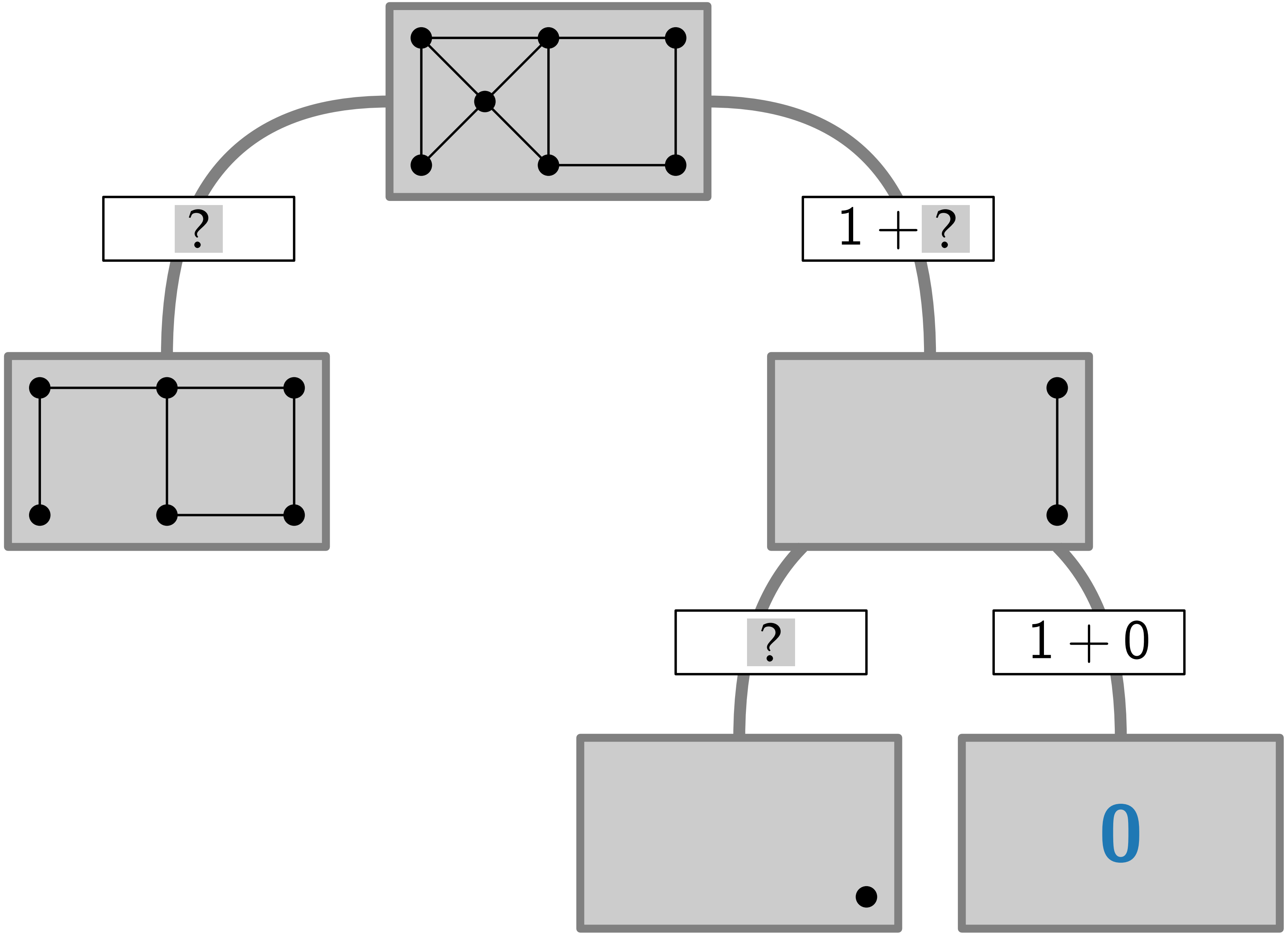
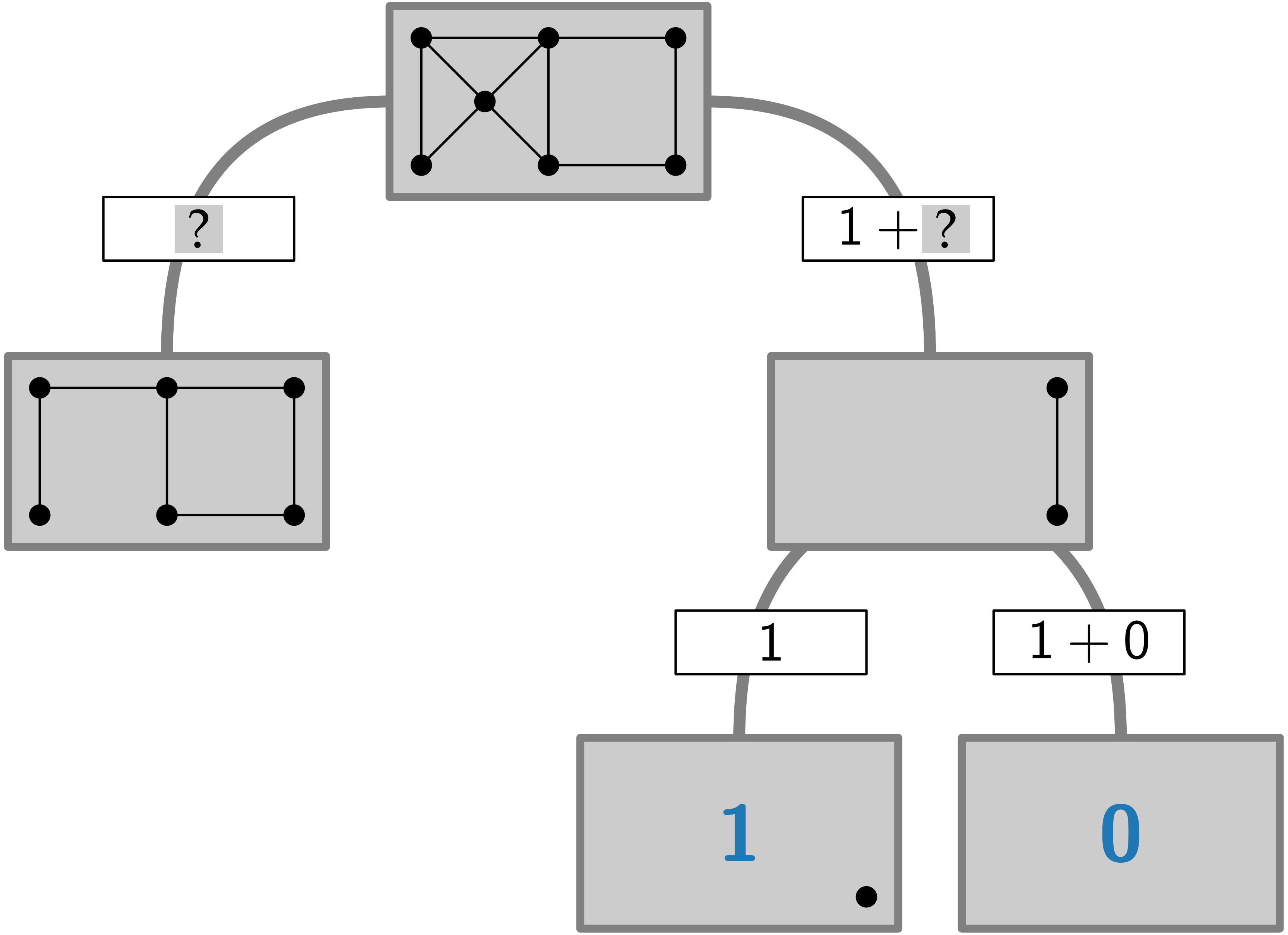
?
1 + ?


?
1 + ?
?
1 + 0
0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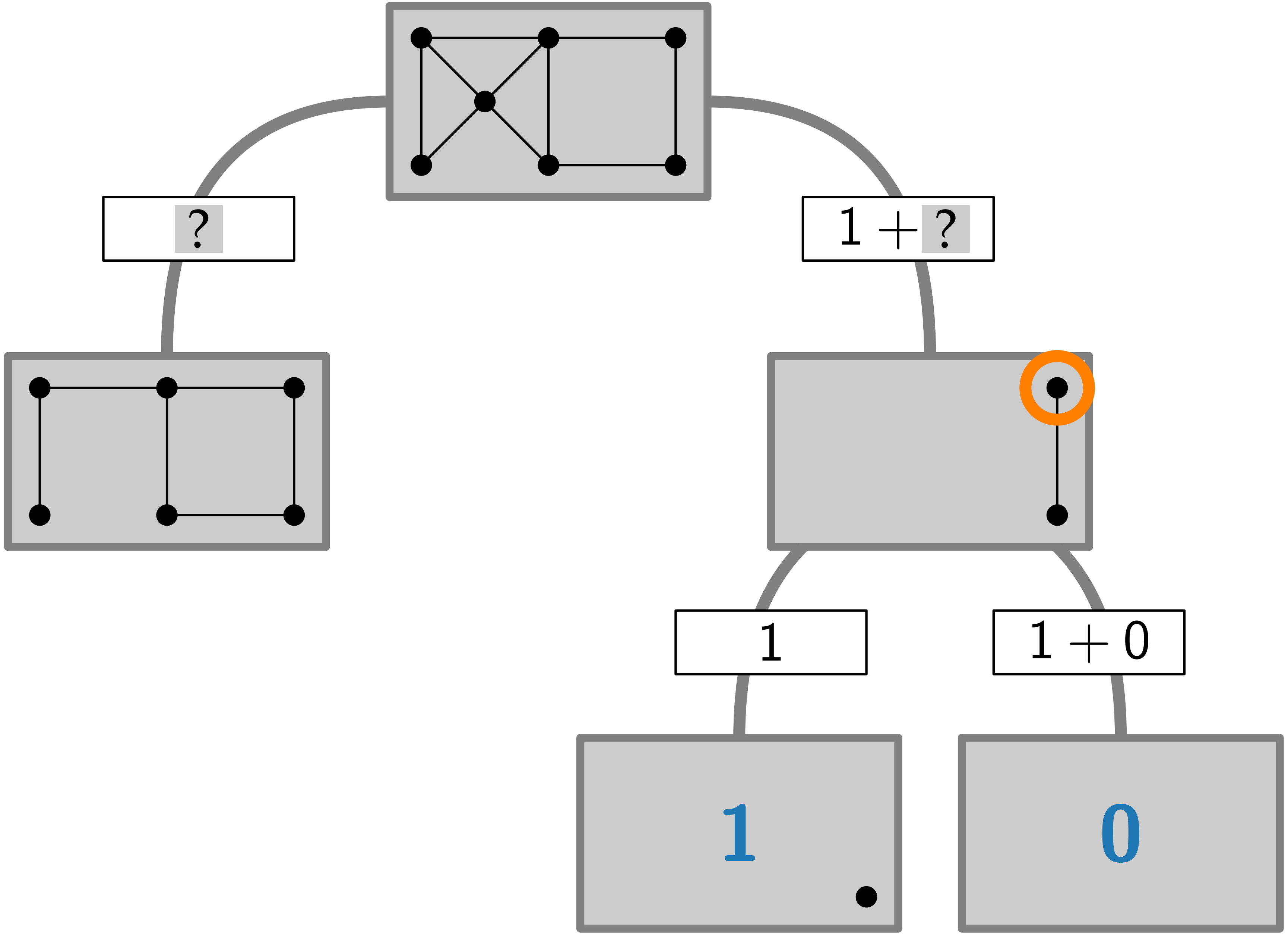
1
1 + 0
1
0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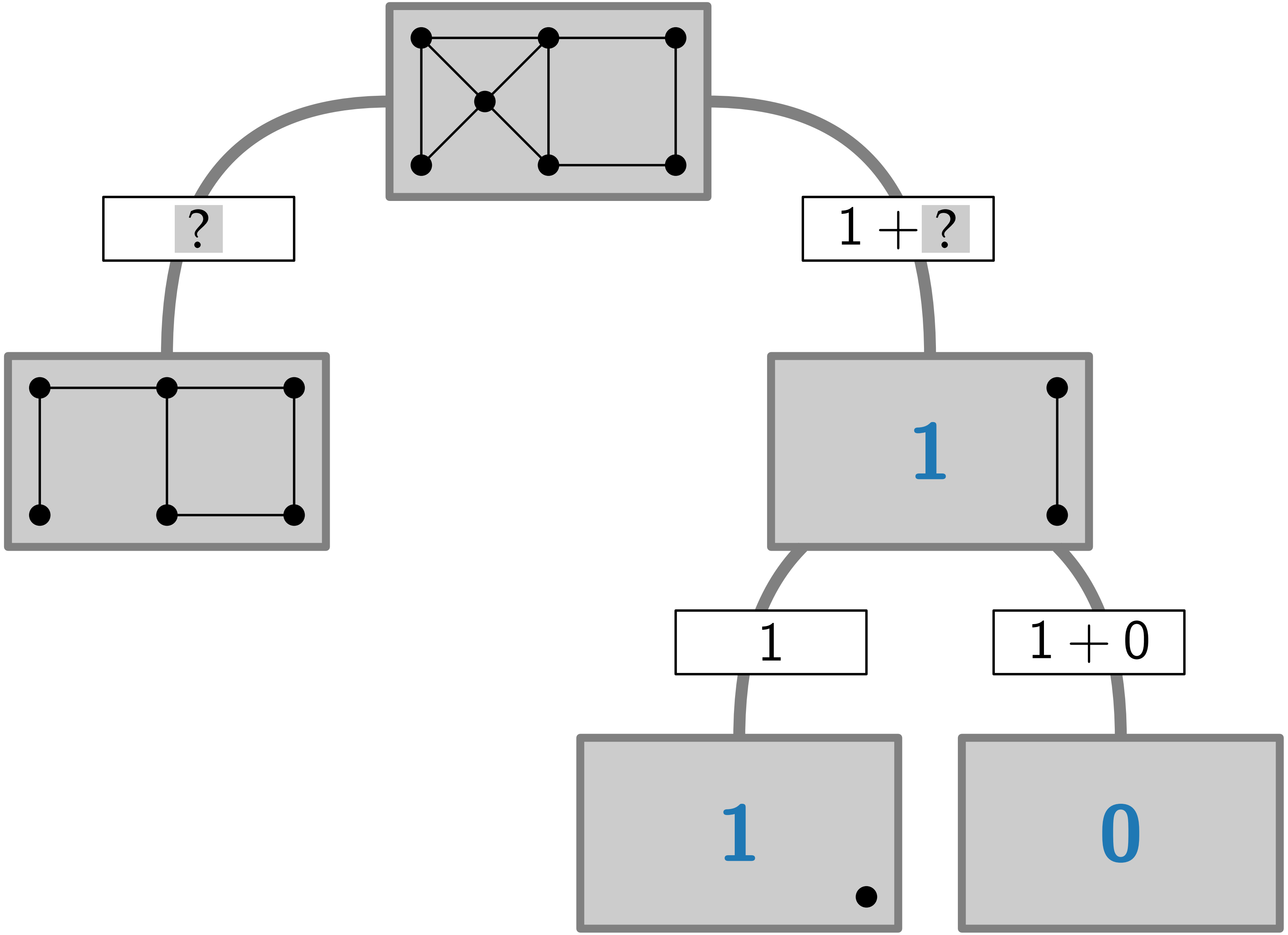
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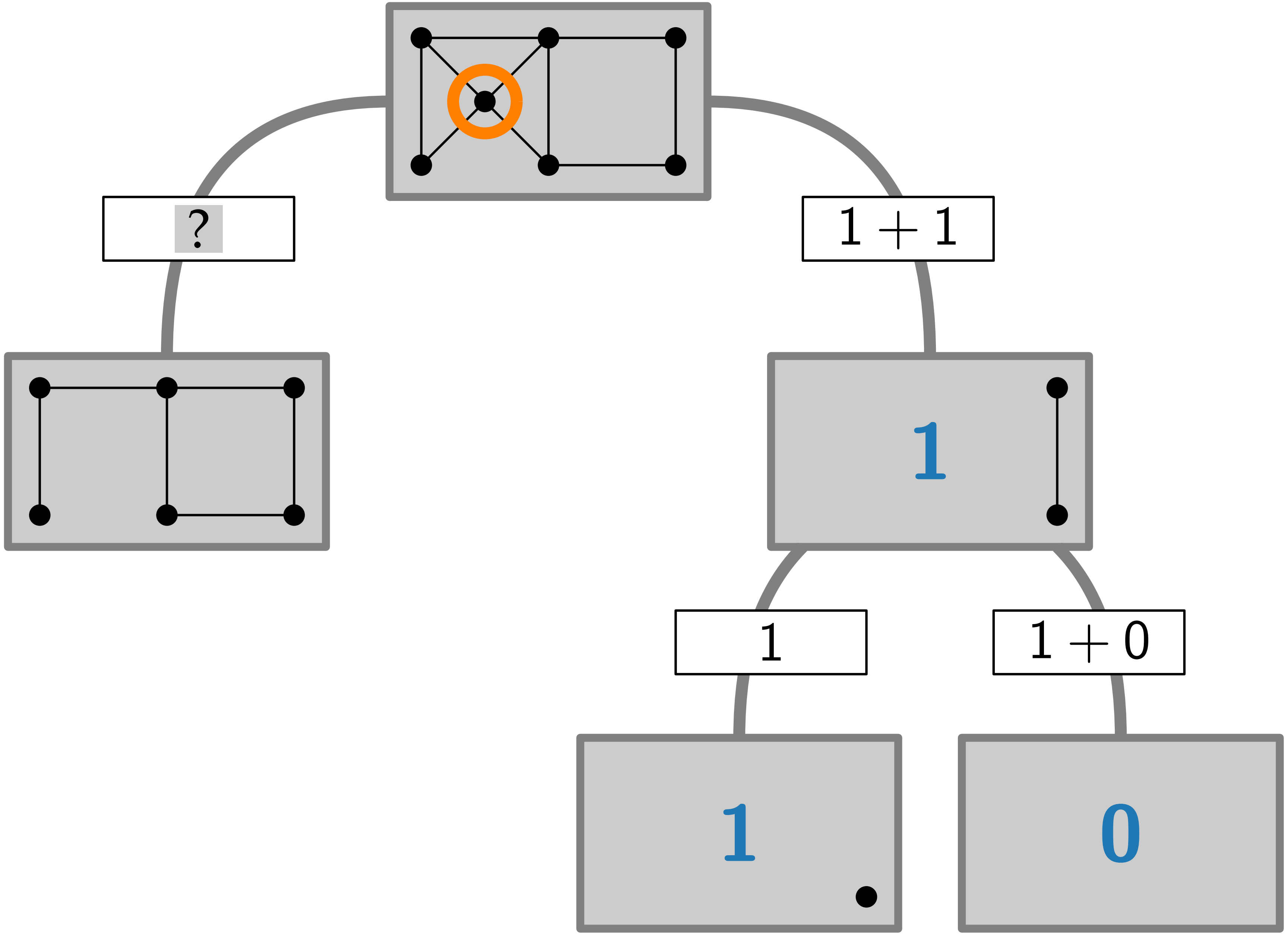
1
1 + 0
1
0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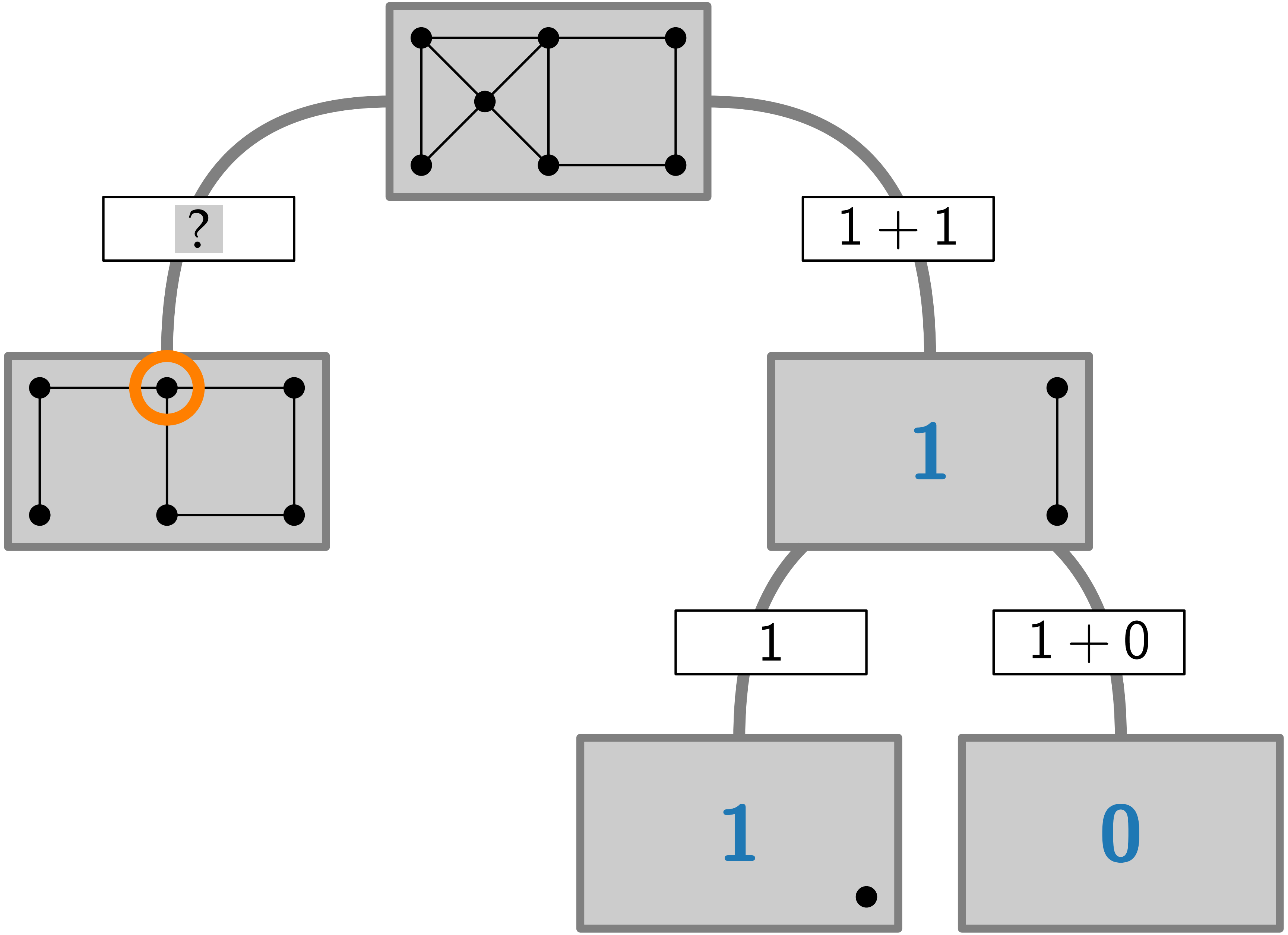
1
1
1 + 0
1
0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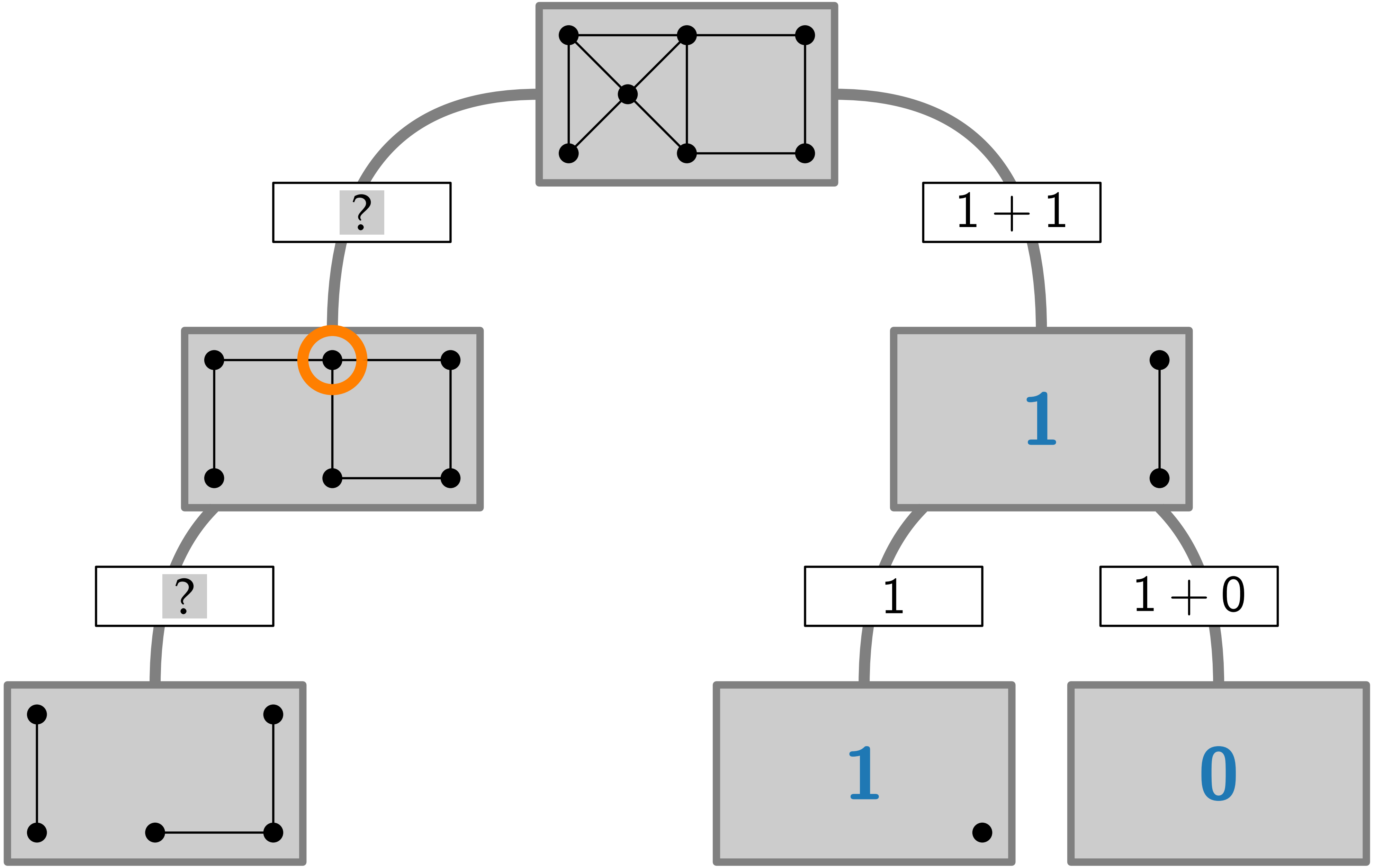
1
1
1 + 0
1
0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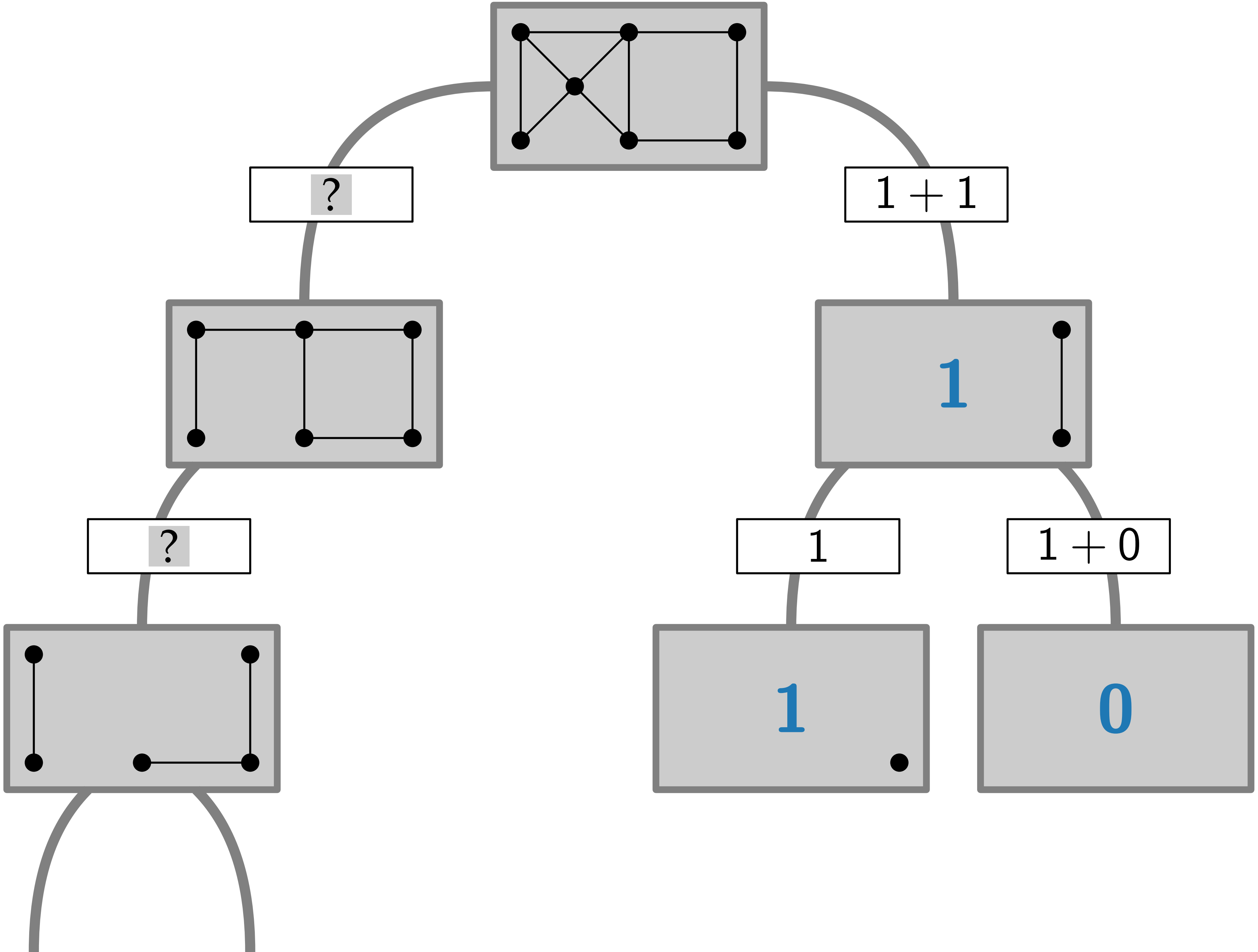
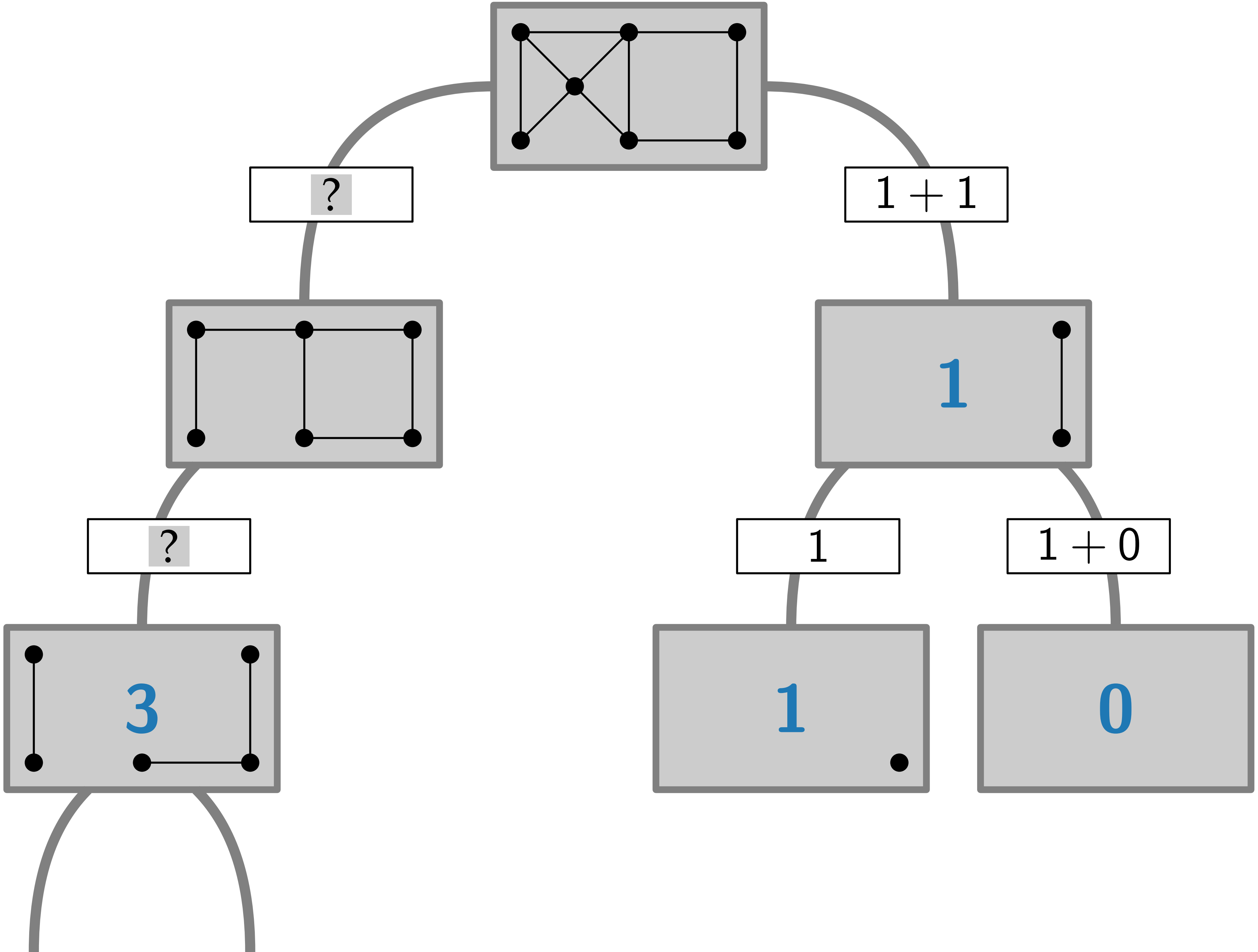
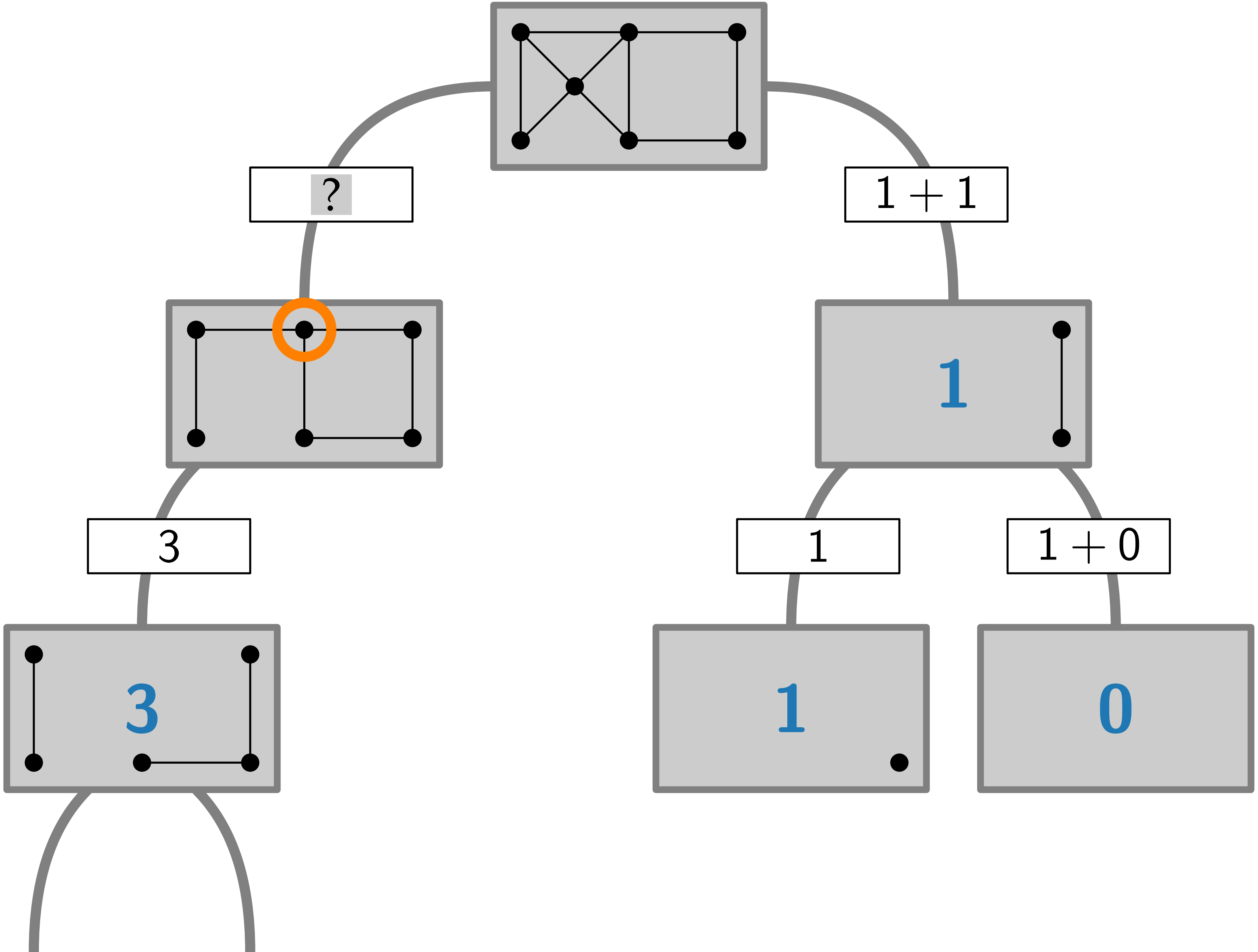
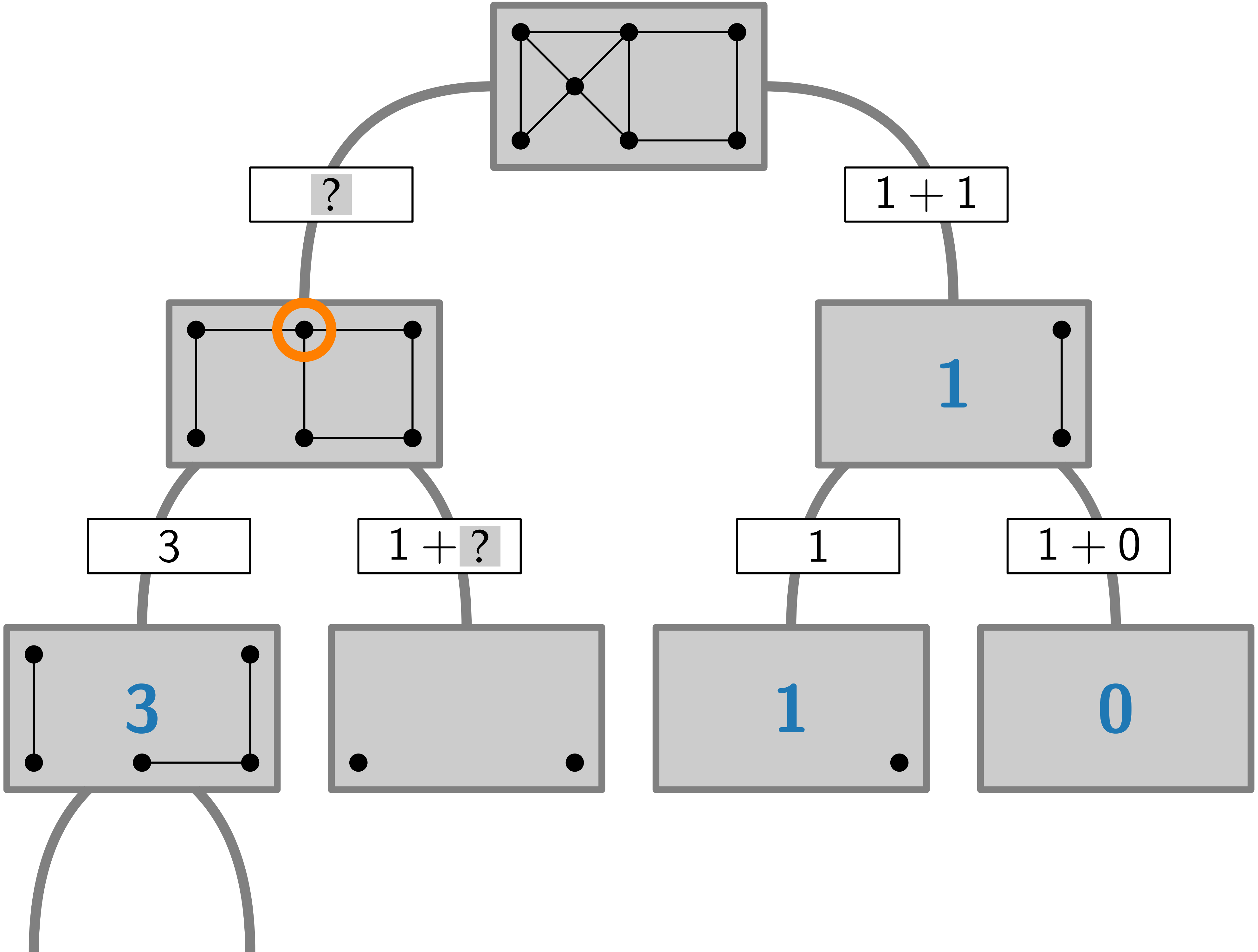
1
?
1
1 + 0
1
0


?
1 + 1
1
?
1
1 + 0
1
0


?
1 + 1
1
?
1
1 + 0
3
1
0


?
1 + 1
1
3
1
1 + 0
3
1
0


?
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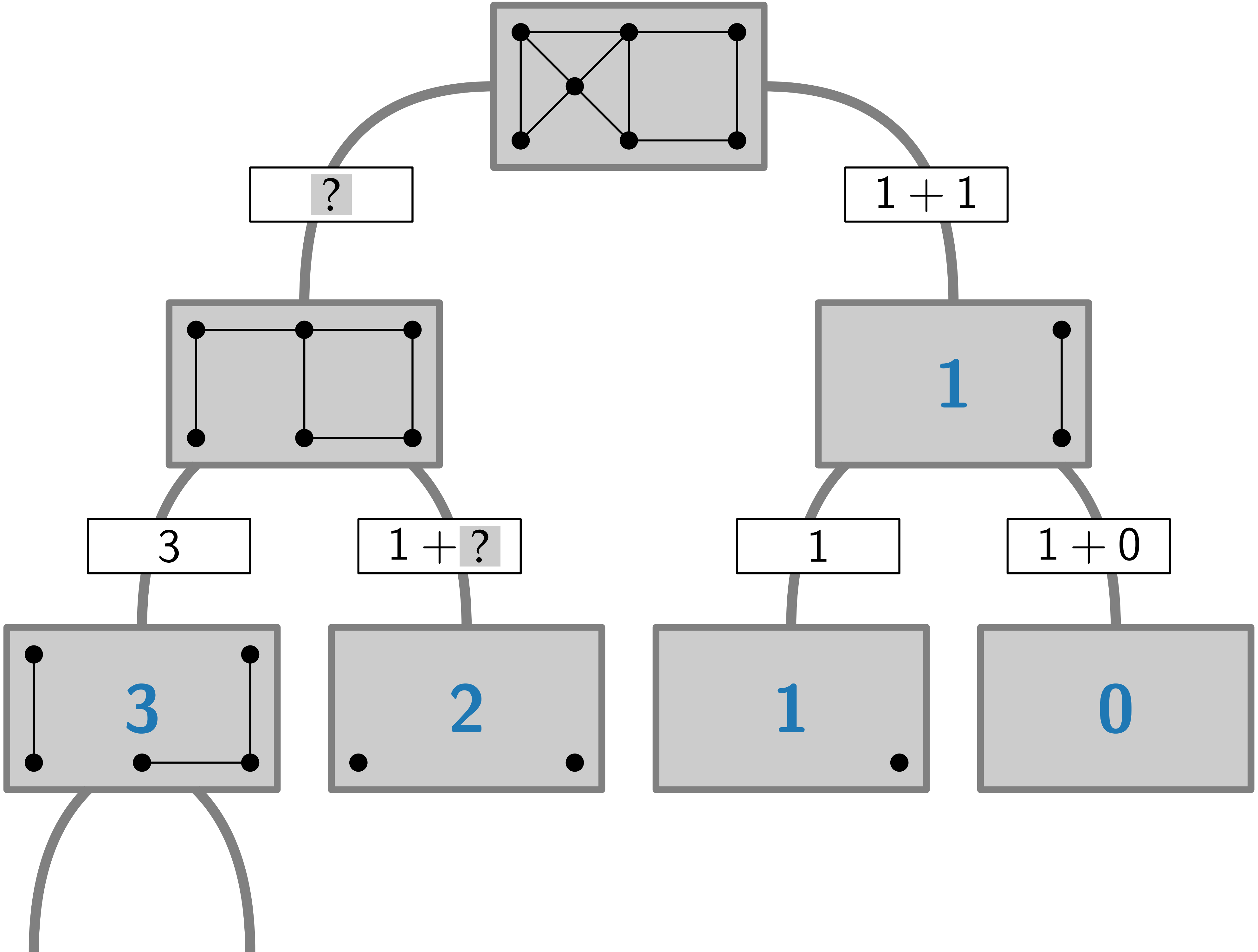
1
3
1 + ?
1
1 + 0
3
1
0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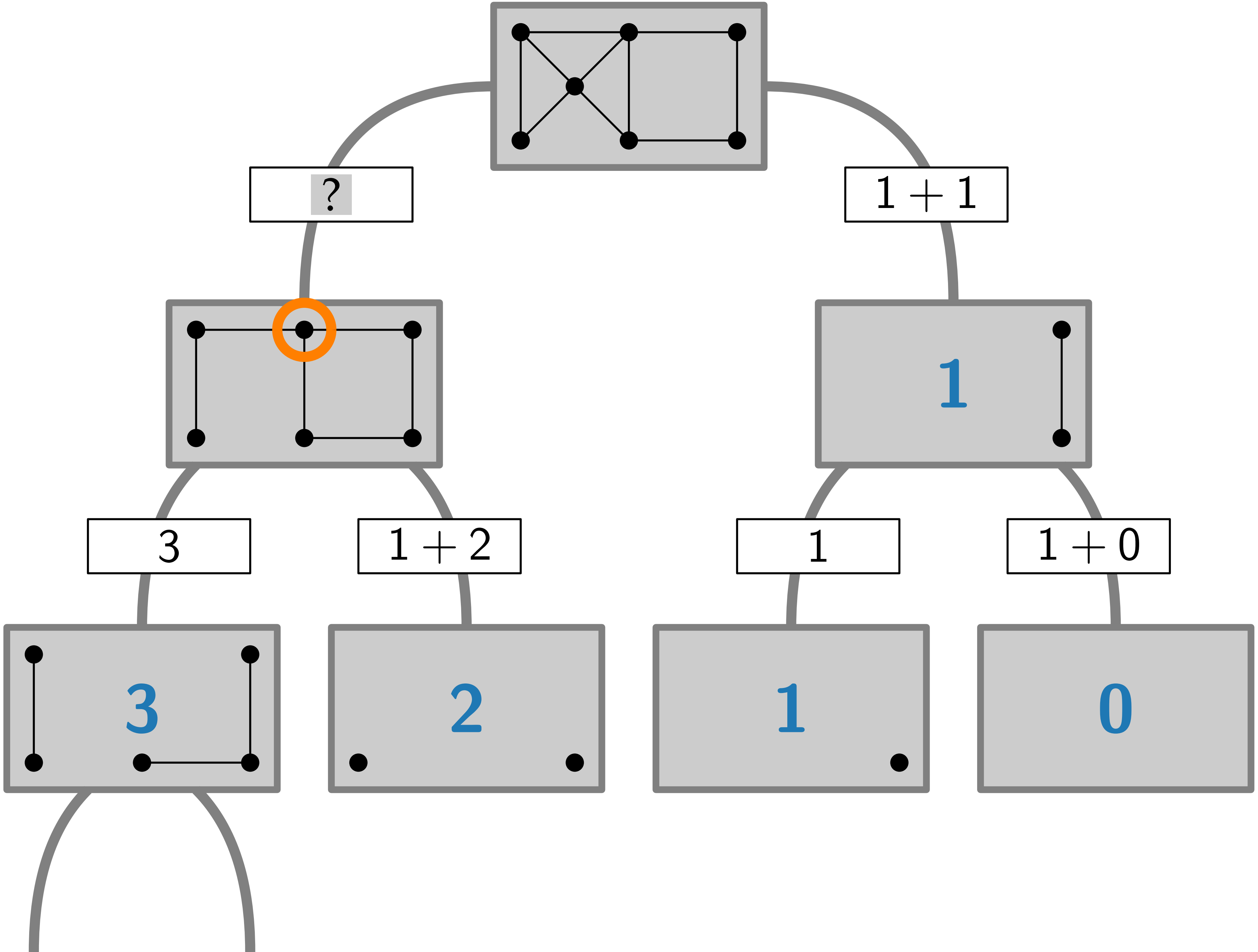
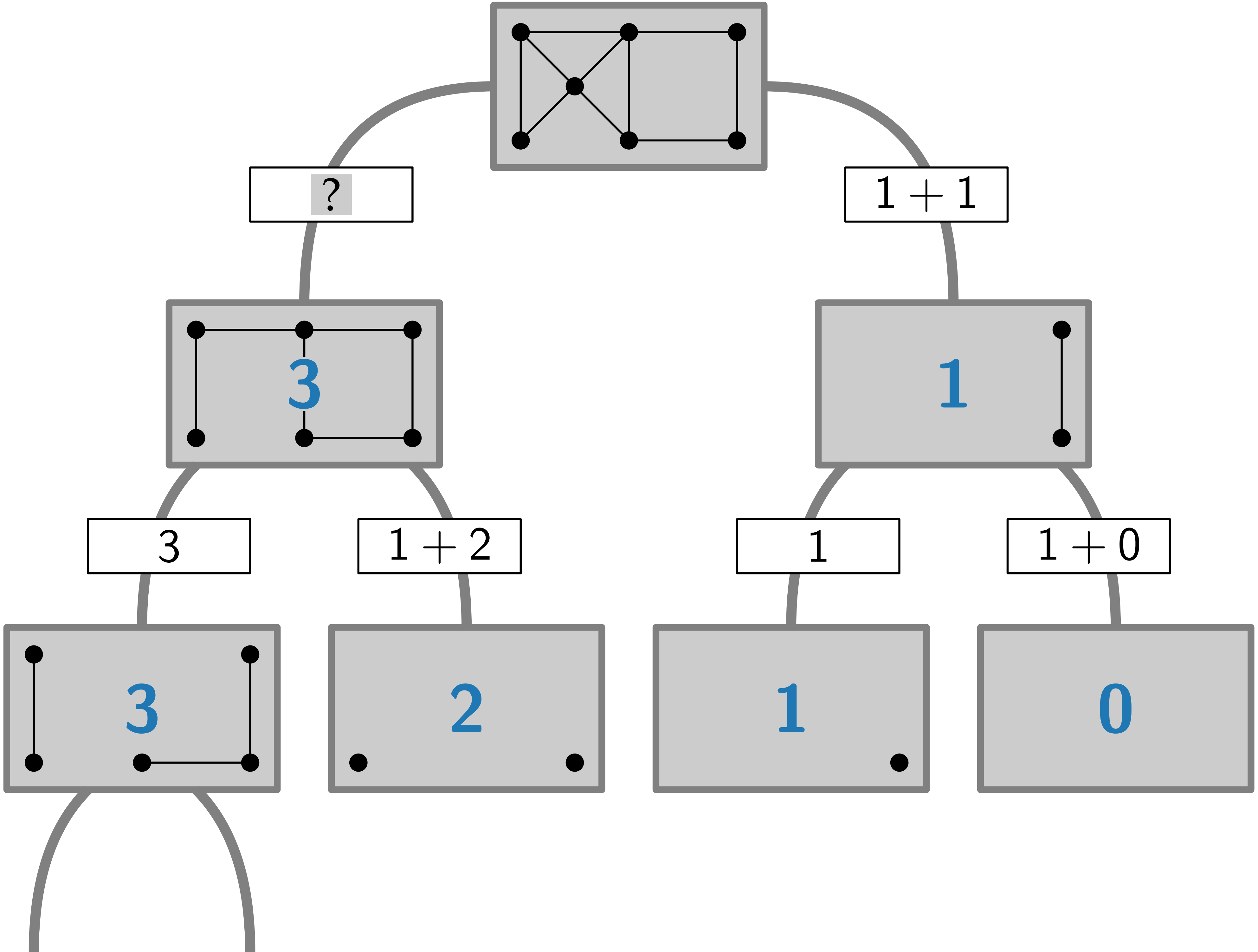
1
3
1 + ?
1
1 + 0
3
2
1
0


?
1 + 1
1
3
1 + 2
1
1 + 0
3
2
1
0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2
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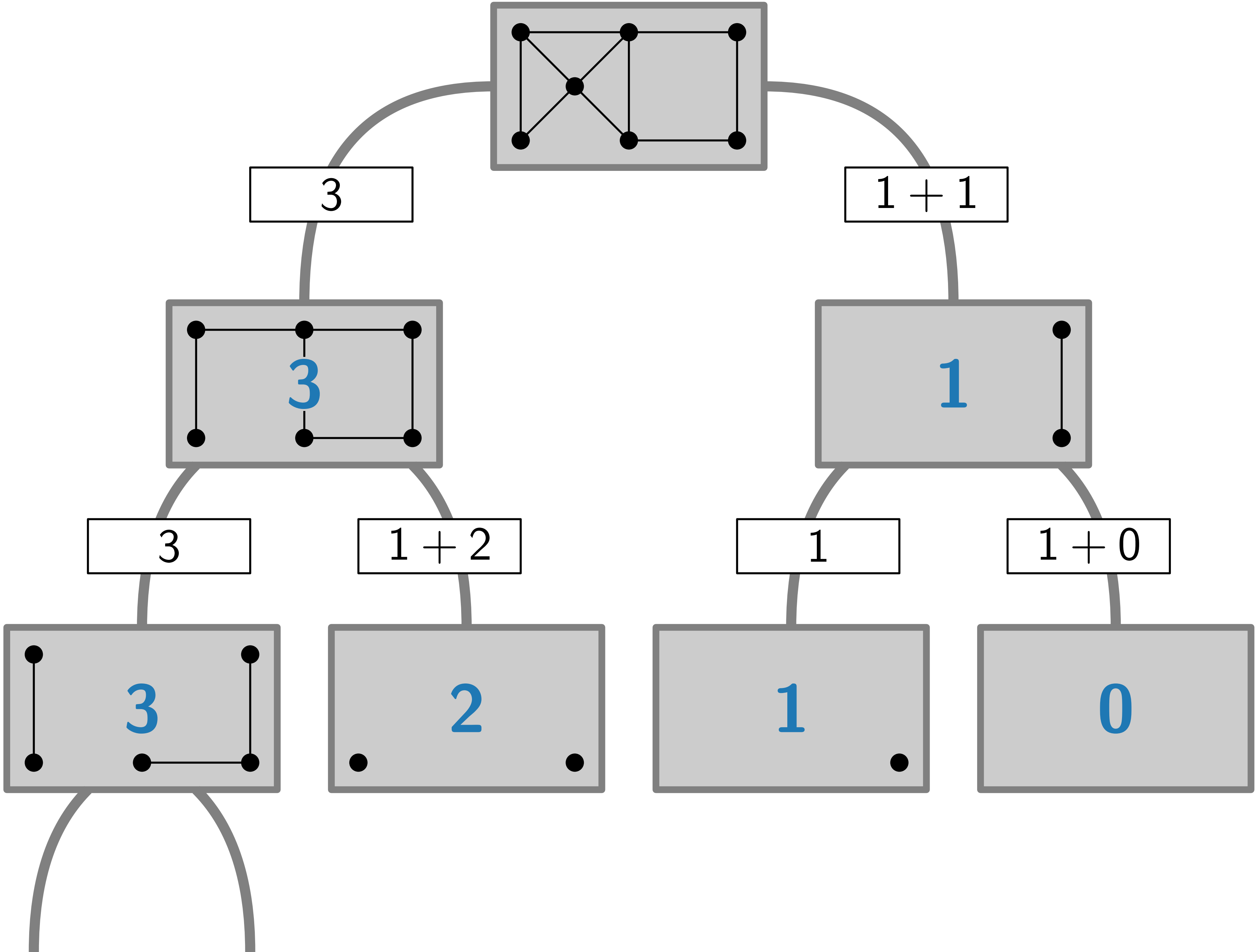
0


3
1 + 1
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1 + 2
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2
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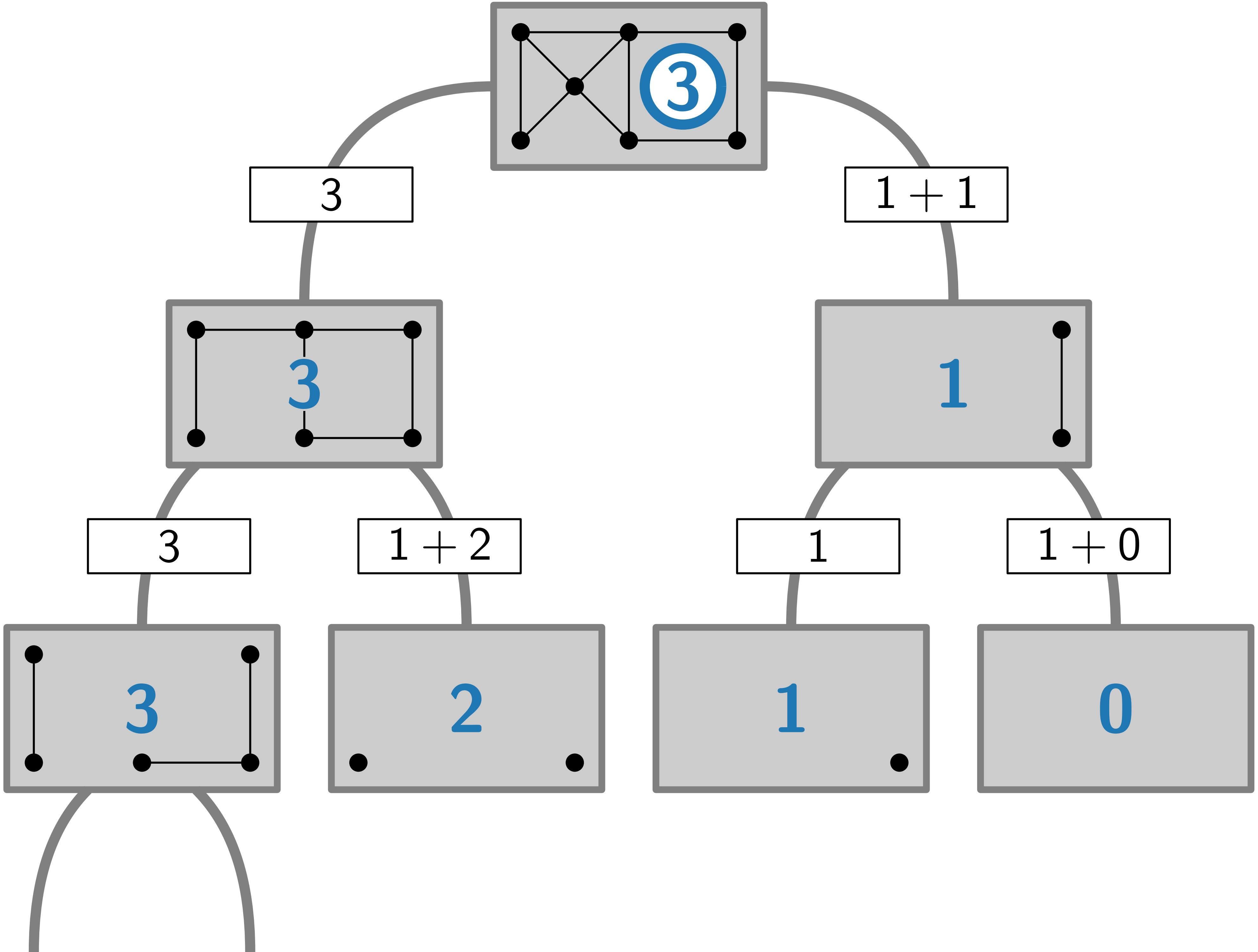
3
3
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1 + 1
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3
1 + 2
1
1 + 0
3
2
1
0

MIS – Smarter Branching

Lemma.
Let U be a maximum independent set in G. Then for each $v \in V$:

1. $v \in U \Rightarrow N(v) \cap U = \emptyset$
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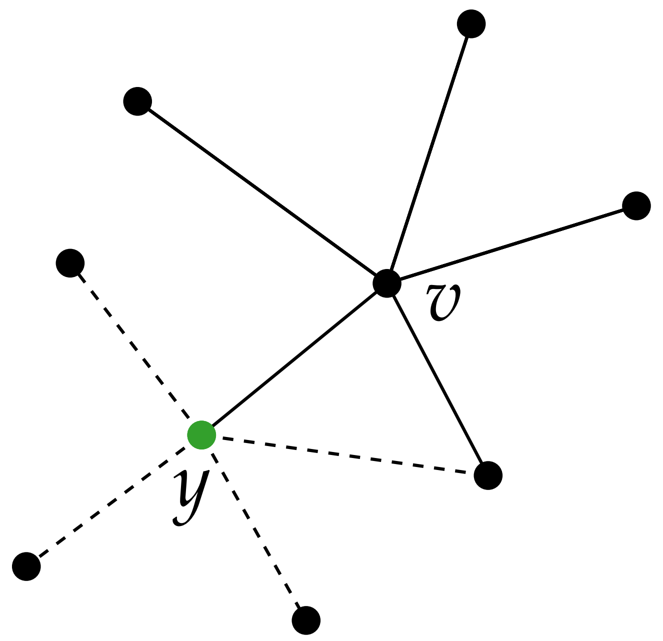
Thus, $N[v] := N(v) \cup \{v\}$ contains some $y \in U$ and no other vertex of $N[y]$ is in U.



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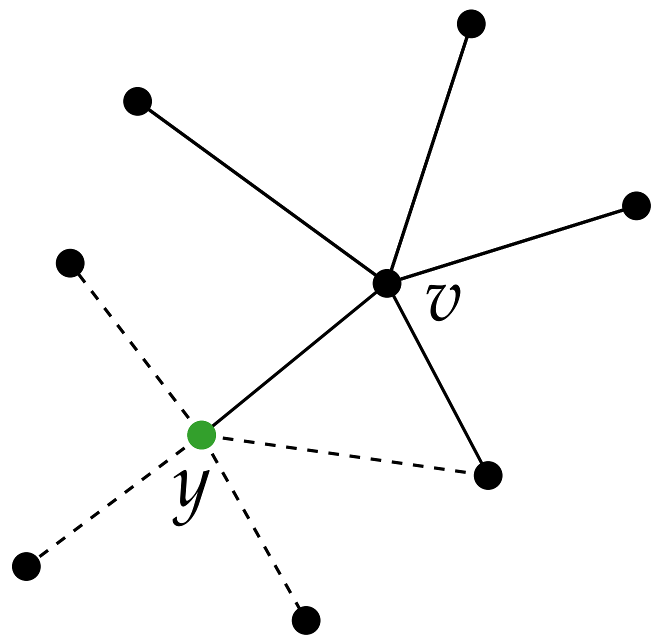

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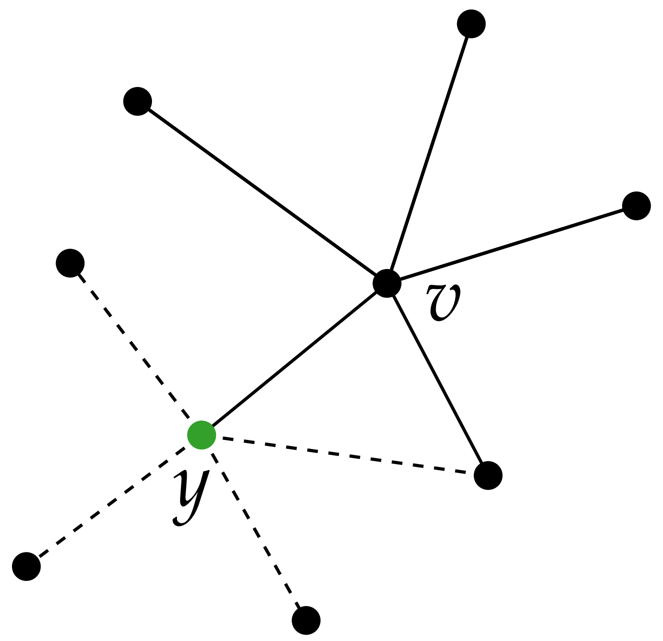

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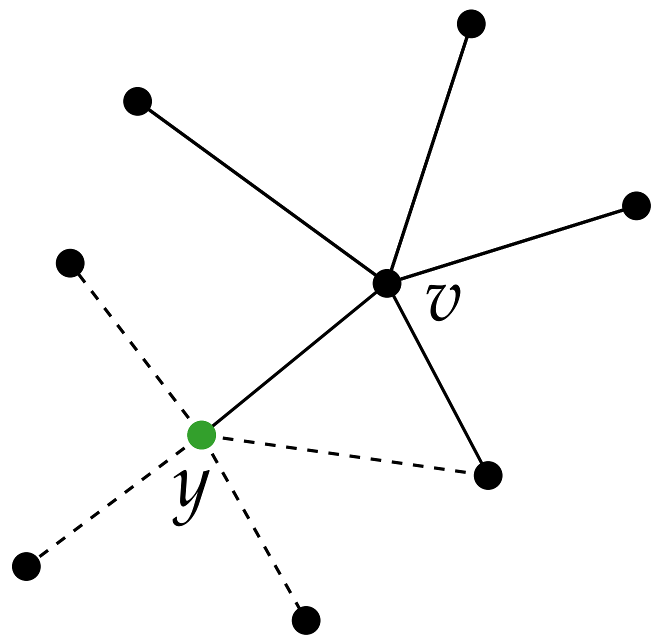

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- Correctness follows from Lemma.
- We prove a runtime of $\mathcal{O}^*(3^{n/3}) = \mathcal{O}^*(1.4423^n)$.

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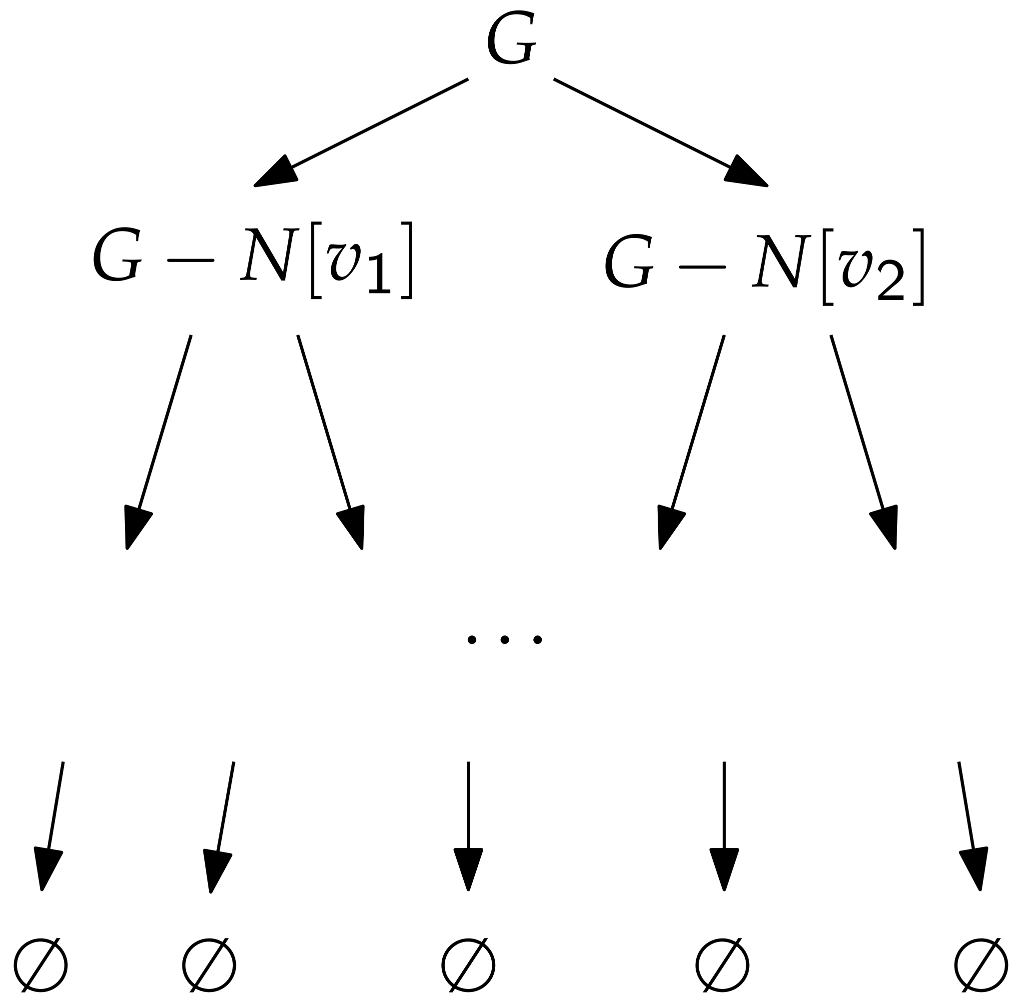
Execution corresponds to a **search tree** whose vertices are labeled with the input of the respective recursive call.



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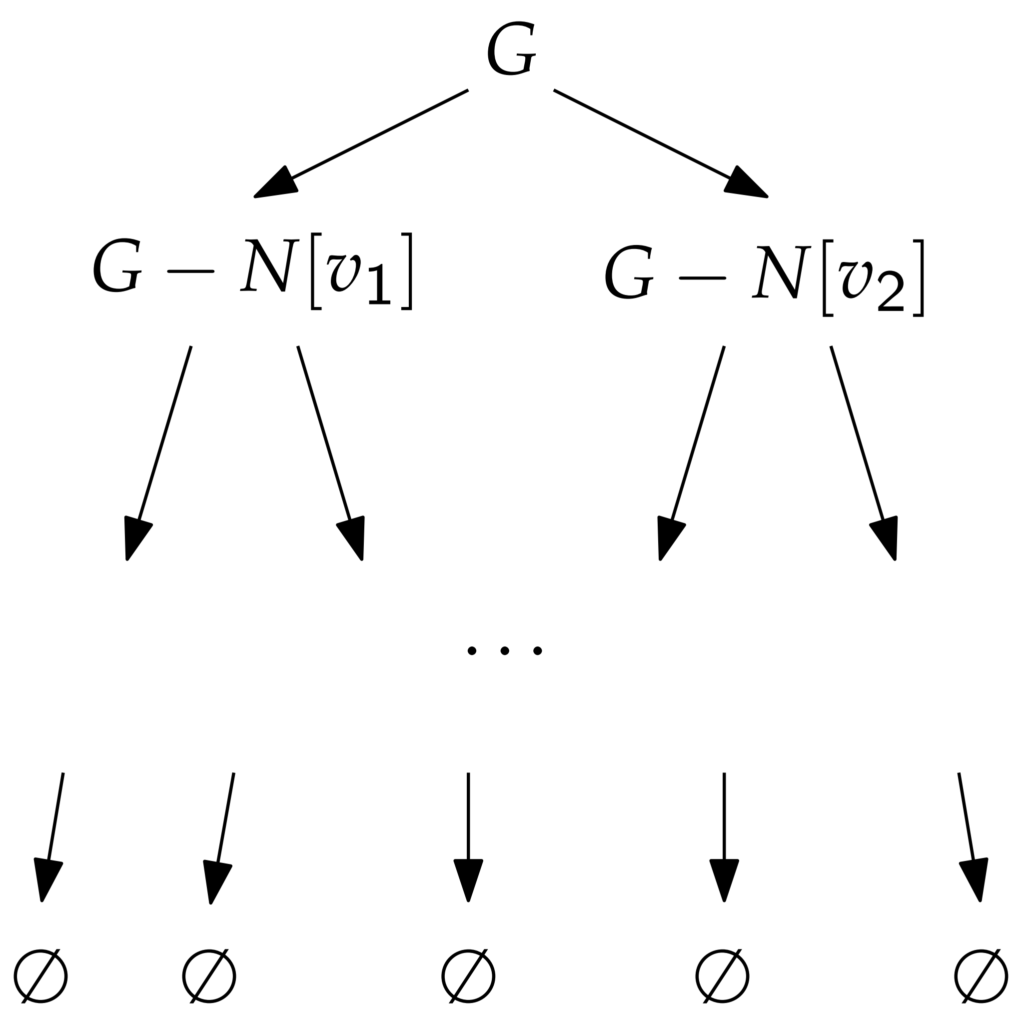

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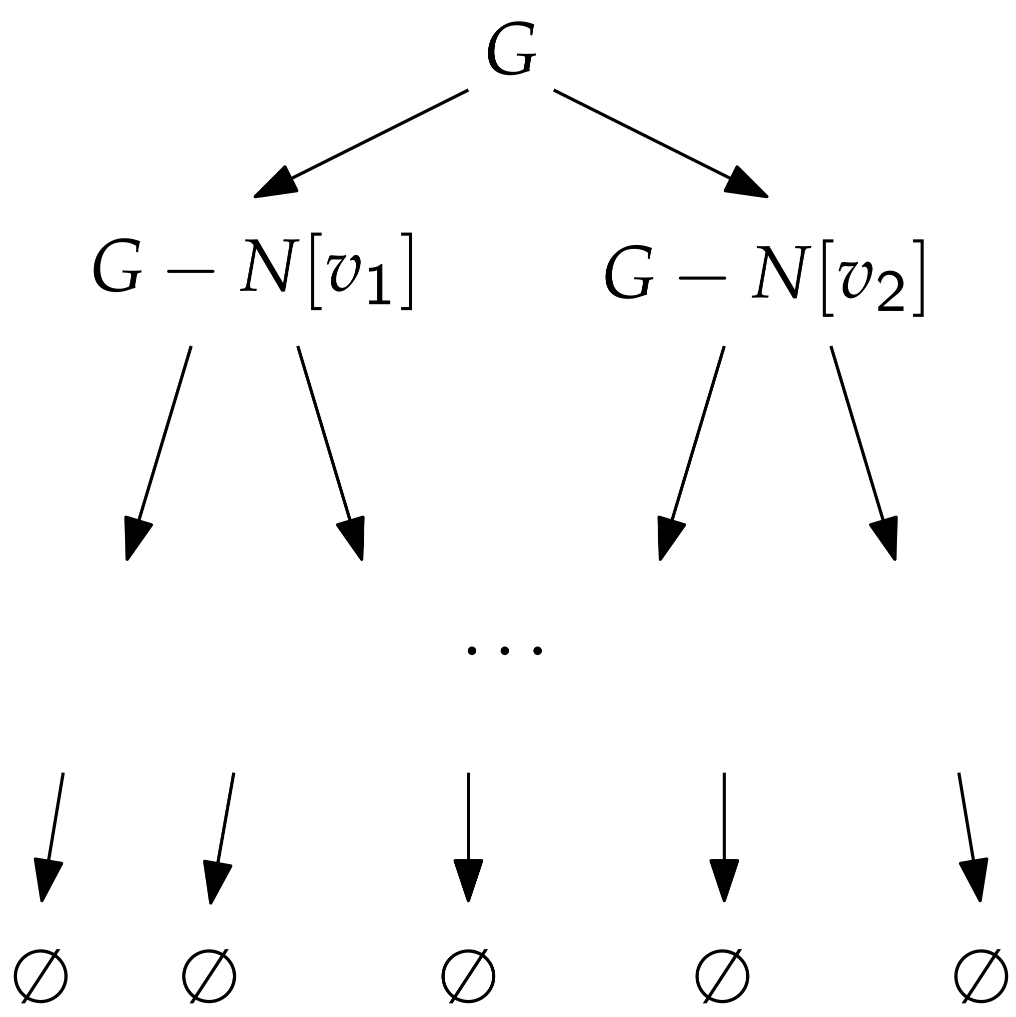

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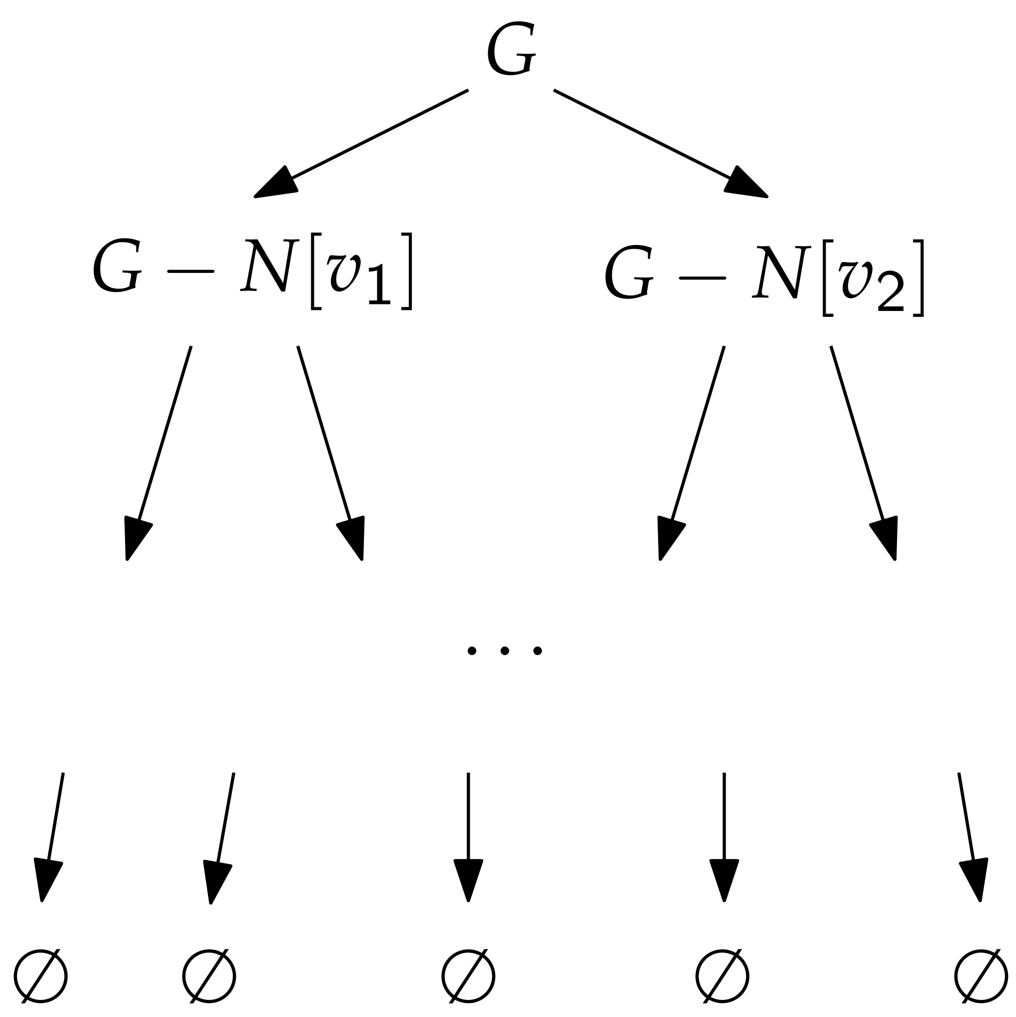
$\rightsquigarrow$ The algorithm's runtime is

$$T(n) \in O^*(nB(n)) = O^*(B(n)).$$



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- Let $B(n)$ be the maximum number of leaves of a search tree for a graph with n vertices.
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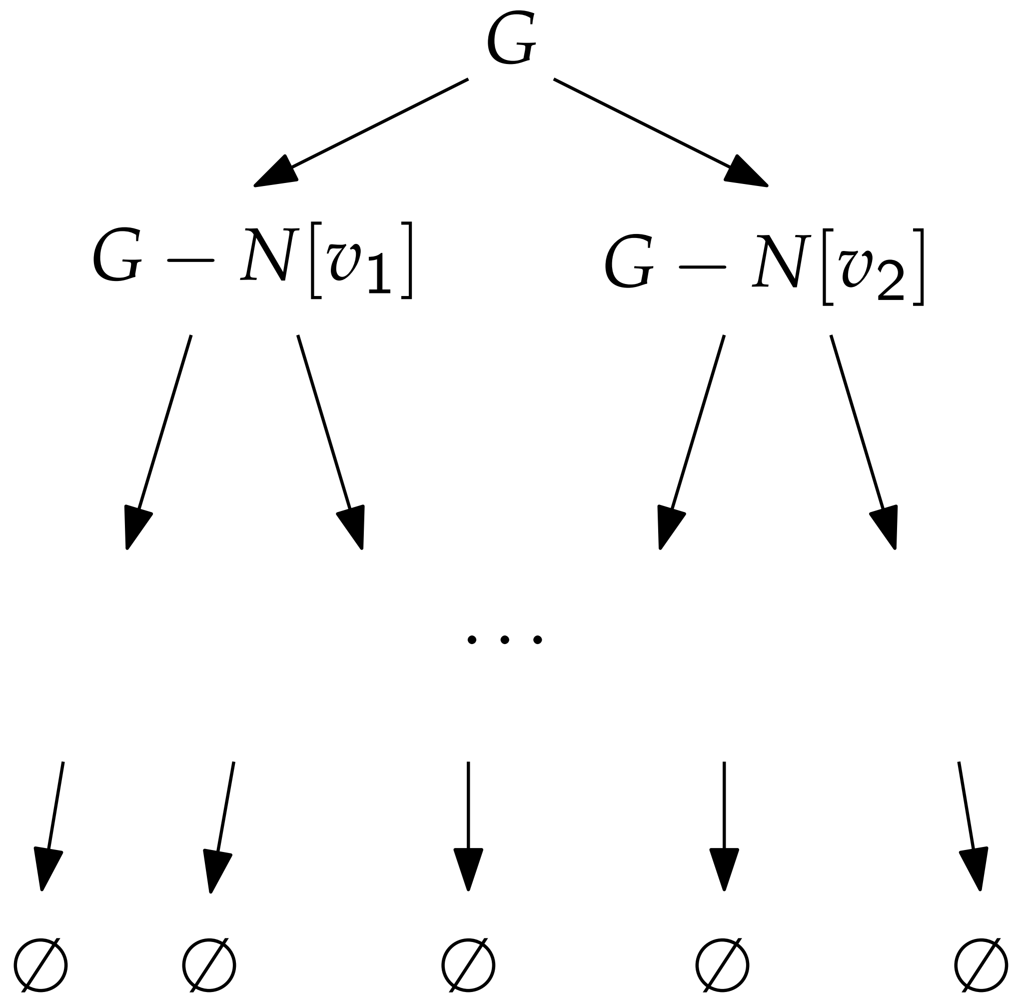
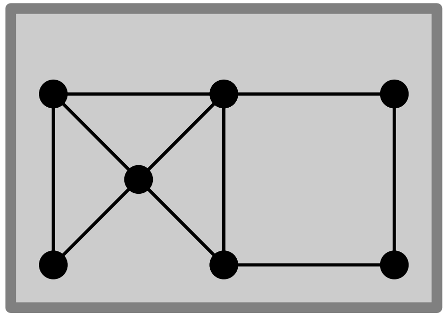
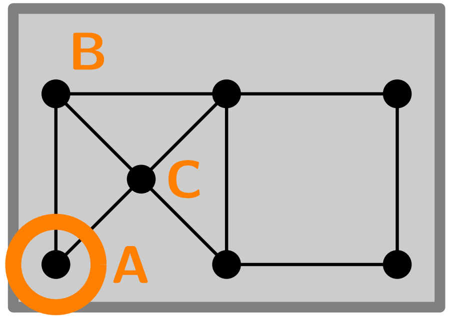
$\rightsquigarrow$ The algorithm's runtime is

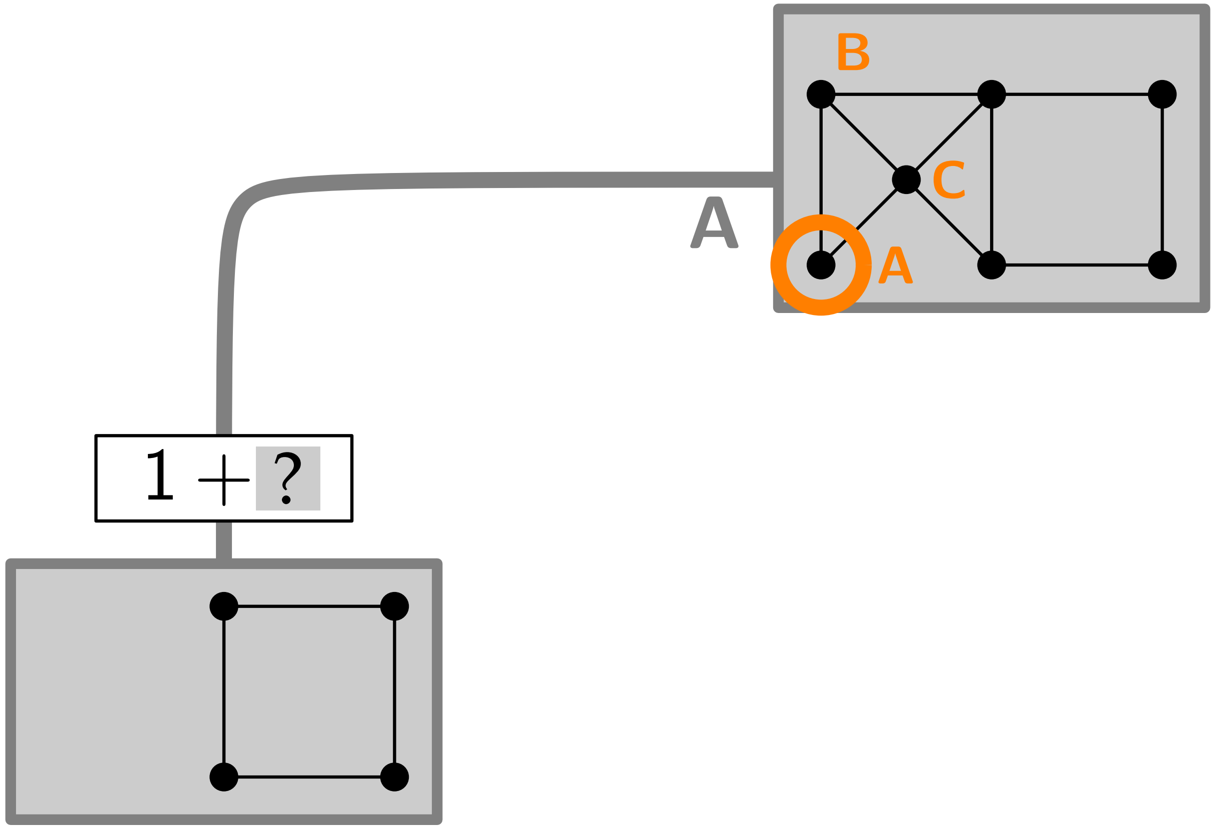
$$T(n) \in O^*(nB(n)) = O^*(B(n)).$$

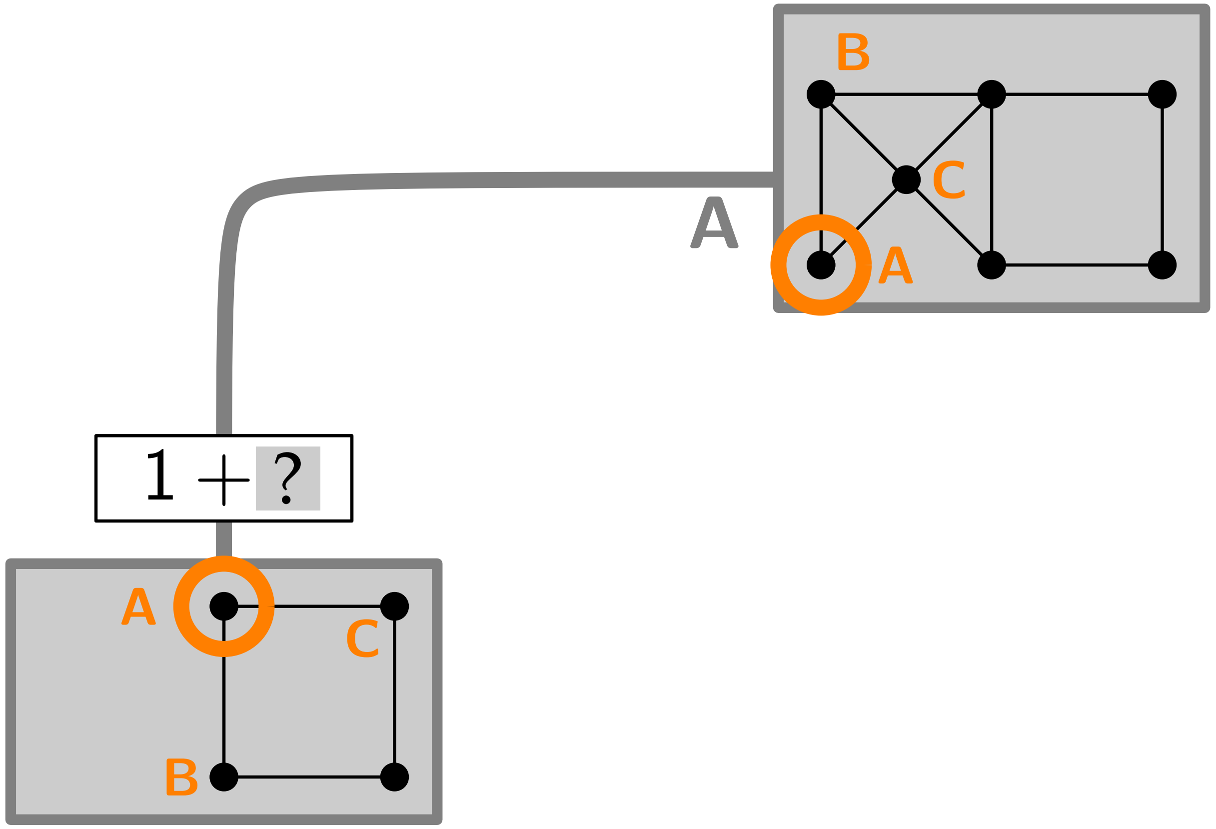
- Let's consider an example run.

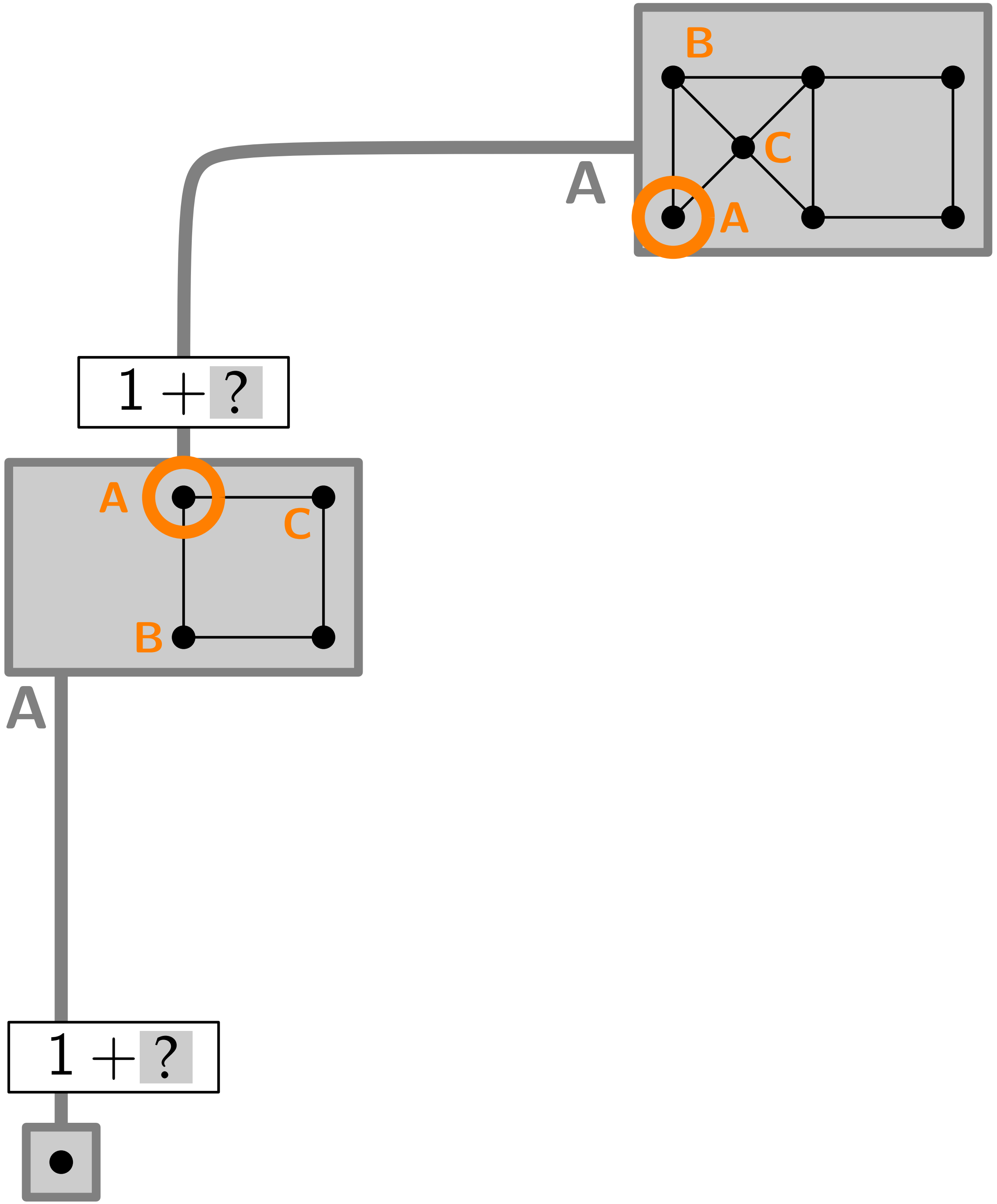


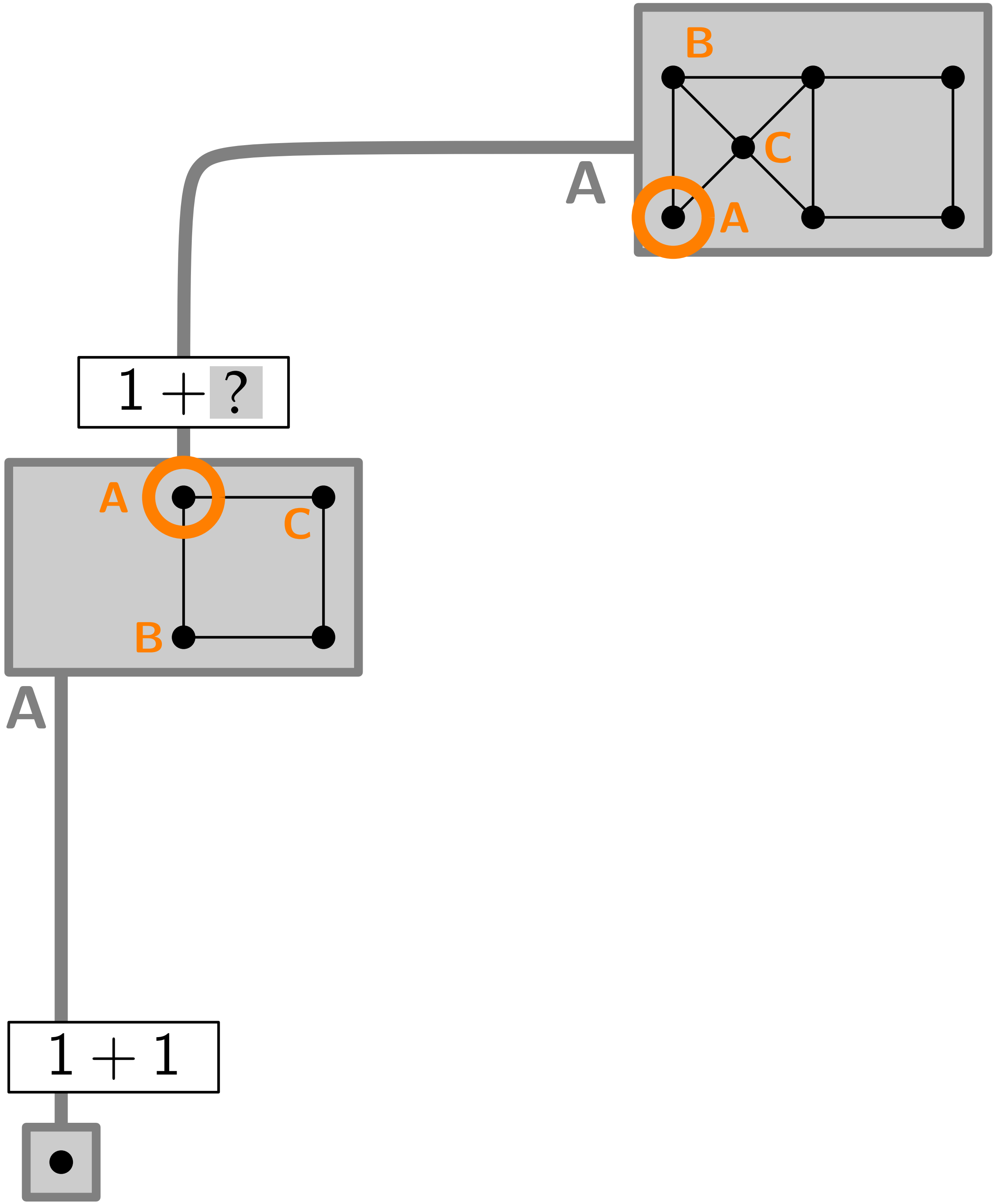


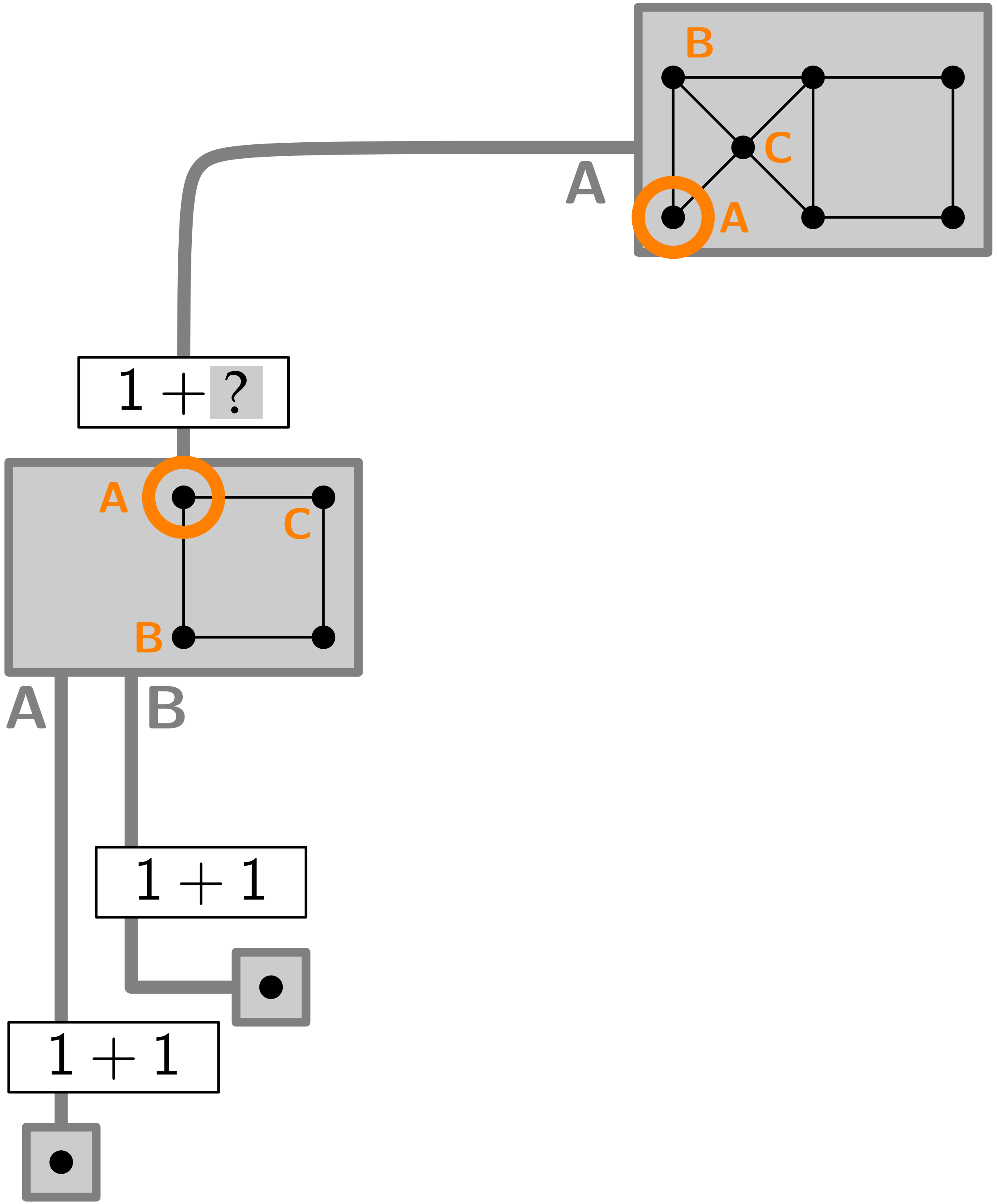

B
C
A

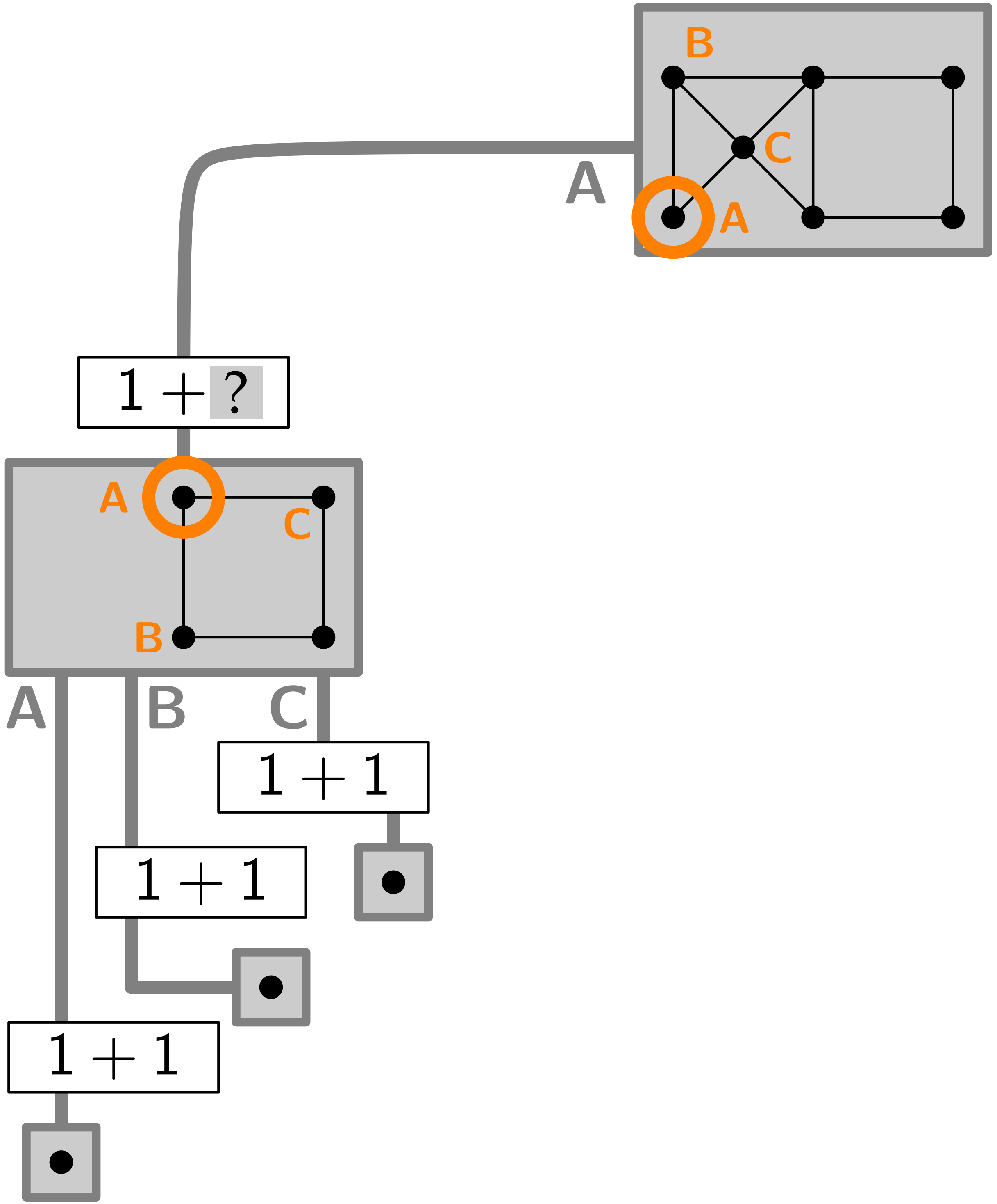

B
C
A
A
1 + ?

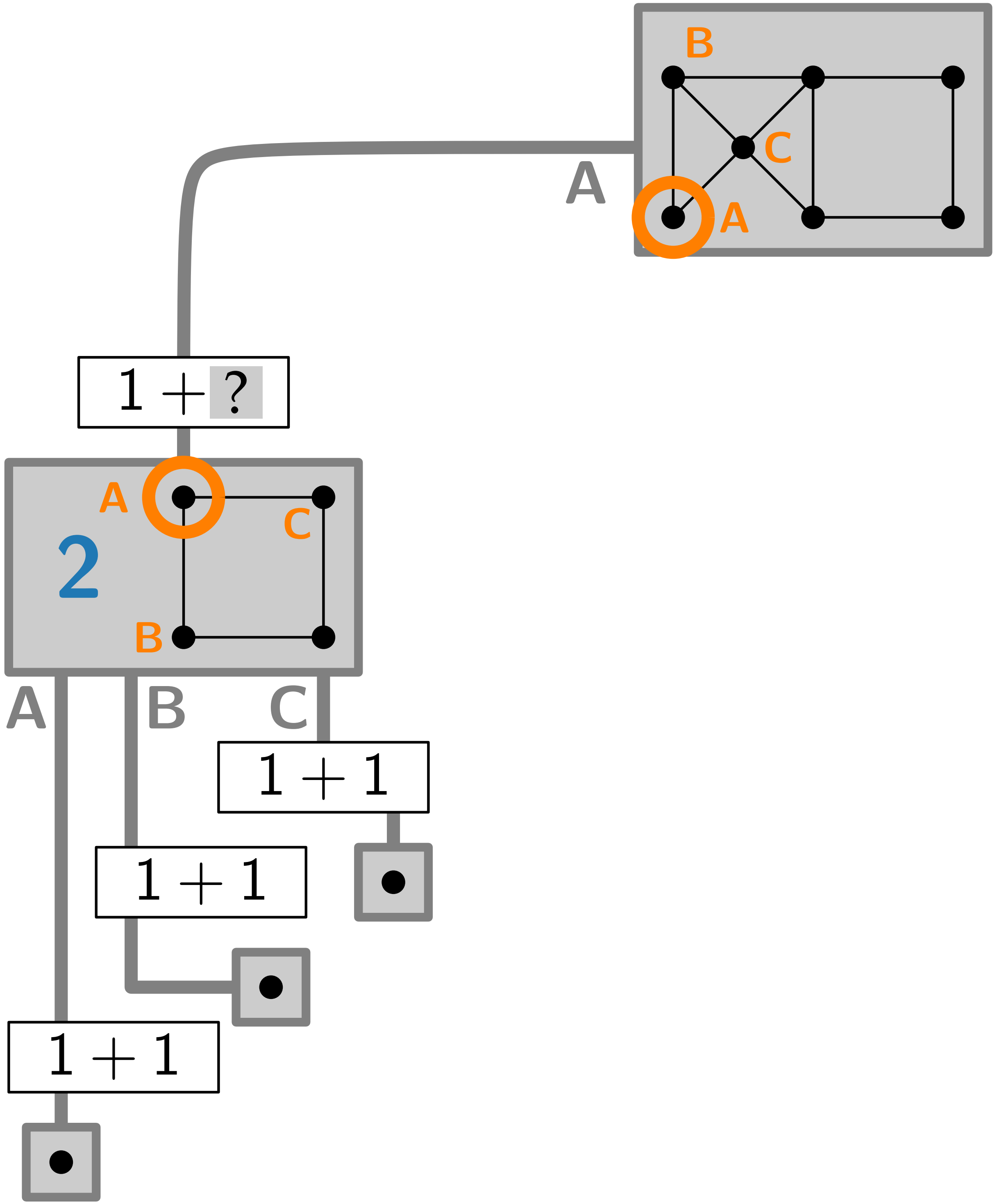

B
C
A
A
$1 + ?$
A
C
B

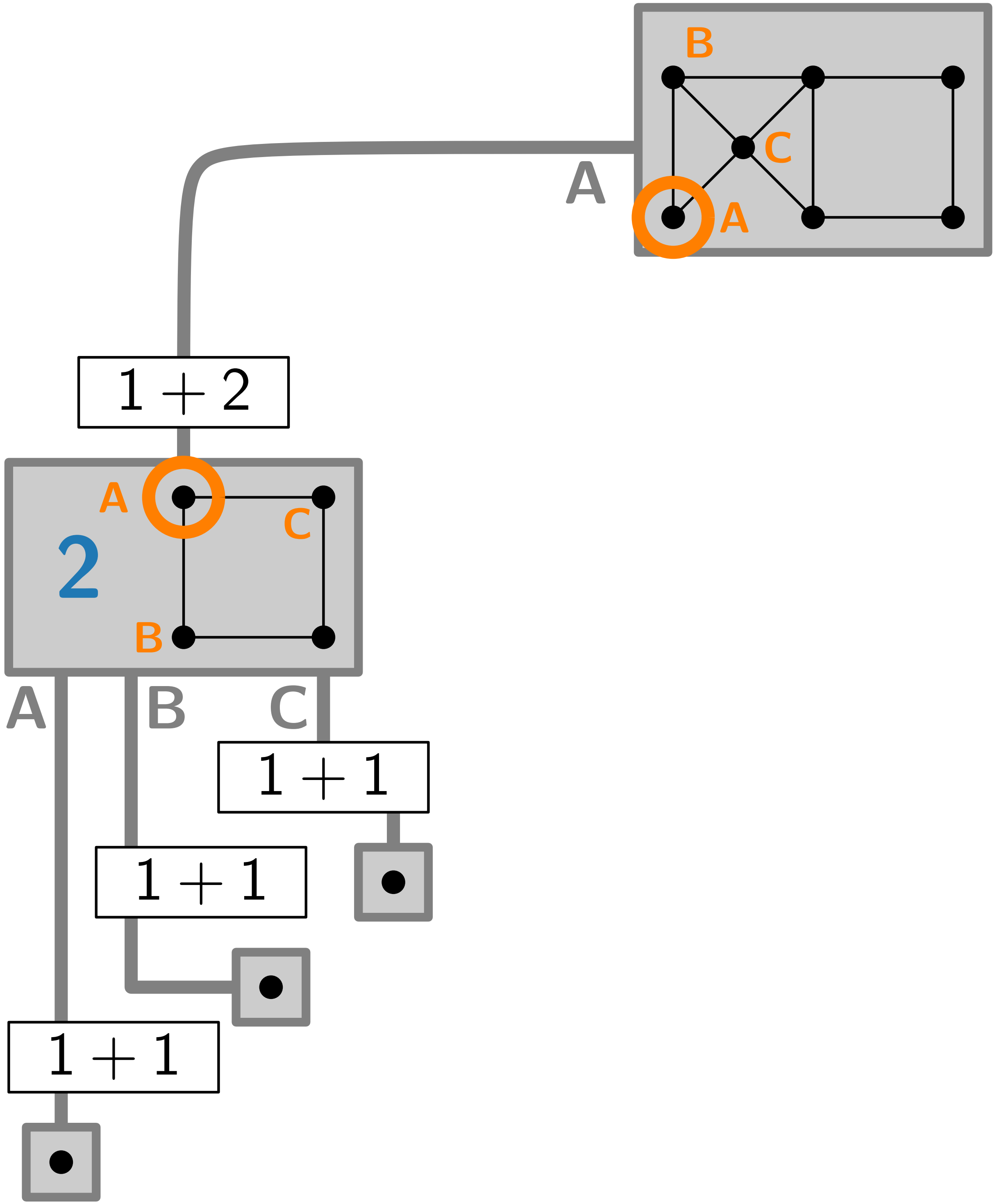

B
C
A
A
1 + ?
A
C
B
A
1 + ?

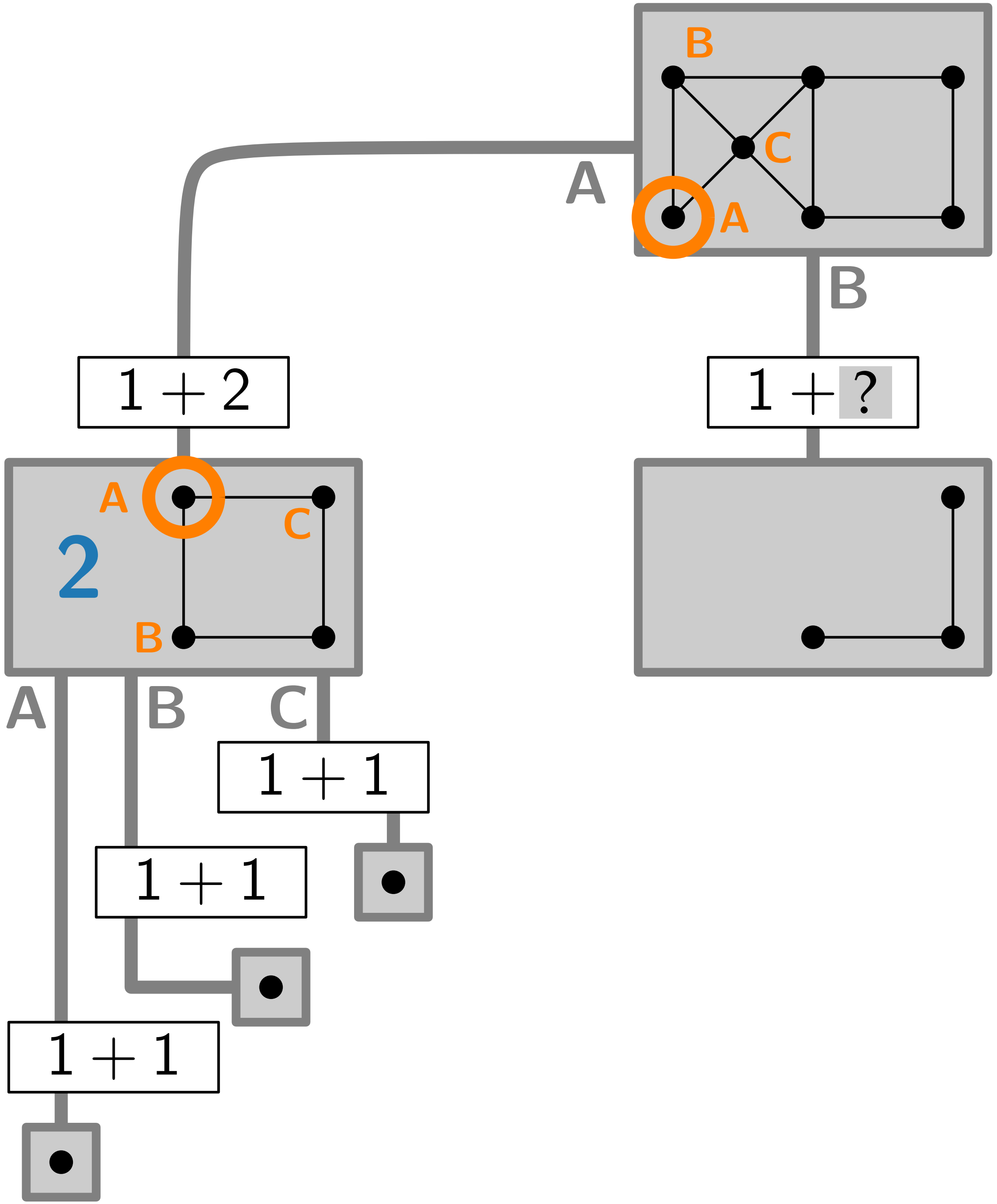

B
C
A
A
1 + ?
A
C
B
A
1 + 1

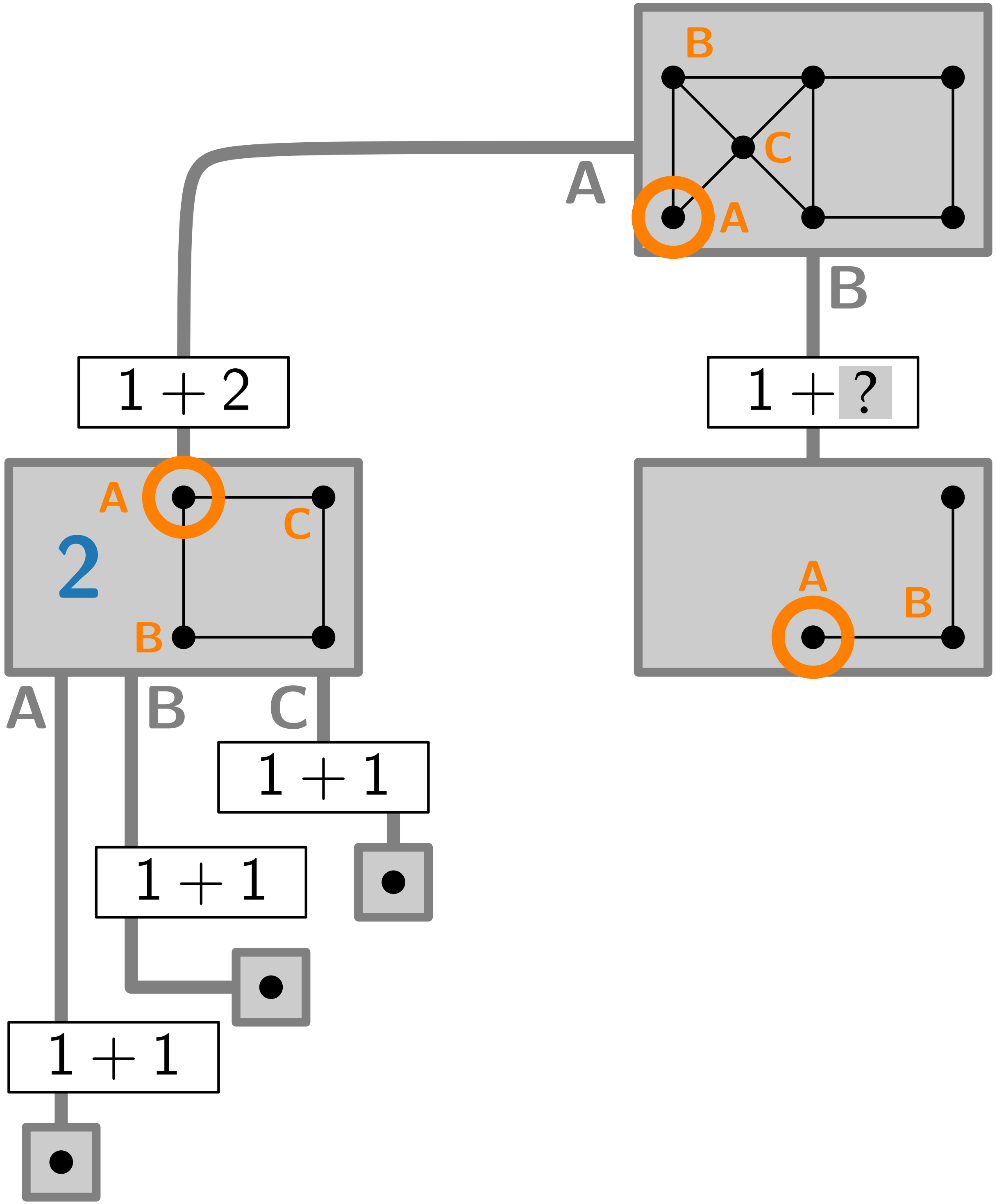

B
C
A
A
1 + ?
A
C
B
A
B
1 + 1
1 + 1

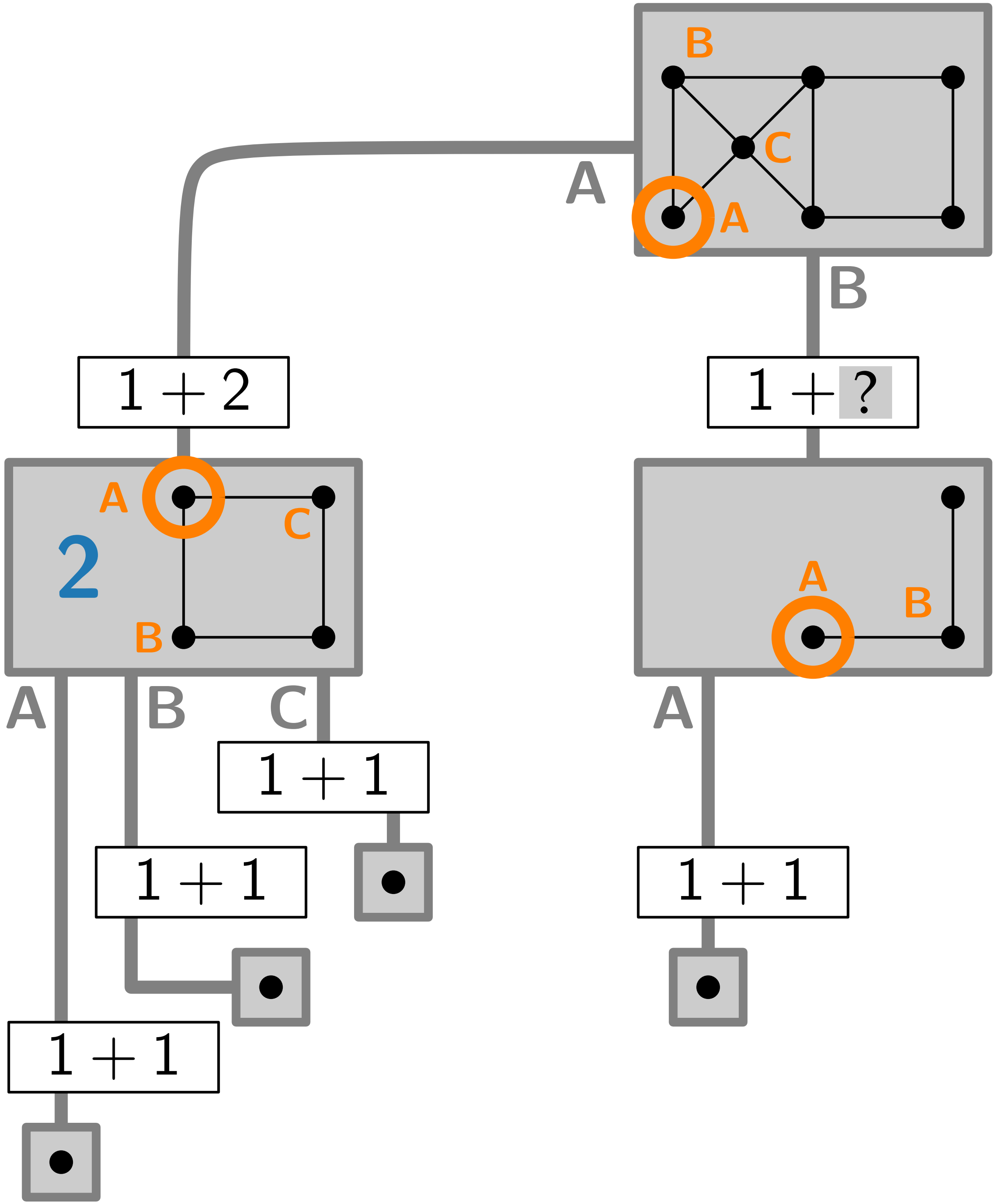

B
C
A
A
1 + ?
A
C
B
A
B
C
1 + 1
1 + 1
1 + 1

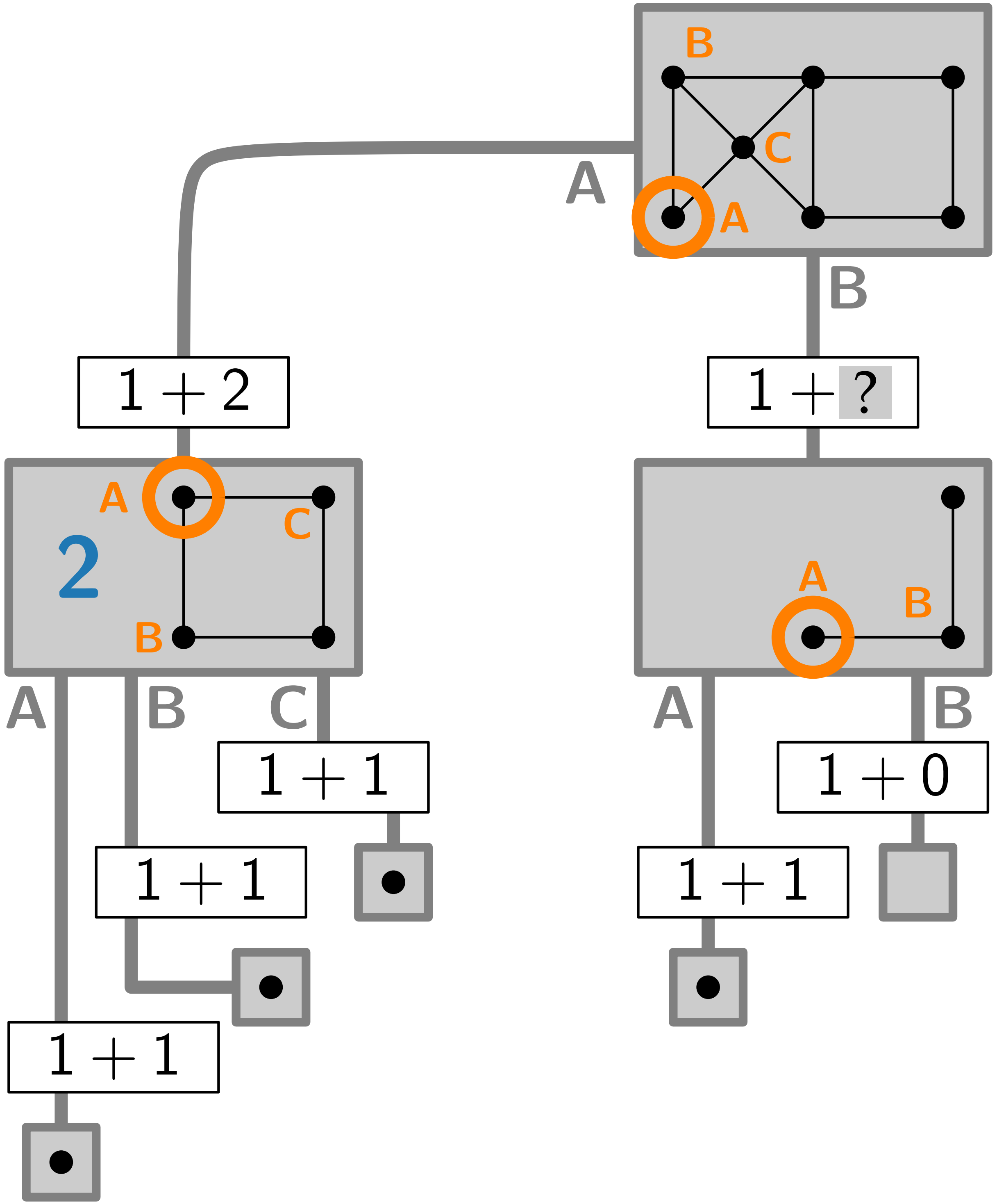

B
C
A
A
1 + ?
A
C
2
B
A
B
C
1 + 1
1 + 1
1 + 1


B
C
A
A
1 + 2
A
C
2
B
A
B
C
1 + 1
1 + 1
1 + 1


B
C
A
A
B
1 + 2
1 + ?
A
C
2
B
A
B
C
1 + 1
1 + 1
1 + 1


B
C
A
A
B
1 + 2
1 + ?
A
C
2
B
A
B
A
B
C
1 + 1
1 + 1
1 + 1


B
C
A
A
B
1 + 2
1 + ?
A
C
2
B
A
B
A
B
C
A
1 + 1
1 + 1
1 + 1
1 + 1



B
C
A
A
B
1 + 2
1 + ?
A
C
2
B
A
B
A
B
C
A
B
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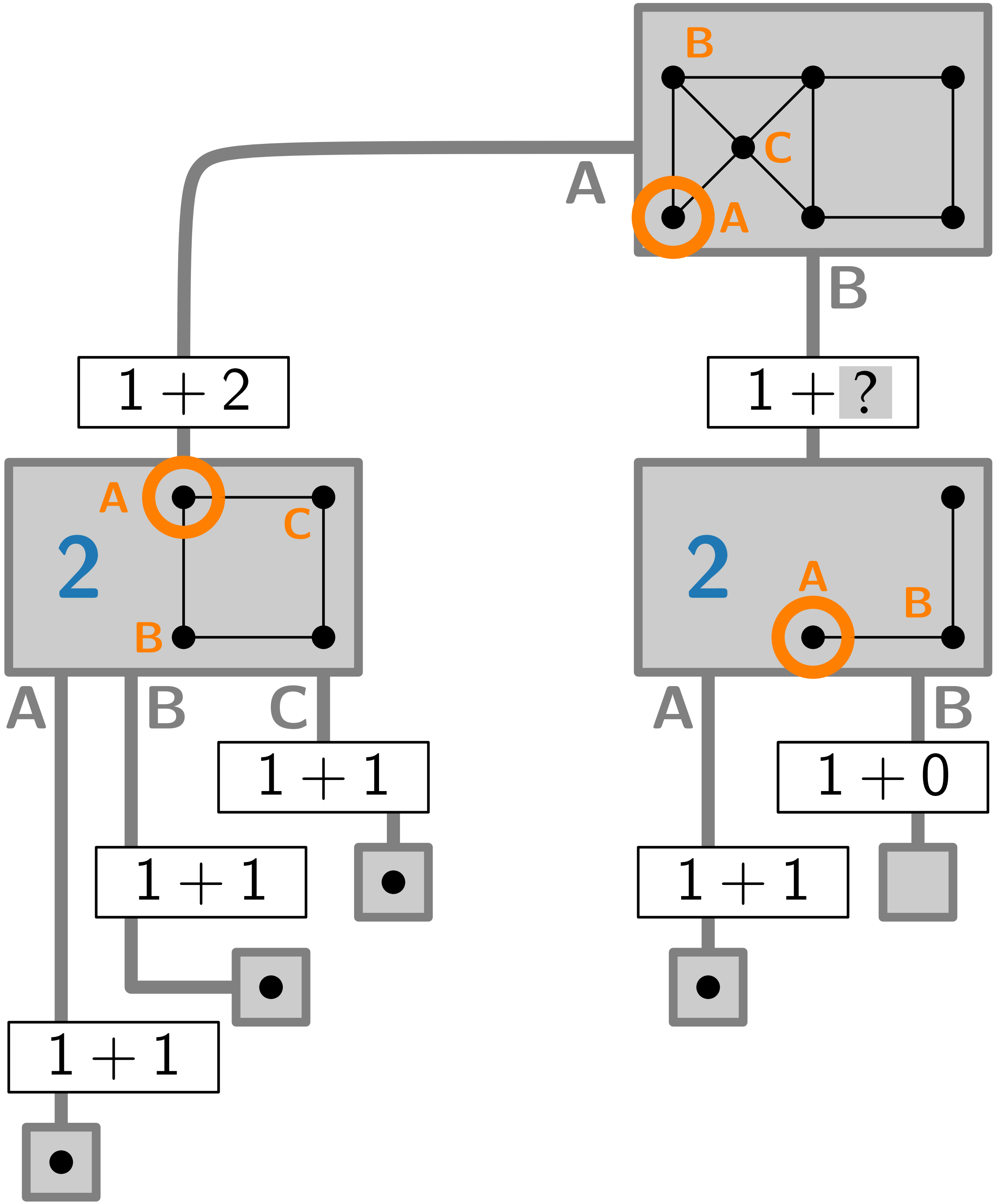
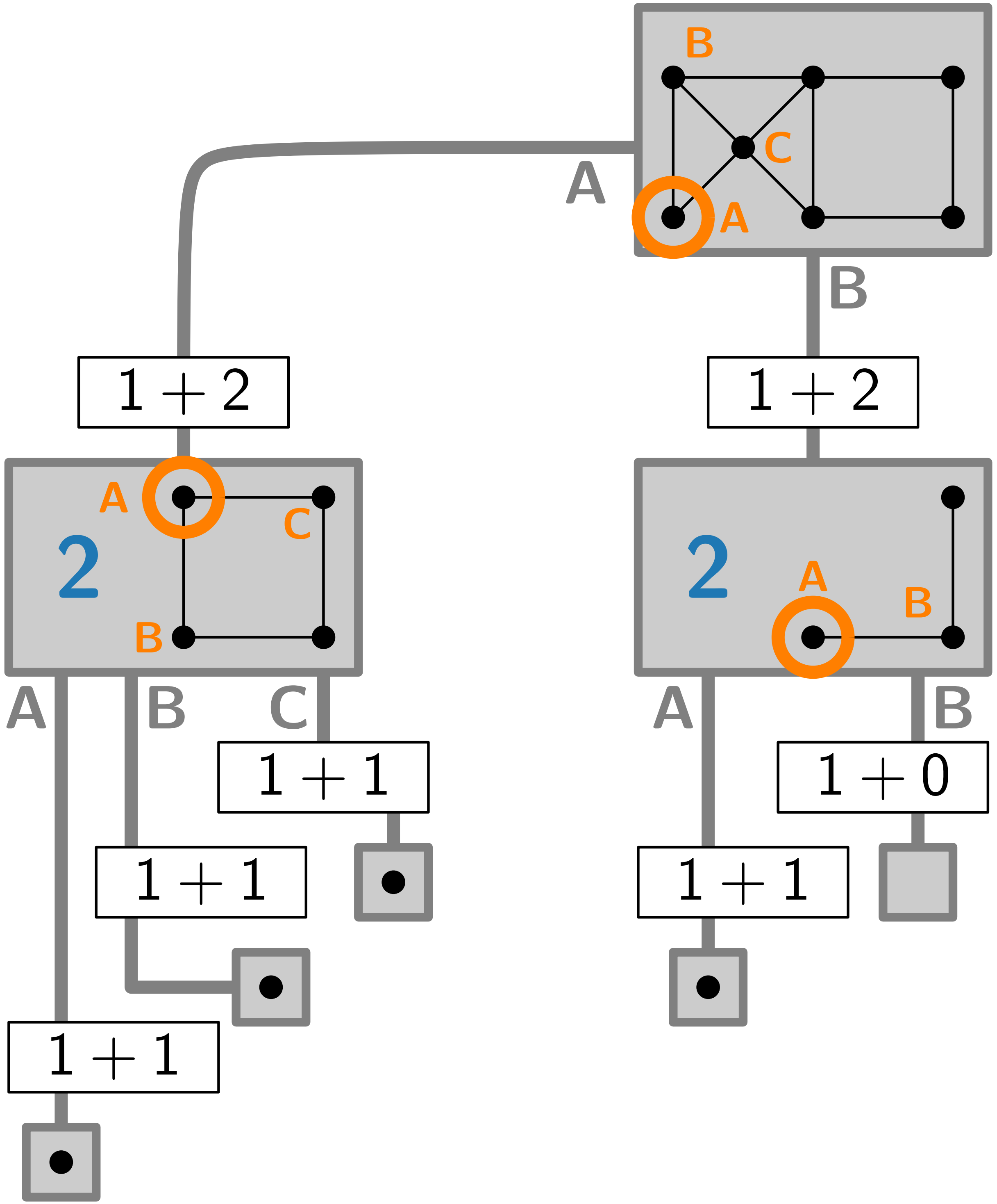
1 + 0
1 + 1
1 + 1
1 + 1



B
C
A
A
B
1 + 2
1 + ?
A
2
C
B
2
A
B
A
B
C
1 + 1
1 + 1
1 + 1
A
B
1 + 0
1 + 1


B
C
A
A
B
1 + 2
1 + 2
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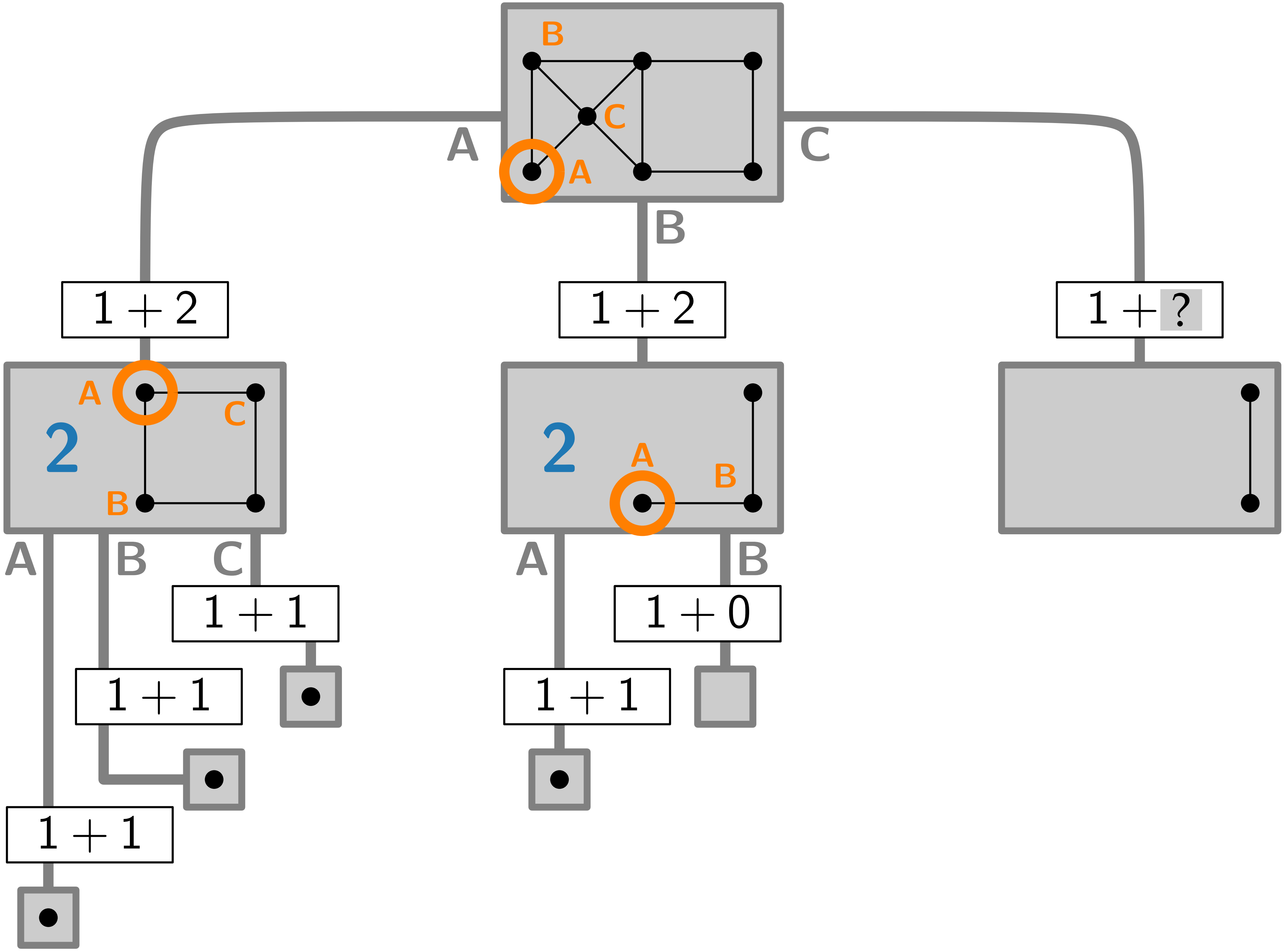
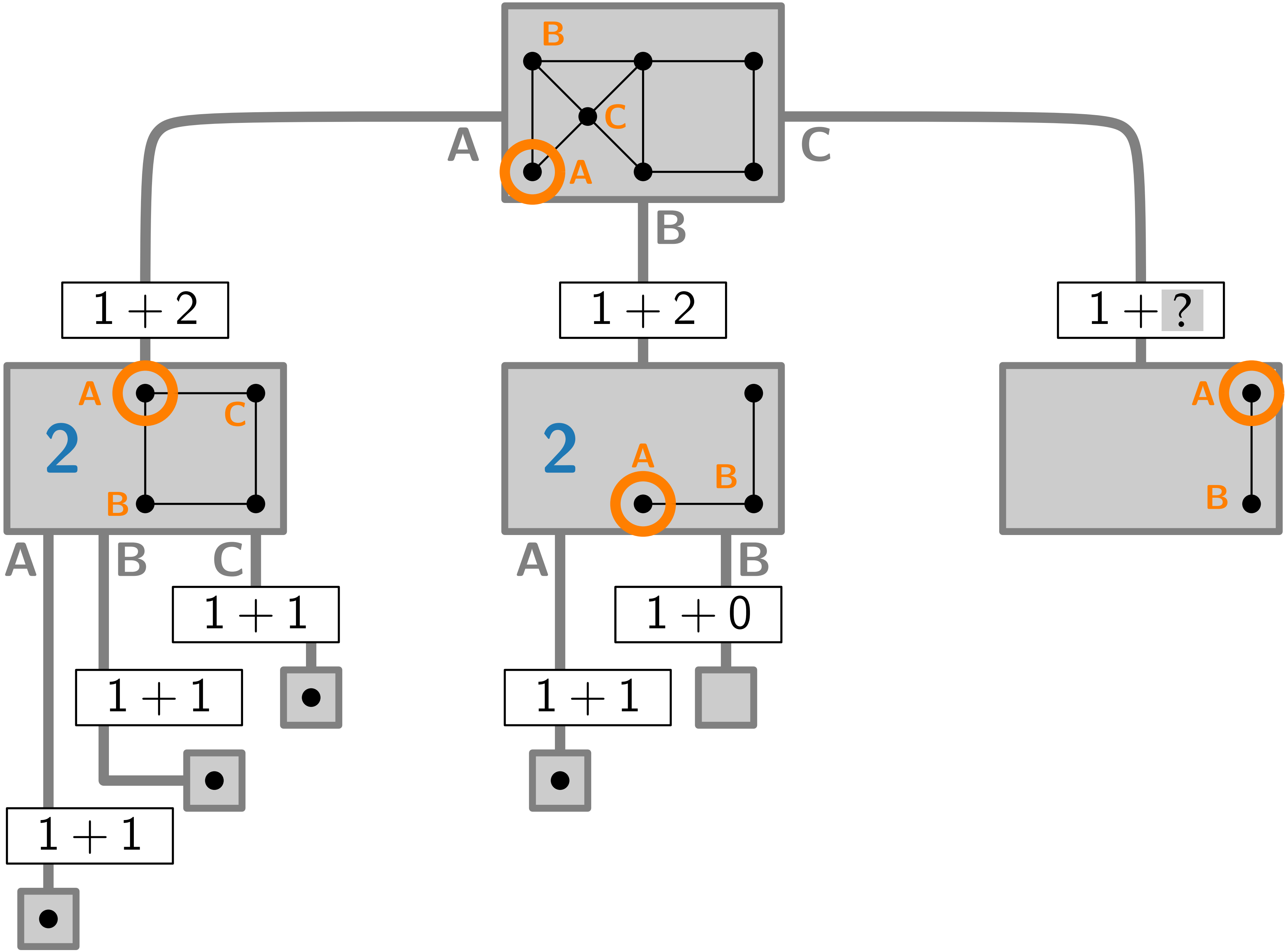
2
B
2
A
B
A
B
C
A
B
1 + 1
1 + 0
1 + 1
1 + 1
1 + 1


B
C
A
A
C
B
1 + 2
1 + 2
1 + ?
A
C
2
B
2
A
B
A
B
C
A
B
1 + 1
1 + 0
1 + 1
1 + 1
1 + 1


B
C
A
A
B
C
1 + 2
1 + 2
1 + ?
A
C
2
B
2
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B
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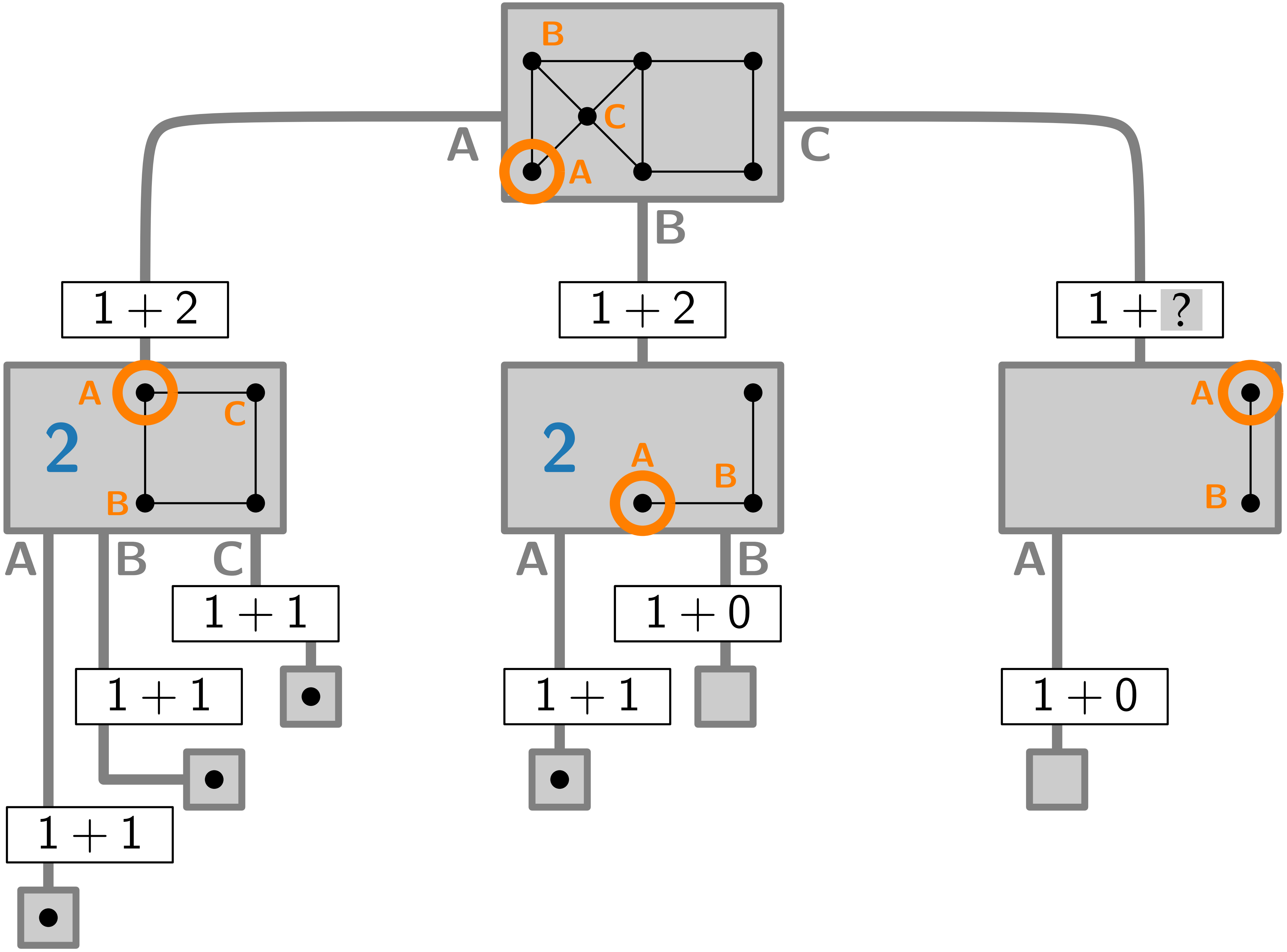
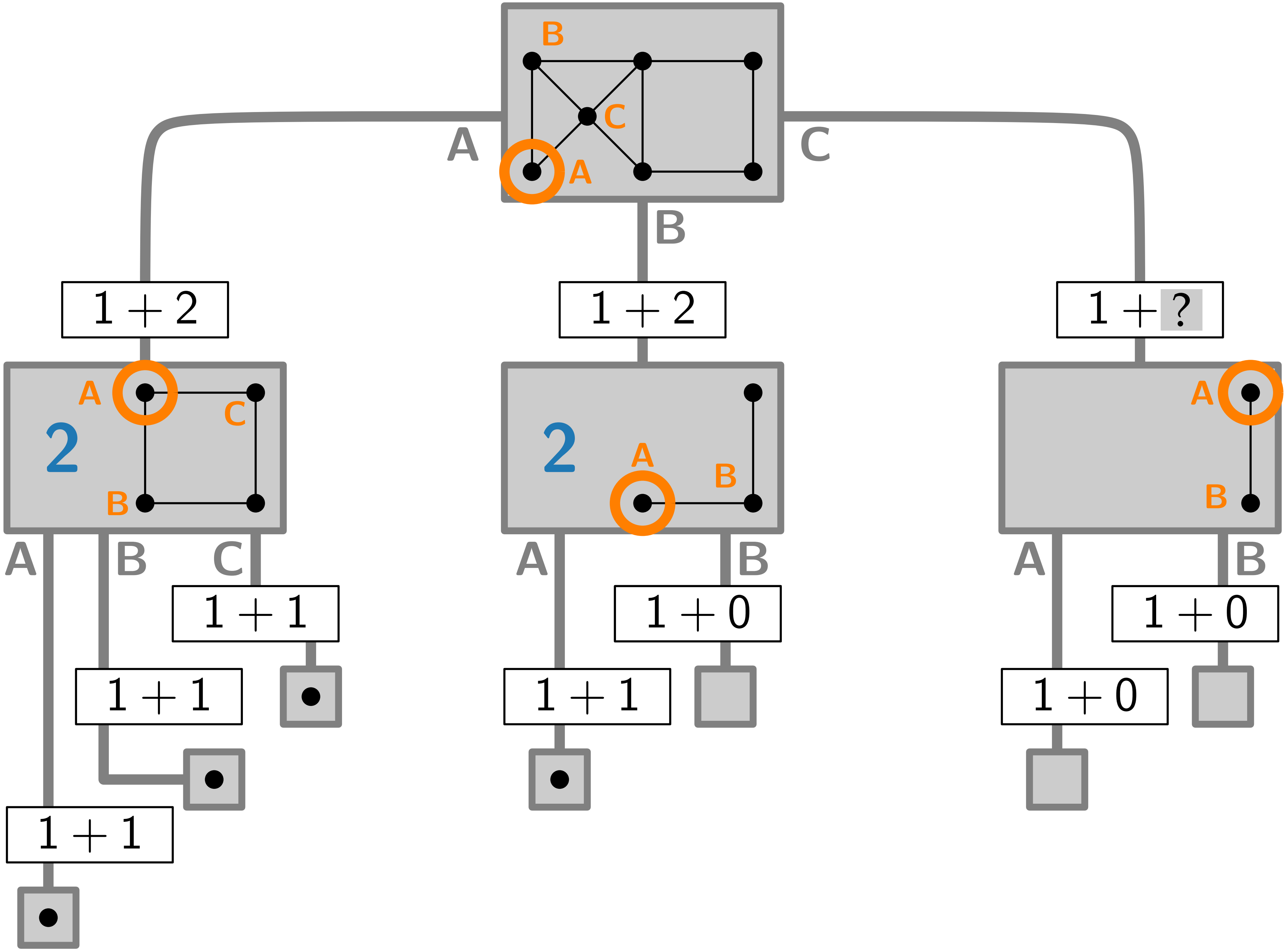
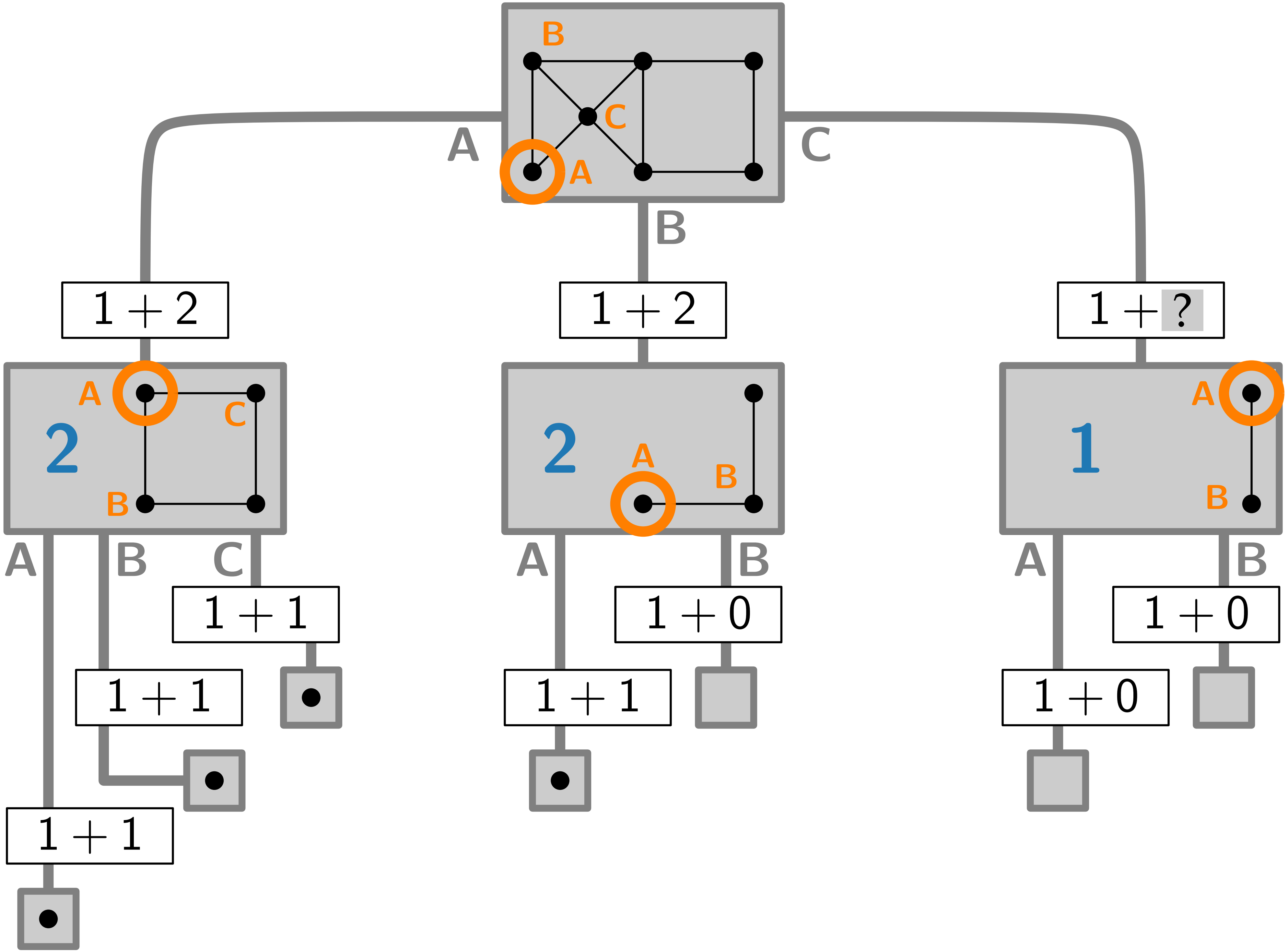
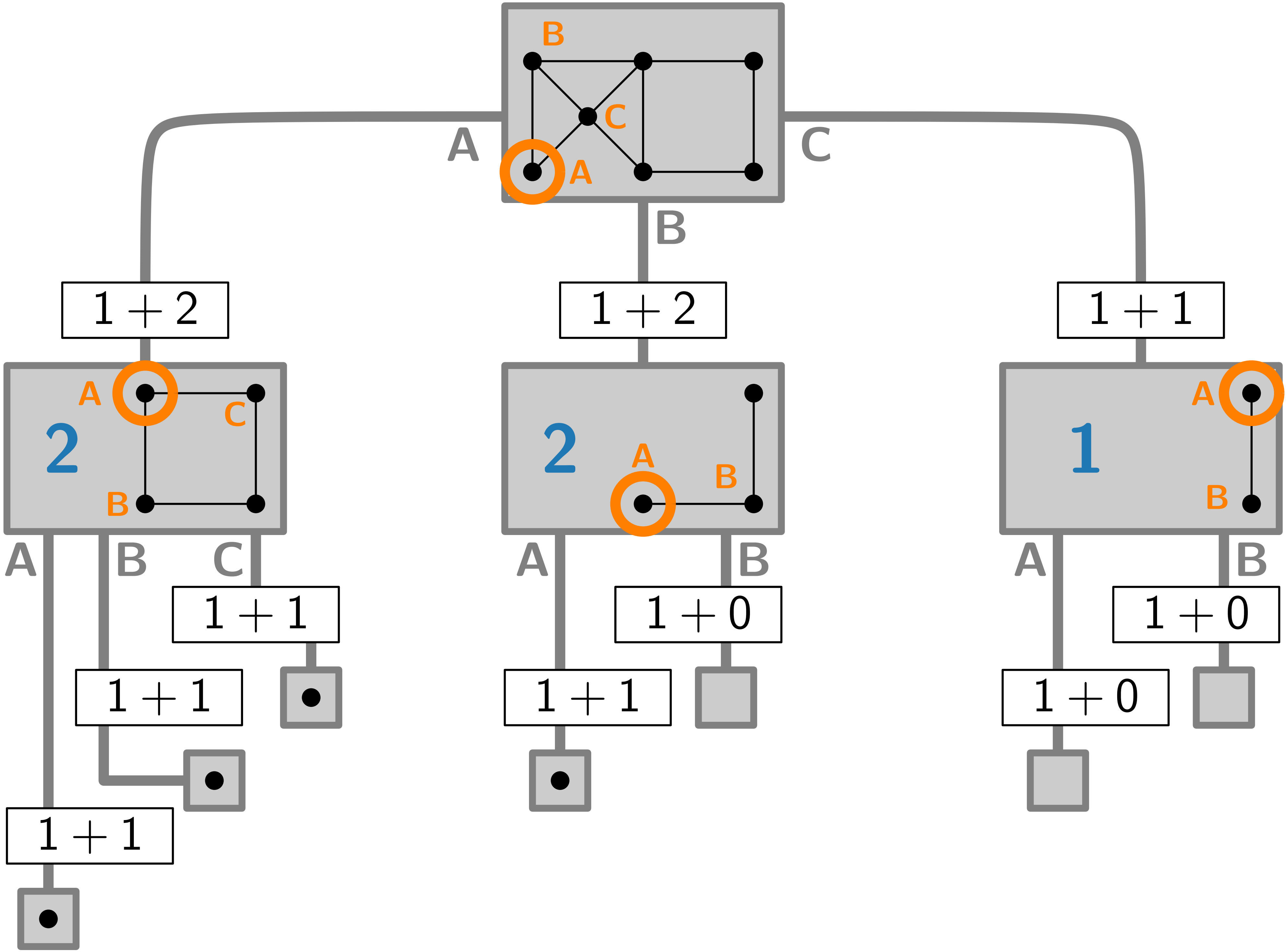
B
A
B
C
A
B
1 + 1
1 + 0
1 + 1
1 + 1
1 + 1


B
C
A
A
B
C
1 + 2
1 + 2
1 + ?
A
C
2
B
2
A
B
A
B
A
B
C
A
B
A
1 + 1
1 + 0
1 + 1
1 + 1
1 + 0
1 + 1


B
C
A
A
B
C
1 + 2
1 + 2
1 + ?
A
C
2
B
2
A
B
A
B
A
B
C
A
B
A
B
1 + 1
1 + 0
1 + 0
1 + 1
1 + 1
1 + 0
1 + 1



B
C
A
A
B
C
1 + 2
1 + 2
1 + ?
A
C
2
B
2
A
B
1
A
B
A
B
C
A
B
A
B
1 + 1
1 + 0
1 + 0
1 + 1
1 + 1
1 + 0
1 + 1



B
C
A
A
B
C
1 + 2
1 + 2
1 + 1
A
C
B
2
2
A
B
1
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B
A
B
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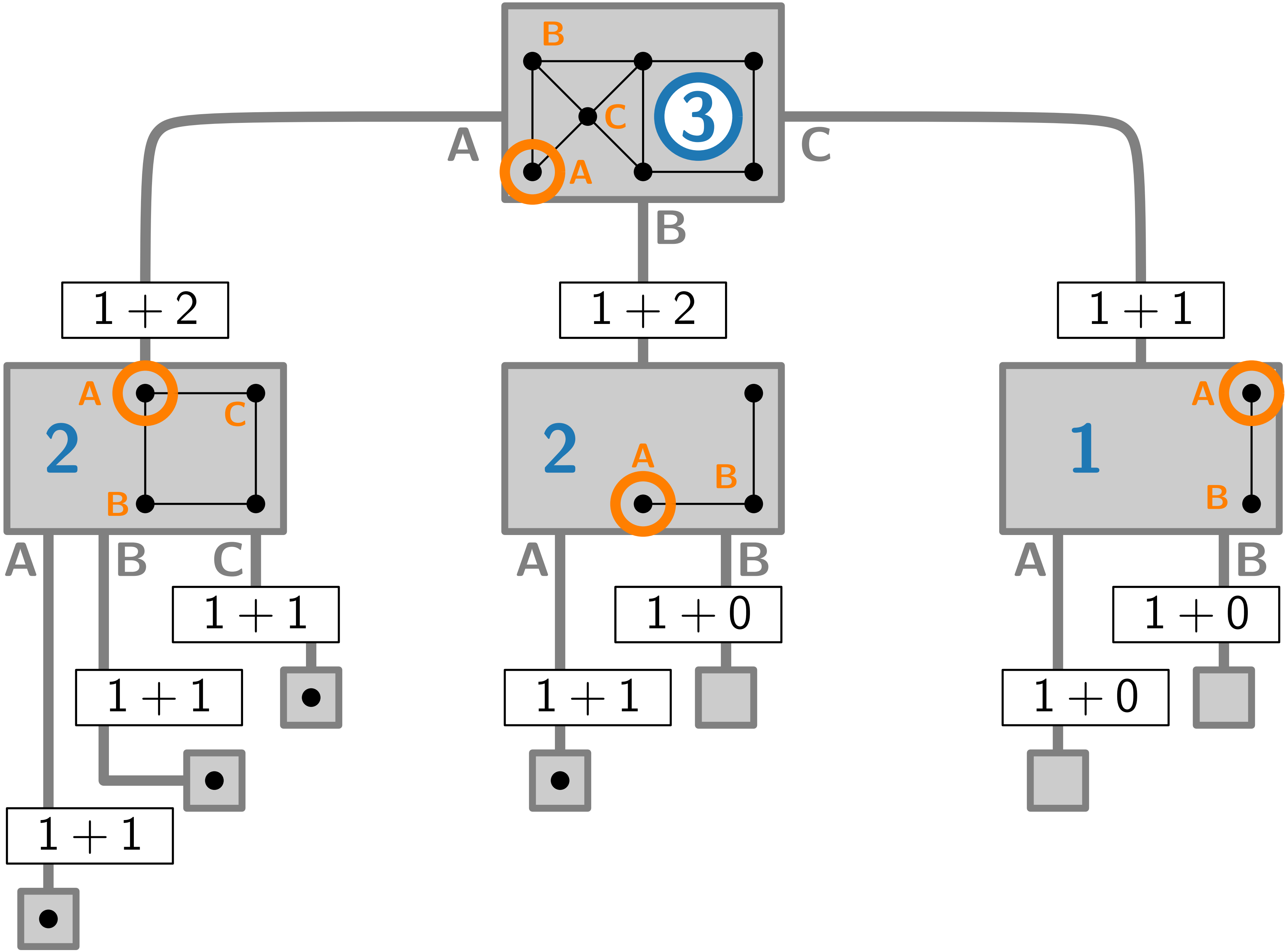
A
B
A
B
1 + 1
1 + 0
1 + 0
1 + 1
1 + 1
1 + 0
1 + 1



B
C
A
3
A
B
C
1 + 2
1 + 2
1 + 1
A
C
B
2
2
A
B
A
B
1
A
B
A
B
C
A
B
A
B
1 + 1
1 + 0
1 + 0
1 + 1
1 + 1
1 + 0
1 + 1

MIS – Runtime Analysis

For a worst-case n-vertex graph G ($n \geq 1$):

$$B(n) \leq \sum_{y \in N[v]} B(n - (\text{deg}(y) + 1))$$

where v is a minimum degree vertex of G, and we note that $B(n') \leq B(n)$ for any $n' \leq n$.

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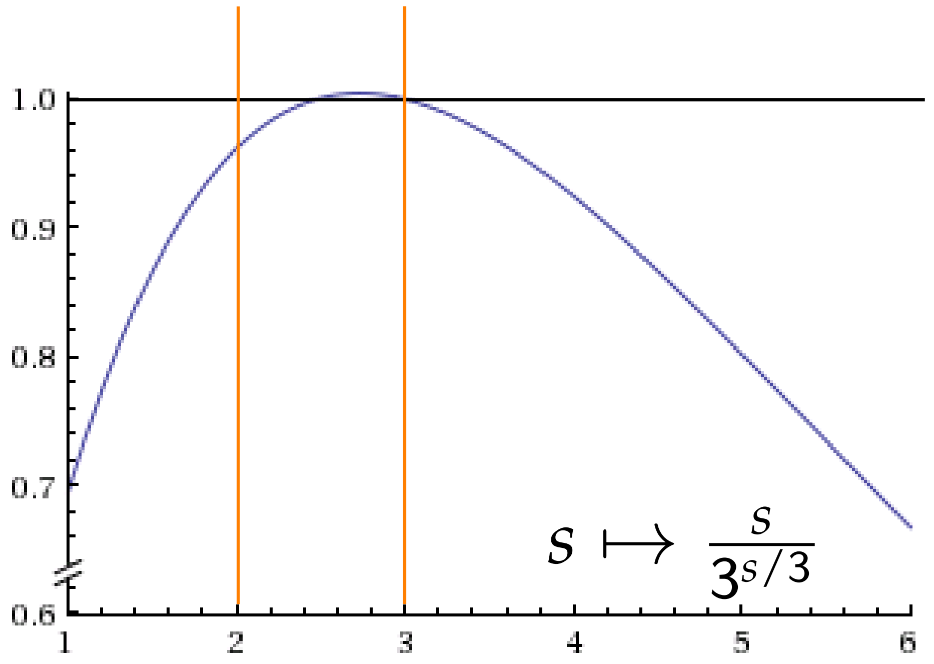

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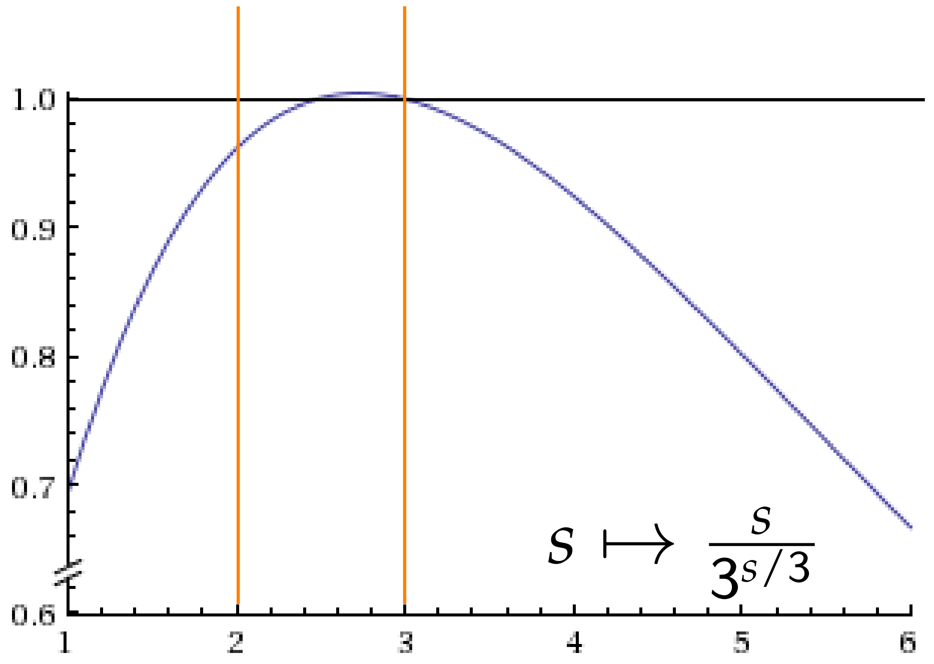
$$B(n) \in O^*(\sqrt[3]{3}^n) \subset O^*(1.44225^n)$$



MIS – Discussion

- Smarter branching leads to $\mathcal{O}^*(1.44225^n)$-time algorithm,
- compared to brute-force, which runs in $\mathcal{O}^*(2^n)$ time.

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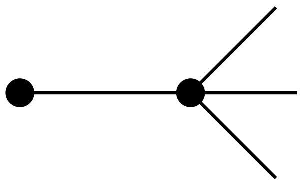
- What vertices can we savely assume are in a MIS?



- Advanced case analysis in [Fomin, Kratsch Ch 2.3] leading to a $\mathcal{O}^*(1.2786^n)$-time algorithm.

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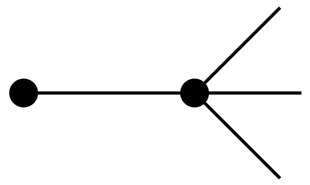

- Advanced case analysis in [Fomin, Kratsch Ch 2.3] leading to a $\mathcal{O}^*(1.2786^n)$-time algorithm.
- **Exercise**: Edge-branching for MIS

Literature

Main source:

- [Fomin, Kratsch Ch1] “Exact Exponential Algorithms”

Referenced papers:

- [ADMV ’15] Classic Nintendo Games are (Computationally) Hard
- [Mann ’17] The Top Eight Misconceptions about NP-Hardness