

Approximation Algorithms

Lecture 11:

MAXSAT via Randomized Rounding

Part I:

Maximum Satisfiability (MAXSAT)

Maximum Satisfiability (MAXSAT)

Given: Boolean variables x_1, \dots, x_n ,
clauses C_1, \dots, C_m with weight w_1, \dots, w_m .

Task: Find an assignment of the variables x_1, \dots, x_n
such that the total weight of the satisfied clauses
is maximized.

Literal: Variable or negation of variable – e.g. $x_1, \overline{x_1}$

Clause: Disjunction of literals – e.g. $x_1 \vee \overline{x_2} \vee x_3$

Length of a clause: Number of literals

Problem is NP-hard since SATISFIABILITY (SAT) is NP-hard:
Is a given formula in conjunctive normal form satisfiable?

E.g. $(x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee \overline{x_4})$.

Approximation Algorithms

Lecture 11:

MAXSAT via Randomized Rounding

Part II:

A Simple Randomized Algorithm

A Simple Randomized Algorithm

Theorem. Independently setting each **variable** to 1 (true) with probability $1/2$ provides an expected $1/2$ -approximation for MAXSAT.

Proof.

Let $Y_j \in \{0, 1\}$ be a random variable for the truth value of **clause** C_j .

Let W be a random variable for the total **weight** of the satisfied **clauses**.

$$E[W] = E \left[\sum_{j=1}^m w_j Y_j \right] = \sum_{j=1}^m w_j E[Y_j] = \sum_{j=1}^m w_j \Pr[C_j \text{ satisfied}]$$

$$l_j := \text{length}(C_j). \Rightarrow \Pr[C_j \text{ satisfied}] = 1 - (1/2)^{l_j} \geq 1/2.$$

$$\text{Thus, } E[W] \geq 1/2 \sum_{j=1}^m w_j \geq \text{OPT}/2. \quad \square$$

Approximation Algorithms

Lecture 11:

MAXSAT via Randomized Rounding

Part III:

Derandomization by Conditional Expectation

Derandomization by Conditional Expectation

Theorem. The previous algorithm can be derandomized, i.e., there is a deterministic $1/2$ -approximation algorithm for MAXSAT.

Proof.

We set x_1 deterministically, but x_2, \dots, x_n randomly.

Namely: set $x_1 = 1 \Leftrightarrow E[W|x_1 = 1] \geq E[W|x_1 = 0]$.

$$E[W] = (E[W|x_1 = 0] + E[W|x_1 = 1])/2. \quad \text{[because of original random choice of } x_1 \text{]}$$

If x_1 was set to $b_1 \in \{0, 1\}$,
then $E[W|x_1 = b_1] \geq E[W] \geq \text{OPT}/2$.

Derandomization by Conditional Expectation

Assume (by induction) that we have set x_1, \dots, x_i to b_1, \dots, b_i such that

$$E[W | x_1 = b_1, \dots, x_i = b_i] \geq \text{OPT}/2$$

Then (similar to the base case):

$$\begin{aligned} & (E[W | x_1 = b_1, \dots, x_i = b_i, x_{i+1} = 0] \\ & + E[W | x_1 = b_1, \dots, x_i = b_i, x_{i+1} = 1]) / 2 \\ & = E[W | x_1 = b_1, \dots, x_i = b_i] \geq \text{OPT}/2 \end{aligned}$$

So we set $x_{i+1} = 1 \Leftrightarrow$

$$\begin{aligned} & E[W | x_1 = b_1, \dots, x_i = b_i, x_{i+1} = 1] \\ & \geq E[W | x_1 = b_1, \dots, x_i = b_i, x_{i+1} = 0] \end{aligned}$$

Derandomization by Conditional Expectation

Thus, the algorithm can be derandomized if the conditional expectation can be computed efficiently!

Consider a partial assignment $x_1 = b_1, \dots, x_i = b_i$ and a clause C_j .

If C_j is already satisfied, then it contributes exactly w_j to $E[W | x_1 = b_1, \dots, x_i = b_i]$.

If C_j is not yet satisfied and contains k unassigned variables, then it contributes exactly $w_j(1 - (1/2)^k)$ to $E[W | x_1 = b_1, \dots, x_i = b_i]$.

The conditional expectation is simply the sum of the contributions from each clause. □

Summary

Using *Conditional Expectation* is a standard procedure with which many randomized algorithms can be derandomized.

Requirement: respective conditional probabilities can be appropriately estimated for each random decision.

The algorithm simply chooses the best option at each step.

Quality of the obtained solution is then at least as high as the expected value.

The algorithm iteratively sets the variables and greedily decides for the locally best assignment.

Global optimization?

Approximation Algorithms

Lecture 11:

MAXSAT via Randomized Rounding

Part IV:

Randomized Rounding

An ILP

$$\begin{aligned} &\text{maximize} && \sum_{j=1}^m w_j z_j \\ &\text{subject to} && \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j \quad \text{for } j = 1, \dots, m \\ &&& y_i \in \{0, 1\}, \quad \text{for } i = 1, \dots, n \\ &&& z_j \in \{0, 1\}, \quad \text{for } j = 1, \dots, m \end{aligned}$$

$$\text{where } C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i \quad \text{for } j = 1, \dots, m.$$

... and its Relaxation

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^m w_j z_j \\ \text{subject to} \quad & \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j \quad \text{for } j = 1, \dots, m \\ & 0 \leq y_i \leq 1, \quad \text{for } i = 1, \dots, n \\ & 0 \leq z_j \leq 1, \quad \text{for } j = 1, \dots, m \end{aligned}$$

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$ for $j = 1, \dots, m$.

Randomized Rounding

Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation. Independently setting each variable x_i to 1 with probability y_i^* provides a $(1 - 1/e)$ -approximation for MAXSAT.

≈ 0.63

Approximation Algorithms

Lecture 11:

MAXSAT via Randomized Rounding

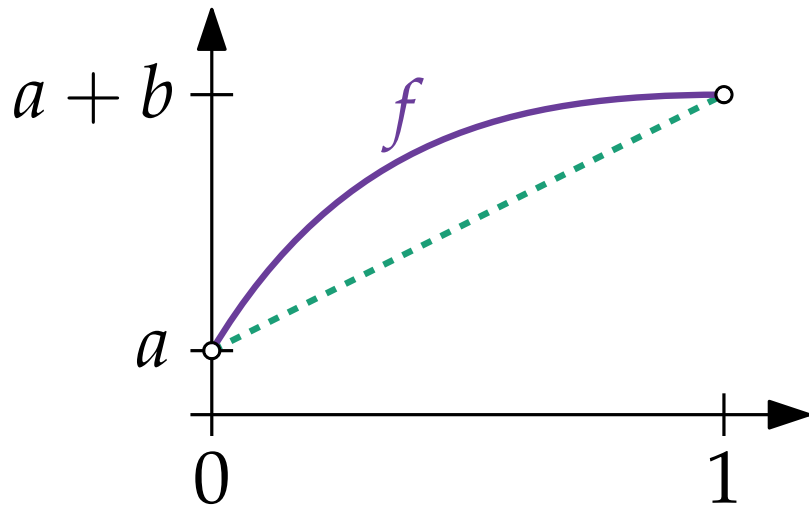
Part V:

Randomized Rounding – Proof

Mathematical Toolkit

Let f be a function that is concave on $[0, 1]$
(i.e. $f''(x) \leq 0$ on $[0, 1]$) with $f(0) = a$ and $f(1) = a + b$

$$\Rightarrow f(x) \geq bx + a \text{ for } x \in [0, 1].$$



Arithmetic–Geometric Mean Inequality (AGMI):

For all non-negative numbers a_1, \dots, a_k :

$$\left(\prod_{i=1}^k a_i \right)^{1/k} \leq \frac{1}{k} \left(\sum_{i=1}^k a_i \right)$$

Randomized Rounding (Proof)

Consider a fixed clause C_j of length l_j . Then we have:

$$\Pr[C_j \text{ not sat.}] = \prod_{i \in P_j} (1 - y_i^*) \prod_{i \in N_j} y_i^*$$

$$\left(\prod_{i=1}^k a_i \right)^{1/k} \leq \frac{1}{k} \left(\sum_{i=1}^k a_i \right)$$

AGMI

$$\begin{aligned} &\leq \left[\frac{1}{l_j} \left(\sum_{i \in P_j} (1 - y_i^*) + \sum_{i \in N_j} y_i^* \right) \right]^{l_j} \\ &= \left[1 - \frac{1}{l_j} \underbrace{\left(\sum_{i \in P_j} y_i^* + \sum_{i \in N_j} (1 - y_i^*) \right)}_{\geq z_j^* \text{ by LP constraints}} \right]^{l_j} \\ &\leq \left(1 - \frac{z_j^*}{l_j} \right)^{l_j} \end{aligned}$$

Randomized Rounding (Proof)

The function $f(z_j^*) = 1 - \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$ is concave on $[0, 1]$.

Thus

$$\Pr[C_j \text{ satisfied}] \geq f(z_j^*) \geq f(1) \cdot z_j^* + f(0)$$

$$\geq \left[1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right] z_j^*$$

$$\geq \left(1 - \frac{1}{e}\right) z_j^*$$

$$1 + x \leq e^x$$

$$x = -\frac{1}{l_j} \Rightarrow 1 - \frac{1}{l_j} \leq e^{-1/l_j}$$

Randomized Rounding (Proof)

Therefore

$$\begin{aligned} E[W] &= \sum_{j=1}^m \Pr[C_j \text{ satisfied}] \cdot w_j \\ &\geq \left(1 - \frac{1}{e}\right) \boxed{\sum_{j=1}^m w_j z_j^*} \text{ LP objective function} \\ &= \left(1 - \frac{1}{e}\right) \text{OPT}_{\text{LP}} \\ &\geq \left(1 - \frac{1}{e}\right) \text{OPT} \end{aligned}$$

□

Theorem. The previous algorithm can be derandomized by the method of conditional expectation.

Approximation Algorithms

Lecture 11:

MAXSAT via Randomized Rounding

Part VI:

Combining the Algorithms

Take the better of the two solutions!

Theorem. The better solution among the randomized algorithm and the randomized LP-rounding algorithm provides a $3/4$ -approximation for MAXSAT .

Proof.

We use another probabilistic argument.

With probability $1/2$, choose the solution of the first algorithm; otherwise the solution of the second algorithm.

The better solution is at least as good as the expectation of the above algorithm.

Take the better of the two solutions!

The probability that clause C_j is satisfied is at least:

$$\frac{1}{2} \left[\underbrace{\left(1 - \left(1 - \frac{1}{l_j} \right)^{l_j} \right)}_{\text{LP-rounding}} + \underbrace{\left(1 - 2^{-l_j} \right)}_{\text{rand. alg.}} \right] z_j^* \geq \underbrace{\frac{3}{4}}_{\text{we claim!}} z_j^*.$$

(The rest follows similarly as in the proofs of the previous two theorems by linearity of expectation.)

For $l_j \in \{1, 2\}$, a simple calculation yields exactly $\frac{3}{4} z_j^*$.

For $l_j \geq 3$, $1 - (1 - 1/l_j)^{l_j} \geq (1 - 1/e)$ and $1 - 2^{-l_j} \geq \frac{7}{8}$.

Thus, we have at least:

$$\frac{1}{2} \left[\left(1 - \frac{1}{e} \right) + \frac{7}{8} \right] z_j^* \approx 0.753 z_j^* \geq \frac{3}{4} z_j^* \quad \square$$

Visualization and Derandomization

- **Randomized alg.** is better for large values of l_j .
 - **Randomized LP-rounding** is better for small values of l_j
- ⇒ higher probability of satisfying clause C_j . $\Pr[C_j \text{ sat.}] / z_j^*$

The **mean** of the two solutions is at least $3/4$ for integer l_j .

The maximum is at least as large as the mean.

This algorithm, too, can be derandomized by conditional expectation.

