Approximation Algorithms

Lecture 12: SteinerForest via Primal–Dual

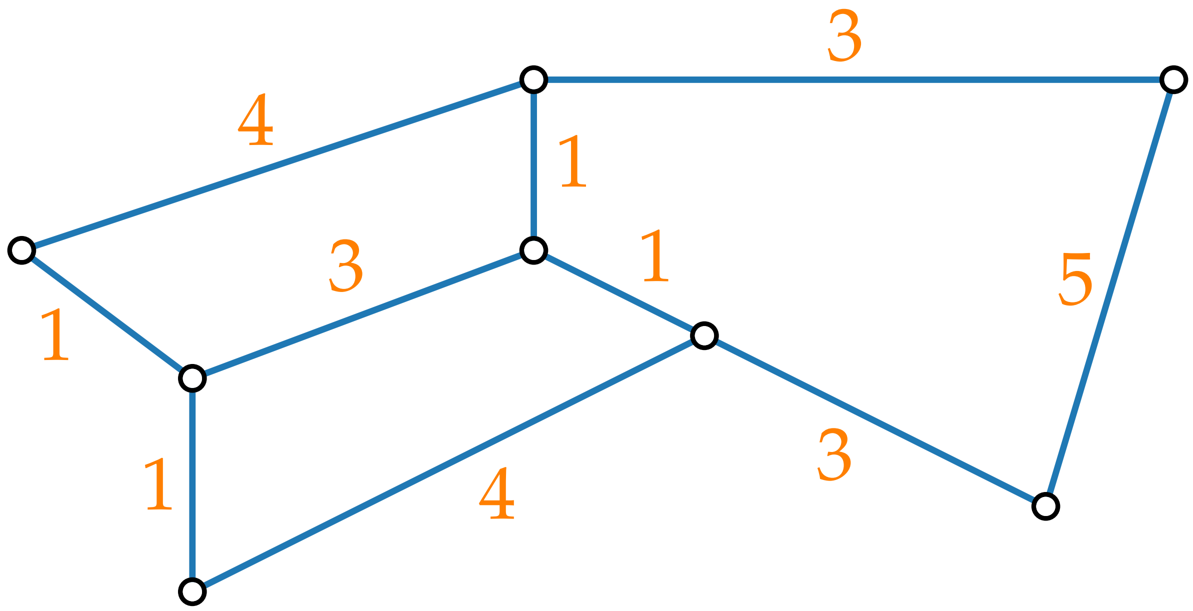
Part I: SteinerForest

Alexander Wolff

Winter 2021/22

SteinerForest

Given: A graph $G = (V, E)$ with edge costs $c \colon E \to \mathbb{N}$ and



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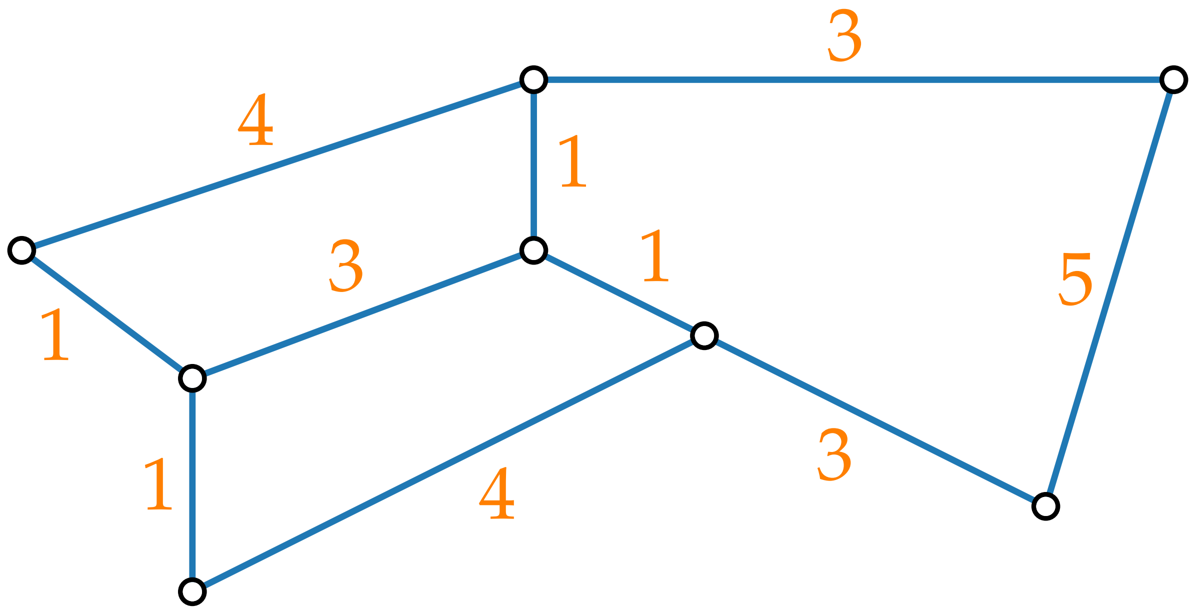
Given: A graph $G = (V, E)$ with edge costs $c\colon E \to \mathbb{N}$ and a set $R = \{(s_1, t_1), \ldots, (s_k, t_k)\}$ of k vertex pairs.



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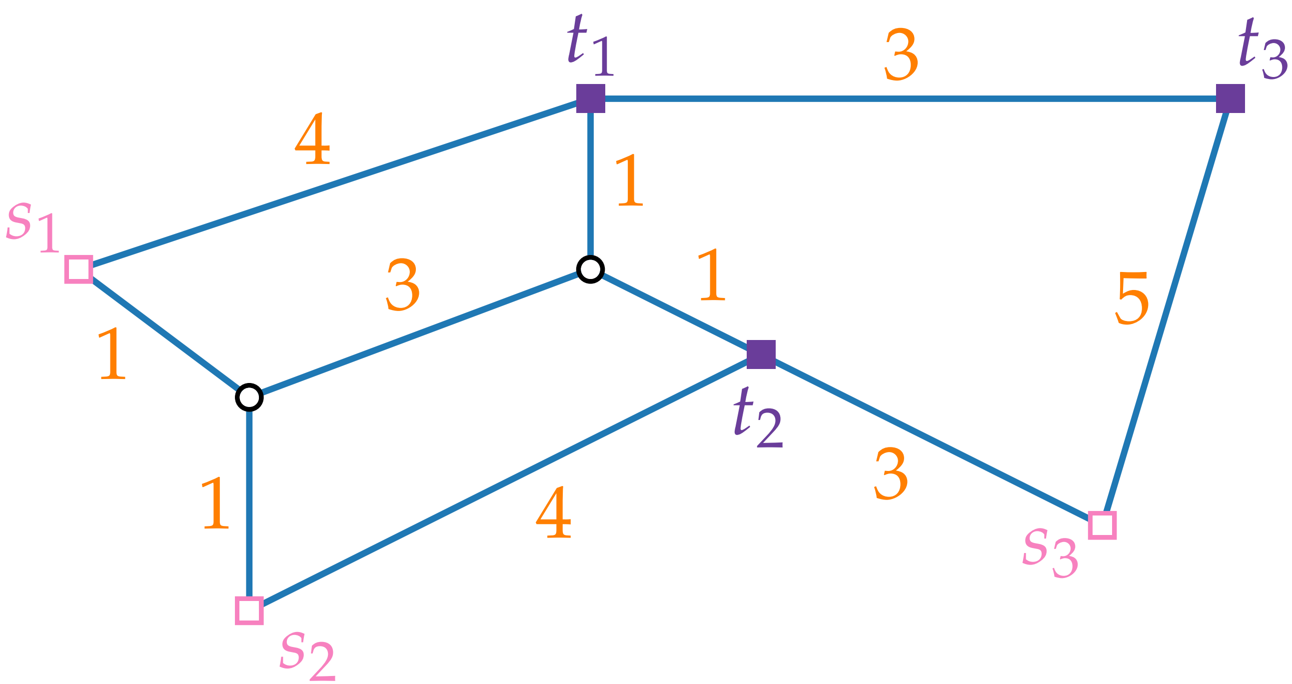

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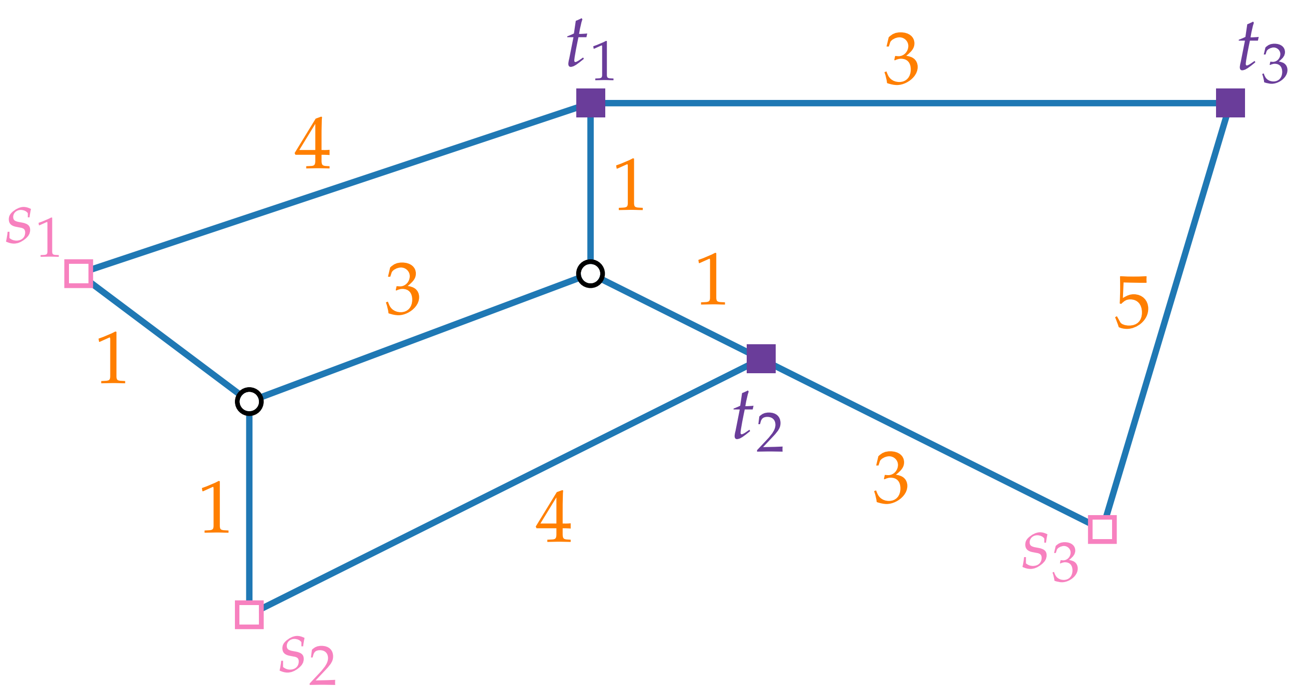
Task: Find an edge set $F \subseteq E$ of minimum total cost $c(F)$



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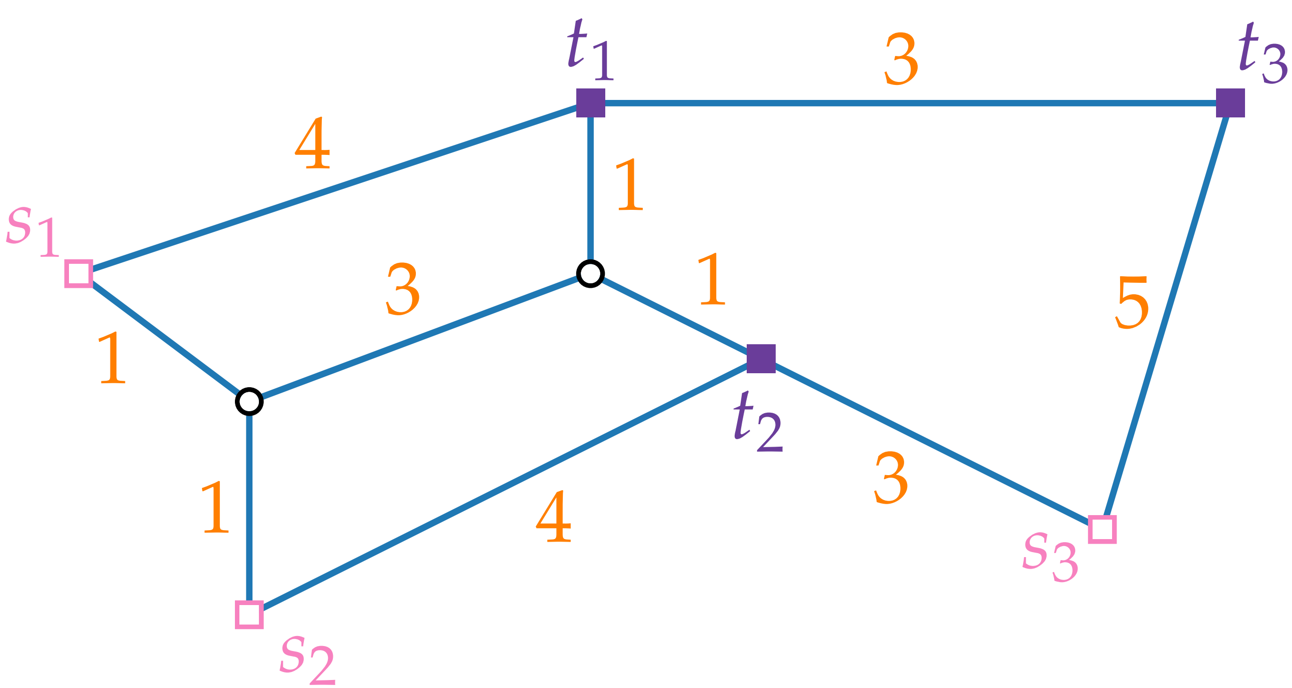
Task: Find an edge set $F \subseteq E$ of minimum total cost $c(F)$ such that the subgraph (V, F) connects every pair (s_i, t_i), $i = 1, \ldots, k$.



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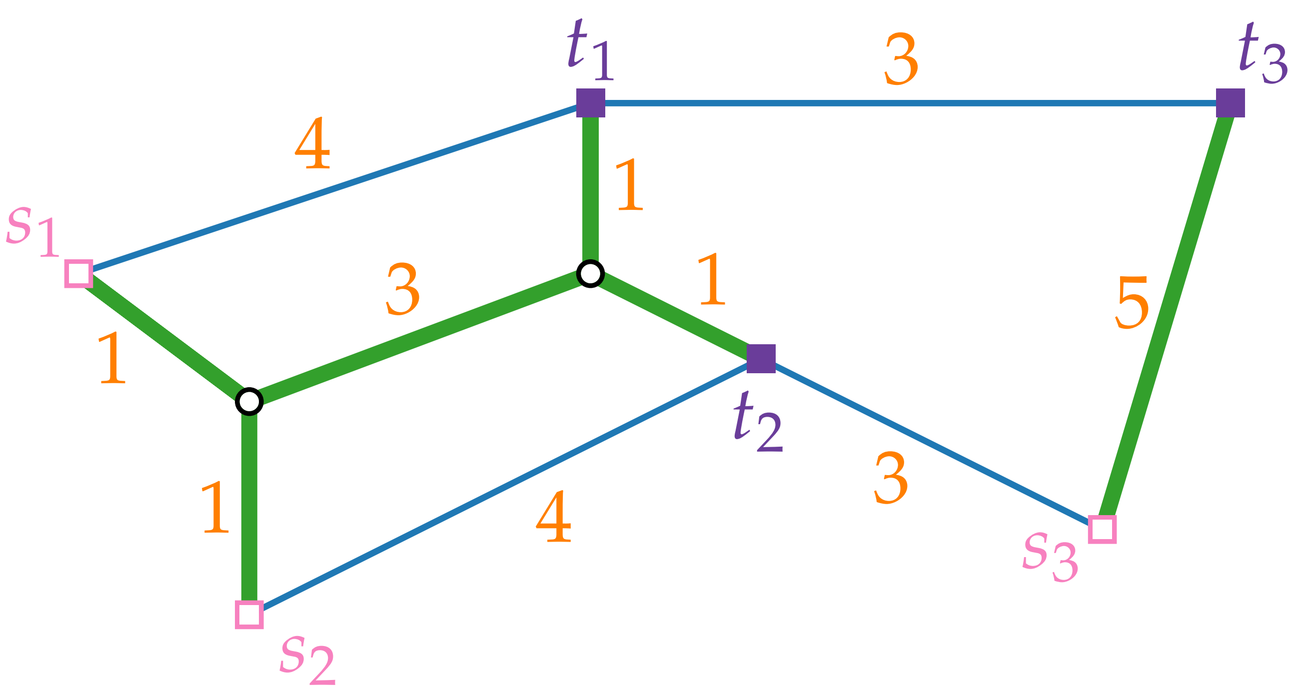

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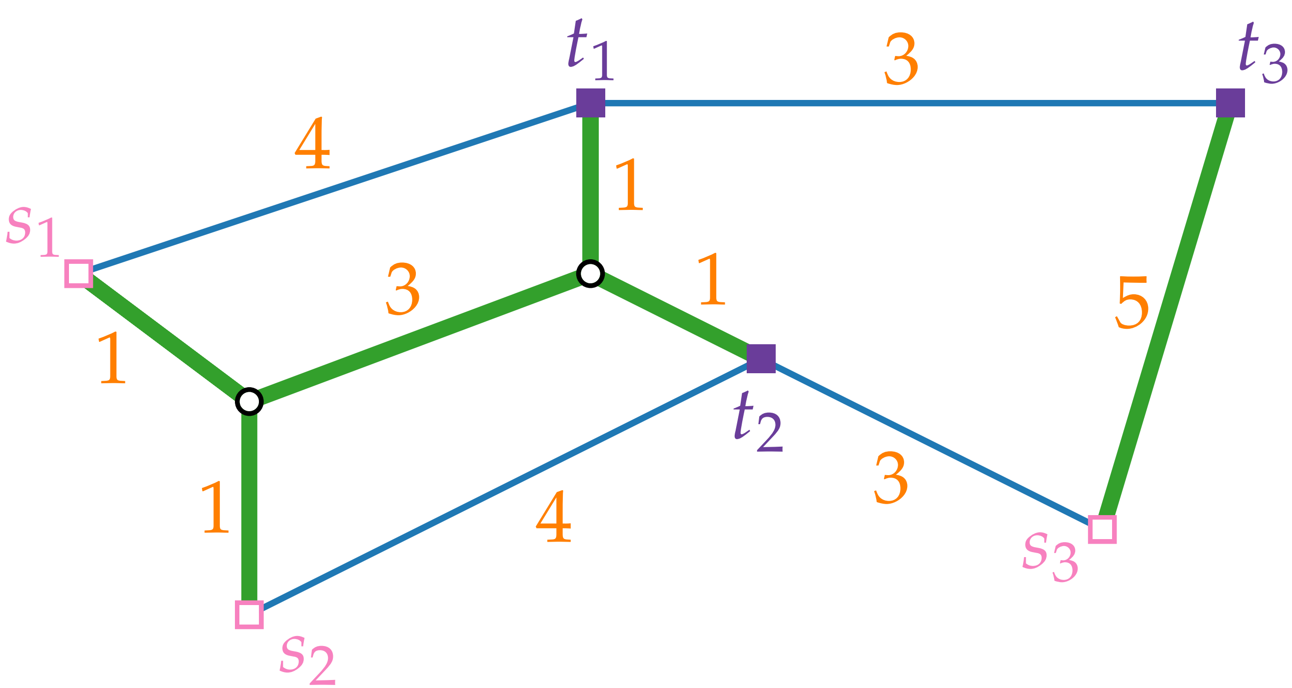

Special cases?

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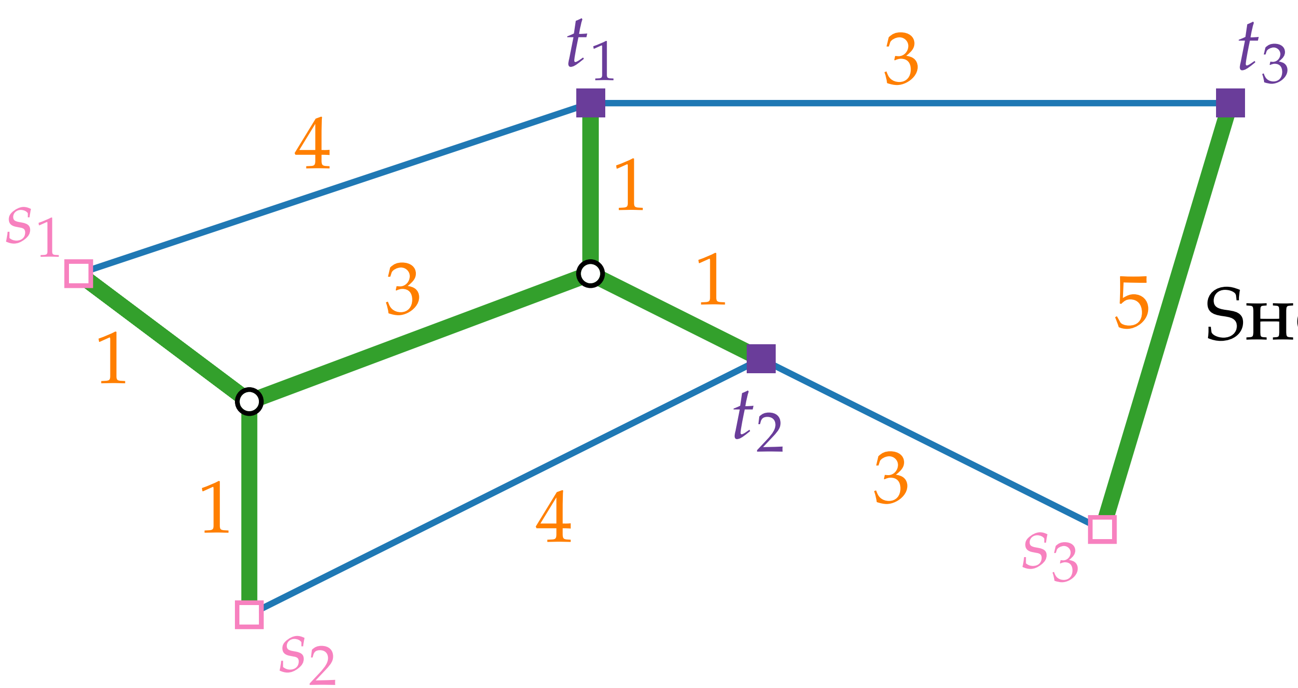

Special cases?

ShortestPath ($R = \{s, t\}$)

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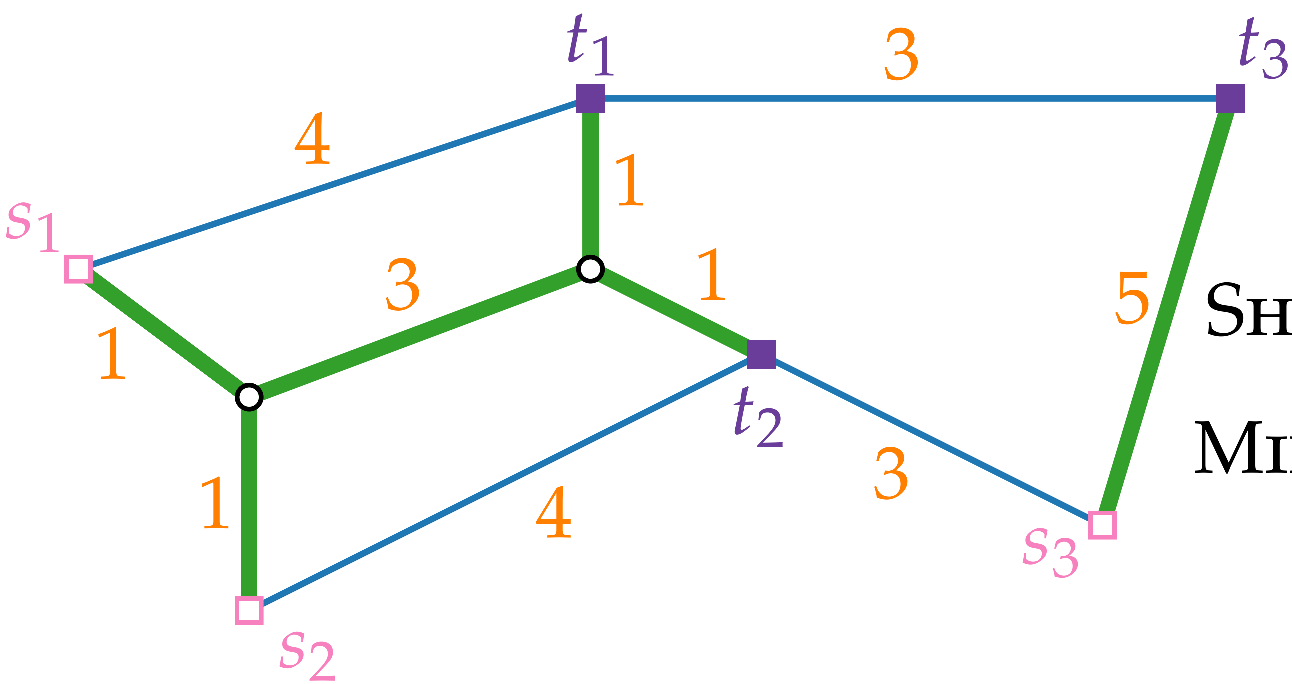

Special cases?

ShortestPath ($R = \{s, t\}$)

MinSpanningTree ($R = E$)

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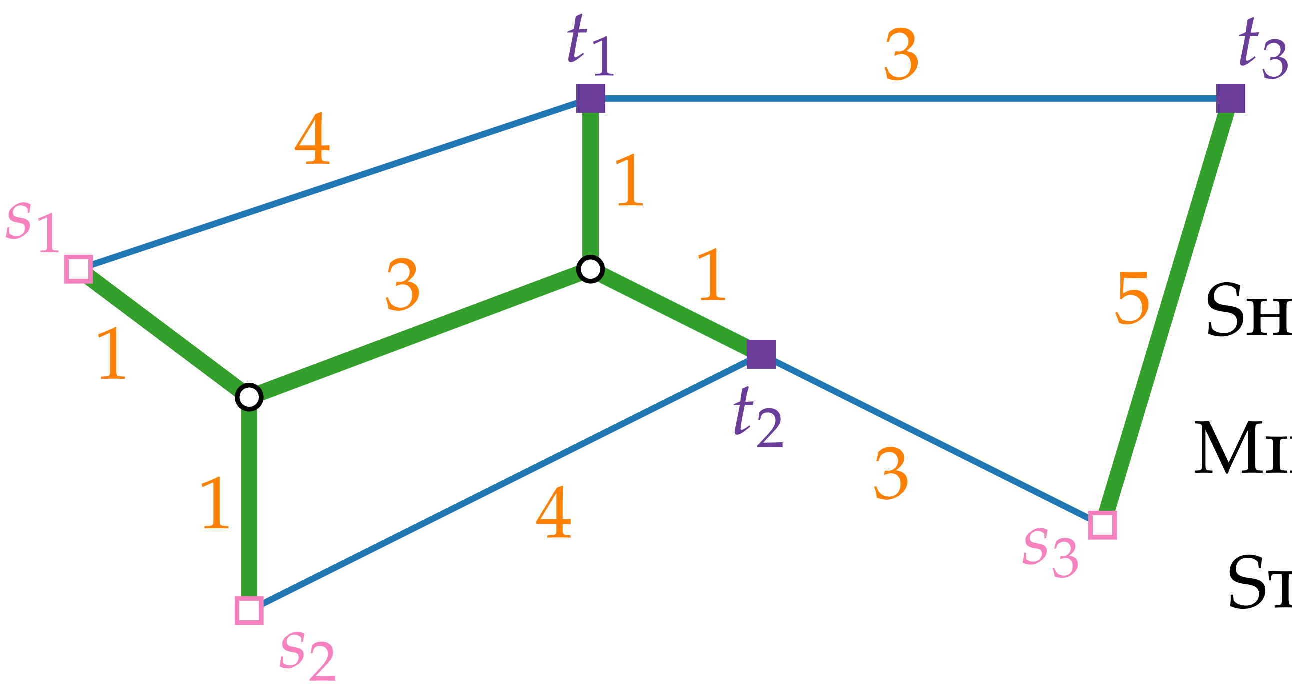


Special cases?

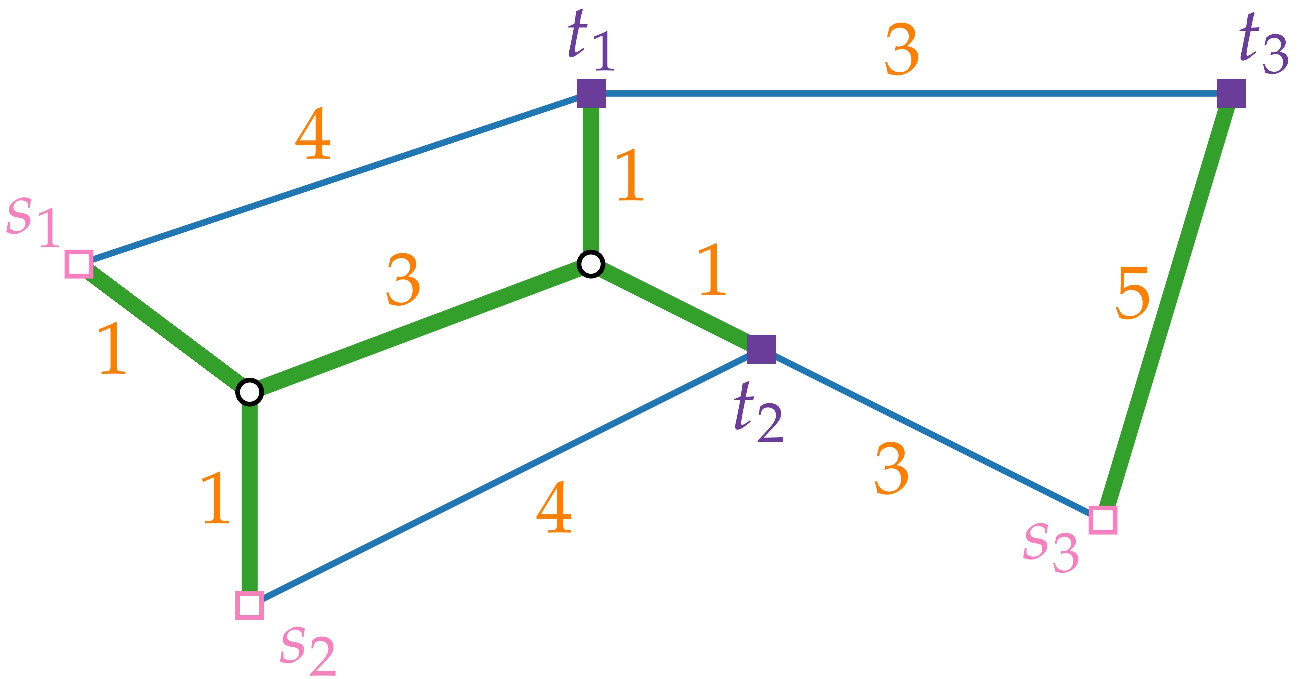
ShortestPath ($R = \{s, t\}$)

MinSpanningTree ($R = E$)

SteinerTree ($R = T \times T$)

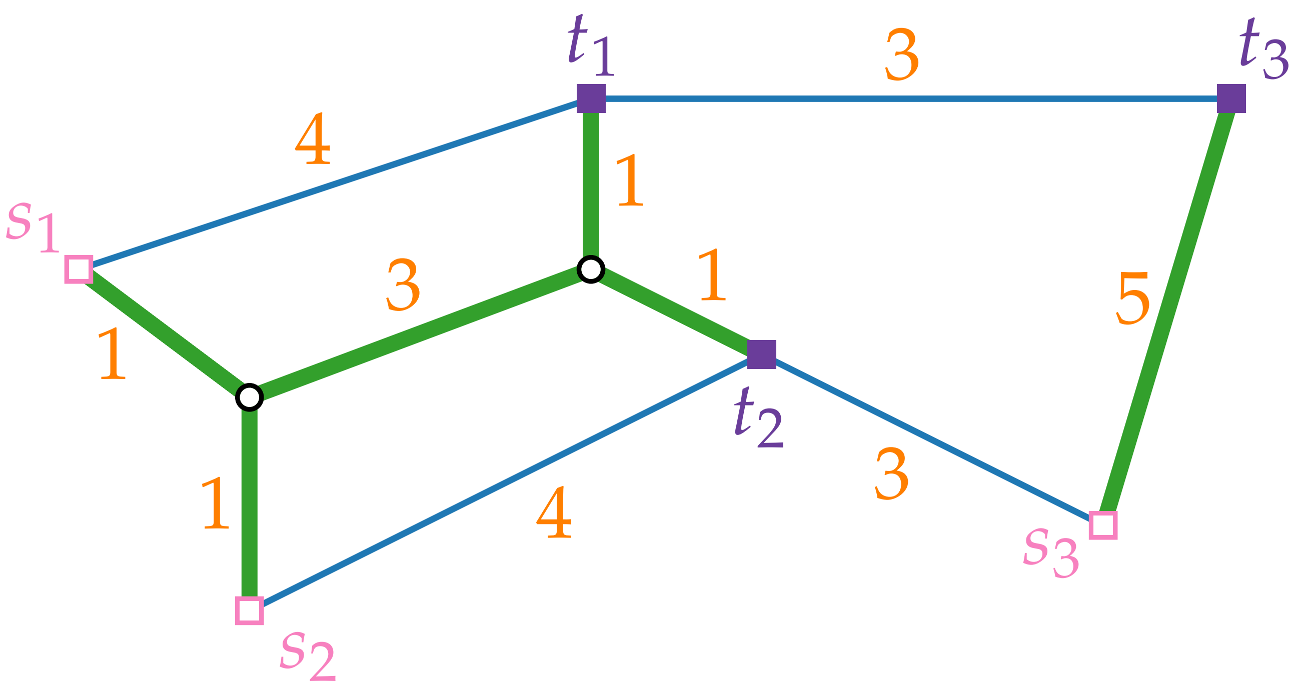
Approaches?

- Merge k shortest s_i–t_i paths



Approaches?

- Merge k shortest s_i–t_i paths
- SteinerTree on the set of terminals



Approaches?

- Merge k shortest s_i–t_i paths
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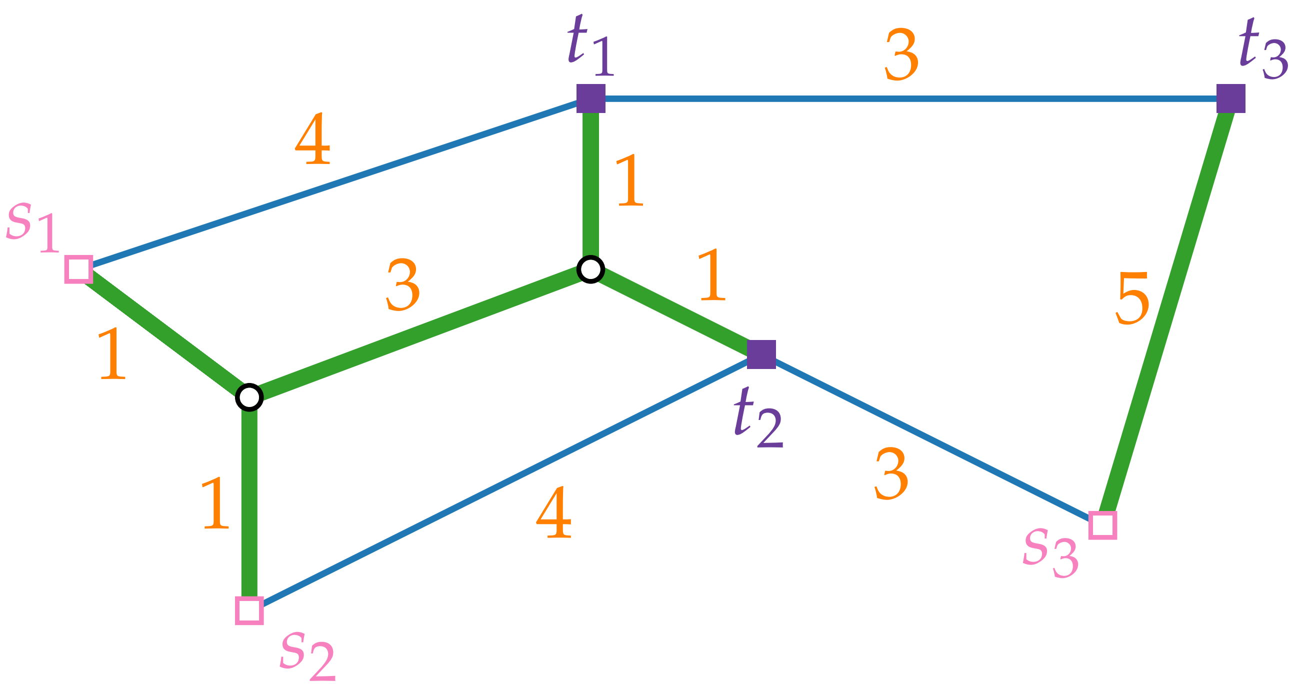
Above approaches perform poorly :-(



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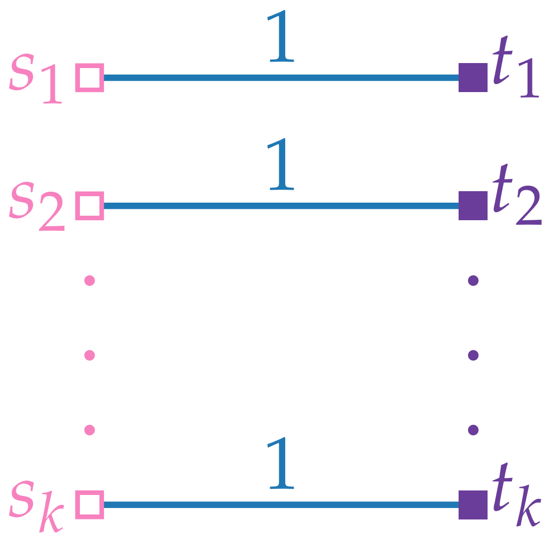

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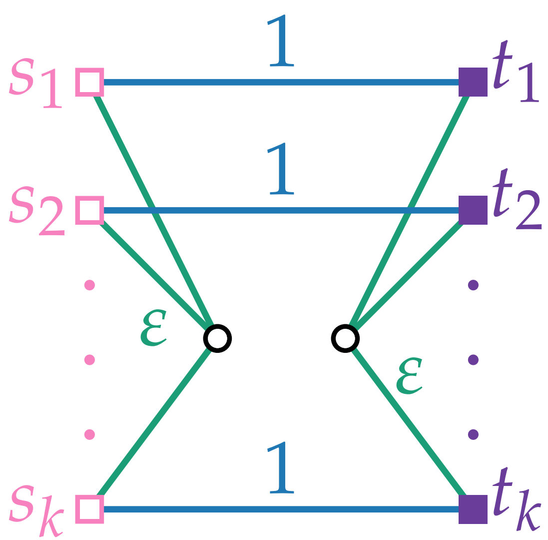

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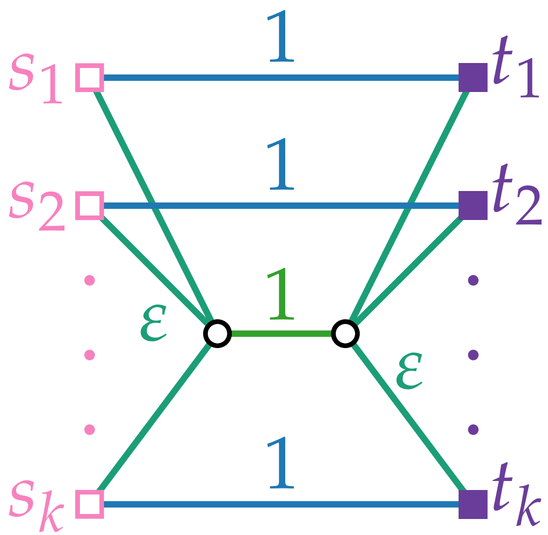

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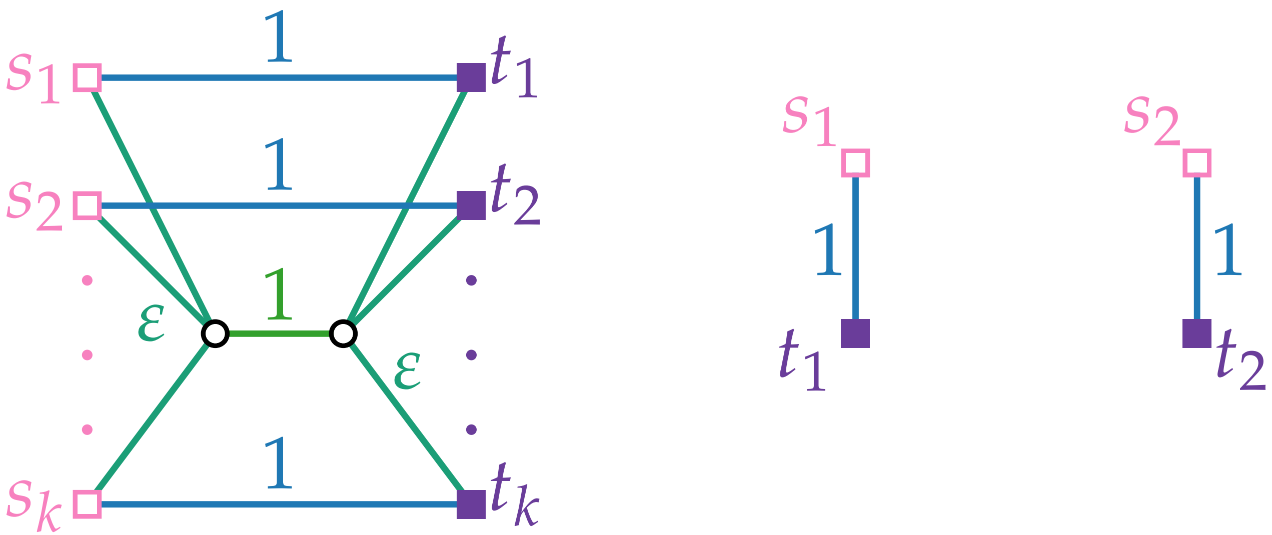

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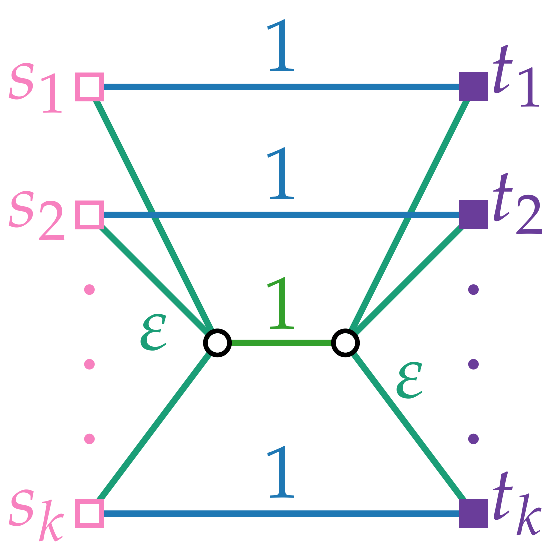

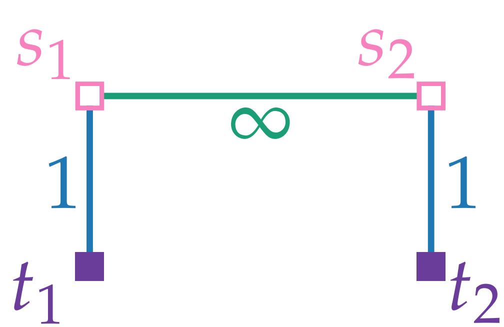

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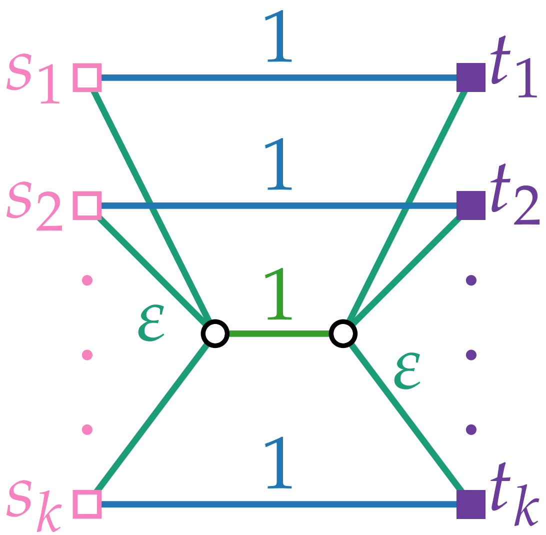
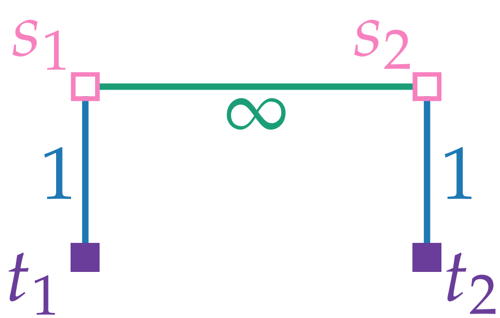
Difficulty:
Which terminals belong to the same tree of the forest?





Approximation Algorithms

Lecture 12:
SteinerForest via Primal–Dual

Part II:
Primal and Dual LP

An ILP

minimize

subject to

An ILP

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subject to

$$x_e \in \{0,1\} \qquad e \in E$$

An ILP

minimize $\sum_{e \in E} c_e x_e$

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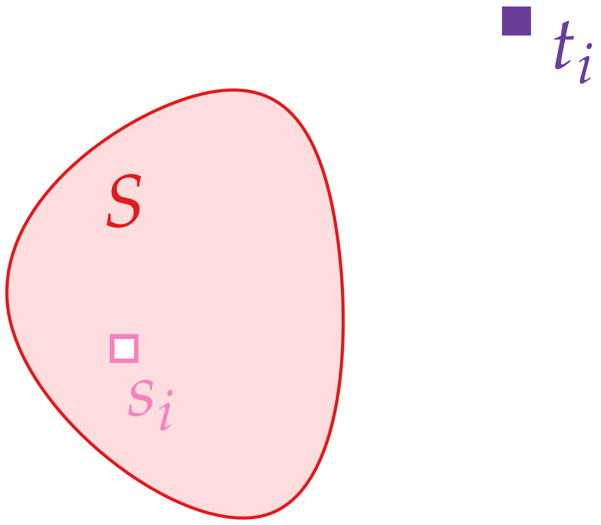

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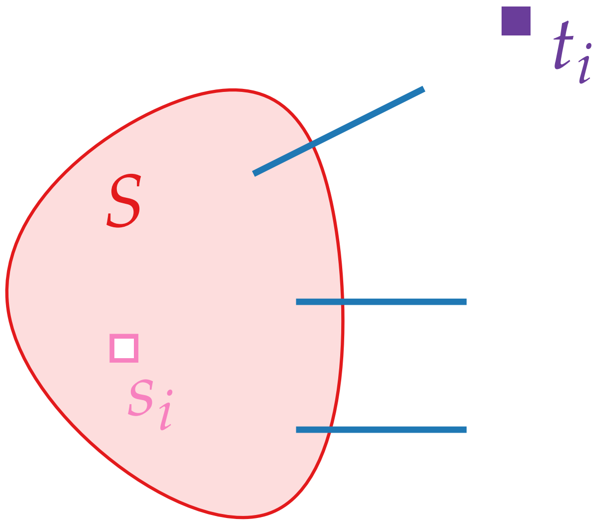

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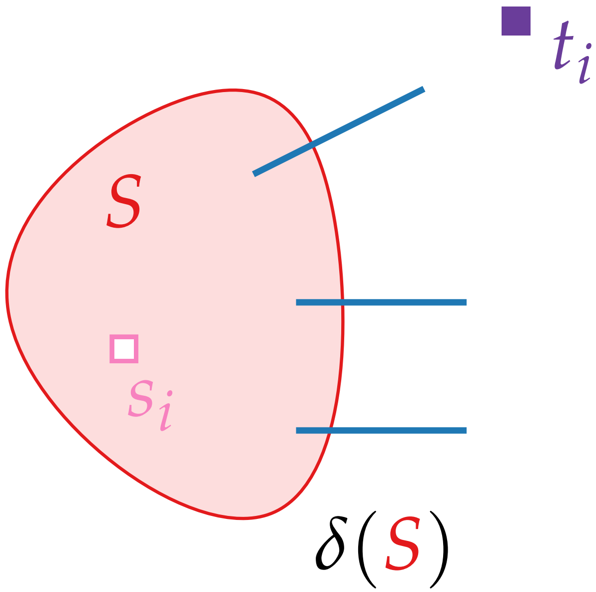

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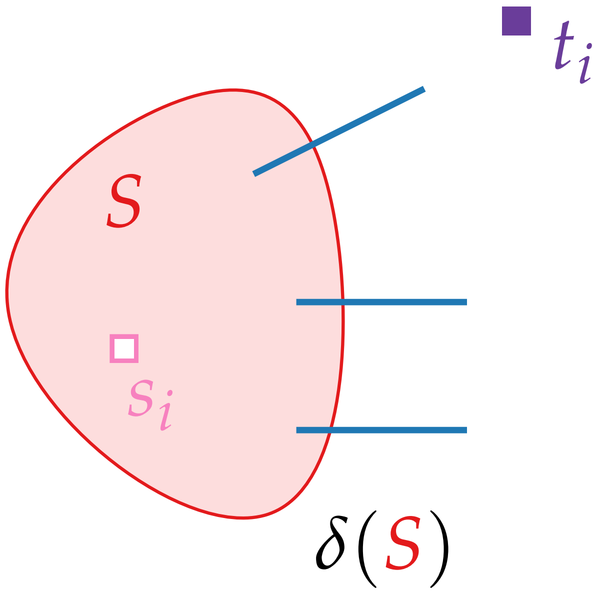
$\delta(S) := \{(u,v) \in E : u \in S \text{ and } v \notin S\}$



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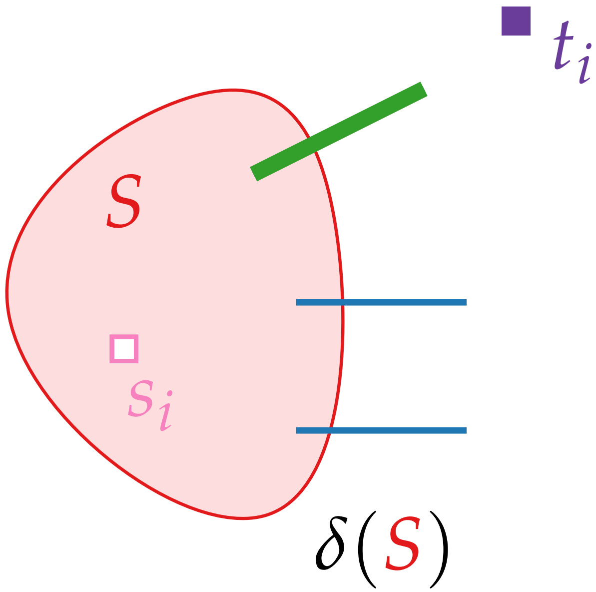

An ILP

$$\begin{aligned} &\textbf{minimize} && \sum_{e \in E} c_e x_e \\ &\textbf{subject to} && \sum_{e \in \delta(S)} x_e \geq 1 \\ & && x_e \in \{0,1\} \qquad e \in E \end{aligned}$$

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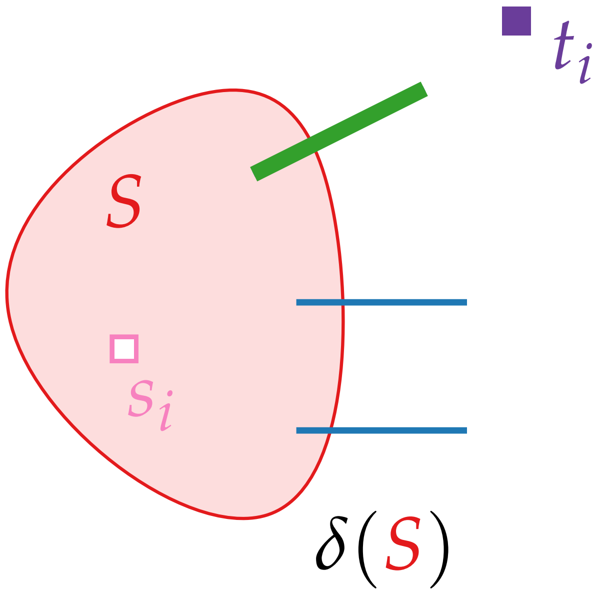

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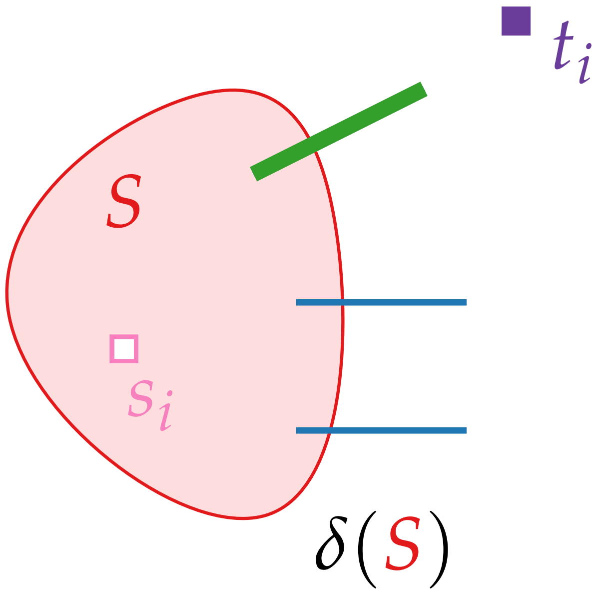

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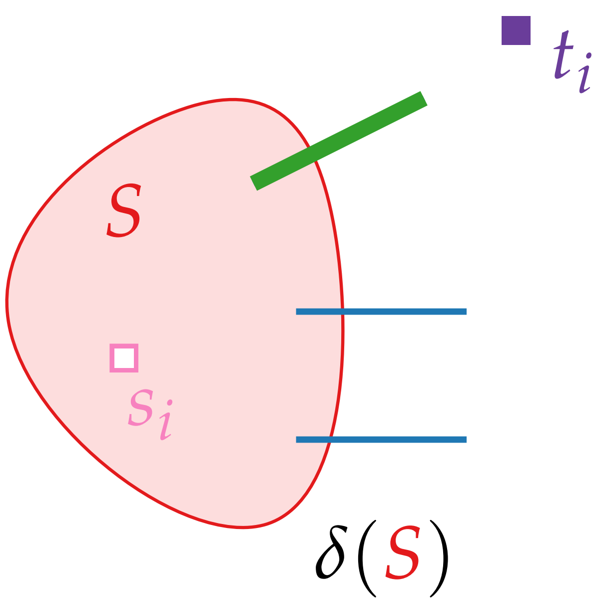

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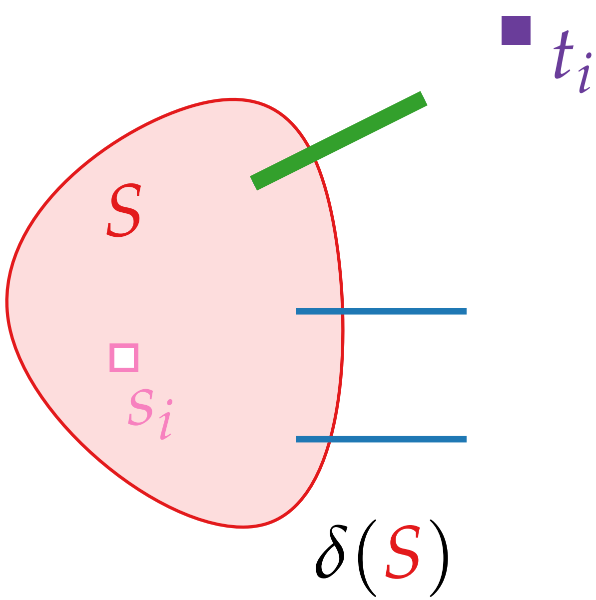
$\rightsquigarrow$ exponentially many constraints!



LP-Relaxation and Dual LP

$$
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\textbf{minimize} & \sum_{e \in E} c_e x_e & \\
\textbf{subject to} & \sum_{e \in \delta(S)} x_e \geq 1 & S \in \mathcal{S}_i,\ i \in \{1, \ldots, k\} \\
 & x_e \geq 0 & e \in E
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$x_e \geq 0 \quad e \in E$

maximize

subject to

$y_S \geq 0 \quad S \in \mathcal{S}_i,\ i \in \{1, \ldots, k\}$

LP-Relaxation and Dual LP

minimize	$\sum_{e \in E} c_e x_e$	
subject to	$\sum_{e \in \delta(S)} x_e \geq 1$	$S \in \mathcal{S}_i,\ i \in \{1, \ldots, k\}\ (y_S)$
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maximize	$\sum_{\substack{S \in \mathcal{S}_i \\ i \in \{1,\ldots,k\}}} y_S$	
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Intuition for the Dual

maximize $\sum_{\substack{S \in \mathcal{S}_i \\ i \in \{1,\dots,k\}}} y_S$

subject to $\sum_{S\,:\, e \in \delta(S)} y_S \leq c_e \qquad e \in E$

$y_S \geq 0 \qquad S \in \mathcal{S}_i,\ i \in \{1, \dots, k\}$

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The graph is a network of **bridges**, spanning the **moats**.

Intuition for the Dual

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\textbf{maximize} \quad & \sum_{\substack{S \in \mathcal{S}_i \\ i \in \{1,\dots,k\}}} y_S \\
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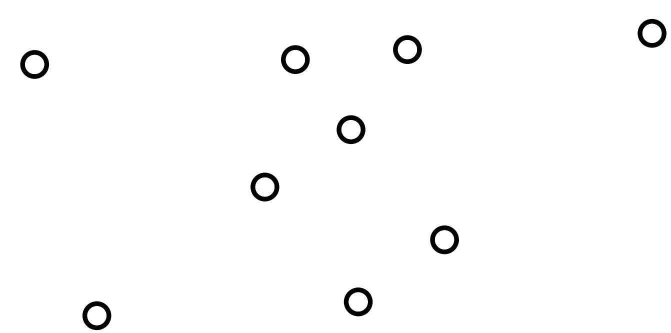

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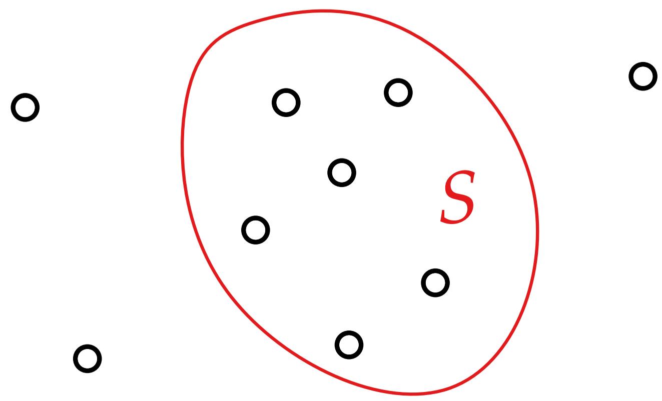

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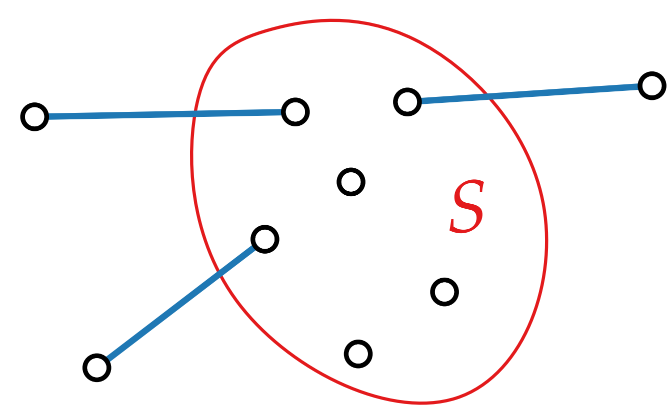

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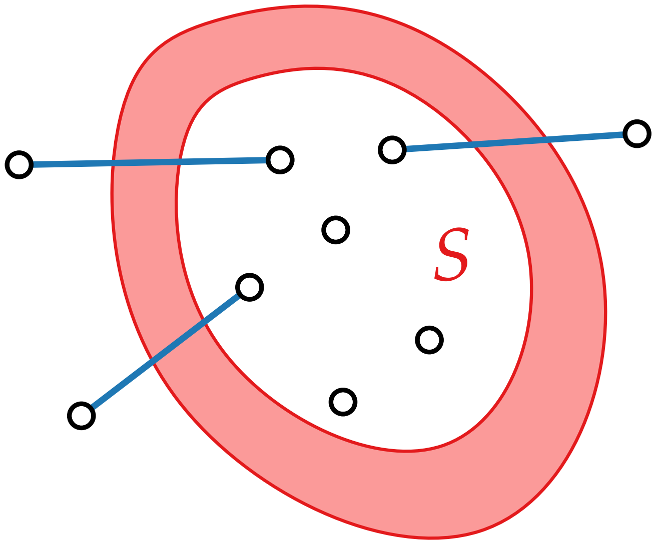

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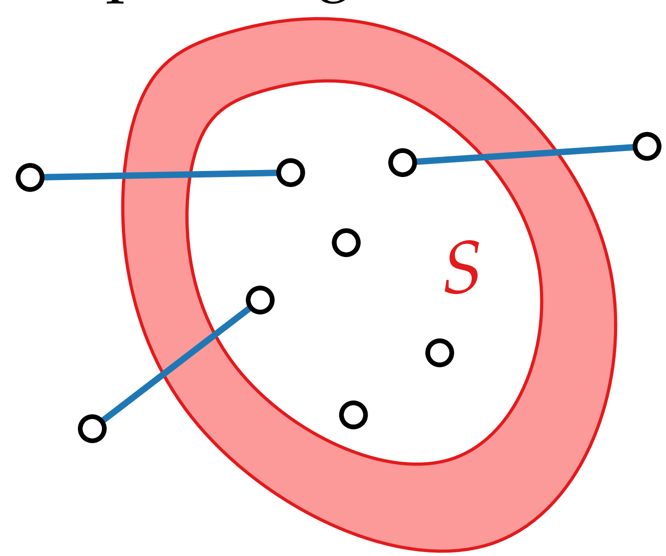

$\delta(S)$ = set of edges / bridges over the moat around S

Intuition for the Dual

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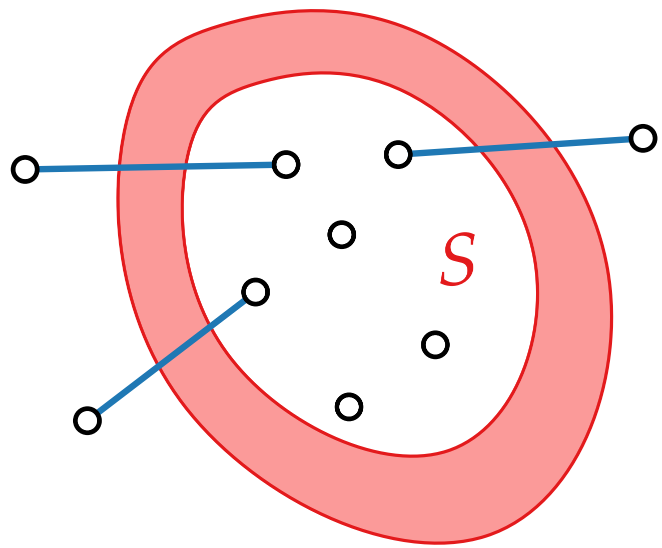

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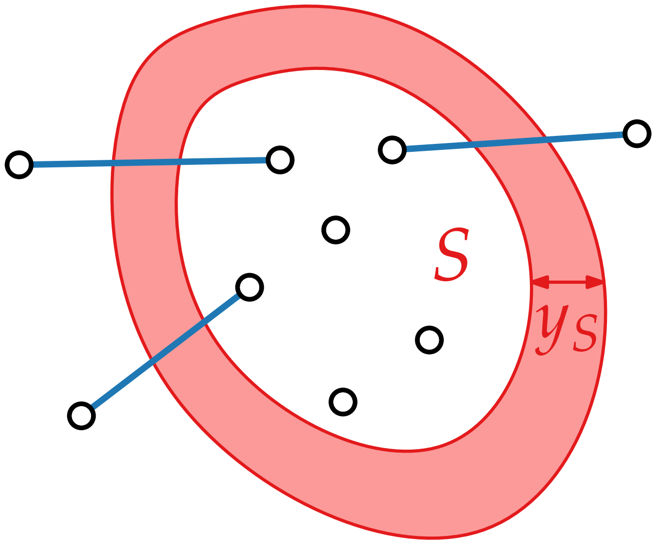

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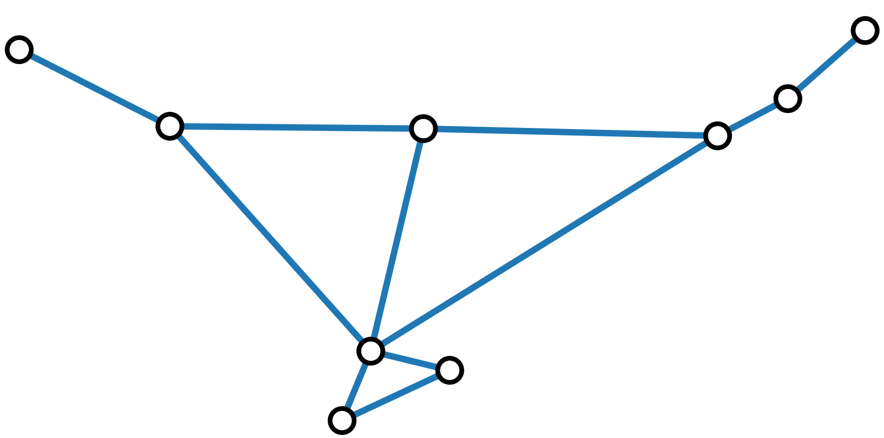

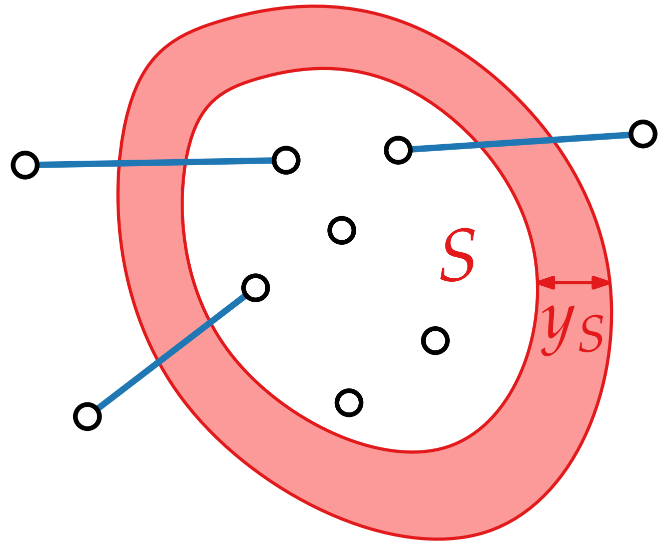

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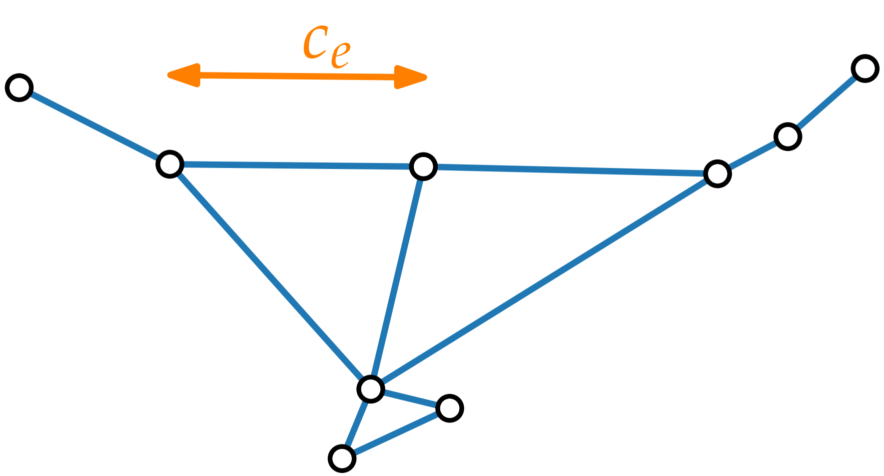

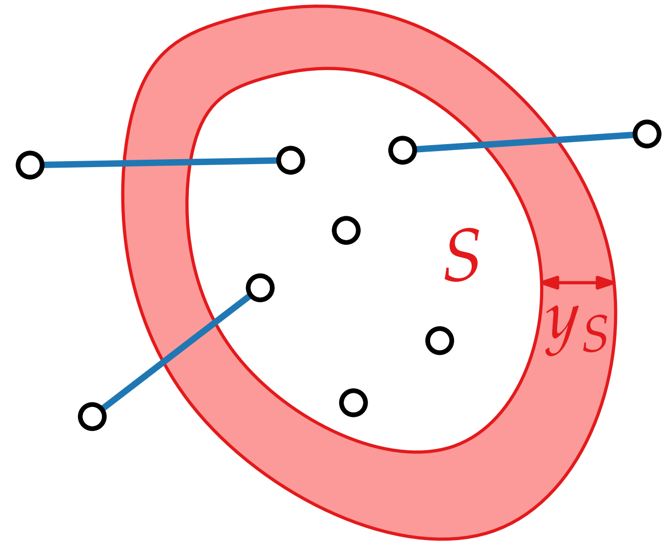

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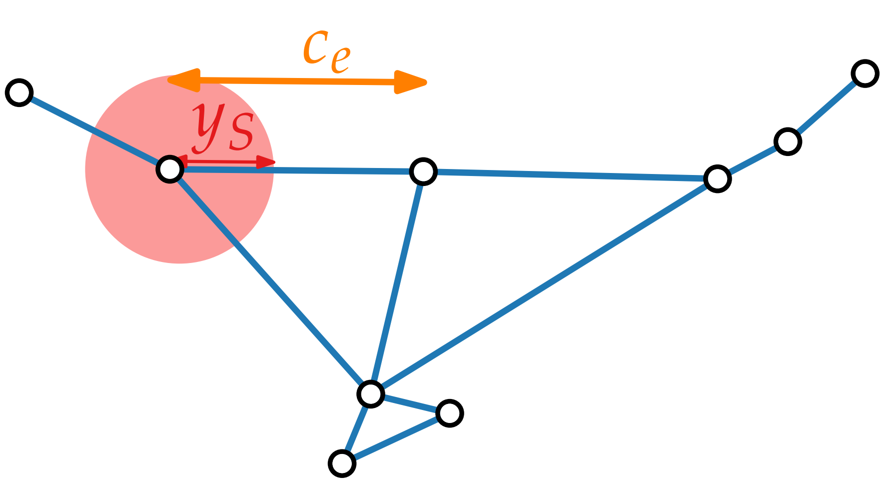

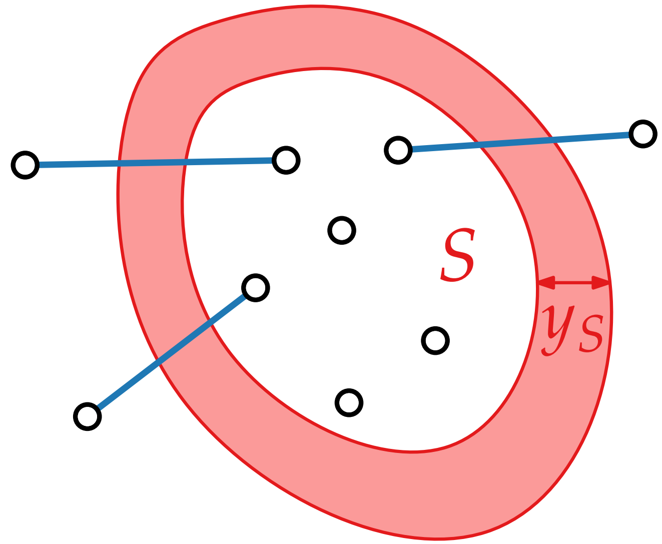

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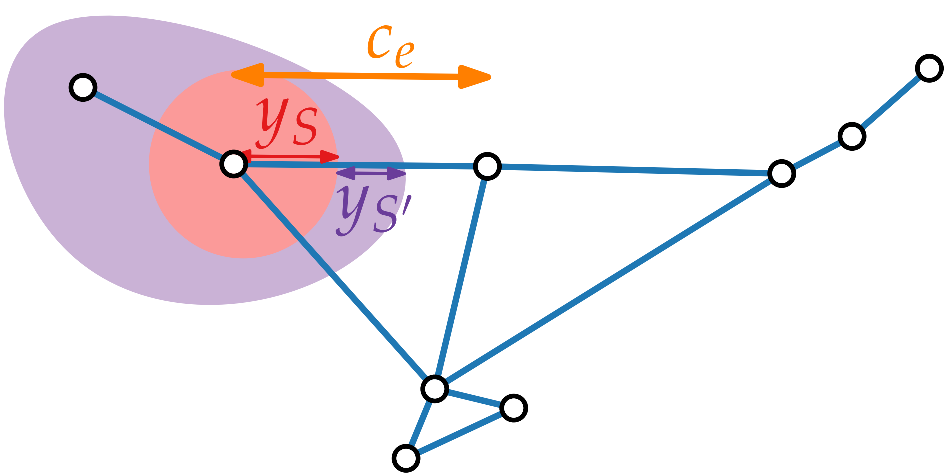

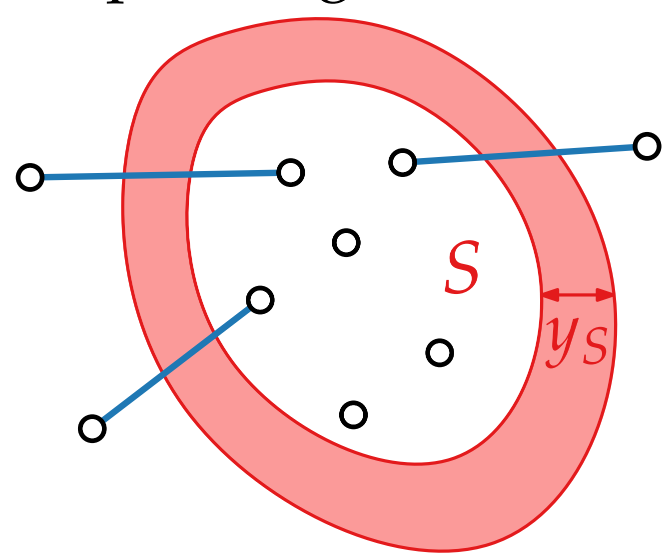

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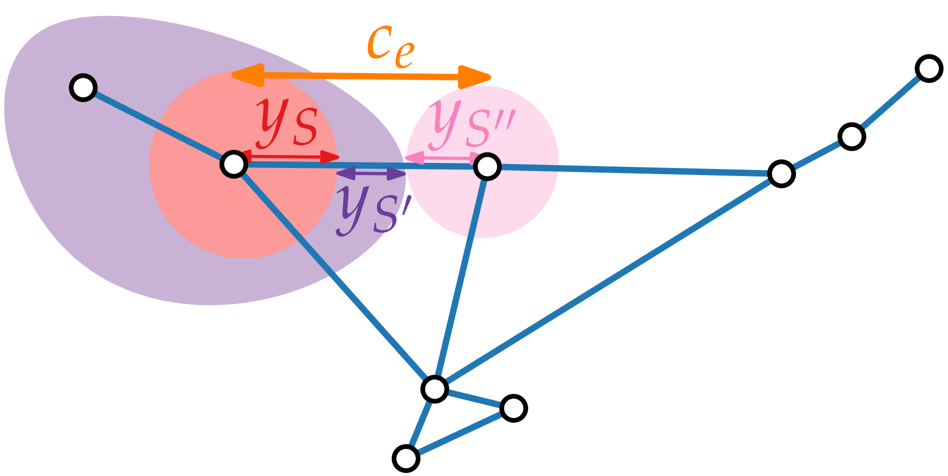

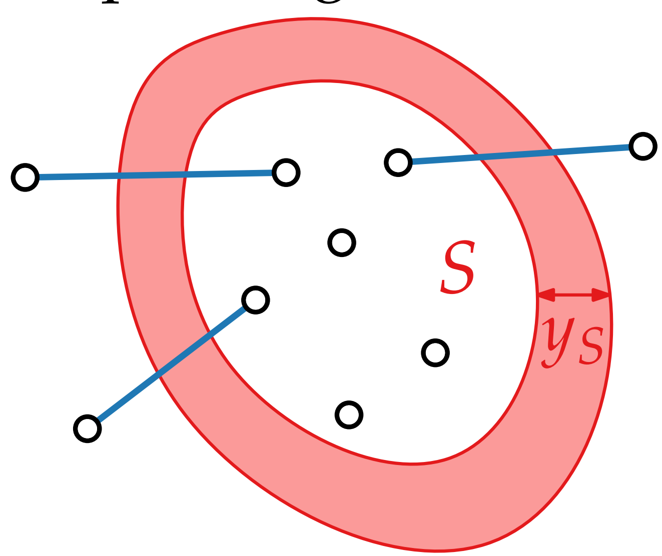

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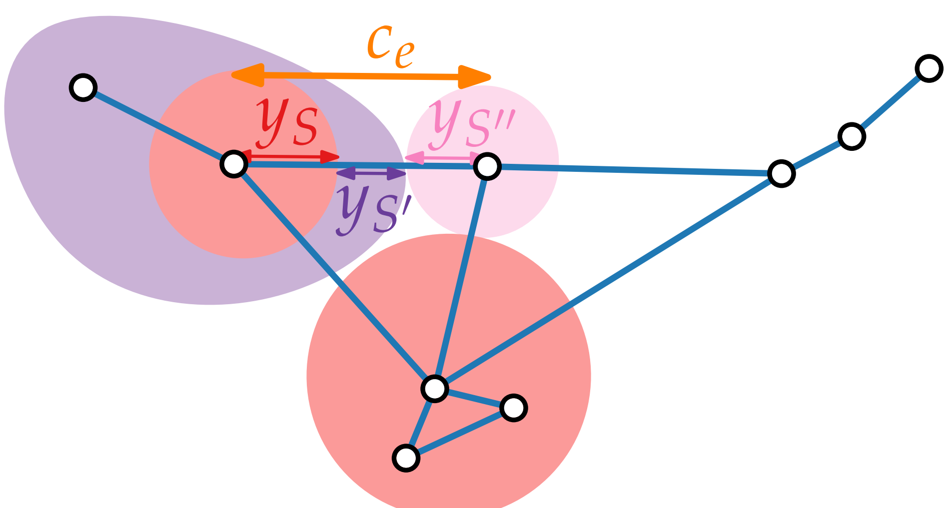

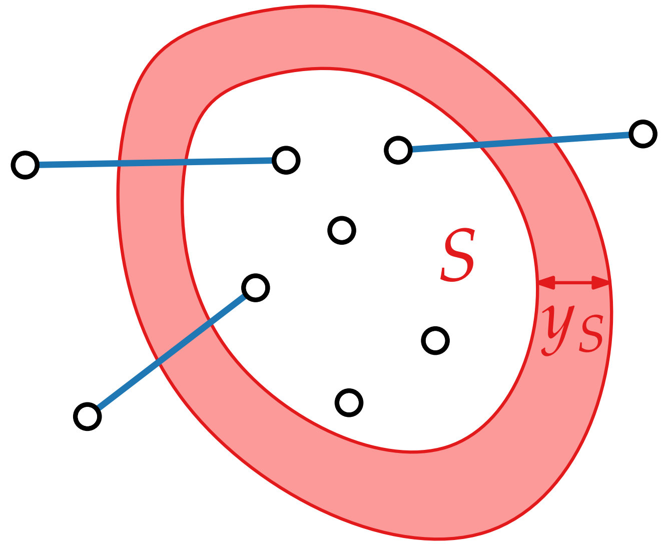

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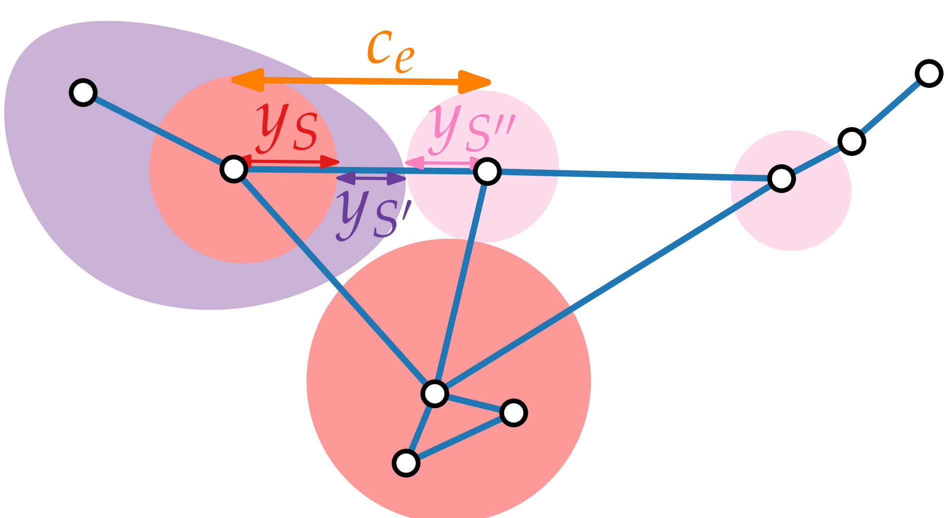

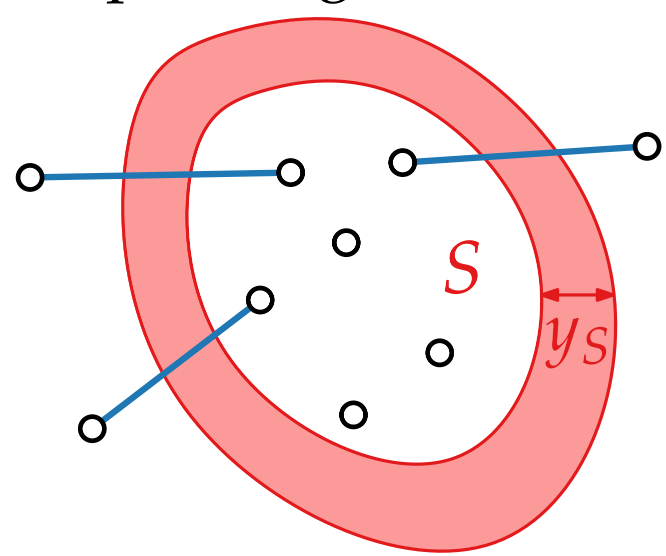

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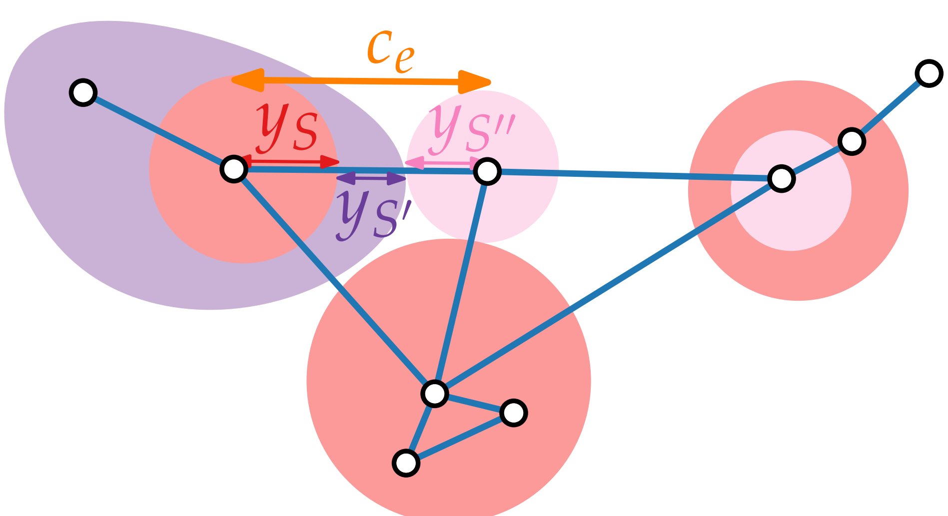

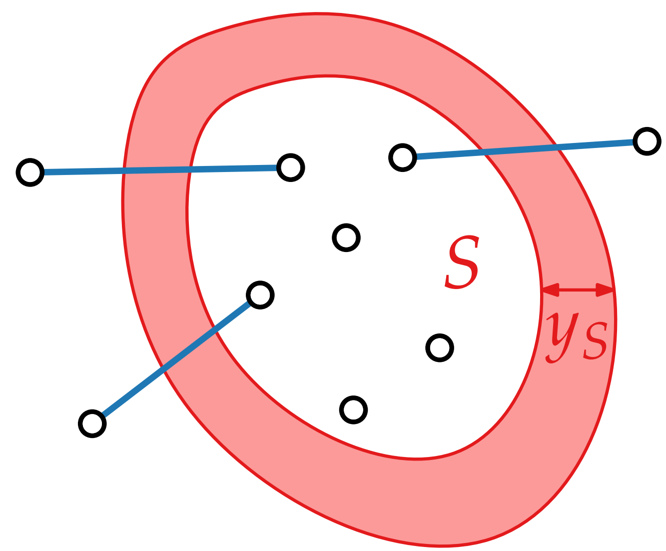

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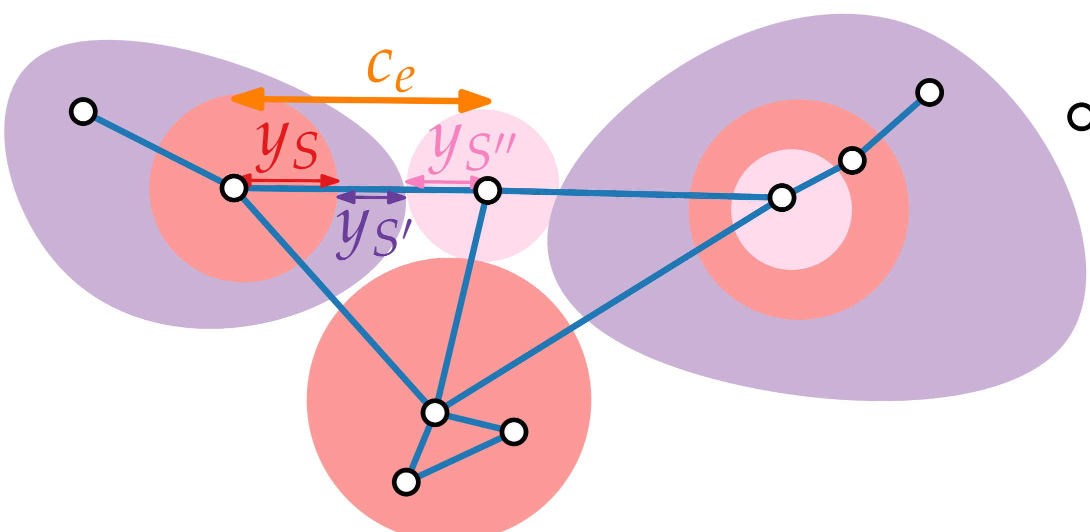

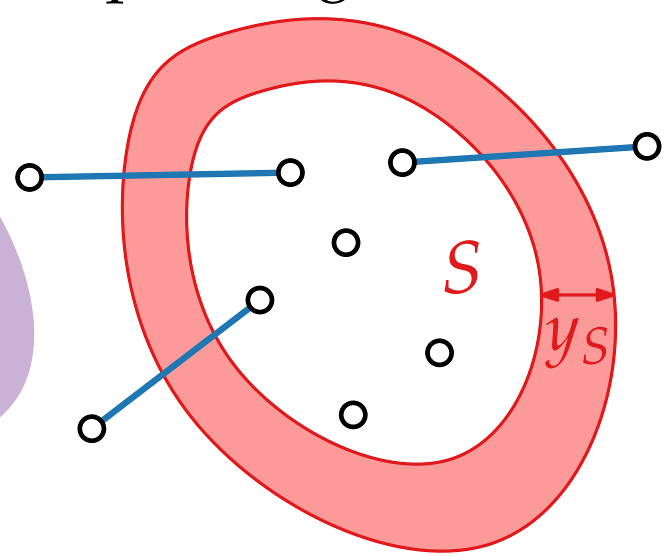

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Approximation Algorithms

Lecture 12:
SteinerForest via Primal–Dual

Part III:
A First Primal–Dual Approach

Complementary Slackness (Rep.)

$$\begin{array}{lrcl} \textbf{minimize} & c^\top x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

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Theorem. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be valid solutions for the primal and dual program (resp.). Then x and y are optimal if and only if the following conditions are met:

Primal CS:
For each $j = 1, \ldots, n$: either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
For each $i = 1, \ldots, m$: either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

A First Primal–Dual Approach

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$\Rightarrow$ pick "critical" edges (and only those)

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Idea: iteratively build a feasible integral primal solution.

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How to find a violated primal constraint? $\left(\sum_{e \in \delta(S)} x_e < 1\right)$

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How do we iteratively improve the dual solution?

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$\rightsquigarrow$ Consider related connected component C!

How do we iteratively improve the dual solution?

$\rightsquigarrow$ Increase y_C (until some edge in $\delta(C)$ becomes critical)!

A First Primal–Dual Approach

PrimalDualSteinerForestNaive(G, c, R)

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$y \leftarrow 0, F \leftarrow \emptyset$

return F

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Running Time?

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Running Time?

Trick: Handle all y_S with $y_S = 0$ implicitly

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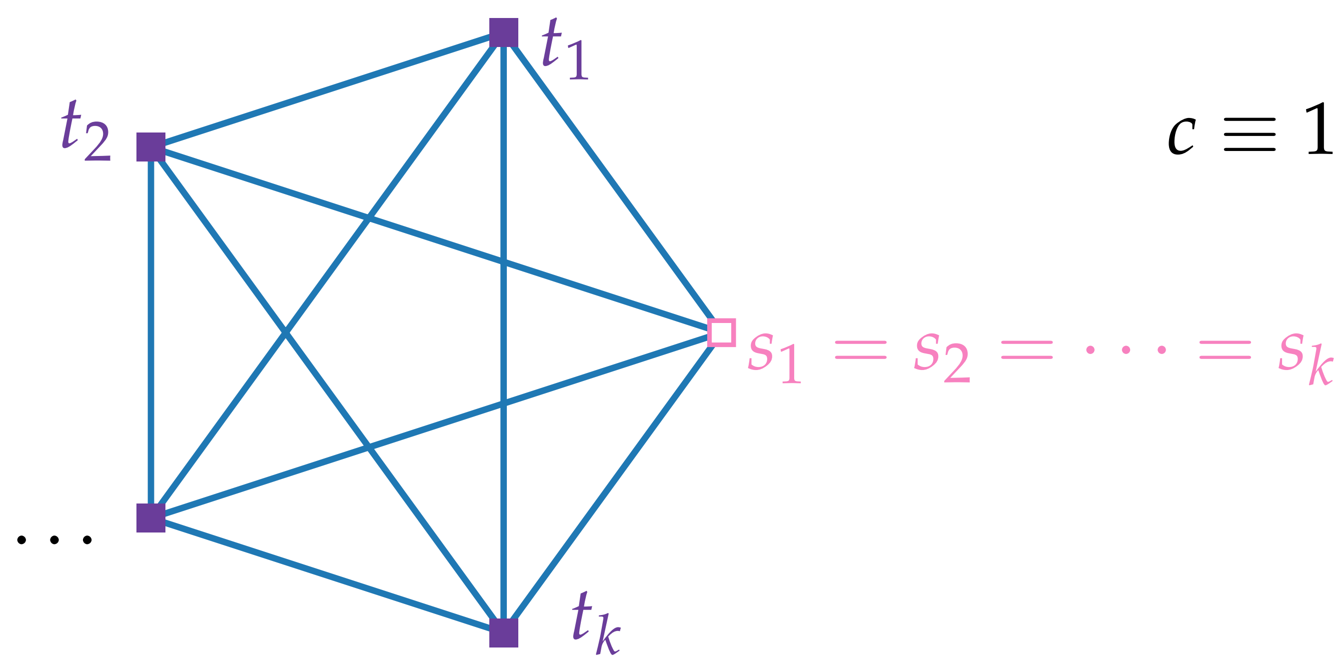

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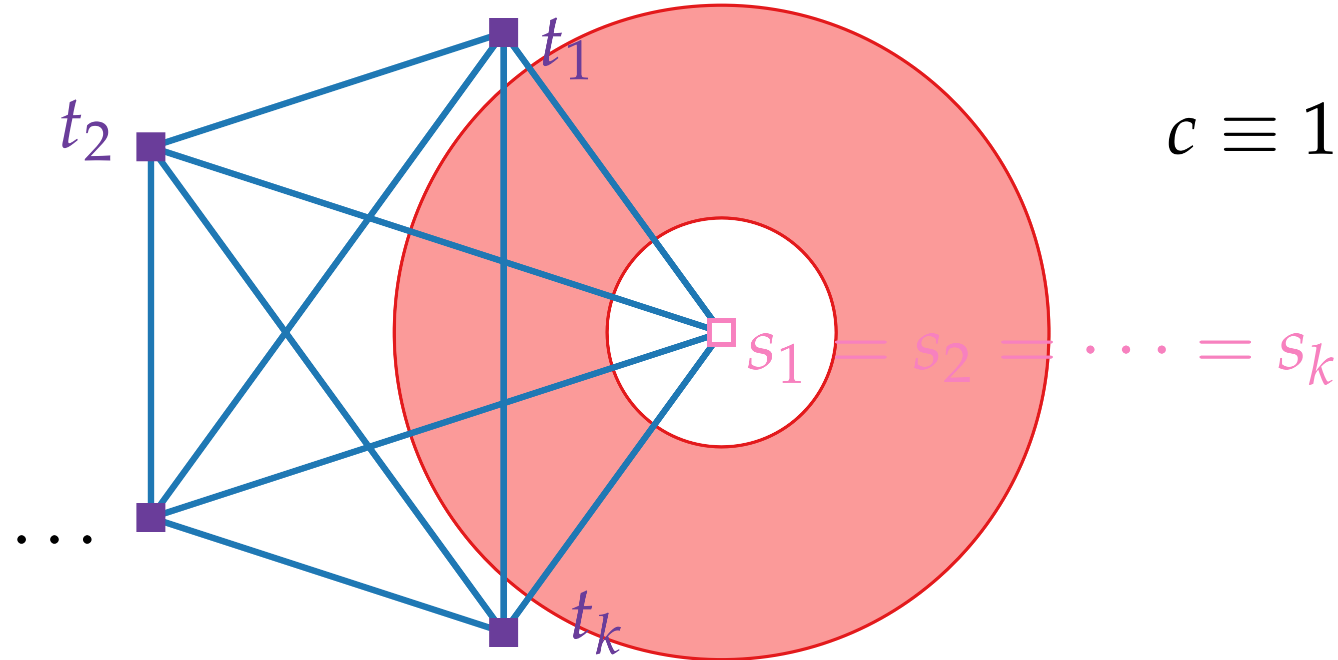

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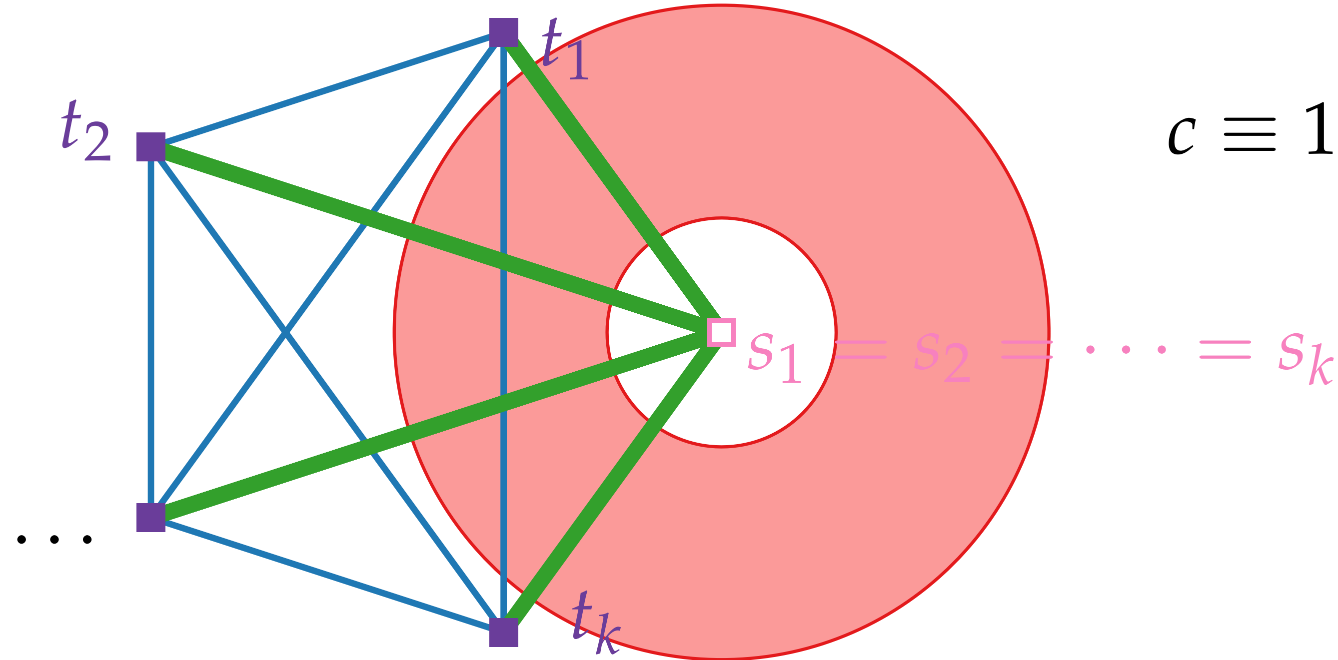

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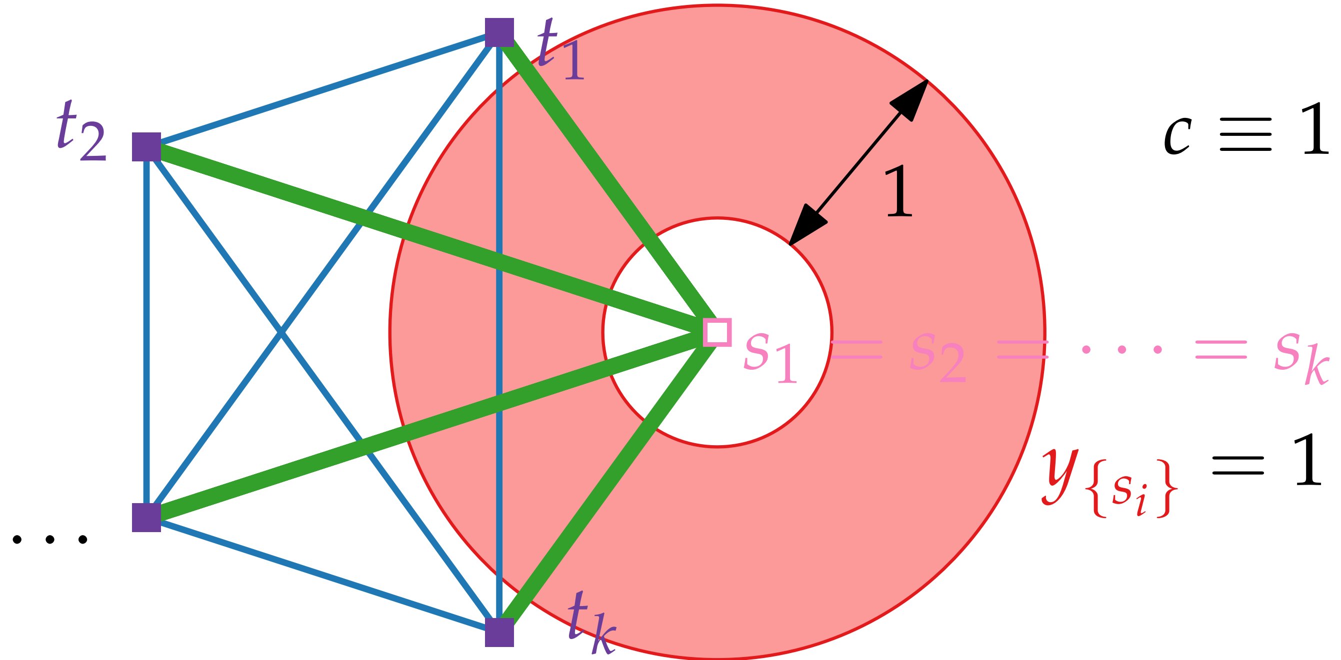

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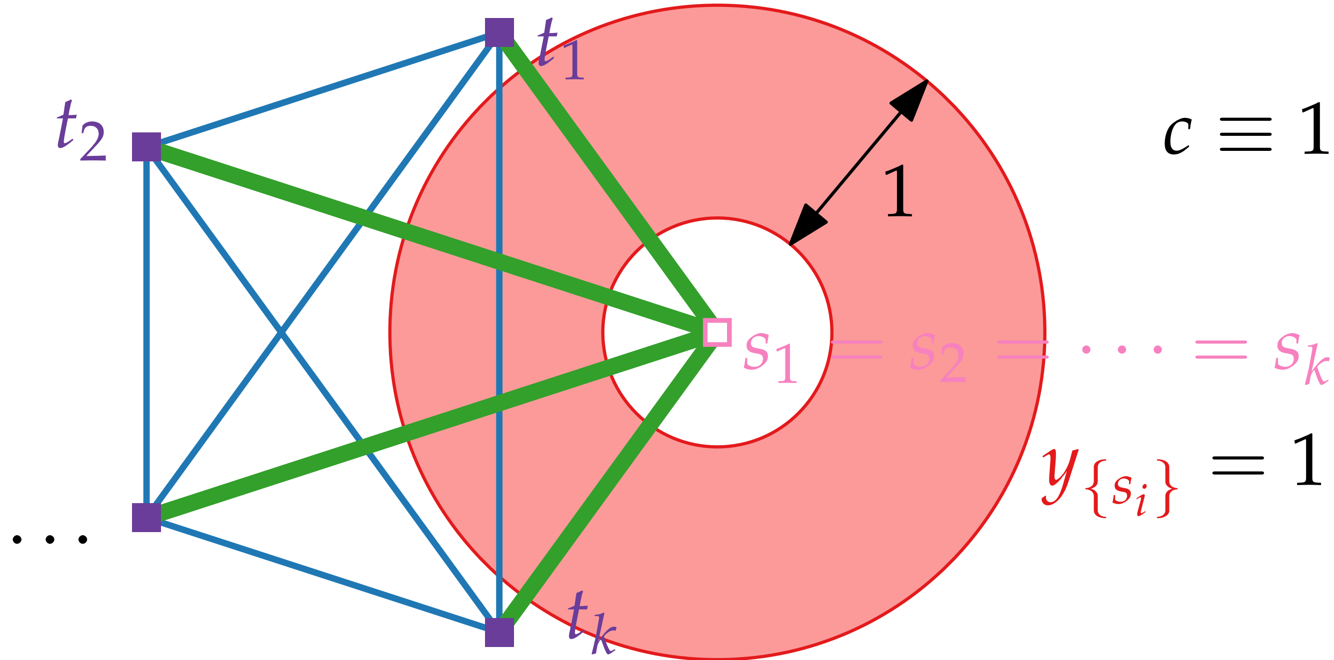

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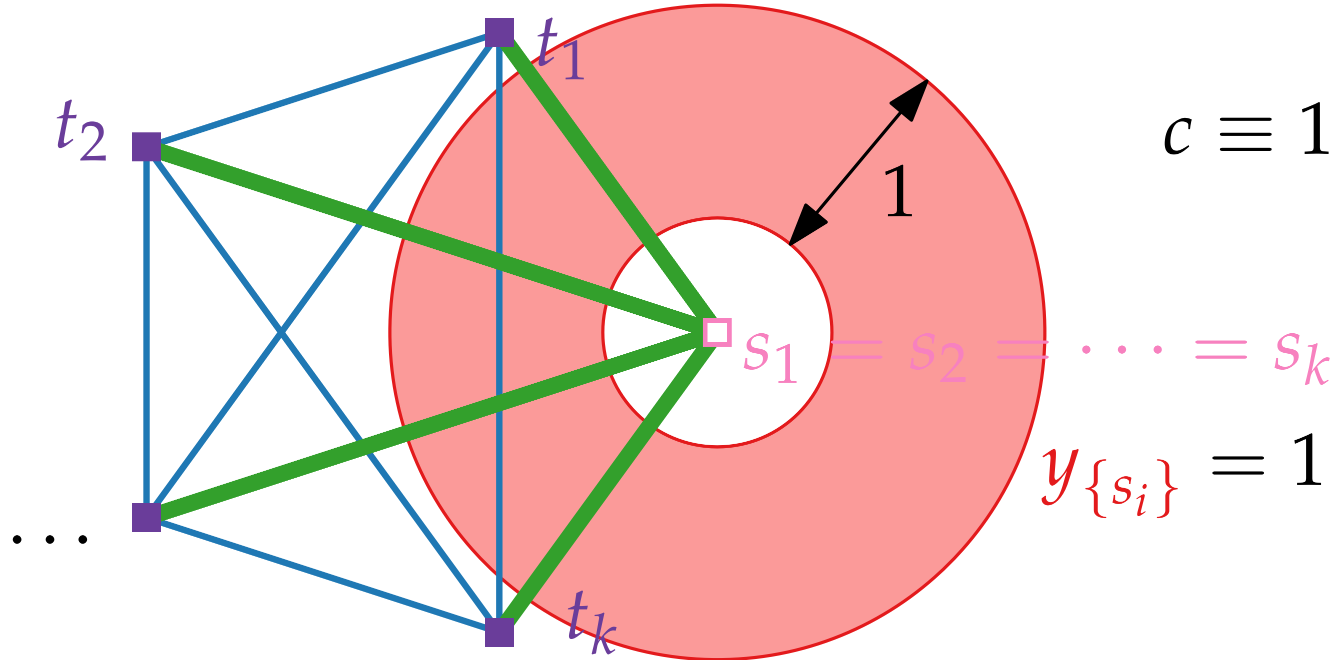
But: Average degree of component is 2!

$\Rightarrow$ Increase y_C for all components C simultaneously!



Approximation Algorithms

Lecture 12:
SteinerForest via Primal–Dual

Part IV:
Primal–Dual with Synchronized Increases

Primal–Dual with Synchronized Increases

PrimalDualSteinerForest(G, c, R)

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while some $(s_i, t_i) \in R$ not connected in (V, F) **do**
    $\ell \leftarrow \ell + 1$

    $F \leftarrow F \cup \{e_\ell\}$

Primal–Dual with Synchronized Increases

PrimalDualSteinerForest(G, c, R)

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// Pruning

return F'

Primal–Dual with Synchronized Increases

```
PrimalDualSteinerForest(G, c, R)
y ← 0, F ← ∅, ℓ ← 0
while some (s_i, t_i) ∈ R not connected in (V, F) do
    ℓ ← ℓ + 1
    𝒞 ← {comp. C in (V, F) with |C ∩ {s_i, t_i}| = 1 for some i}
    Increase y_C for all C ∈ 𝒞 simultaneously
        until Σ_{S : e_ℓ ∈ δ(S)} y_S = c_{e_ℓ} for some e_ℓ ∈ δ(C), C ∈ 𝒞.
    F ← F ∪ {e_ℓ}
F' ← F
// Pruning
for j ← ℓ down to 1 do

return F'
```

Primal–Dual with Synchronized Increases

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    $F \leftarrow F \cup \{e_\ell\}$
$F' \leftarrow F$
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for $j \leftarrow \ell$ **down to** 1 **do**
    if $F' \setminus \{e_j\}$ is feasible solution **then**

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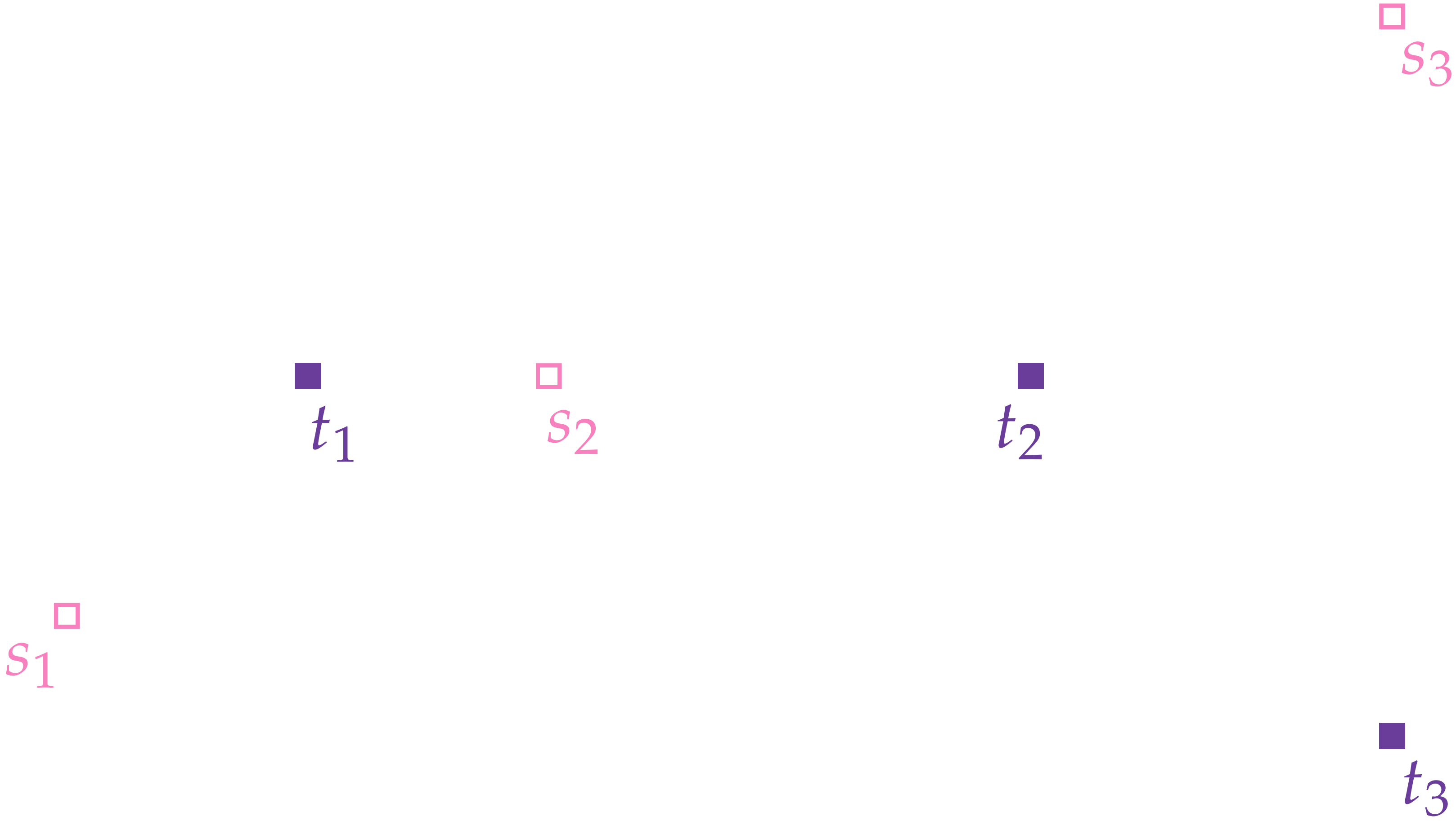
Illustration

$G = K_6$ with Euclidean edge costs



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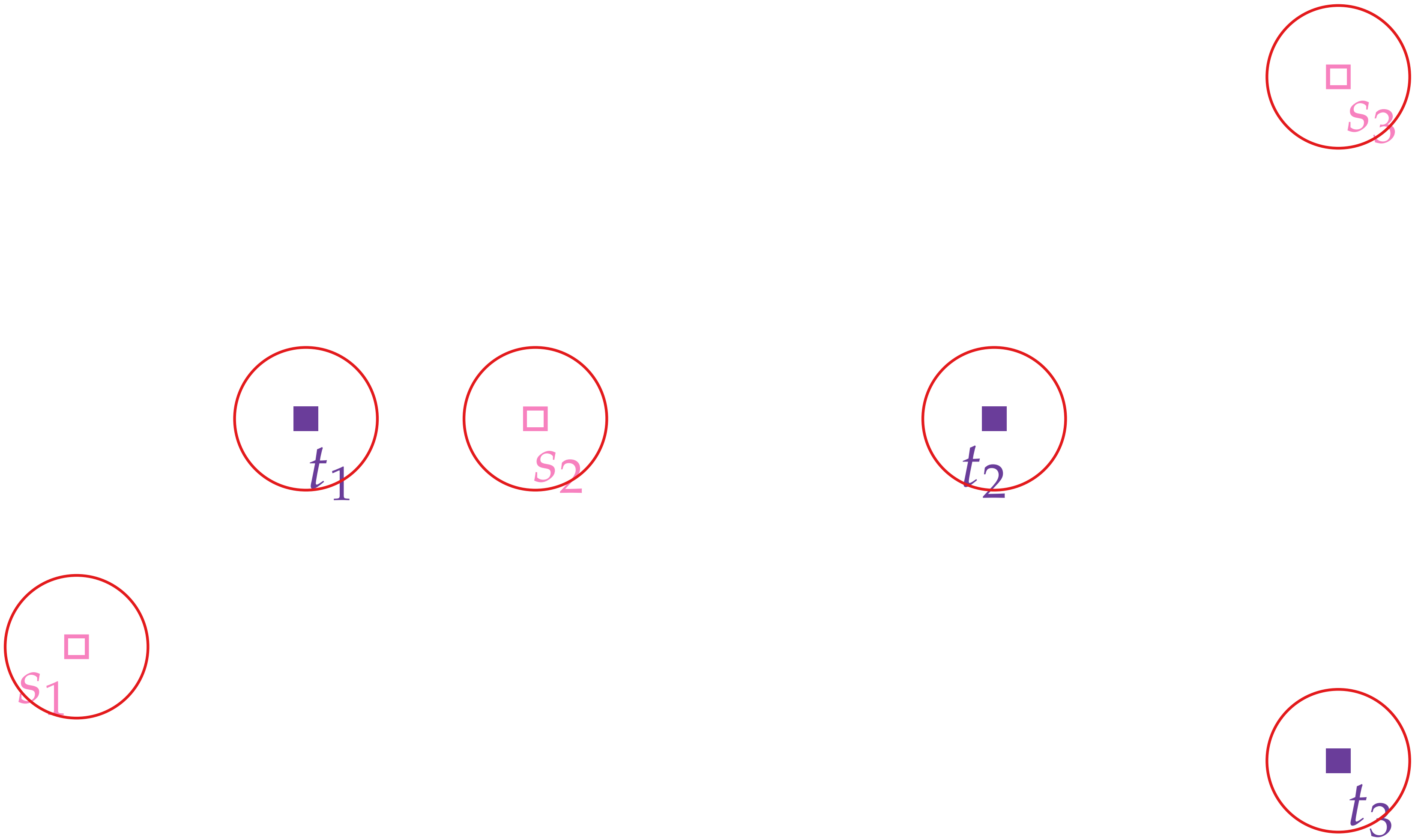

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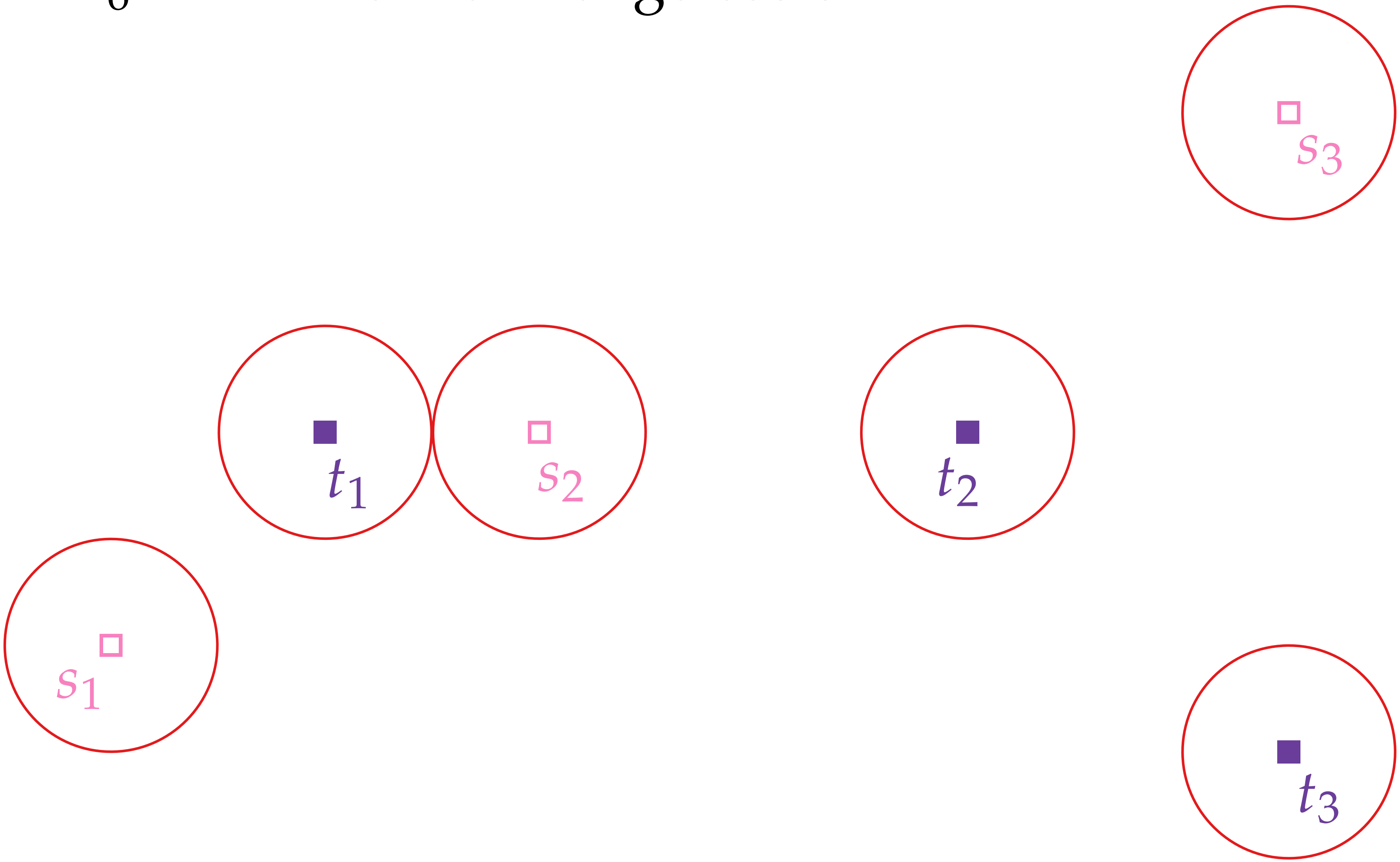

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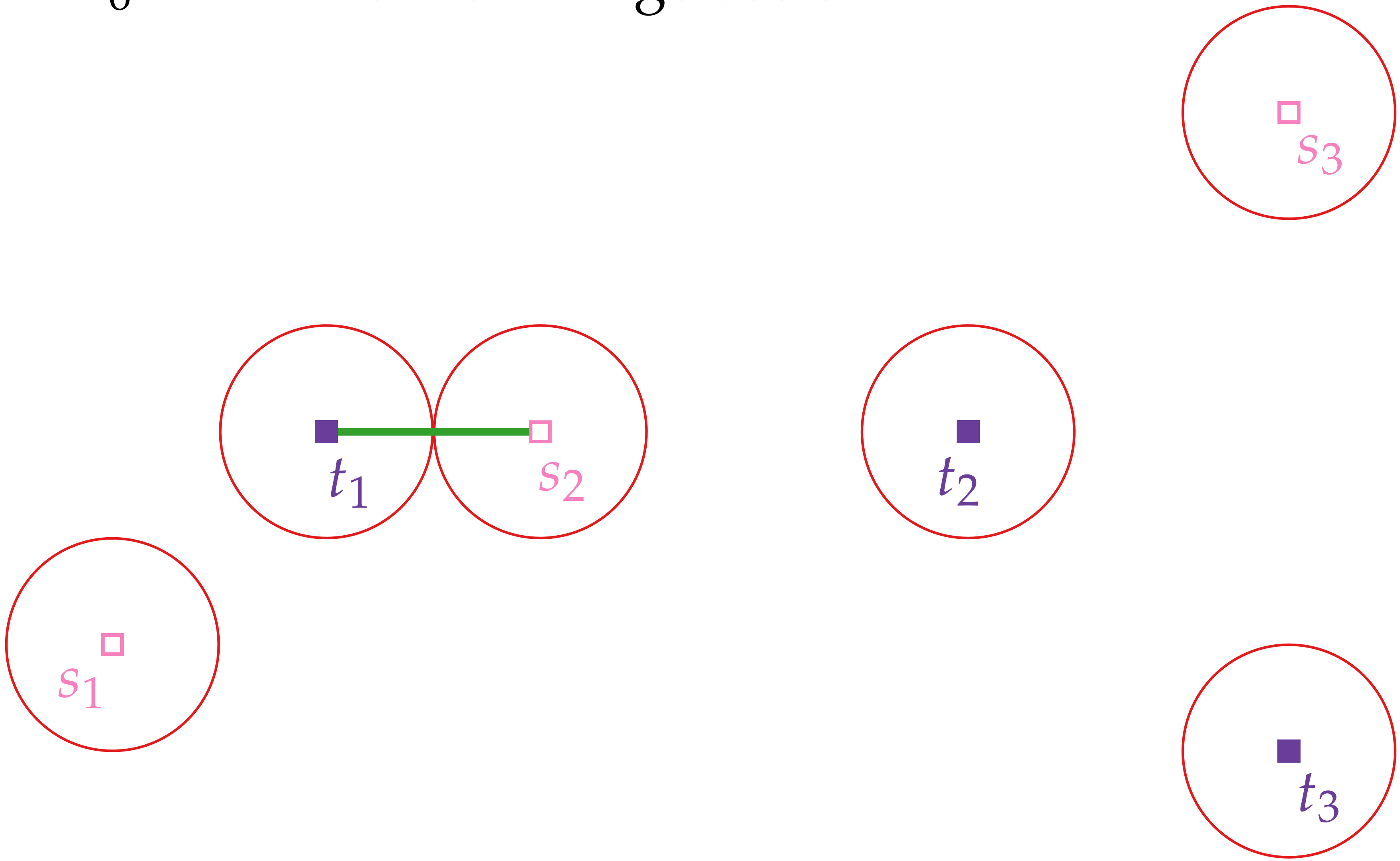

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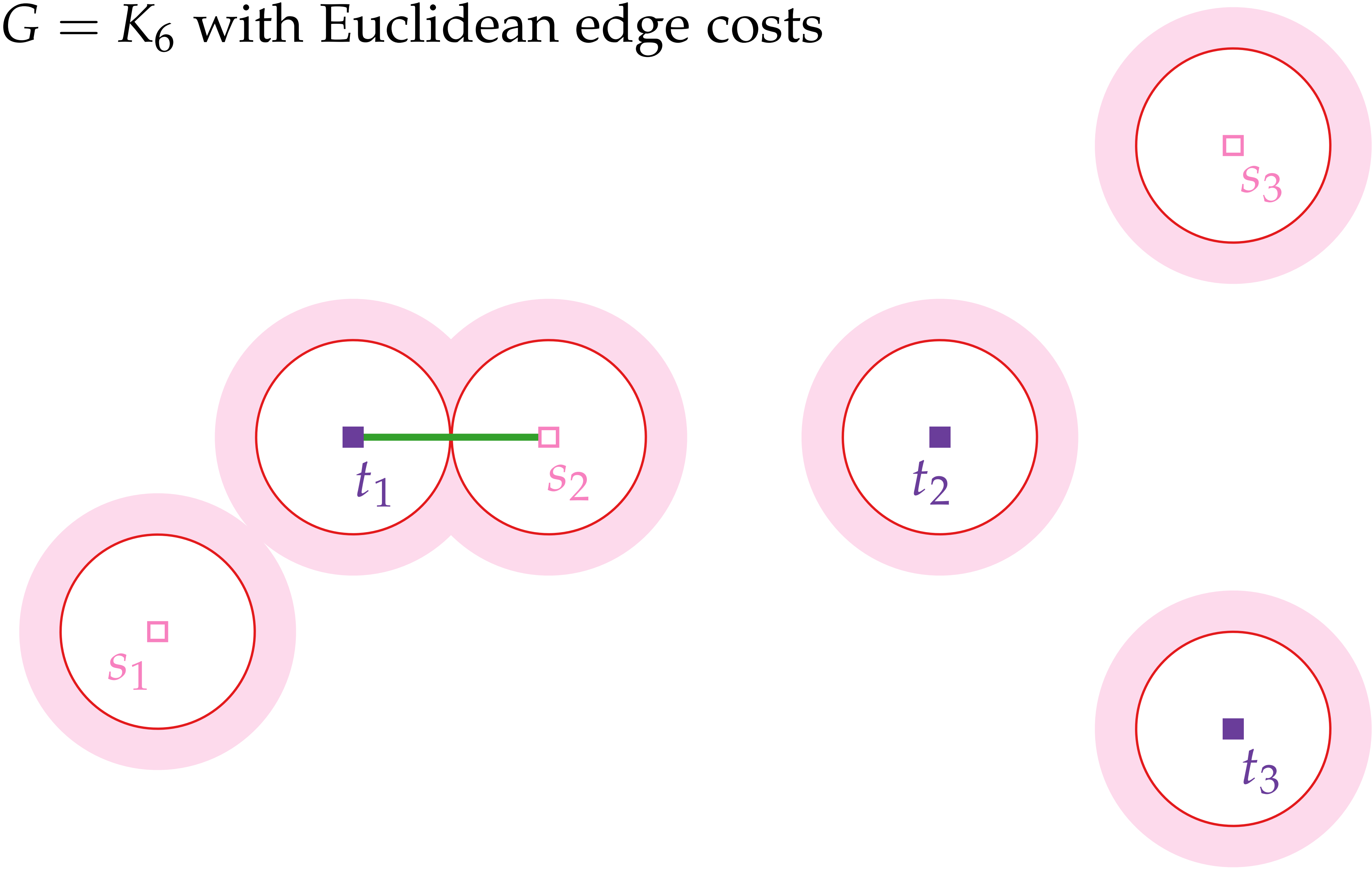

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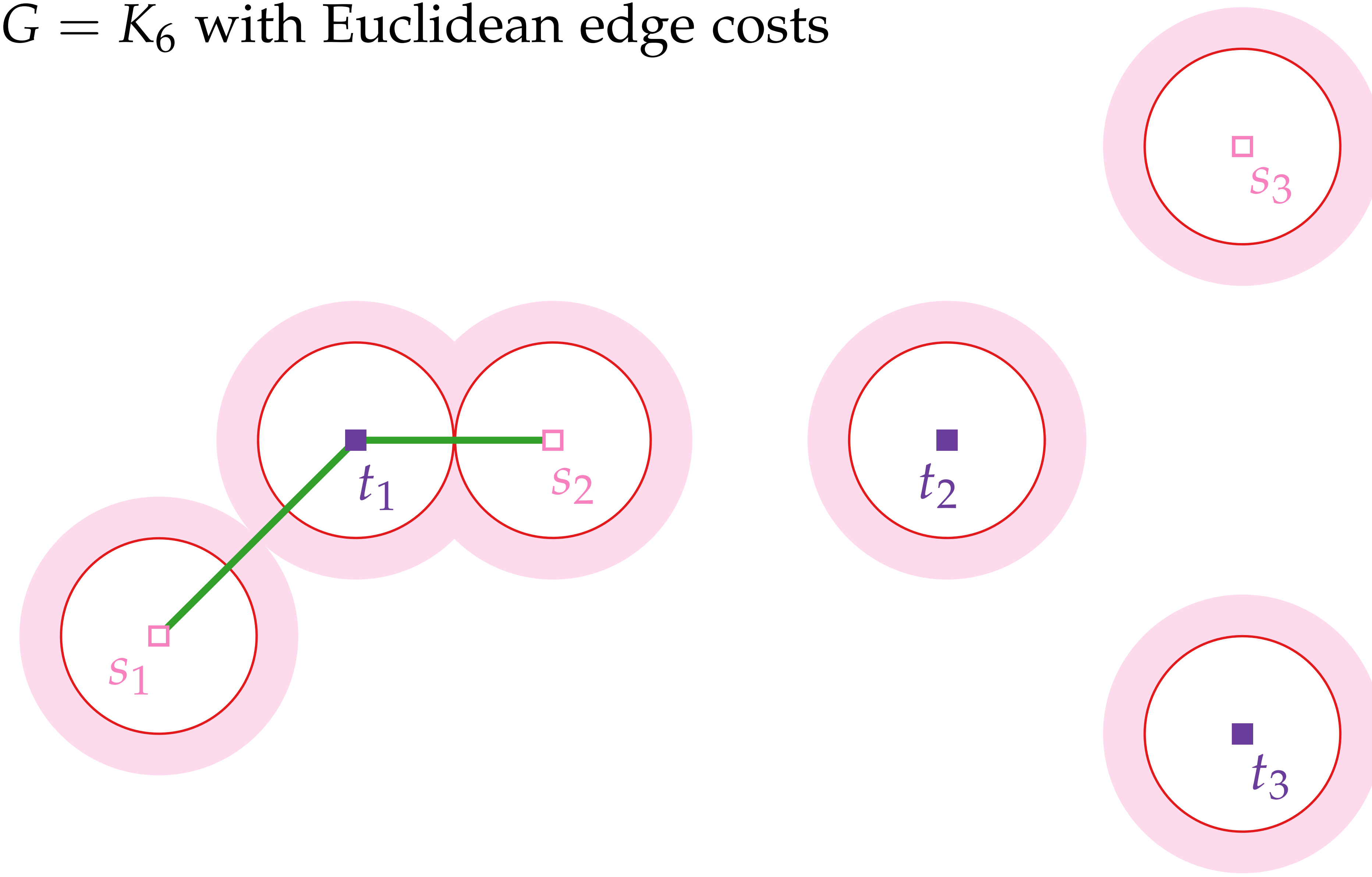

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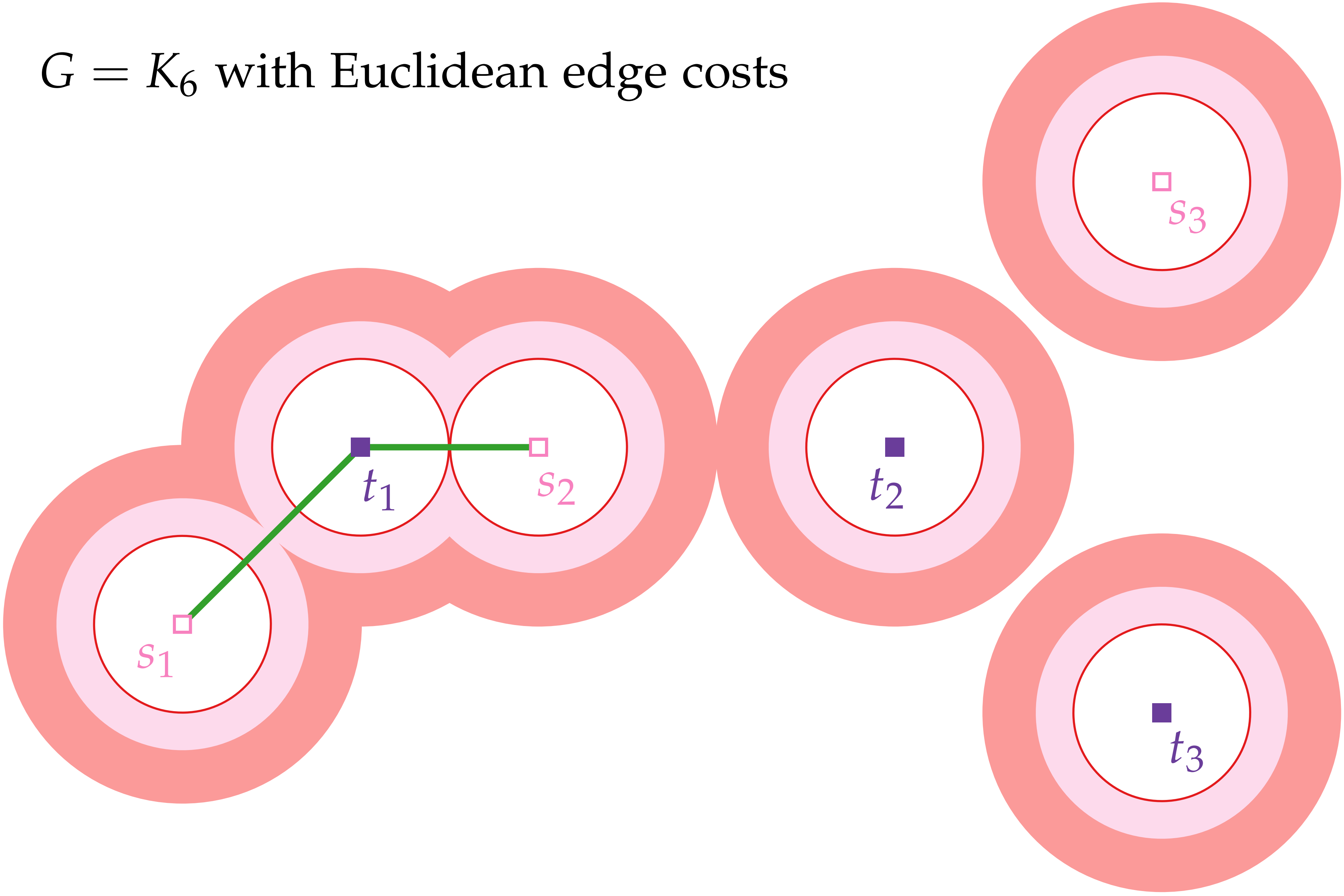

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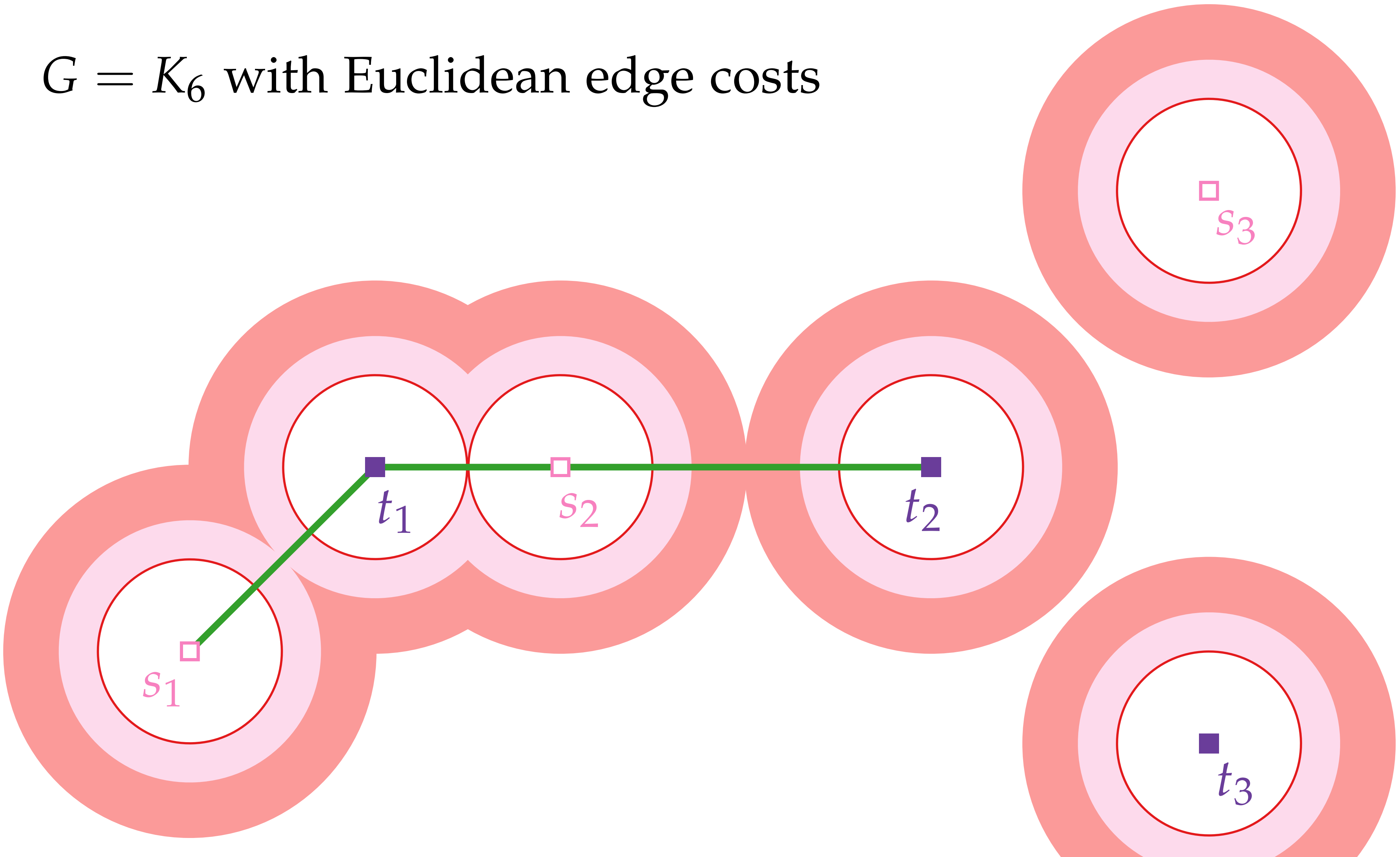

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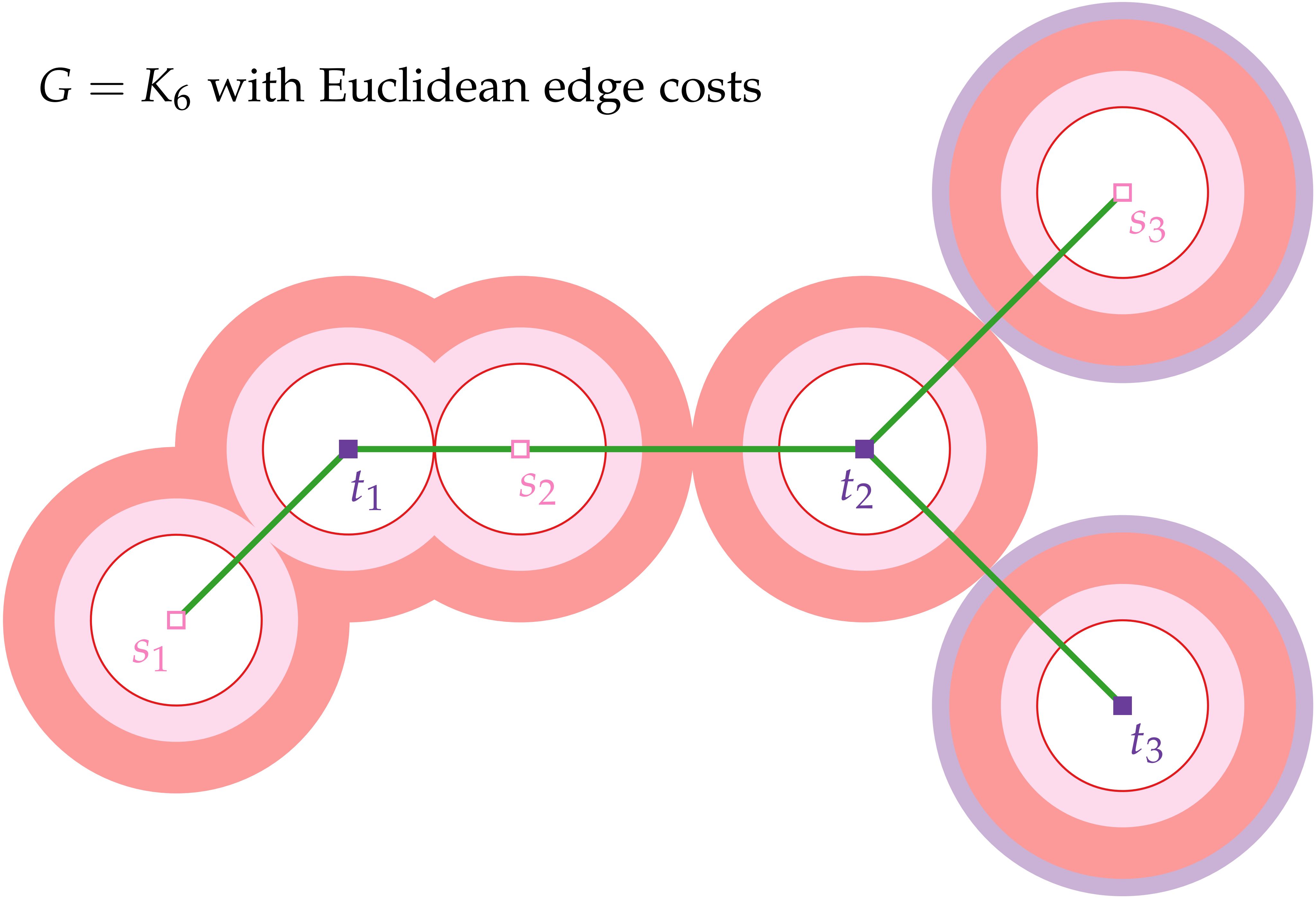

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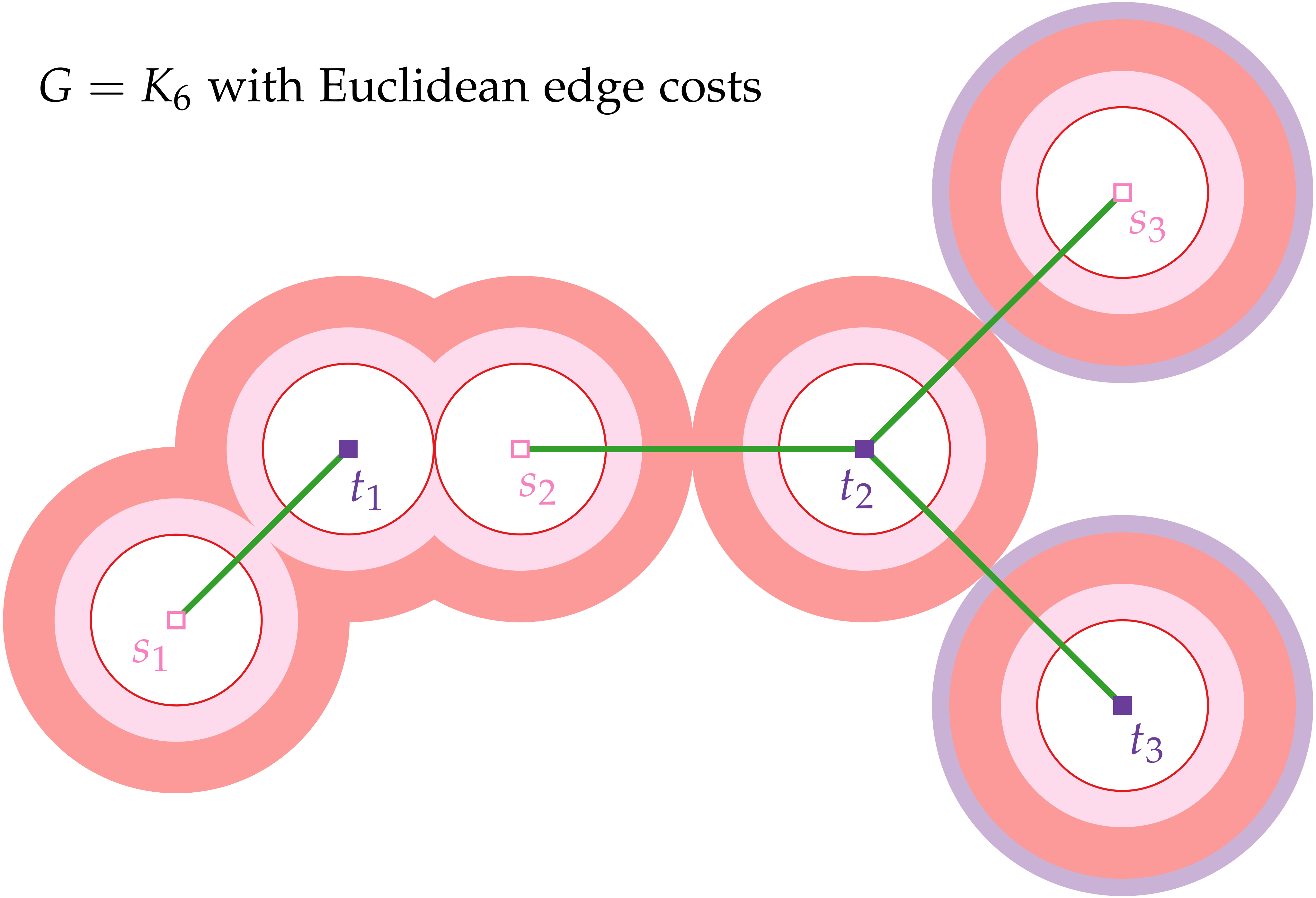

Approximation Algorithms

Lecture 12:
SteinerForest via Primal–Dual

Part V:
Structure Lemma

Structure Lemma

Lemma. For the set $\mathcal{C}$ in any iteration of the algorithm:

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$$\sum_{C \in \mathcal{C}} |\delta(C) \cap F'| \leq$$

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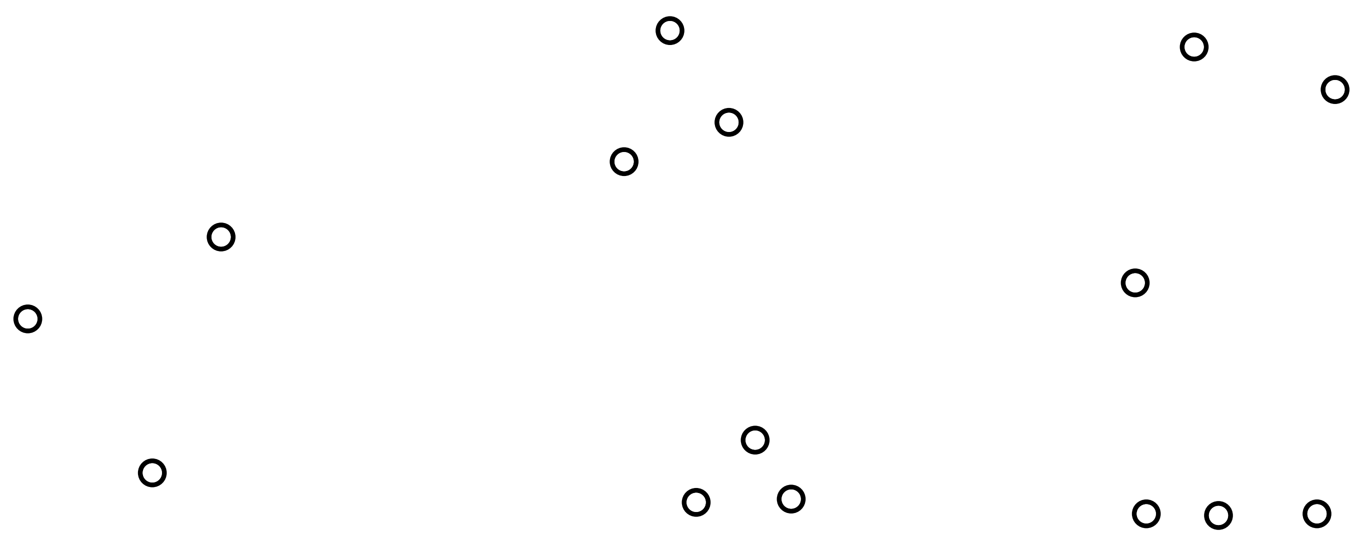

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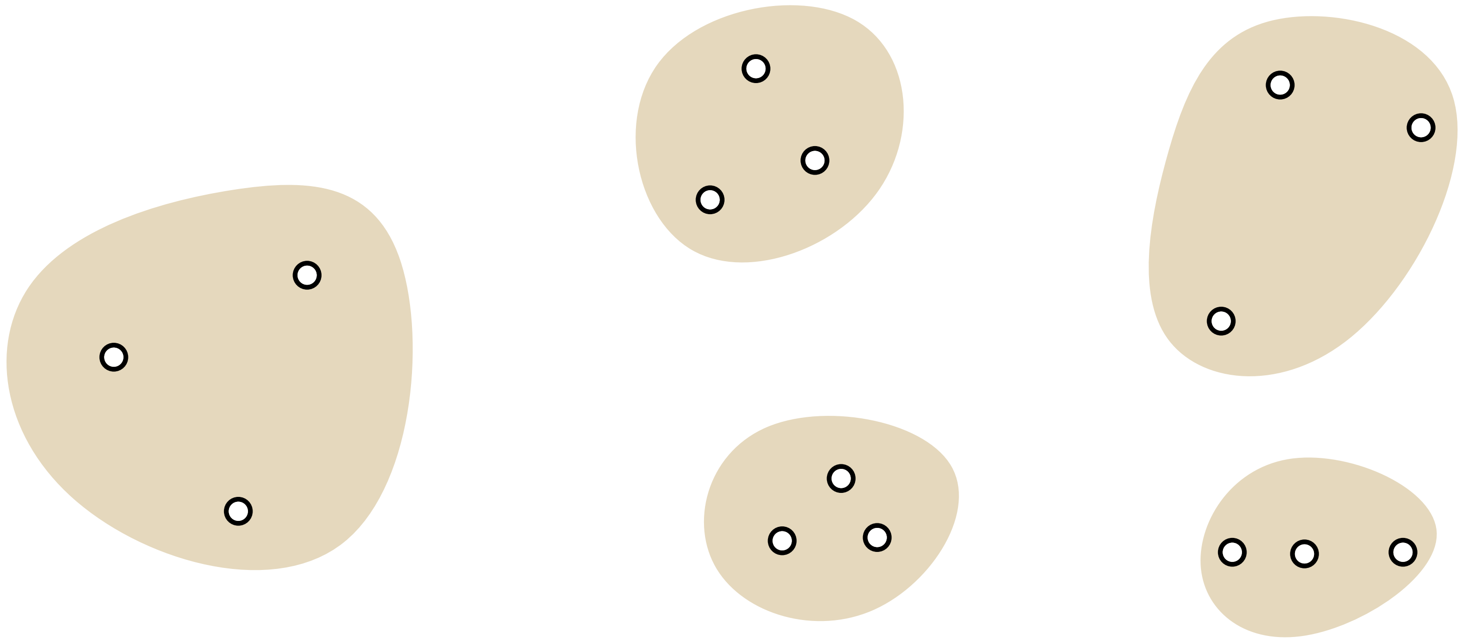

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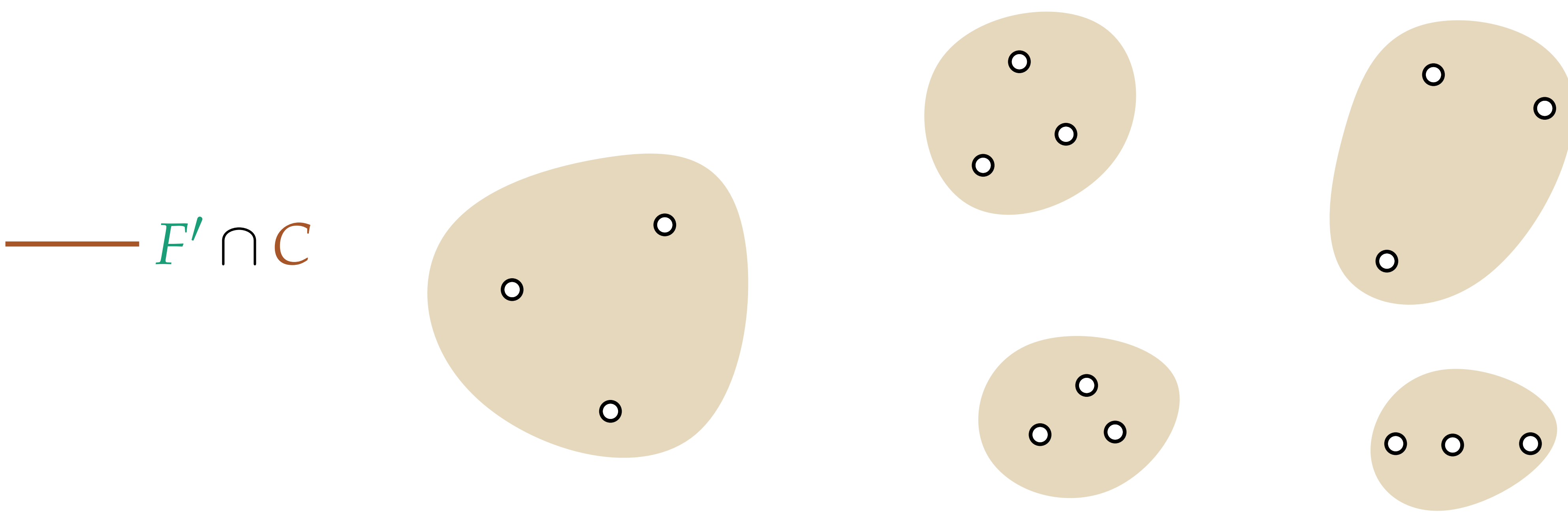

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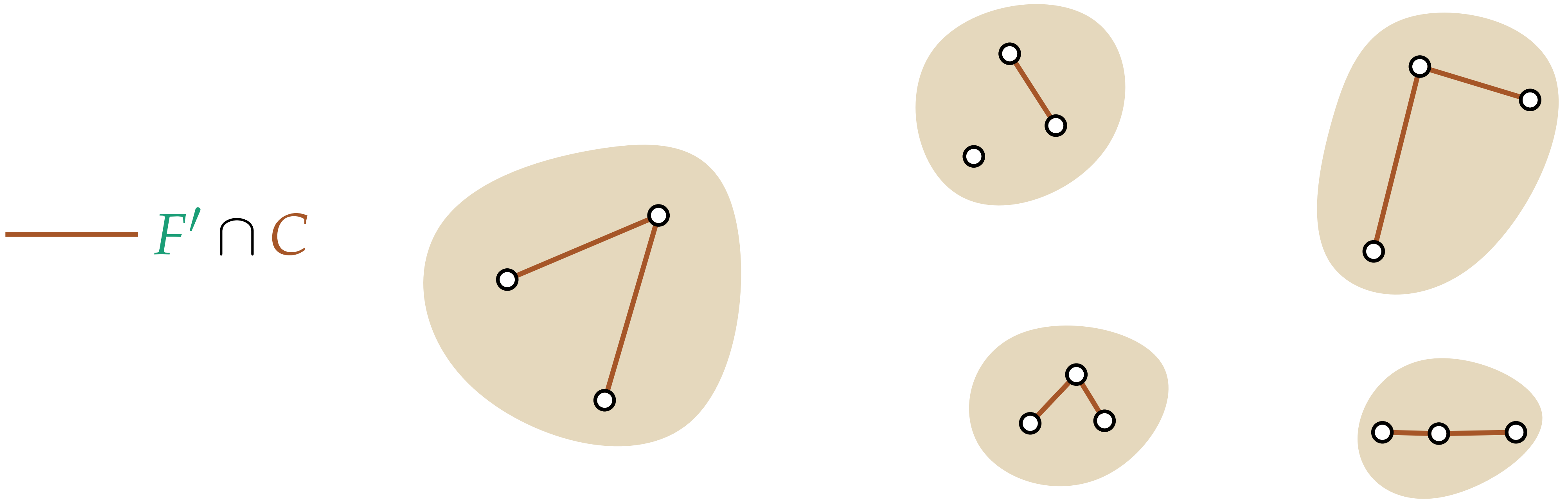

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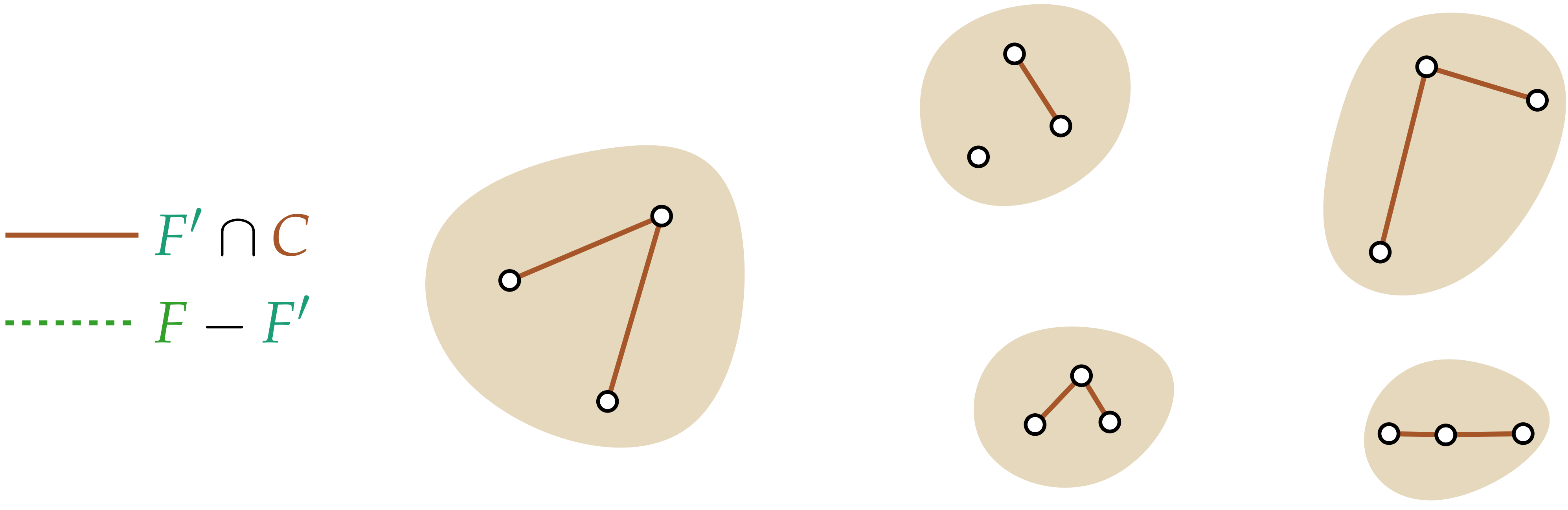

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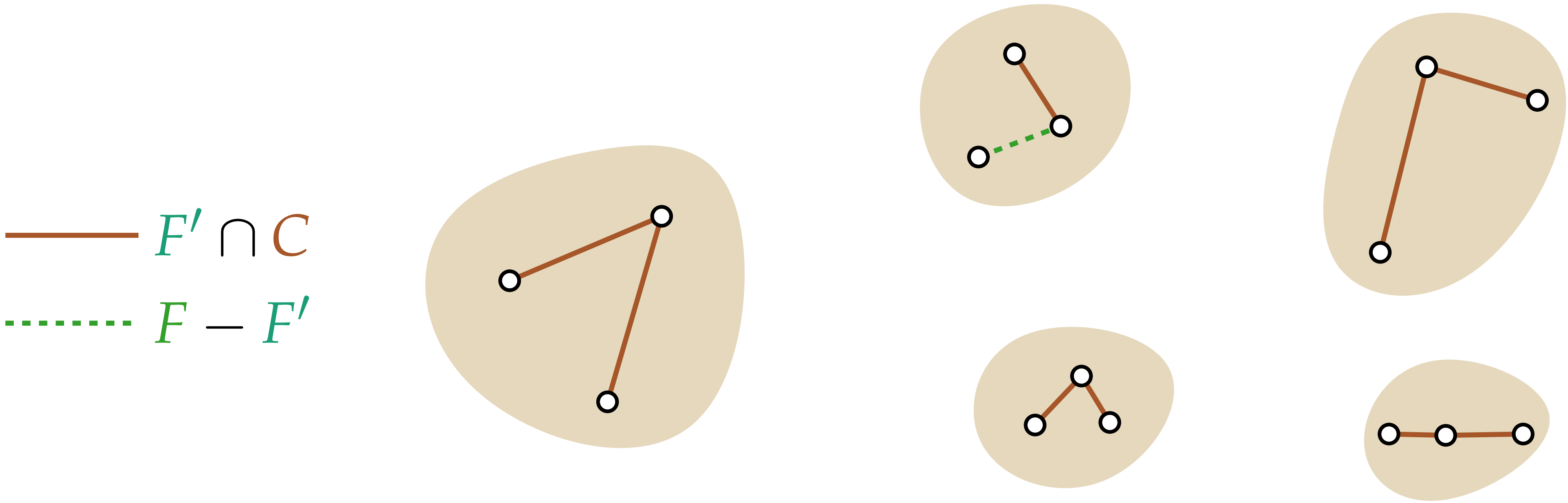

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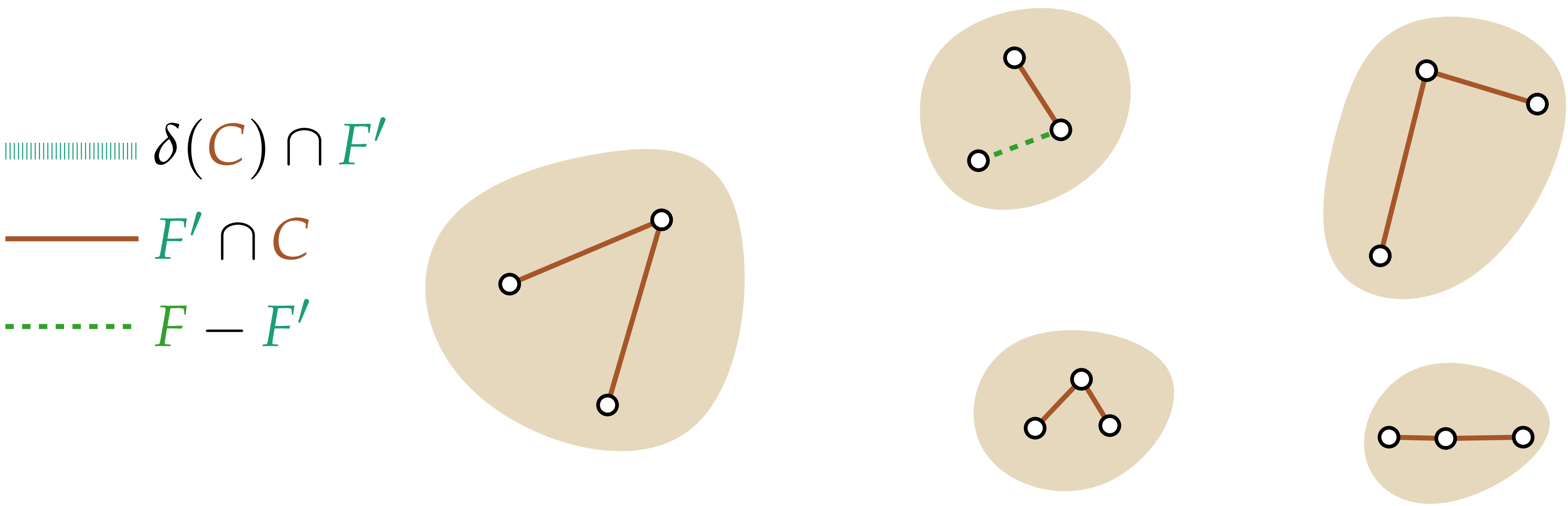

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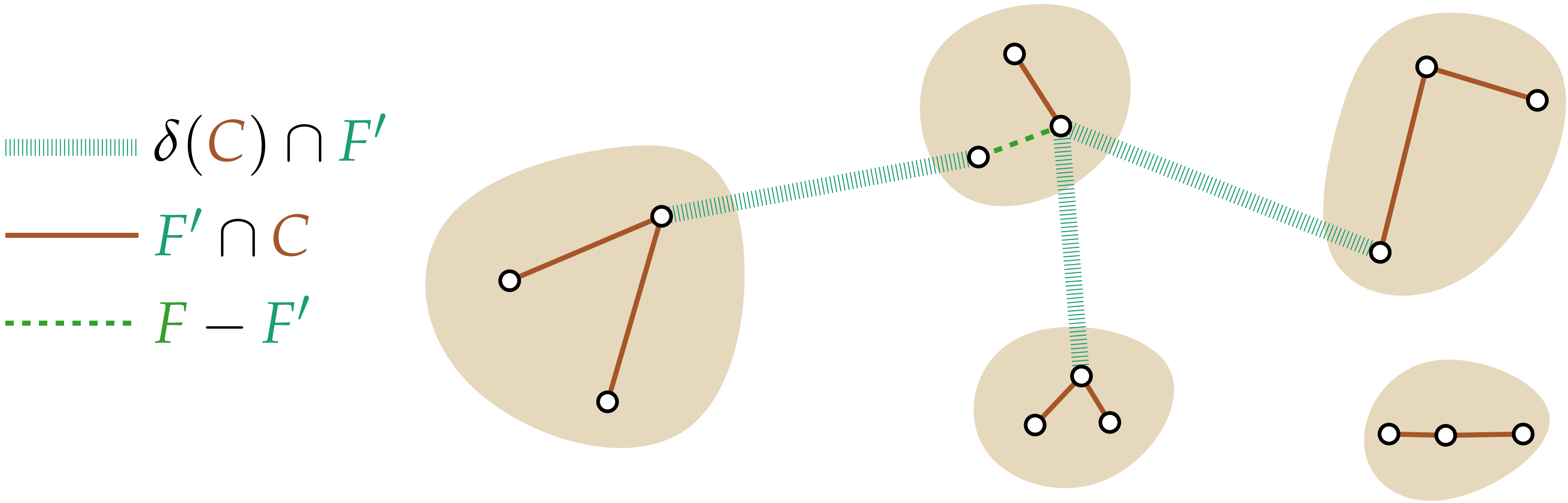

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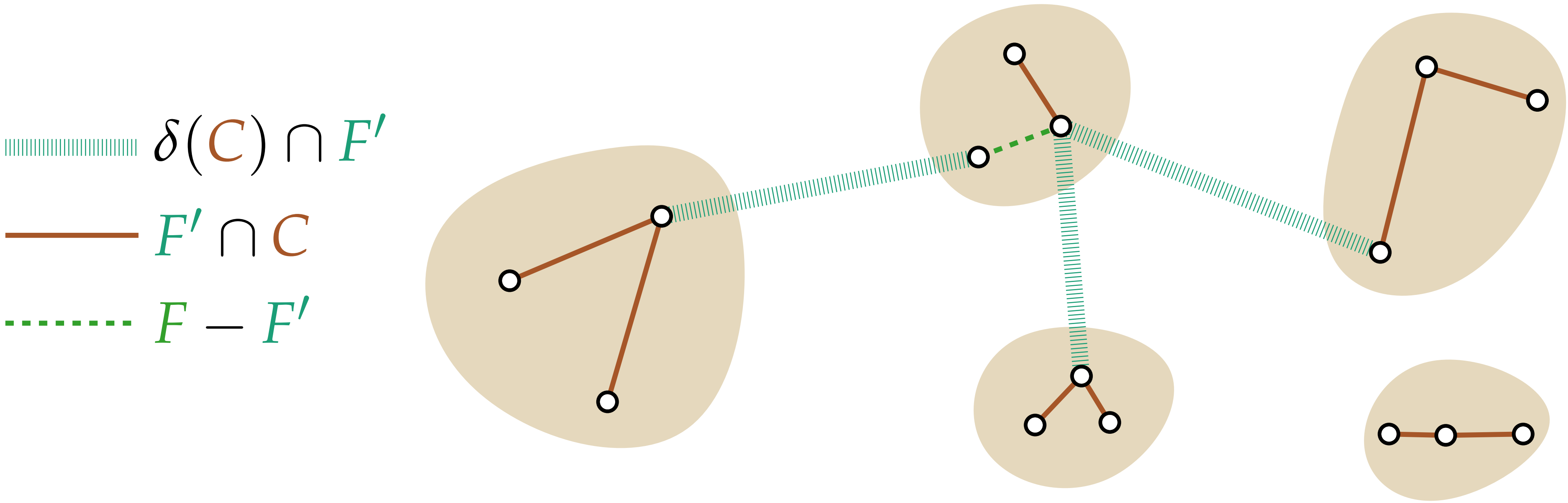
Proof. First the intuition…
Every connected component C of F is a forest in F'.
$\rightsquigarrow$ average degree $\leq$



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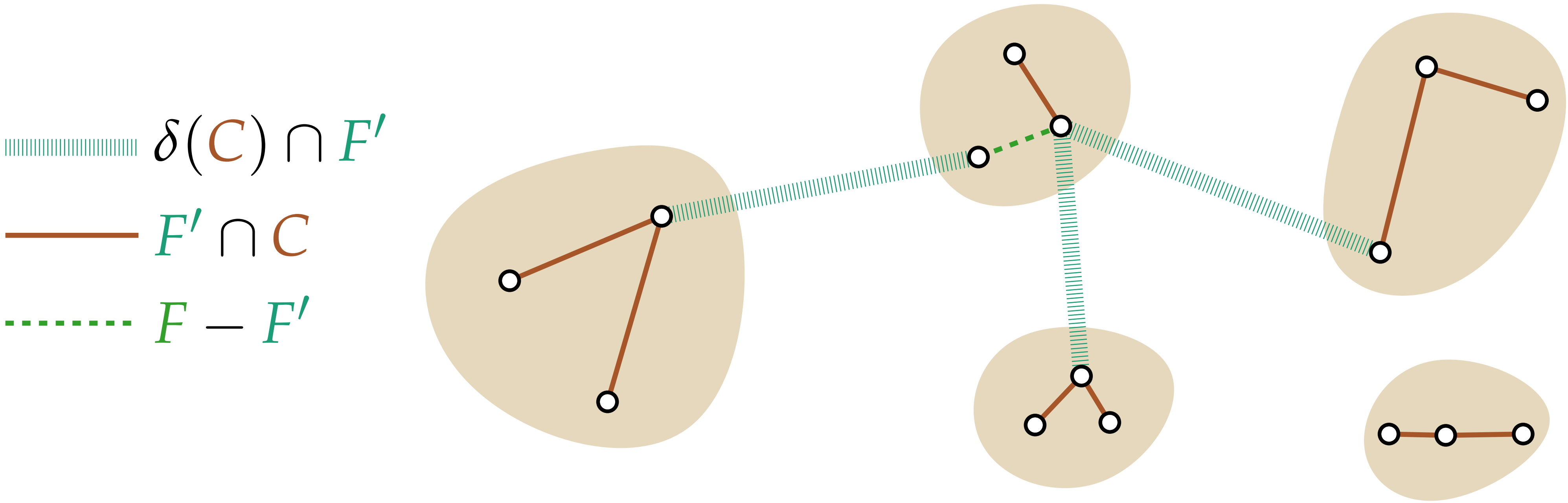
Every connected component C of F is a forest in F'.

$\rightsquigarrow$ average degree ≤ 2



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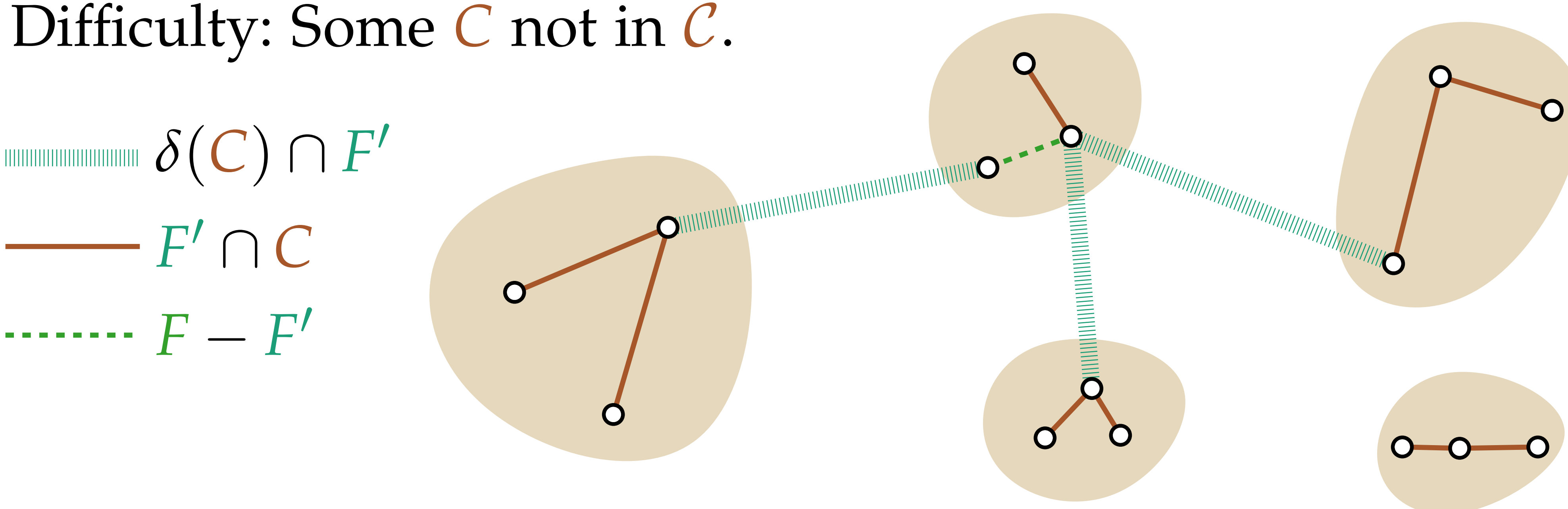
Difficulty: Some C not in $\mathcal{C}$.



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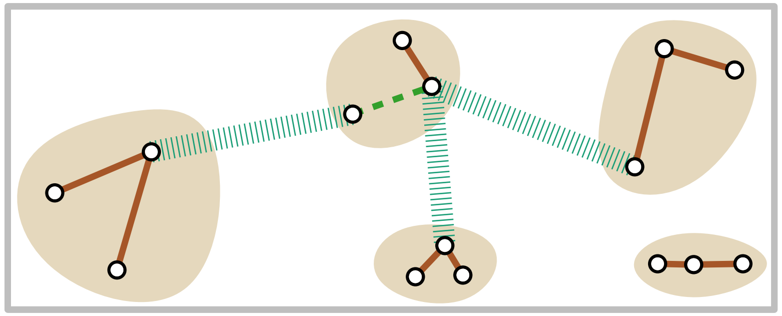

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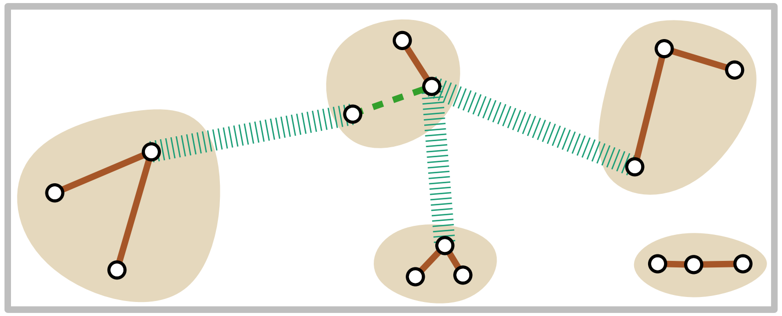
For $i = 1, \ldots, \ell$, consider i-th iteration (when e_i was added to F).



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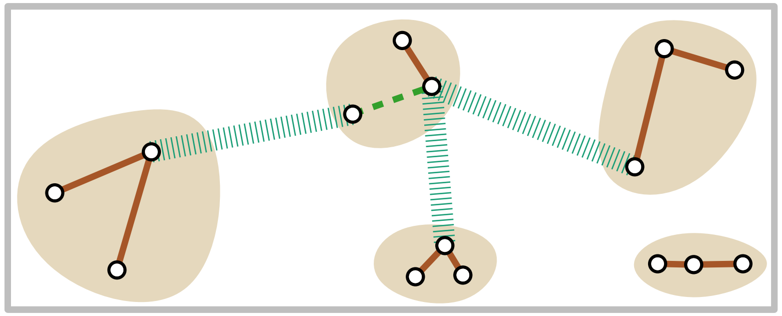

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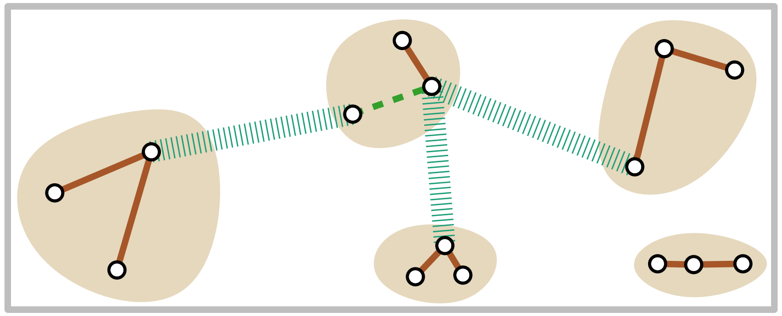

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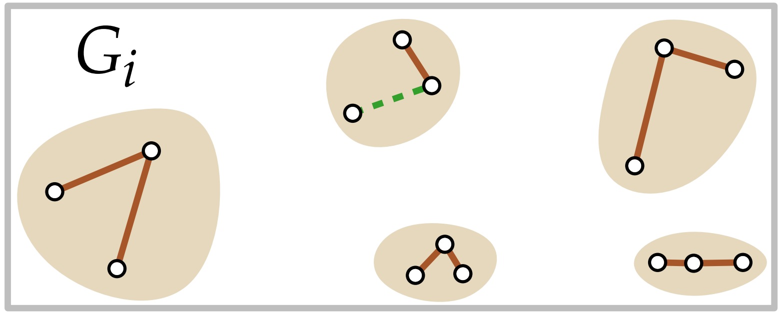

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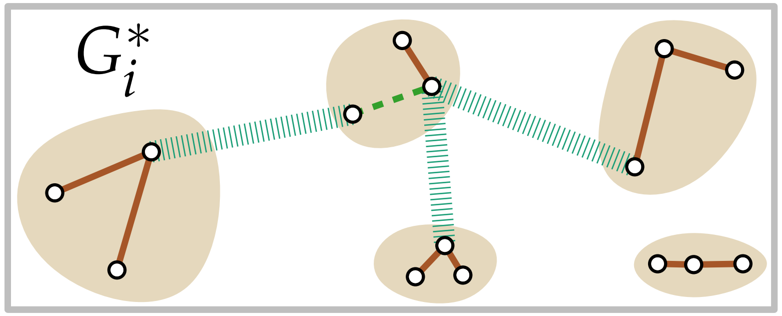

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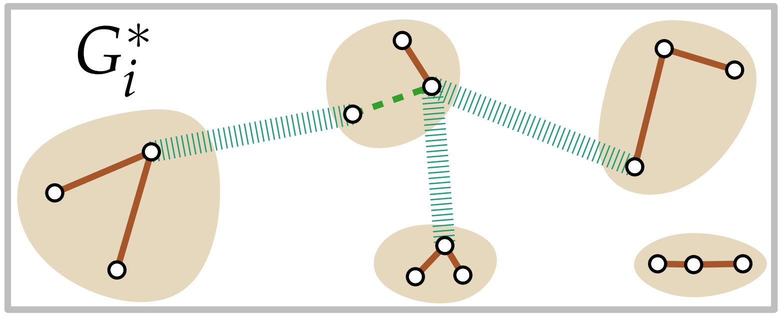
Contract every component C of G_i in G_i^* to a single vertex $\rightsquigarrow G_i'$.



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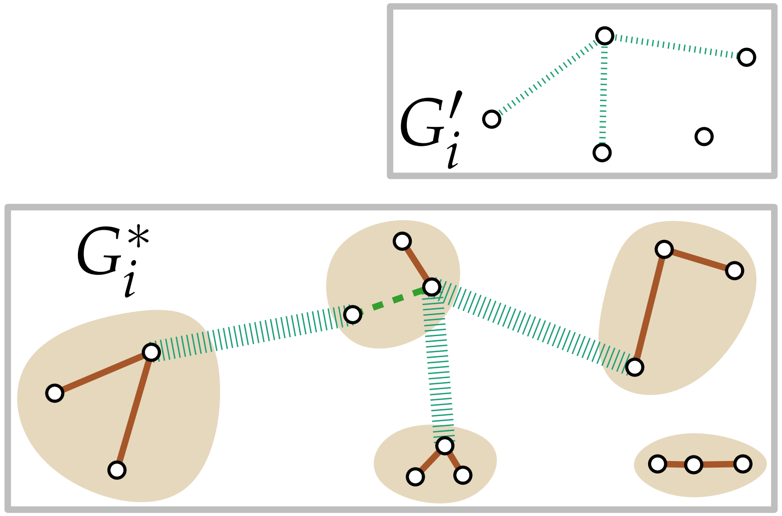

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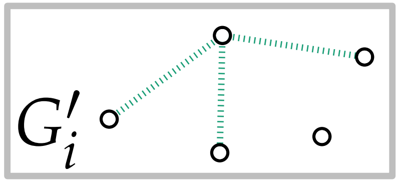

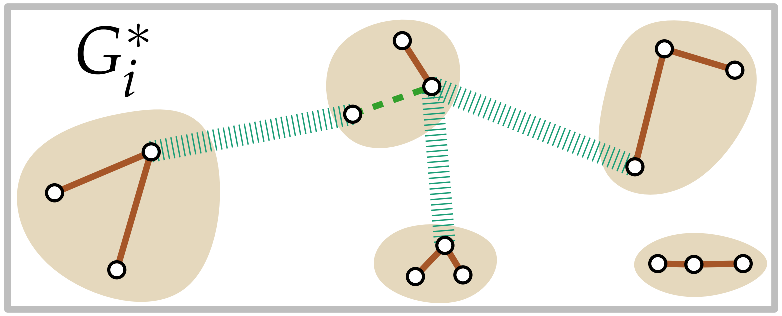

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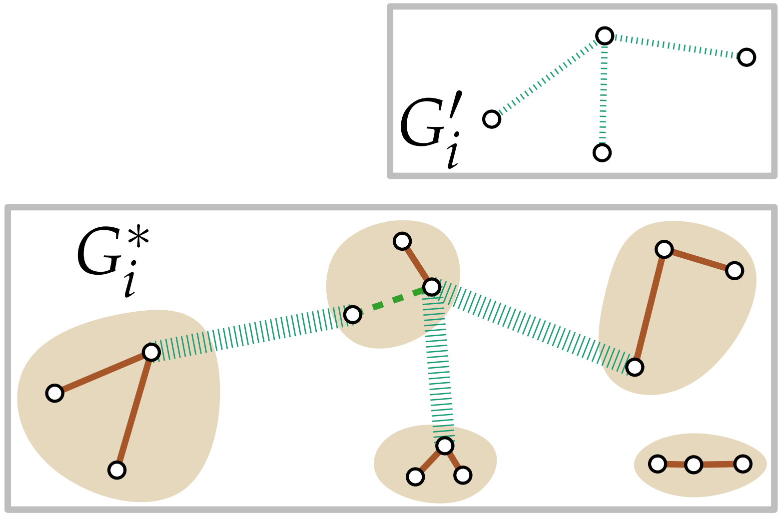

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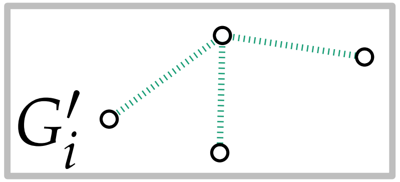
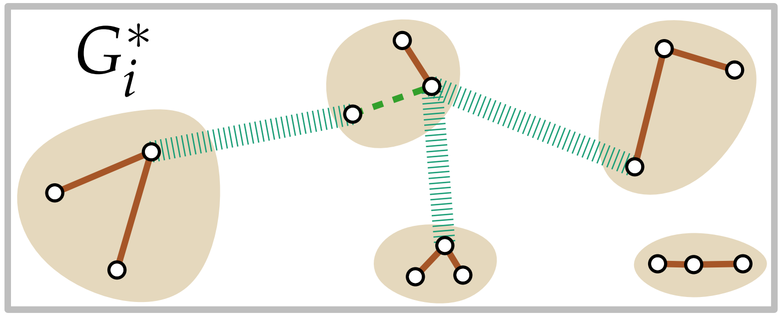
Claim. G_i' is a forest.





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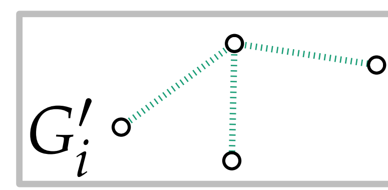

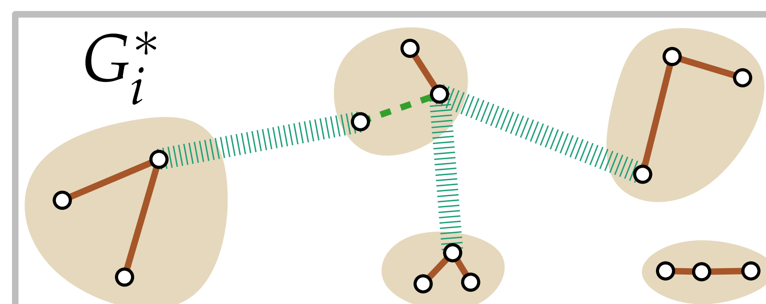

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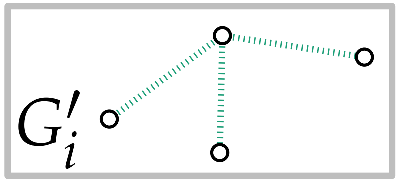

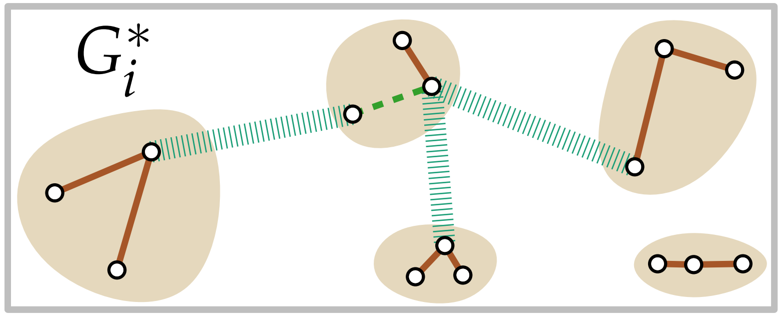

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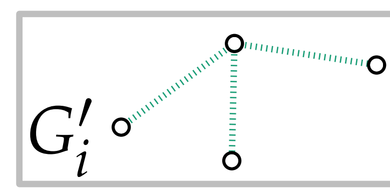

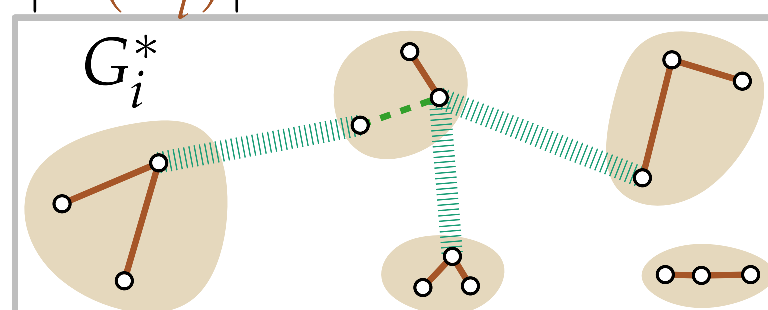

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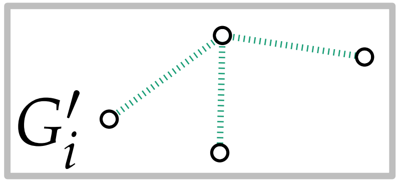

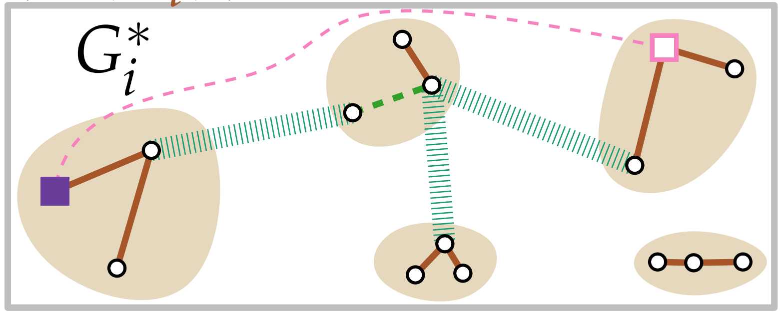

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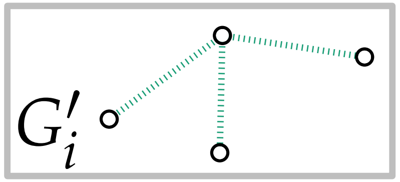

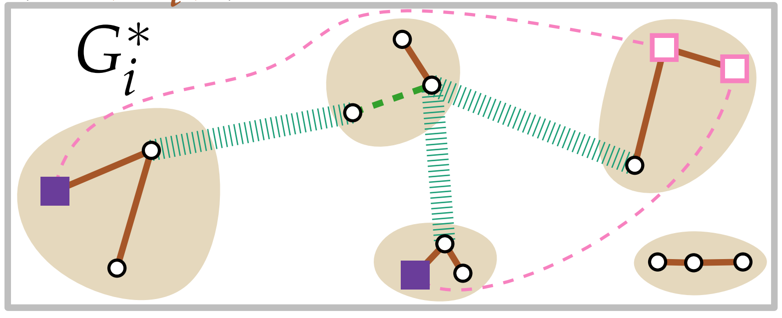

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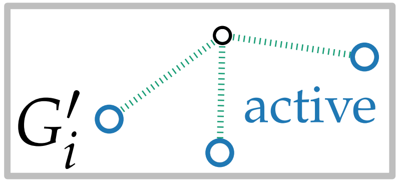

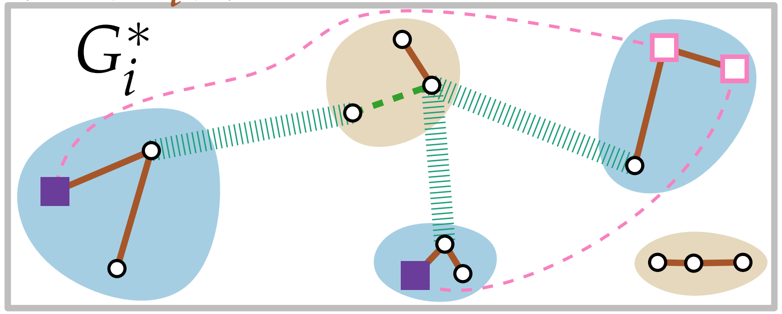

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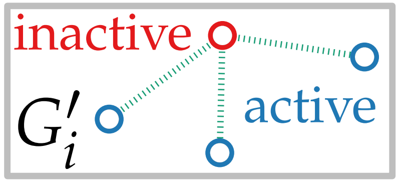

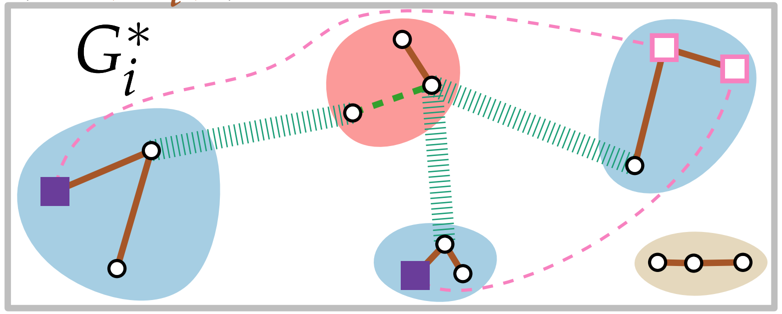

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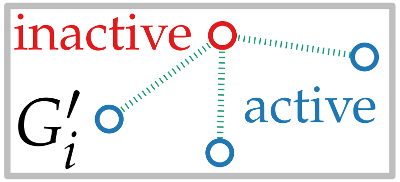
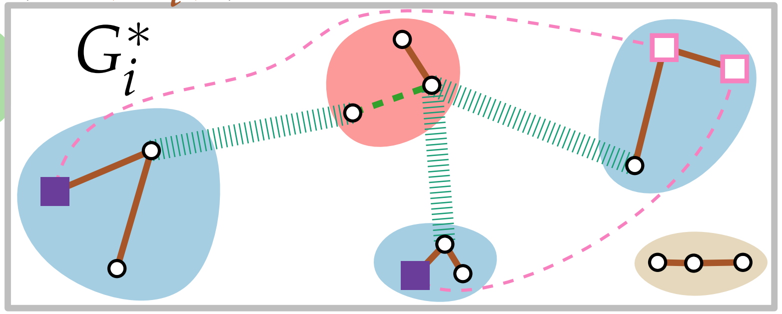


Claim. Inactive vertices have degree ≥ 2.



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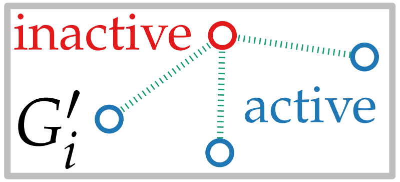

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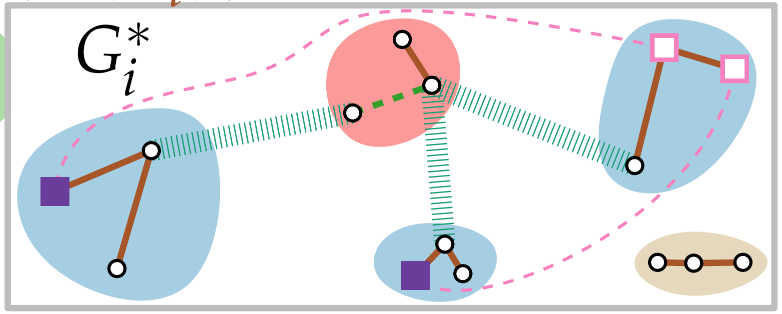

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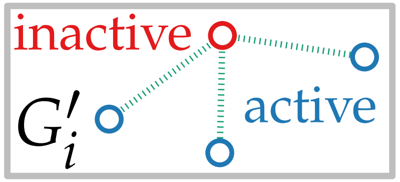

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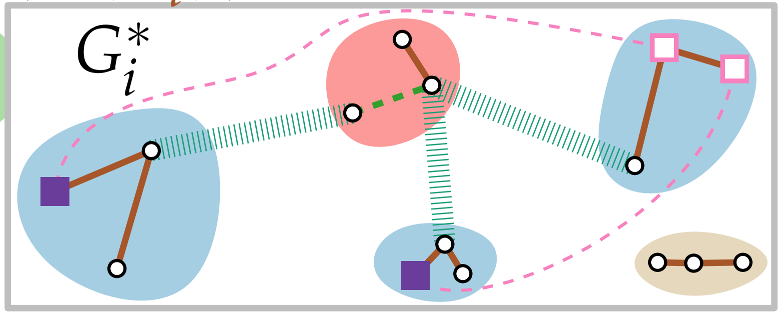

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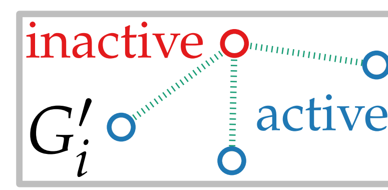

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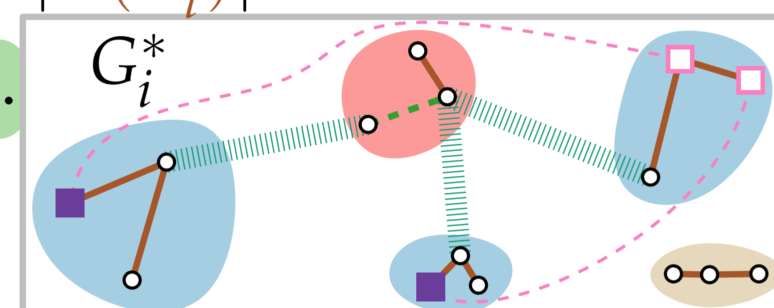

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Approximation Algorithms

Lecture 12:
SteinerForest via Primal–Dual

Part VI:
Analysis

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Theorem. The Primal–Dual algorithm with synchronized increases yields a 2-approximation for SteinerForest.

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As mentioned before,

$$\sum_{e \in F'} c_e \overset{\text{CS}}{=} \sum_{e \in F'} \sum_{S \colon e \in \delta(S)} y_S = \sum_{S} |\delta(S) \cap F'| \cdot y_S.$$

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Structure lemma $\Rightarrow (*)$ also holds after the current iteration. □

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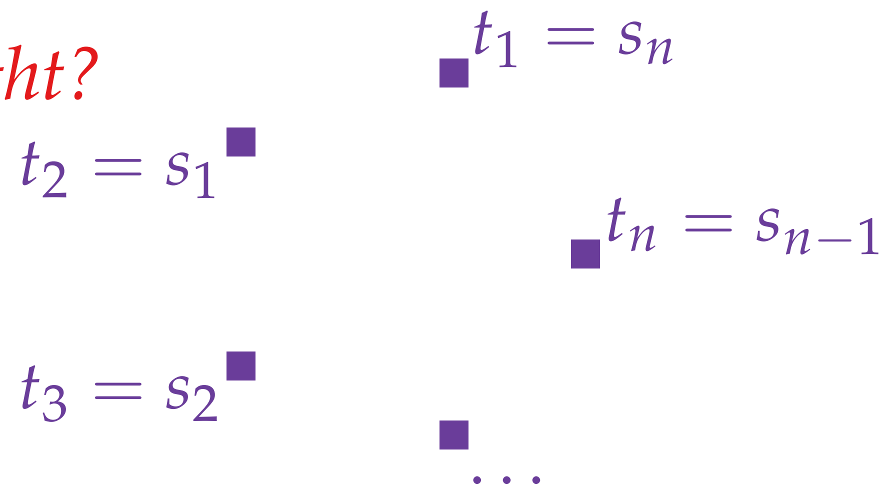

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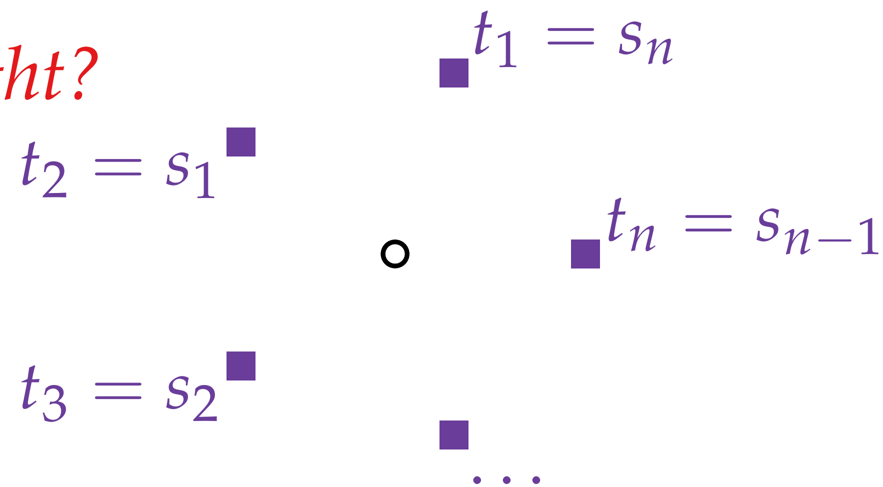

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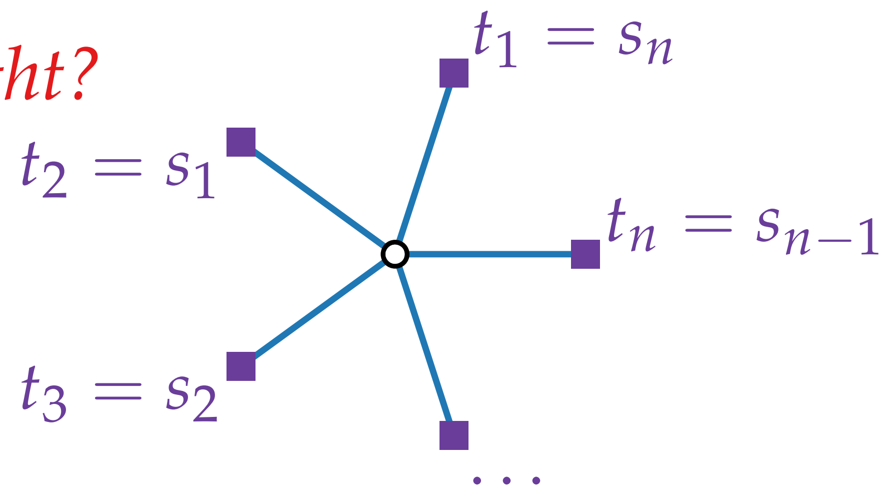

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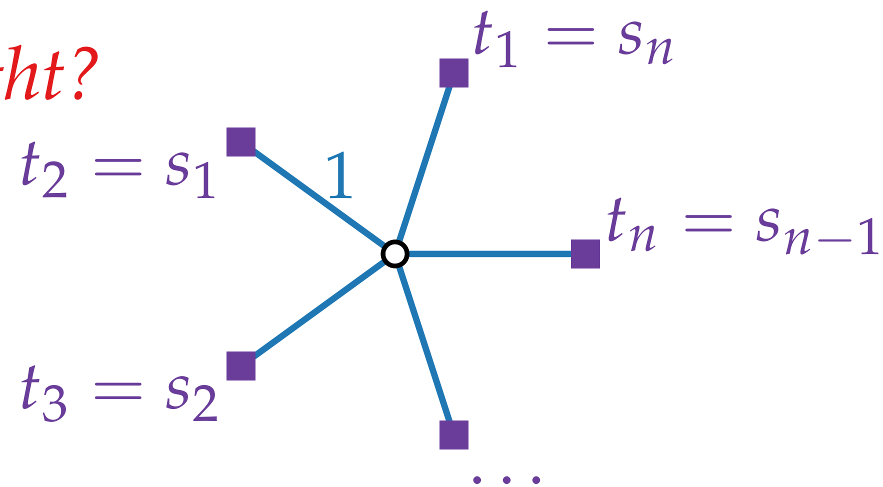

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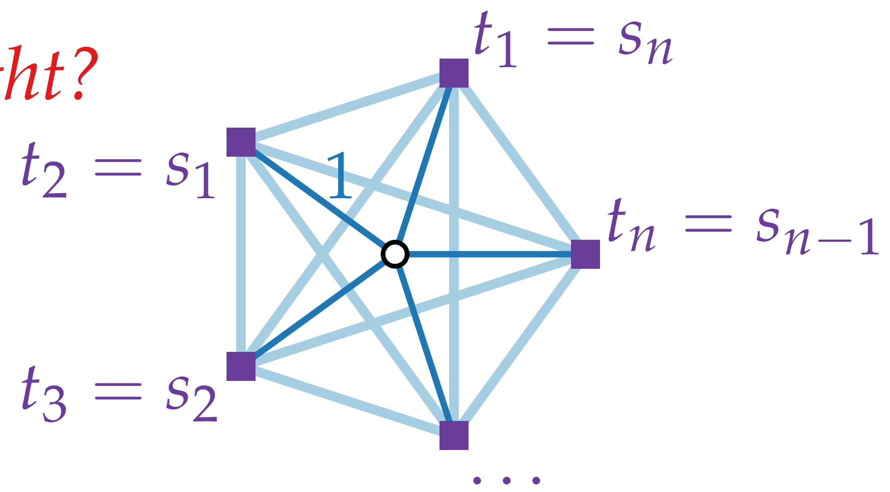

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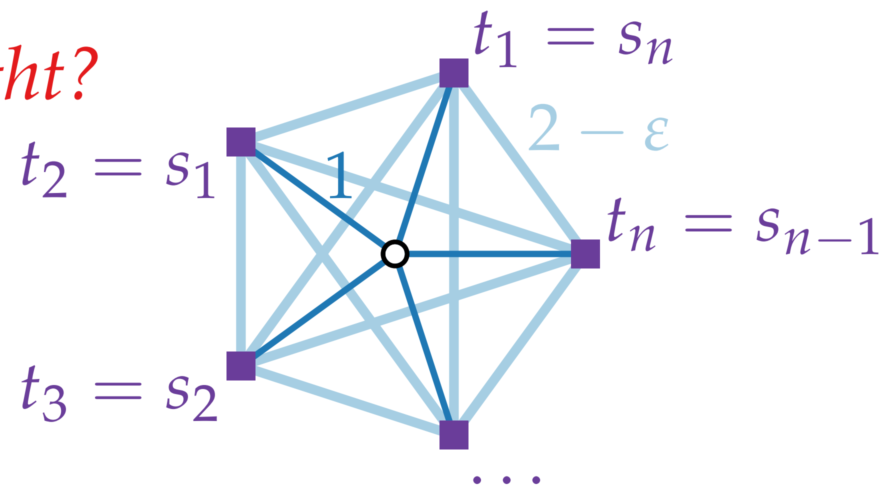

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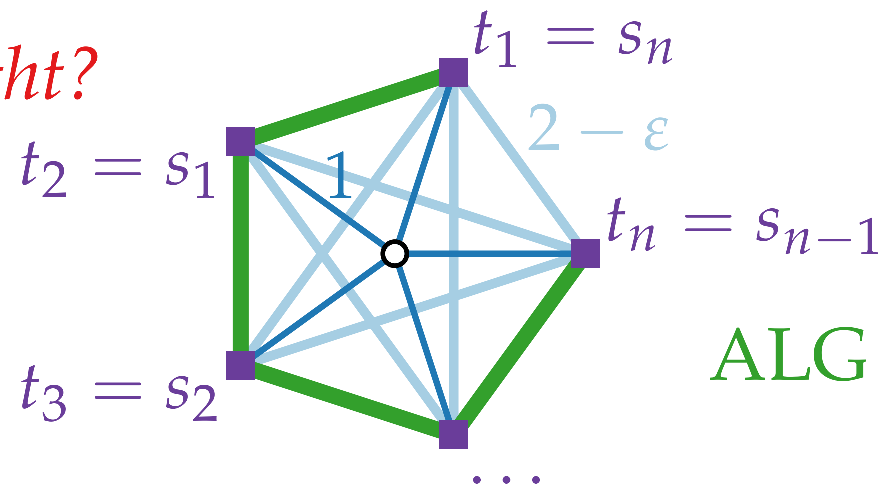

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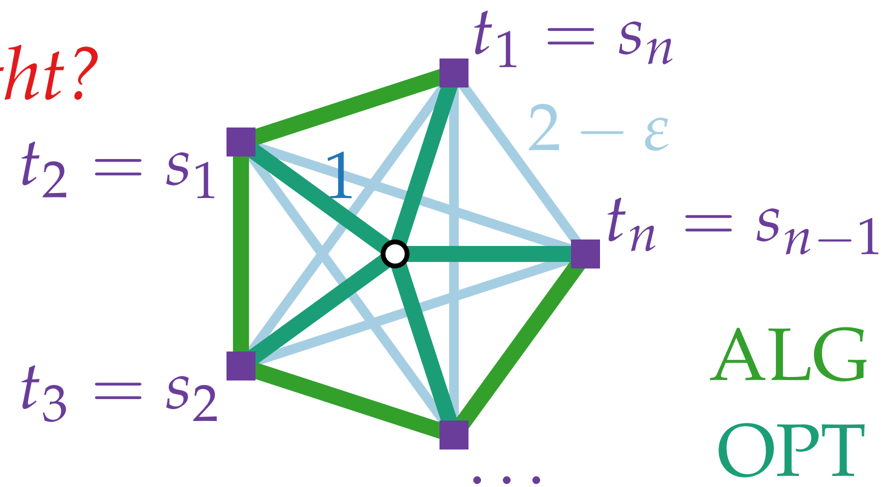

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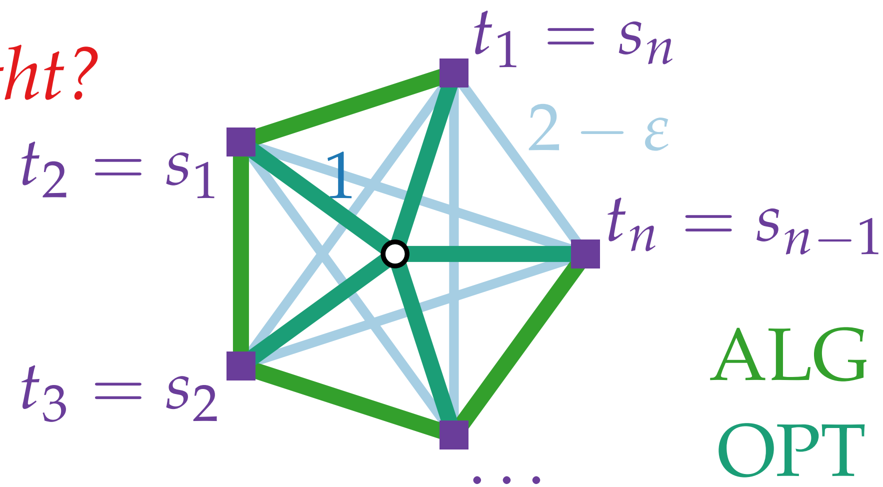

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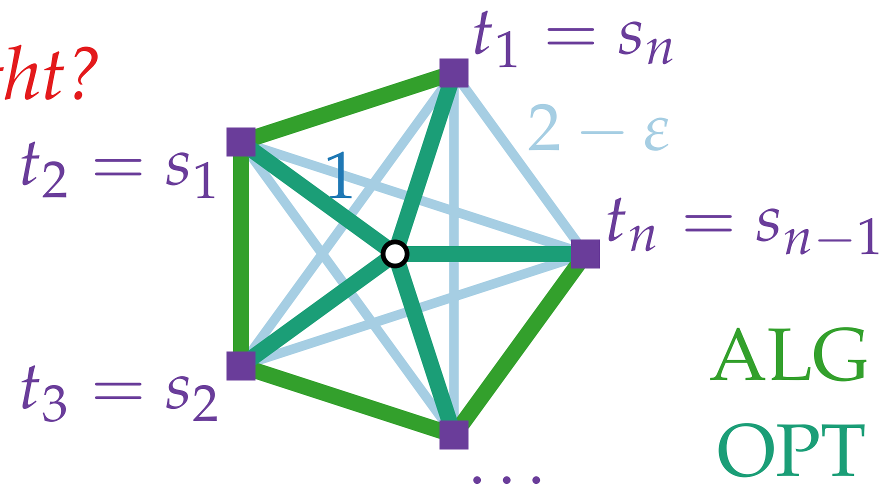

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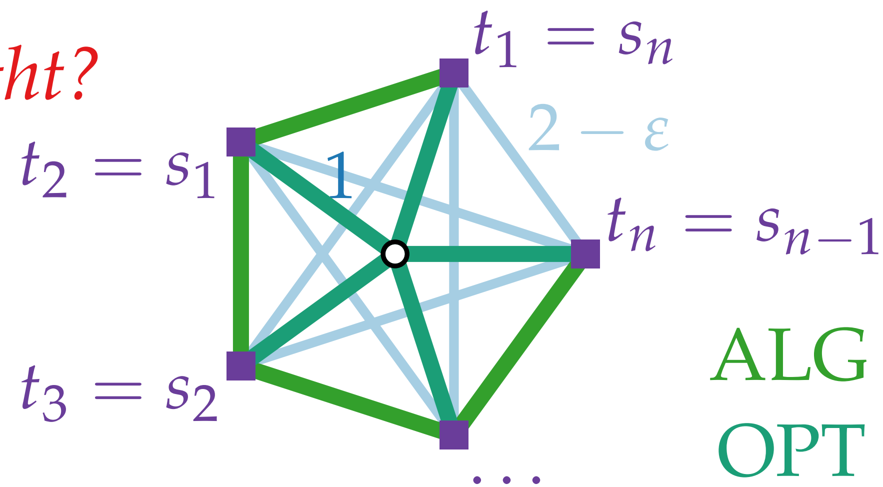

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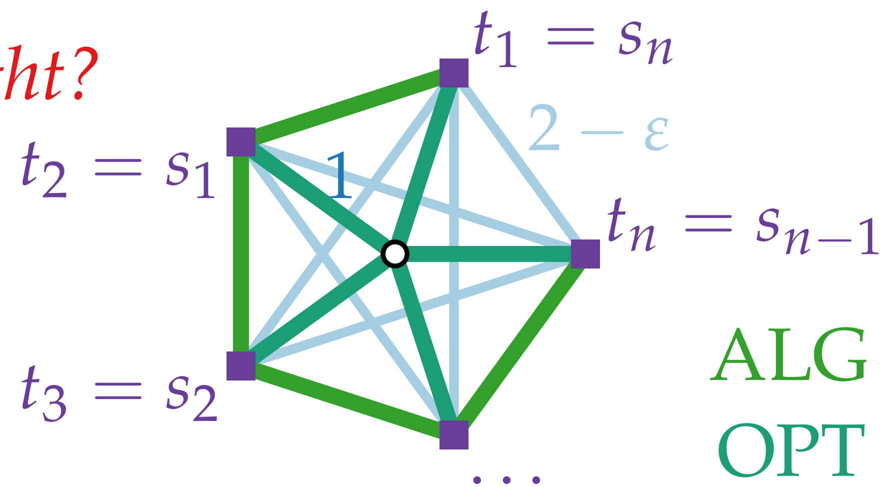

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SteinerForest (as SteinerTree) cannot be approximated within factor $\frac{96}{95} \approx 1.0105$ (unless $P = NP$). [Chlebík, Chlebíková '08]