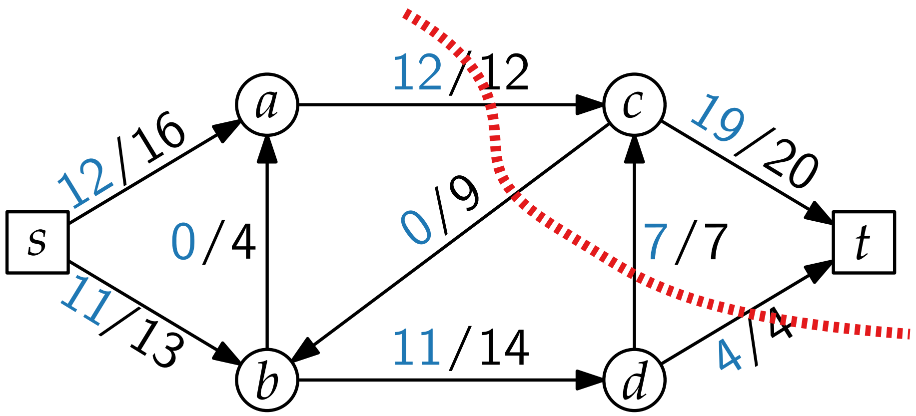
Julius-Maximilians-UNIVERSITÄT WÜRZBURG

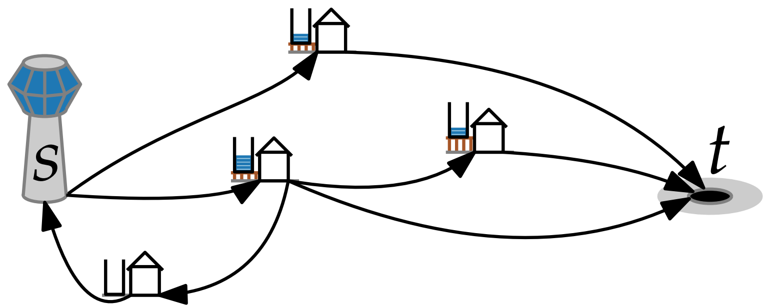
Advanced Algorithms

Maximum Flow Problem

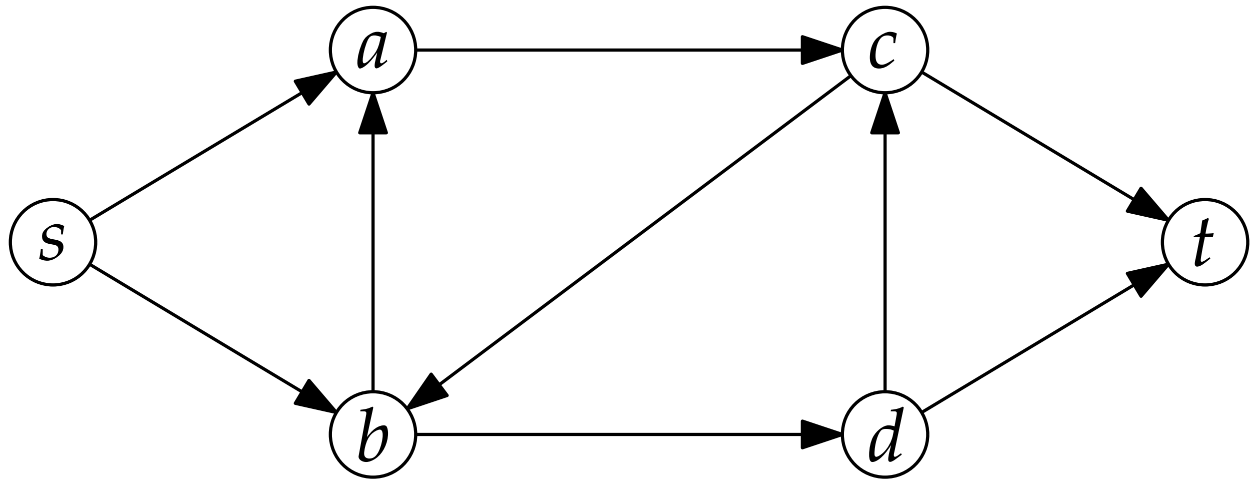
Push–Relabel Algorithm

Alexander Wolff · WS 2022





Flow Networks



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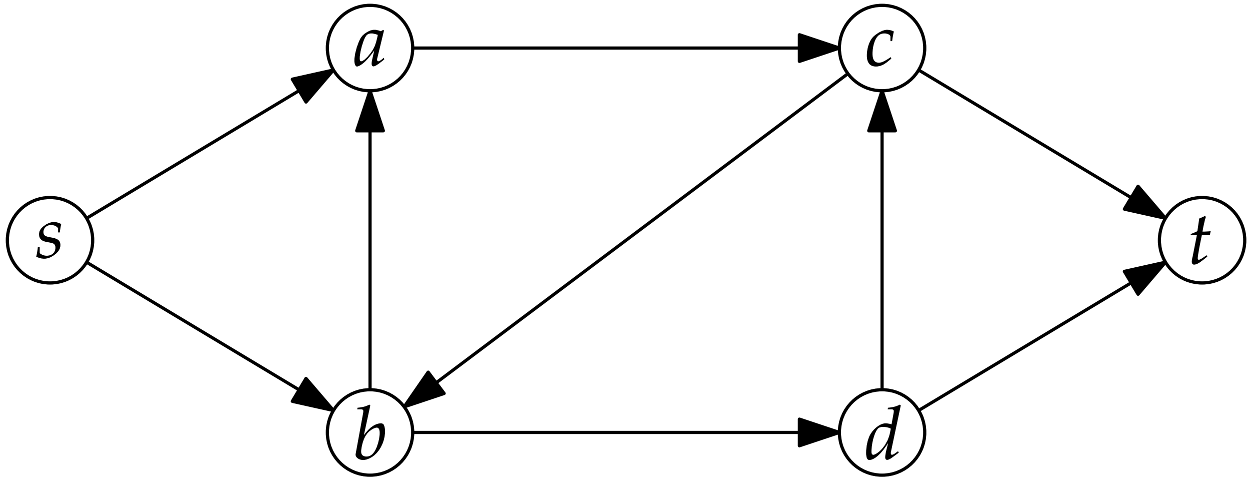

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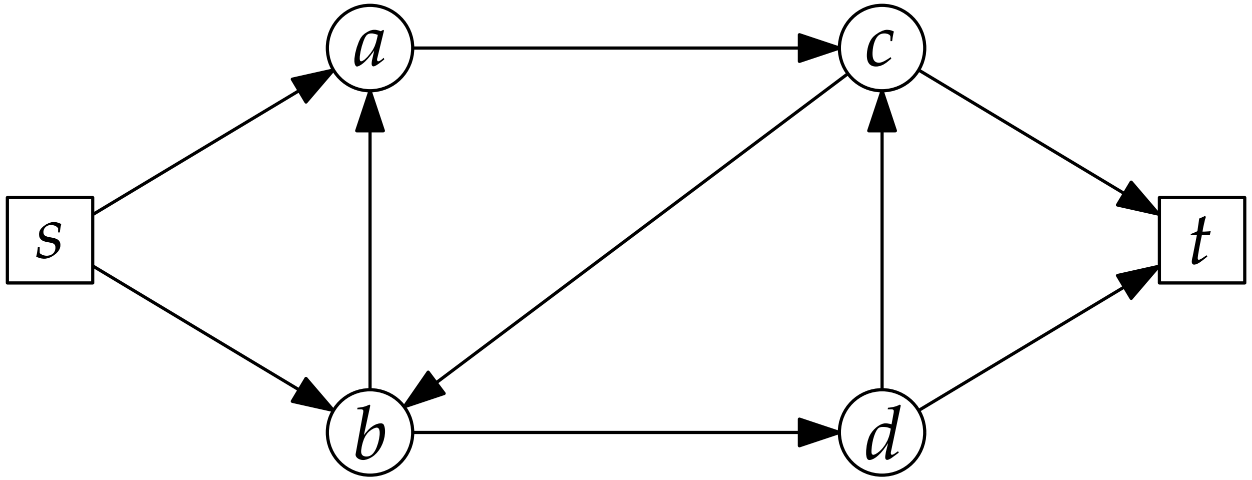

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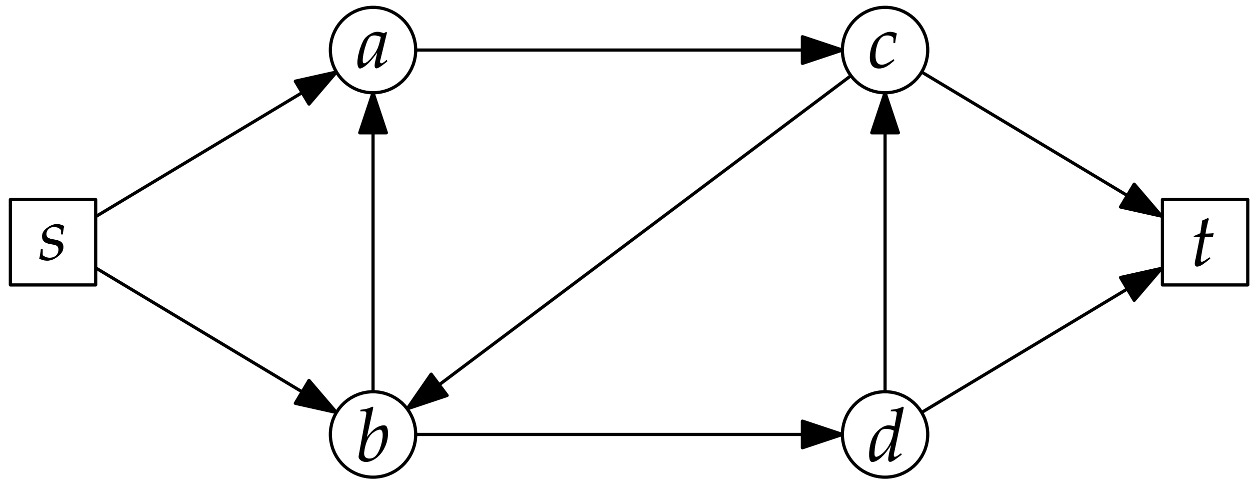

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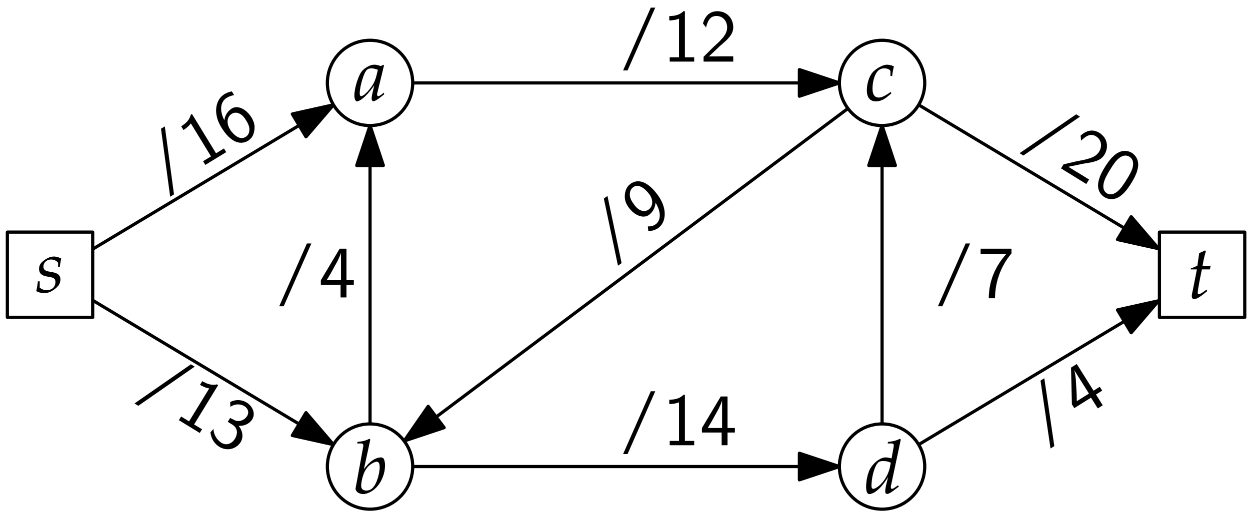

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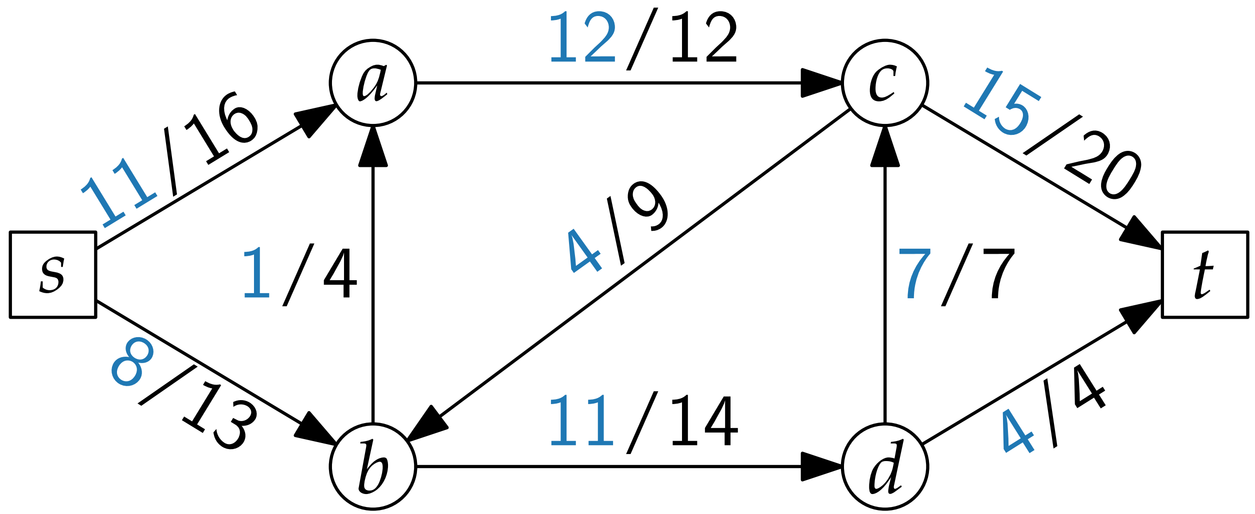

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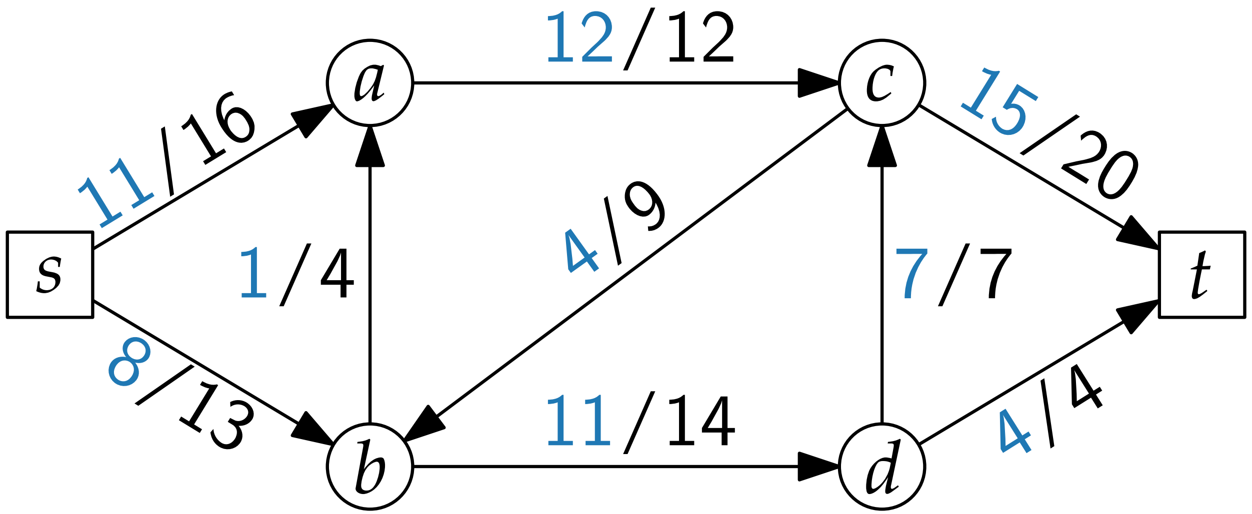

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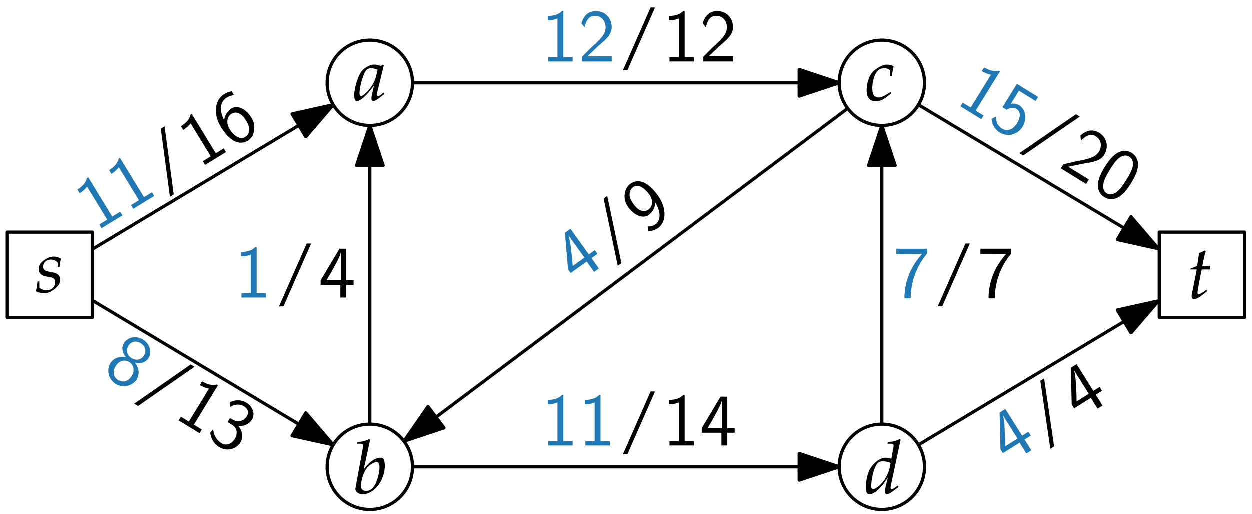

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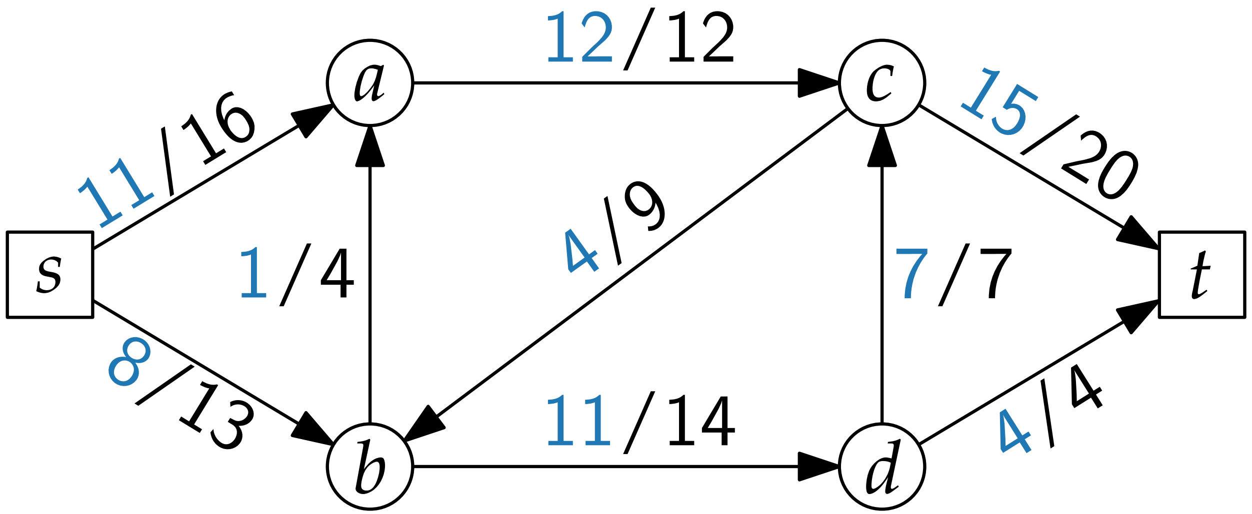

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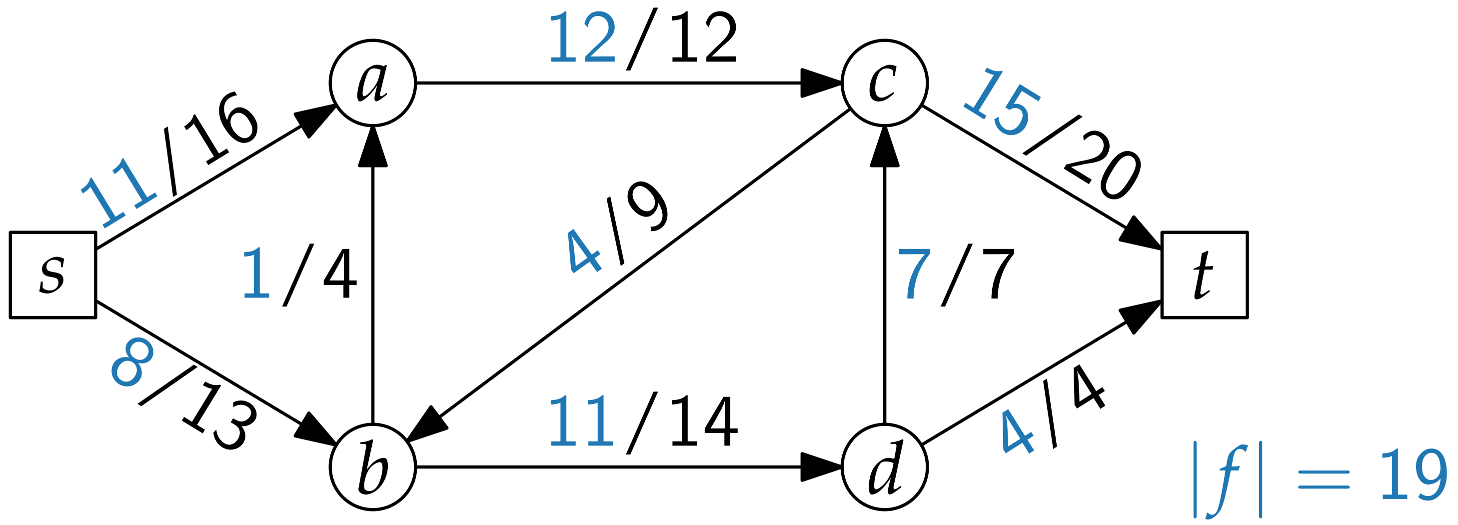

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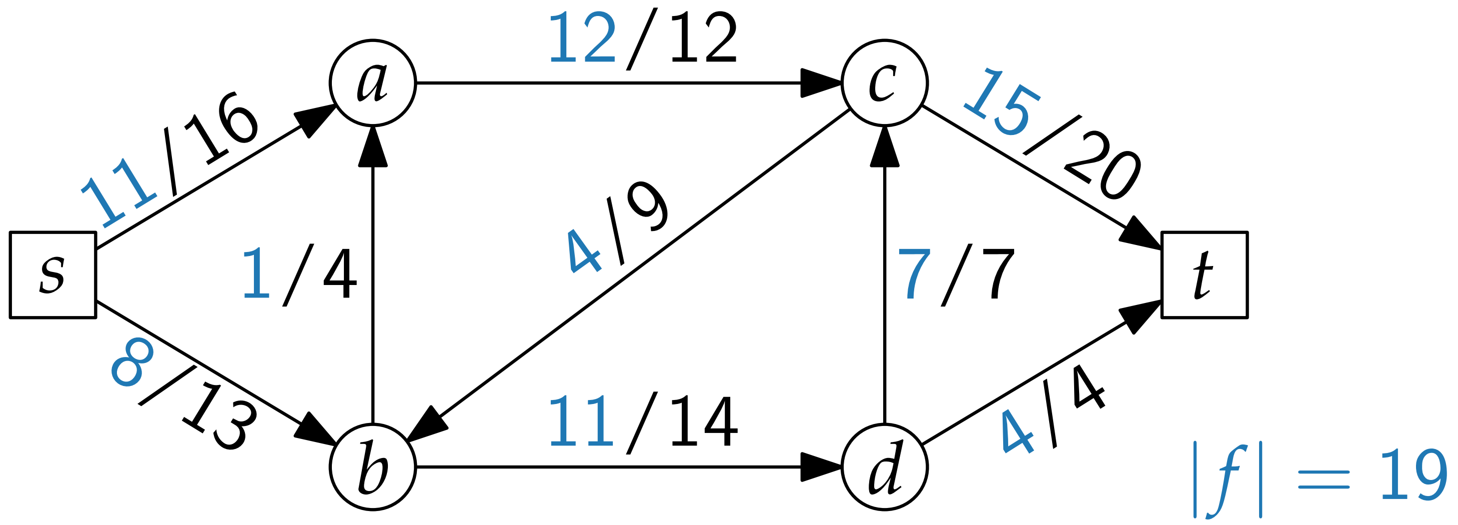

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Given a flow network G with source s and sink t, find an s–t flow of maximum value.

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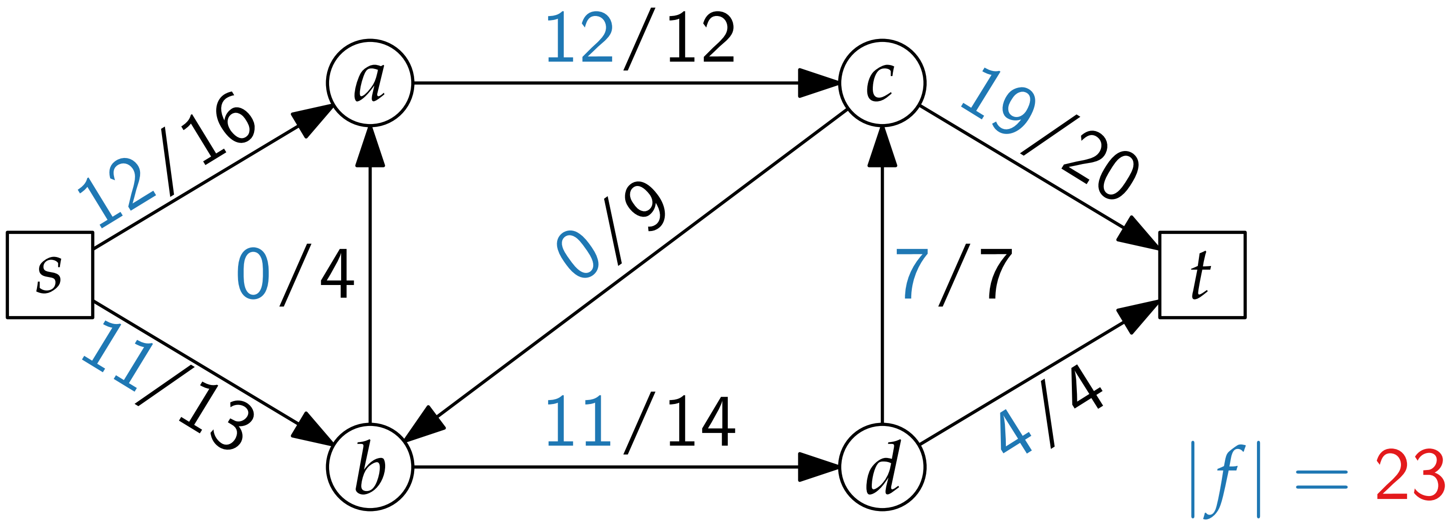

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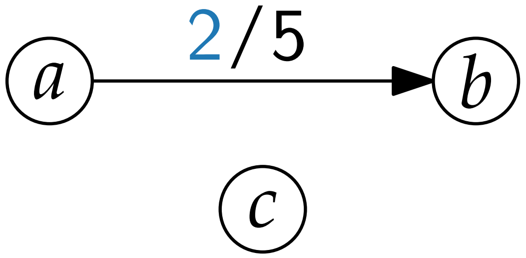

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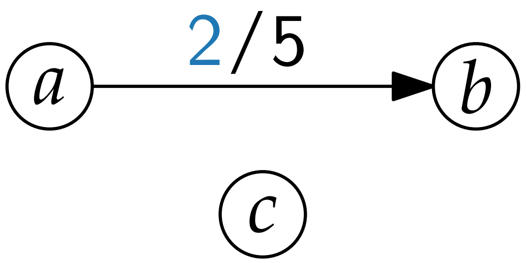
Given G and f, the **residual capacity** c_f for a pair $u, v \in V$ is

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \\ f(v,u) & \text{if } (v,u) \in E \\ 0 & \text{otherwise.} \end{cases}$$



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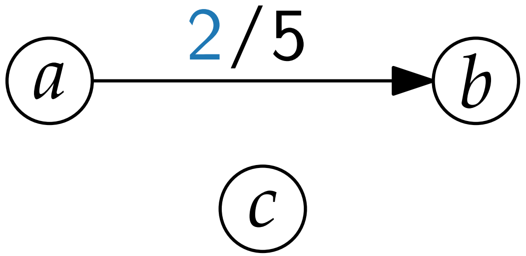

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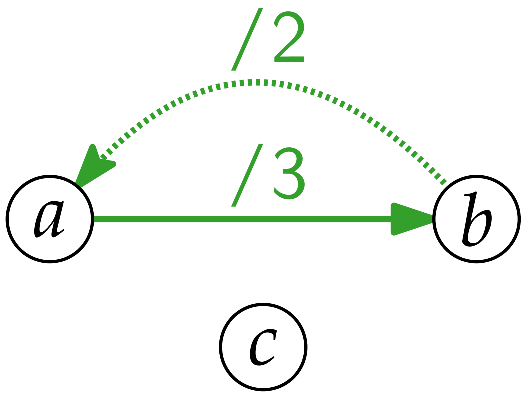

Residual Networks & Augmenting Paths

The **residual network** $G_f = (V, E_f)$ for a flow network G with s–t flow f has

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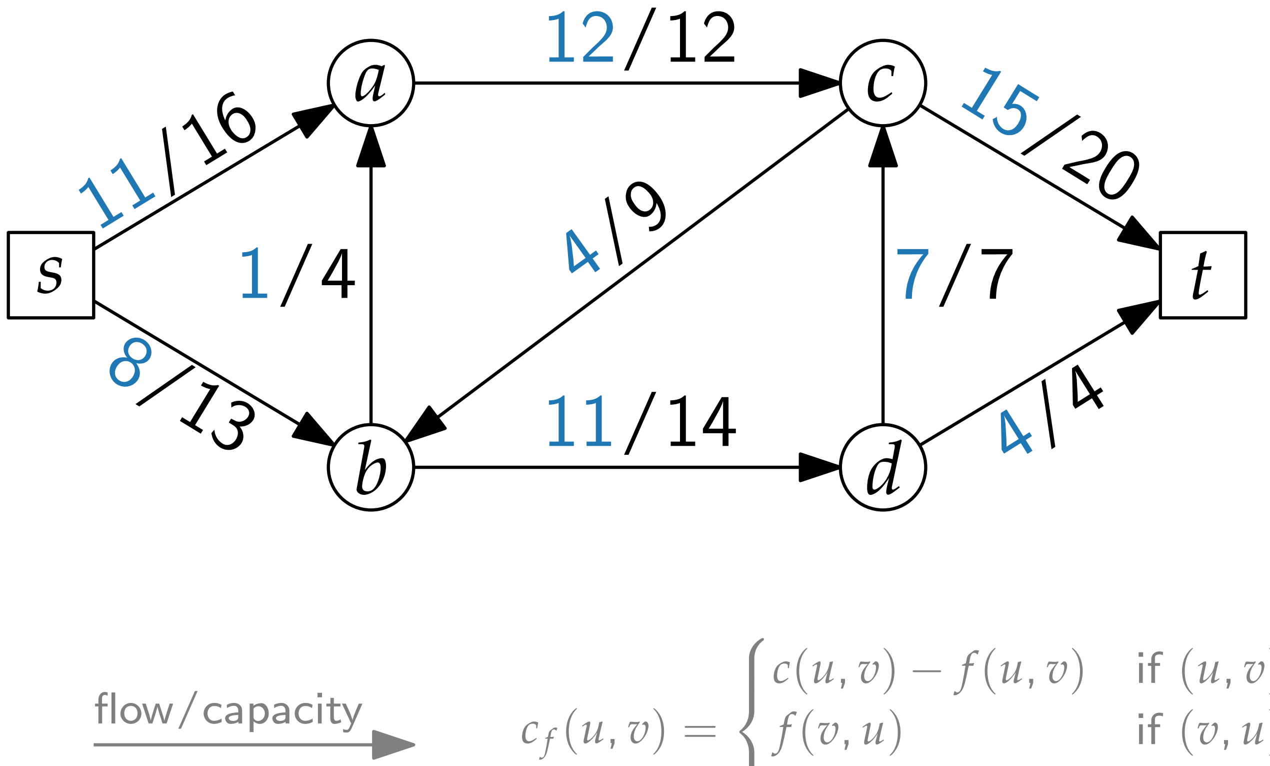

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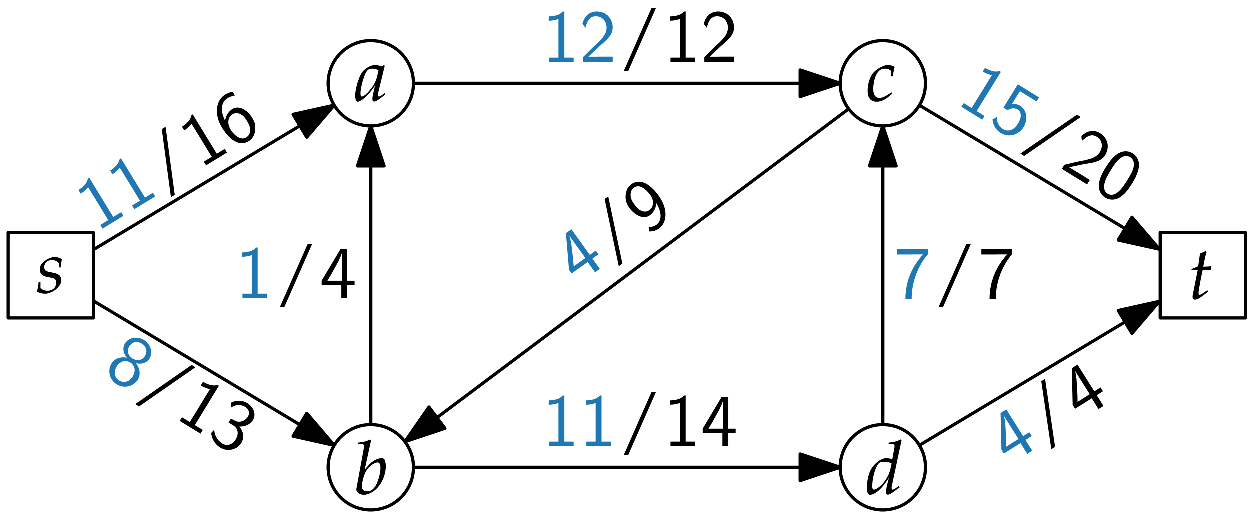

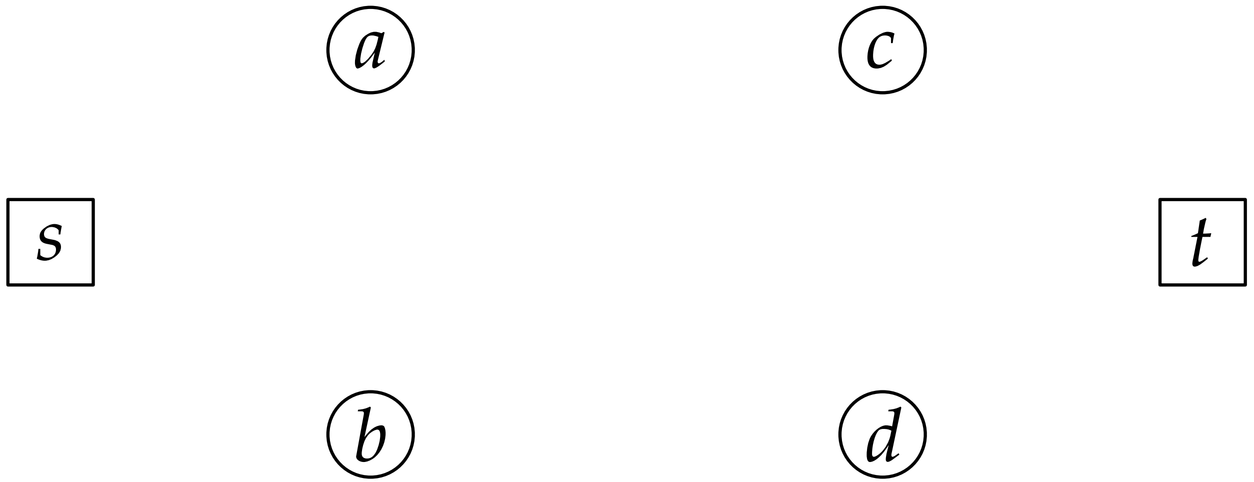

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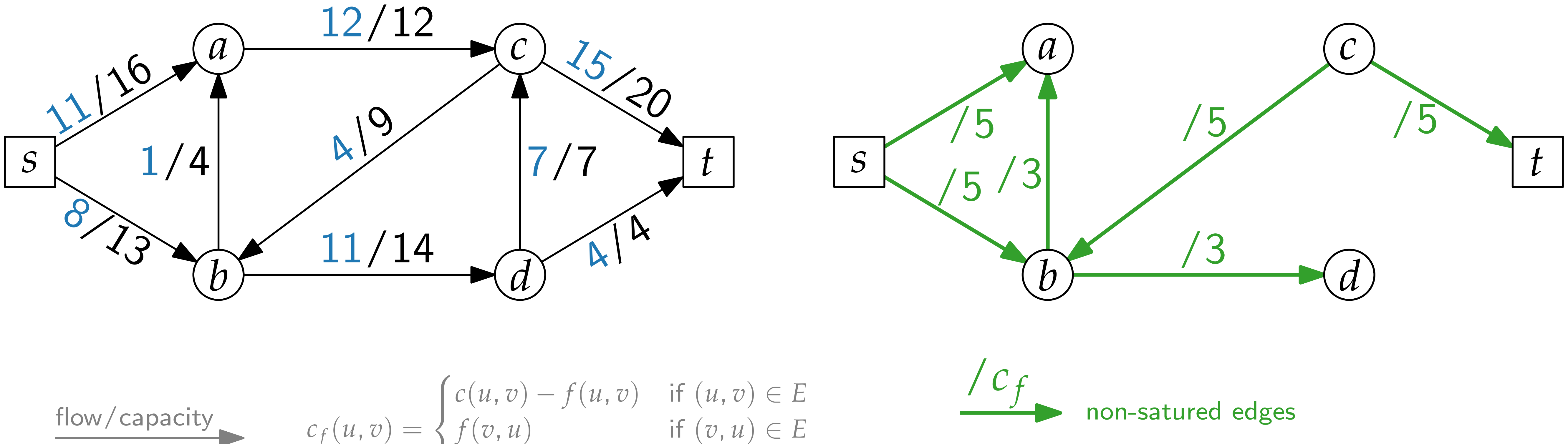

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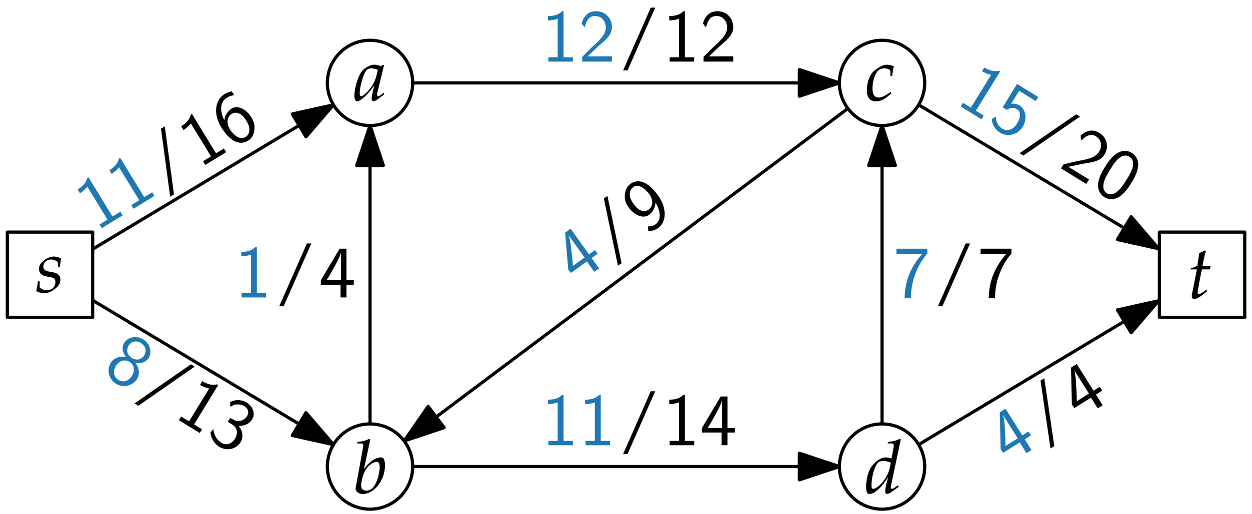

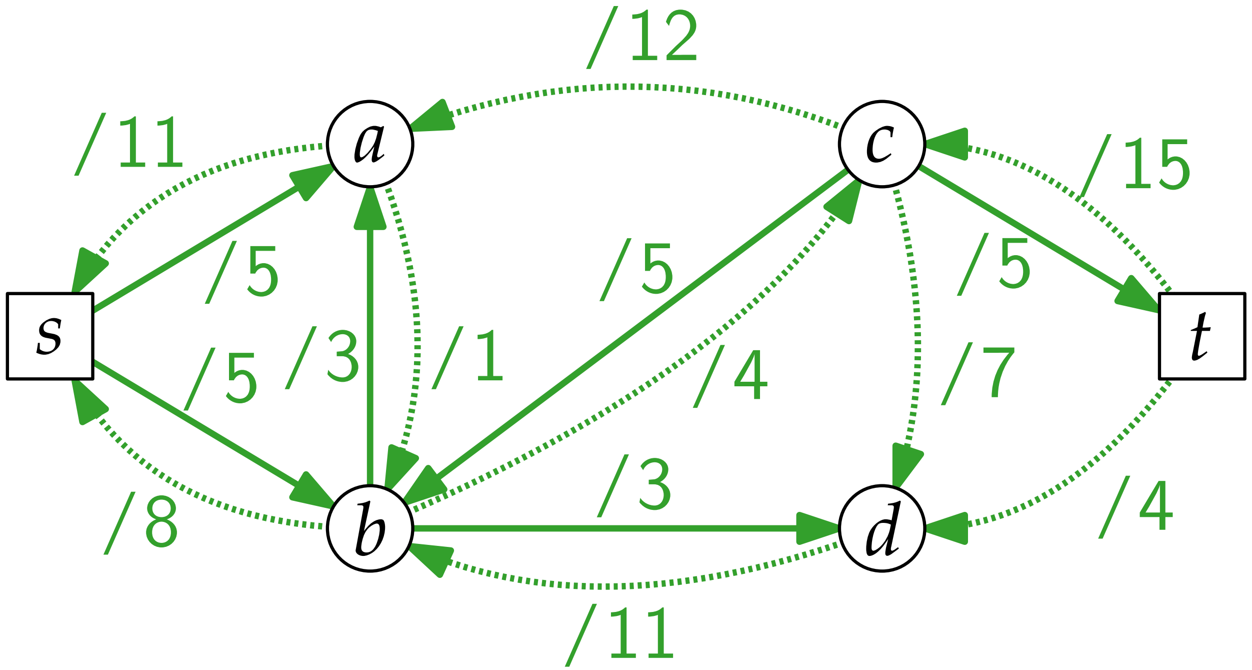

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$/c_f$ non-satured edges

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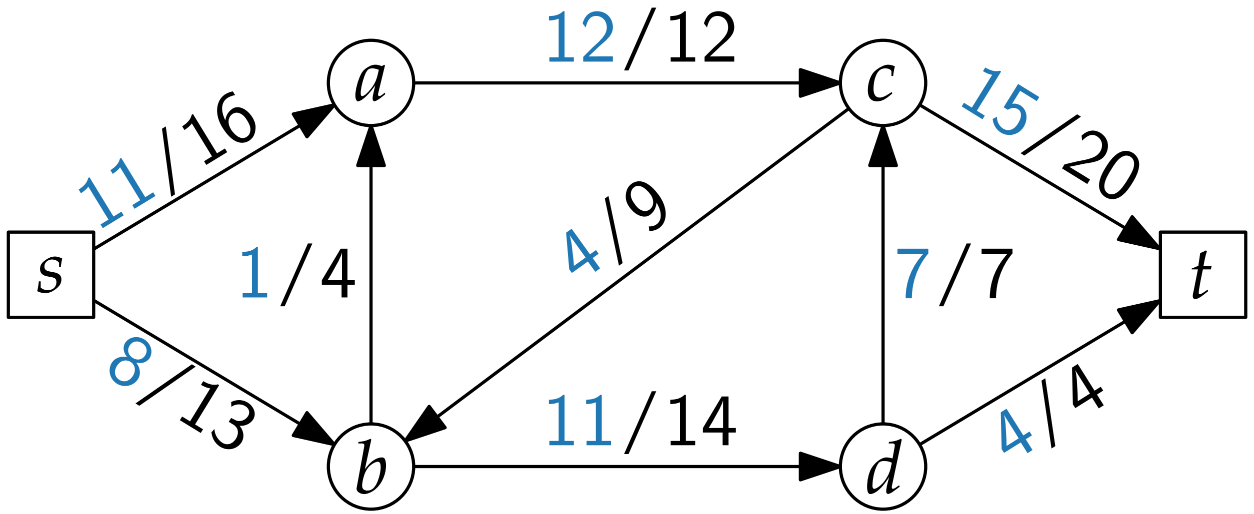

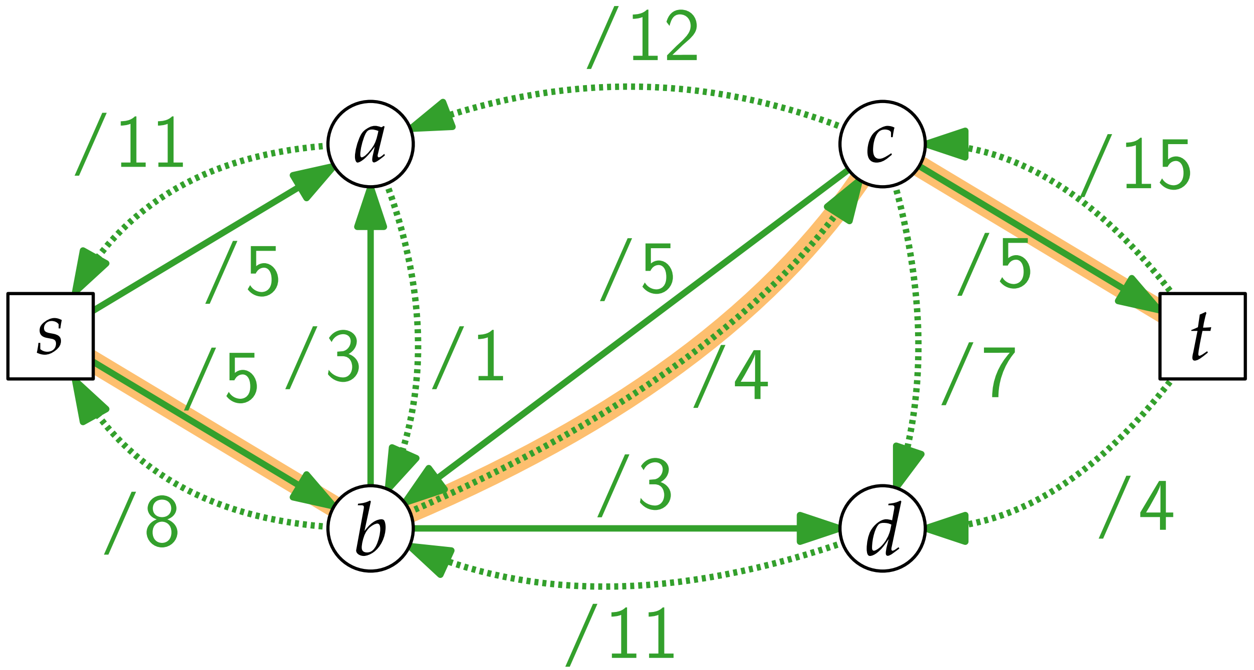

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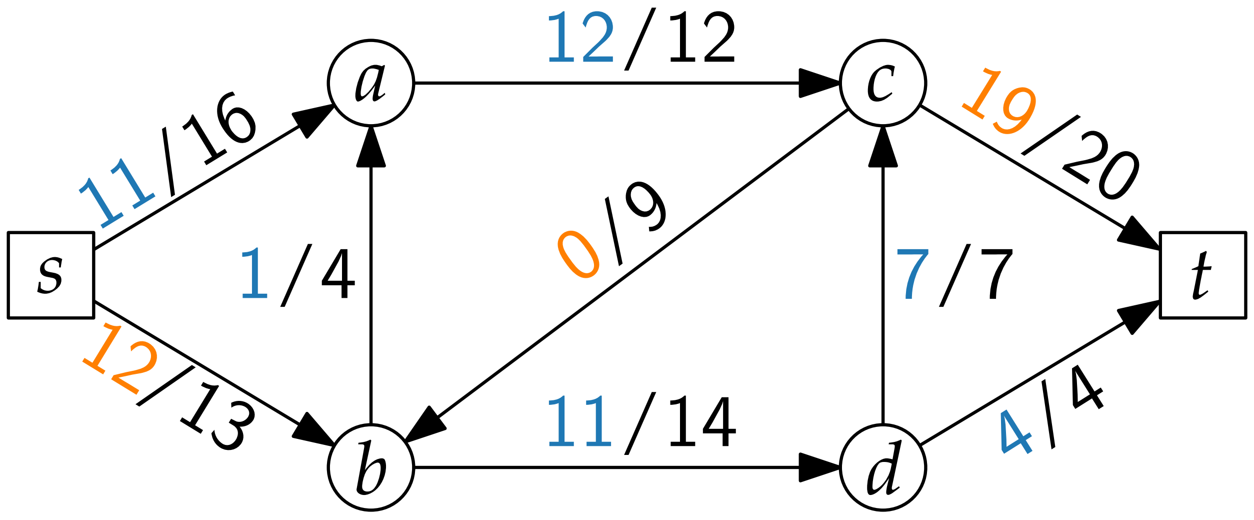

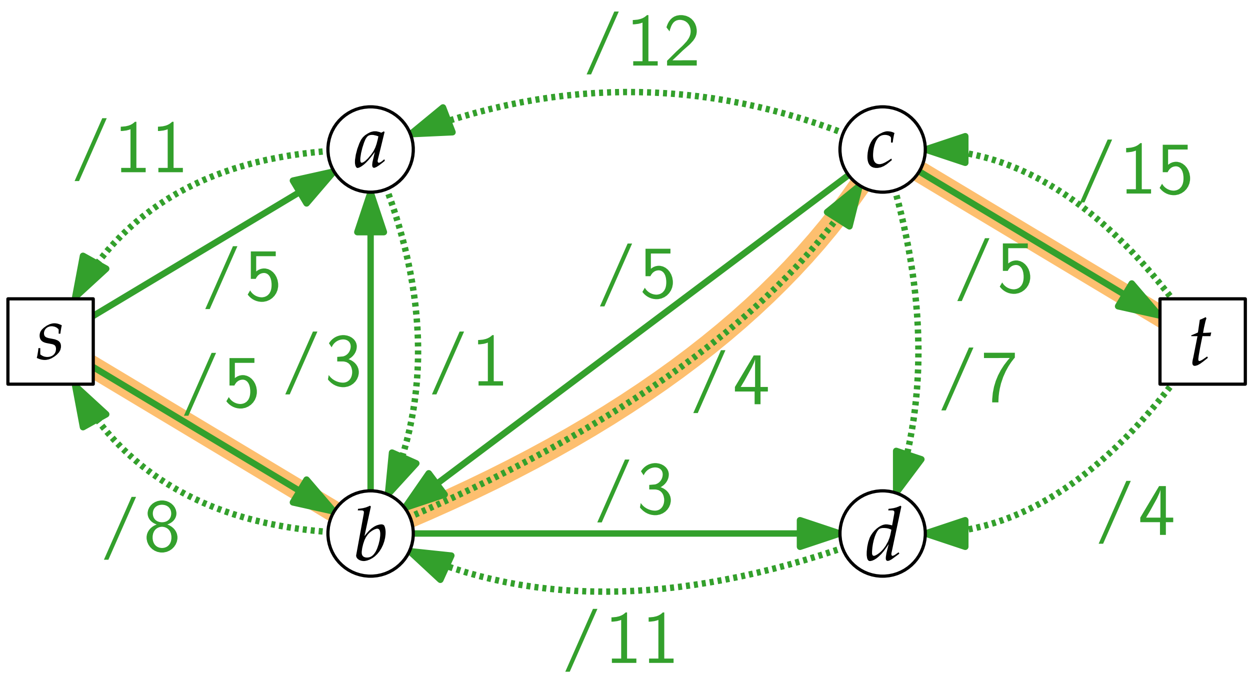

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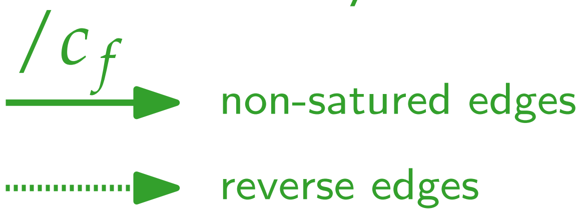

The Algorithms of Ford–Fulkerson and Edmonds–Karp

FordFulkerson$(G = (V, E), c, s, t)$

```
foreach uv ∈ E do                              } initialising zero flow
    f_uv ← 0
while G_f contains augmenting path p do
    Δ ← min_{uv∈p} c_f(uv)                     } residual capacity of p
    foreach uv ∈ p do                          ⎫
        if uv ∈ E then                         ⎪
            f_uv ← f_uv + Δ                    ⎬ augmentation along p
        else                                   ⎪
            f_vu ← f_vu − Δ                    ⎭
return f                                       } return max flow
```

The Algorithms of Ford–Fulkerson and Edmonds–Karp

EdmondsKarp

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while G_f contains shortest augmenting path p **do**

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    if $uv \in E$ **then**

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foreach $uv \in E$ **do** } initialising zero flow
  $f_{uv} \leftarrow 0$

                shortest
while G_f contains augmenting path p **do**
  $\Delta \leftarrow \min_{uv \in p} c_f(uv)$ } residual capacity of p
  foreach $uv \in p$ **do** } augmentation along p
    if $uv \in E$ **then**
      $f_{uv} \leftarrow f_{uv} + \Delta$
    else
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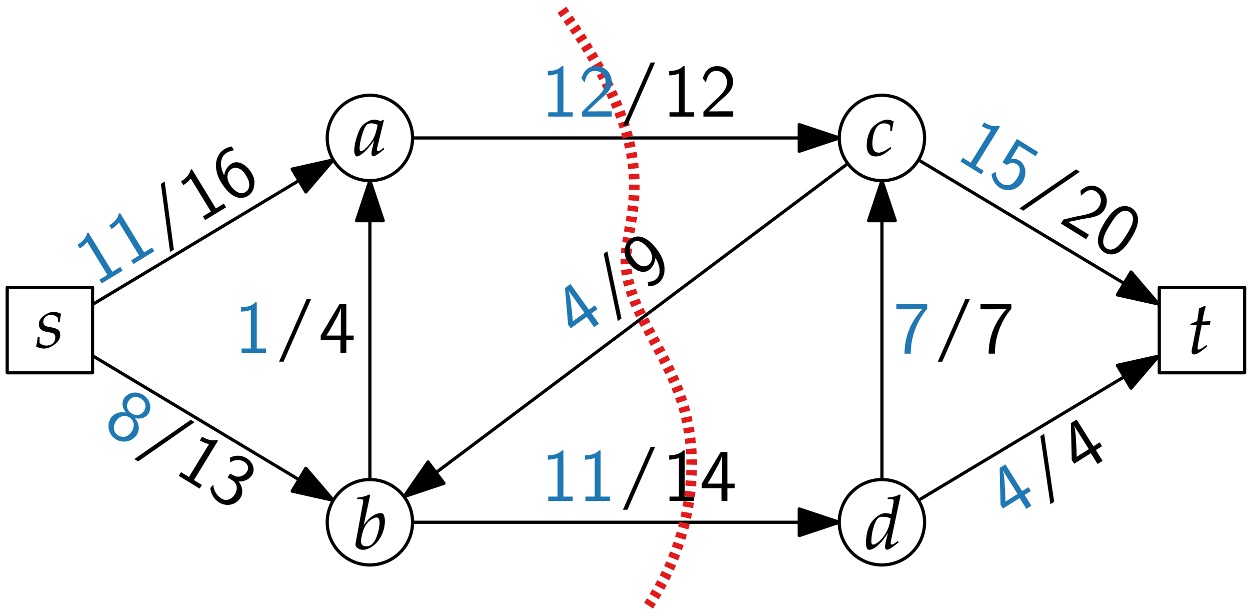
return f } return max flow

- Ford–Fulkerson runs in $\mathcal{O}(|E| \cdot |f^\star|)$ and Edmonds–Karp in $\mathcal{O}(|V| \cdot |E|^2)$ time.

The Max-Flow Min-Cut Theorem



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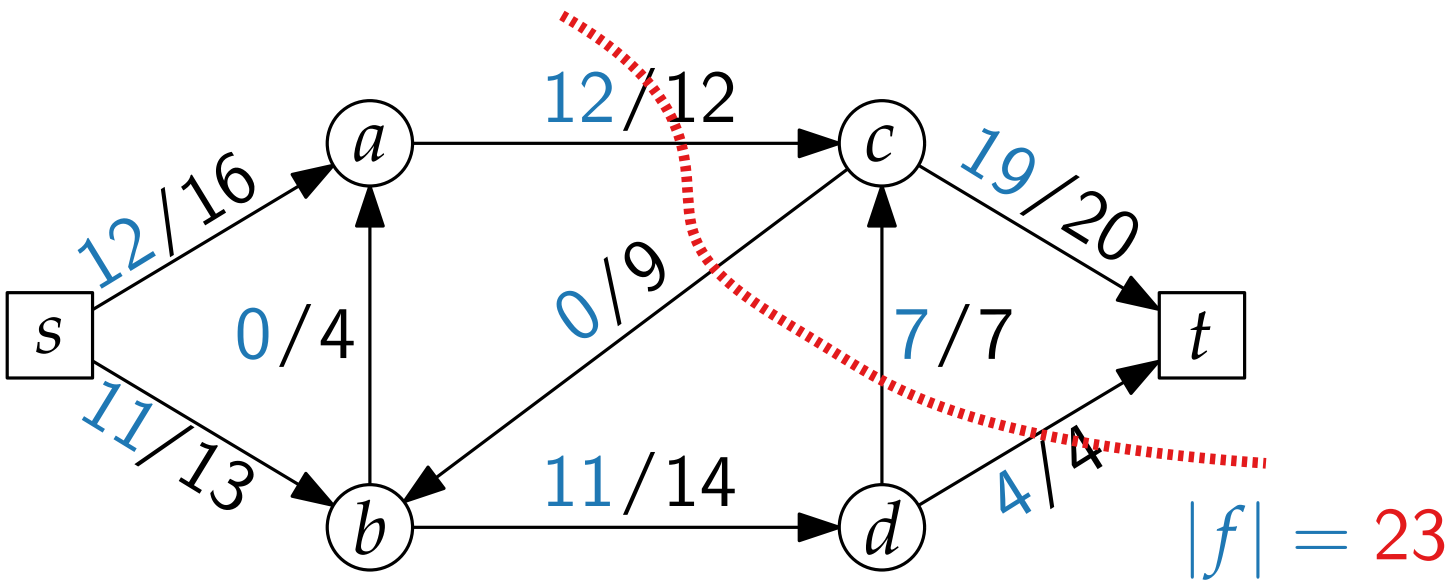
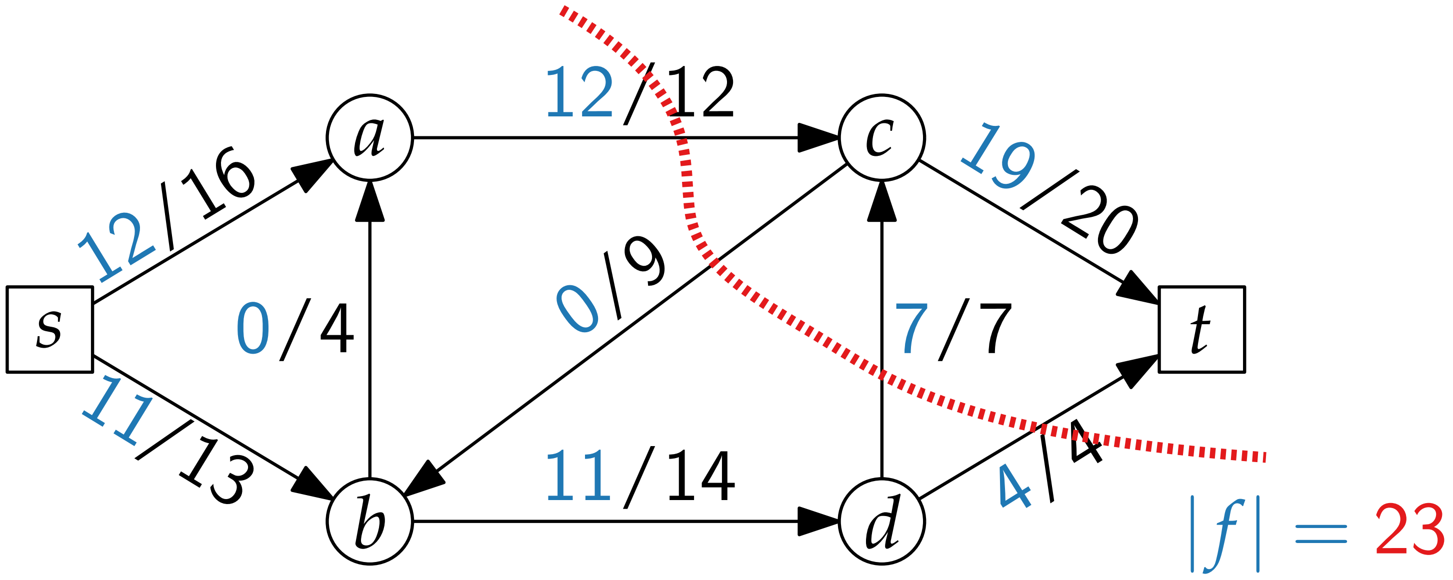


The Max-Flow Min-Cut Theorem

Theorem.
For an s–t flow f in a flow network G, the following conditions are equivalent:
- f is a maximum s–t flow in G.
- G_f contains no augmenting paths.
- $|f| = c(S, T)$, which is the capacity of some s–t cut (S, T) of G.



The Push–Relabel Idea

A New Approach to the Maximum-Flow Problem

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Massachusetts Institute of Technology, Cambridge, Massachusetts

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ROBERT E. TARJAN

Princeton University, Princeton, New Jersey, and AT&T Bell Laboratories, Murray Hill, New Jersey

Abstract. All previously known efficient maximum-flow algorithms work by finding augmenting paths, either one path at a time (as in the original Ford and Fulkerson algorithm) or all shortest-length augmenting paths at once (using the layered network approach of Dinic). An alternative method based on the *preflow* concept of Karzanov is introduced. A preflow is like a flow, except that the total amount

for the next phase. Our algorithm abandons the idea of finding a flow in each phase and also abandons the idea of global phases. Instead, our algorithm maintains a preflow in the original network and pushes local flow excess toward the sink along what it estimates to be shortest paths in the residual graph. This pushing of flow changes the residual graph, and paths to the sink may become saturated. Excess that cannot be moved to the sink is returned to the source, also along estimated shortest paths. Only when the algorithm terminates does the preflow become a flow, and then it is a maximum flow.

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The Push–Relabel Idea

A New Approach to the Maximum-Flow Problem

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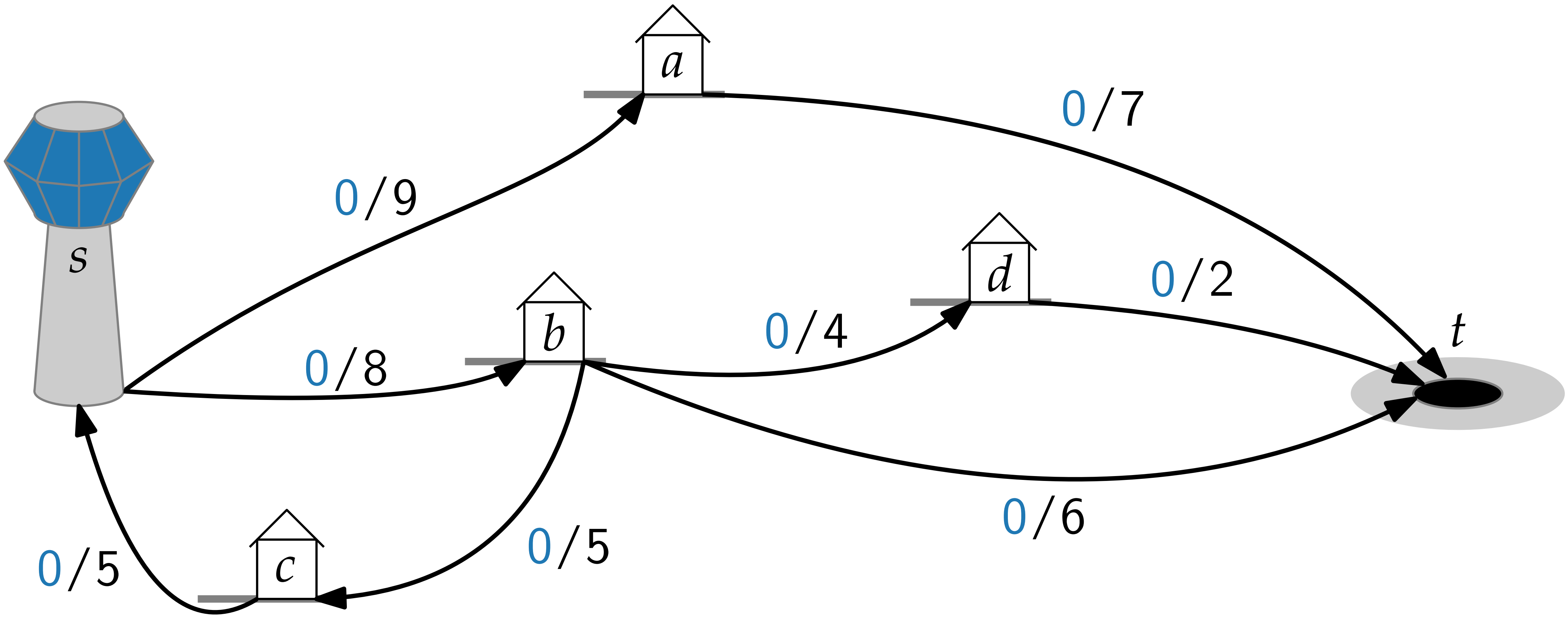

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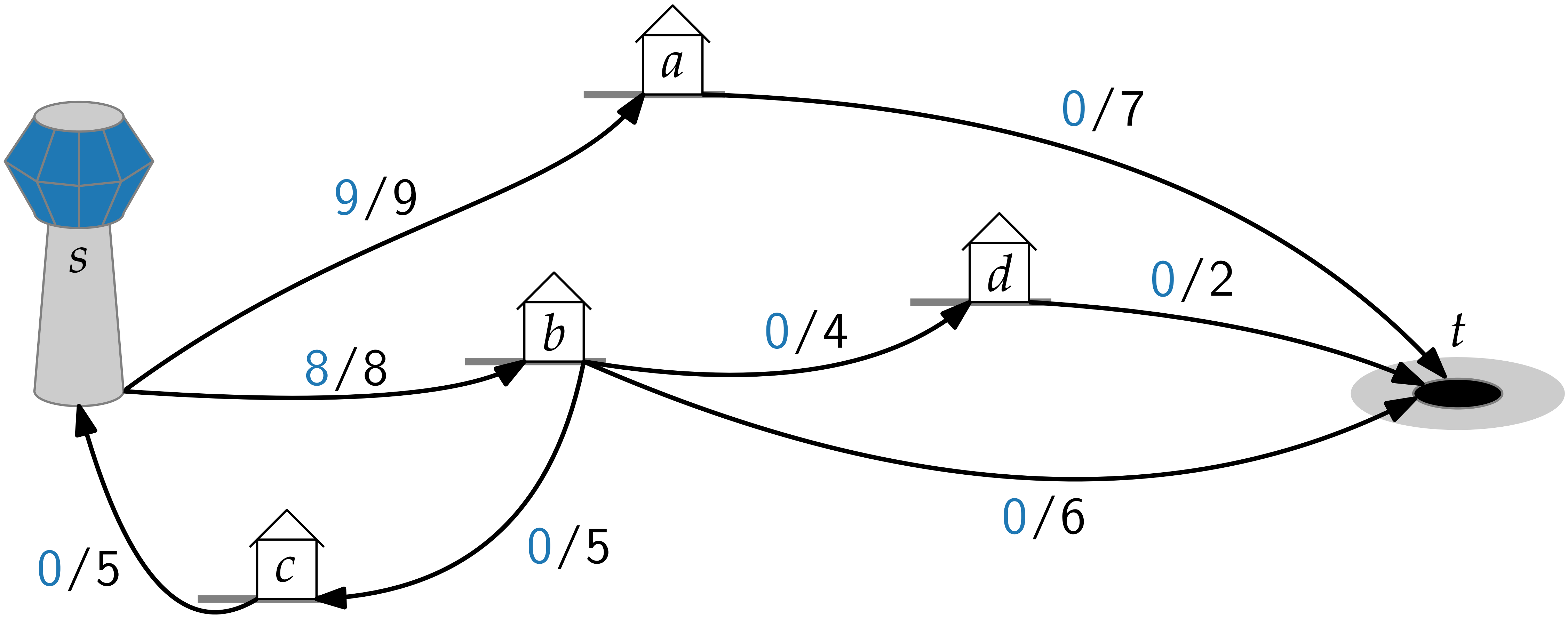

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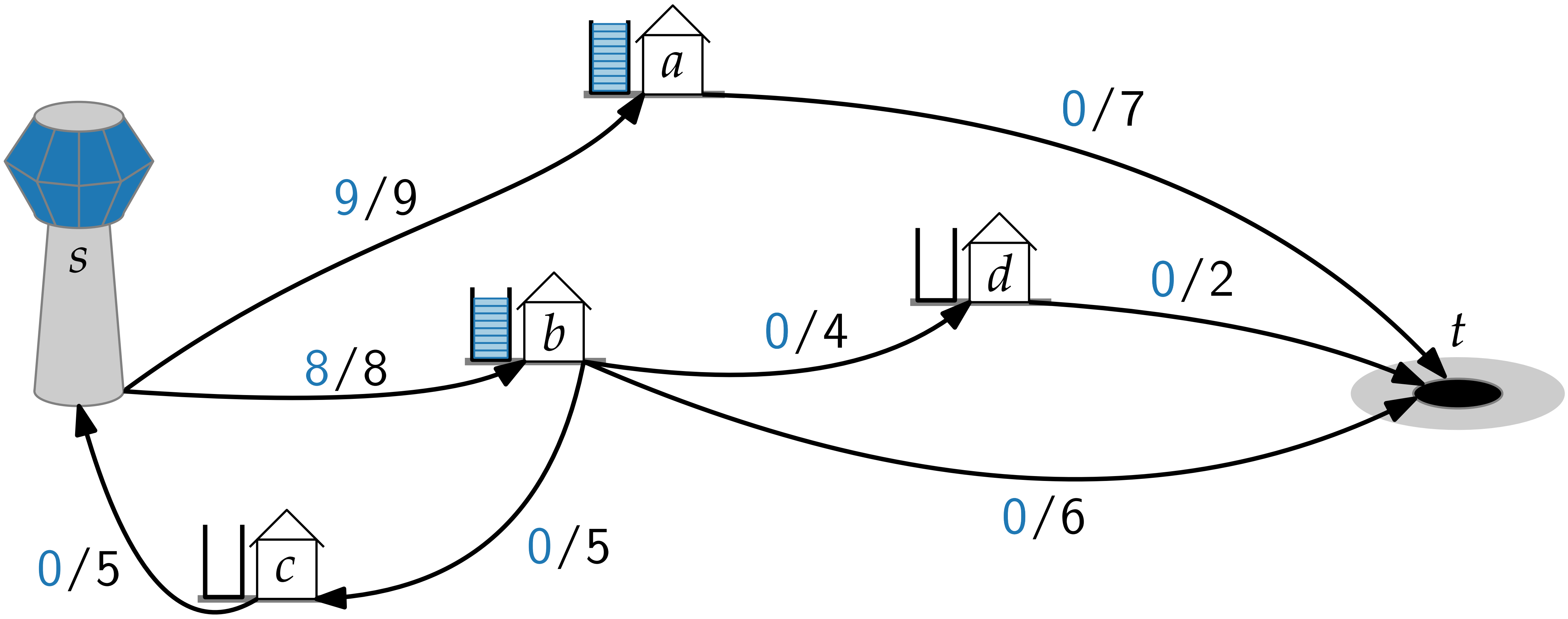

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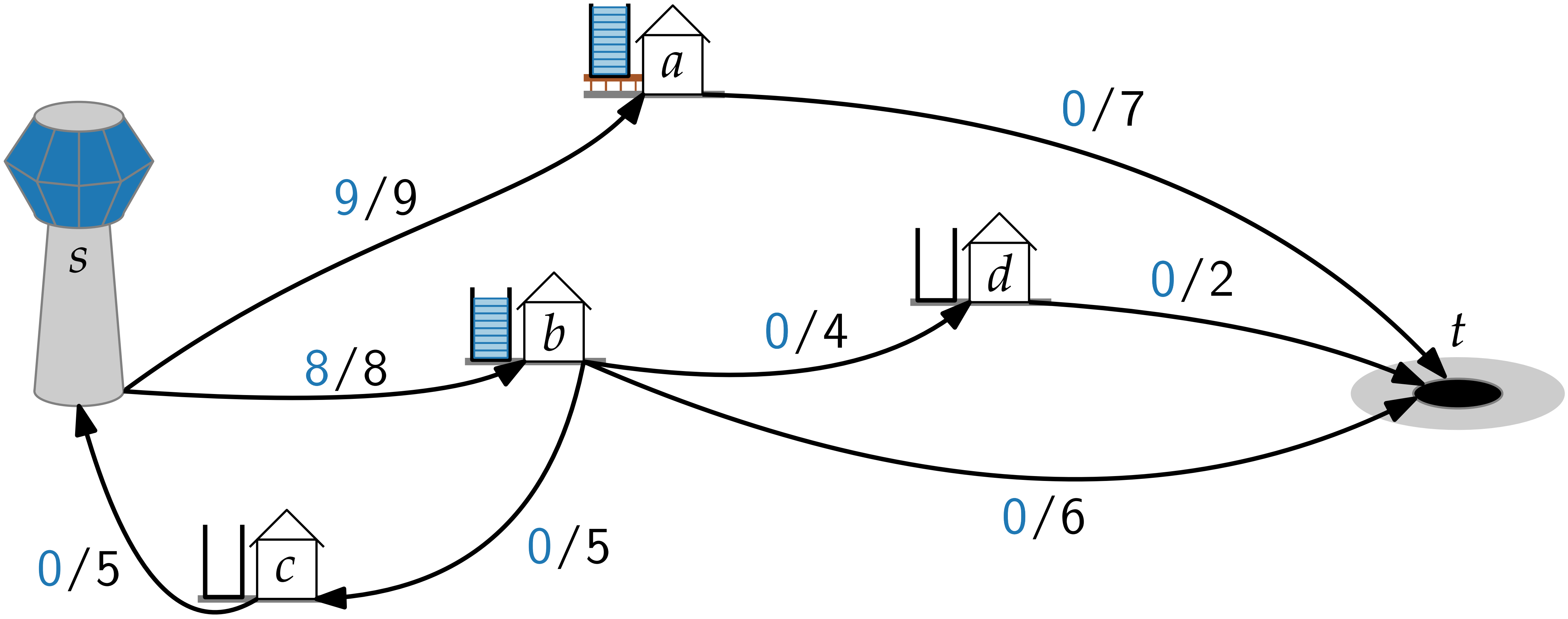

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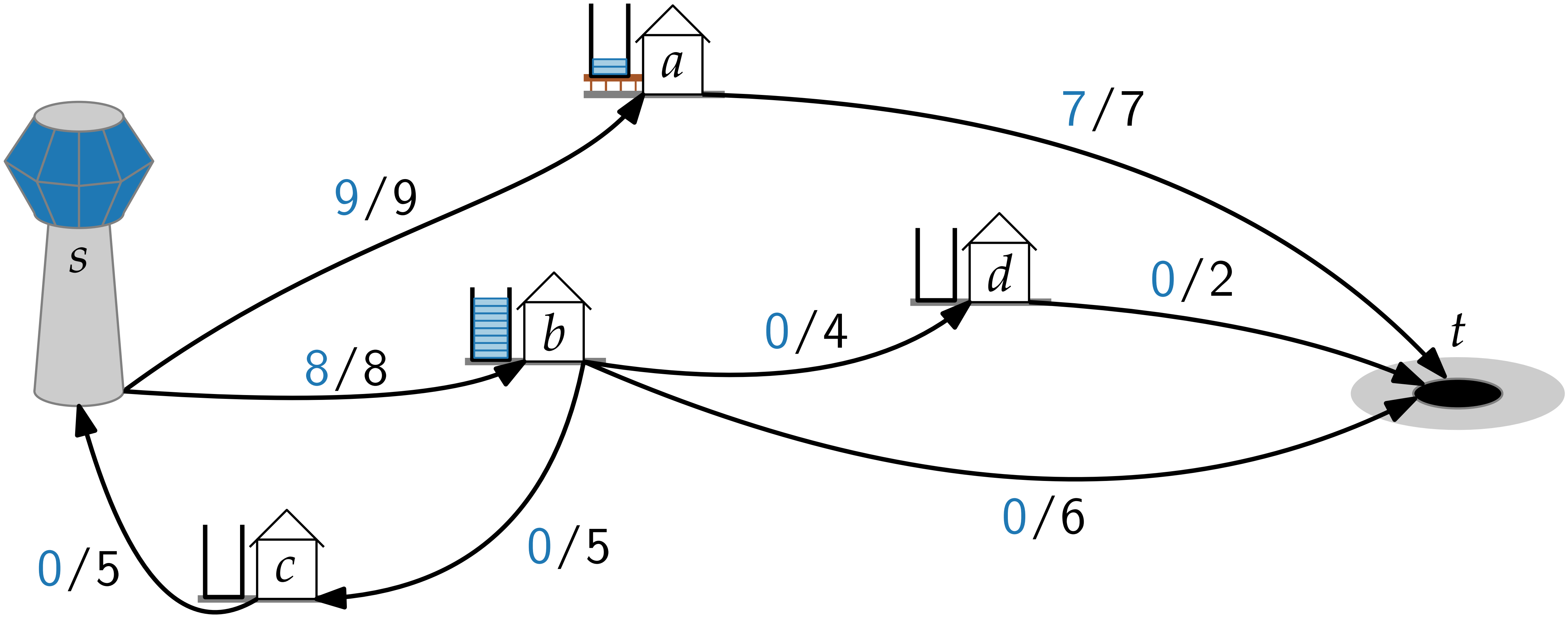

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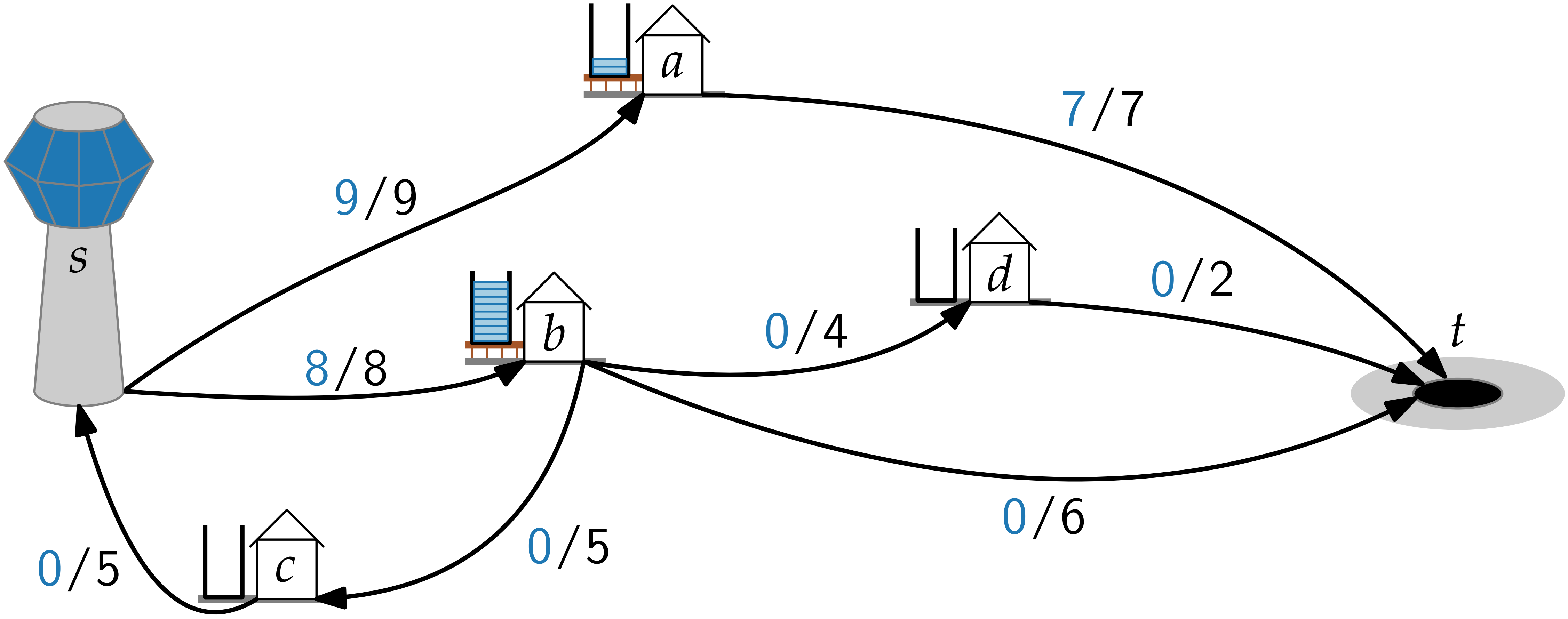

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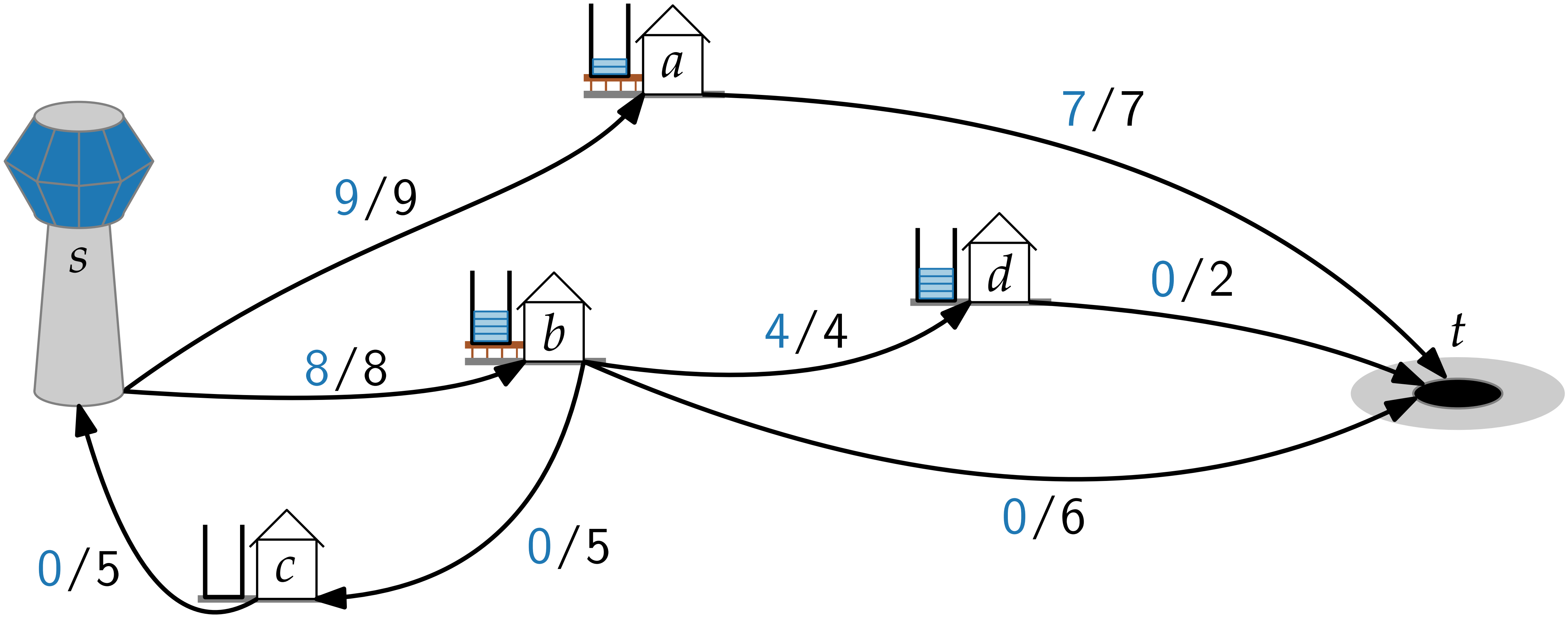

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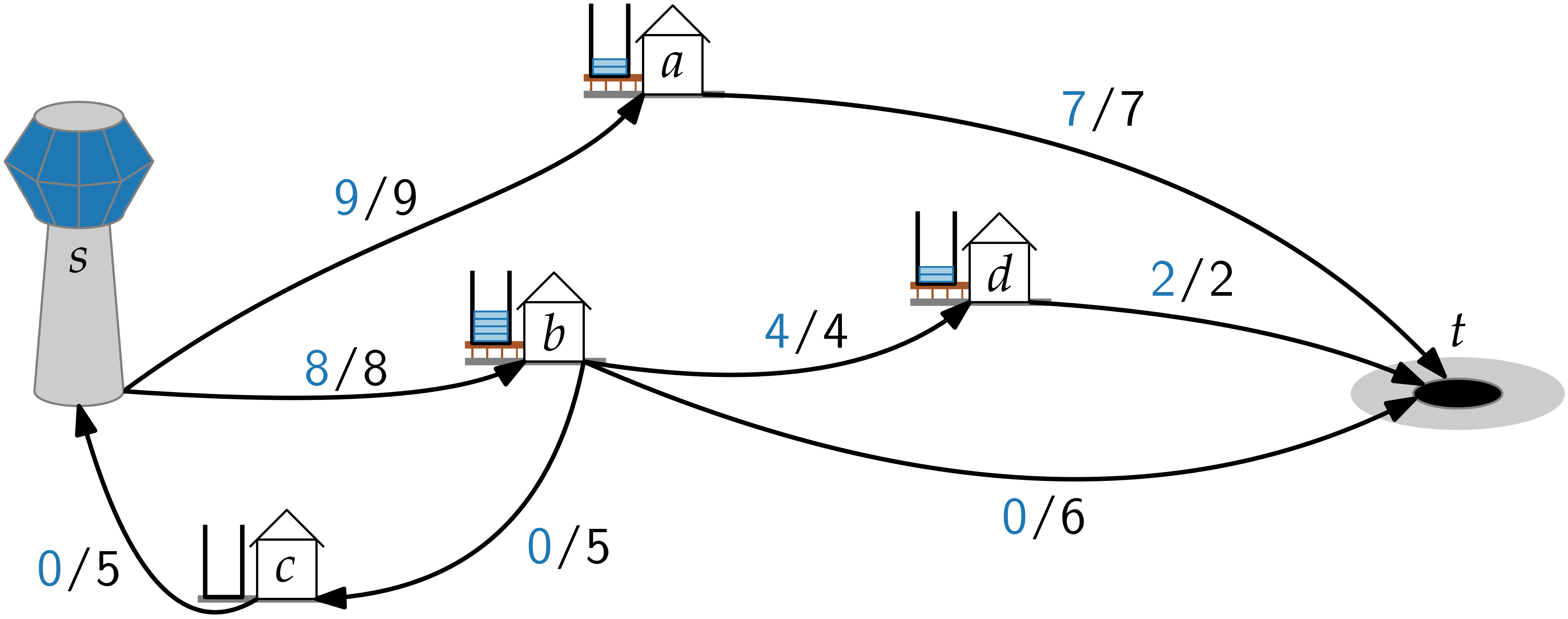

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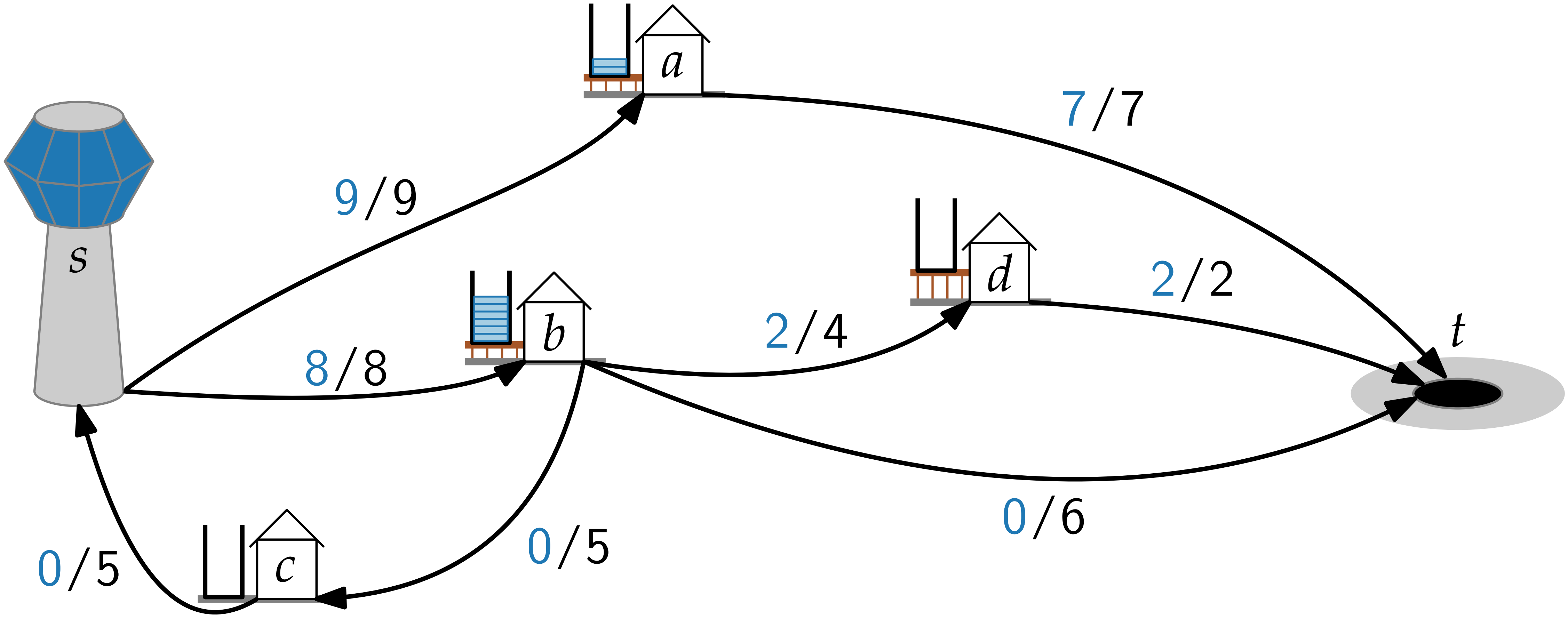

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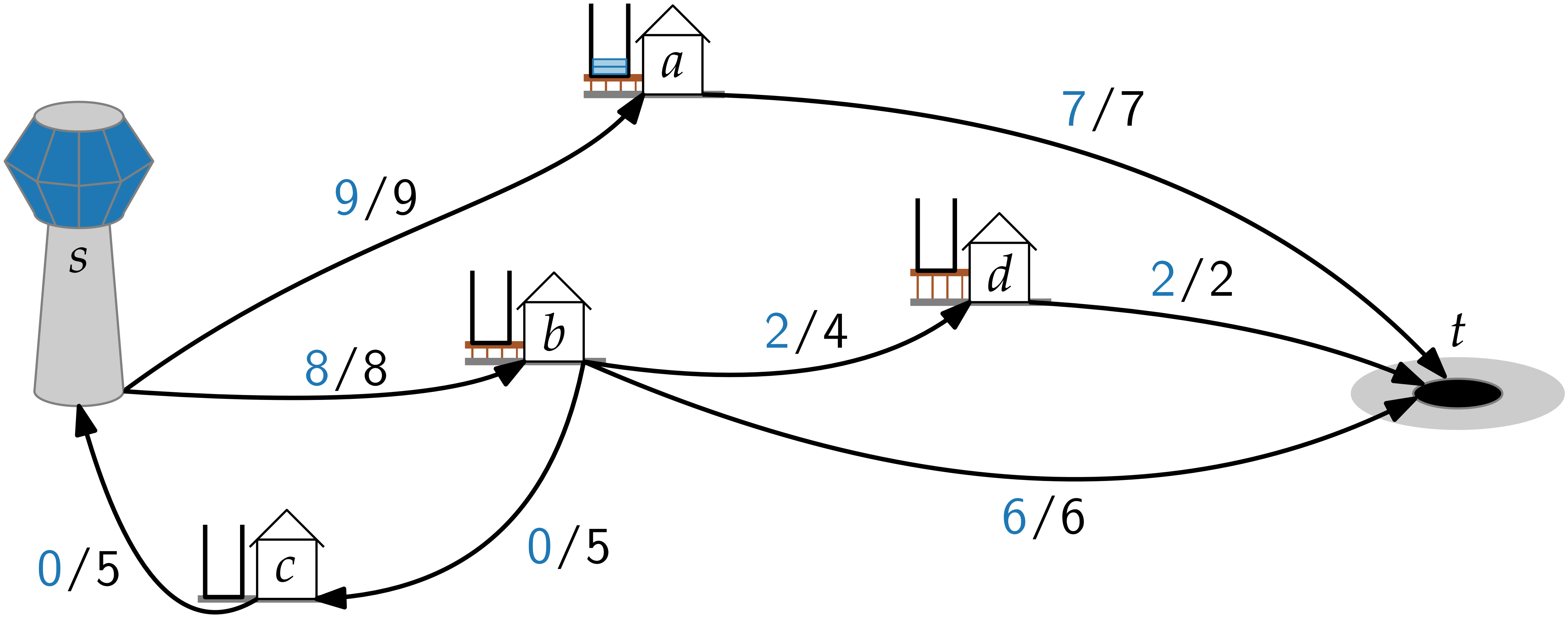

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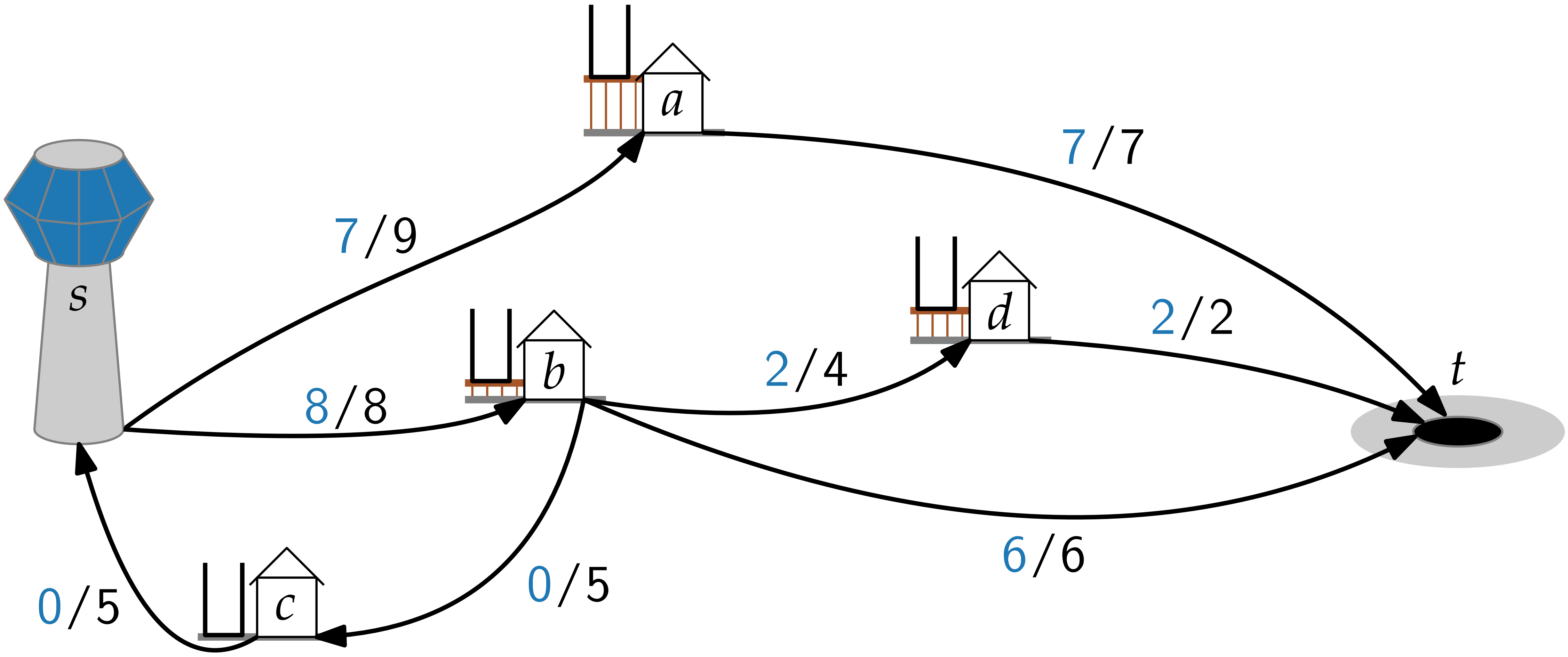
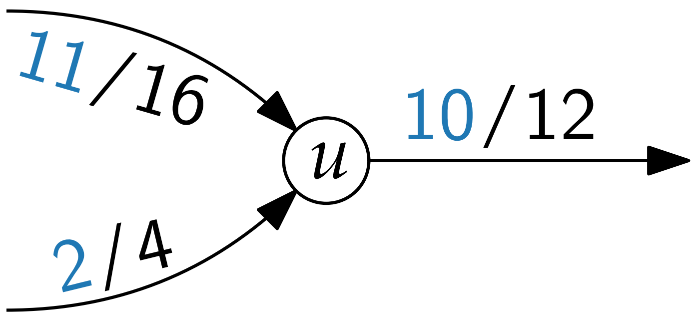


Preflow, Excess Flow, and Height

A **preflow** in G is a real-value function $f\colon V \times V \to \mathbb{R}$ that satisfies the capacity constraint and, for each $u \in V \setminus \{s\}$,

- $\sum_{v \in V} f(v,u) - \sum_{v \in V} f(u,v) \geq 0$.



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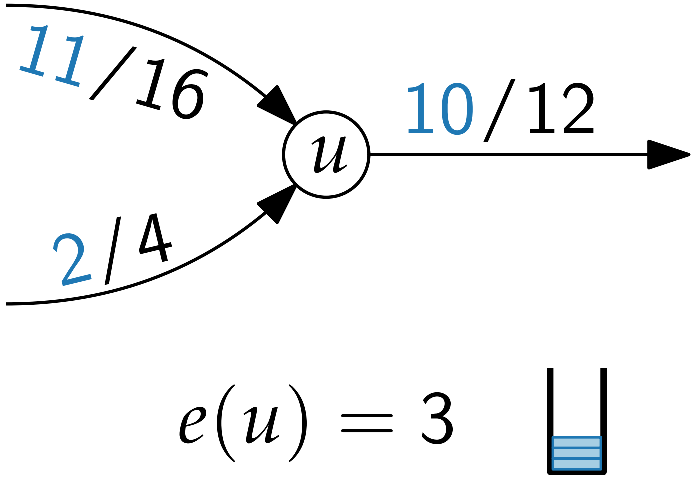

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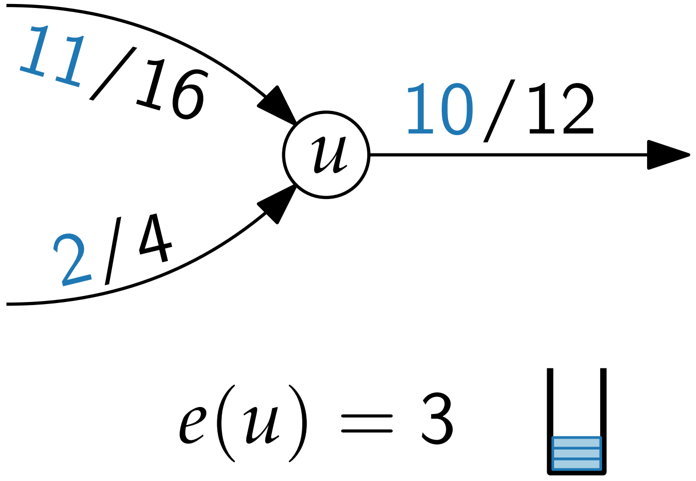

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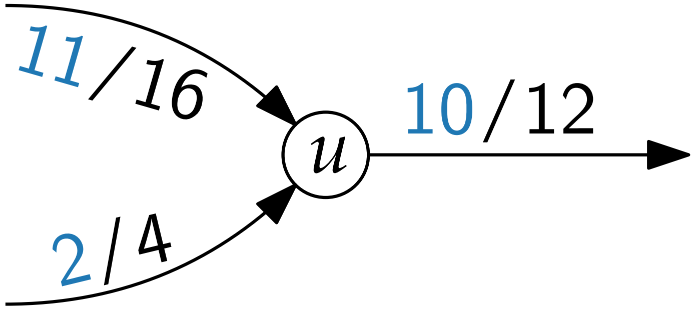

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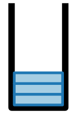

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For a flow network G with preflow f, a **height function** is a function $h\colon V \to \mathbb{N}$ such that

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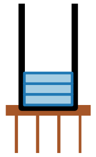

The PUSH Operation

PUSH(u, v)

Condition: u is overflowing, $c_f(u, v) > 0$, and $h(u) = h(v) + 1$

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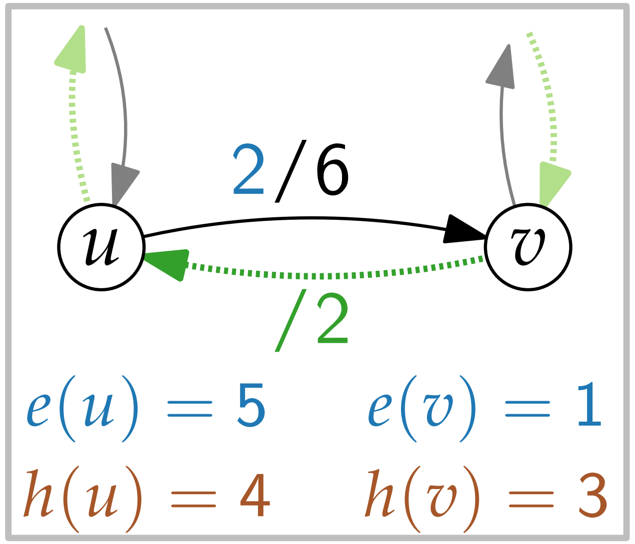

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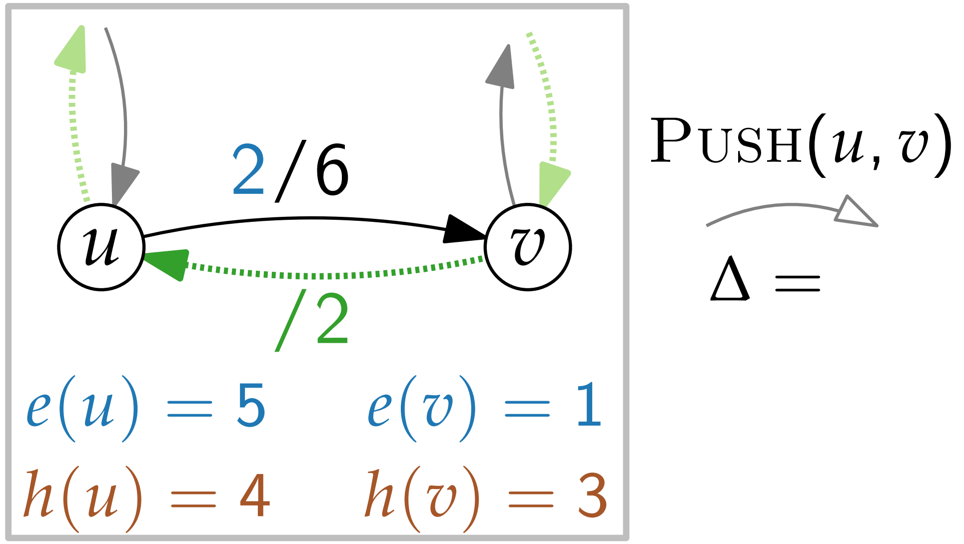

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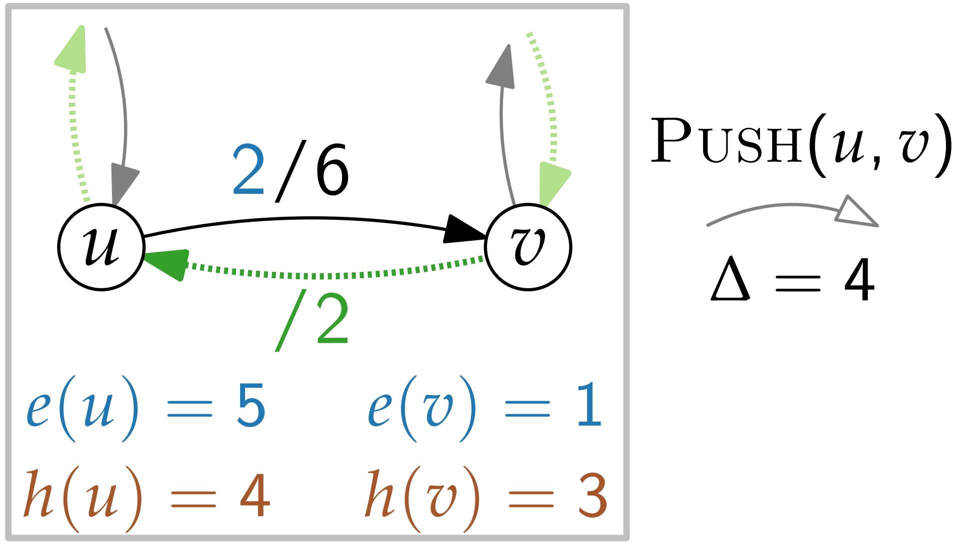

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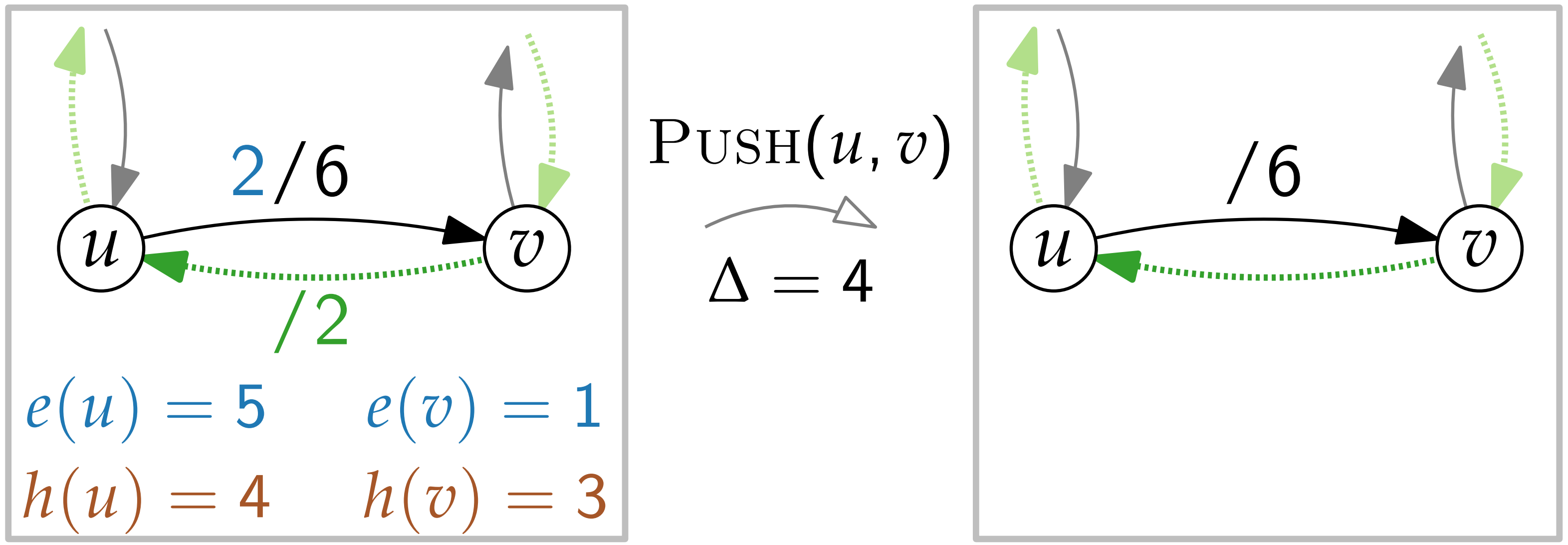

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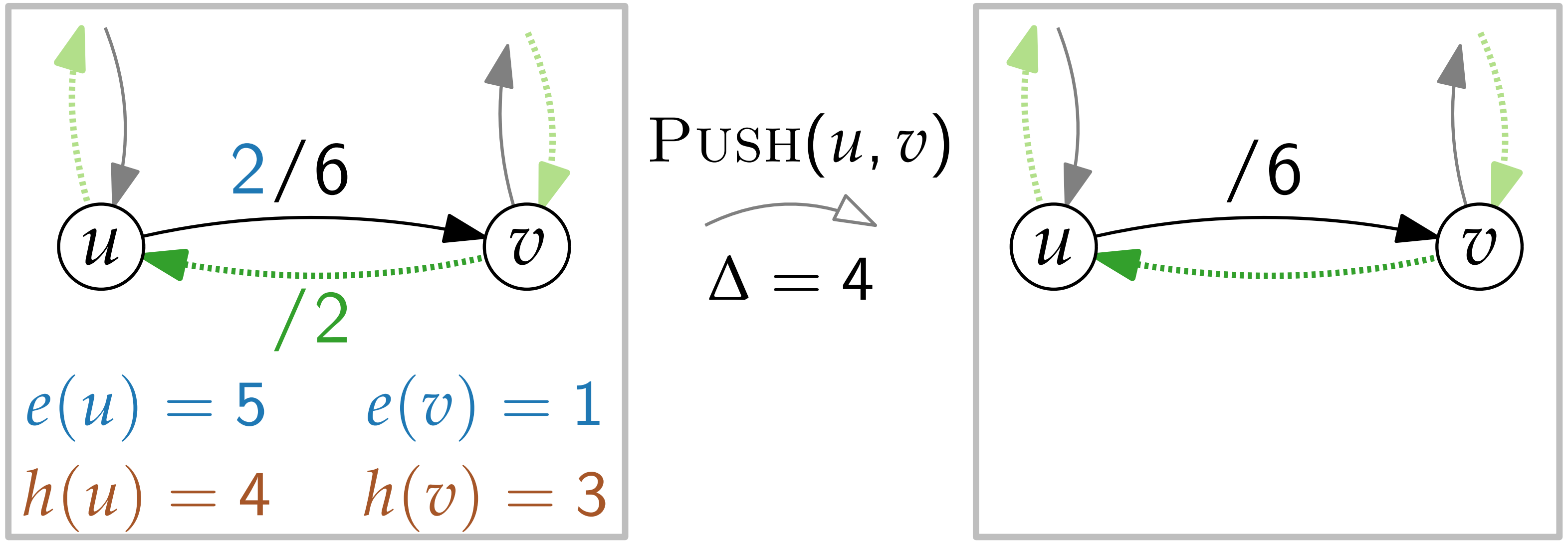

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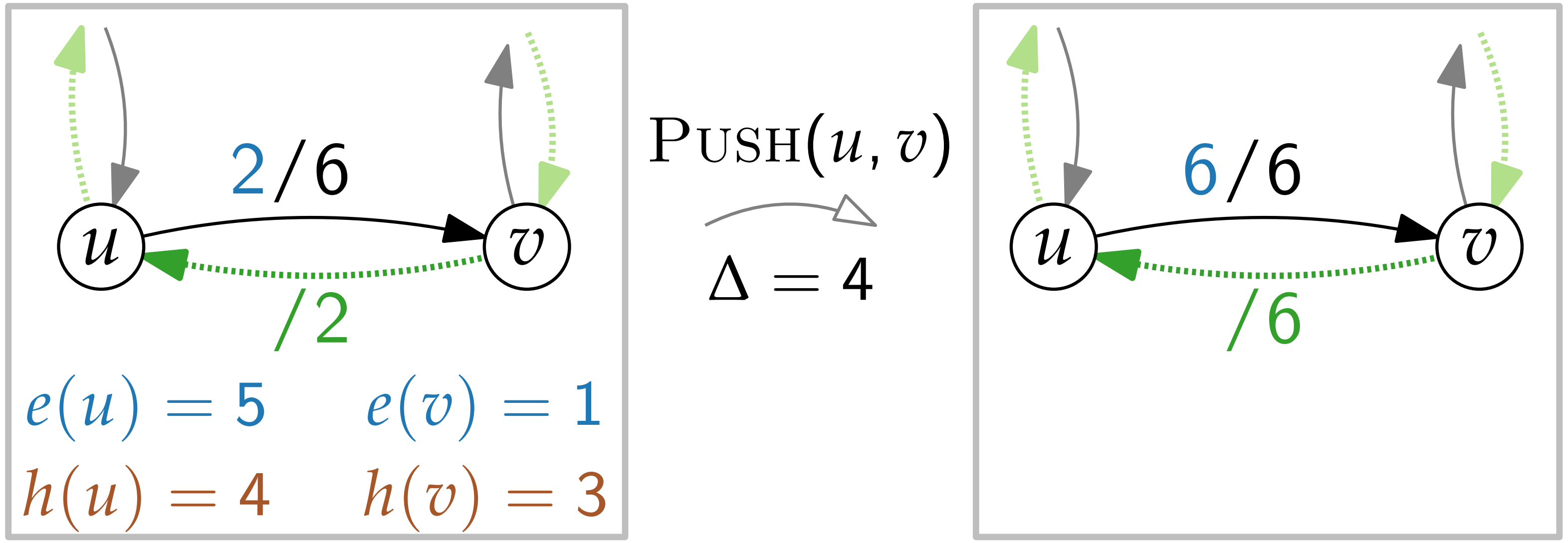

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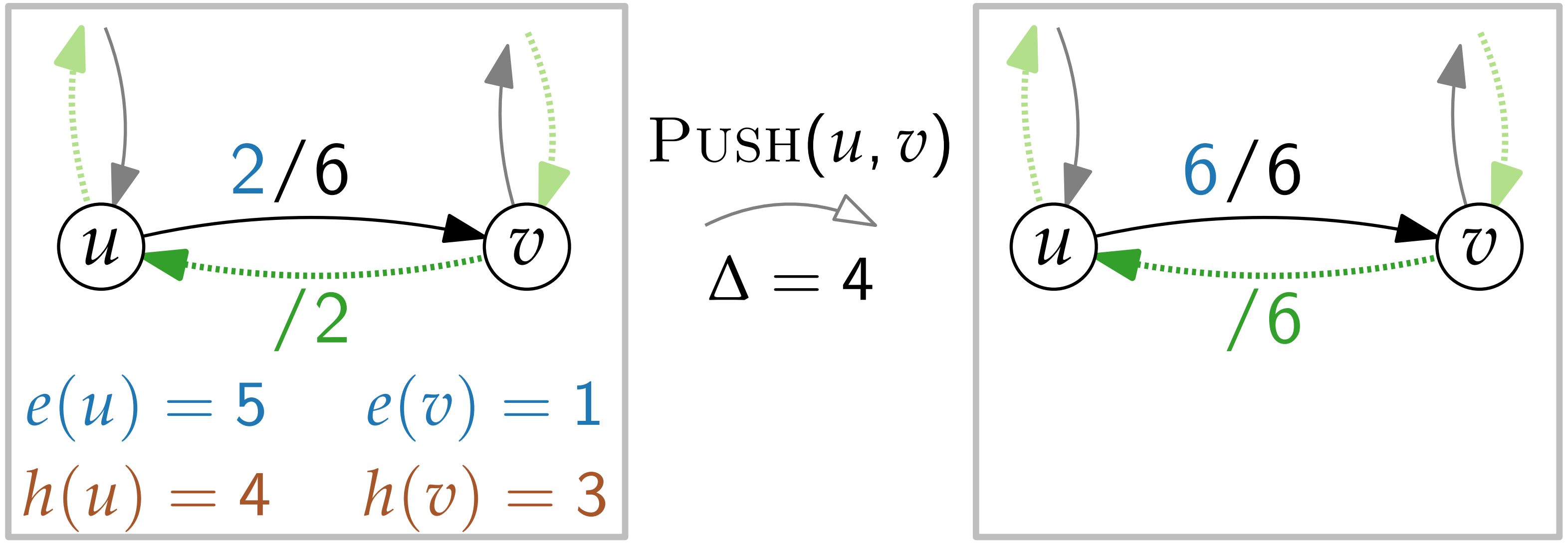

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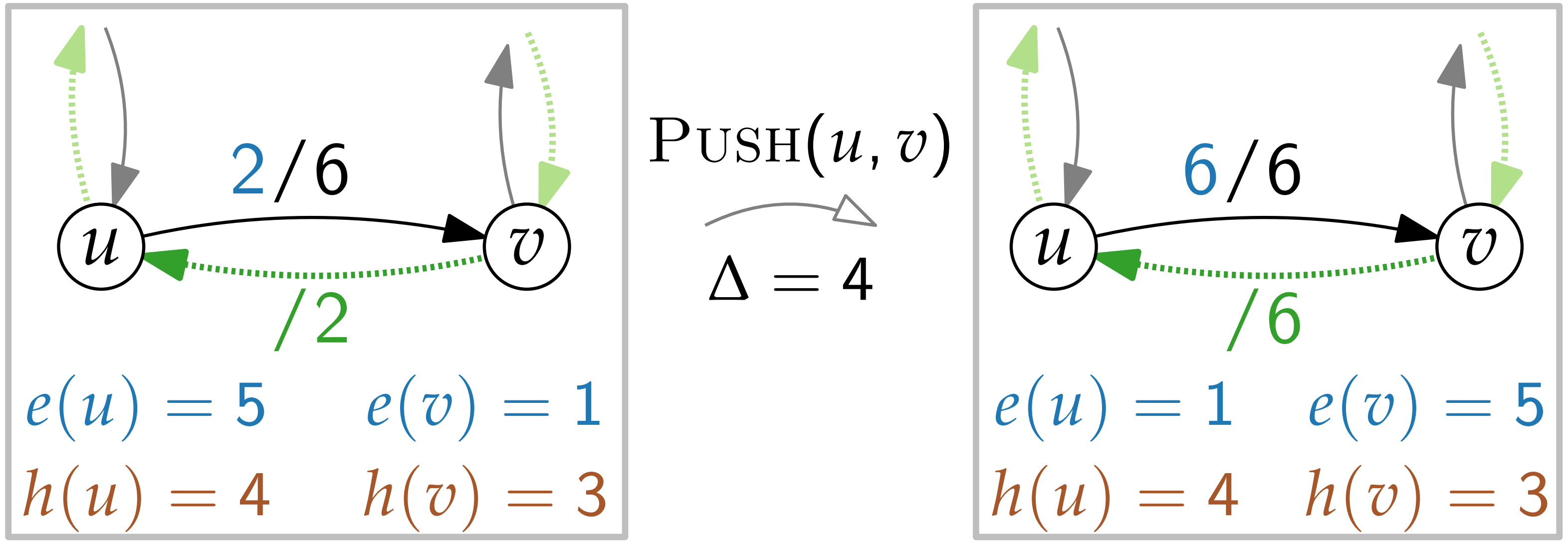

The Relabel Operation

Relabel(u)

Condition: u is overflowing and $h(u) \leq h(v)$ for every $v \in V$ with $(u, v) \in E_f$

Effect: Increase the height of u

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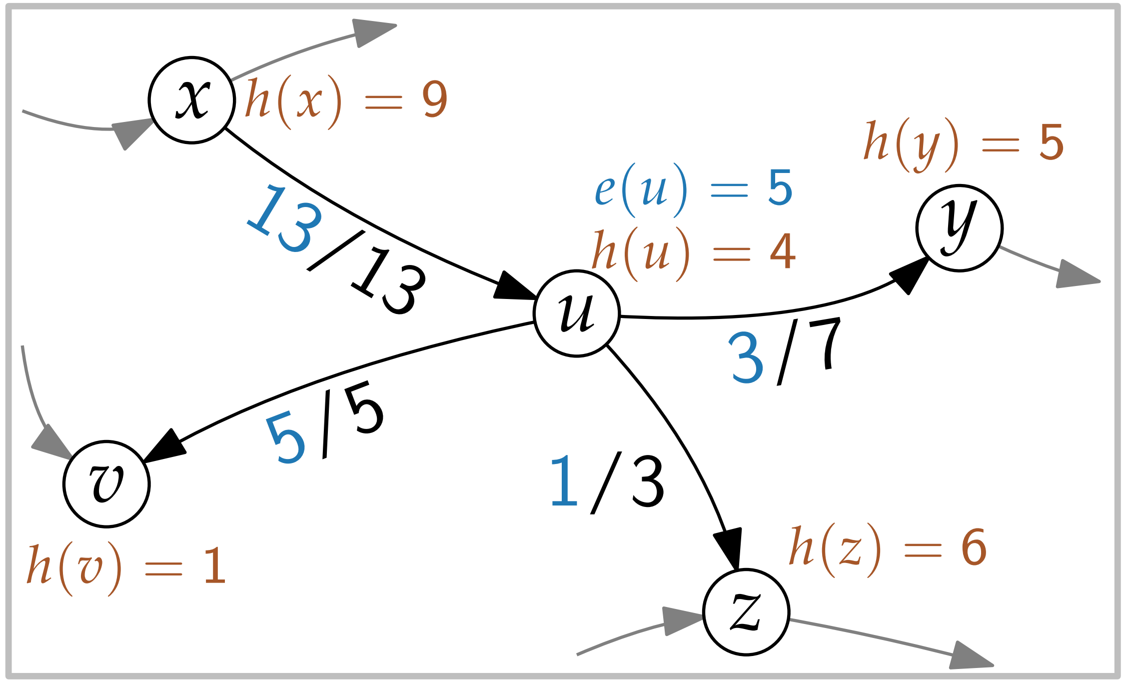

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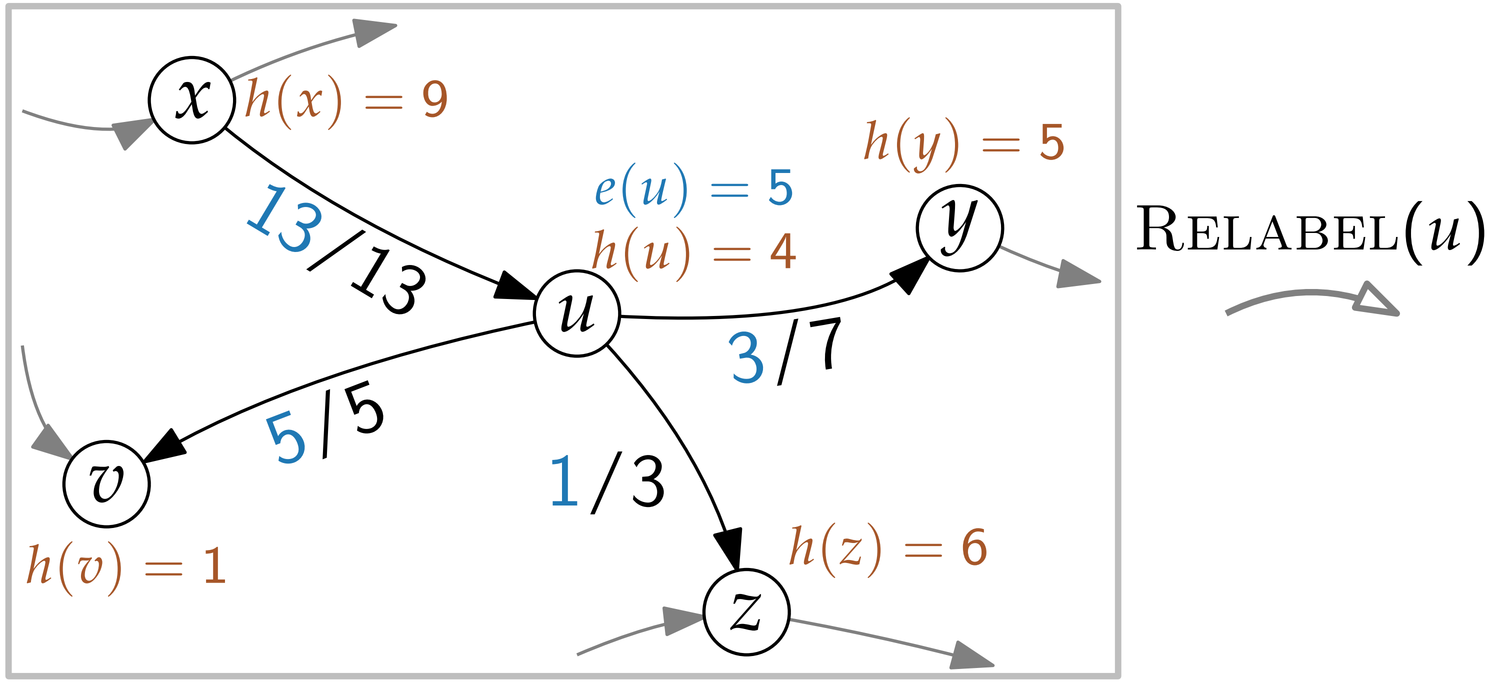

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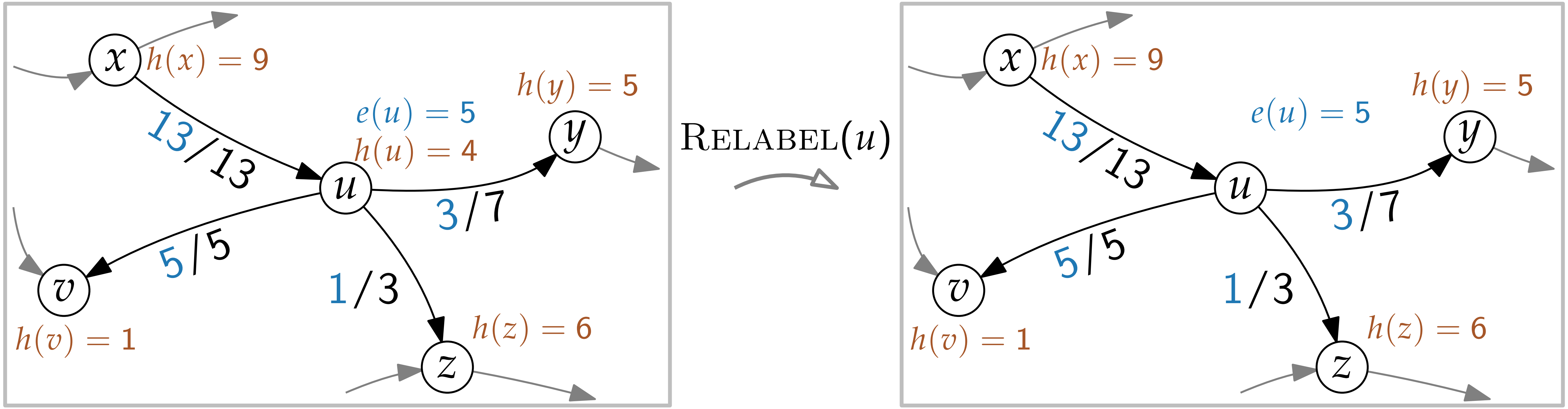

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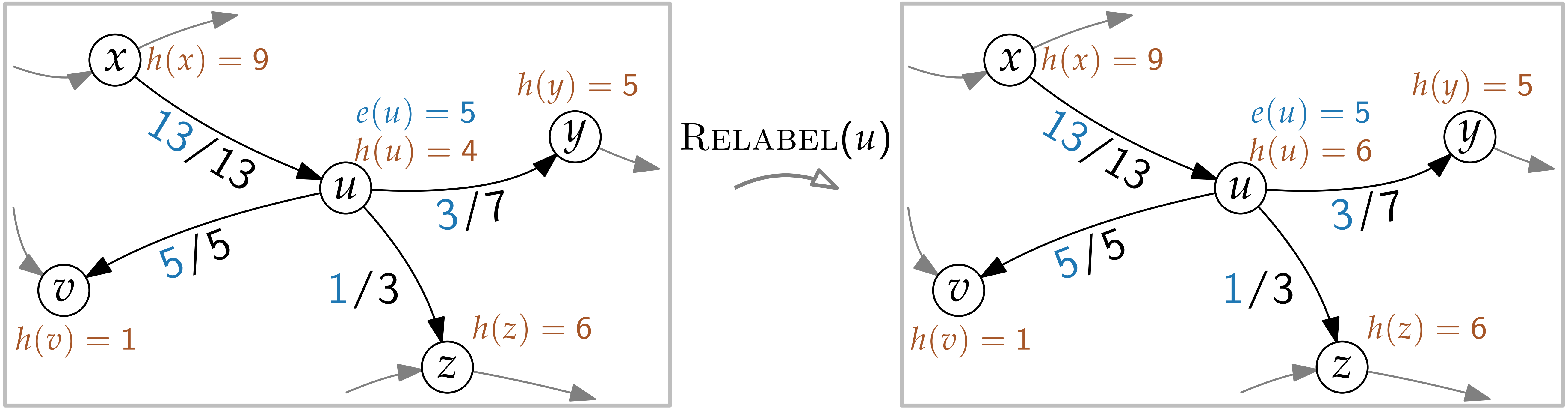

The Push-Relabel Algorithm

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InitPreflow(G, s)

while $\exists$ applicable Push or Relabel operation x **do**

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foreach $v \in V$ **do** $h(v) \leftarrow 0$; $e(v) \leftarrow 0$
$h(s) \leftarrow |V|$
foreach $(u, v) \in E$ **do** $f(u, v) \leftarrow 0$
foreach v such that $(s, v) \in E$ **do**
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- initializes heights
- pushes max flow over every edge that leaves s

Correctness

Part 1.

If the algorithm terminates, the preflow is a maximum flow.

- If an overflowing vertex exists, the algorithm can continue.
- The algorithm maintains f as a preflow and h as a height function.
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Part 2.

The algorithm terminates and the heights stay finite.

- Find upper bound on heights.
- Find upper bound for the number of calls to $\textsc{Relabel}$.
- Find upper bound for the number of calls to $\textsc{Push}$.

Continuation

Lemma 1.
If a vertex u is overflowing, either a push or a relabel operation applies to u.

Height function:

- $h(s) = |V|$
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Proof.
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If no push operation is valid for $(u, v) \in E_f$, then
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Therefore, $\textsc{Relabel}(u)$ is applicable.

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Condition: u is overflowing and $h(u) \leq h(v)$ $\forall v \in V$ with $(u, v) \in E_f$
$h(u) \leftarrow 1 + \min\{h(v) : (u, v) \in E_f\}$

Maintaining the Preflow

Lemma 2.
The push-relabel algorithm maintains a preflow f.

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- $h(s) = |V|$
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$\textsc{Push}(u, v)$

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The push-relabel algorithm maintains a preflow f.

Proof.

- InitPreflow initialises a preflow f. ✓
- Relabel(u) doesn't affect f. ✓
- Push(u, v) maintains f as a preflow. ✓

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During the execution of the push–relabel algorithm, there is no path from s to t in G_f.

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But since $k < |V|$, it follows that $h(s) < |V|$. ✗

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Theorem 5.
When the push–relabel algorithm terminates, the computed preflow f is a maximum flow.

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  $\Rightarrow$ By the Max-Flow Min-Cut Theorem, the flow f is a maximum flow.

Correctness

Part 1. ✓

If the algorithm terminates, the preflow is maximum flow.

- If an overflowing vertex exists, the algorithm can continue.
- The algorithm maintains f as a preflow and h as a height function.
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Part 2.

The algorithm terminates and the heights stay finite.

- Find upper bound on heights.
- Find upper bound for the number of calls to $\textsc{Relabel}$.
- Find upper bound for the number of calls to $\textsc{Push}$.

Reachability of the Source in the Residual Graph

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For every overflowing vertex v,
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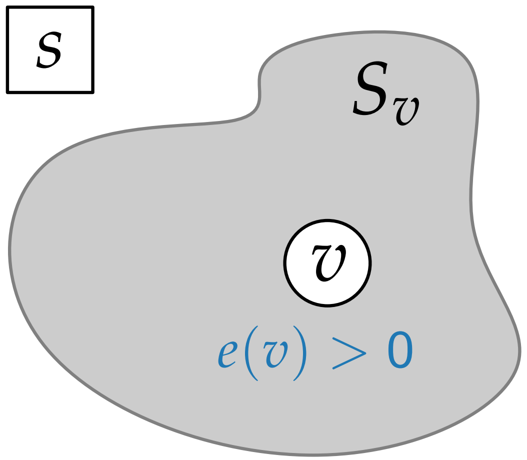

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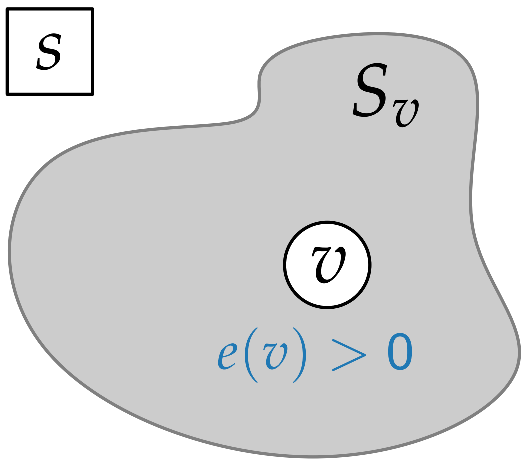

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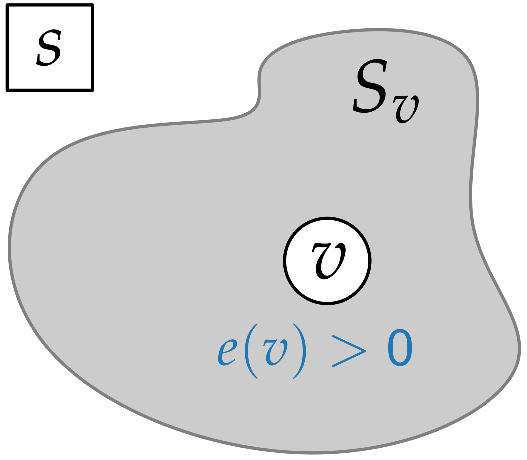

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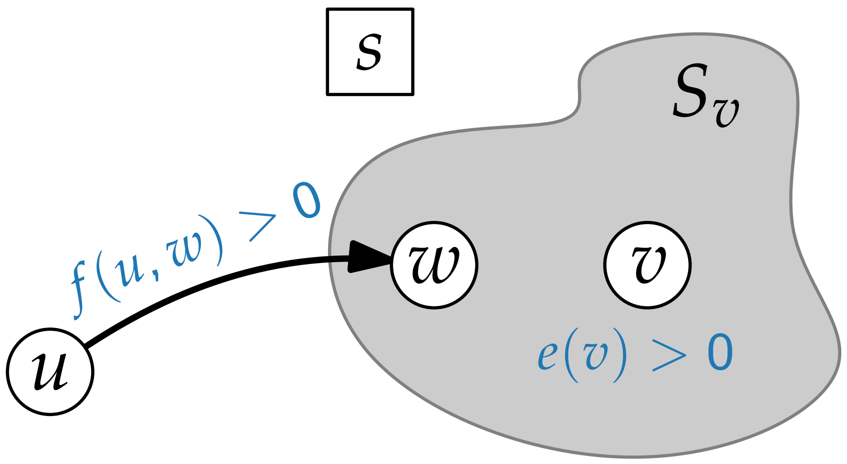

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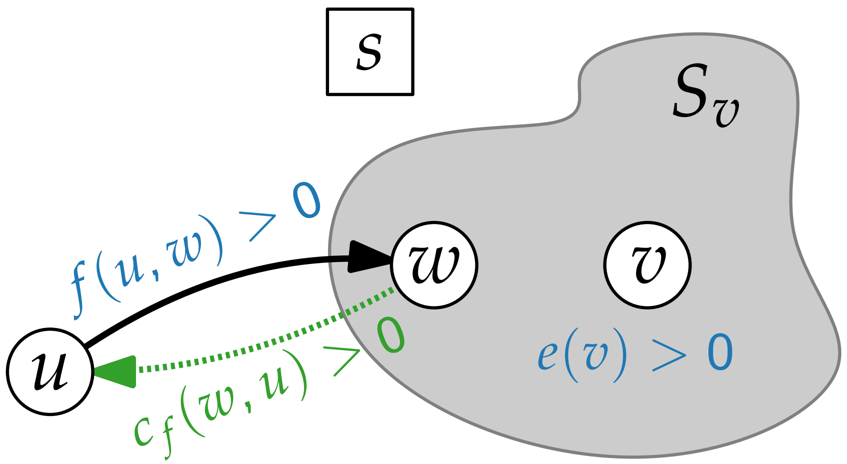

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- There is an edge (u, w) with $u \notin S_v, w \in S_v$ and $f(u, w) > 0$.
- But then $c_f(w, u) > 0$, meaning u is reachable from v. ✗

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Lemma 7.
During the push–relabel algorithm,
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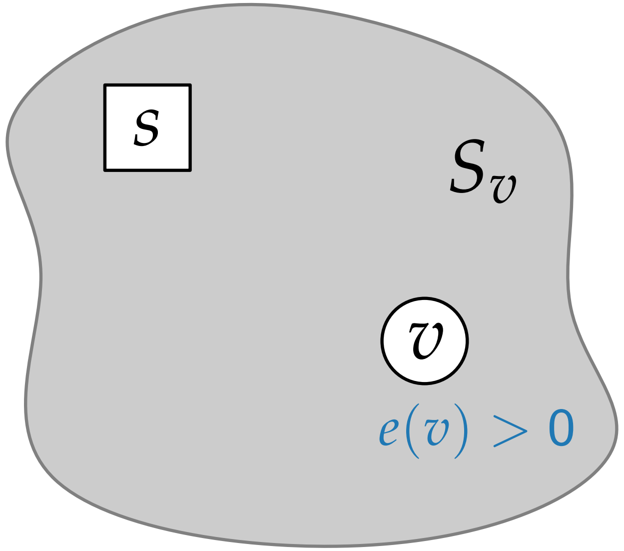

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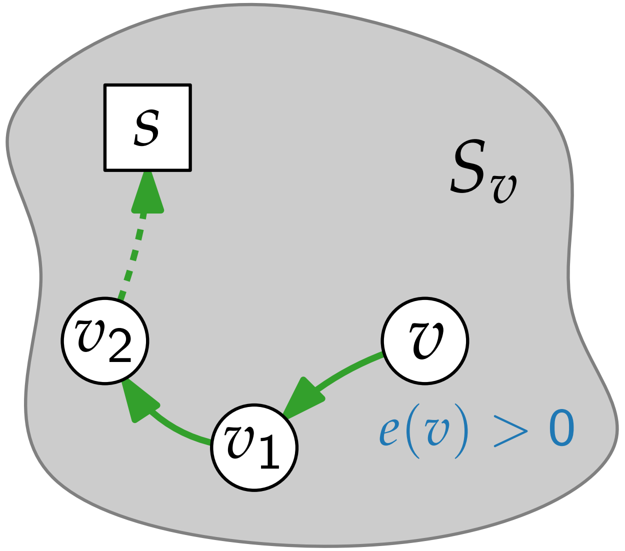

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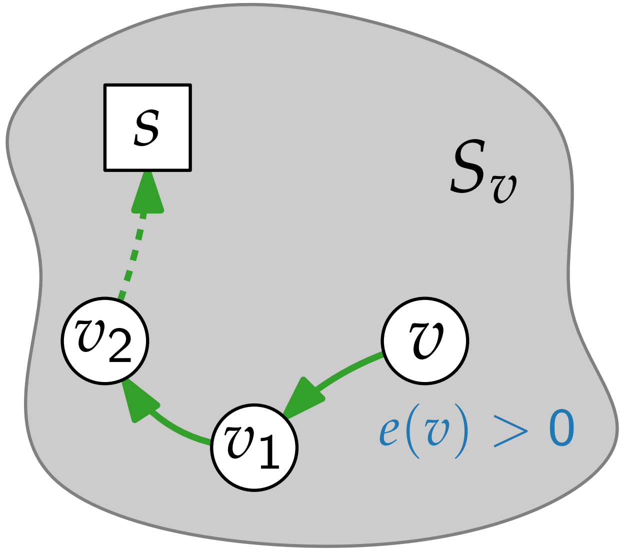

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- Since $k \leq |V| - 1$, we have $h(v) \leq h(s) + k \leq 2|V| - 1$.

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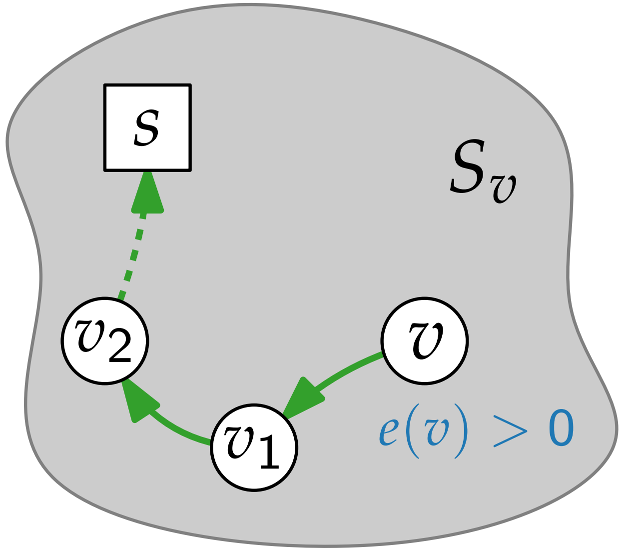
- $h(t) = 0$
- $h(u) \leq h(v) + 1 \quad \forall (u, v) \in E_f$

RELABEL(u)

Condition: u is overflowing and $h(u) \leq h(v)$ $\forall v \in V$ with $(u, v) \in E_f$
$h(u) \leftarrow 1 + \min\{h(v) : (u, v) \in E_f\}$



Upper Bounds on the Height and # Relabel Operations

Lemma 7.
During the push–relabel algorithm, we have $h(v) \leq 2|V| - 1$ for all $v \in V$.

Proof.

- Statement holds after initialisation.
- Let v be an overflowing vertex that is relabeled.
- By Lemma 6, there is a path $v = v_0, v_1, \ldots, v_k = s$ in G_f.
- Then $h(v_i) \leq h(v_{i+1}) + 1$ for $0 \leq i \leq k - 1$.
- Since $k \leq |V| - 1$, we have $h(v) \leq h(s) + k \leq 2|V| - 1$.

Corollary 8.
The push–relabel algorithm executes at most $2|V|^2$ Relabel operations.

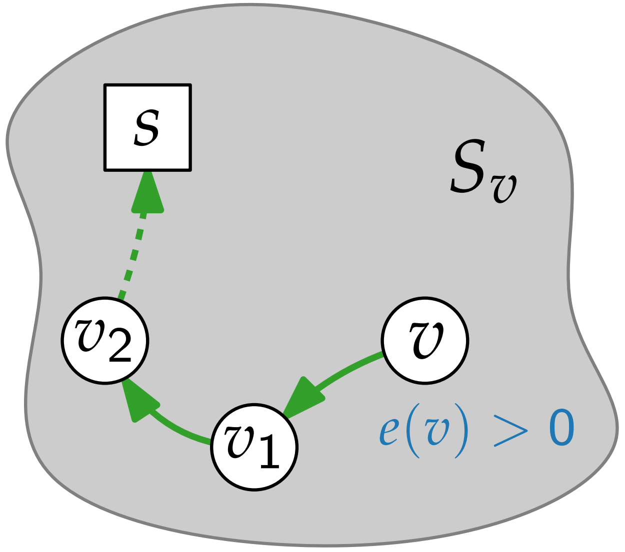
Height function:

- $h(s) = |V|$
- $h(t) = 0$
- $h(u) \leq h(v) + 1 \quad \forall (u, v) \in E_f$

Relabel(u)

Condition: u is overflowing and $h(u) \leq h(v)$ $\forall v \in V$ with $(u, v) \in E_f$

$h(u) \leftarrow 1 + \min\{h(v) \colon (u, v) \in E_f\}$



Saturating and Unsaturating $\textsc{Push}$ Operations

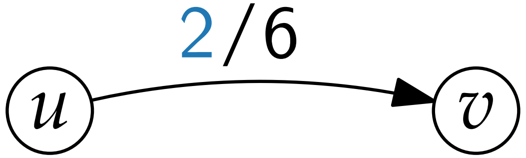
The operation $\textsc{Push}(u, v)$ is

- **saturating** if afterwards $c_f(u, v) = 0$,

Saturating and Unsaturating PUSH Operations

The operation PUSH(u, v) is

- **saturating** if afterwards $c_f(u, v) = 0$,



Saturating and Unsaturating Push Operations

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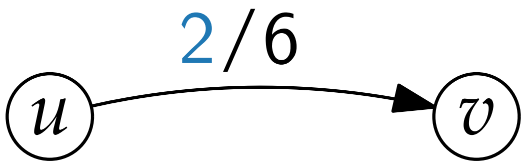

$\textsc{Push}(u, v)$

$\Delta = 4$

Saturating and Unsaturating PUSH Operations

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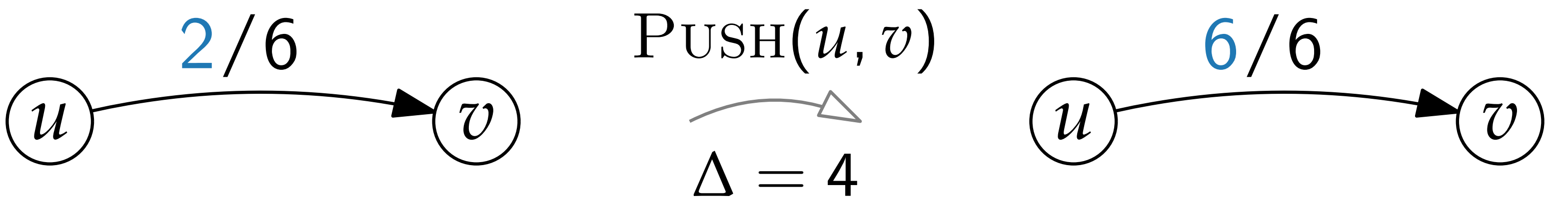

Saturating and Unsaturating PUSH Operations

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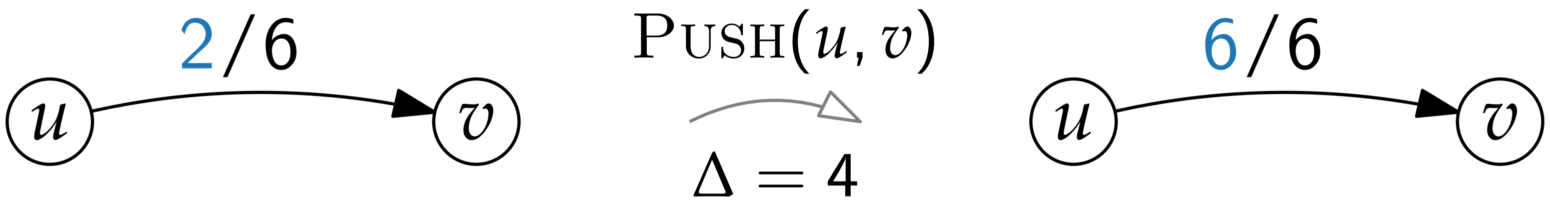

- and **unsaturating** otherwise.

Saturating and Unsaturating Push Operations

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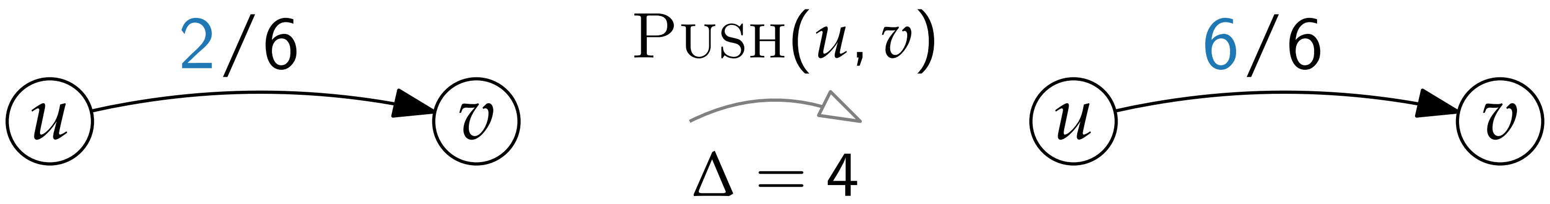
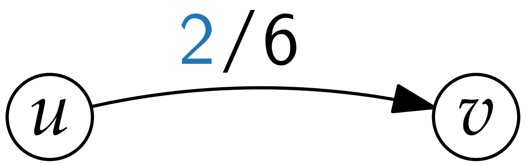


- and **unsaturating** otherwise.



Saturating and Unsaturating PUSH Operations

The operation PUSH(u, v) is

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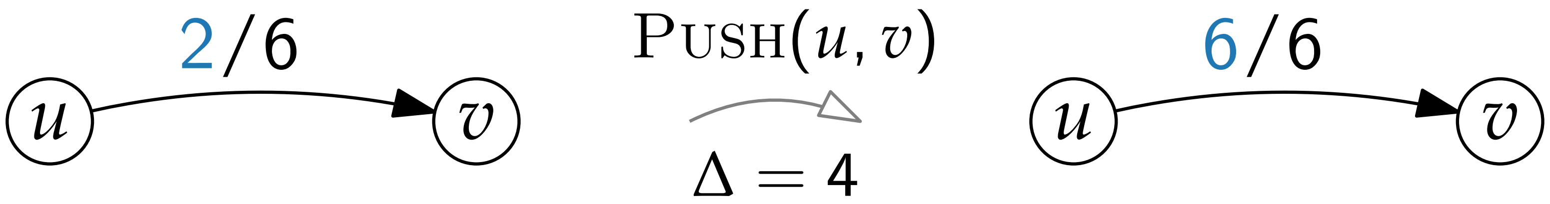
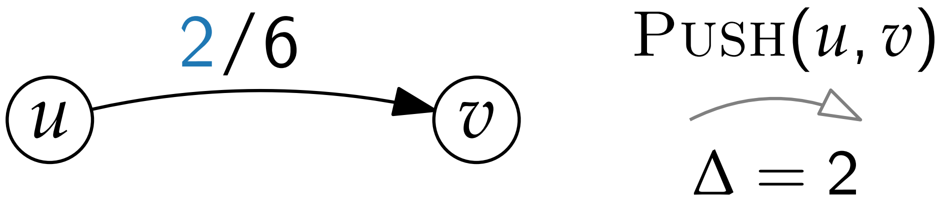


- and **unsaturating** otherwise.



Saturating and Unsaturating PUSH Operations

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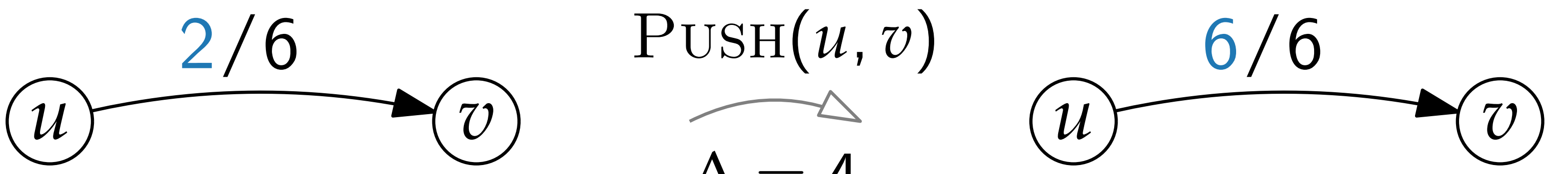
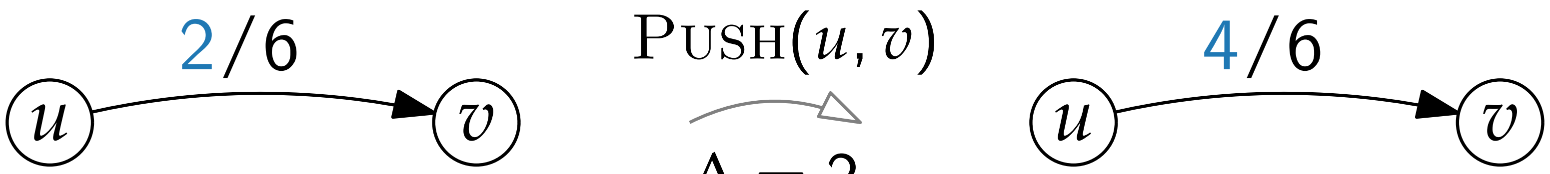


- and **unsaturating** otherwise.



Upper Bound on the Number of Saturating Push Operations

Lemma 9.
The push–relabel algorithm executes at most $2|V| \cdot |E|$ saturating Push operations.

Push(u, v)

Condition: u is overflowing, $c_f(u, v) > 0$, and $h(u) = h(v) + 1$
$\Delta \leftarrow \min(e(u), c_f(u, v))$
if $(u, v) \in E$ **then**
  $f(u, v) \leftarrow f(u, v) + \Delta$
else
  $f(v, u) \leftarrow f(v, u) + \Delta$
$e(u) \leftarrow e(u) - \Delta$
$e(v) \leftarrow e(v) + \Delta$

Upper Bound on the Number of Saturating Push Operations

Lemma 9.
The push–relabel algorithm executes at most $2|V| \cdot |E|$ saturating Push operations.

Proof.

- Consider saturating Push(u, v)
  - $h(u) = h(v) + 1$

Push(u, v)

Condition: u is overflowing, $c_f(u, v) > 0$, and $h(u) = h(v) + 1$
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    $f(v, u) \leftarrow f(v, u) + \Delta$
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$e(v) \leftarrow e(v) + \Delta$

Push(u, v)
...

Upper Bound on the Number of Saturating PUSH Operations

Lemma 9.
The push–relabel algorithm executes at most $2|V| \cdot |E|$ saturating PUSH operations.

Proof.

- Consider saturating PUSH(u, v)
  - $h(u) = h(v) + 1$
- For another saturating PUSH(u, v), first PUSH(v, u) necessary
  - $h(v) = h(u) + 1$ necessary

PUSH(u, v)

Condition: u is overflowing, $c_f(u, v) > 0$, and $h(u) = h(v) + 1$

```
Δ ← min(e(u), c_f(u, v))
if (u, v) ∈ E then
    f(u, v) ← f(u, v) + Δ
else
    f(v, u) ← f(v, u) + Δ
e(u) ← e(u) − Δ
e(v) ← e(v) + Δ
```

PUSH(u, v)
...
PUSH(v, u)

Upper Bound on the Number of Saturating $\textsc{Push}$ Operations

Lemma 9.
The push–relabel algorithm executes at most $2|V| \cdot |E|$ saturating $\textsc{Push}$ operations.

$\textsc{Push}(u, v)$
Condition: u is overflowing, $c_f(u, v) > 0$, and $h(u) = h(v) + 1$
$\Delta \leftarrow \min(e(u), c_f(u, v))$
if $(u, v) \in E$ **then**
  $f(u, v) \leftarrow f(u, v) + \Delta$
else
  $f(v, u) \leftarrow f(v, u) + \Delta$
$e(u) \leftarrow e(u) - \Delta$
$e(v) \leftarrow e(v) + \Delta$

Proof.

- Consider saturating $\textsc{Push}(u, v)$
  - $h(u) = h(v) + 1$
- For another saturating $\textsc{Push}(u, v)$, first $\textsc{Push}(v, u)$ necessary
  - $h(v) = h(u) + 1$ necessary
- After another saturating $\textsc{Push}(u, v)$, both $h(u)$ and $h(v)$ have increased by at least two.

$\textsc{Push}(u, v)$
...
$\textsc{Push}(v, u)$
...
$\textsc{Push}(u, v)$
...

Upper Bound on the Number of Saturating Push Operations

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- For another saturating $\textsc{Push}(u, v)$, first $\textsc{Push}(v, u)$ necessary
  - $h(v) = h(u) + 1$ necessary
- After another saturating $\textsc{Push}(u, v)$, both $h(u)$ and $h(v)$ have increased by at least two.
- But by Lemma 6, $h(u) \leq 2|V| - 1$ and $h(v) \leq 2|V| - 1$.

$\textsc{Push}(u, v)$

Condition: u is overflowing, $c_f(u, v) > 0$, and $h(u) = h(v) + 1$
$\Delta \leftarrow \min(e(u), c_f(u, v))$
if $(u, v) \in E$ **then**
  $f(u, v) \leftarrow f(u, v) + \Delta$
else
  $f(v, u) \leftarrow f(v, u) + \Delta$
$e(u) \leftarrow e(u) - \Delta$
$e(v) \leftarrow e(v) + \Delta$

$\textsc{Push}(u, v)$
...
$\textsc{Push}(v, u)$
...
$\textsc{Push}(u, v)$
...

Upper Bound on the Number of Saturating PUSH Operations

Lemma 9.
The push–relabel algorithm executes at most $2|V| \cdot |E|$ saturating PUSH operations.

PUSH(u, v)
Condition: u is overflowing, $c_f(u, v) > 0$, and $h(u) = h(v) + 1$
$\Delta \leftarrow \min(e(u), c_f(u, v))$
if $(u, v) \in E$ **then**
  $f(u, v) \leftarrow f(u, v) + \Delta$
else
  $f(v, u) \leftarrow f(v, u) + \Delta$
$e(u) \leftarrow e(u) - \Delta$
$e(v) \leftarrow e(v) + \Delta$

Proof.

- Consider saturating PUSH(u, v)
  - $h(u) = h(v) + 1$
- For another saturating PUSH(u, v), first PUSH(v, u) necessary
  - $h(v) = h(u) + 1$ necessary
- After another saturating PUSH(u, v), both $h(u)$ and $h(v)$ have increased by at least two.
- But by Lemma 6, $h(u) \leq 2|V| - 1$ and $h(v) \leq 2|V| - 1$.
- There are at most $2|V| - 1$ saturated PUSH operations for edge (u, v).

PUSH(u, v)
...
PUSH(v, u)
...
PUSH(u, v)
...

Upper Bound on the Number of Unsaturating Push Ops

Lemma 10.
The push–relabel algorithm executes at most $4|V|^2 \cdot |E|$ unsaturating Push ops.

Push(u, v)

Condition: u is overflowing, $c_f(u, v) > 0$, and $h(u) = h(v) + 1$
$\Delta \leftarrow \min(e(u), c_f(u, v))$
if $(u, v) \in E$ **then**
  $f(u, v) \leftarrow f(u, v) + \Delta$
else
  $f(v, u) \leftarrow f(v, u) + \Delta$
$e(u) \leftarrow e(u) - \Delta$
$e(v) \leftarrow e(v) + \Delta$

Upper Bound on the Number of Unsaturating Push Ops

Lemma 10.
The push–relabel algorithm executes at most $4|V|^2 \cdot |E|$ unsaturating Push ops.

Proof.

- Consider $\mathcal{H} = \sum_{v \in V \setminus \{s,t\},\ v \text{ overflowing}} h(v)$.
- After initialization and at the end $\mathcal{H} = 0$.

Push(u, v)

Condition: u is overflowing, $c_f(u,v) > 0$, and $h(u) = h(v) + 1$
$\Delta \leftarrow \min(e(u), c_f(u,v))$
if $(u,v) \in E$ **then**
  $f(u,v) \leftarrow f(u,v) + \Delta$
else
  $f(v,u) \leftarrow f(v,u) + \Delta$
$e(u) \leftarrow e(u) - \Delta$
$e(v) \leftarrow e(v) + \Delta$

Upper Bound on the Number of Unsaturating Push Ops

Lemma 10.
The push–relabel algorithm executes at most $4|V|^2 \cdot |E|$ unsaturating Push ops.

Proof.

- Consider $\mathcal{H} = \sum_{v \in V \setminus \{s,t\},\ v \text{ overflowing}} h(v)$.
- After initialization and at the end $\mathcal{H} = 0$.
- A saturating Push increases $\mathcal{H}$ by at most $2|V| - 1$.
- By Lemma 9, all saturating Push operations increase $\mathcal{H}$ by at most $(2|V| - 1) \cdot 2|V| \cdot |E|$.

Push(u, v)

Condition: u is overflowing, $c_f(u,v) > 0$, and $h(u) = h(v) + 1$
$\Delta \leftarrow \min(e(u), c_f(u,v))$
if $(u,v) \in E$ **then**
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Upper Bound on the Number of Unsaturating PUSH Ops

Lemma 10.
The push–relabel algorithm executes at most $4|V|^2 \cdot |E|$ unsaturating PUSH ops.

PUSH(u, v)
- **Condition:** u is overflowing, $c_f(u,v) > 0$, and $h(u) = h(v) + 1$
- $\Delta \leftarrow \min(e(u), c_f(u,v))$
- **if** $(u,v) \in E$ **then**
  - $f(u,v) \leftarrow f(u,v) + \Delta$
- **else**
  - $f(v,u) \leftarrow f(v,u) + \Delta$
- $e(u) \leftarrow e(u) - \Delta$
- $e(v) \leftarrow e(v) + \Delta$

Proof.

- Consider $\mathcal{H} = \sum_{v \in V \setminus \{s,t\},\ v \text{ overflowing}} h(v)$.
- After initialization and at the end $\mathcal{H} = 0$.
- A saturating PUSH increases $\mathcal{H}$ by at most $2|V| - 1$.
- By Lemma 9, all saturating PUSH operations increase $\mathcal{H}$ by at most $(2|V| - 1) \cdot 2|V| \cdot |E|$.
- By Lemma 7, all RELABEL operations increase $\mathcal{H}$ by at most $(2|V| - 1) \cdot |V|$.

Upper Bound on the Number of Unsaturating Push Ops

Lemma 10.
The push–relabel algorithm executes at most $4|V|^2 \cdot |E|$ unsaturating Push ops.

Push(u, v)

Condition: u is overflowing, $c_f(u,v) > 0$, and $h(u) = h(v) + 1$
$\Delta \leftarrow \min(e(u), c_f(u,v))$
if $(u,v) \in E$ **then**
  $f(u,v) \leftarrow f(u,v) + \Delta$
else
  $f(v,u) \leftarrow f(v,u) + \Delta$
$e(u) \leftarrow e(u) - \Delta$
$e(v) \leftarrow e(v) + \Delta$

Proof.

- Consider $\mathcal{H} = \sum_{v \in V \setminus \{s,t\},\ v \text{ overflowing}} h(v)$.
- After initialization and at the end $\mathcal{H} = 0$.
- A saturating Push increases $\mathcal{H}$ by at most $2|V| - 1$.
- By Lemma 9, all saturating Push operations increase $\mathcal{H}$ by at most $(2|V| - 1) \cdot 2|V| \cdot |E|$.
- By Lemma 7, all Relabel operations increase $\mathcal{H}$ by at most $(2|V| - 1) \cdot |V|$.
- An unsaturating Push(u, v) decreases $\mathcal{H}$ by at least 1 since $h(u) - h(v) \geq 1$.

Termination of the Algorithm

Theorem 5.
When the push–relabel algorithm terminates, the computed preflow f is a maximum flow.

Theorem 11.
The push–relabel algorithm terminates after $\mathcal{O}(|V|^2|E|)$ valid $\textsc{Push}$ or $\textsc{Relabel}$ ops.

Proof.

- Follows by Corollary 8 and Lemmas 9+10.

Implementation

The actual running time depends on the selection order of the overflowing vertices:

- **FIFO implementation:**
  Pick overflowing vertex by *first-in-first-out* principle: $\mathcal{O}(|V|^3)$ running time.
  with dynamic trees: $\mathcal{O}(|V||E| \log \frac{|V|^2}{|E|})$
- **Highest label:**
  For $\textsc{Push}$ select highest overflowing vertex: $\mathcal{O}(|V|^2|E|^{\frac{1}{2}})$
- **Excess scaling:**
  For $\textsc{Push}(u, v)$ choose edge (u, v) such that u is overflowing, $e(u)$ is *sufficiently high* and $e(v)$ *sufficiently small*: $\mathcal{O}(|E| + |V|^2 \log C)$, where $C = \max_{(u,v)\in E} c(u, v)$

Discussion

- The push–relabel method offers an alternative framework to the Ford–Fulkerson method to develop algorithms that solve the maximum flow problem.
- Push–relabel algorithms are regarded as benchmarks for maximum flow algorithms.
- In practice, heuristics are used to improve the performance of push–relabel algorithms. Any ideas?
- The algorithm can be extended to solve the minimum-cost flow problem.

Literature

Main source:

- [CLRS Ch26] $\leftarrow$ Cormen et al. “Introduction to Algorithms”

Original paper:

- [Goldberg, Tarjan ’88] A new approach to the maximum-flow problem

Links:

- Animations of the max-flow algorithms by Ford–Fulkerson and Edmonds–Karp: https://visualgo.net/en/maxflow