Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part I:
Maximum Satisfiability (MaxSat)

Alexander Wolff

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such that the total weight of the *satisfied* clauses
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E.g. $(x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee \overline{x_3} \vee x_4) \wedge (x_1 \vee \overline{x_4})$.

Approximation Algorithms

Lecture 11:
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Part II:
A Simple Randomized Algorithm

A Simple Randomized Algorithm

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Thus, $E[W] \geq 1/2 \sum_{j=1}^{m} w_j \geq \text{OPT}/2$. □

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Part III:
Derandomization by Conditional Expectation

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Theorem. The previous algorithm can be derandomized, i.e., there is a deterministic 1/2-approximation algorithm for MaxSat.

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Consider a partial assignment $x_1 = b_1, \ldots, x_i = b_i$ and a clause C_j.

If C_j is already satisfied, then it contributes exactly to $E[W | x_1 = b_1, \ldots, x_i = b_i]$.

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Consider a partial assignment $x_1 = b_1, \ldots, x_i = b_i$ and a clause C_j.

If C_j is already satisfied, then it contributes exactly w_j to $E[W|x_1 = b_1, \ldots, x_i = b_i]$.

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If C_j is not yet satisfied and contains k unassigned variables, then it contributes exactly to $E[W|x_1 = b_1, \ldots, x_i = b_i]$.

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The conditional expectation is simply the sum of the contributions from each clause. □

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Global optimization?

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Part IV:
Randomized Rounding

An ILP

maximize

subject to

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$ for $j = 1, \ldots, m$.

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$$y_i \in \{0,1\}, \qquad \text{for } i = 1,\ldots,n$$

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maximize $\displaystyle\sum_{j=1}^{m} w_j z_j$

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where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$ for $j = 1, \ldots, m$.

An ILP

$$\textbf{maximize} \quad \sum_{j=1}^{m} w_j z_j$$

$$\textbf{subject to} \quad \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \qquad \text{for } j = 1, \ldots, m$$

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... and its Relaxation

$$\begin{aligned}
&\textbf{maximize} && \sum_{j=1}^{m} w_j z_j && \\
&\textbf{subject to} && \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j && \text{for } j = 1, \ldots, m \\
& && 0 \leq y_i \leq 1, && \text{for } i = 1, \ldots, n \\
& && 0 \leq z_j \leq 1, && \text{for } j = 1, \ldots, m
\end{aligned}$$

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$ for $j = 1, \ldots, m$.

Randomized Rounding

Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation.

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≈ 0.63

Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part V:
Randomized Rounding – Proof

Mathematical Toolkit

Let f be a function that is concave on $[0, 1]$

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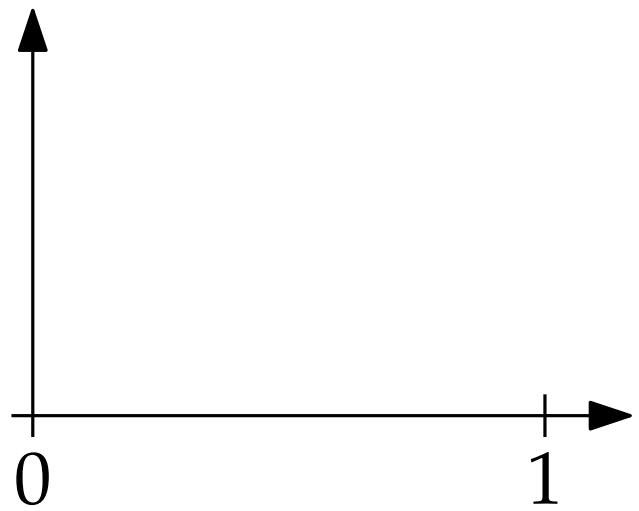

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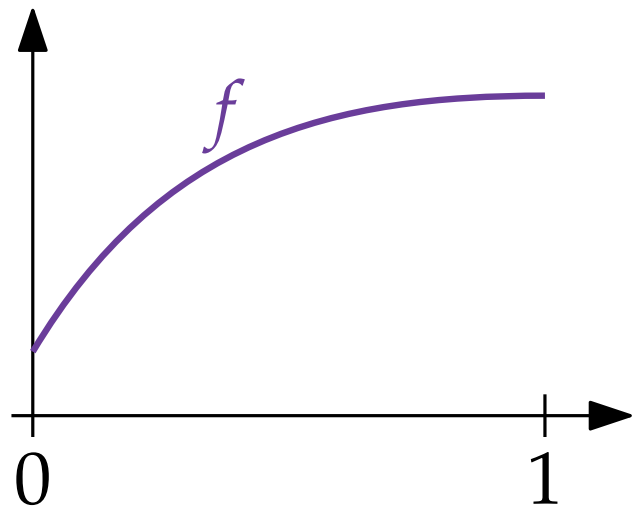

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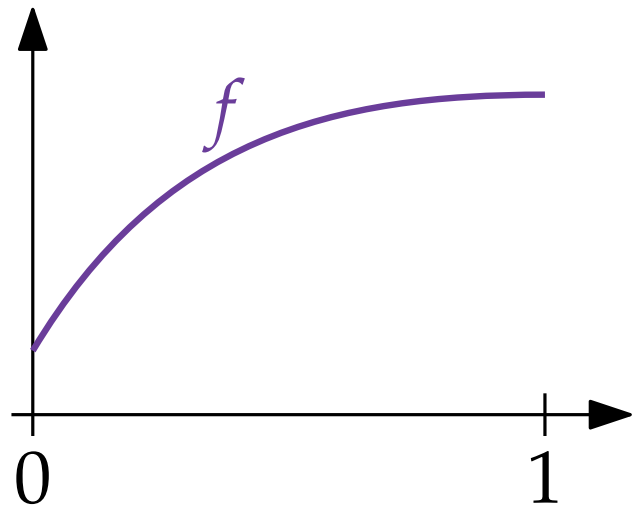

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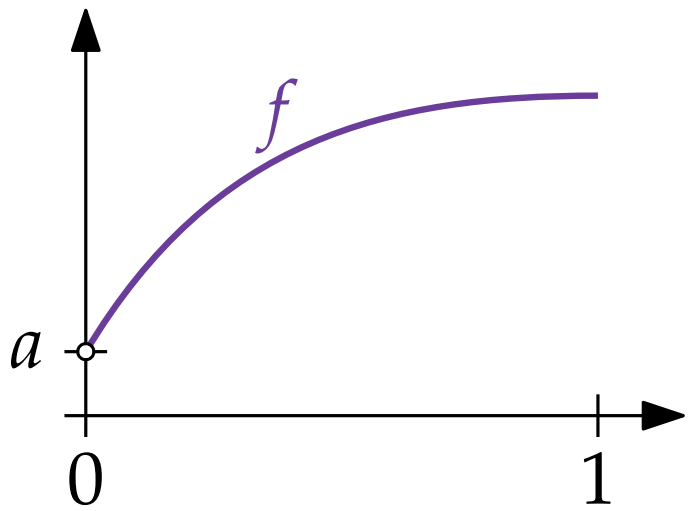

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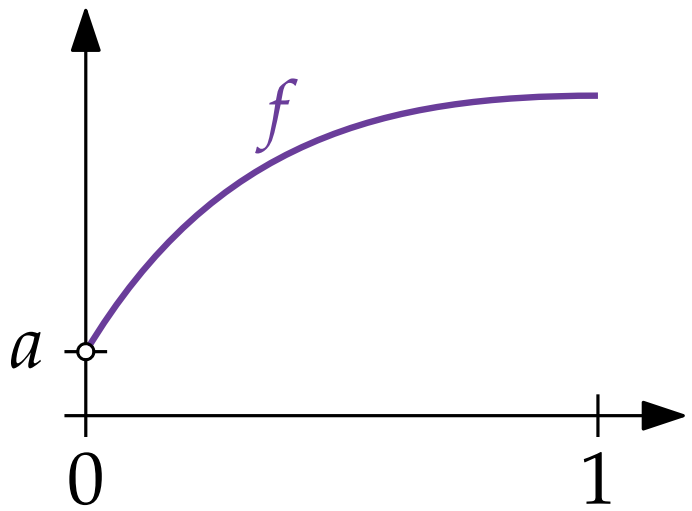

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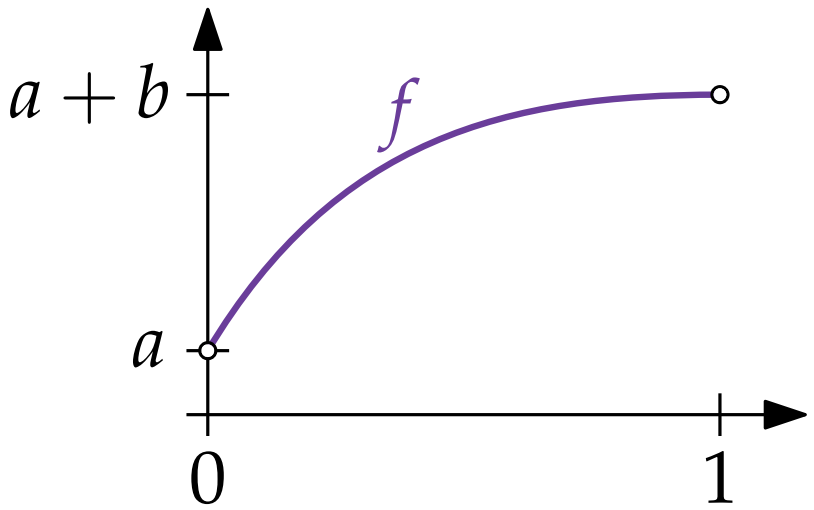

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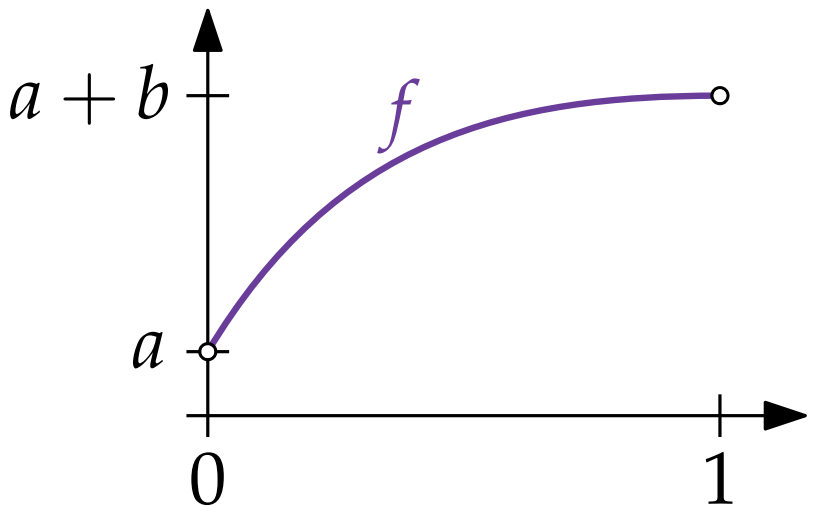

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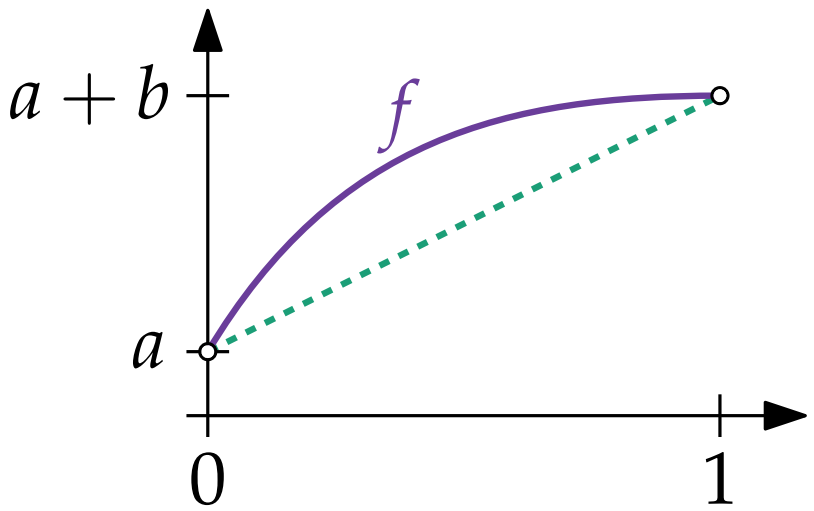

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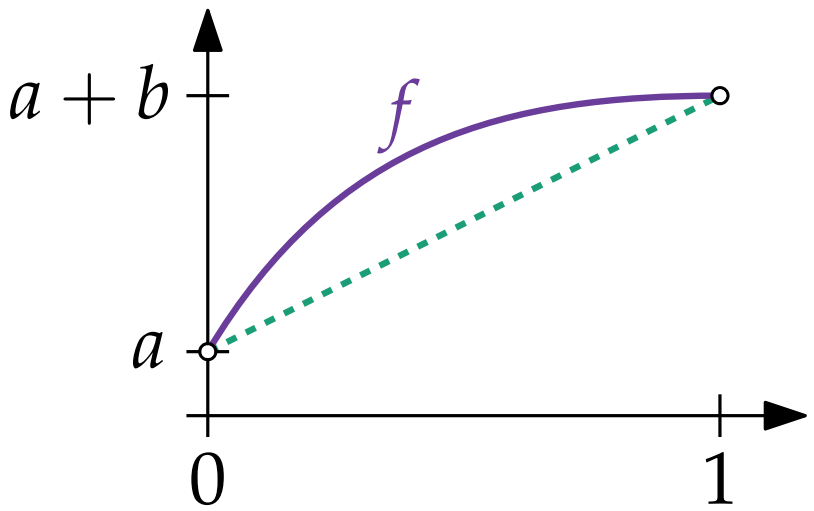

Arithmetic–Geometric Mean Inequality (AGMI):

Mathematical Toolkit

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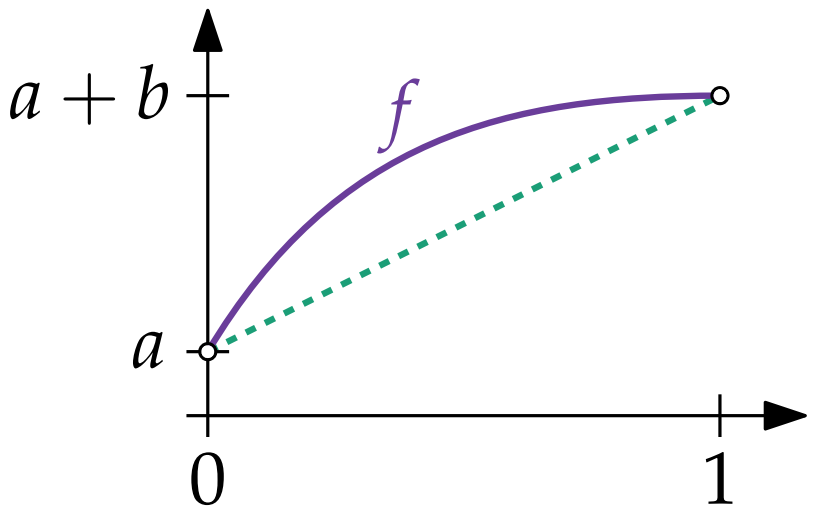

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For all non-negative numbers $a_1, \ldots, a_k$:

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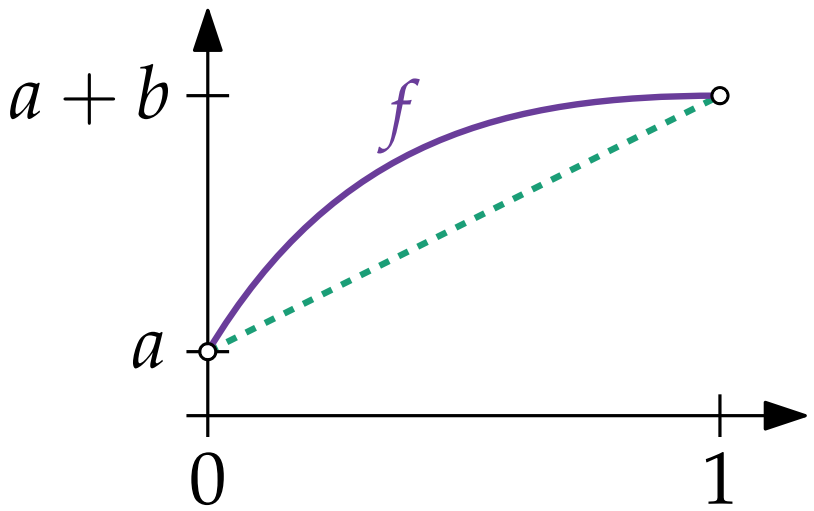

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For all non-negative numbers $a_1, \ldots, a_k$:

$$\left(\prod_{i=1}^{k} a_i \right)^{1/k} \leq$$

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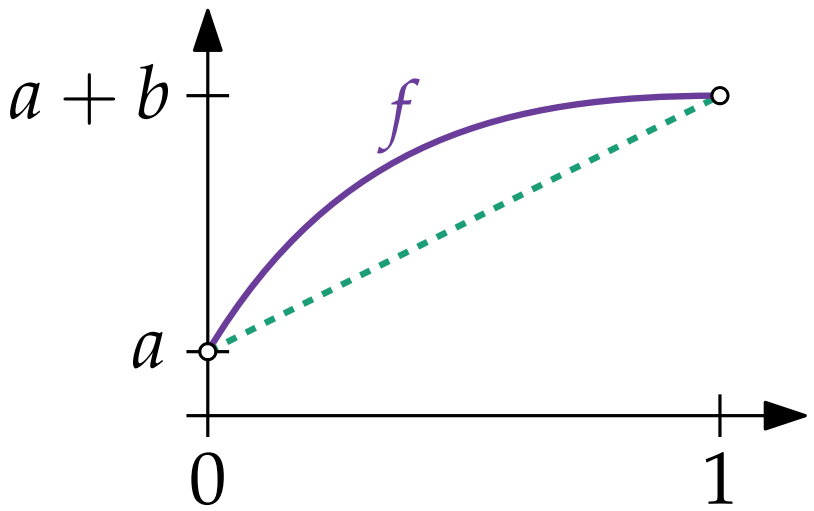

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Randomized Rounding (Proof)

Consider a fixed clause C_j of length l_j. Then we have:

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$$\leq \left(1 - \frac{z_j^*}{l_j} \right)^{l_j}$$

Randomized Rounding (Proof)

The function $f(z_j^*) = 1 - \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$ is concave on $[0, 1]$.

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&\geq \left(1 - \frac{1}{e}\right) z_j^*
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Randomized Rounding (Proof)

Therefore

$$E[W] = \sum_{j=1}^{m} \Pr[C_j \text{ satisfied}] \cdot w_j$$

$$\geq$$

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□

Theorem. The previous algorithm can be derandomized by the method of conditional expectation.

Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part VI:
Combining the Algorithms

Take the better of the two solutions!

Theorem. The better solution among the randomized algorithm and the randomized LP-rounding algorithm provides a -approximation for MaxSat.

Take the better of the two solutions!

Theorem. The better solution among the randomized algorithm and the randomized LP-rounding algorithm provides a $3/4$-approximation for MaxSat.

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Proof.

We use another probabilistic argument.
With probability $1/2$, choose the solution of the first algorithm; otherwise the solution of the second algorithm.

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Proof.

We use another probabilistic argument.
With probability $1/2$, choose the solution of the first algorithm; otherwise the solution of the second algorithm.

The better solution is at least as good as the expectation of the above algorithm.

Take the better of the two solutions!

The probability that clause C_i is satisfied is at least:

Take the better of the two solutions!

The probability that clause C_j is satisfied is at least:

$$\frac{1}{2}\left[\qquad\qquad + \qquad\qquad\right].$$

Take the better of the two solutions!

The probability that clause C_j is satisfied is at least:

$$\frac{1}{2}\left[\underbrace{\left(1-\left(1-\frac{1}{l_j}\right)^{l_j}\right)z_j^*}_{\text{LP-rounding}}+\qquad\right].$$

Take the better of the two solutions!

The probability that clause C_j is satisfied is at least:

$$\frac{1}{2}\left[\underbrace{\left(1-\left(1-\frac{1}{l_j}\right)^{l_j}\right)z_j^*}_{\text{LP-rounding}}+\underbrace{\left(1-2^{-l_j}\right)}_{\text{rand. alg.}}\right].$$

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The probability that clause C_j is satisfied is at least:

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For $l_j \in \{1,2\}$, a simple calculation yields exactly $\frac{3}{4}z_j^*$.

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For $l_j \in \{1,2\}$, a simple calculation yields exactly $\frac{3}{4}z_j^*$.

For $l_j \geq 3$, $\quad 1-(1-1/l_j)^{l_j} \geq$ $\quad$ and $1-2^{-l_j} \geq$

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For $l_j \geq 3$, $\quad 1-(1-1/l_j)^{l_j} \geq (1-1/e)$ and $1-2^{-l_j} \geq$

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For $l_j \geq 3$, $\quad 1-(1-1/l_j)^{l_j} \geq (1-1/e)$ and $1-2^{-l_j} \geq \frac{7}{8}$.

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Thus, we have at least:

$$\frac{1}{2}\left[\left(1-\frac{1}{e}\right)+\frac{7}{8}\right] z_j^* \approx$$

□

Take the better of the two solutions!

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$$\frac{1}{2}\left[\left(1-\frac{1}{e}\right)+\frac{7}{8}\right]z_j^* \approx 0.753 z_j^* \geq$$

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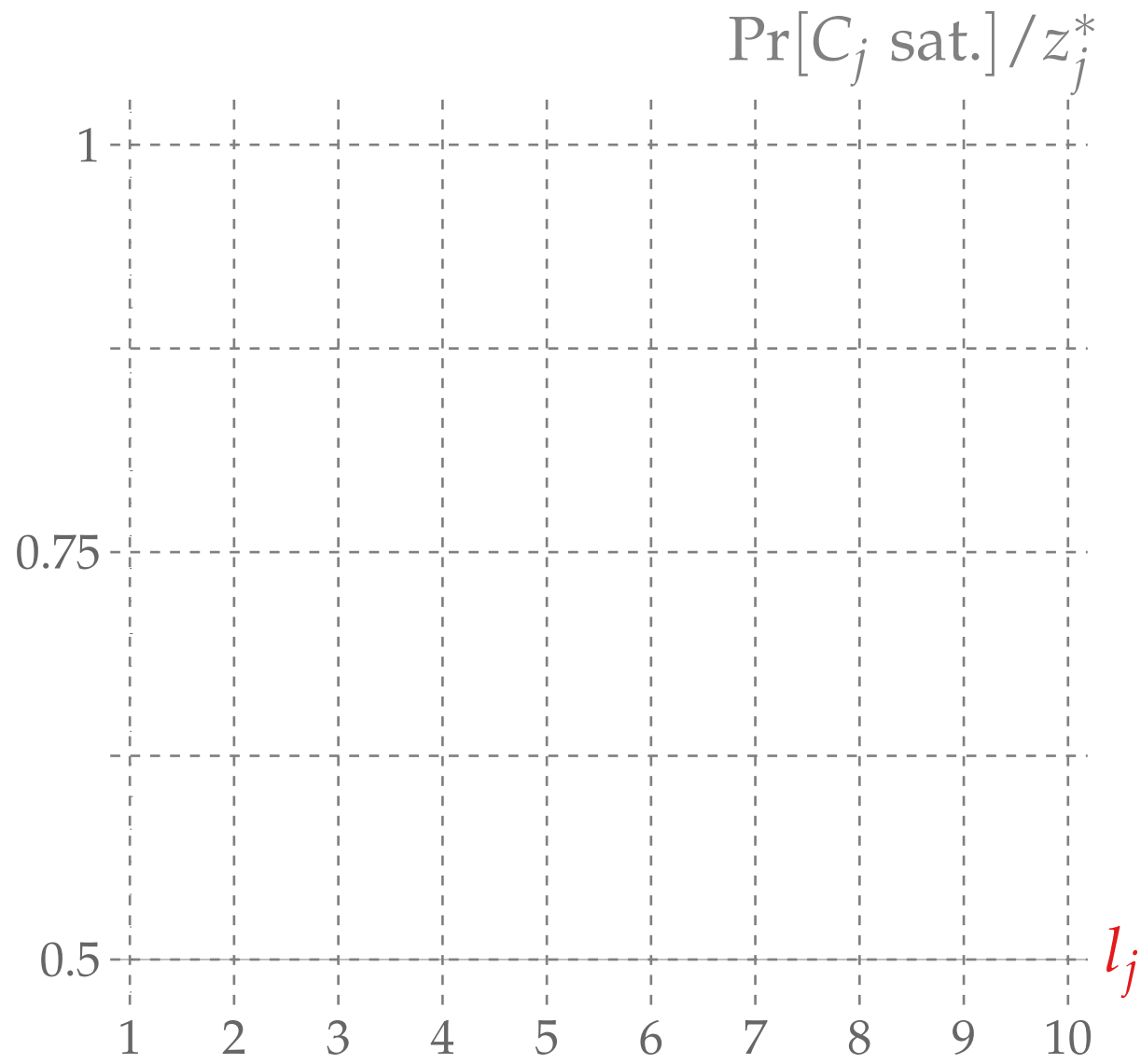
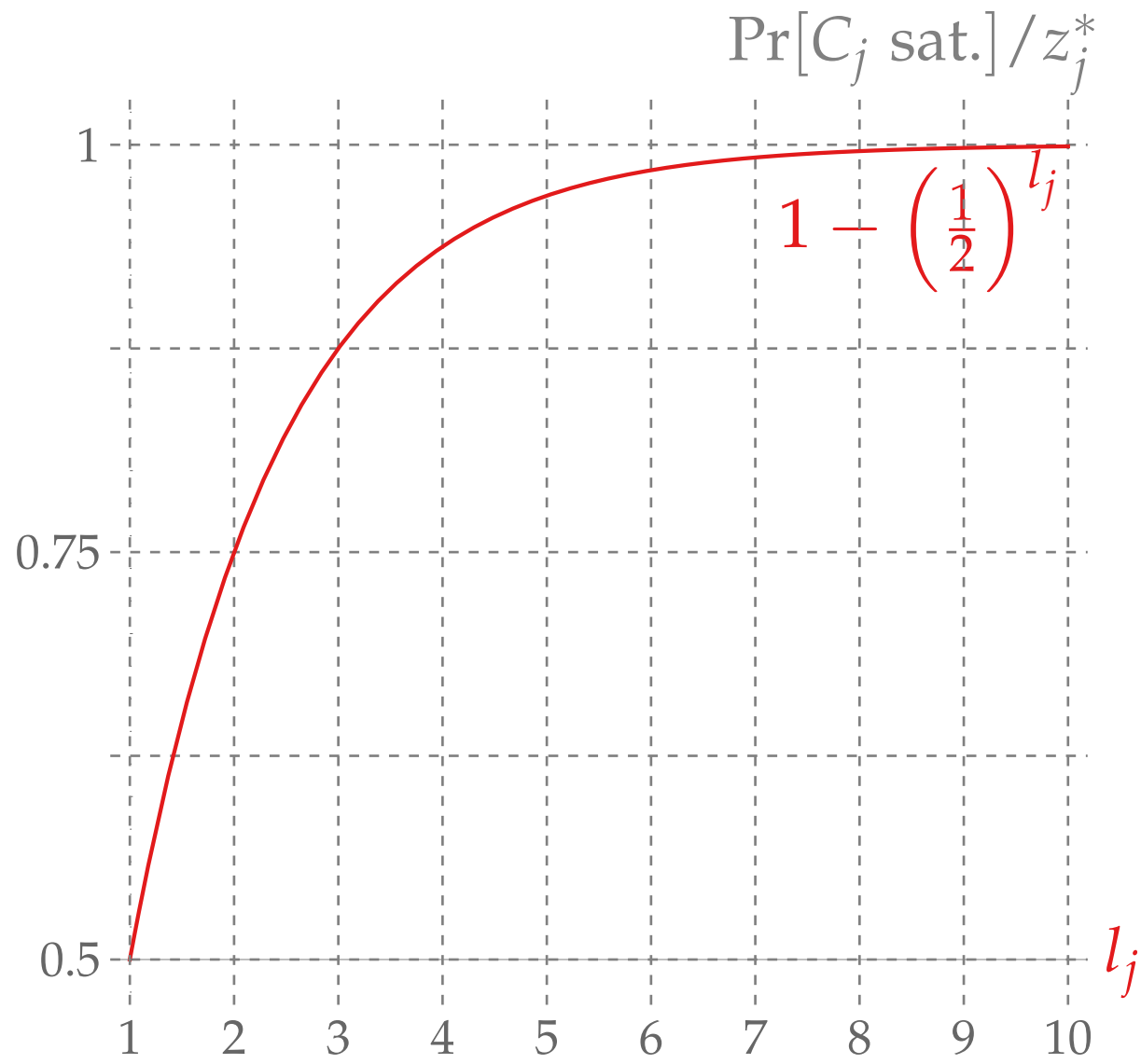
□

Visualization and Derandomization

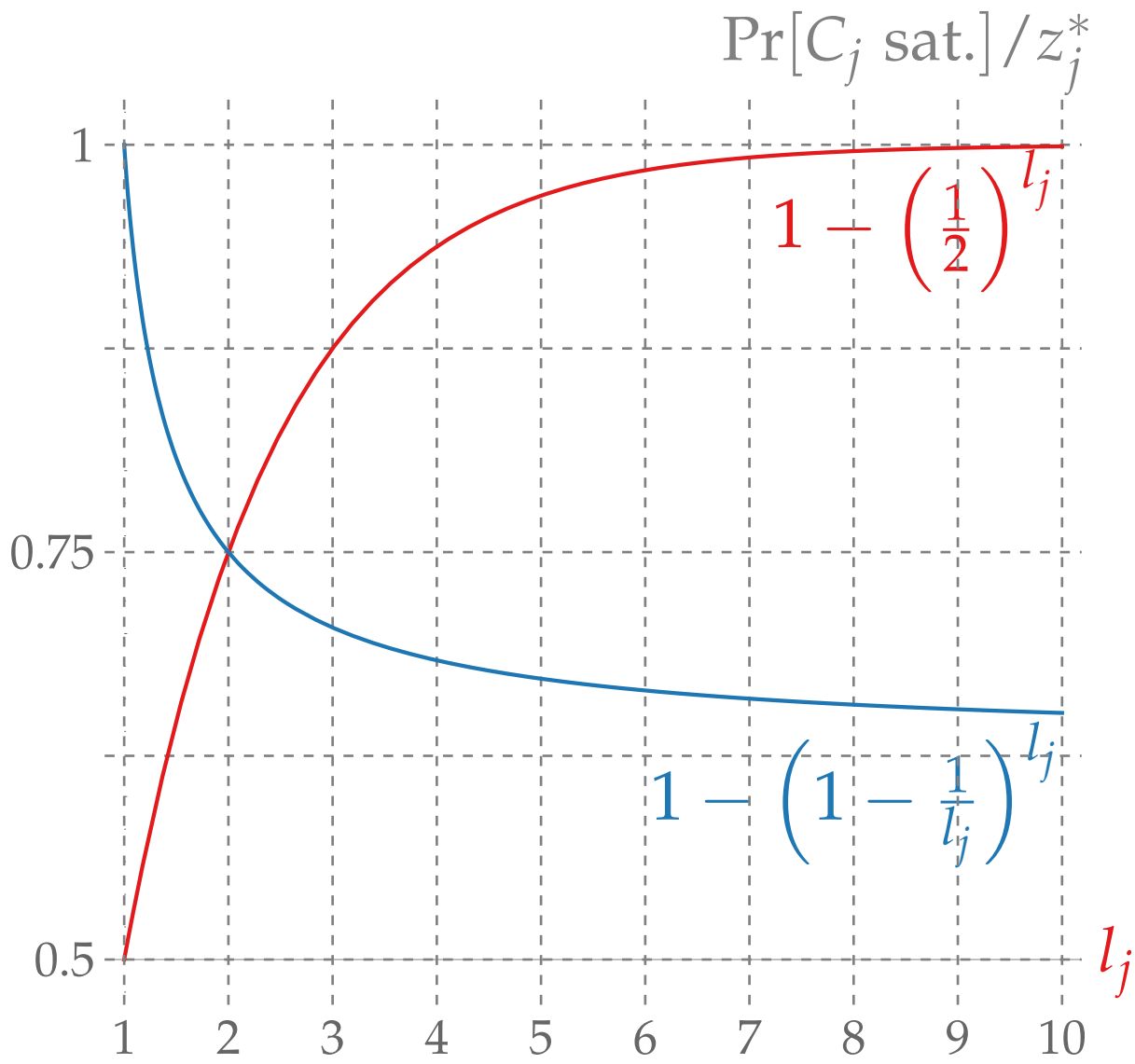
Visualization and Derandomization



Visualization and Derandomization

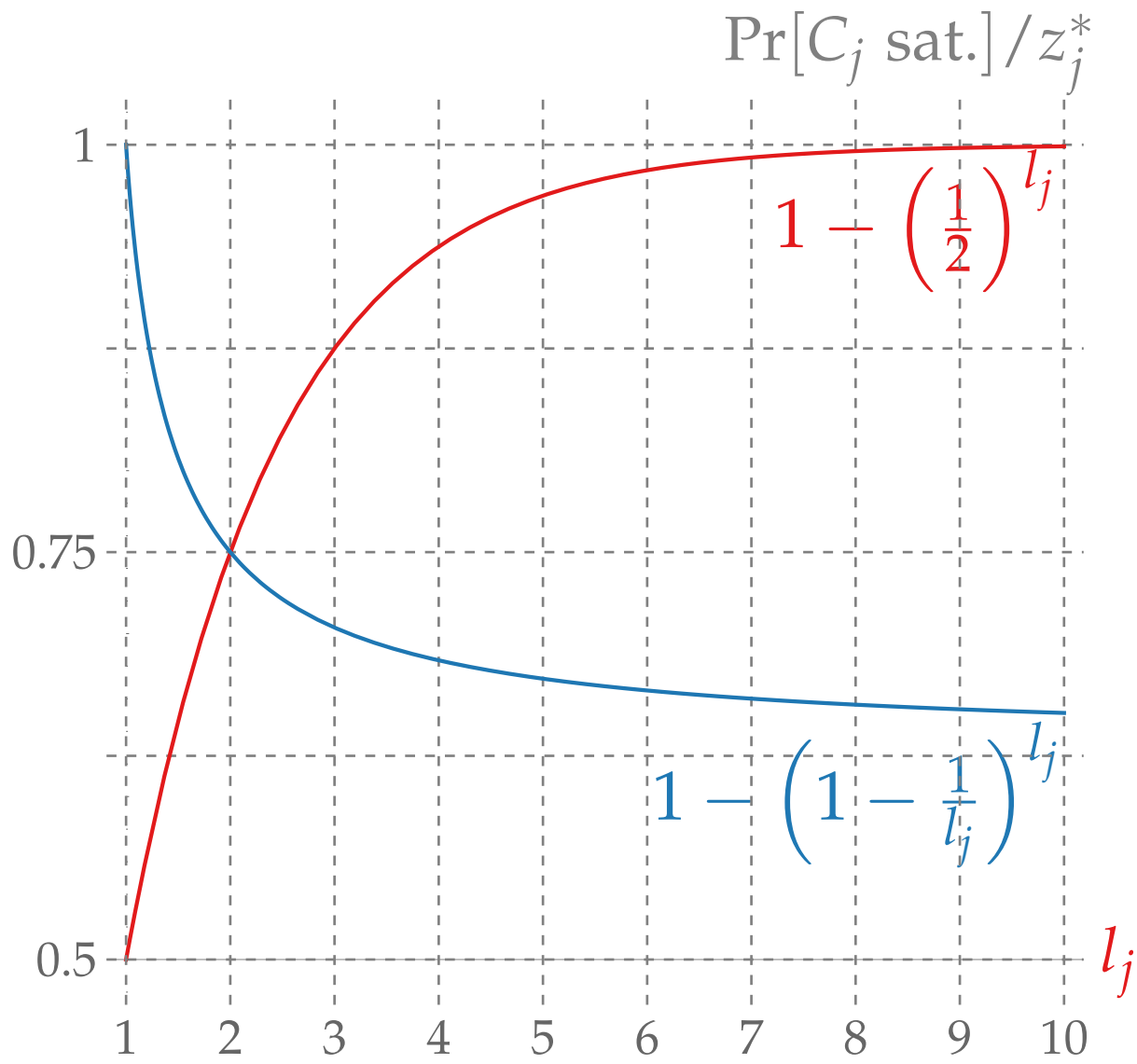


Visualization and Derandomization



Visualization and Derandomization

– Randomized alg. is better for large values of l_j.



Visualization and Derandomization

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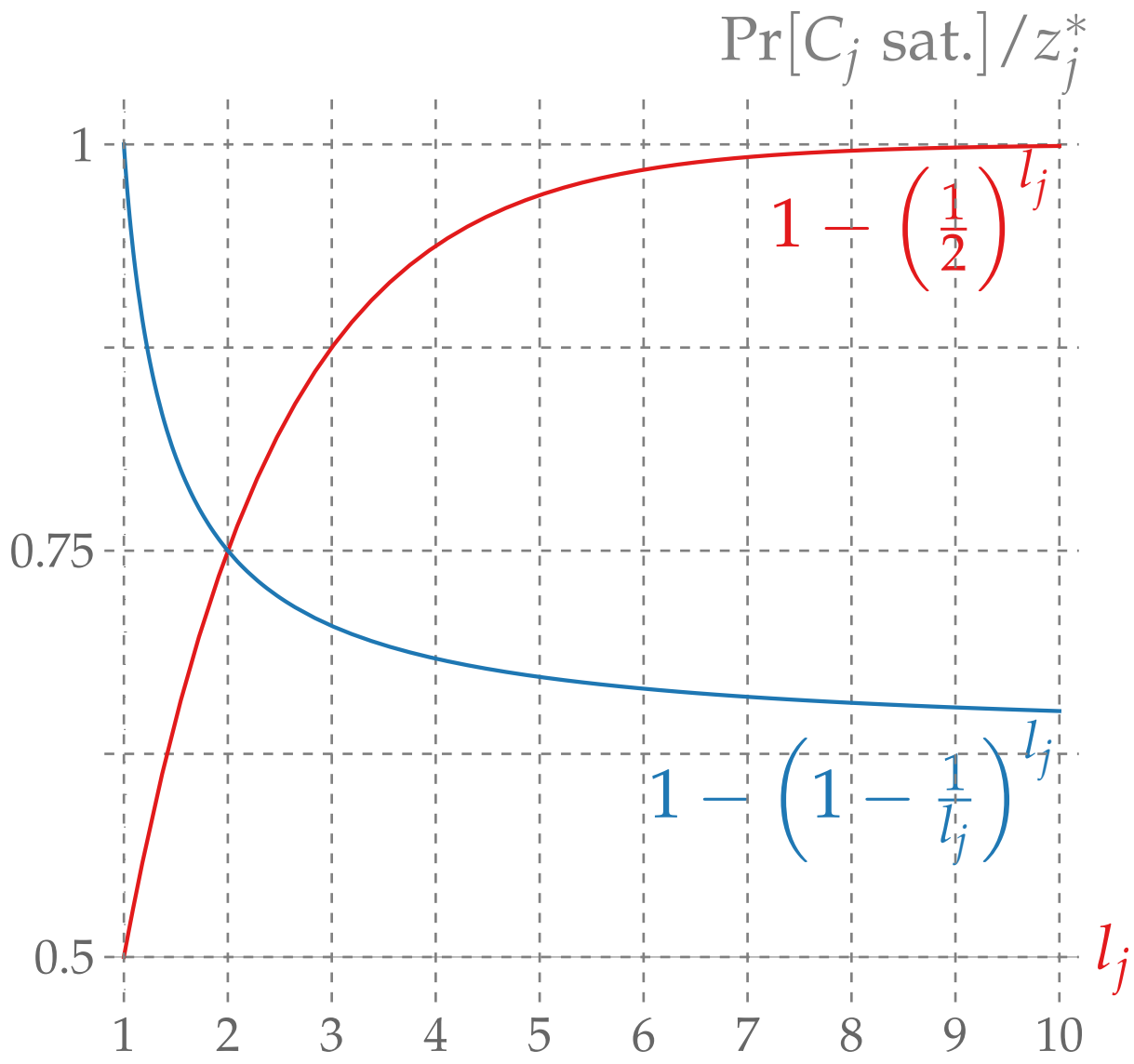

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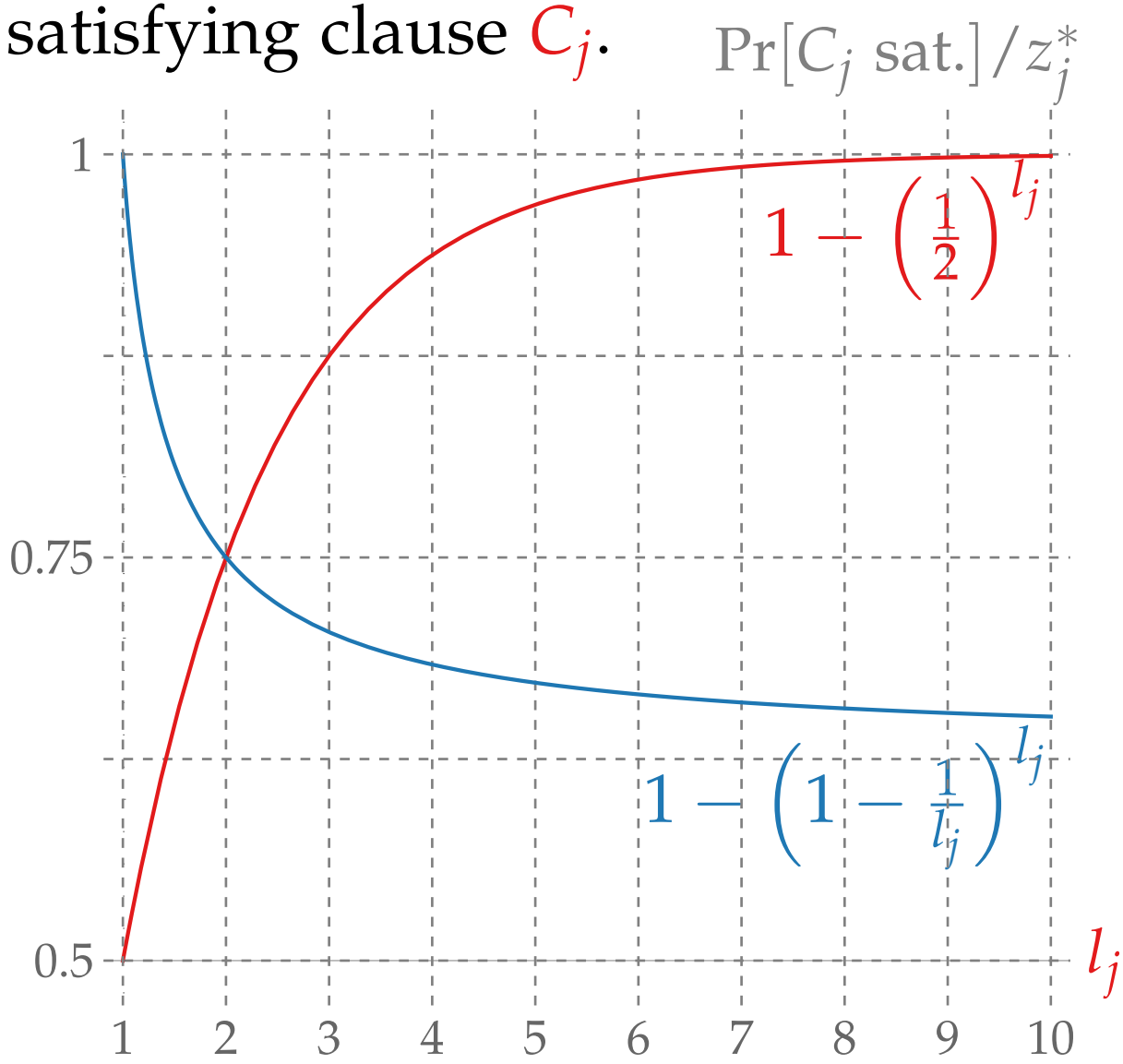
$\Rightarrow$ higher probability of satisfying clause C_j.



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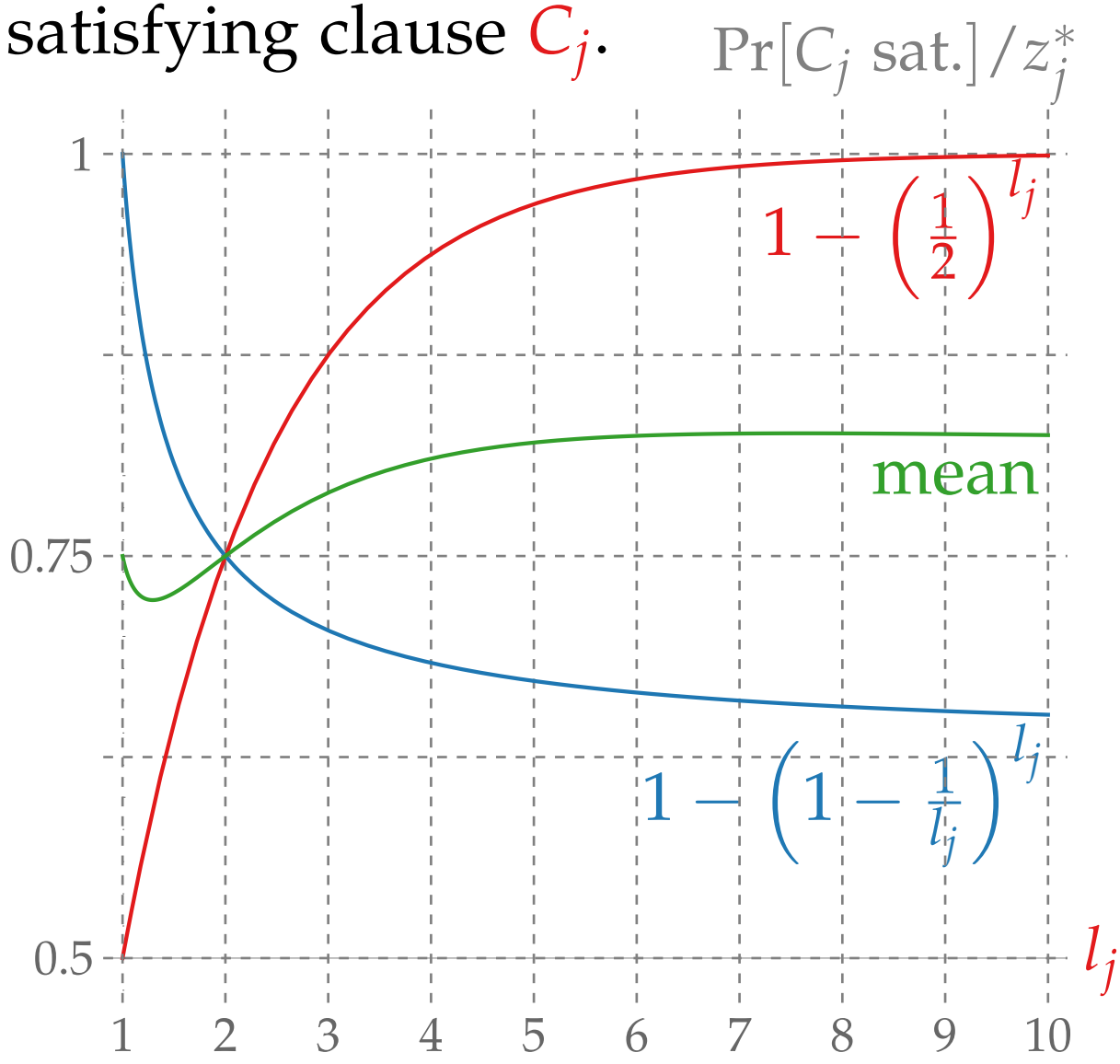

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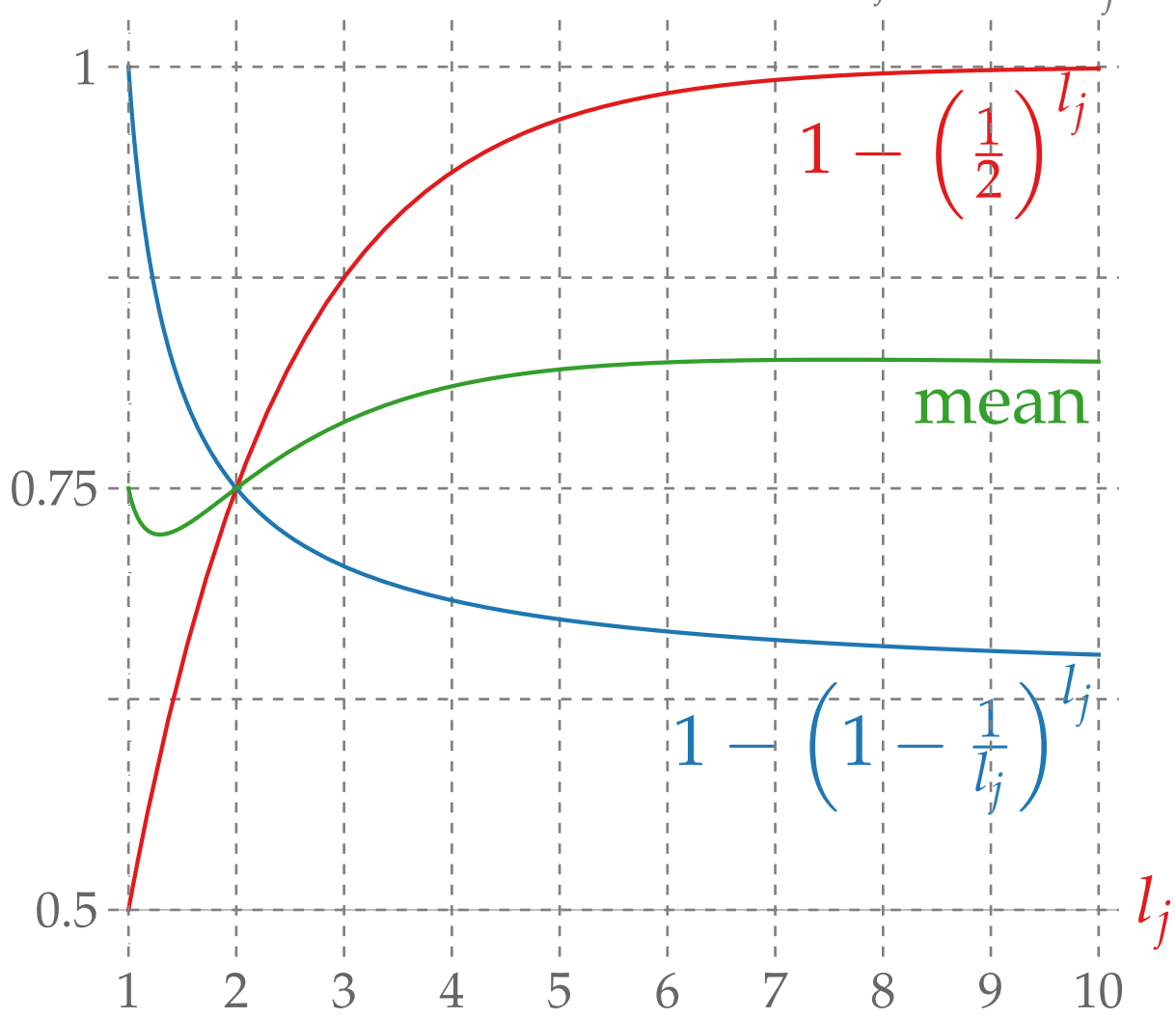
The mean of the two solutions is at least 3/4 for *integer* l_j.



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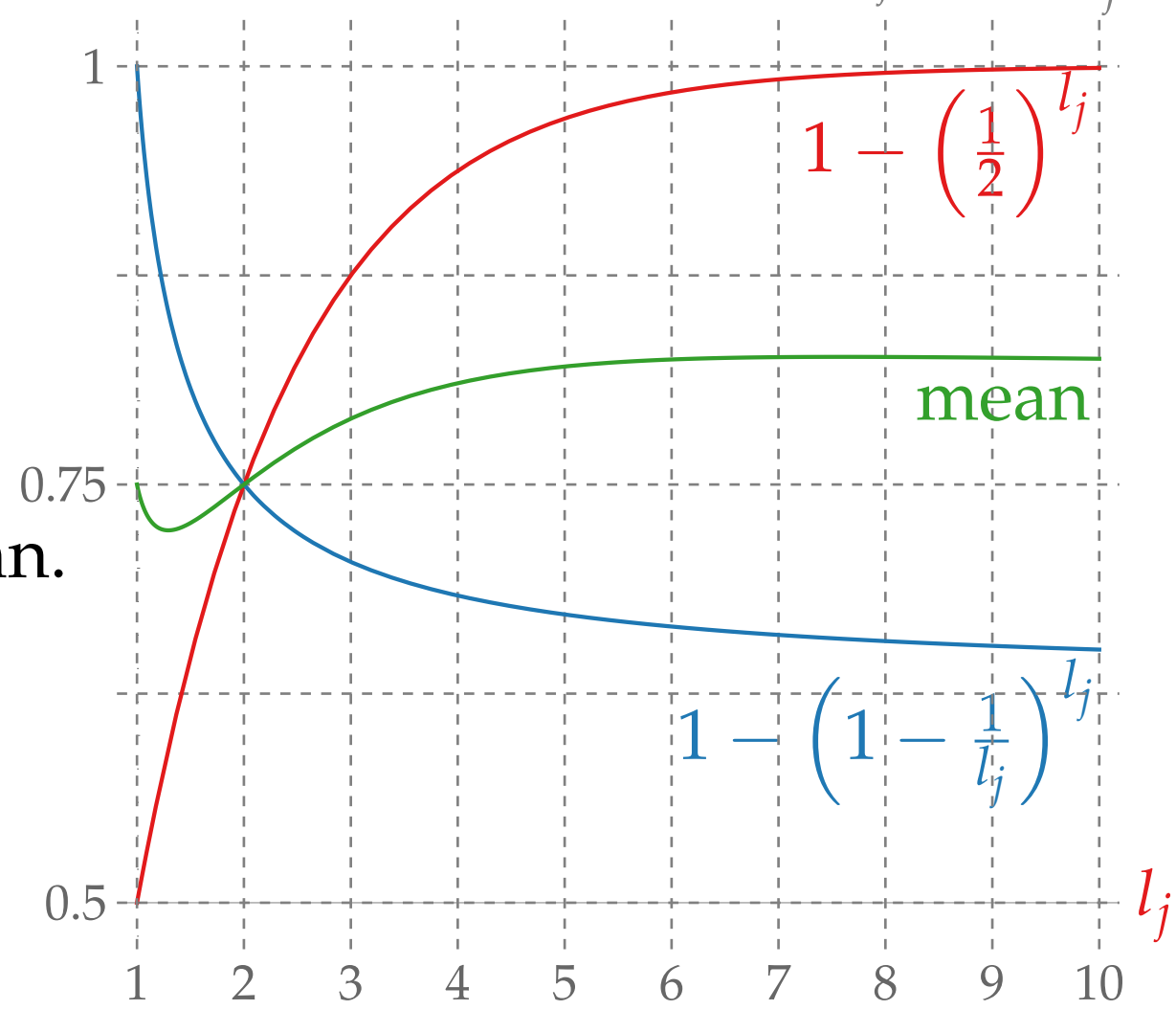
The maximum is at least as large as the mean.



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The maximum is at least as large as the mean.

This algorithm, too, can be derandomized by conditional expectation.