Approximation Algorithms

Lecture 9:
An Approximation Scheme for Euclidean TSP

Part I:
The Traveling Salesman Problem

Alexander Wolff

Winter term 2022/23

Traveling Salesman Problem (TSP)

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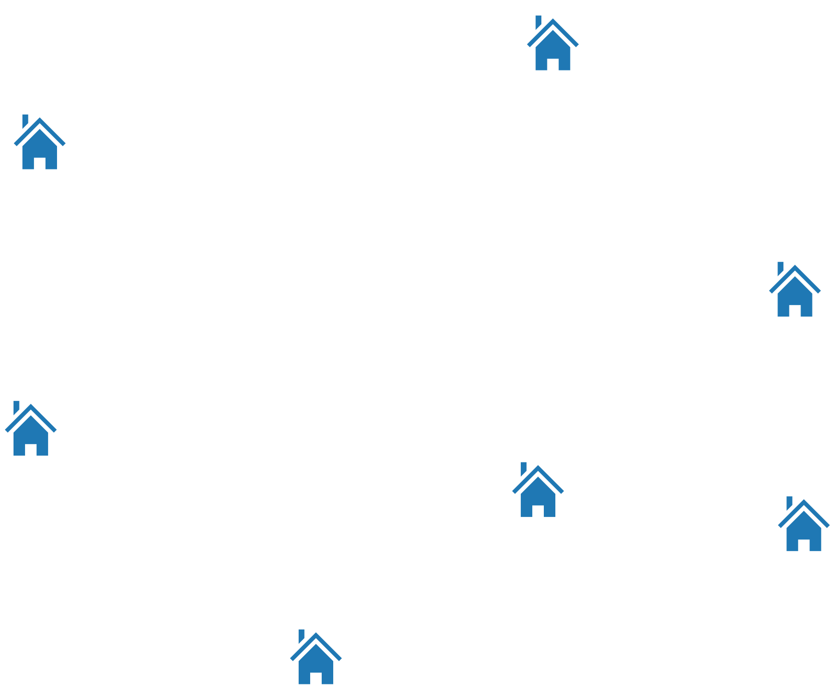

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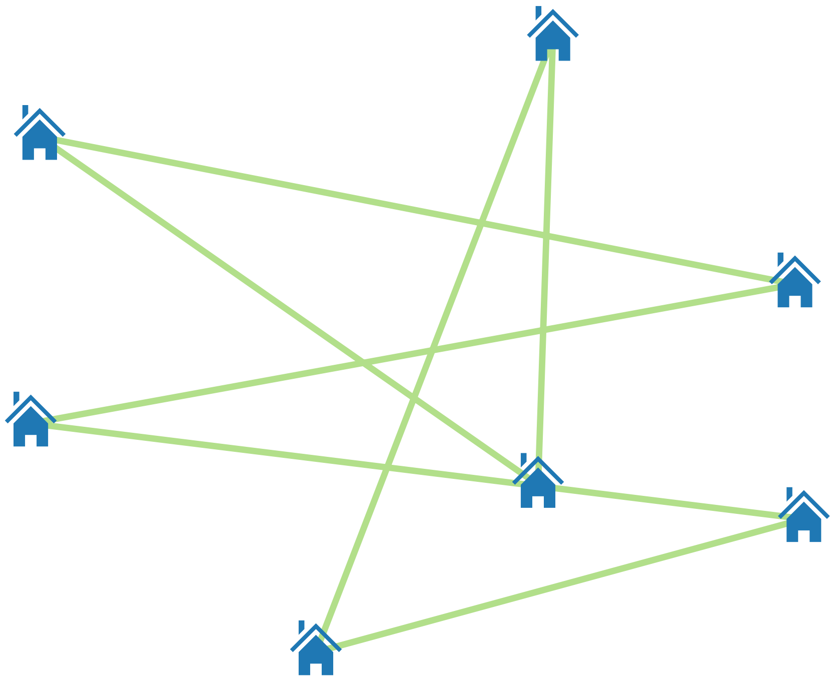

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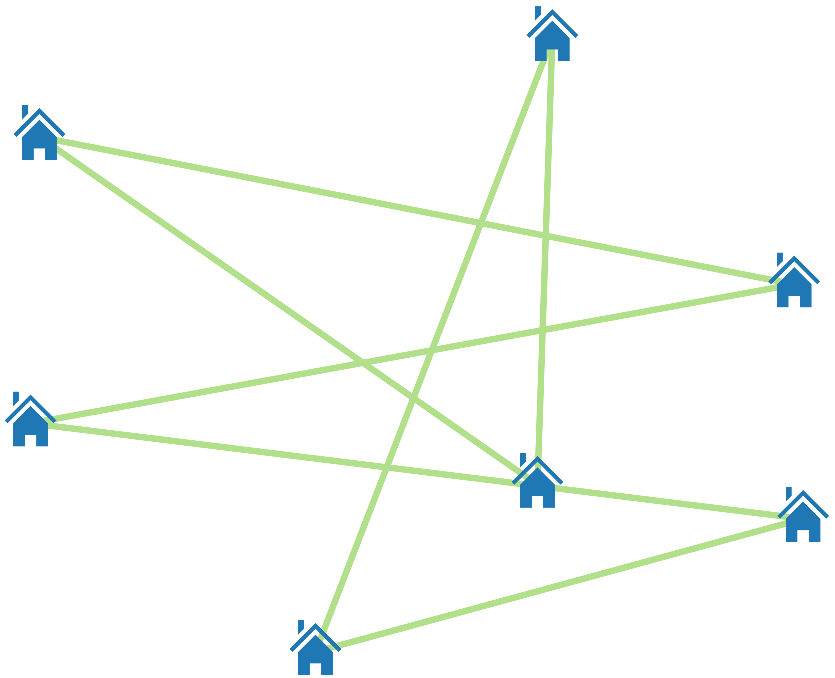

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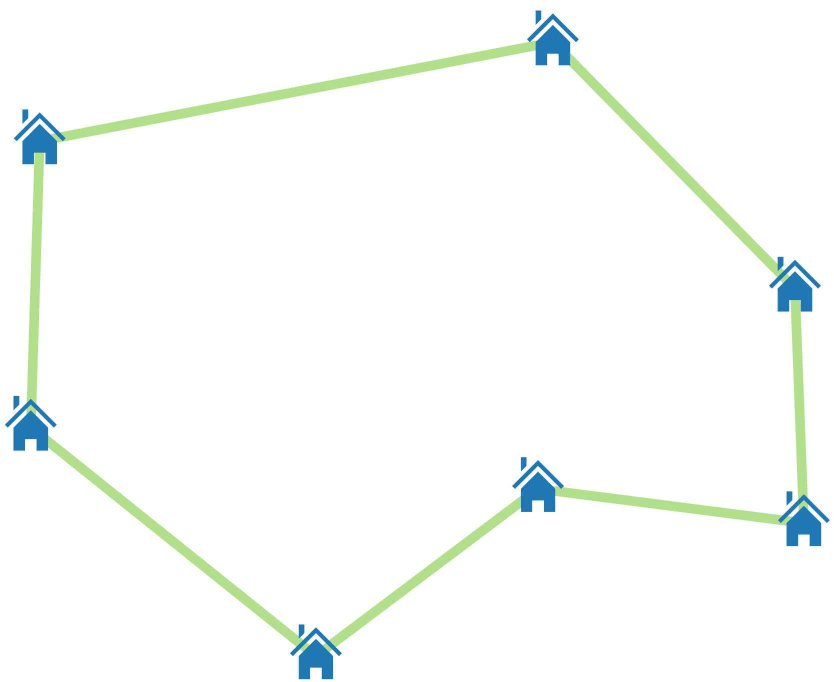

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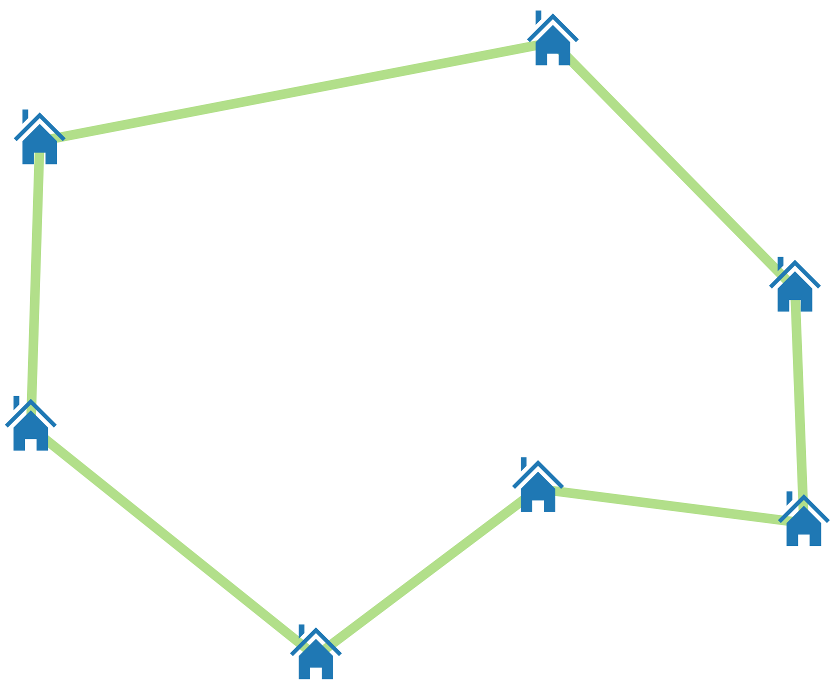

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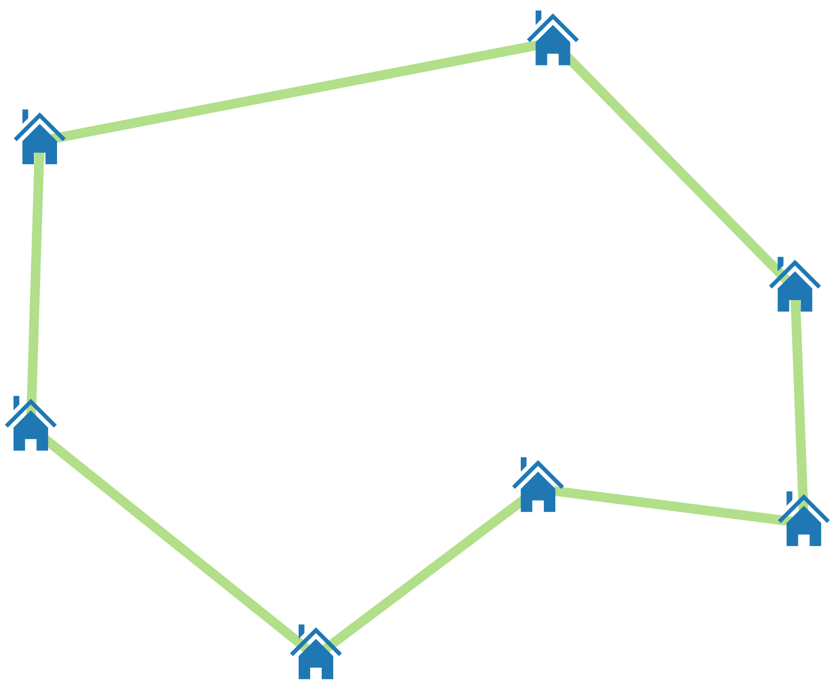

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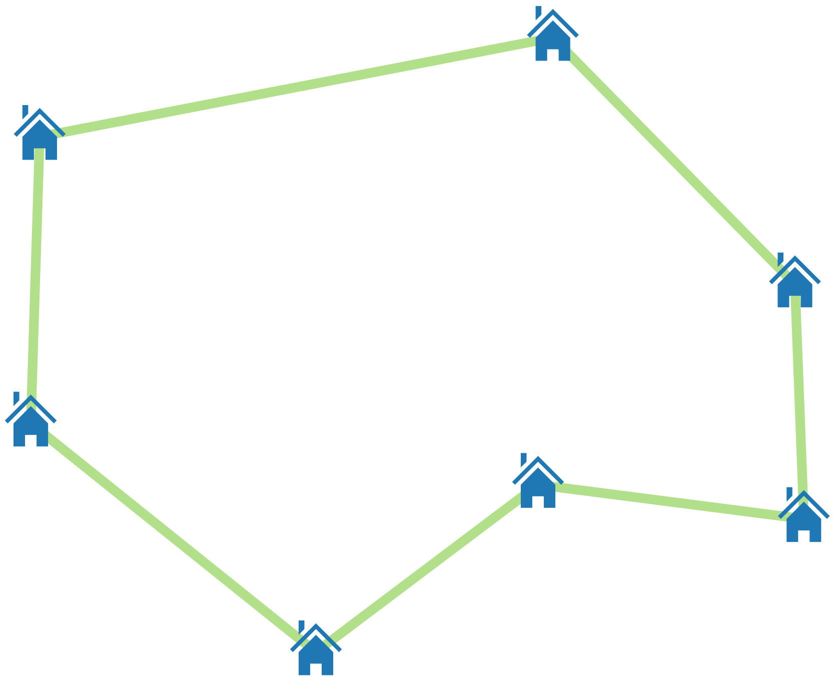
Distance between two points?



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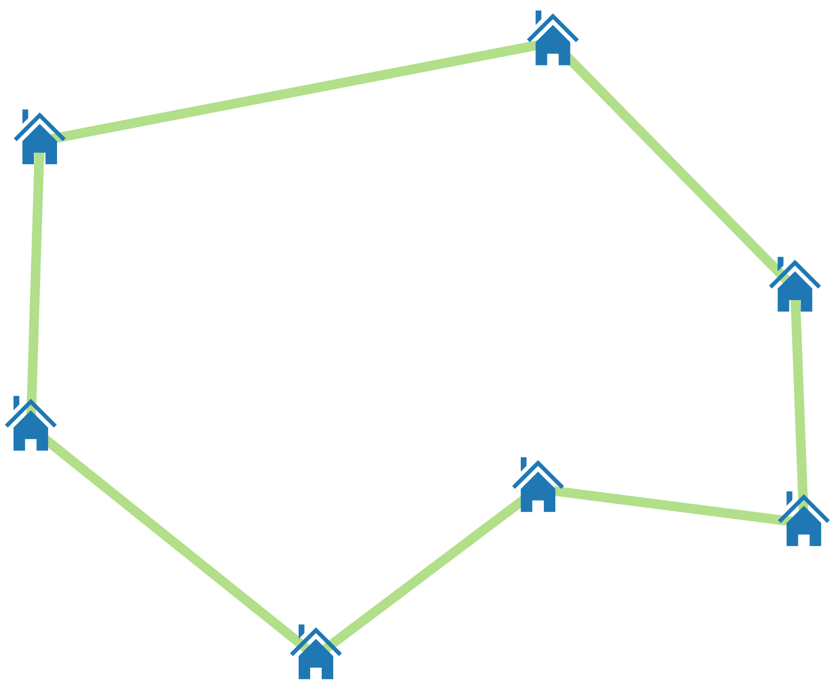

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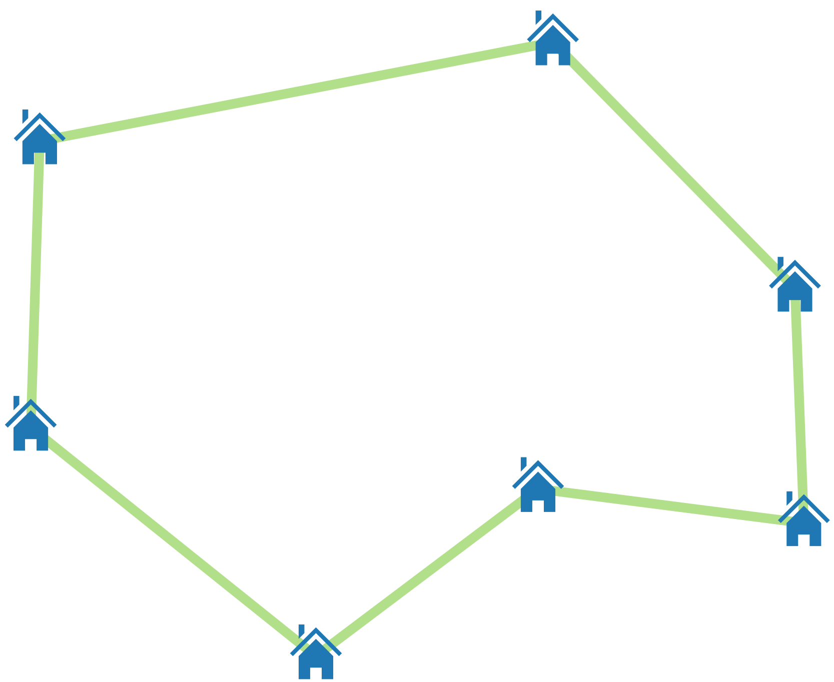

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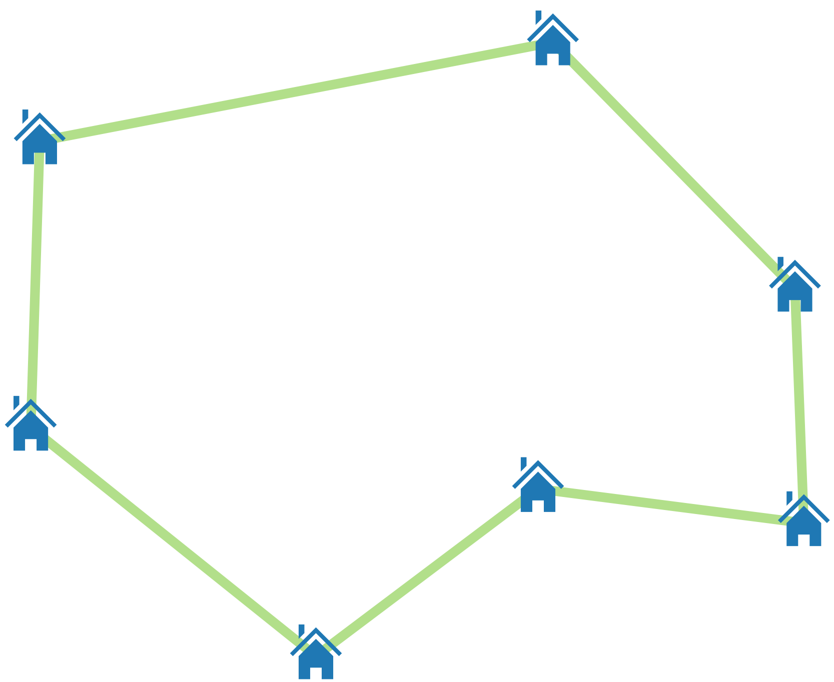
Let's assume that the salesman flies $\Rightarrow$ Euclidean distances.



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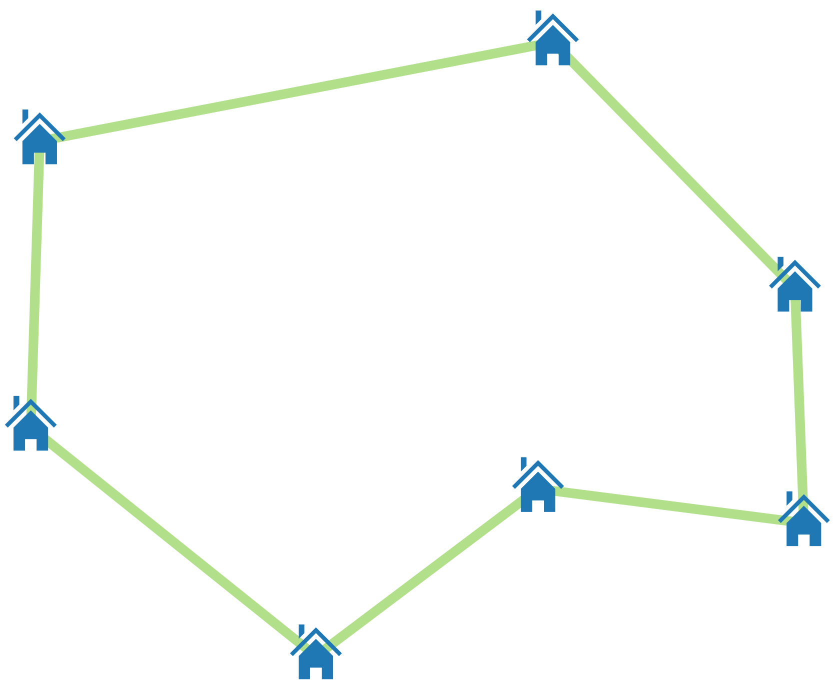

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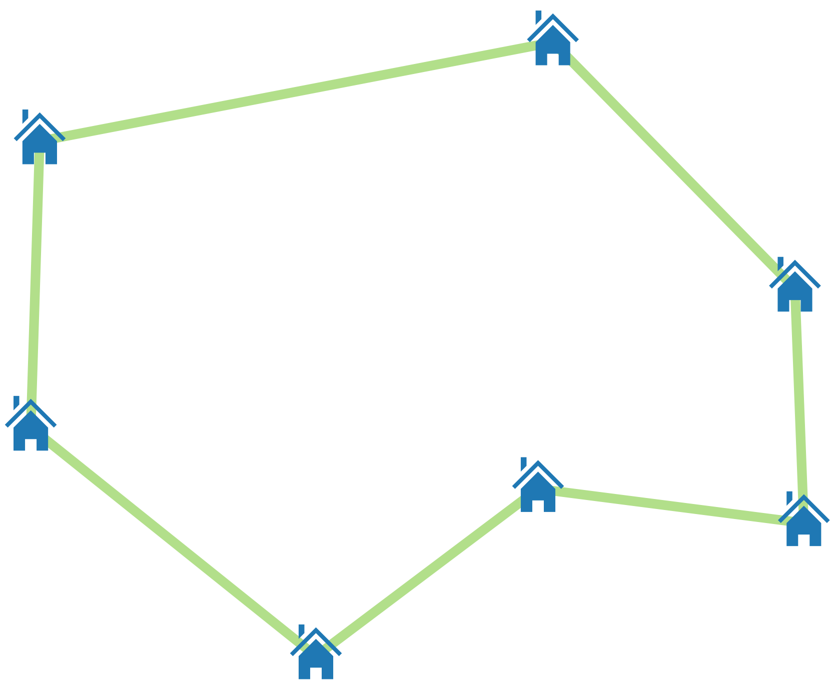

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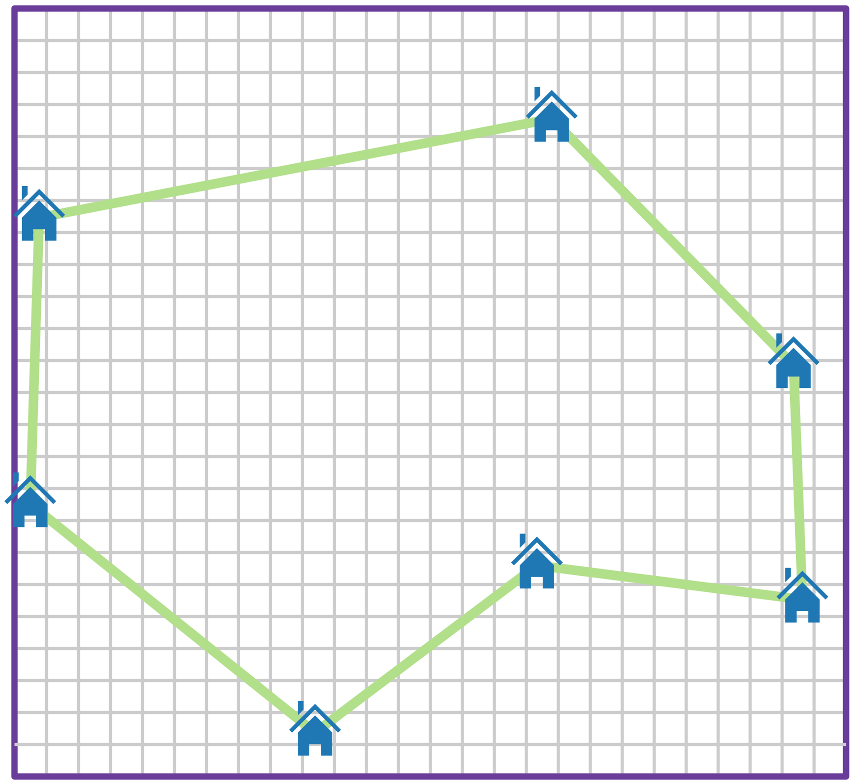

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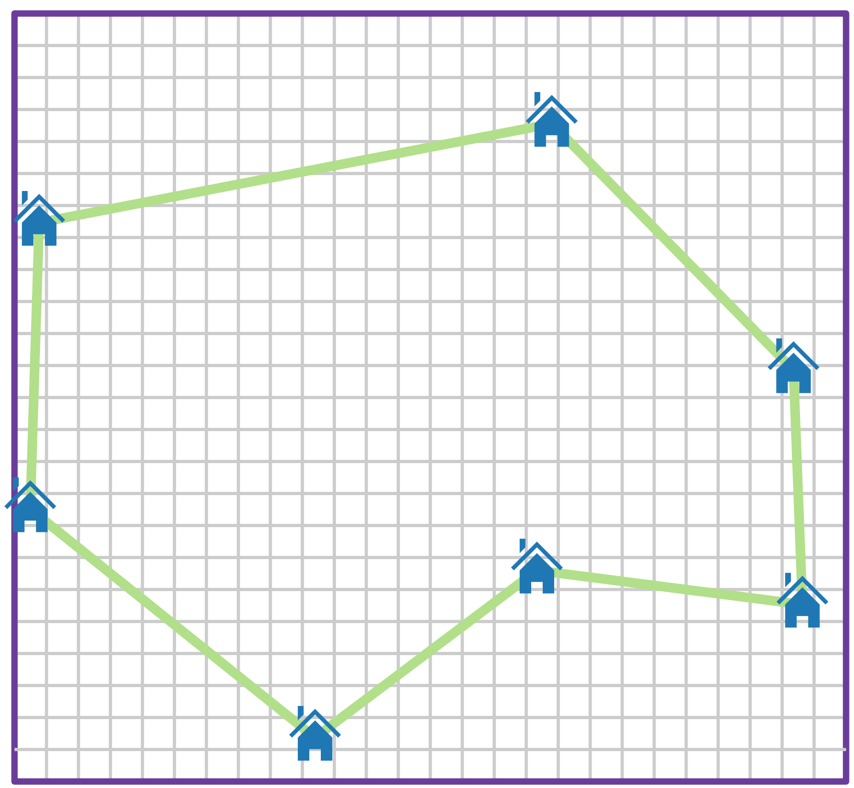

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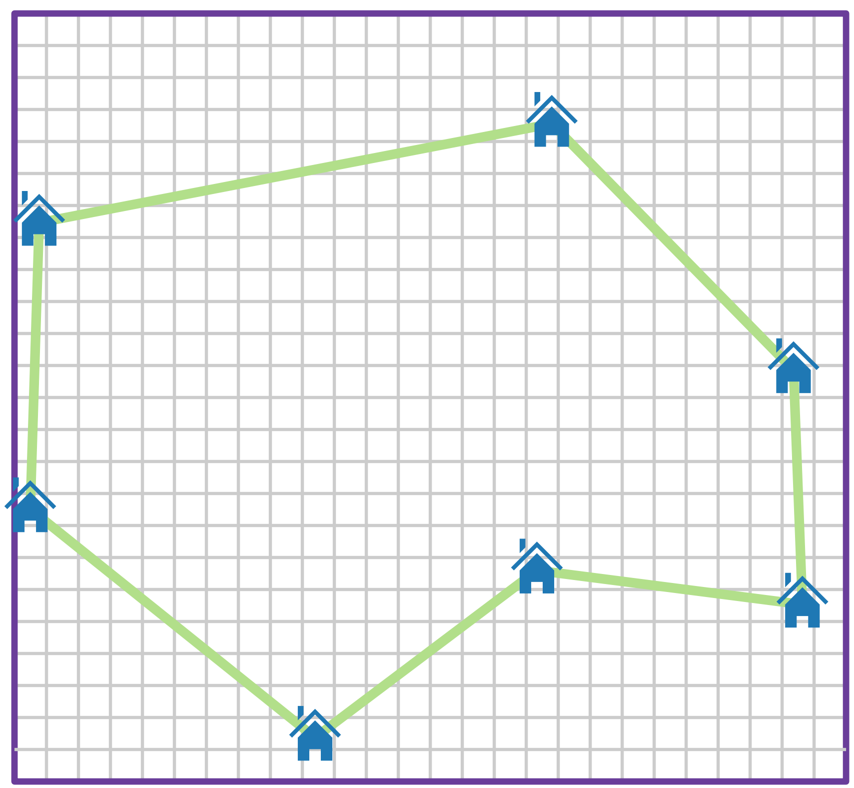

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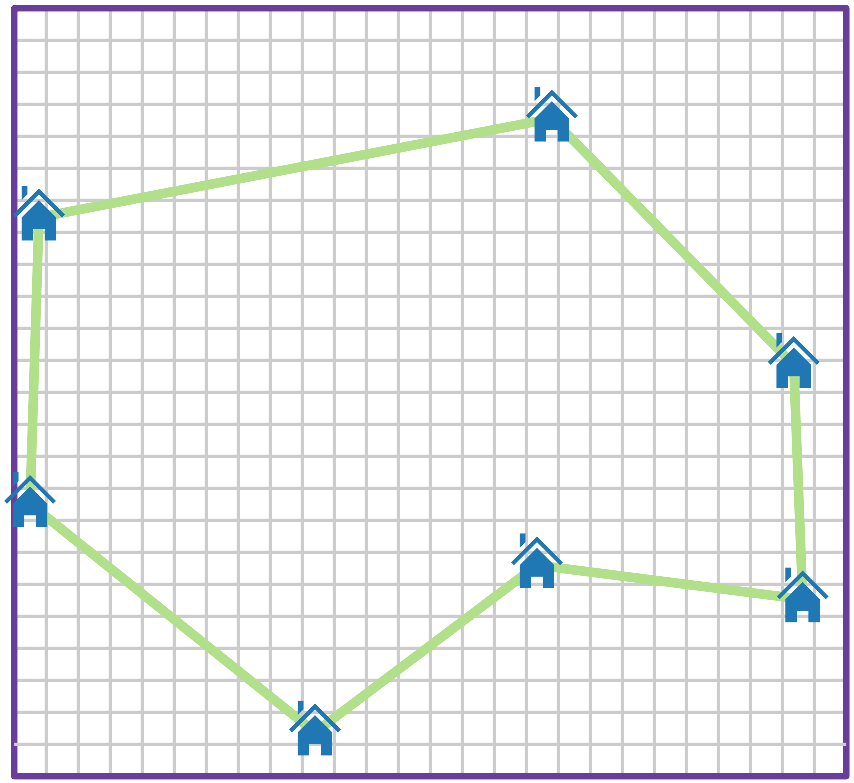

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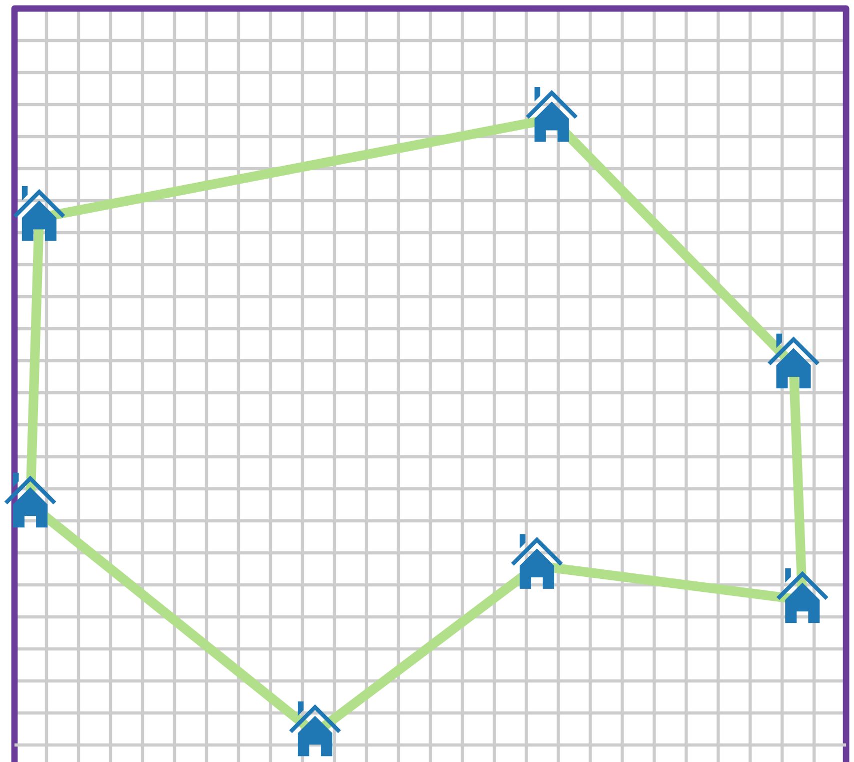

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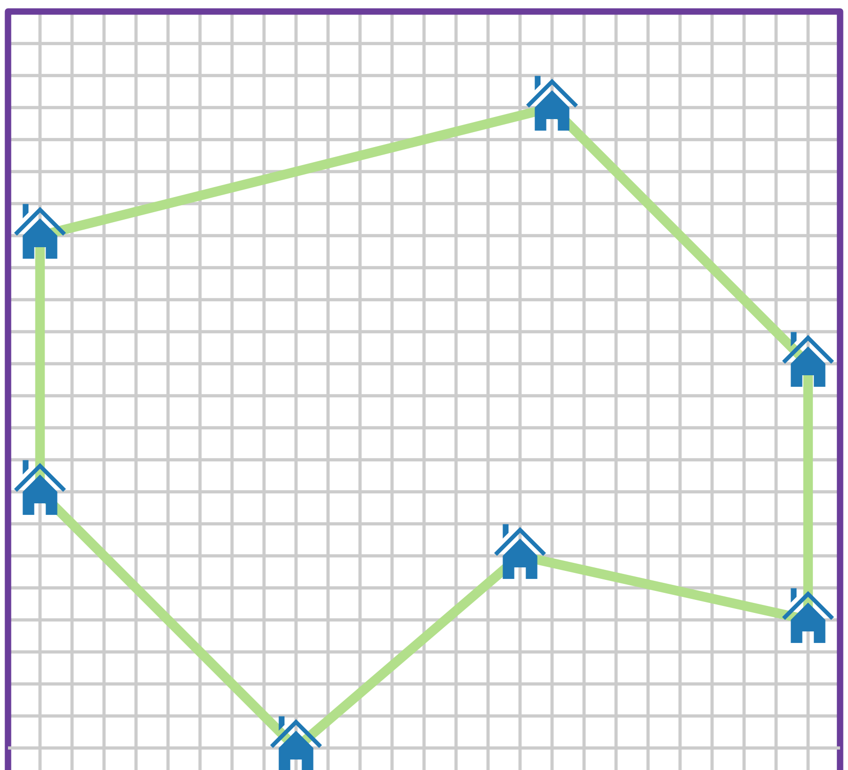

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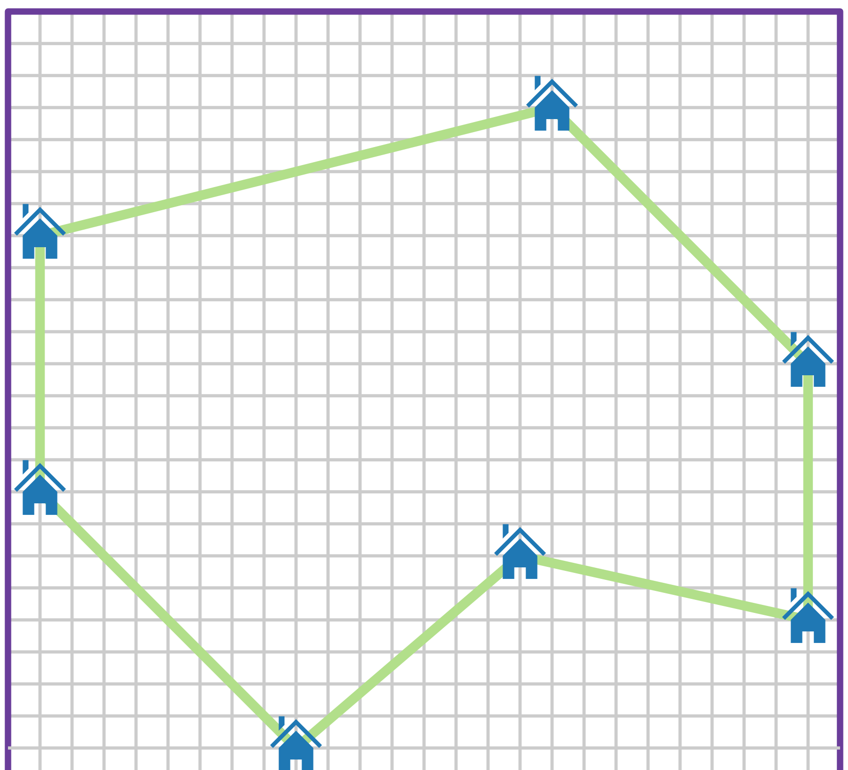

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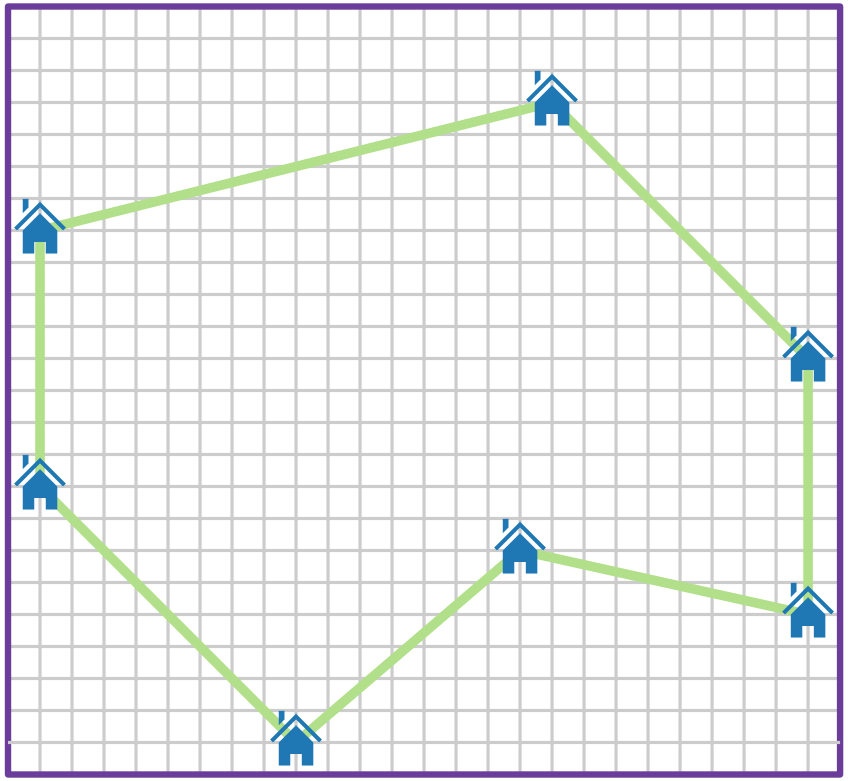

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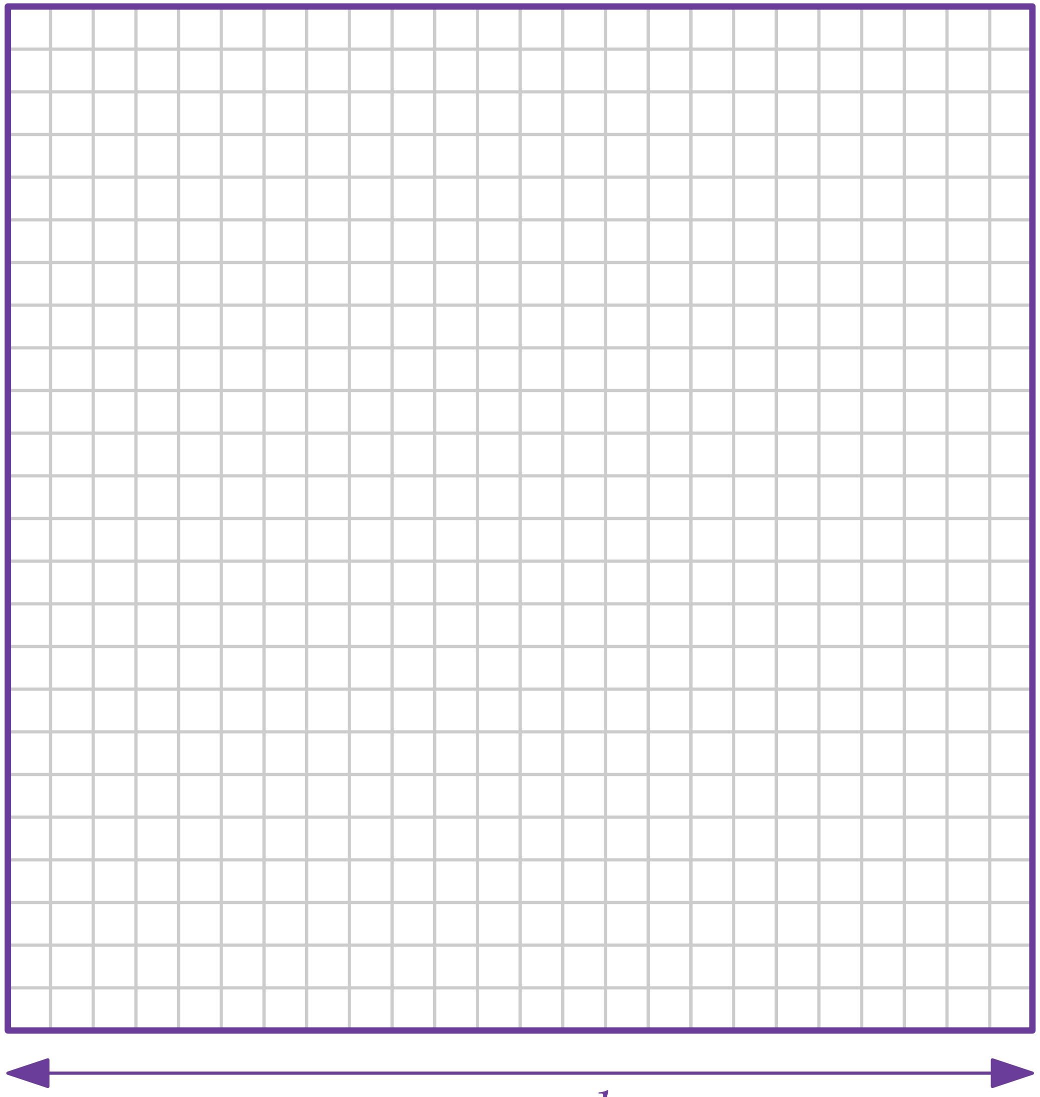
Goal: $(1+\varepsilon)$-approximation!

Approximation Algorithms

Lecture 9:
A PTAS for Euclidean TSP

Part II:
Dissection

Basic Dissection



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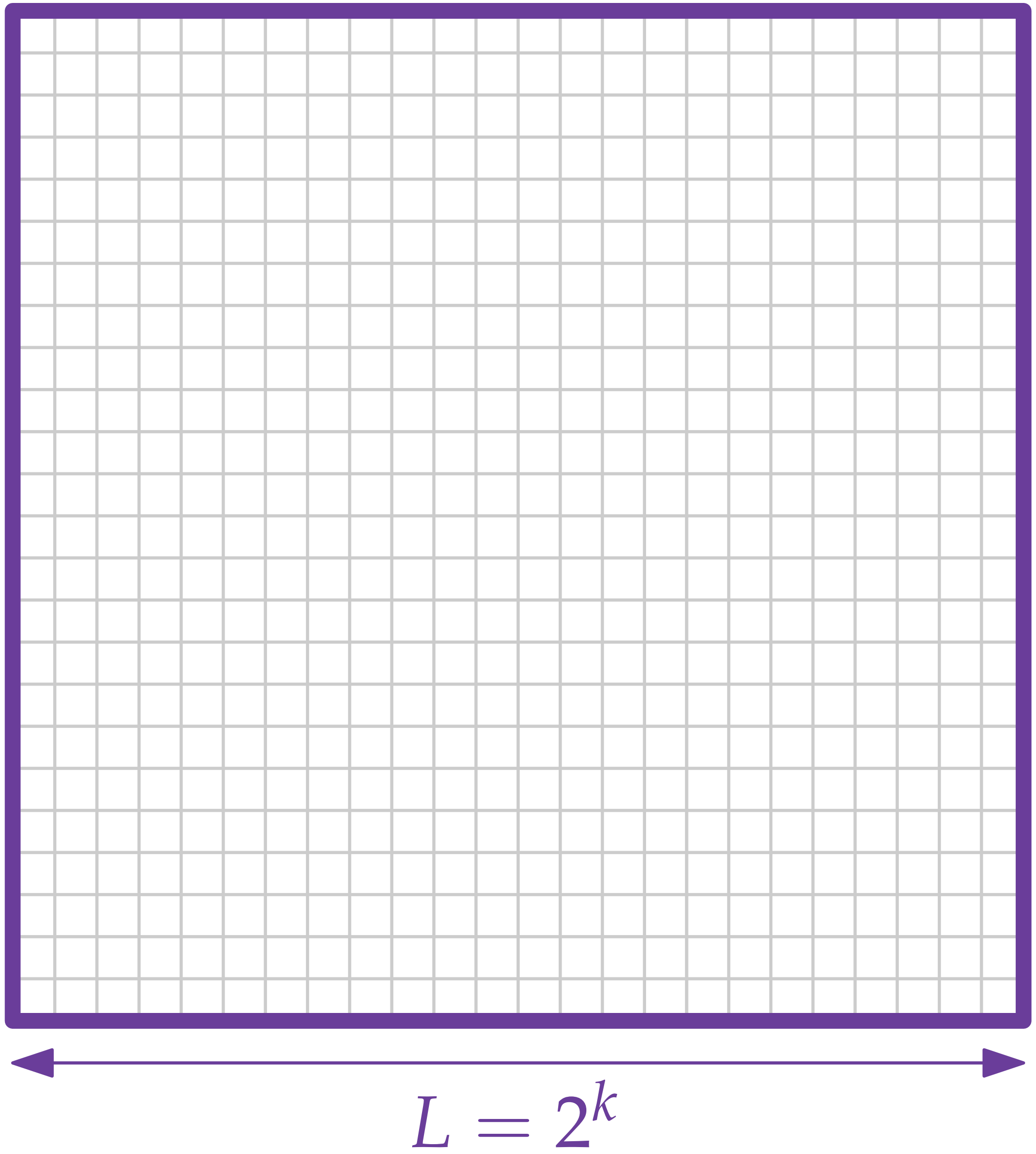
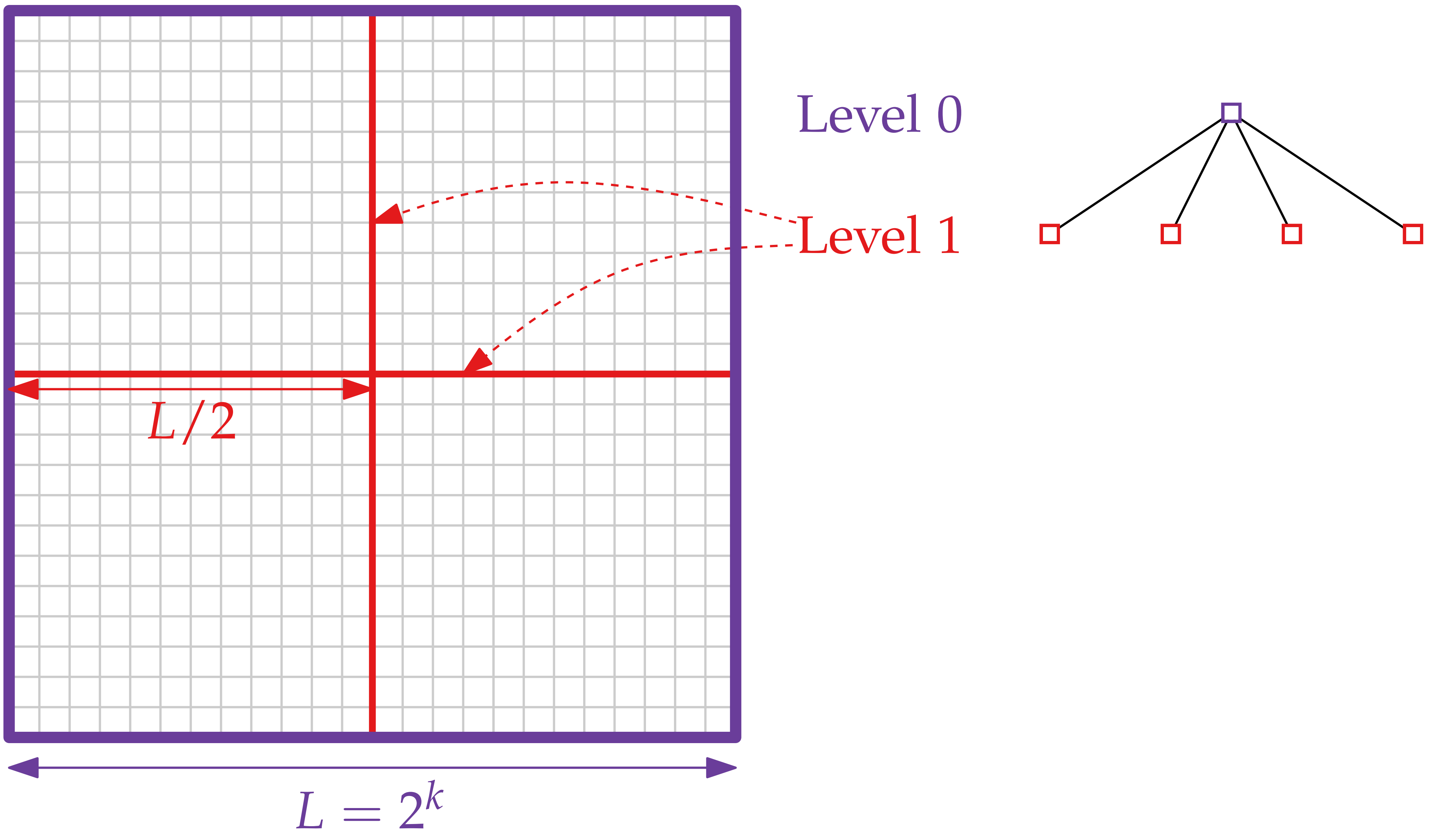
Basic Dissection



Level 0 ▫

Basic Dissection



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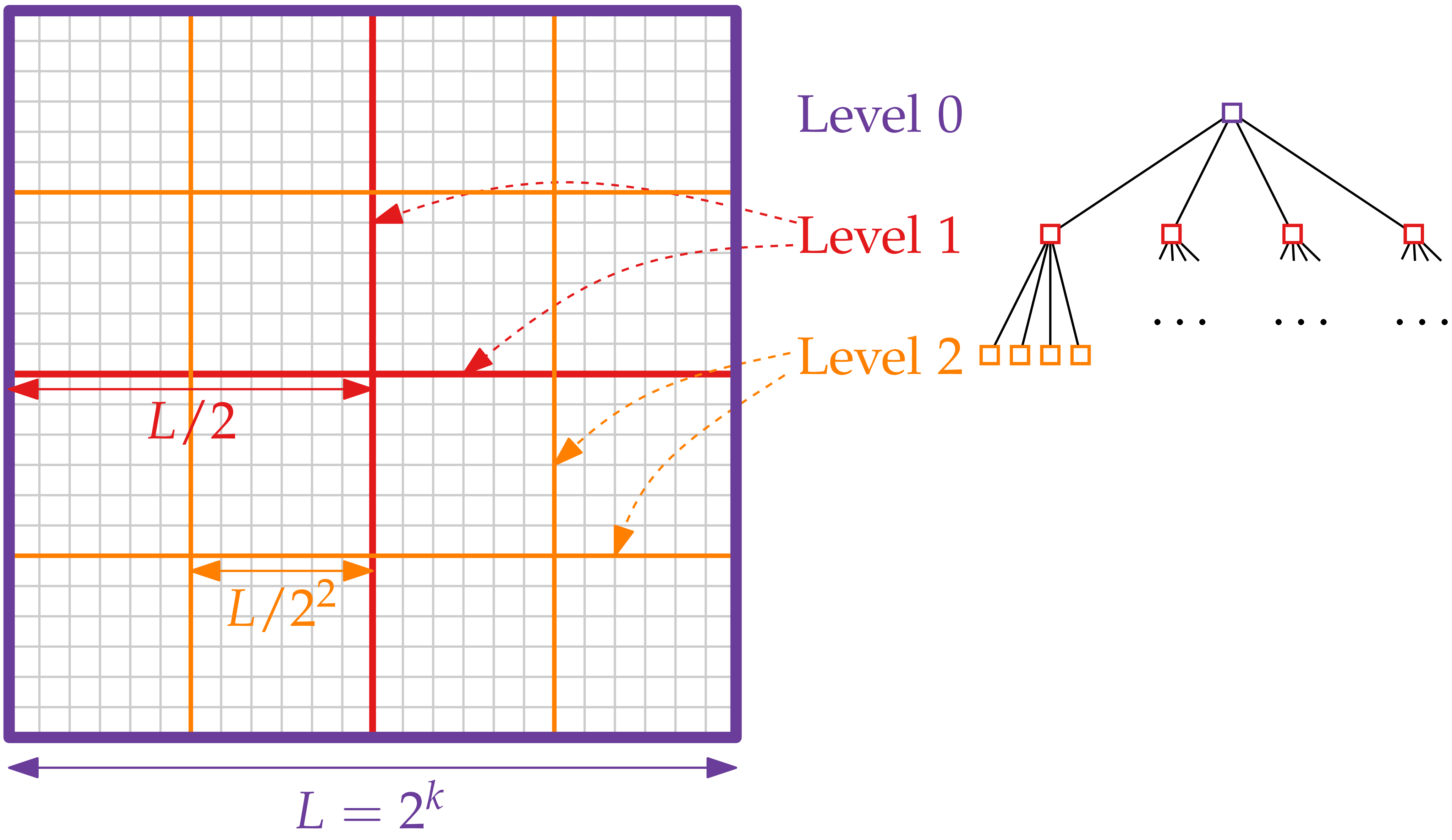

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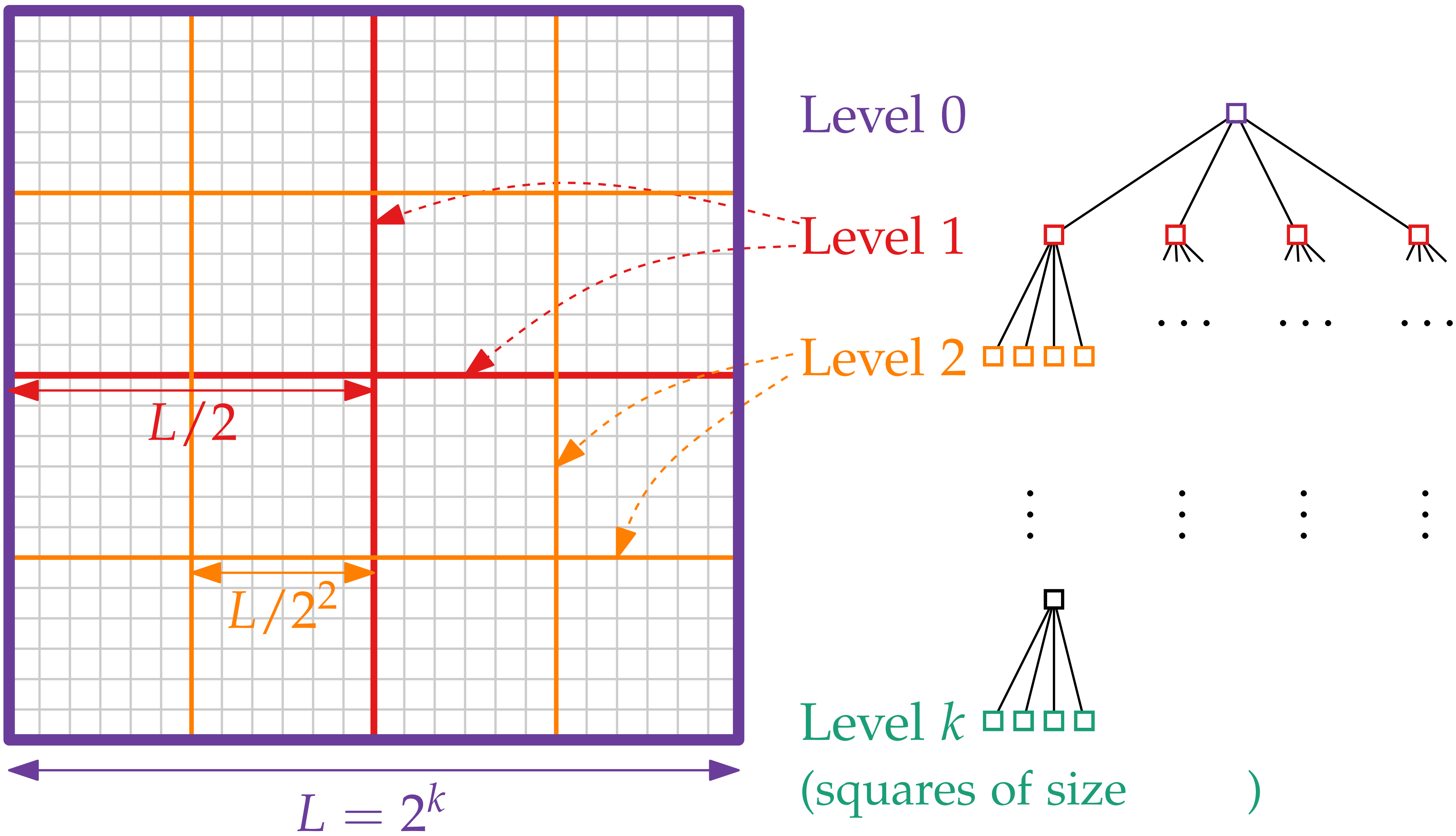

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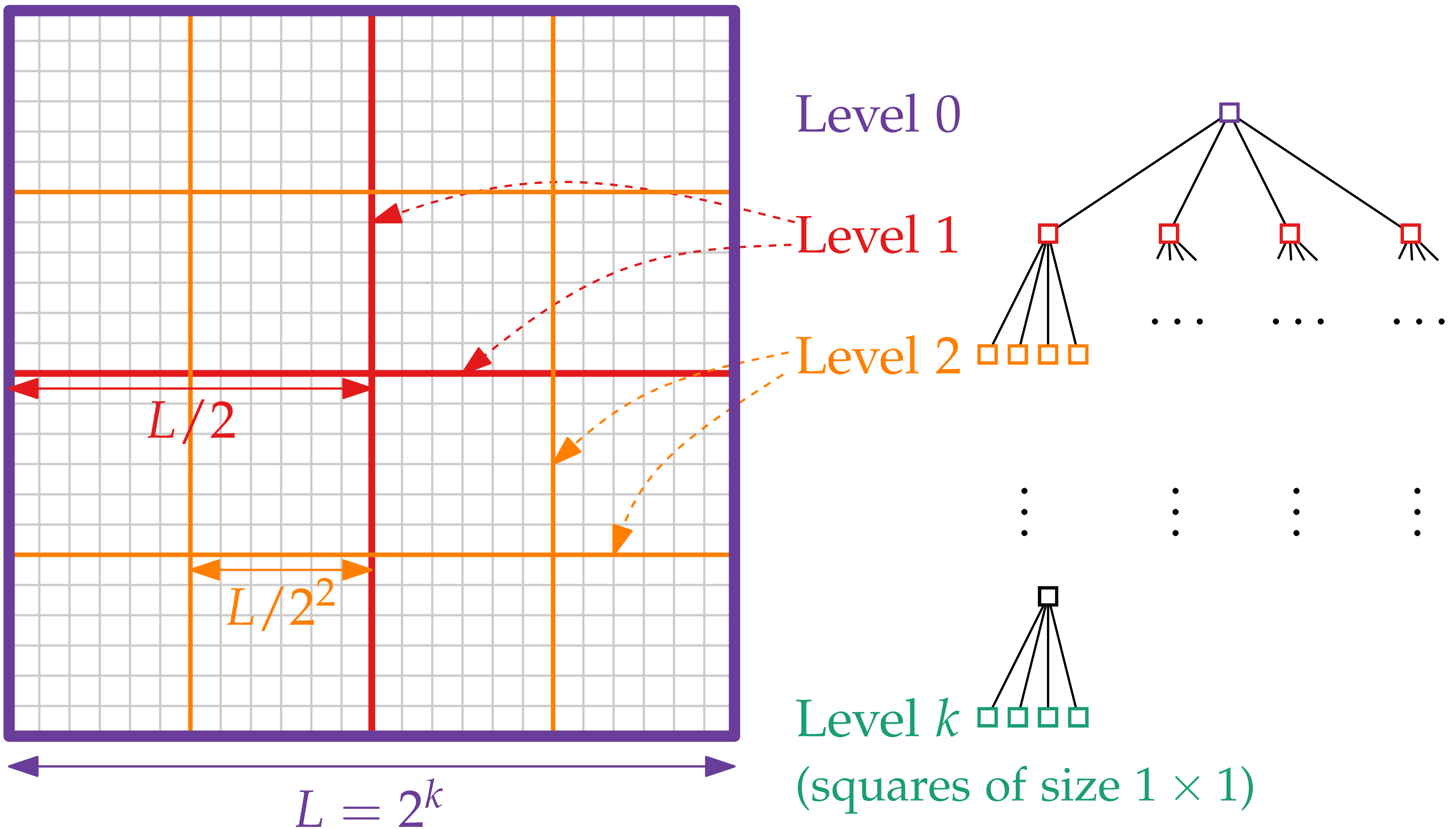
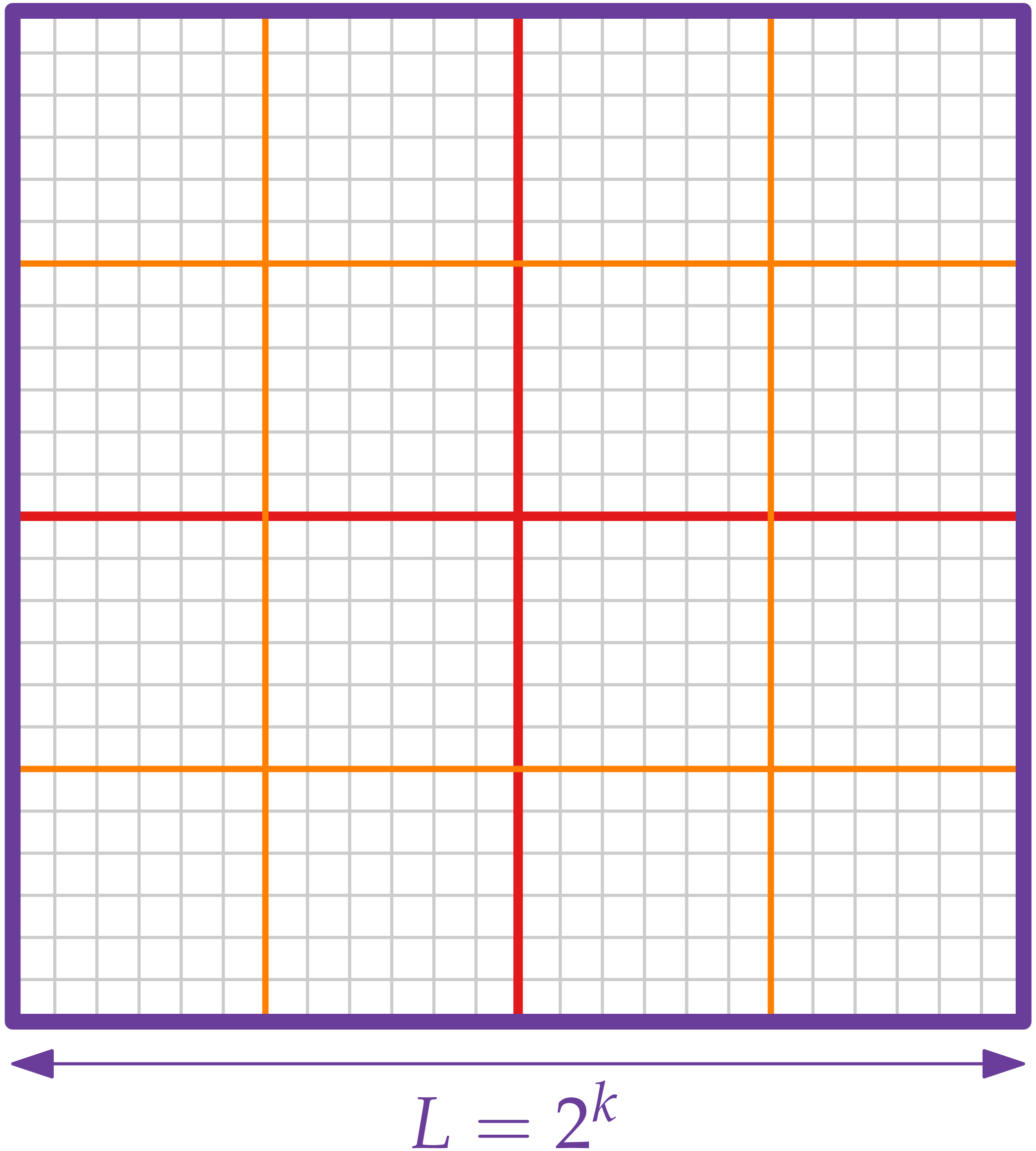


Portals



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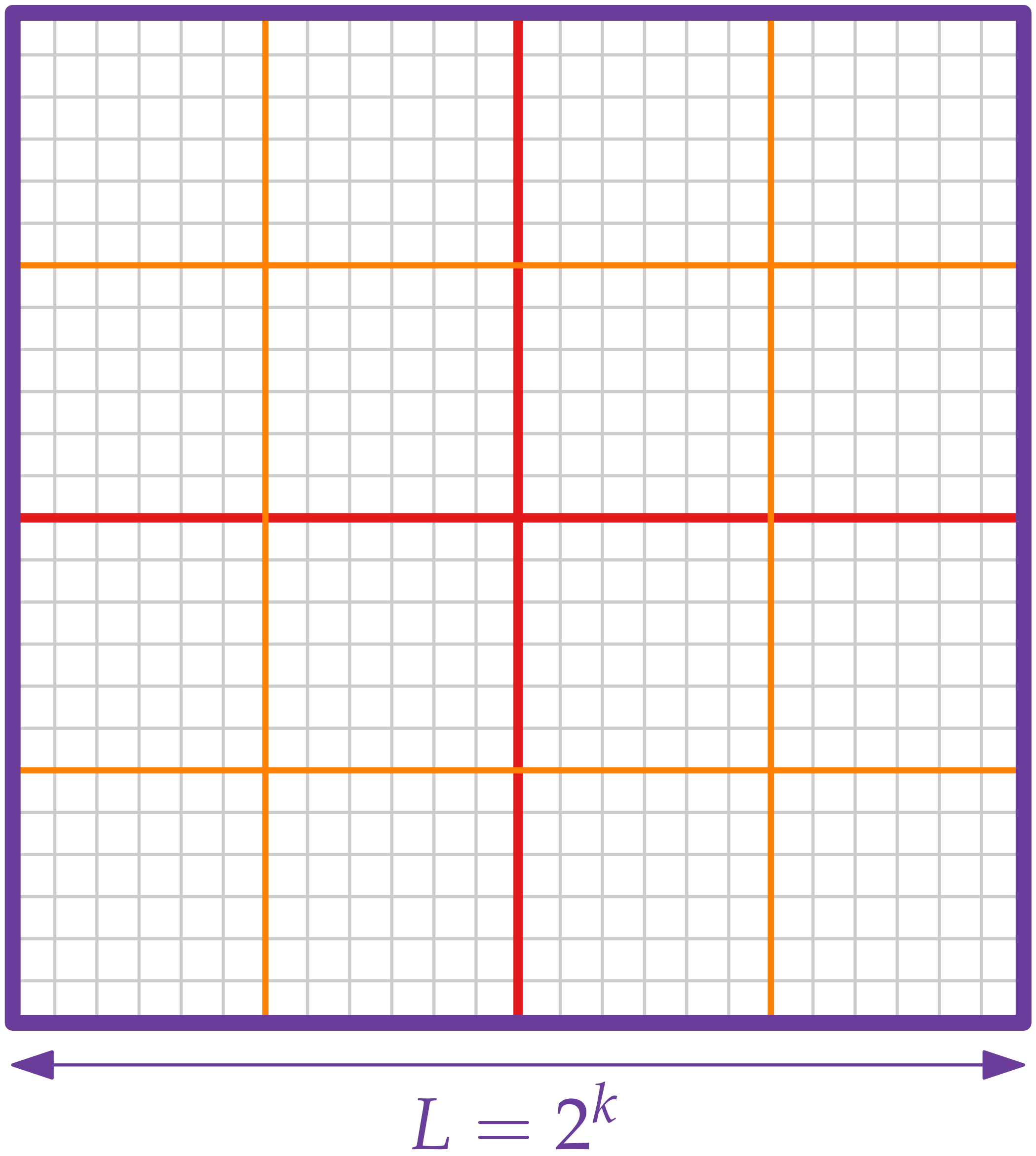

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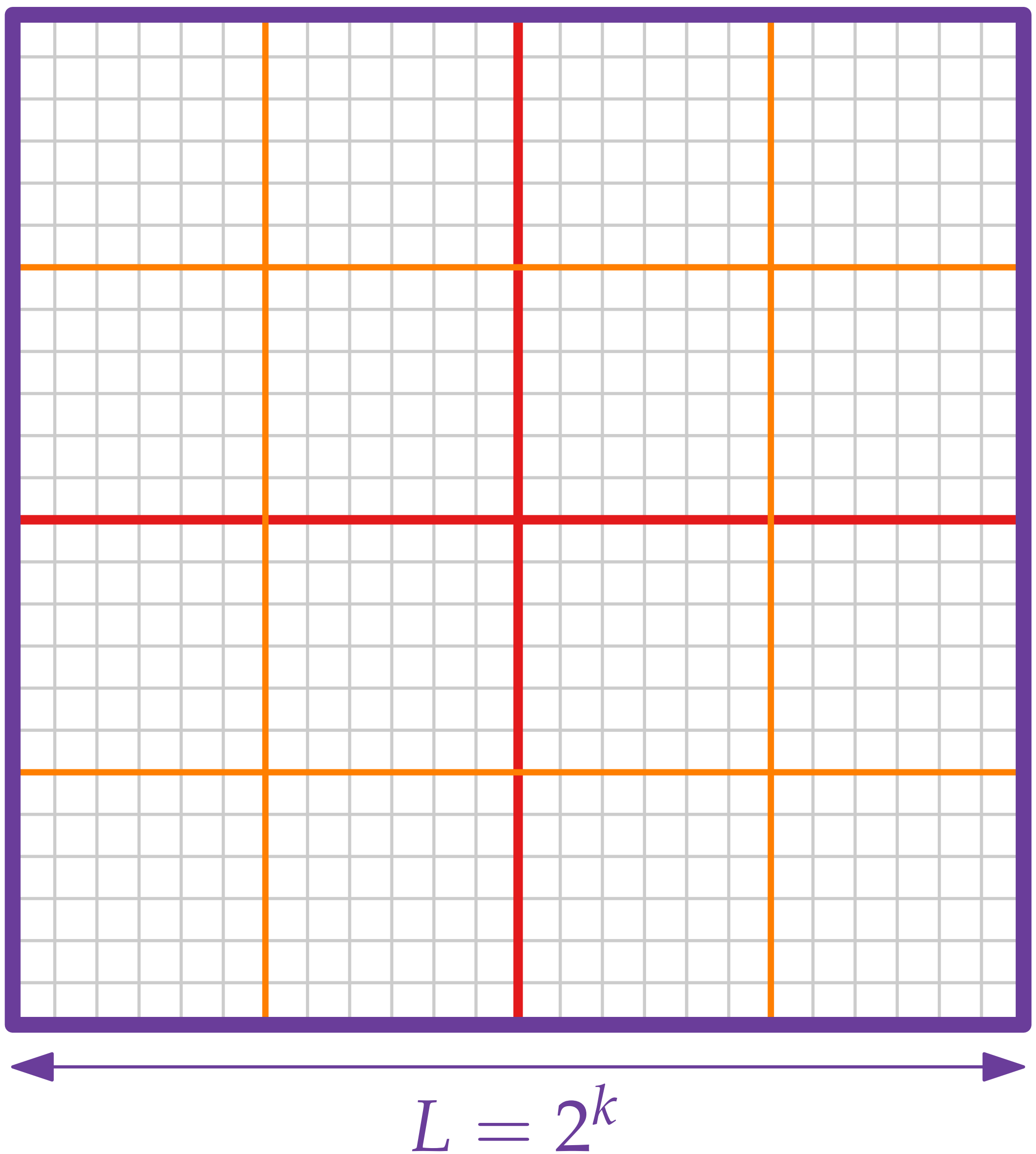

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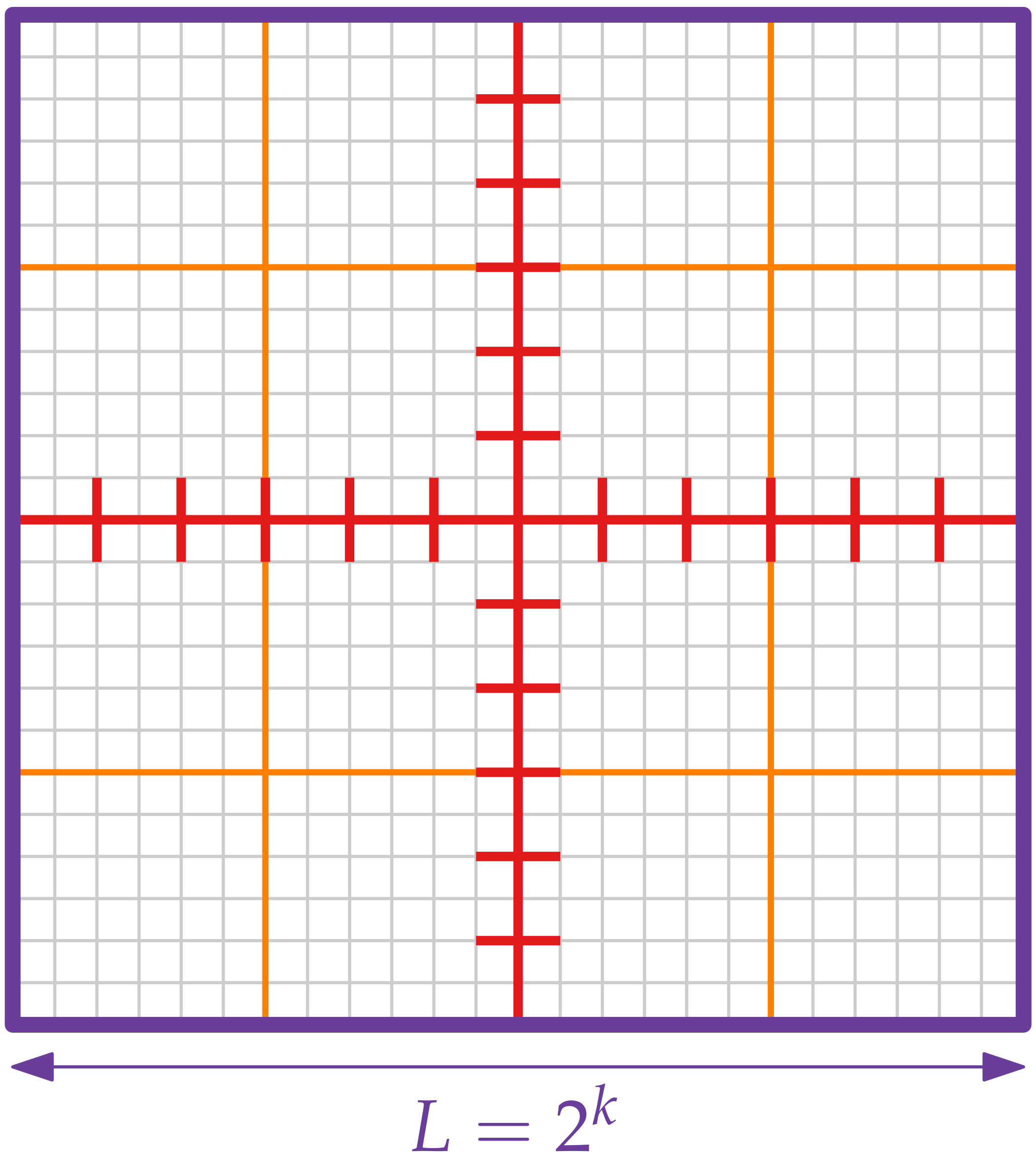

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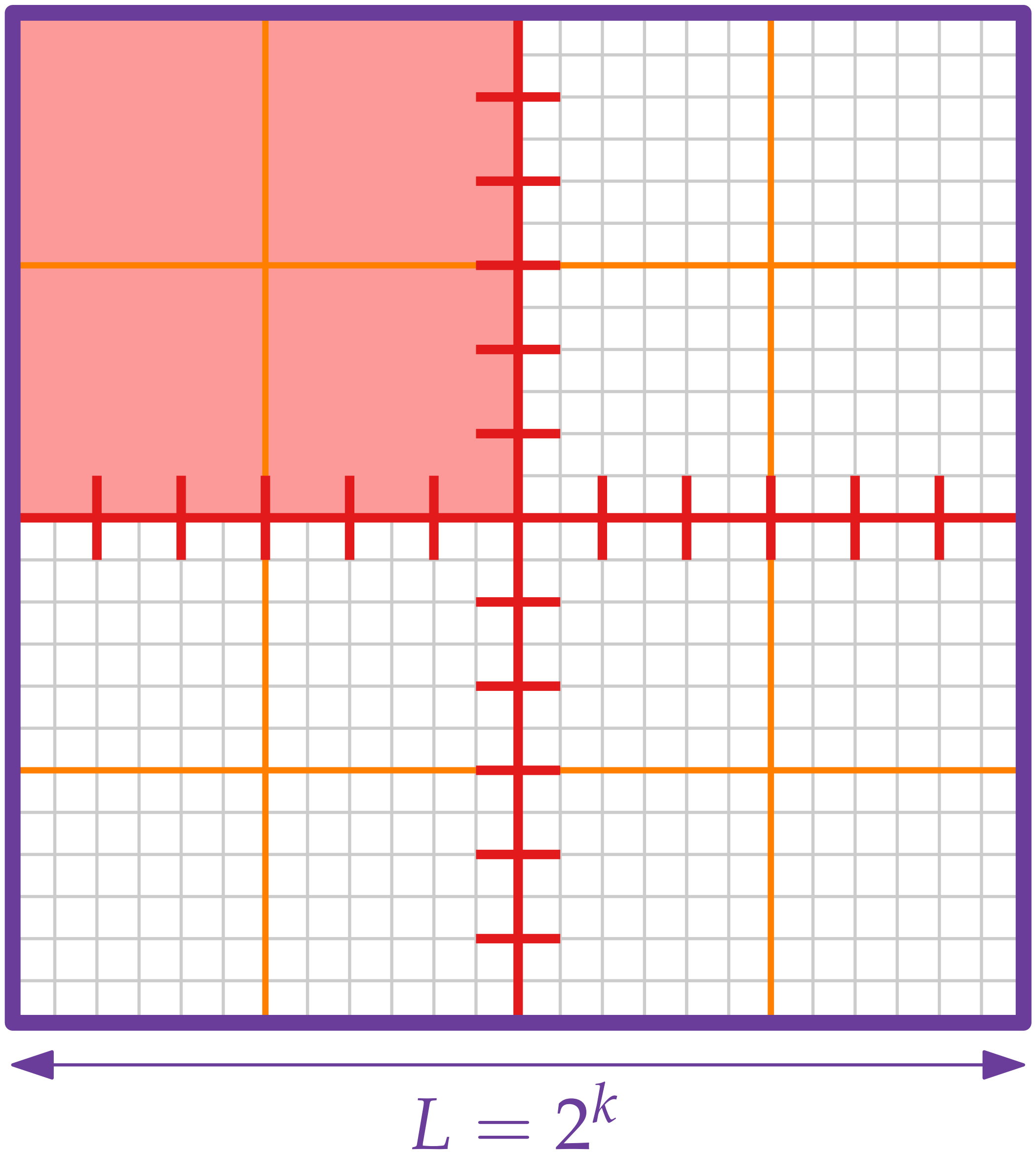

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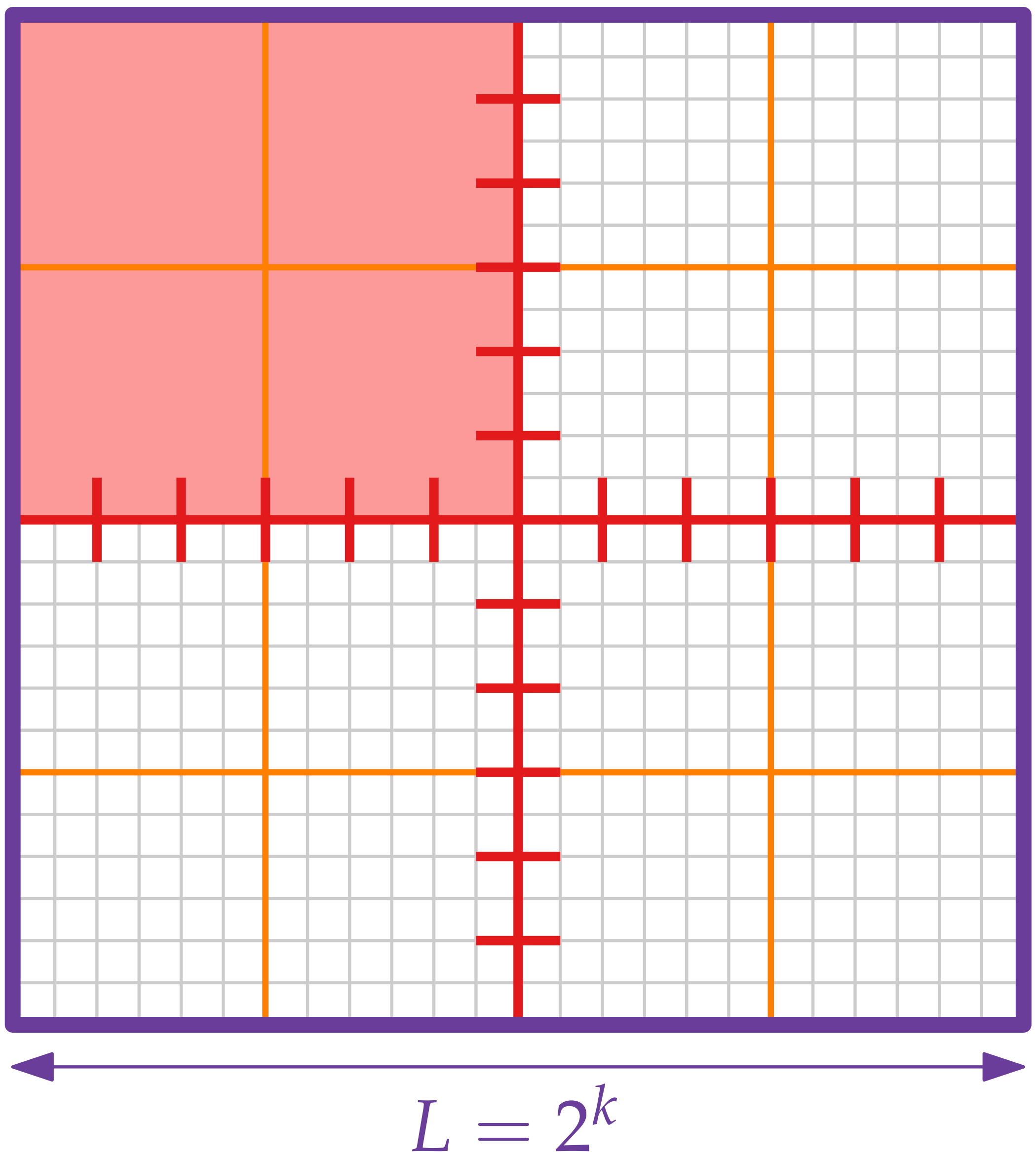

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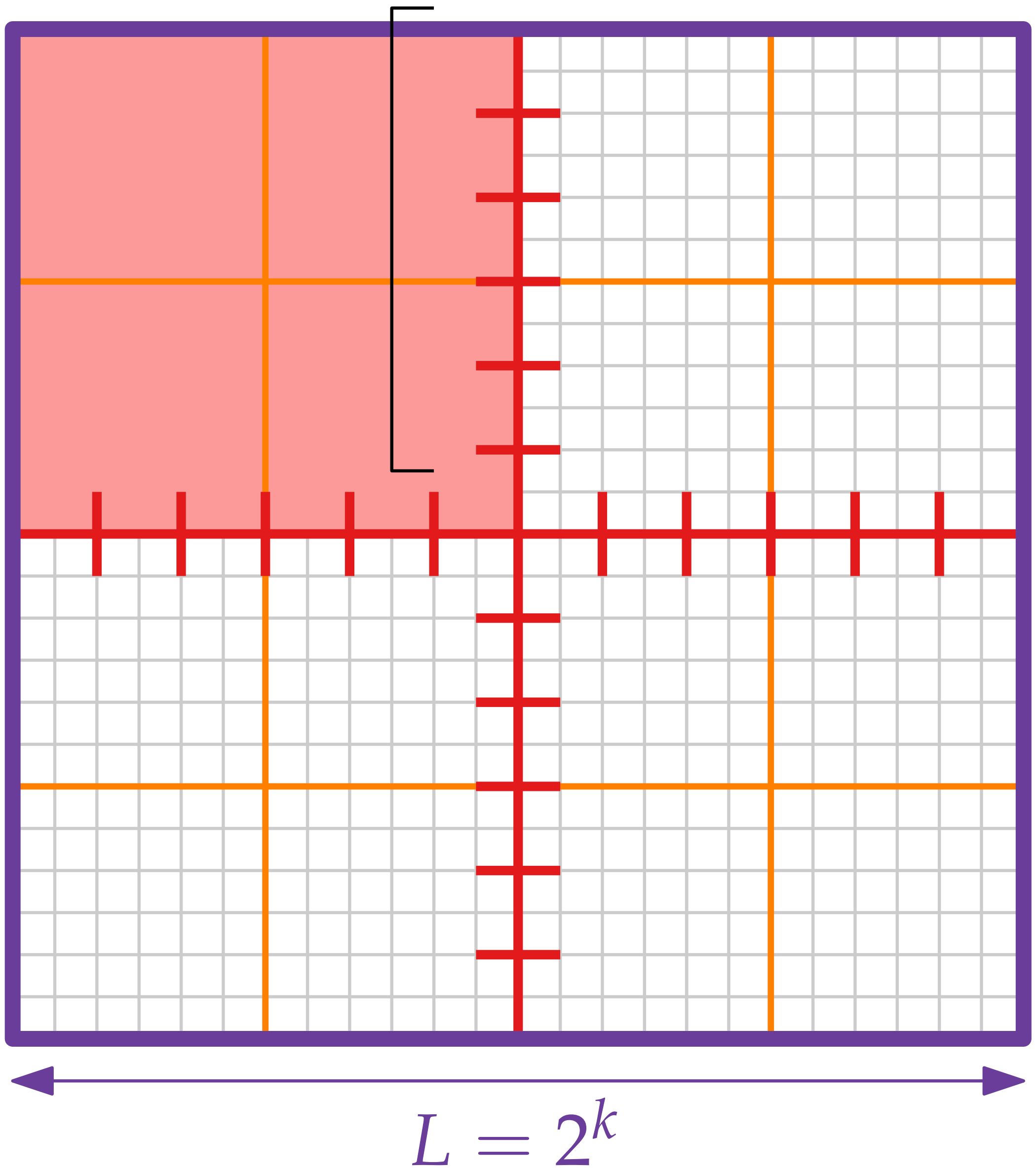

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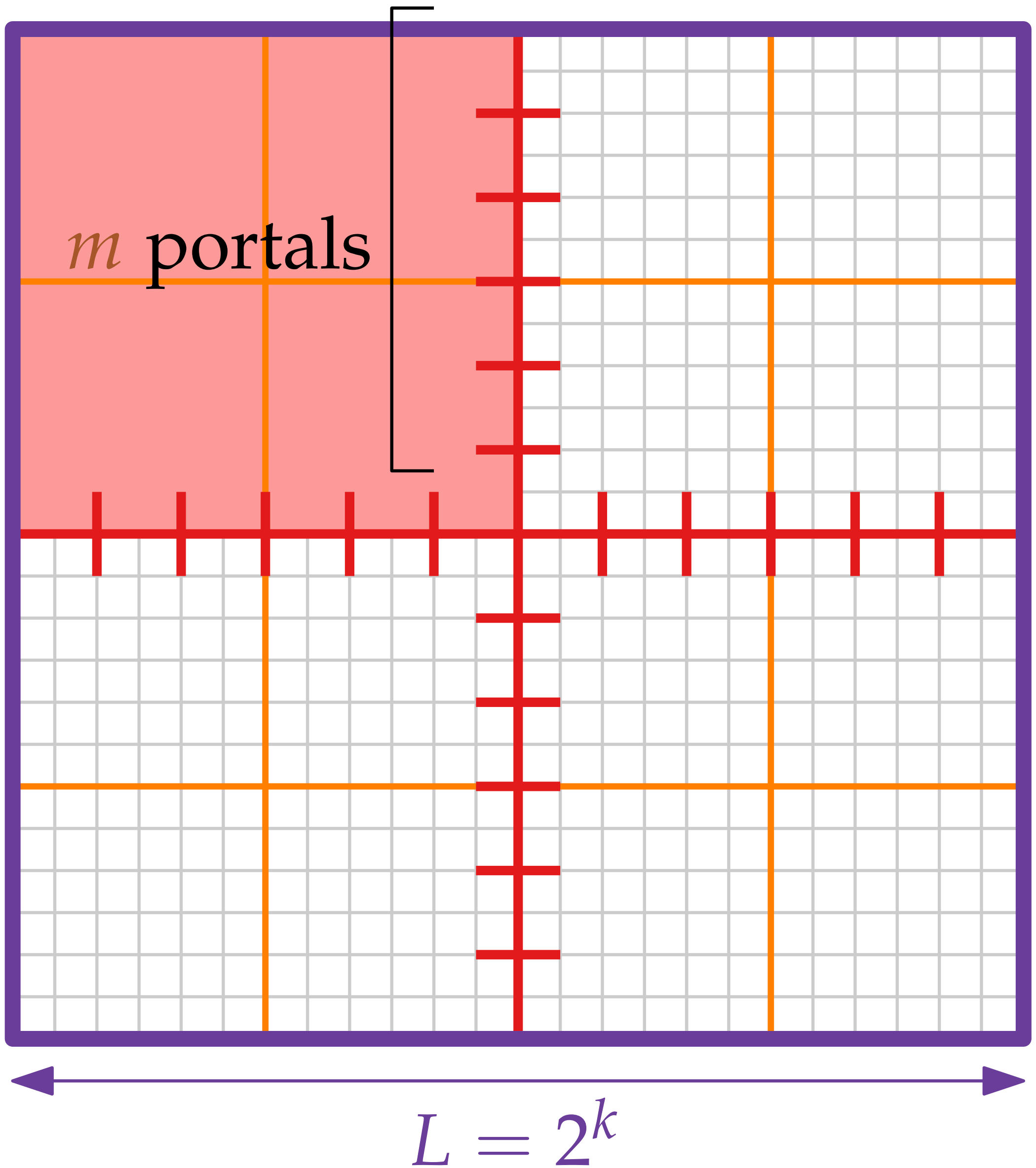

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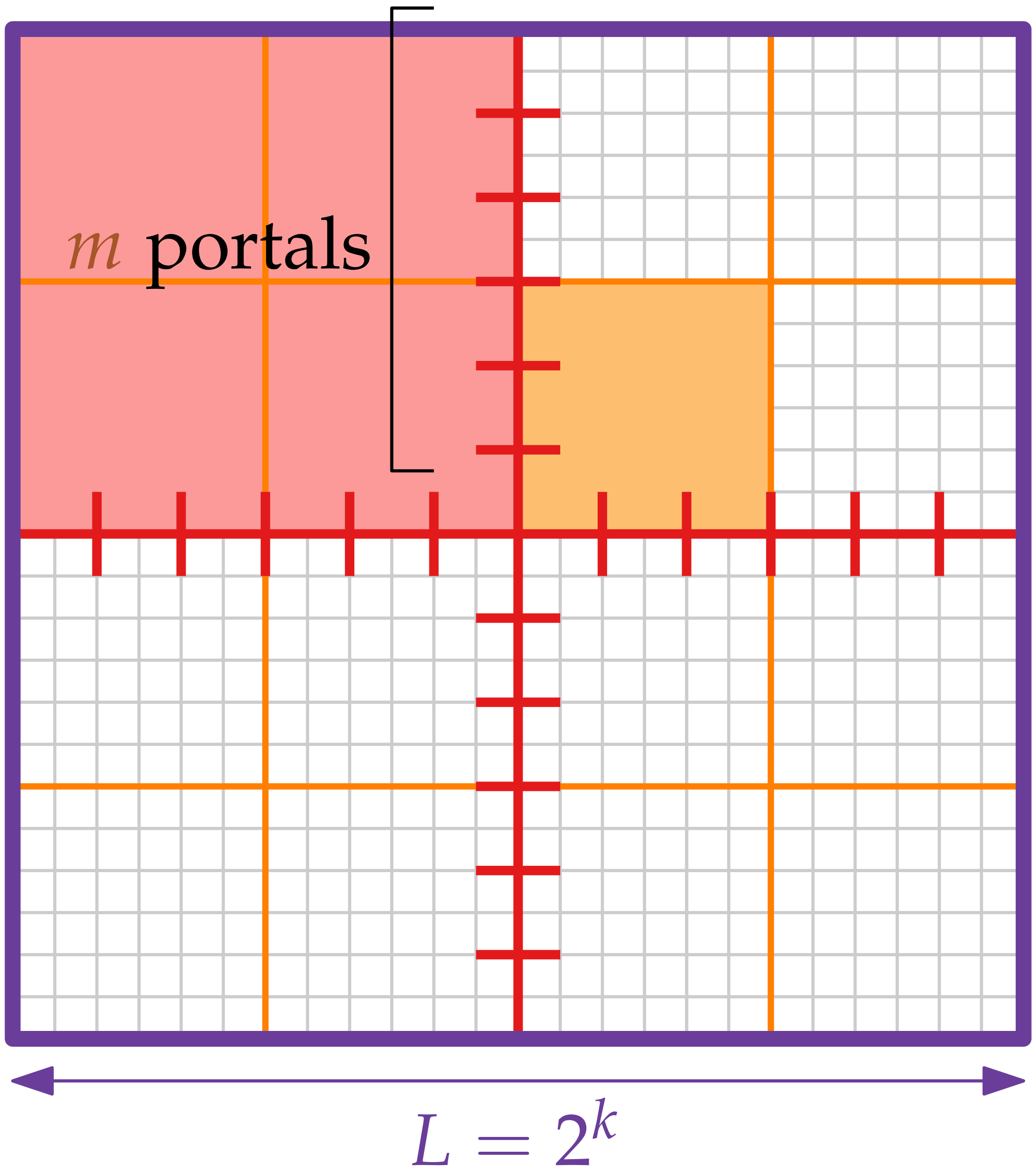

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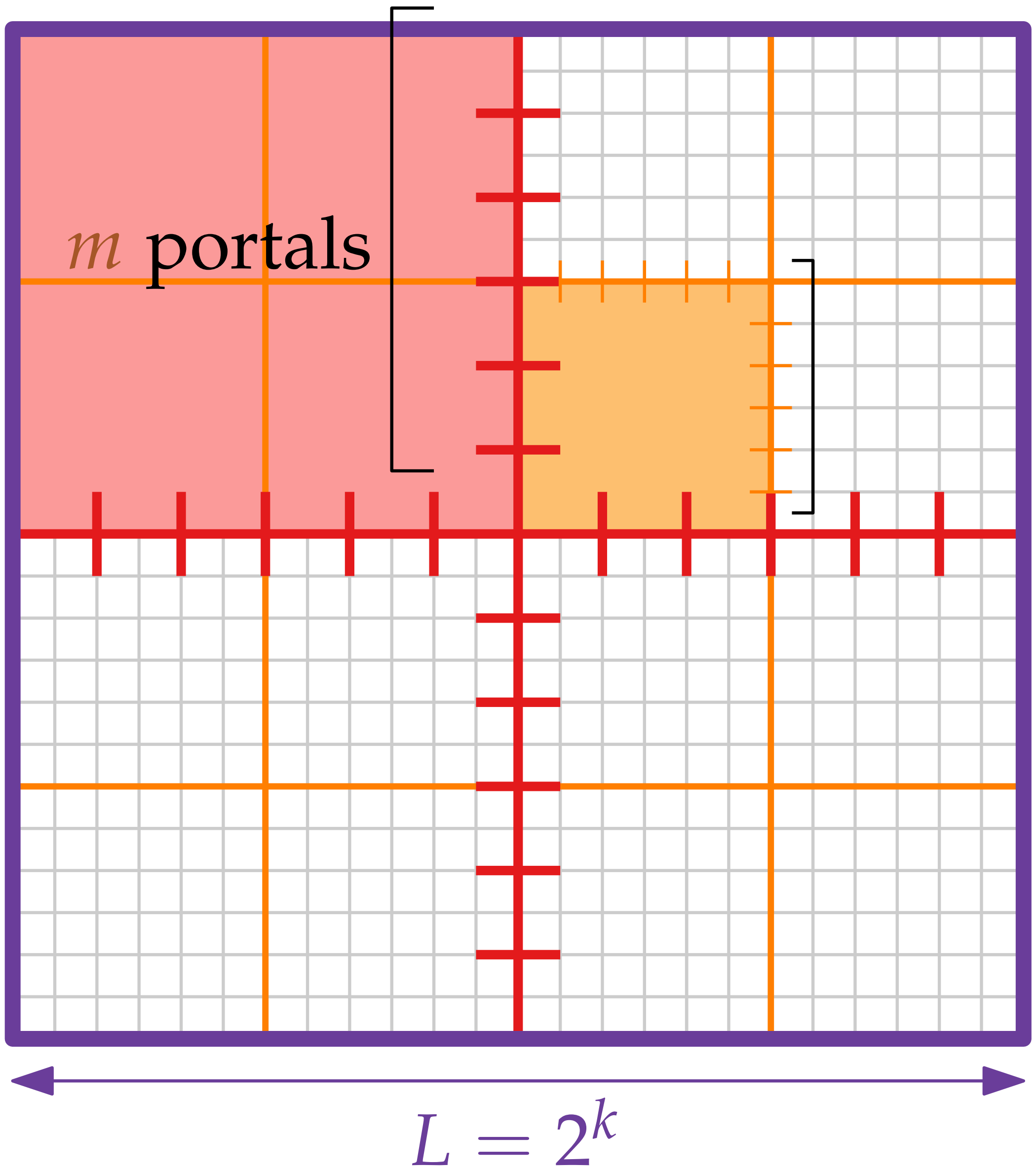

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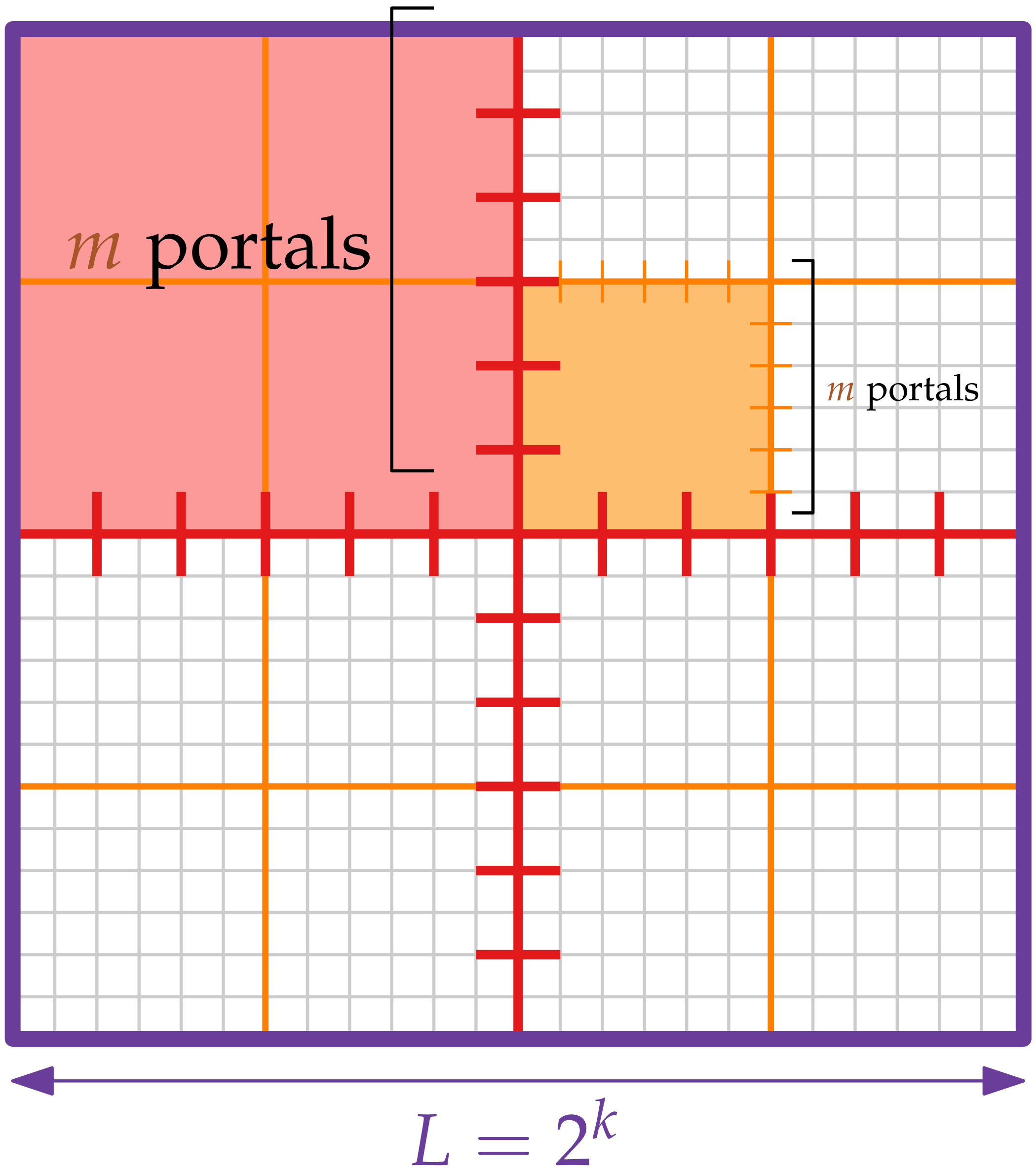

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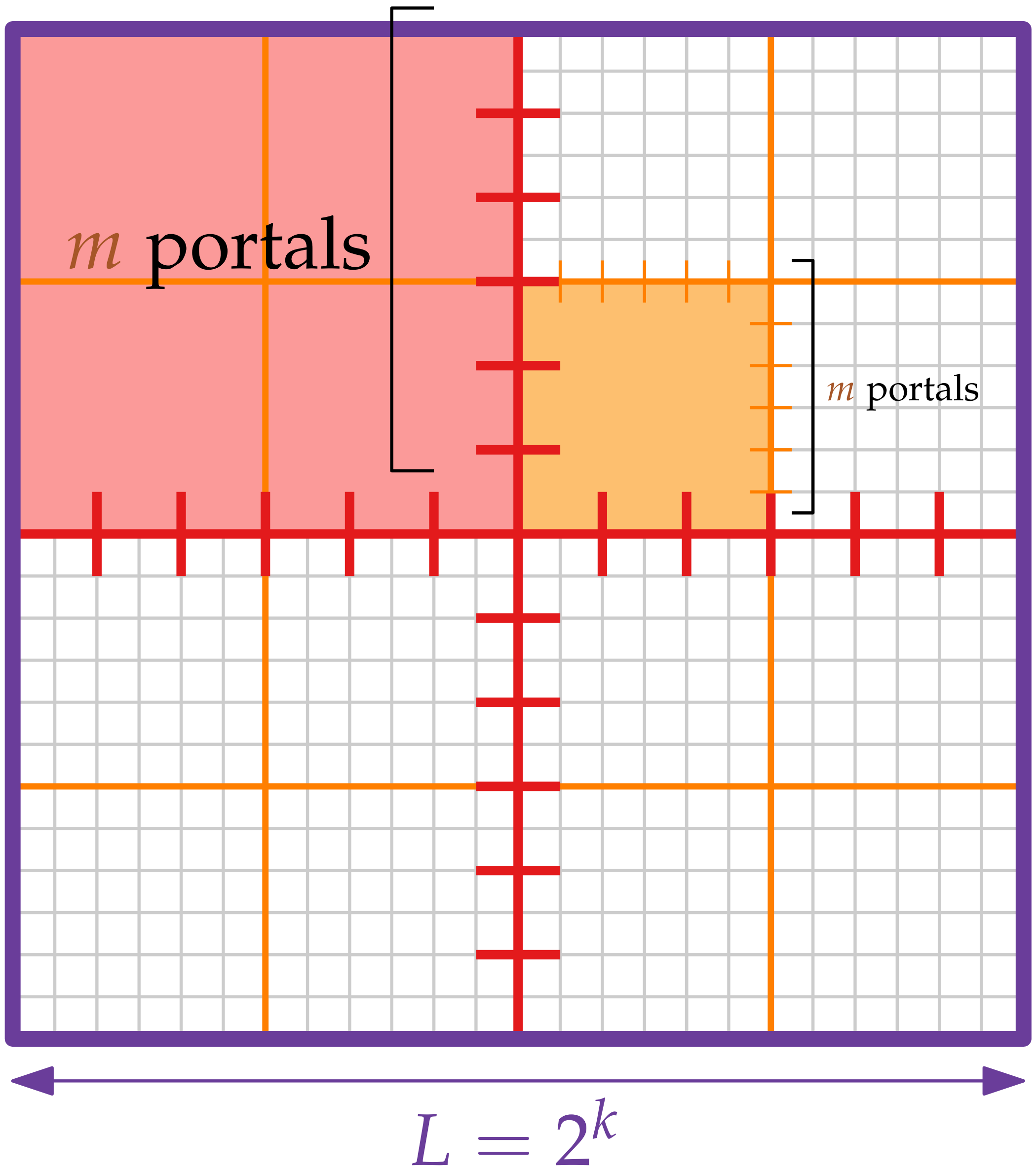

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- A level-i square has $\leq 4m$ portals on its boundary.

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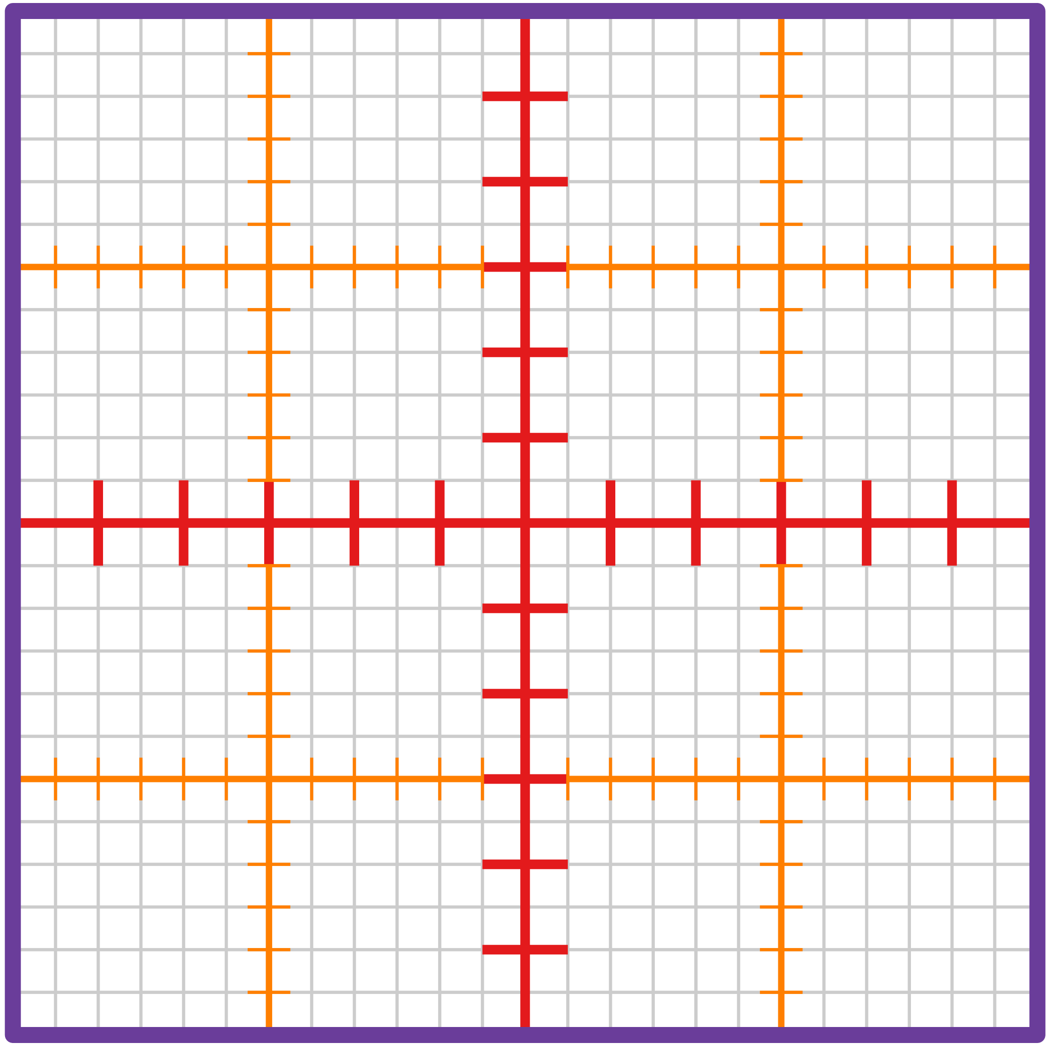
Part III:
Well-Behaved Tours

Well-Behaved Tours



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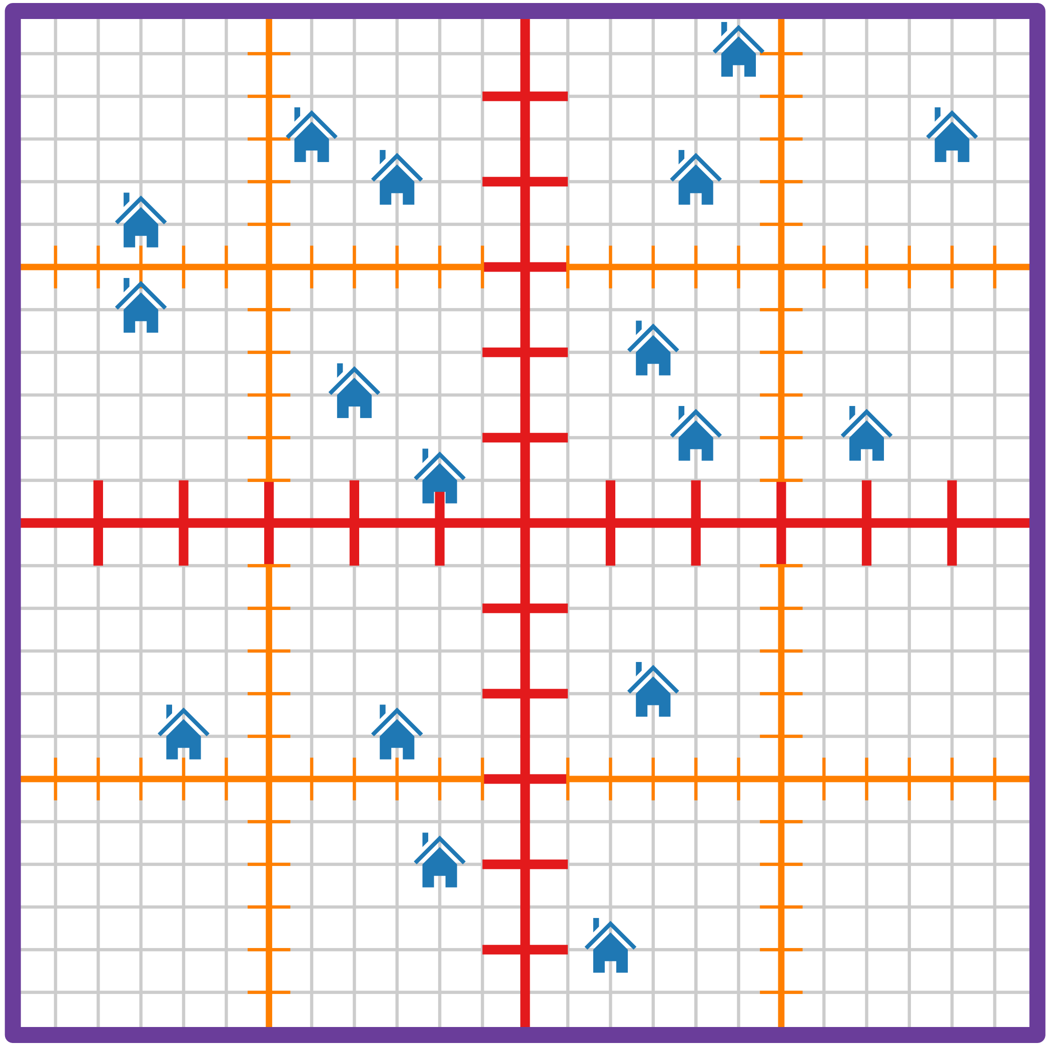

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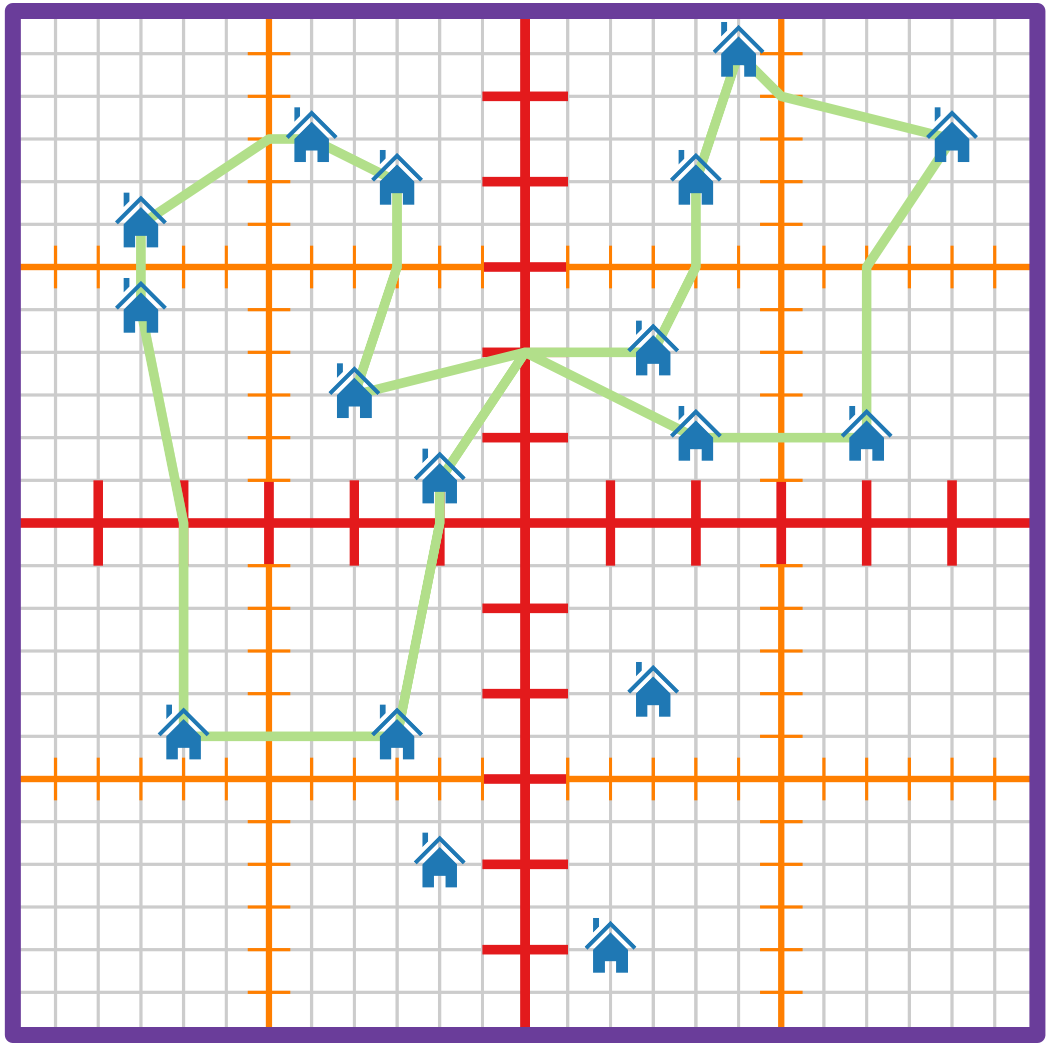

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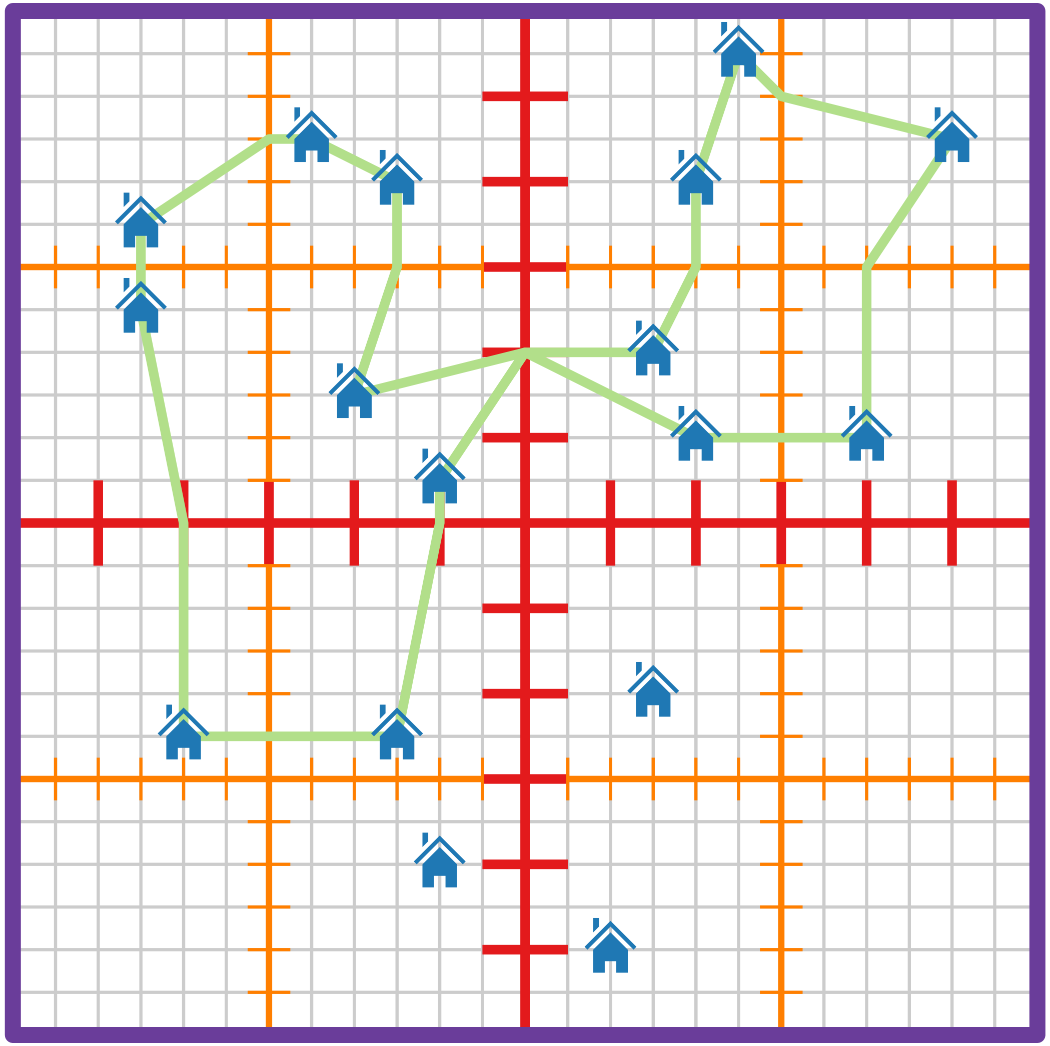

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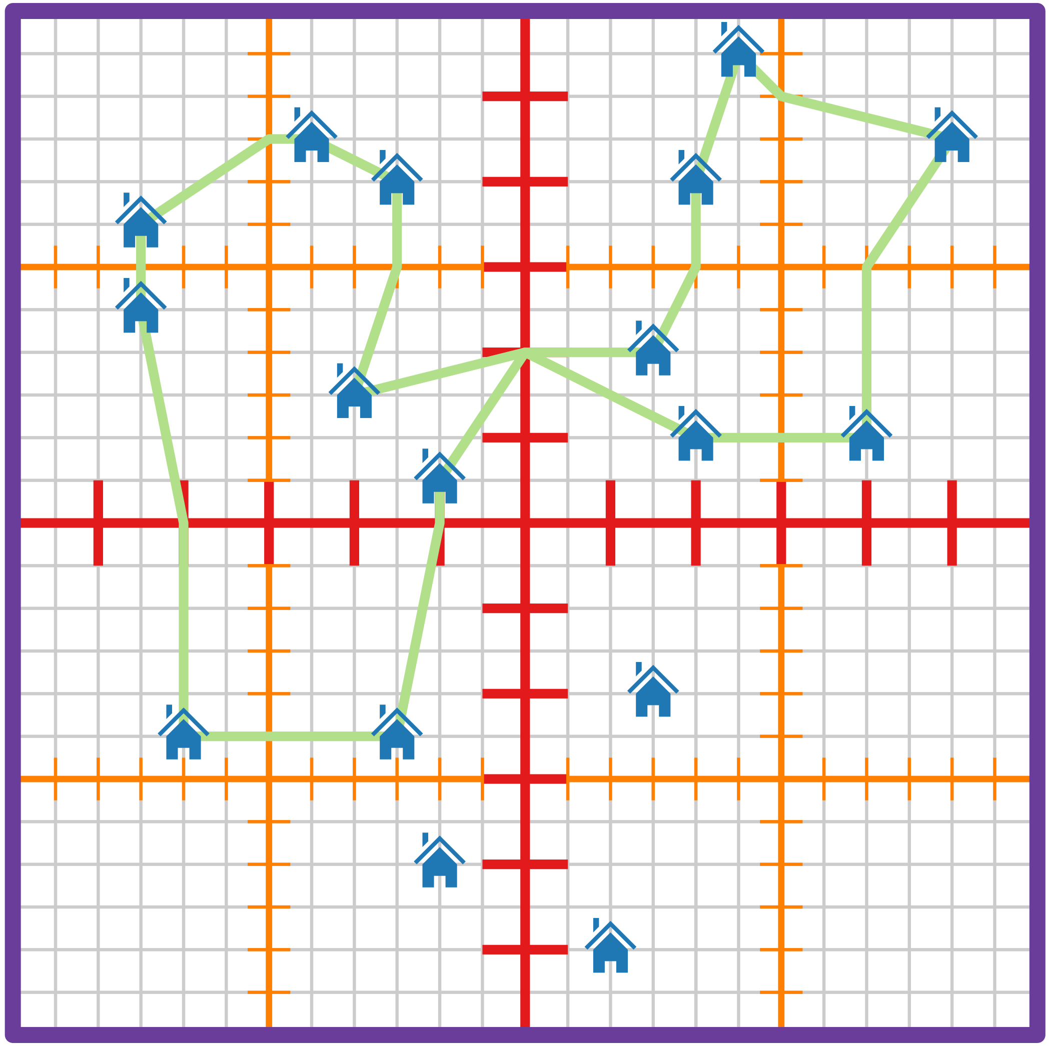

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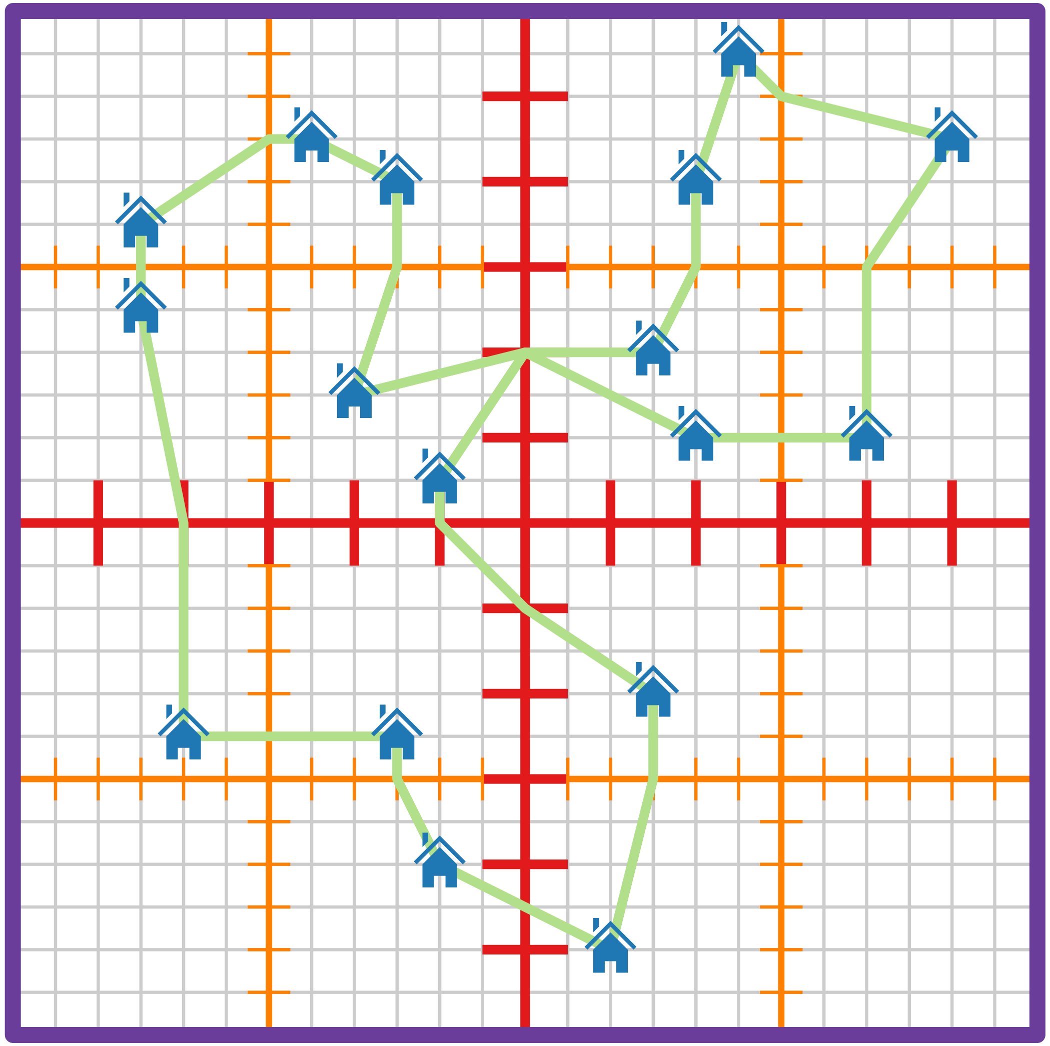

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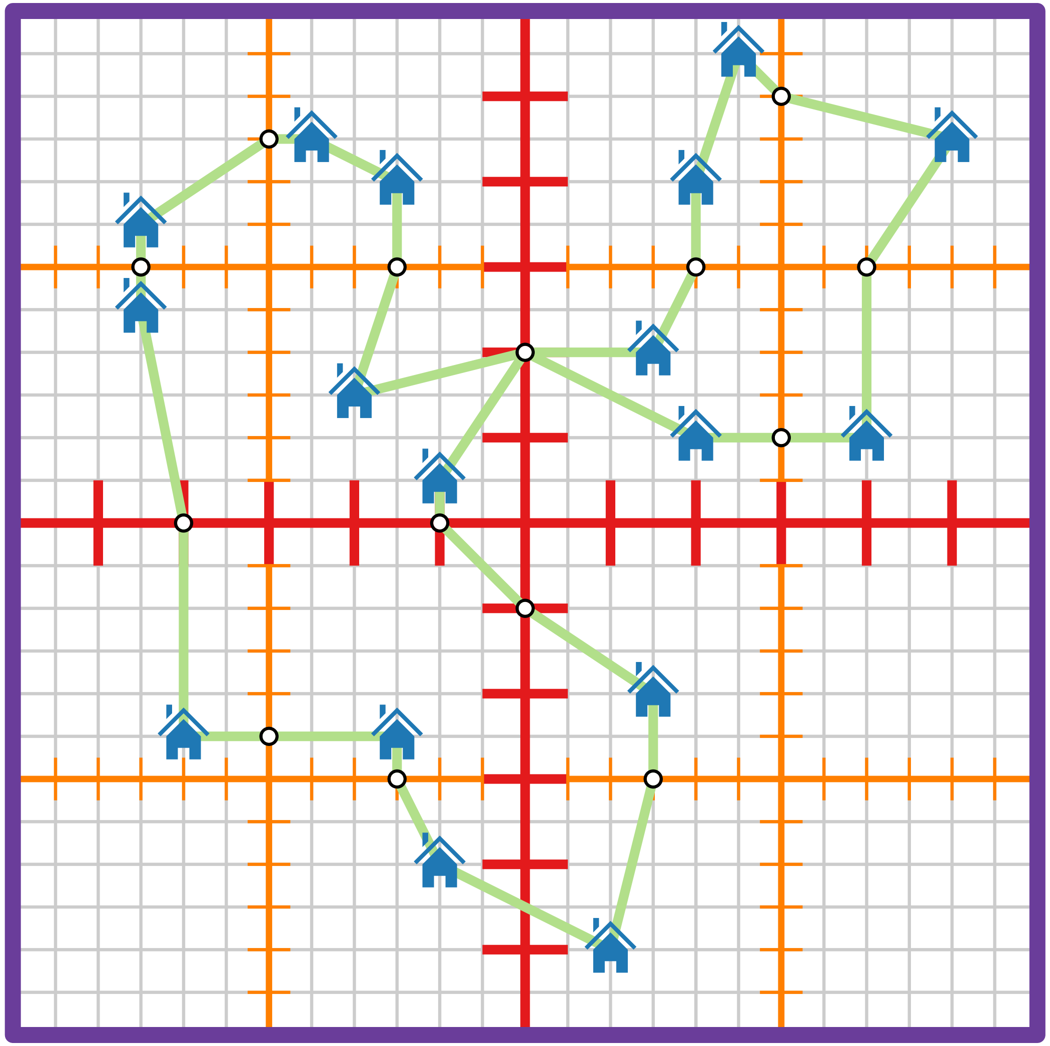
Well-Behaved Tours



A tour is *well-behaved* if

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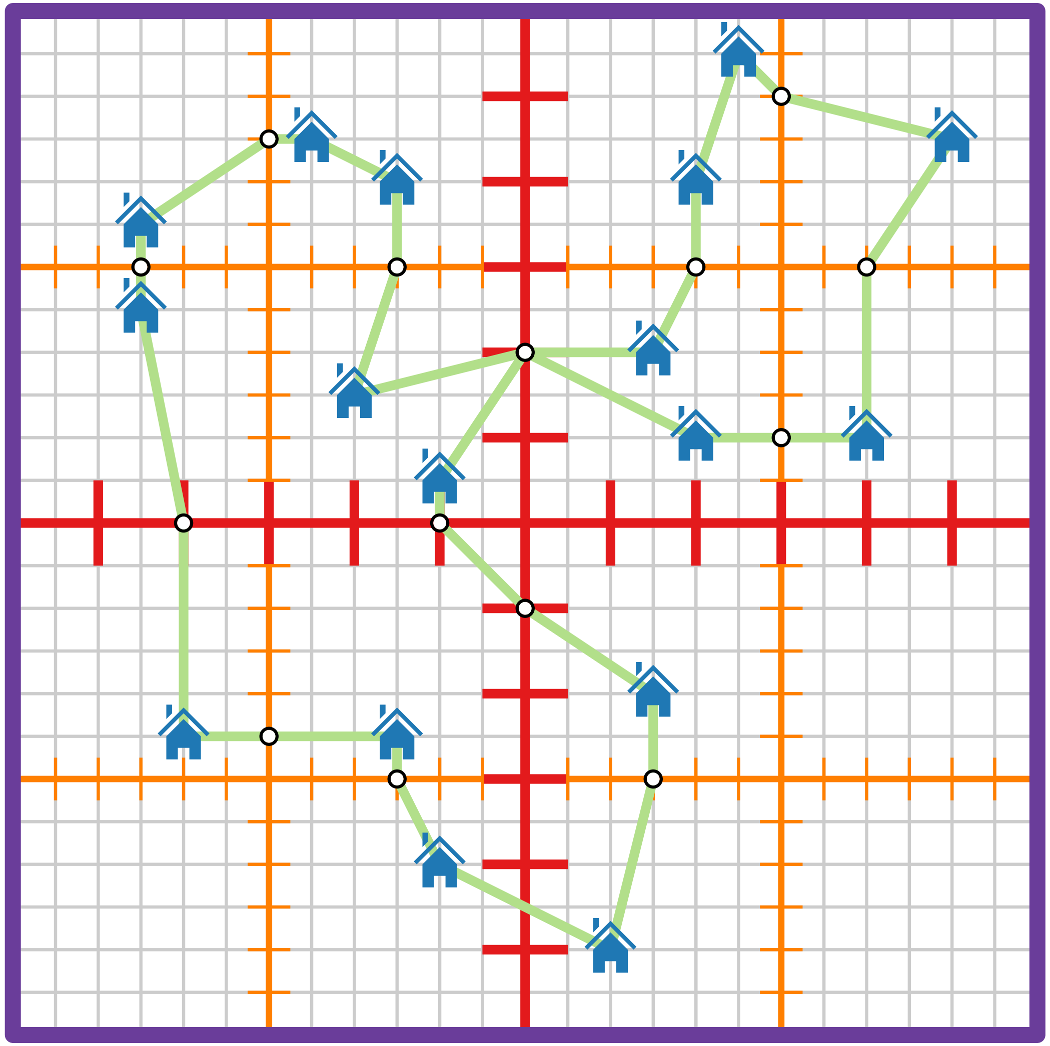
Well-Behaved Tours



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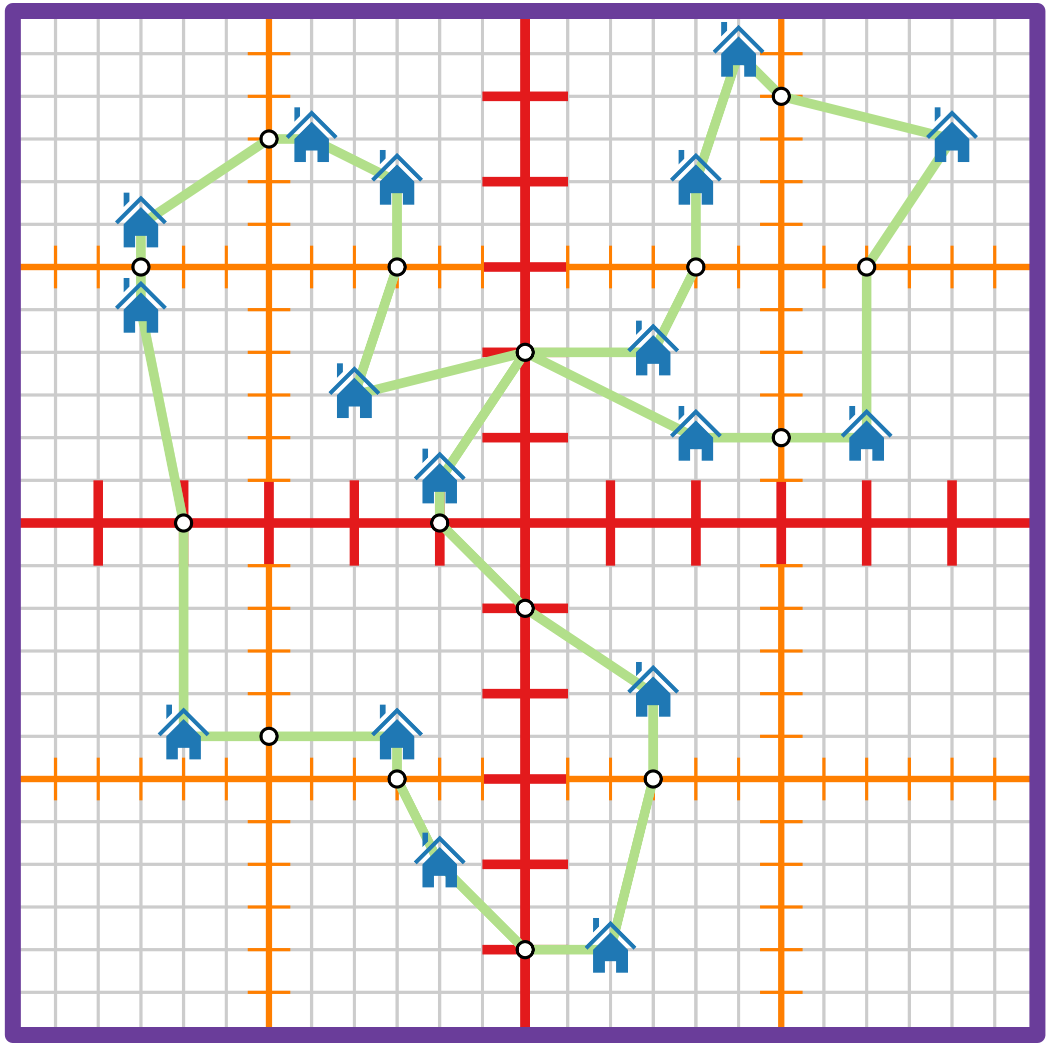

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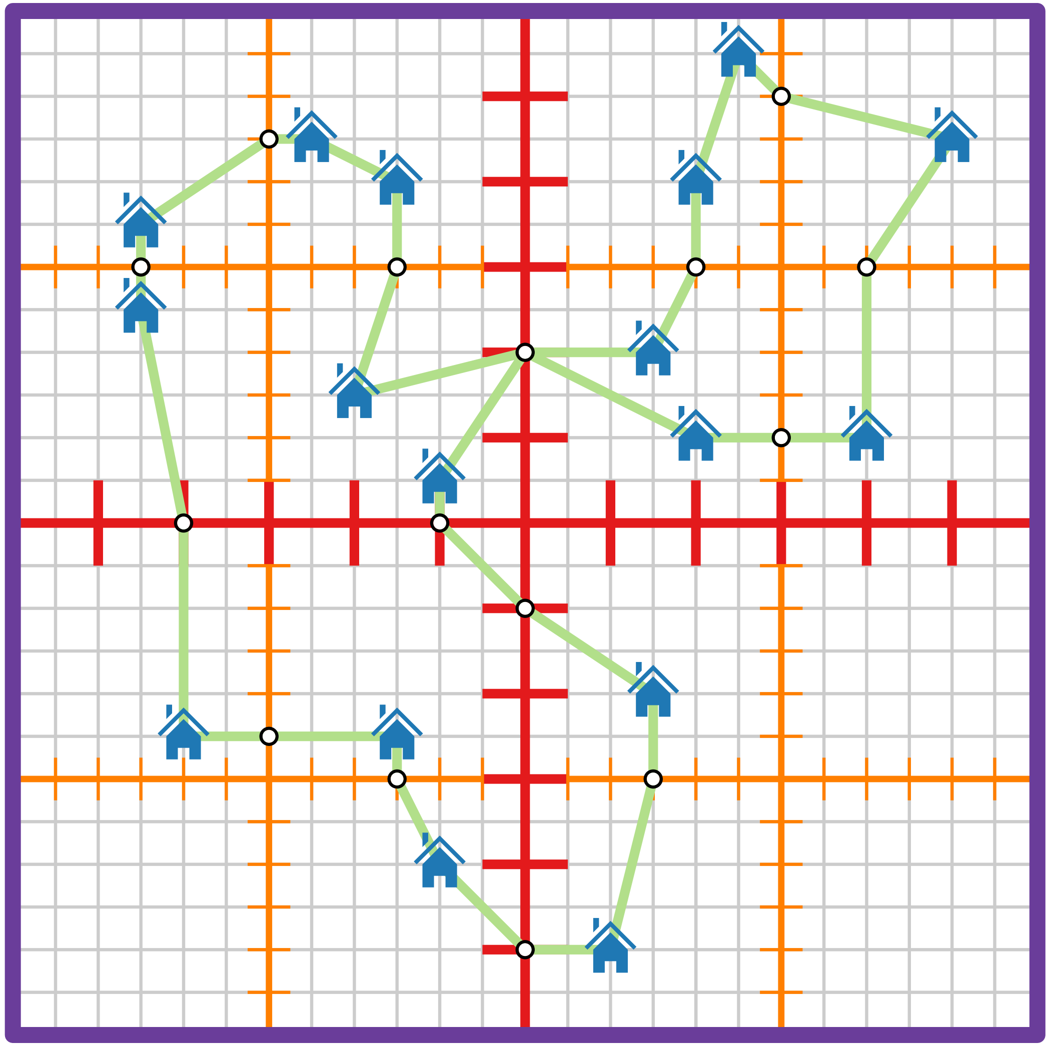

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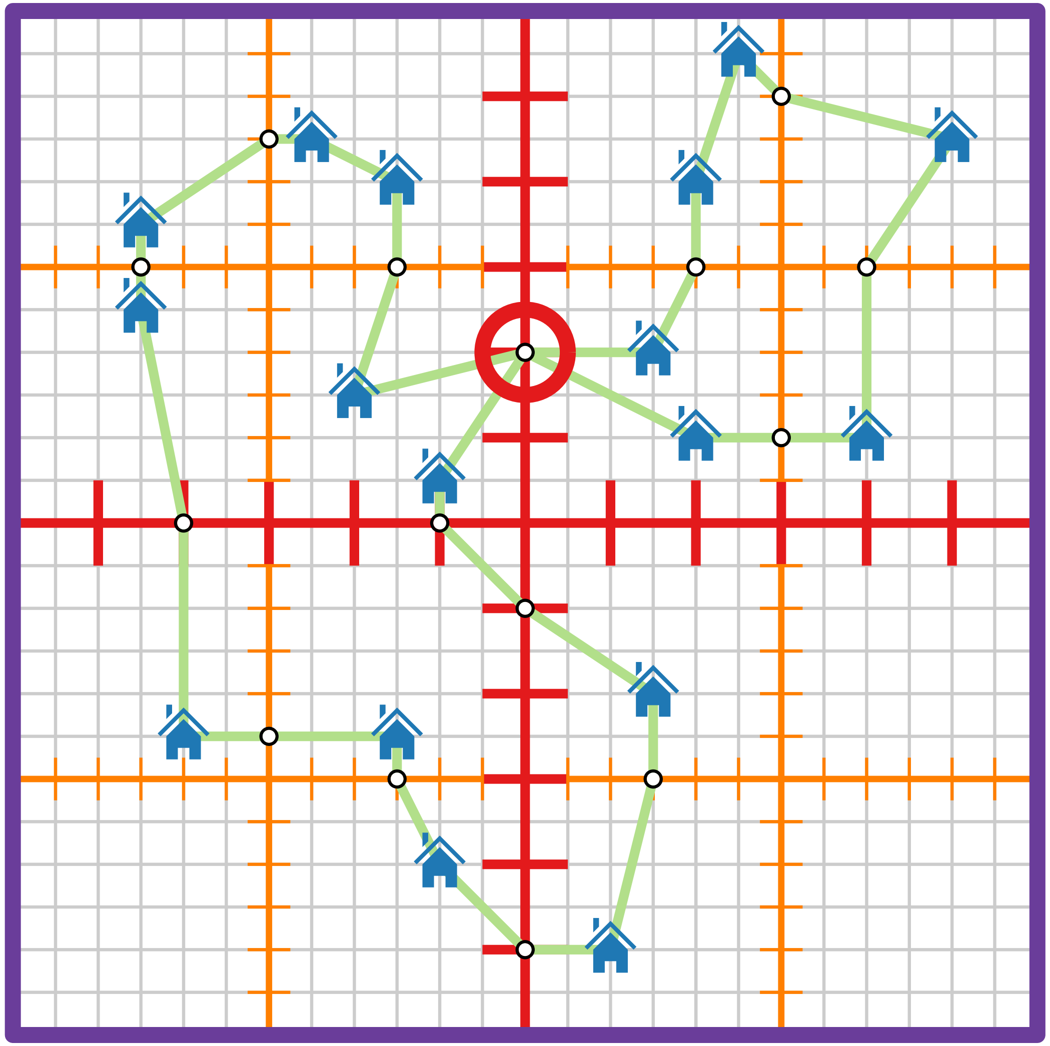

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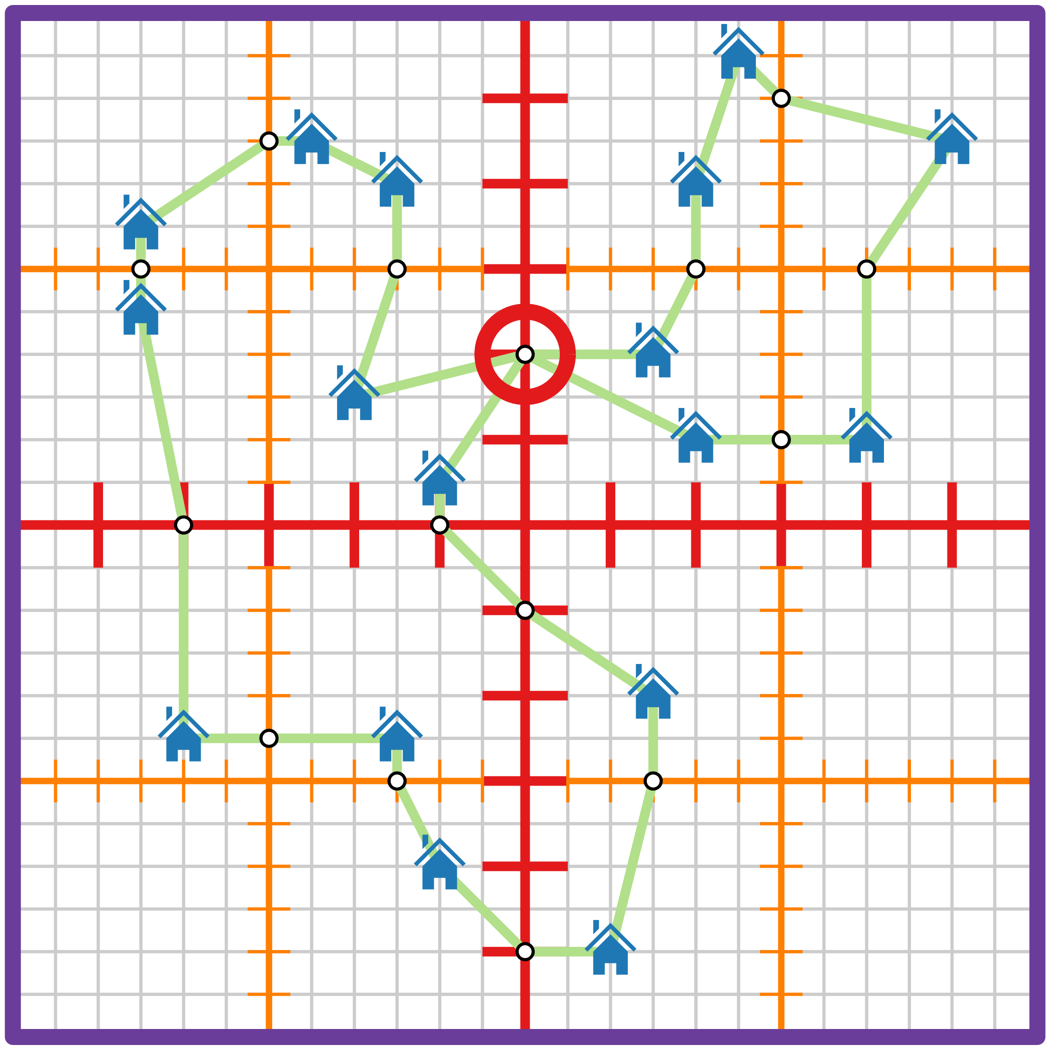

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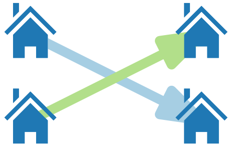
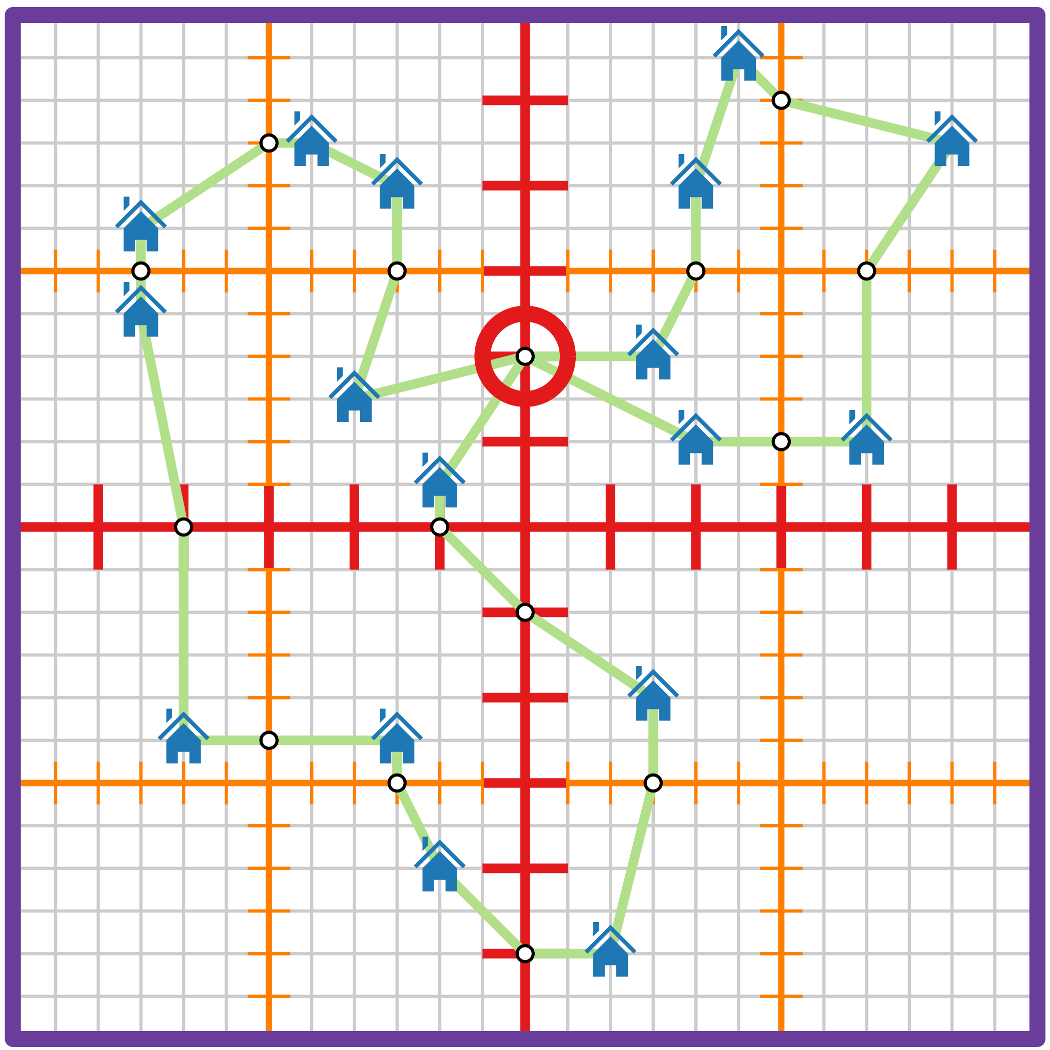
Crossing



Well-Behaved Tours



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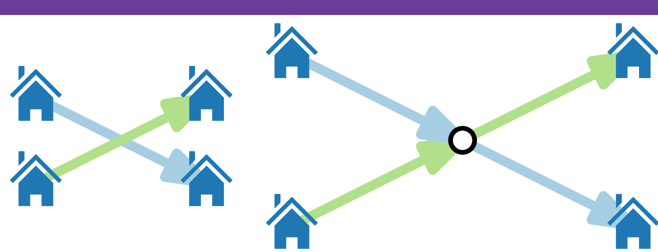

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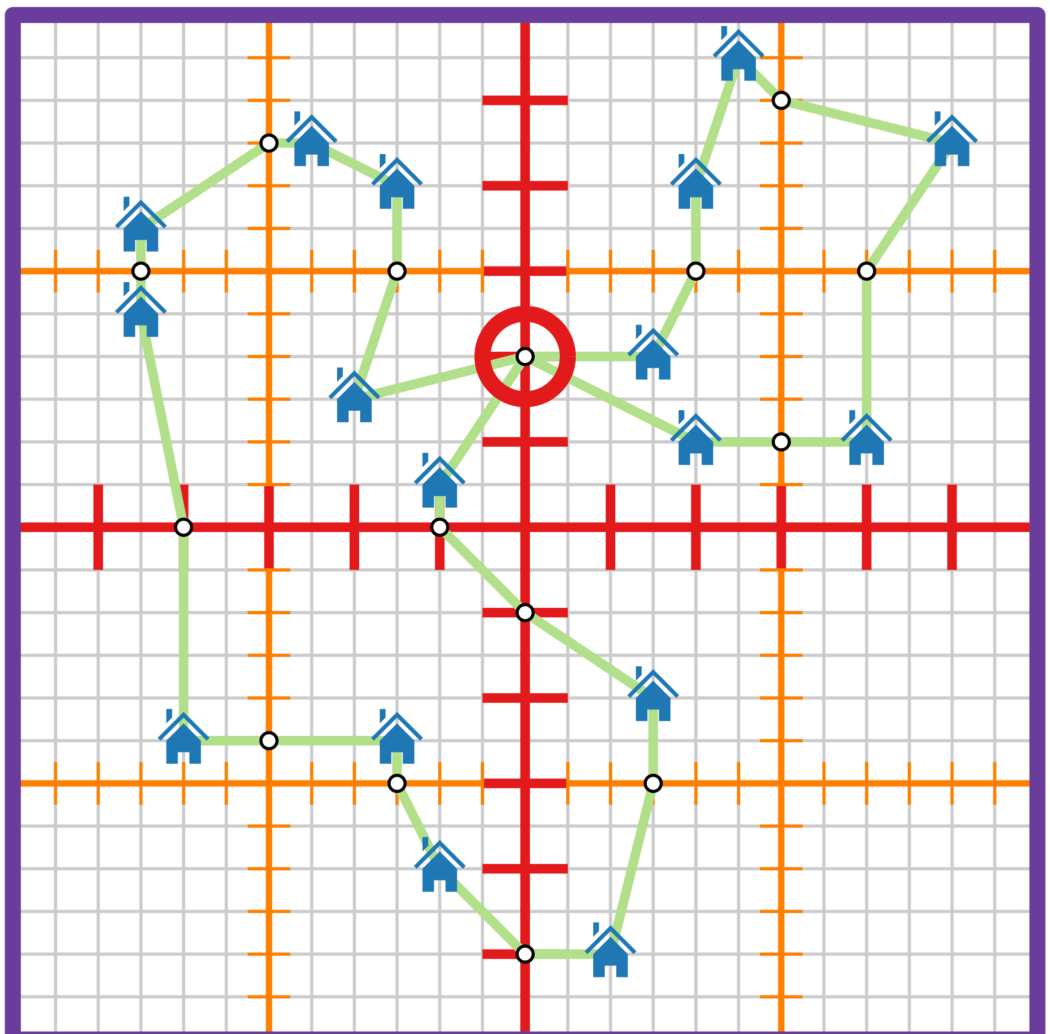

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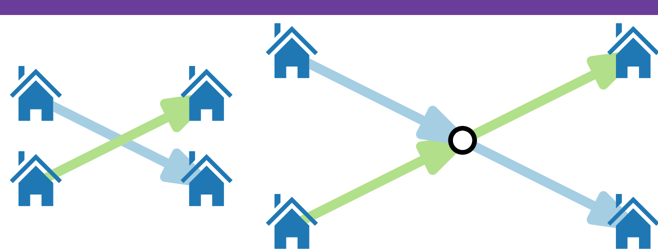
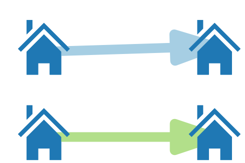
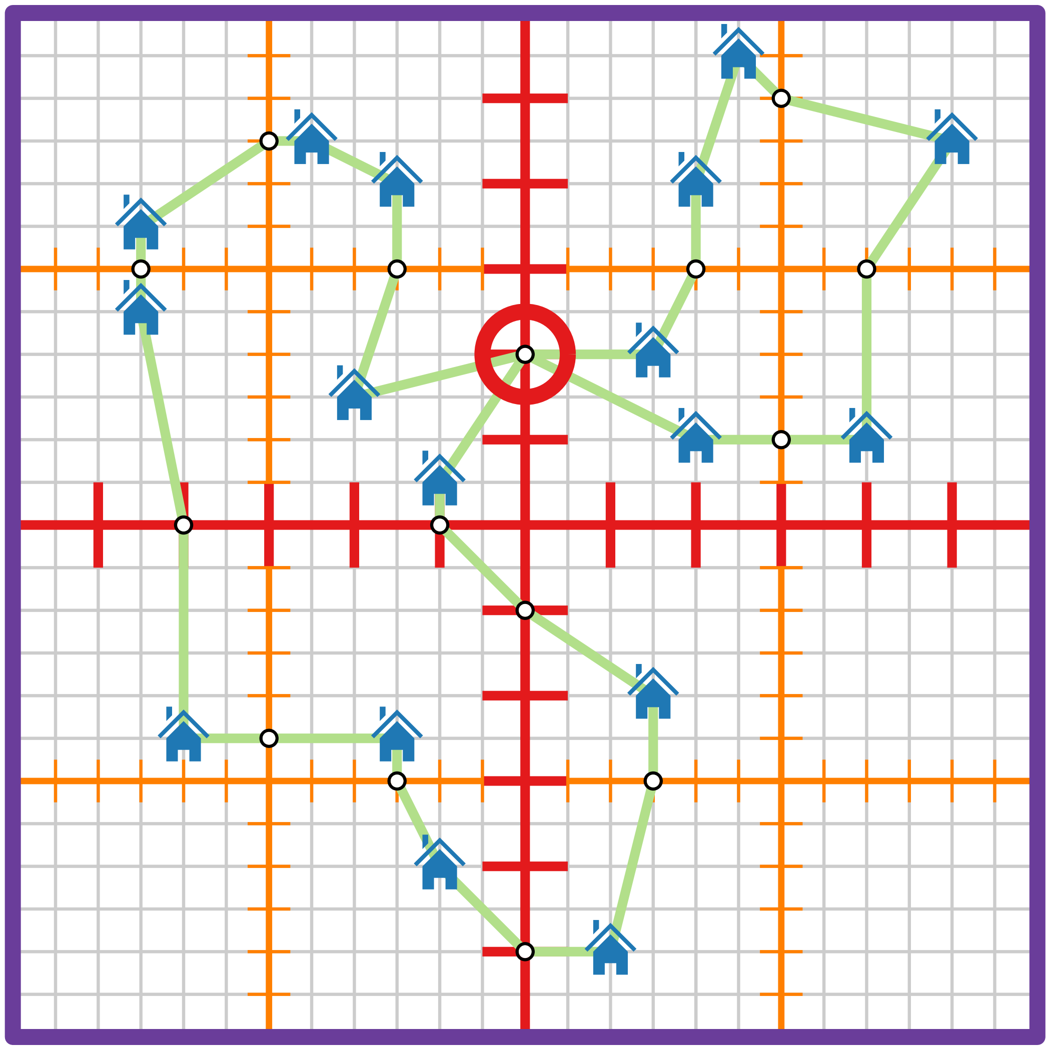


No crossing



Well-Behaved Tours



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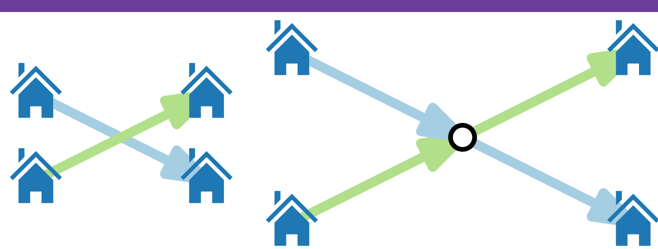

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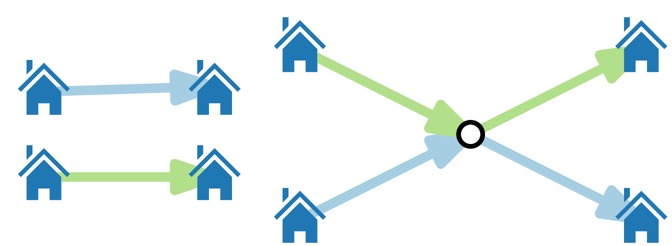

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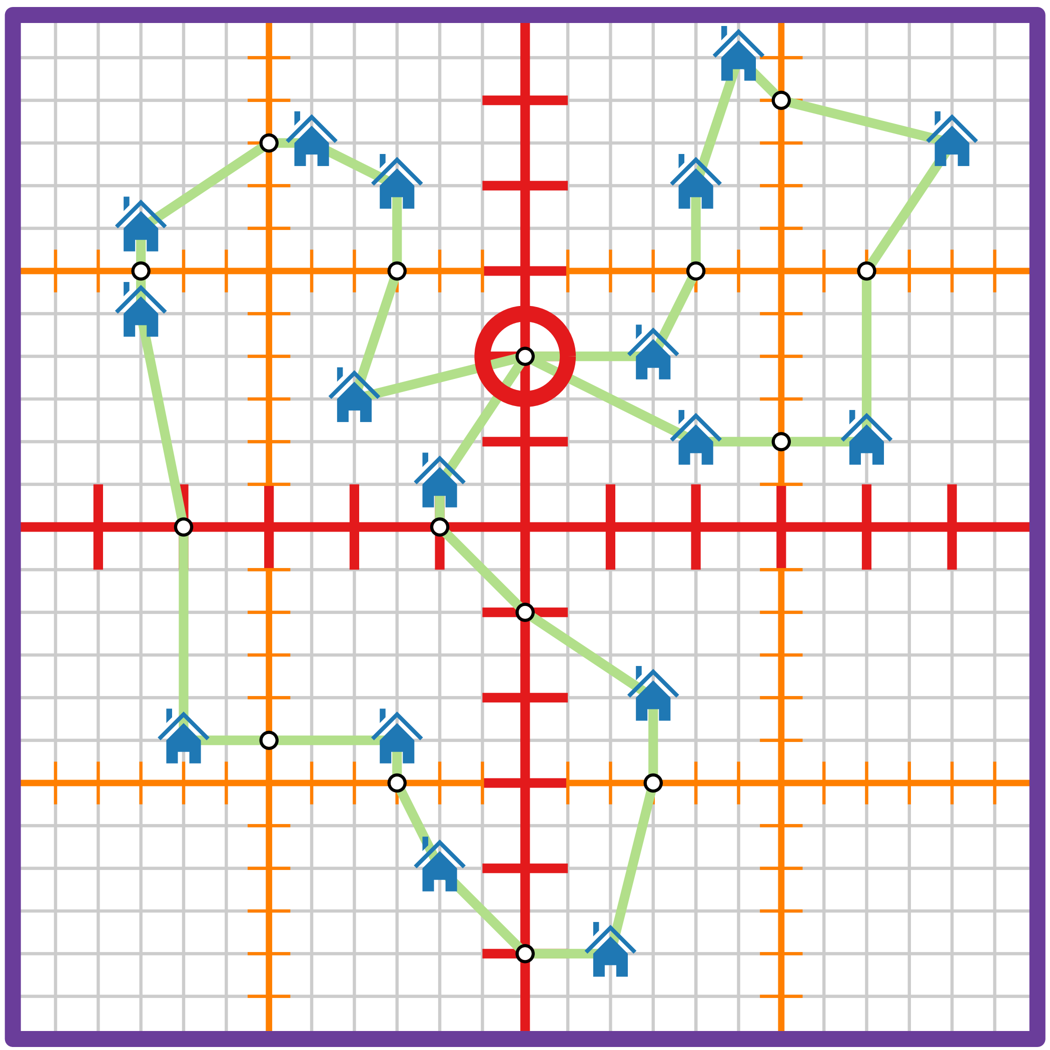

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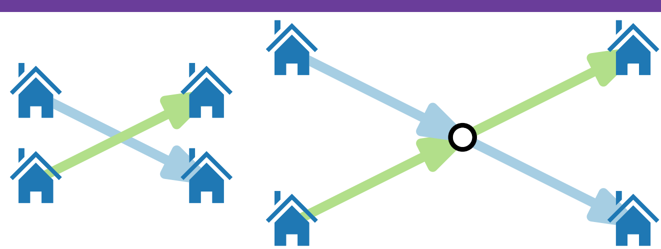
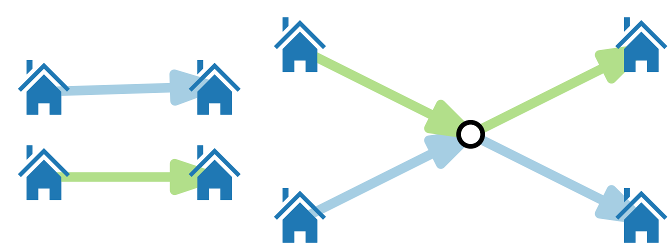
- it is crossing-free.

W.l.o.g. (homework):
No portal visited more than twice

Crossing



No crossing



Computing a Well-Behaved Tour

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Lemma. An optimal well-behaved tour can be computed in $2^{O(m)} = n^{O(1/\varepsilon)}$ time.

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Sketch.

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- Compute sub-structure of an optimal tour for each square in the dissection tree.

Computing a Well-Behaved Tour

Lemma. An optimal well-behaved tour can be computed in $2^{O(m)} = n^{O(1/\varepsilon)}$ time.

Sketch.

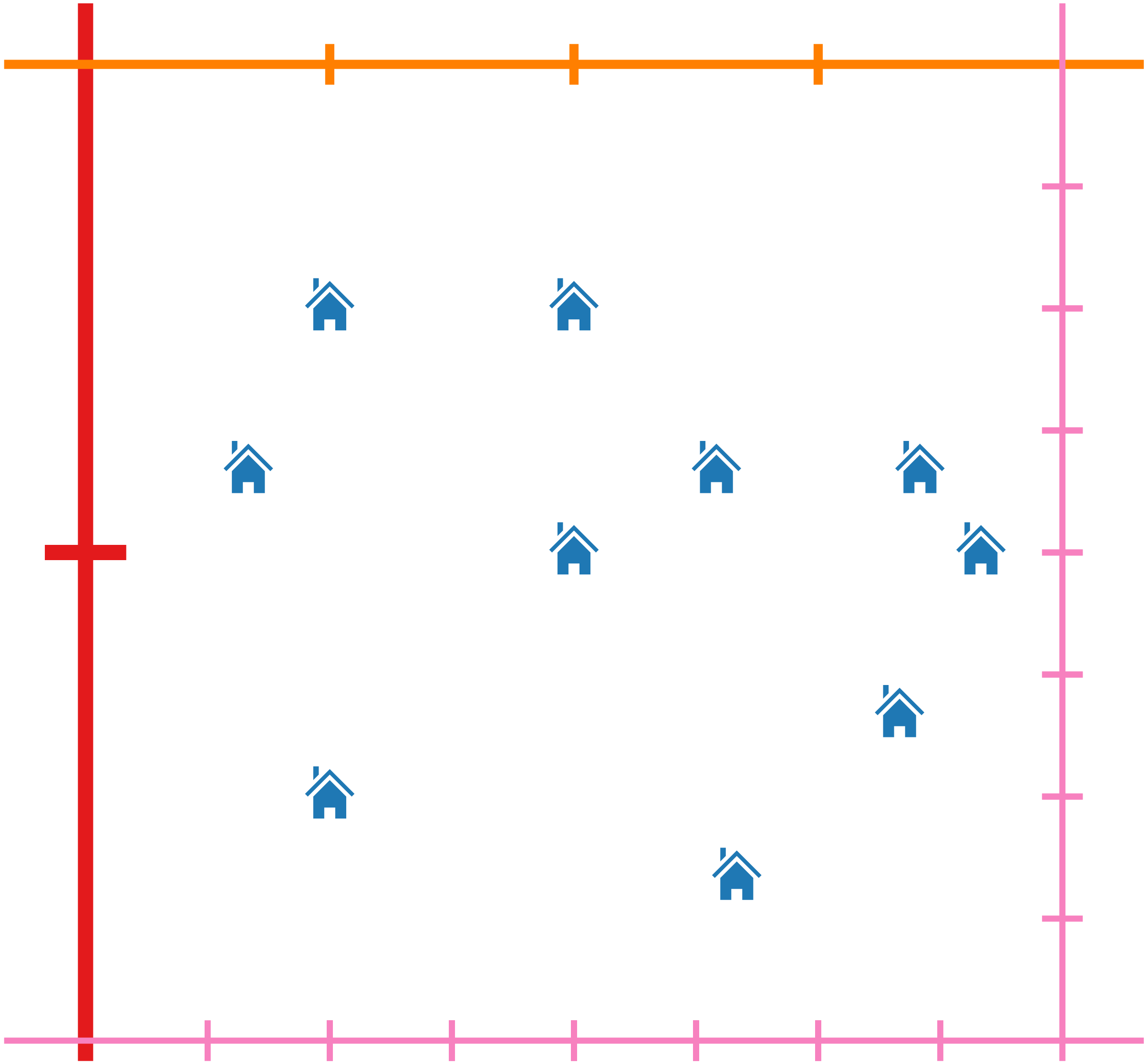
- Dynamic programming!
- Compute sub-structure of an optimal tour for each square in the dissection tree.
- These solutions can be efficiently propagated bottom-up through the dissection tree.

Approximation Algorithms

Lecture 9:
A PTAS for Euclidean TSP

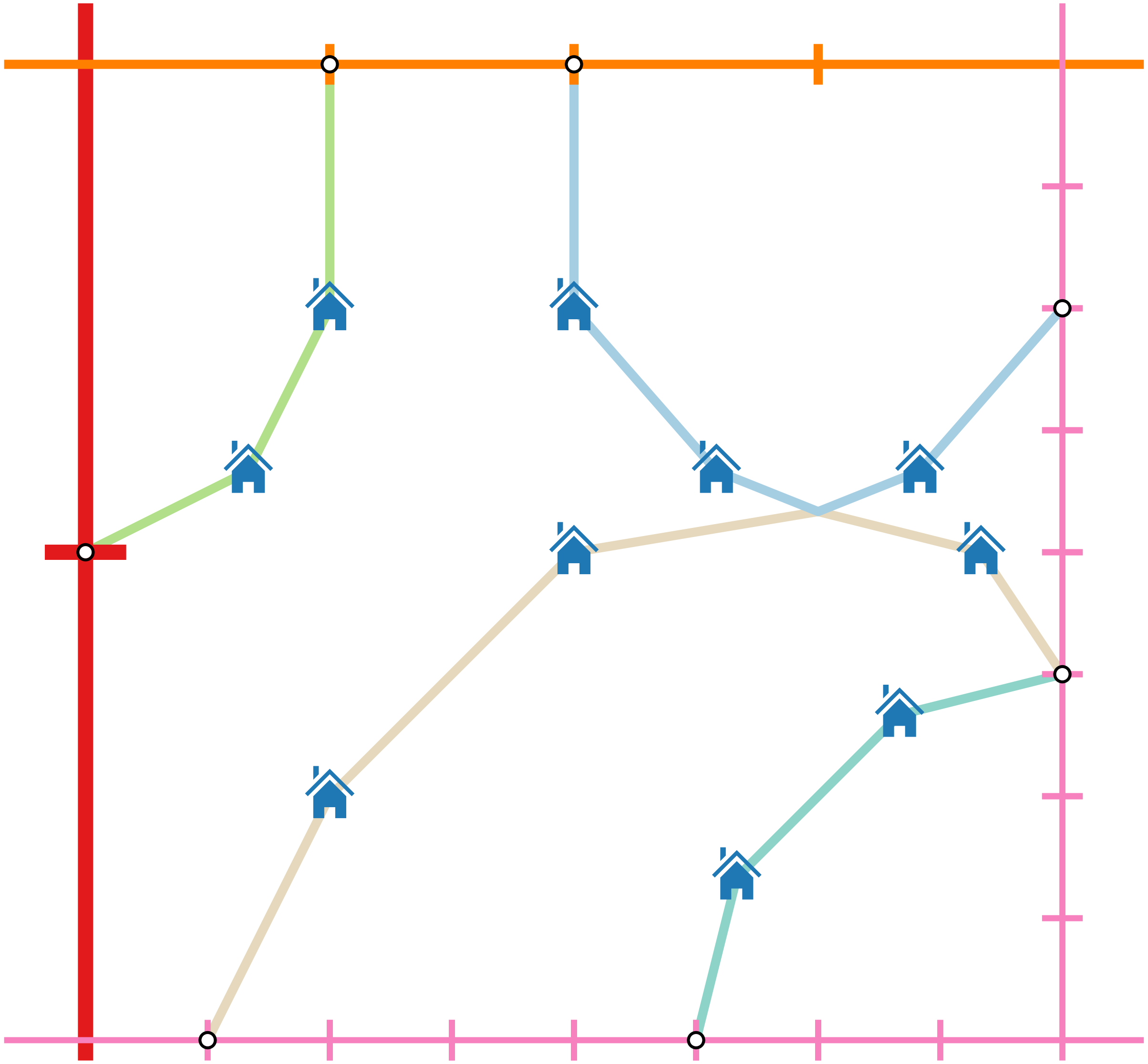
Part IV:
Dynamic Program

Dynamic Program (I)



Each well-behaved tour induces the following in each square Q of the dissection:

Dynamic Program (I)



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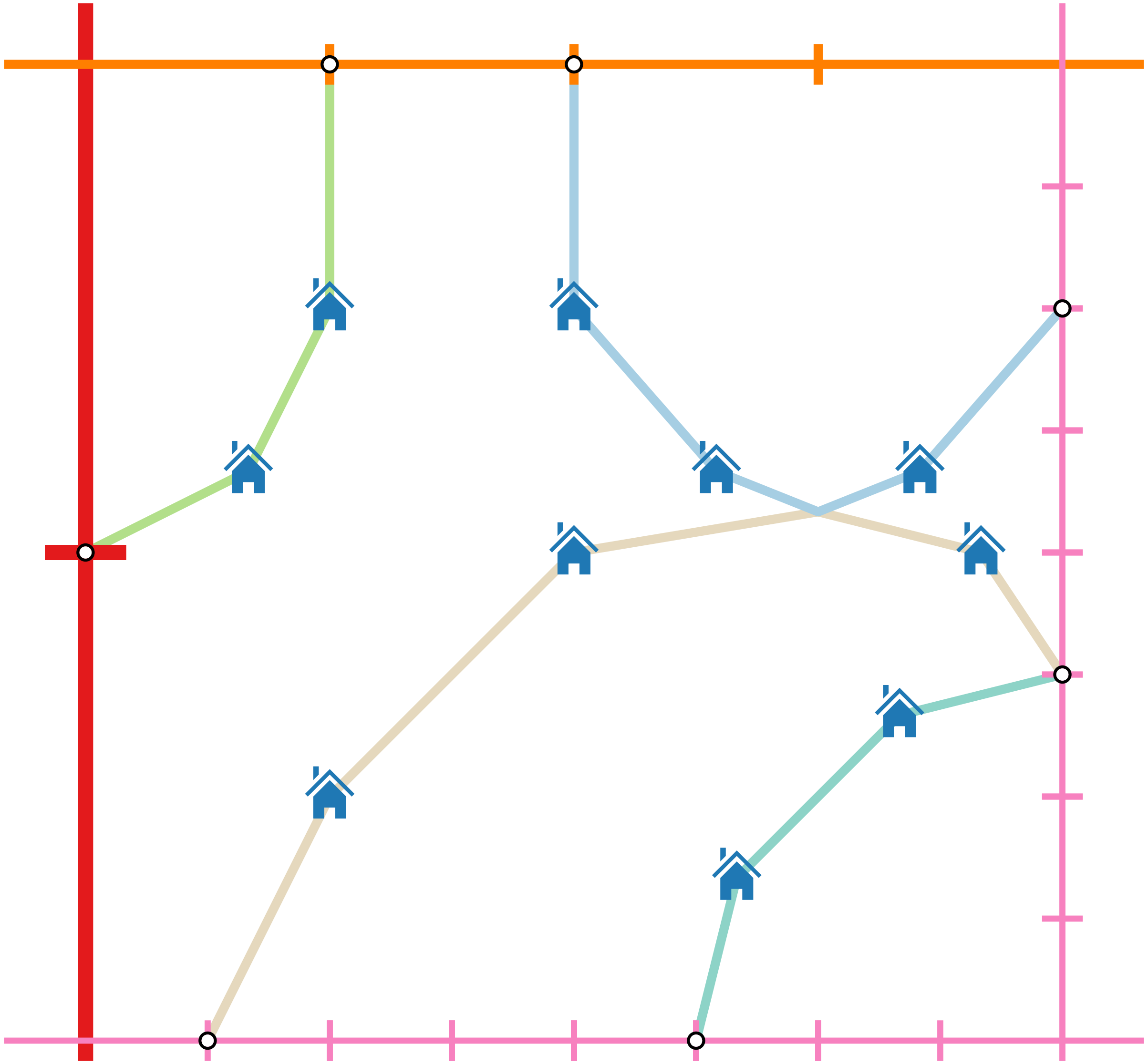

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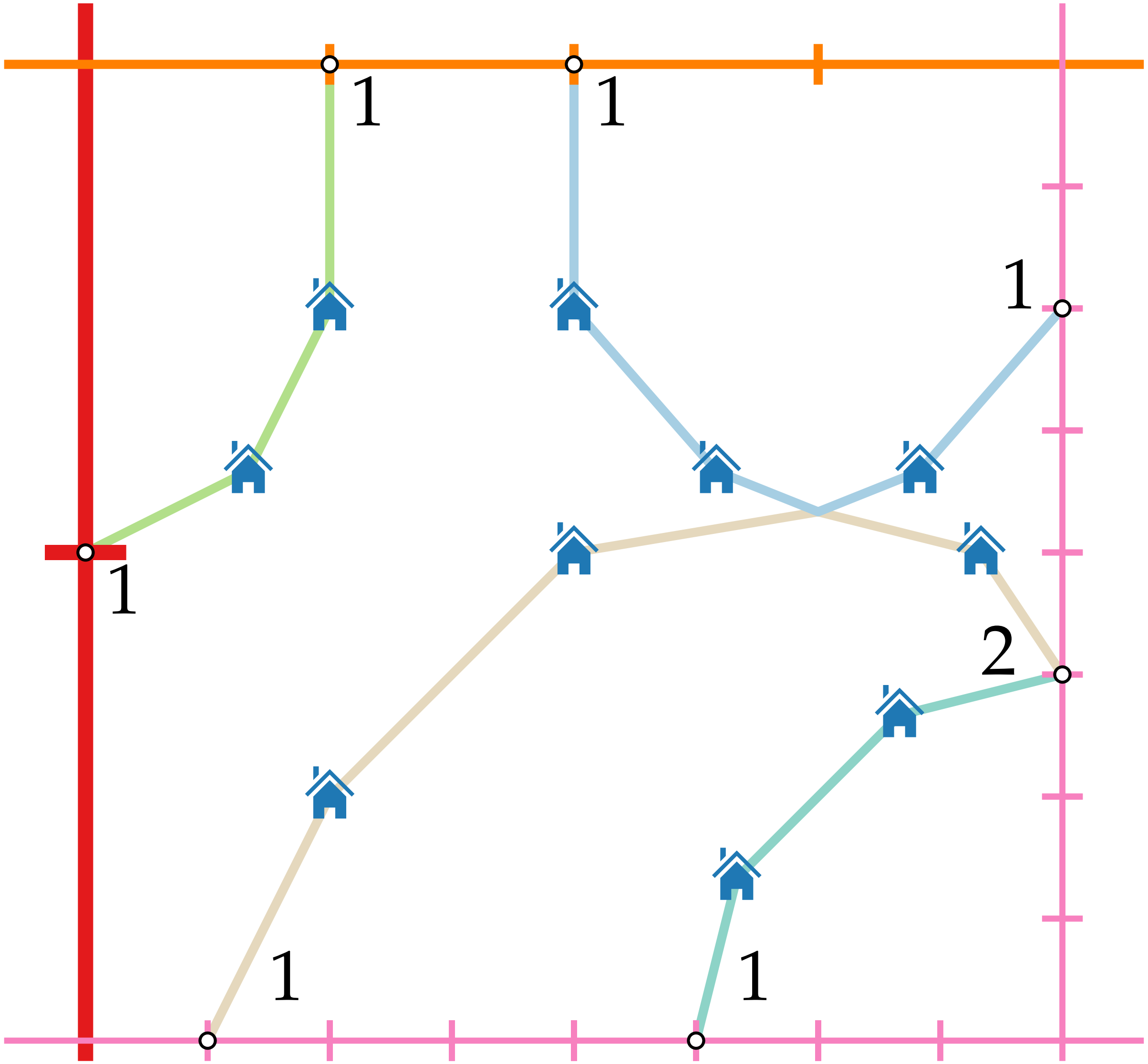

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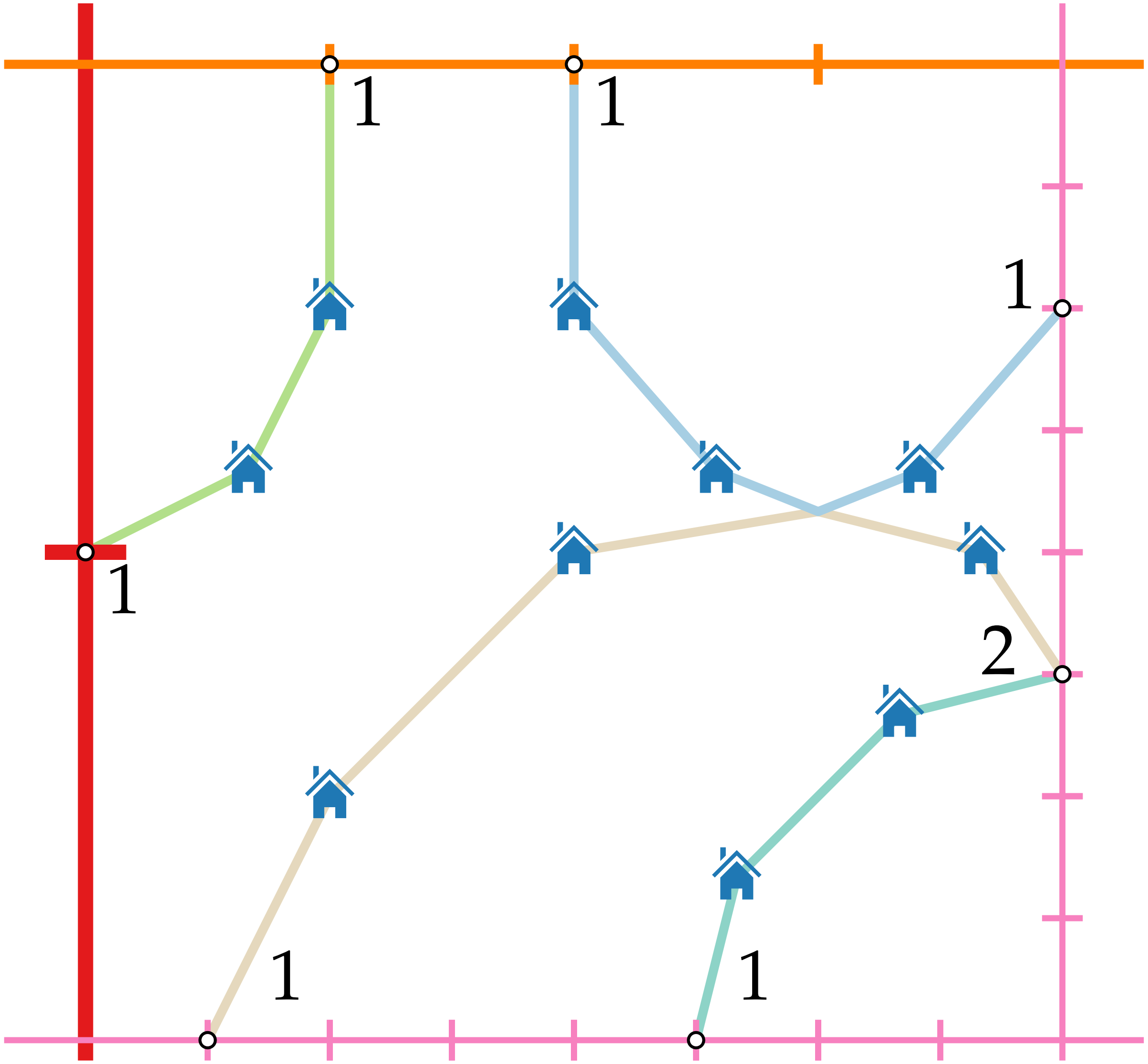

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$\Rightarrow$ max. possibilities

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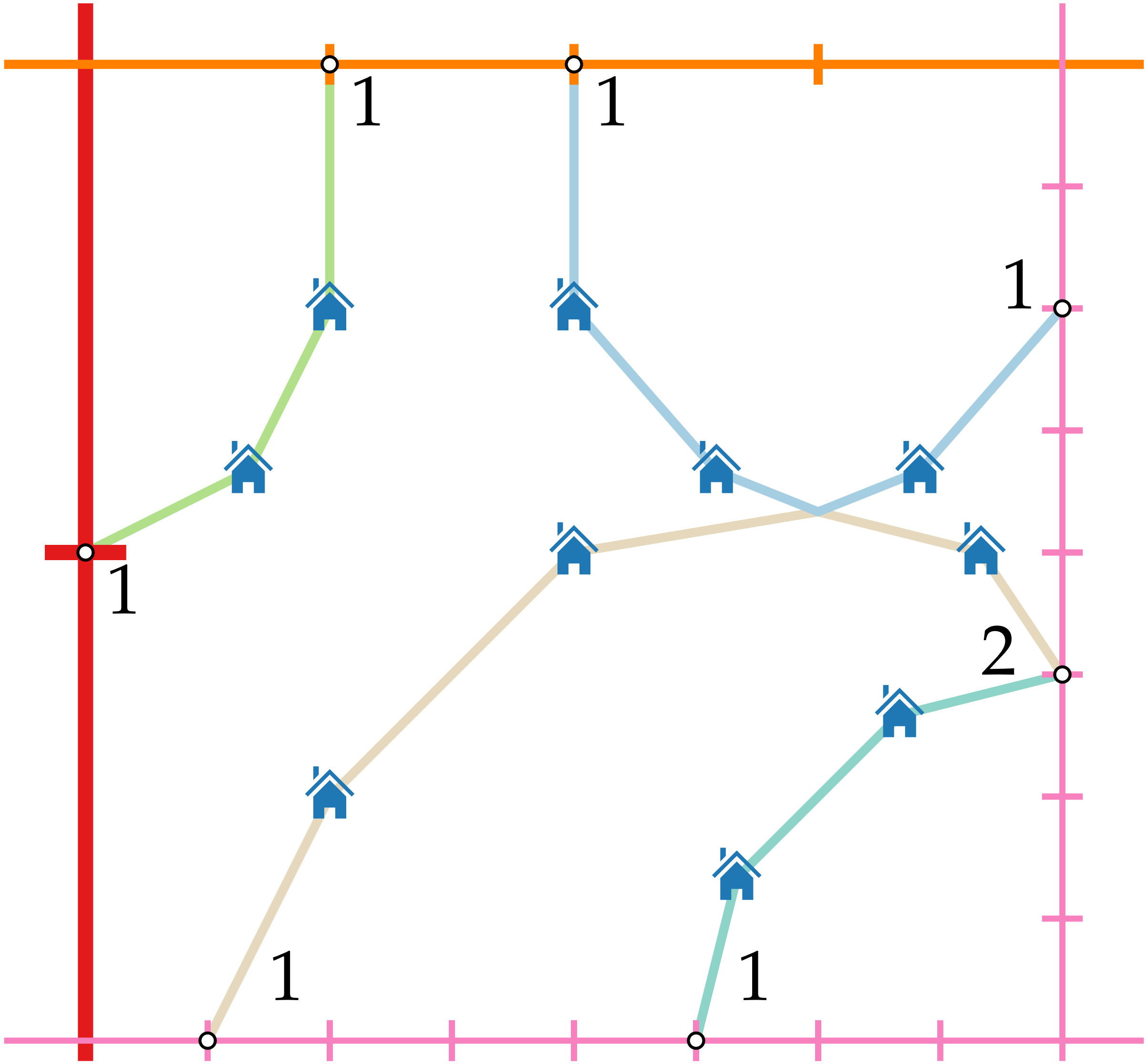

Each well-behaved tour induces the following in each square Q of the dissection:

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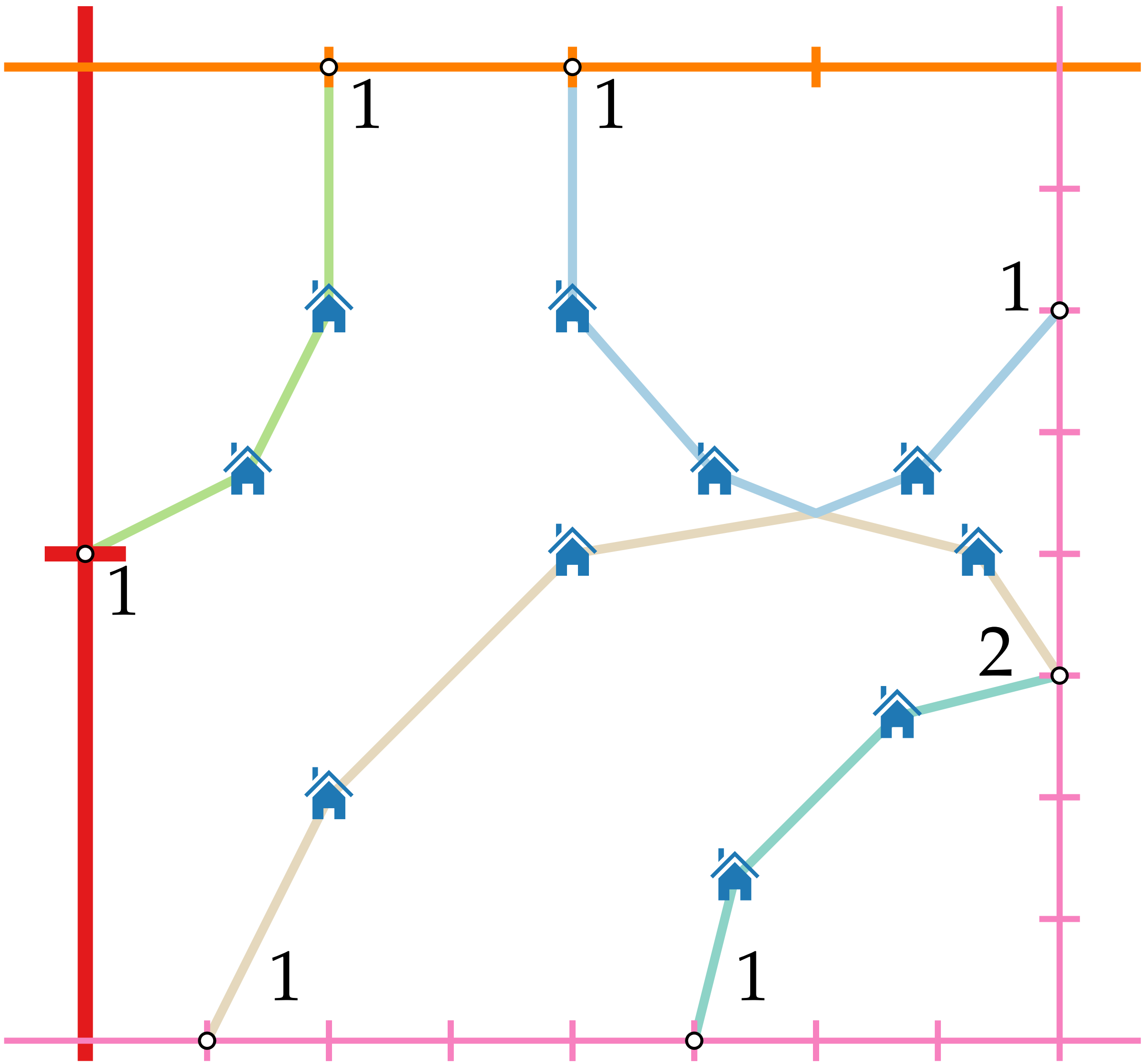

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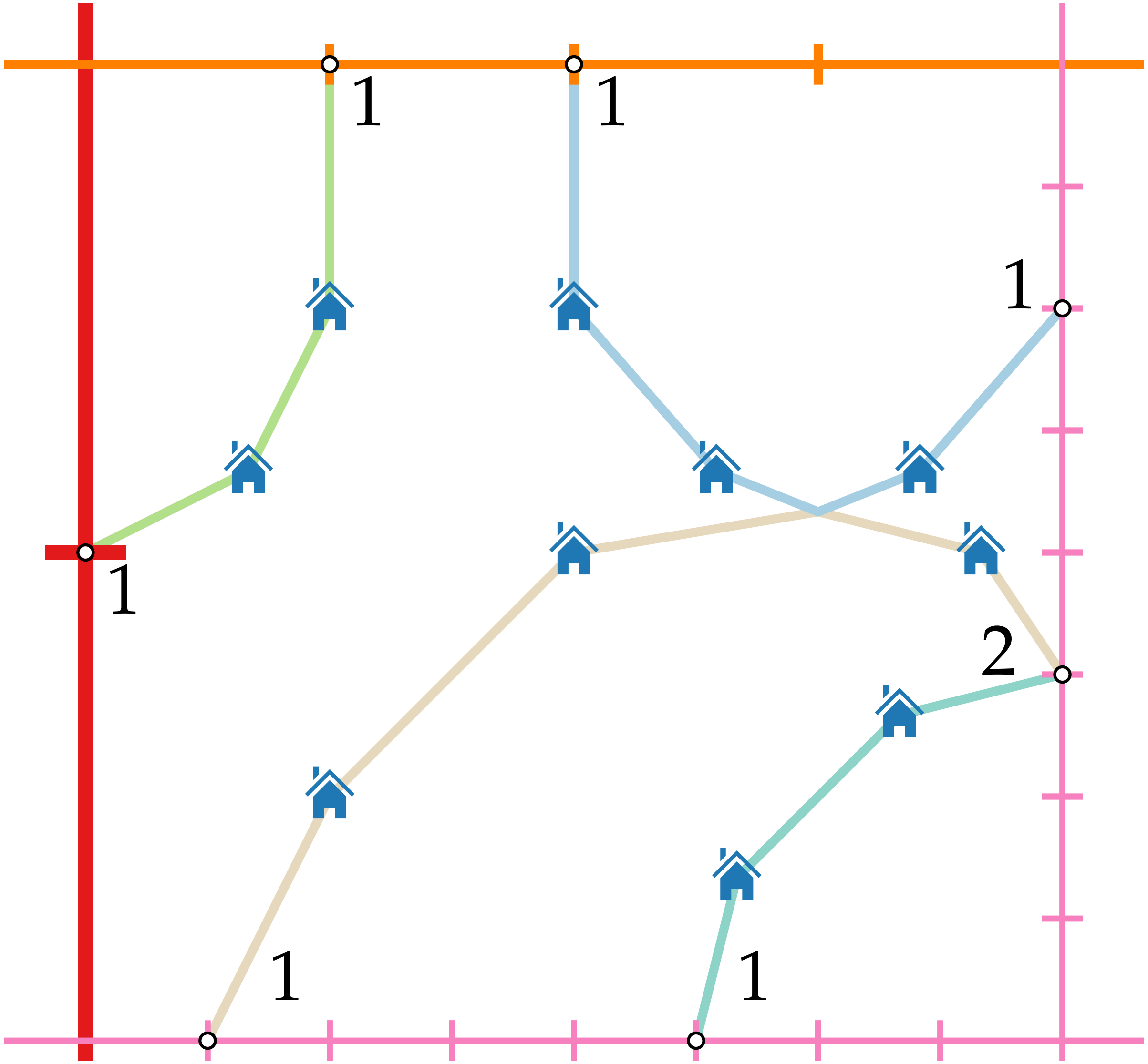

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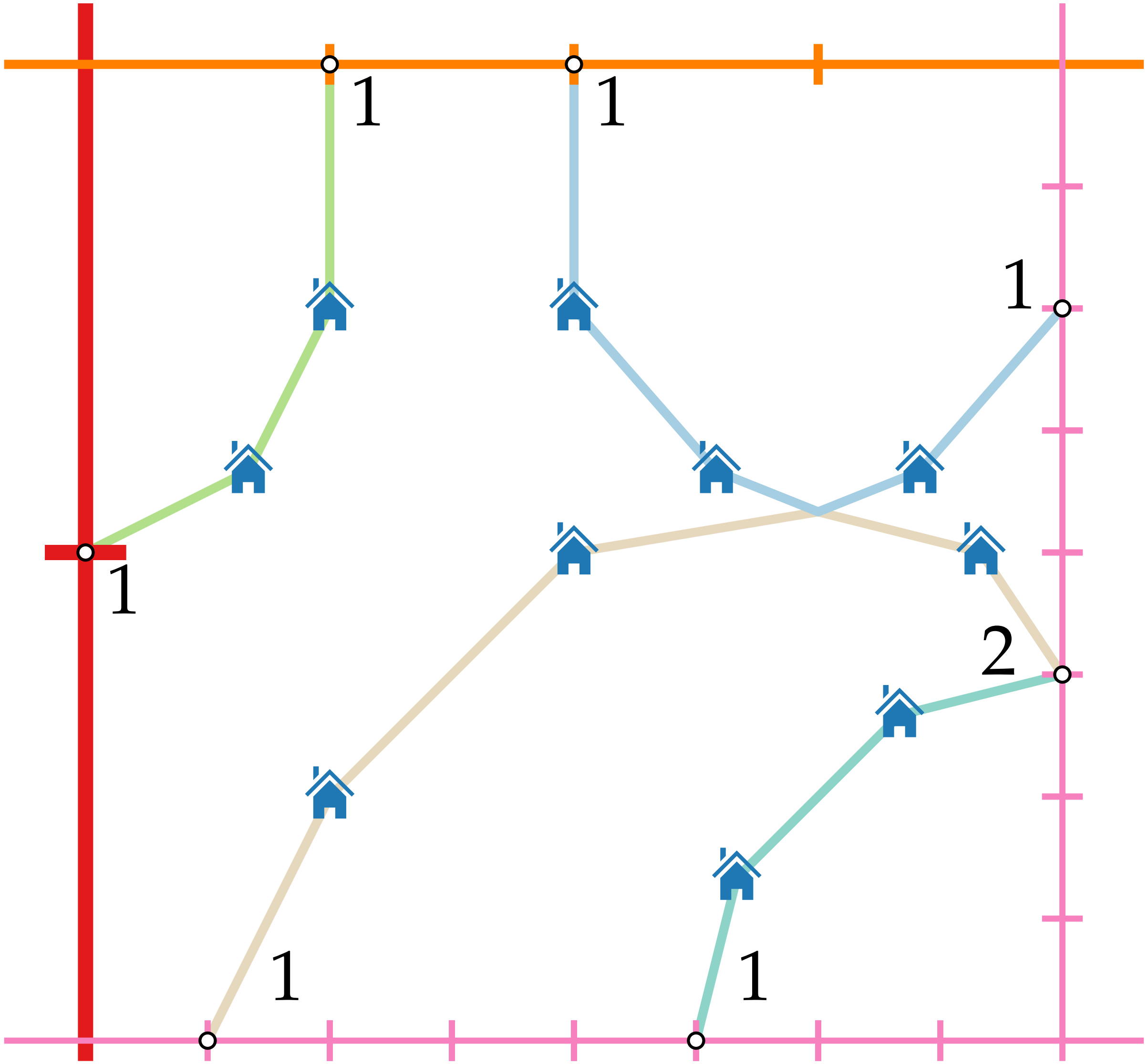

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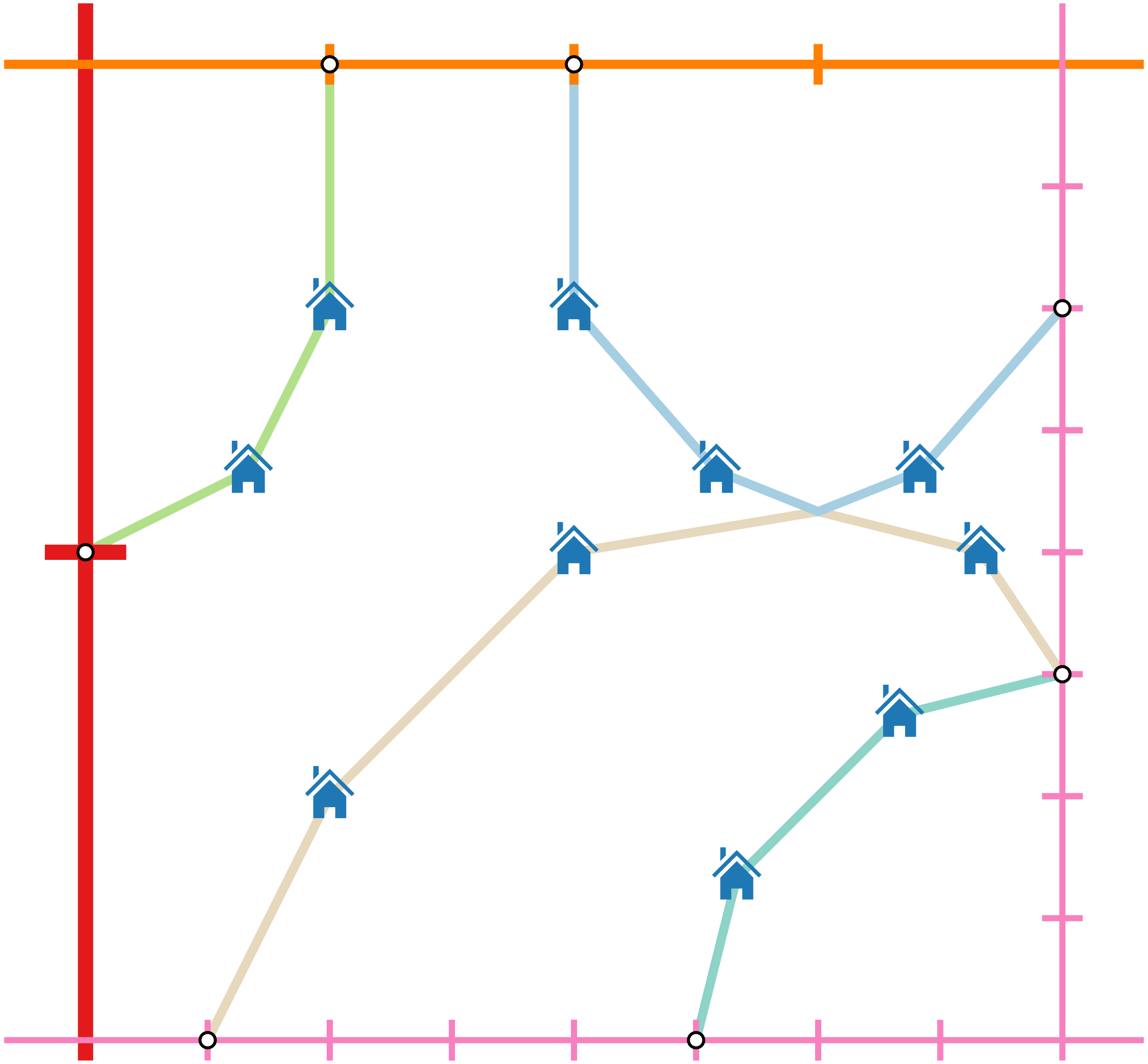

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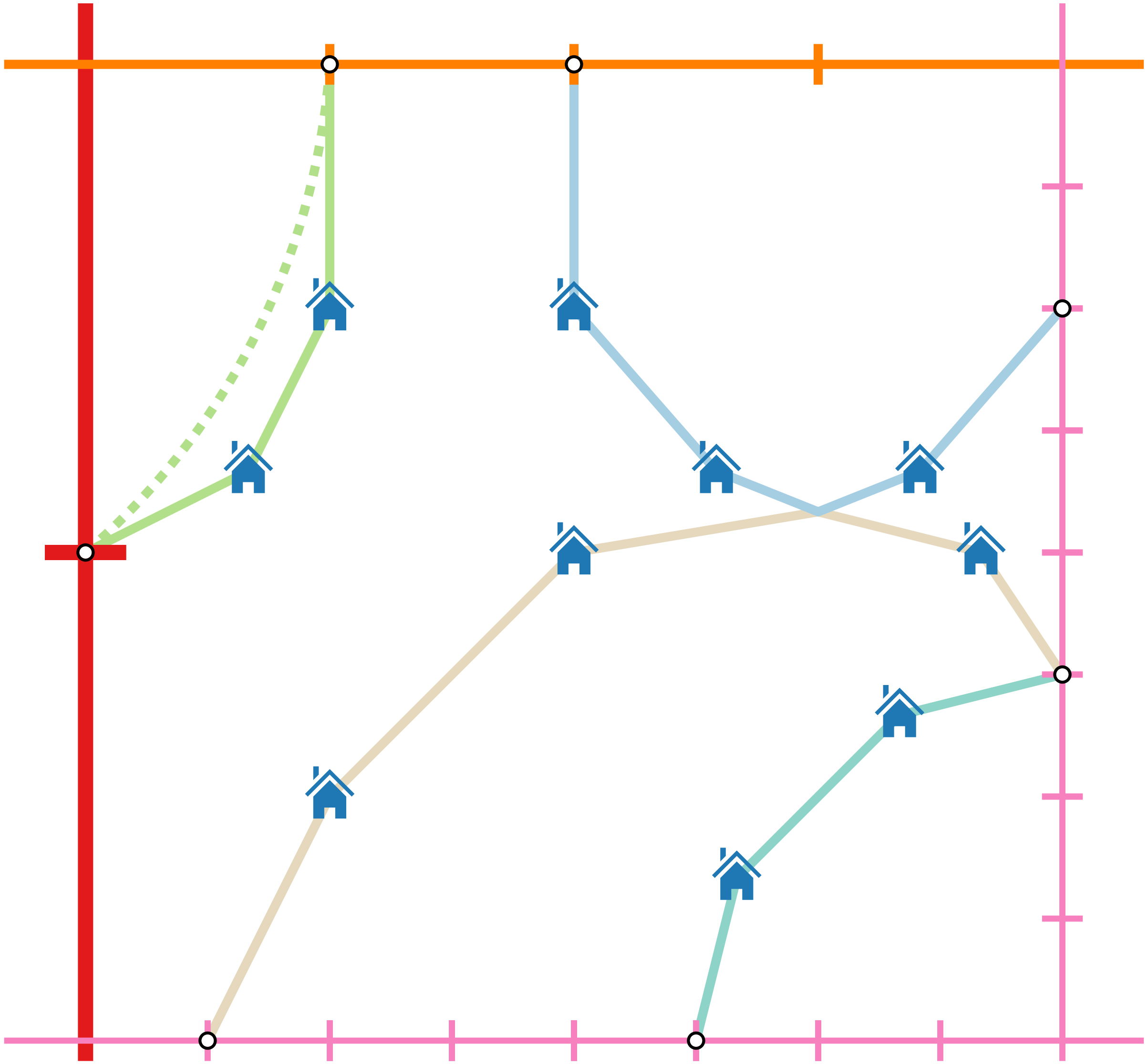

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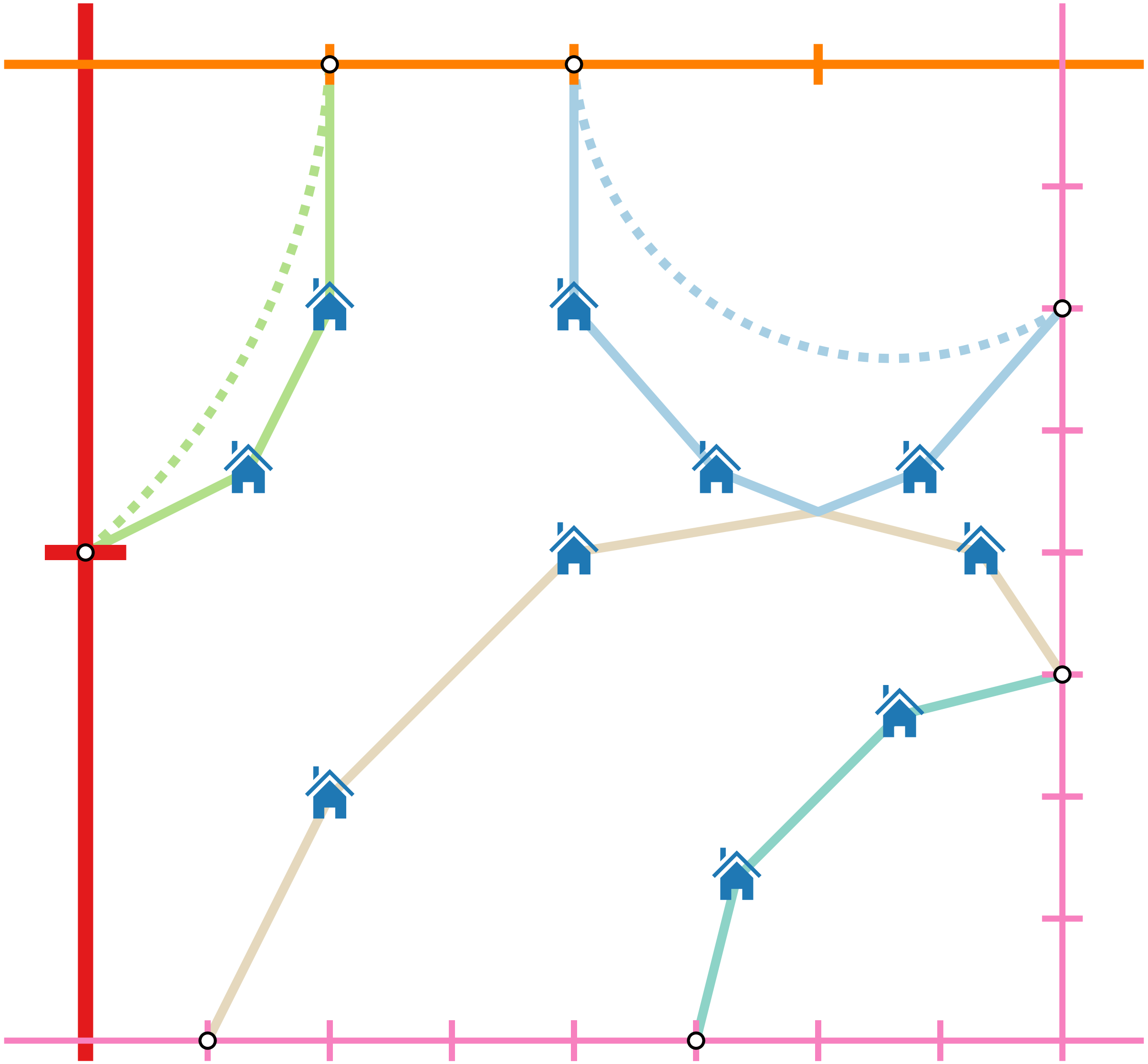

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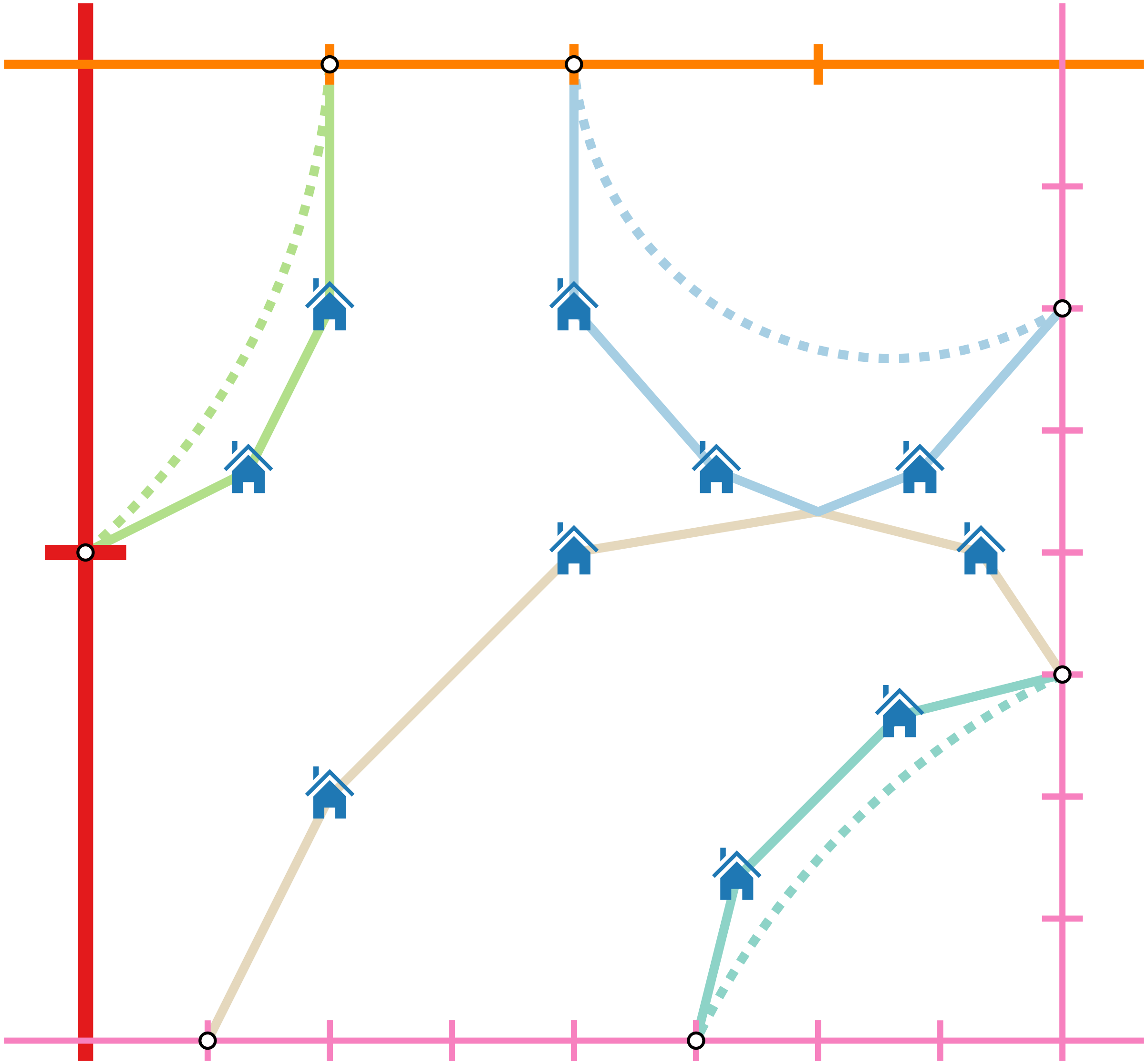

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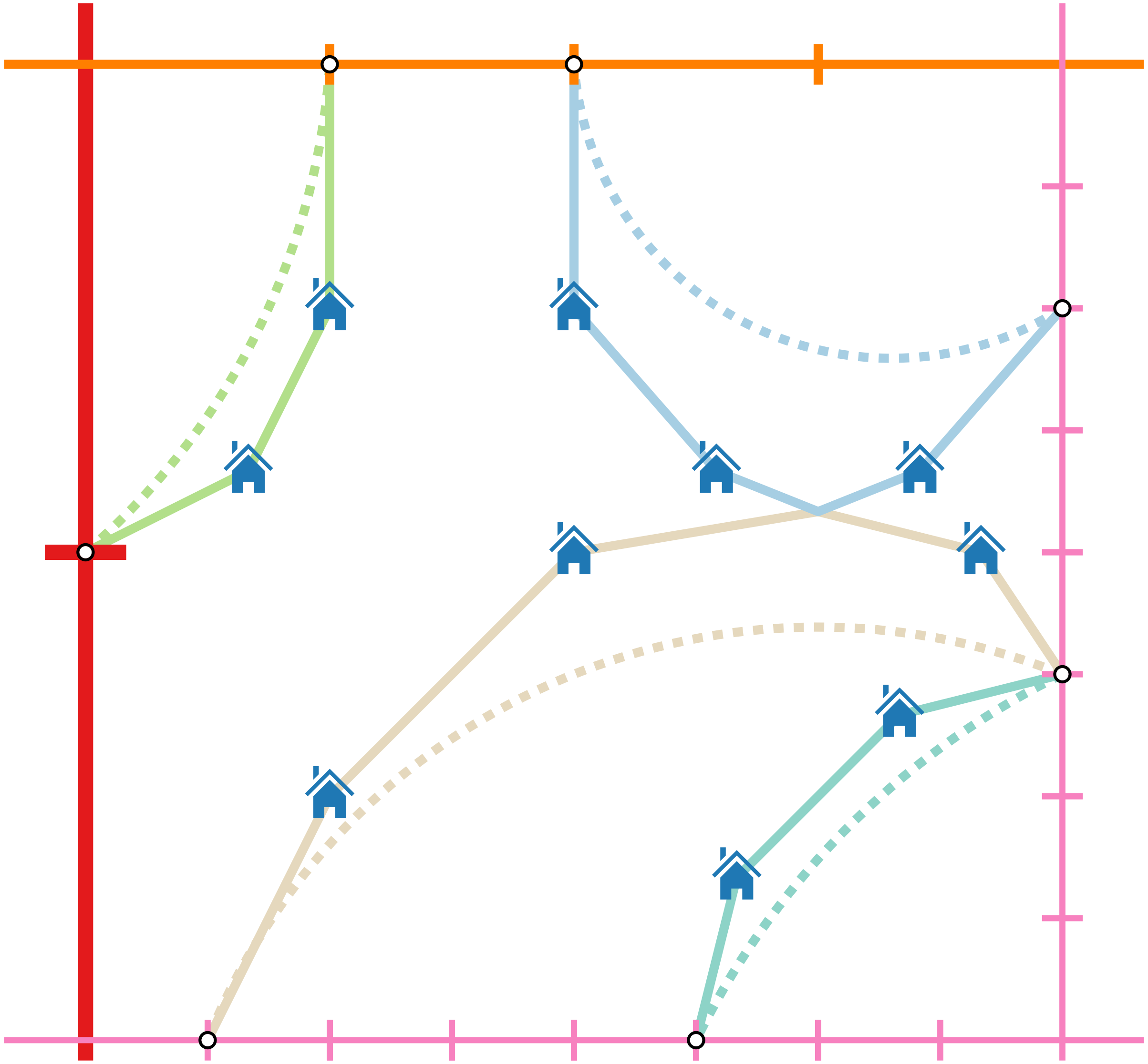

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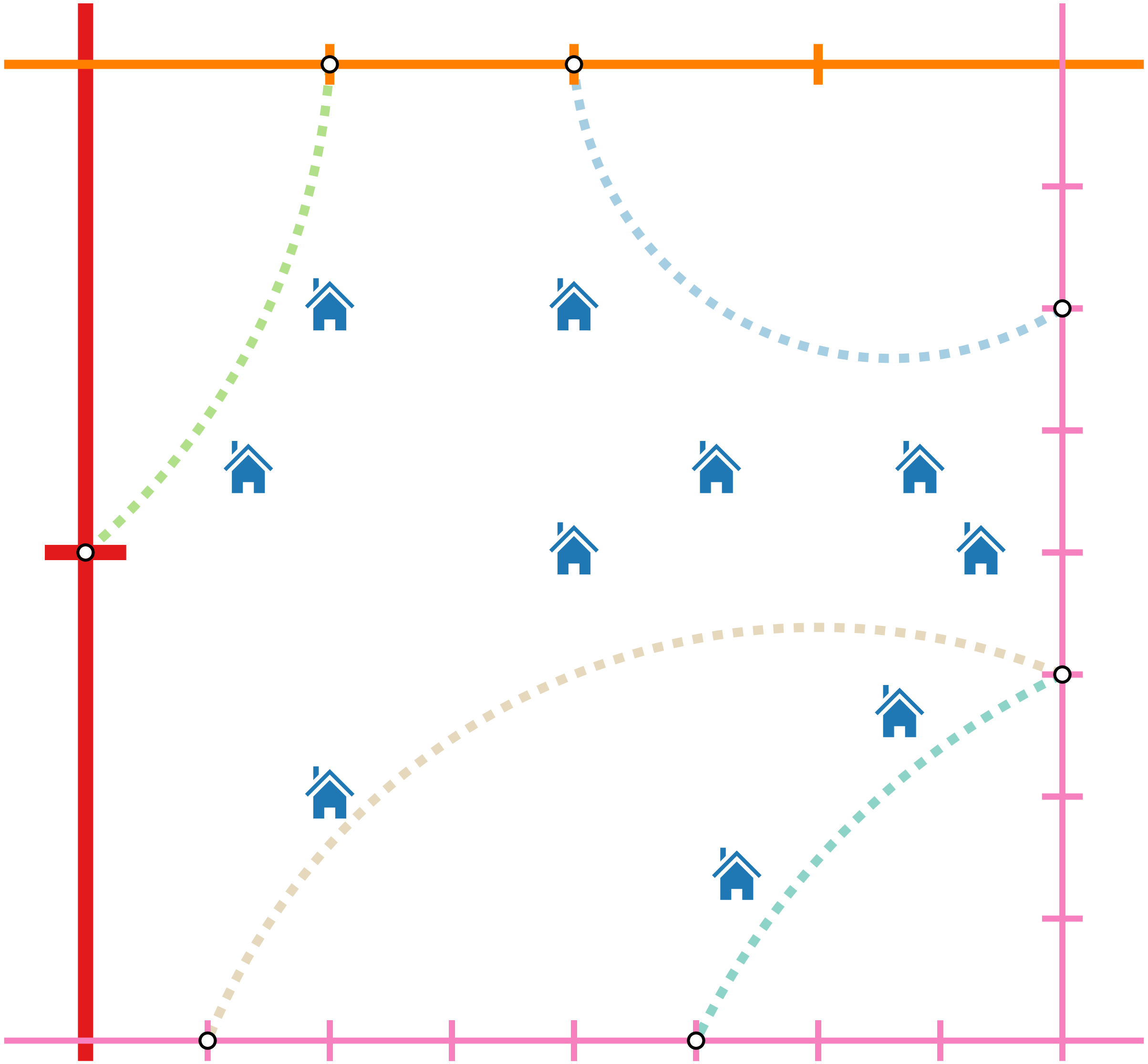

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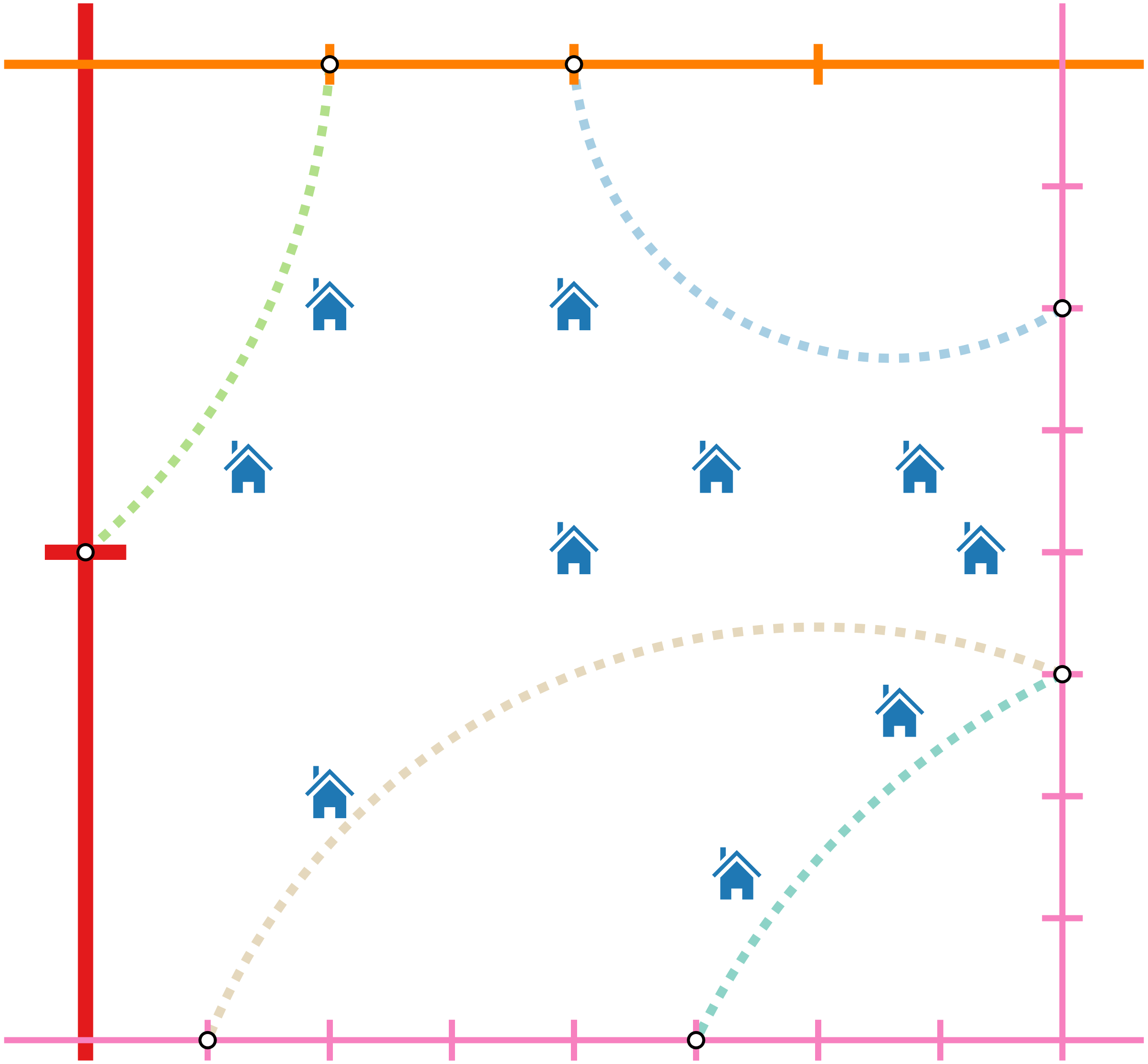

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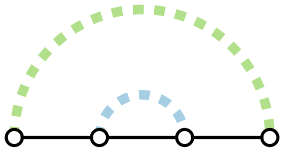

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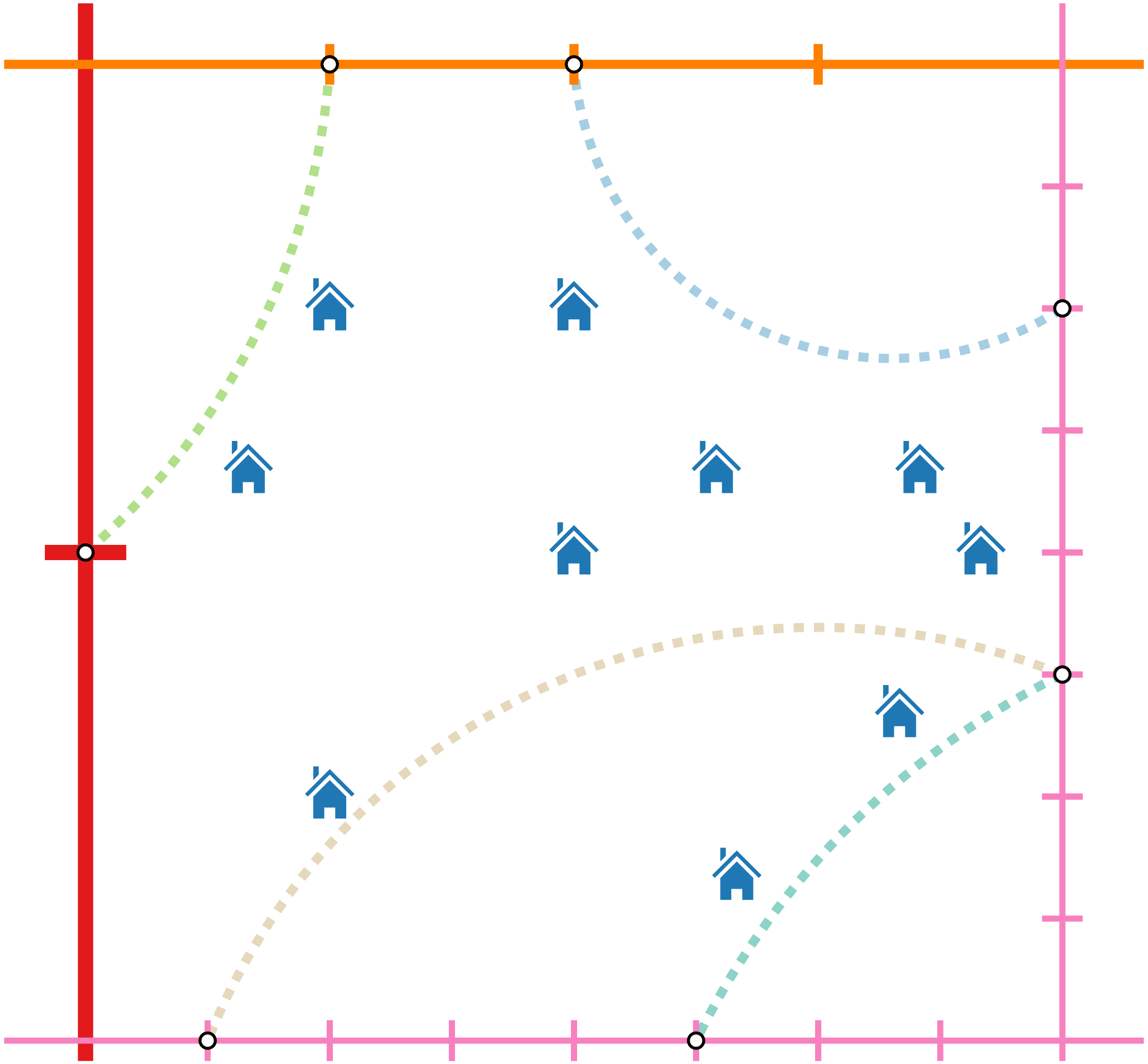

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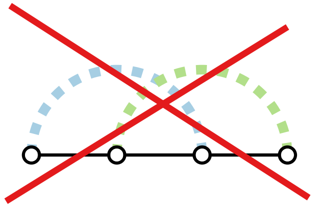

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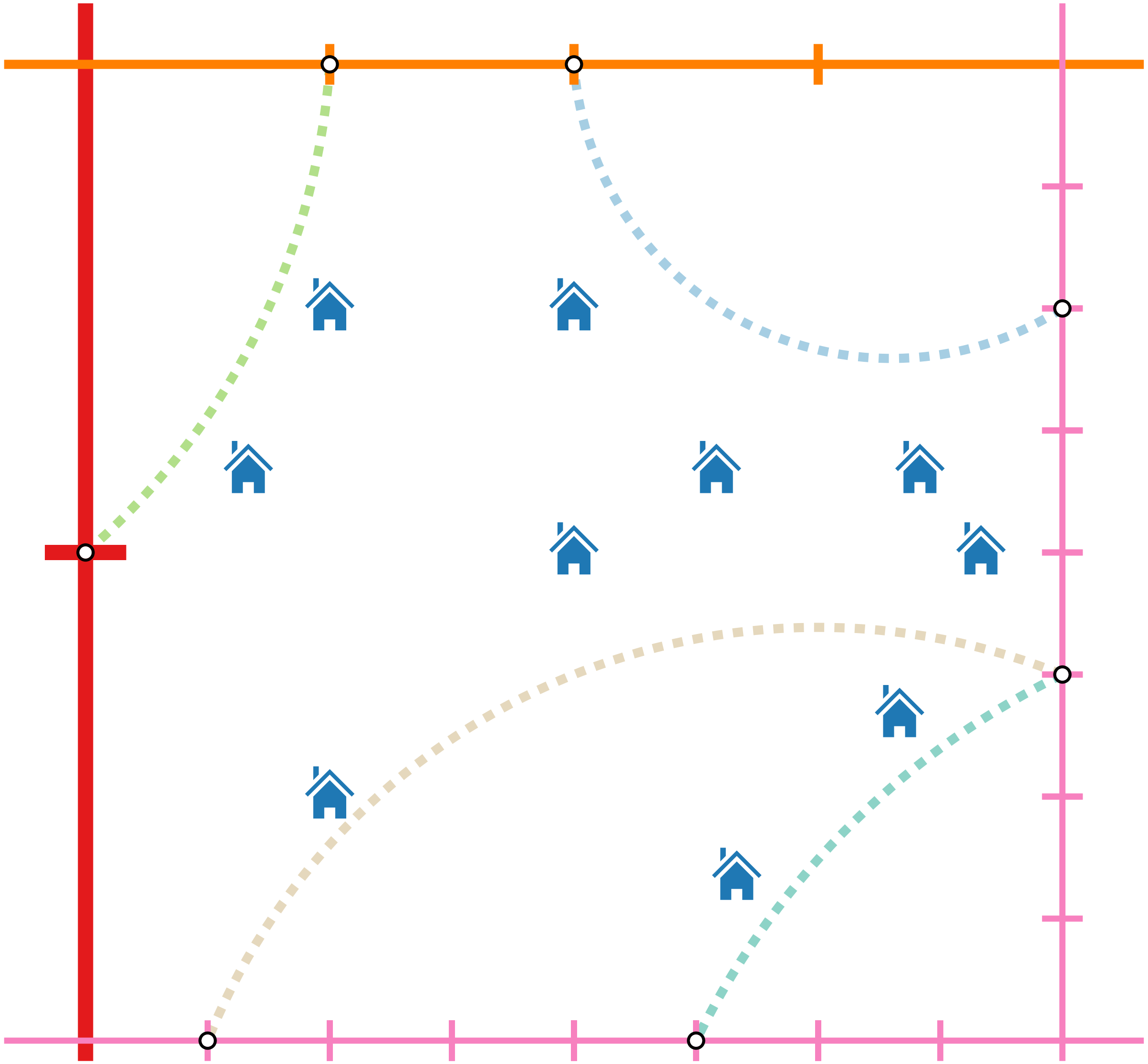

⇒ max.

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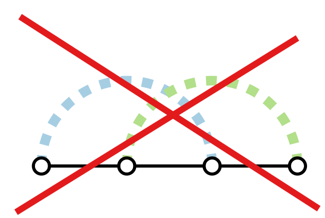

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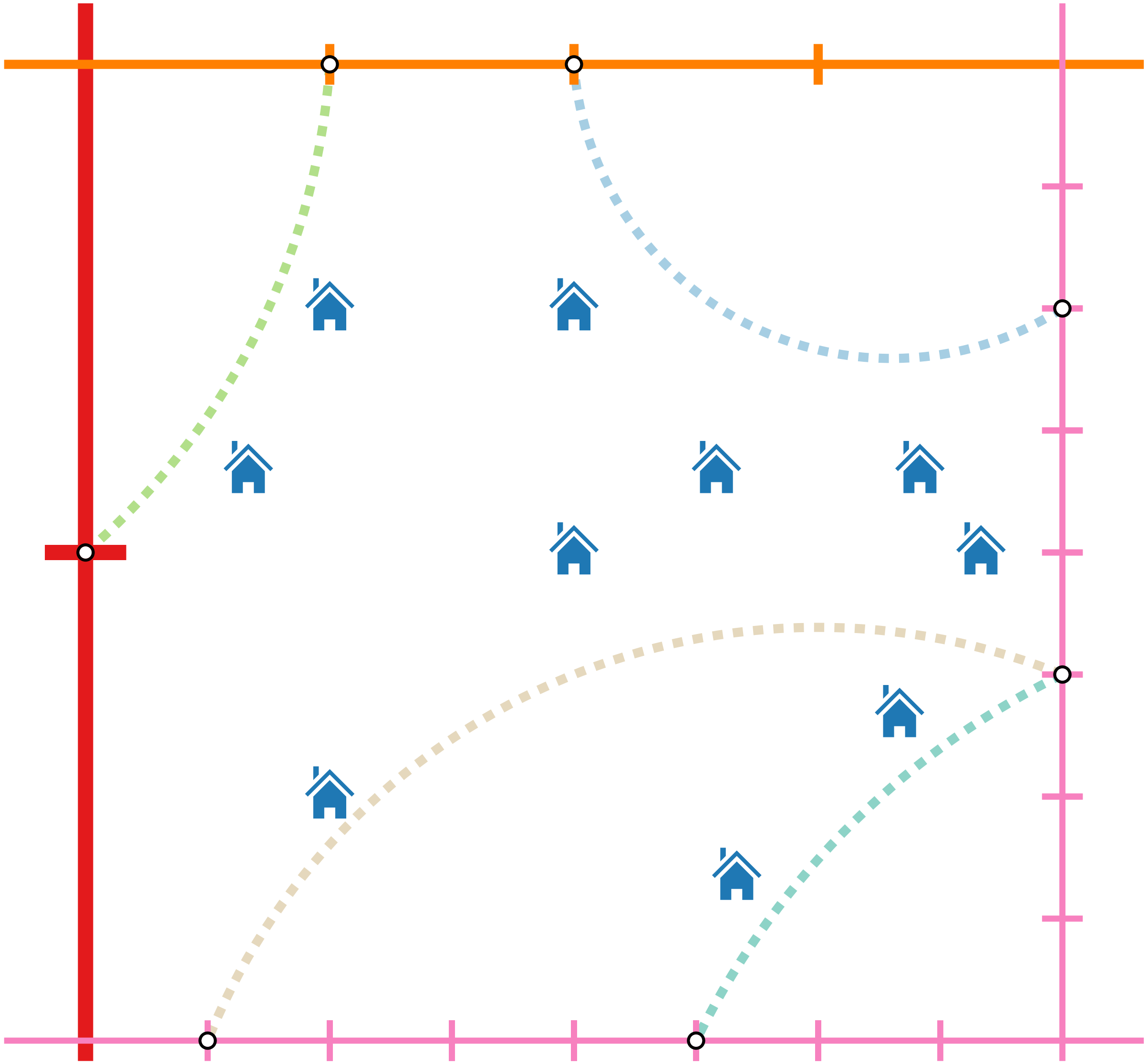

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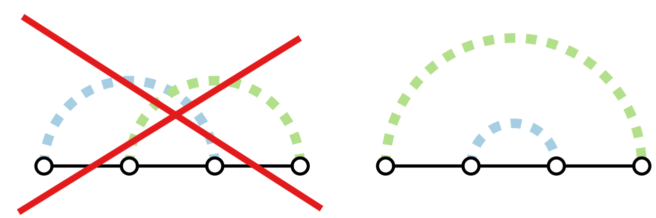
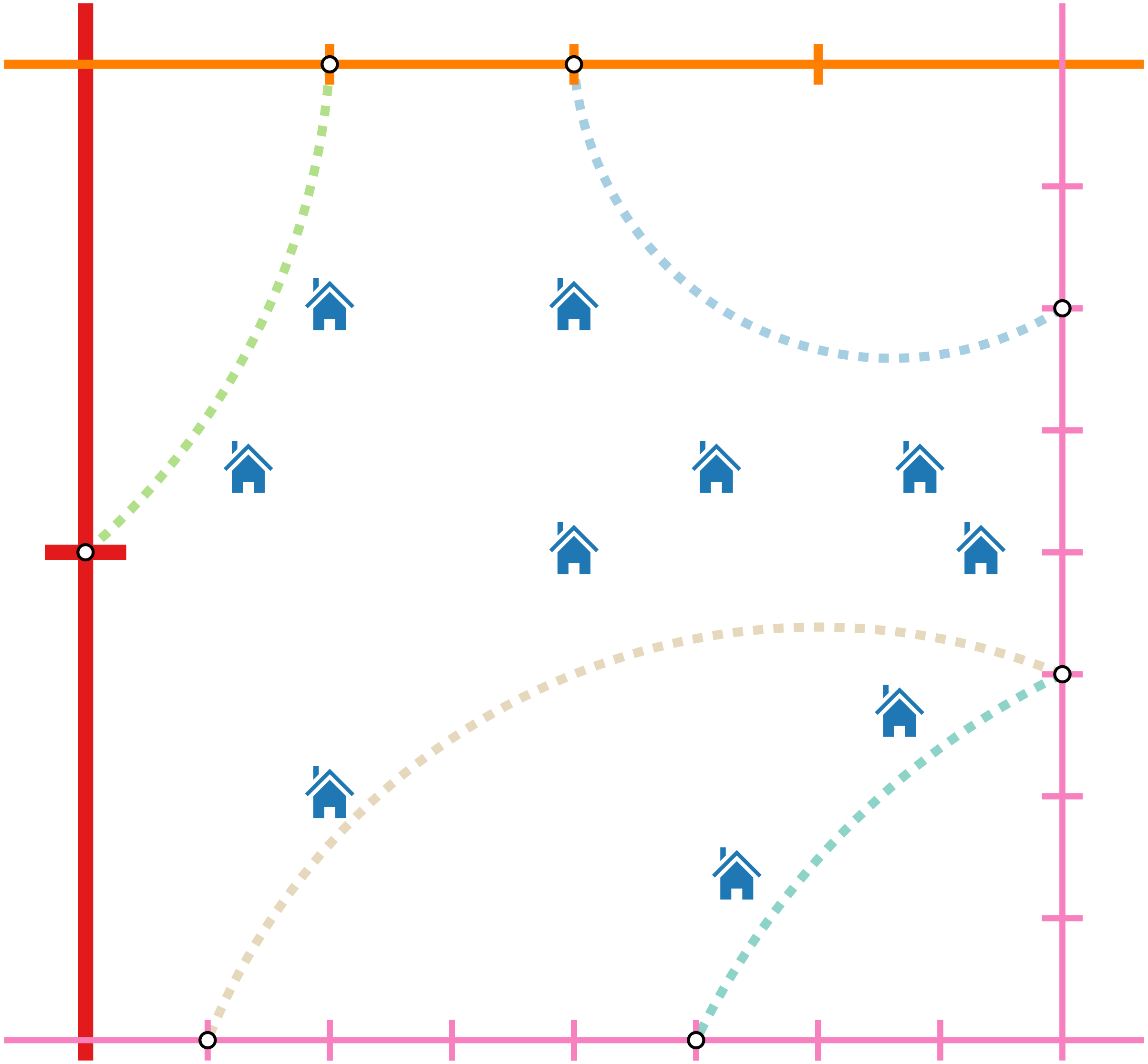


$\Rightarrow$ max. $\underbrace{}_{\text{\#visit vectors}}$

Dynamic Program (I)



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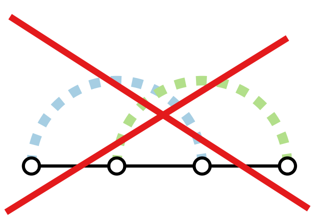
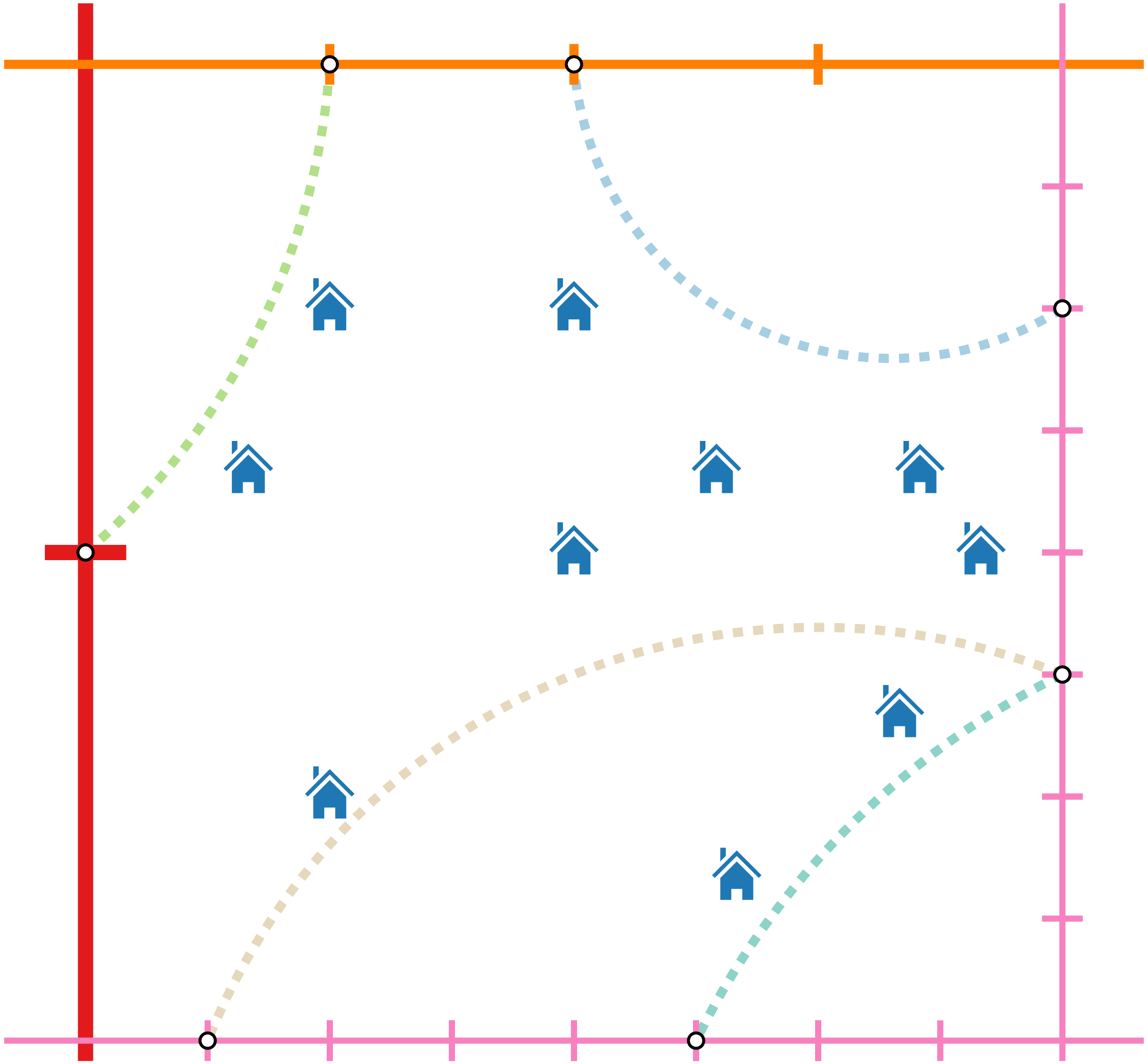


$\Rightarrow$ max. $\underbrace{\qquad}_{\#\text{visit vectors}} \times \underbrace{\qquad}_{\#\text{realizable pairings}}$

Dynamic Program (I)



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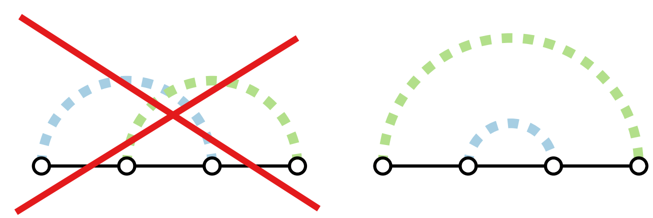
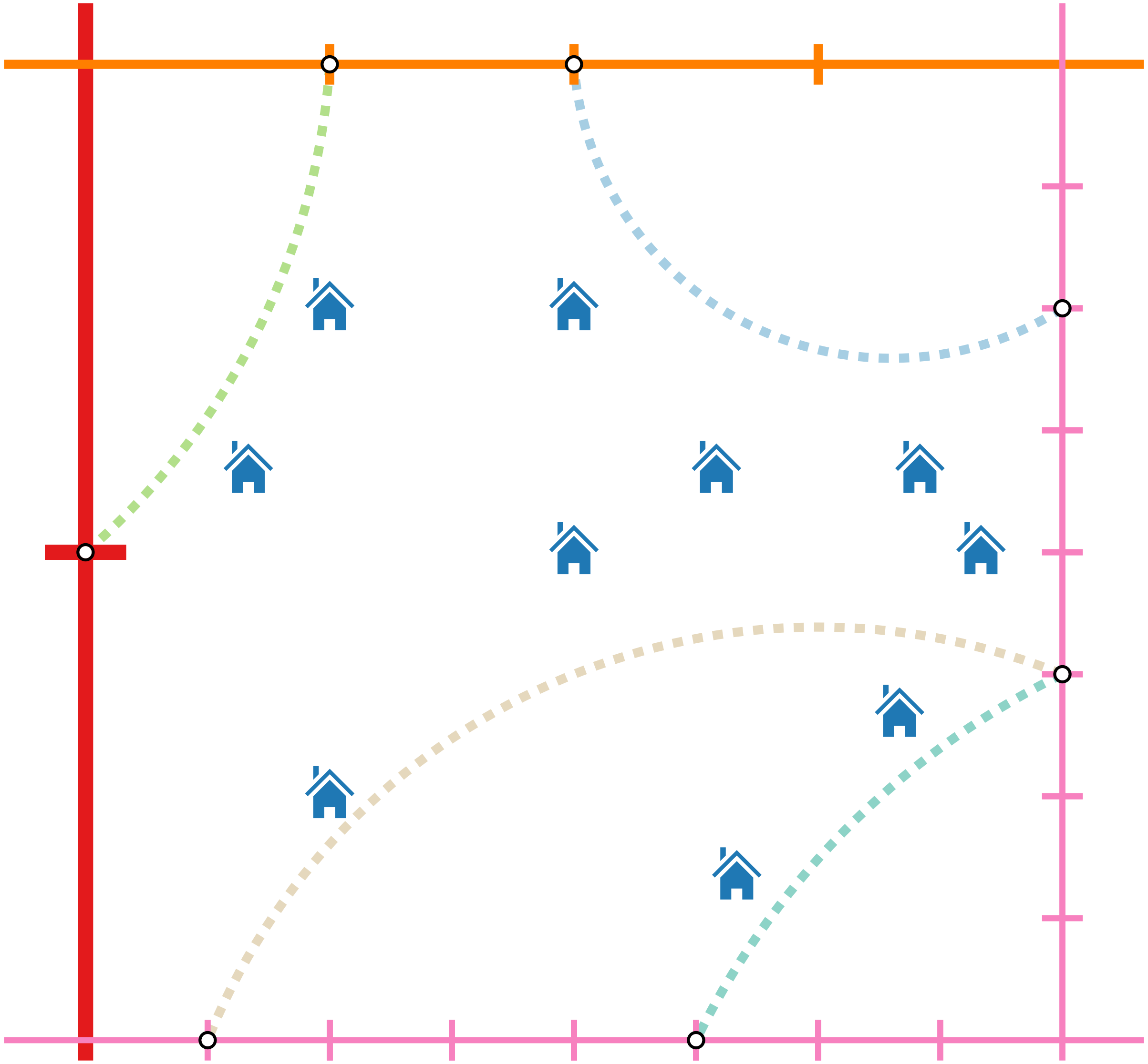


$\Rightarrow$ max. $\underbrace{n^{O(1/\varepsilon)}}_{\text{\#visit vectors}} \times \underbrace{\quad}_{\text{\#realizable pairings}}$

Dynamic Program (I)



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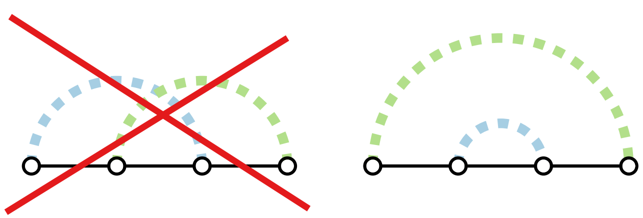
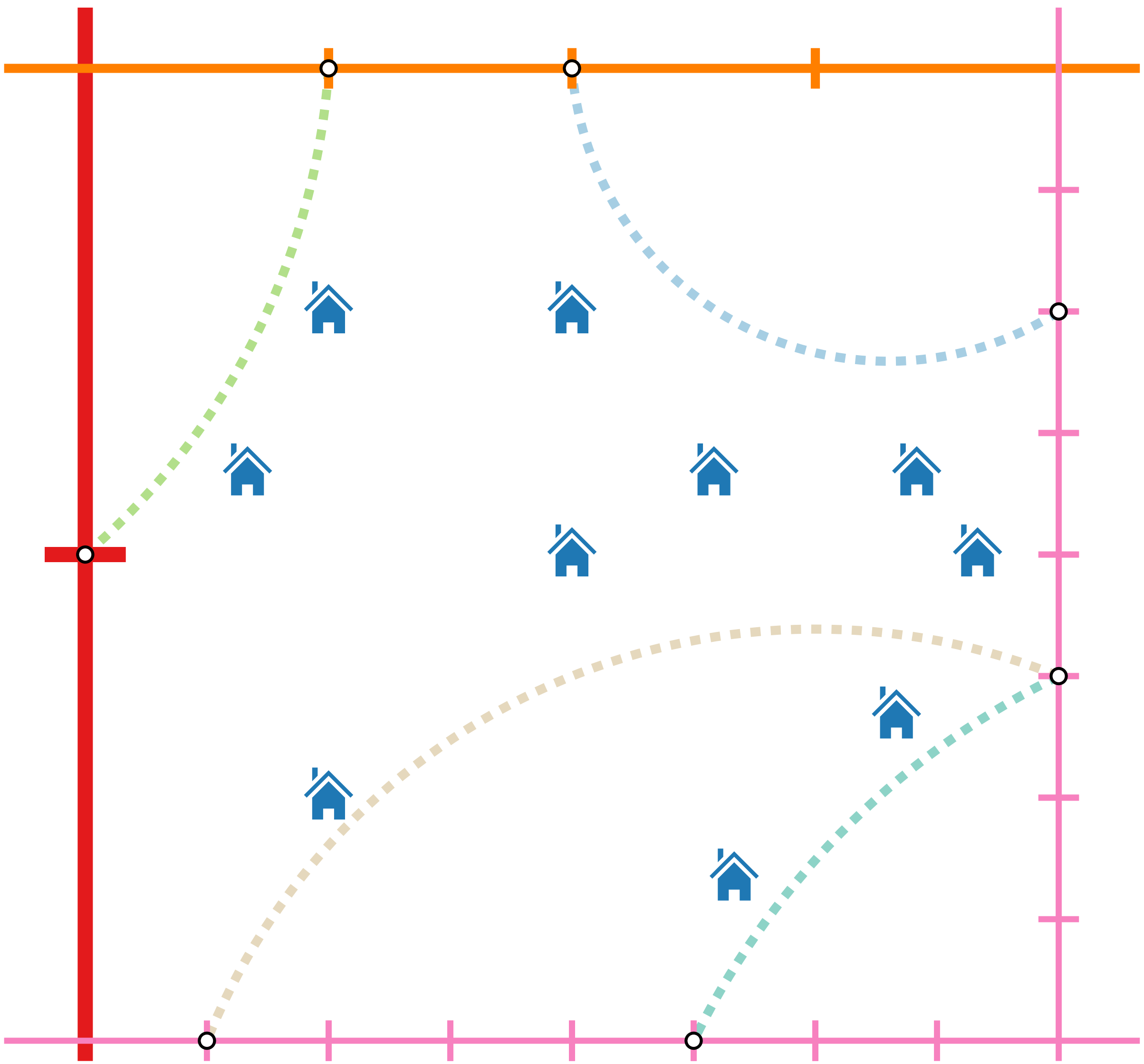


$$\Rightarrow \text{max. } \underbrace{n^{O(1/\varepsilon)}}_{\text{\#visit vectors}} \times \underbrace{2^{O(m)}}_{\text{\#realizable pairings}} =$$

Dynamic Program (I)



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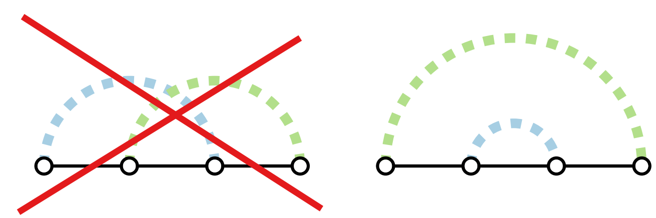

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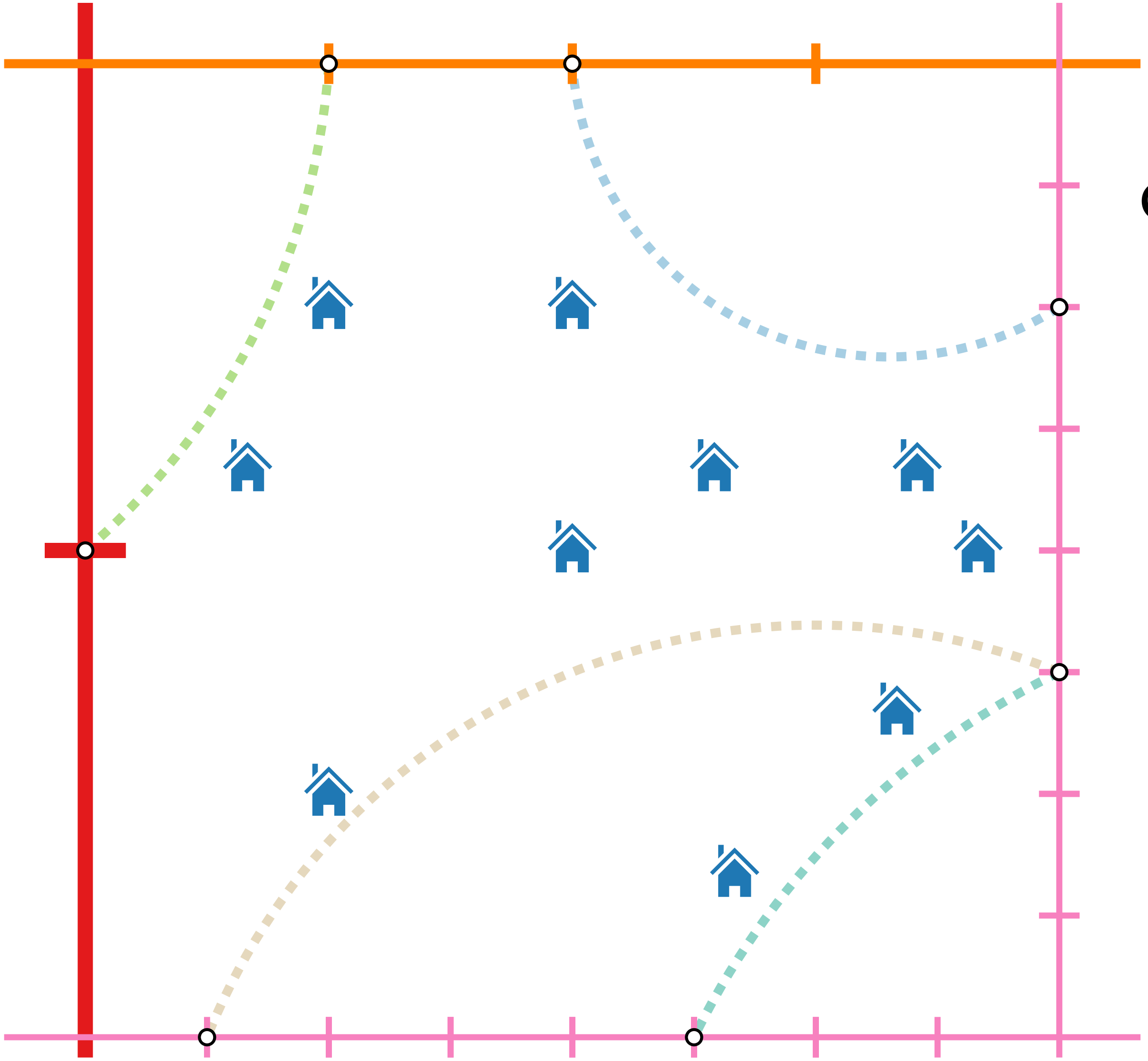
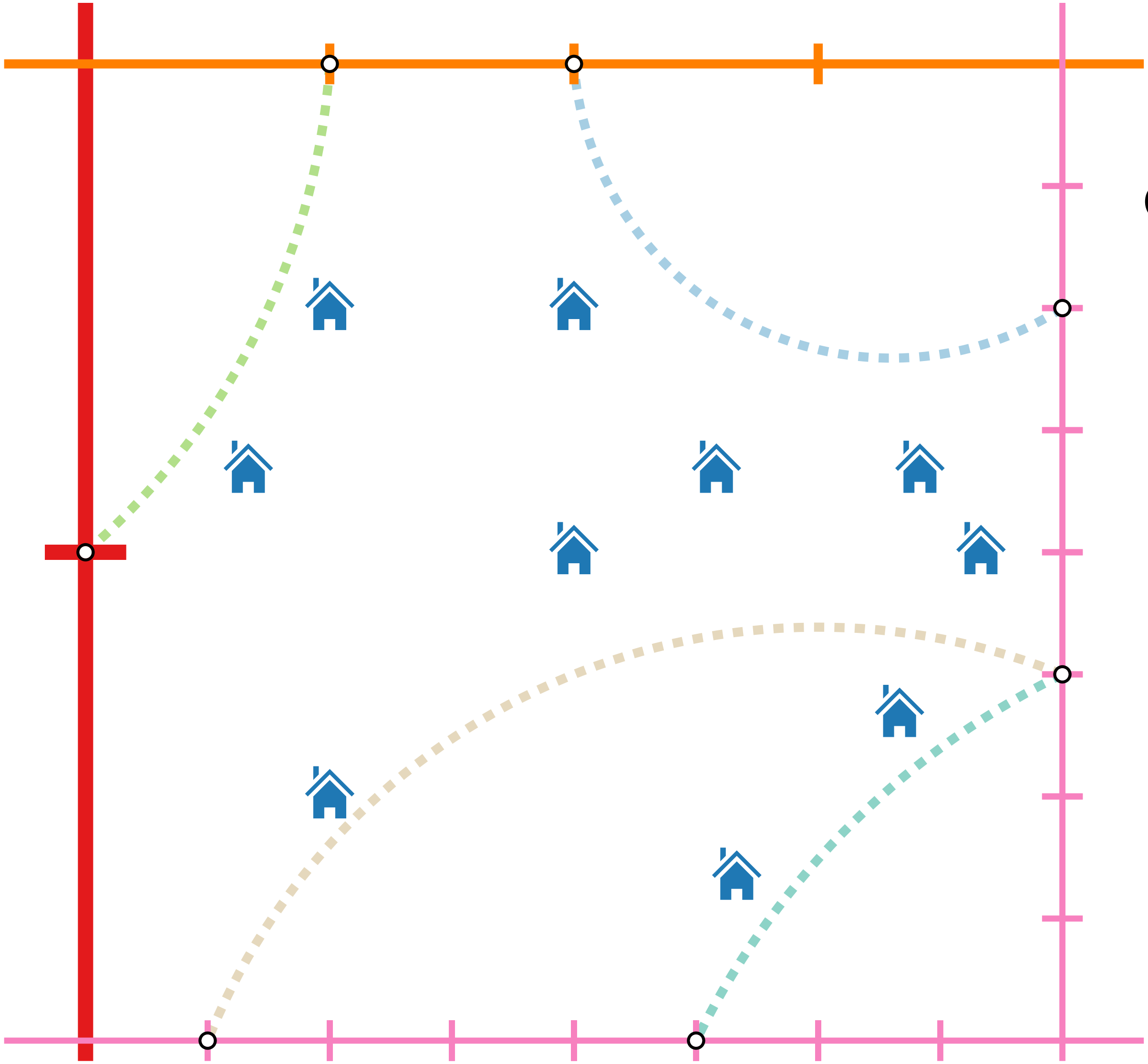
Dynamic Program (II)



Compute

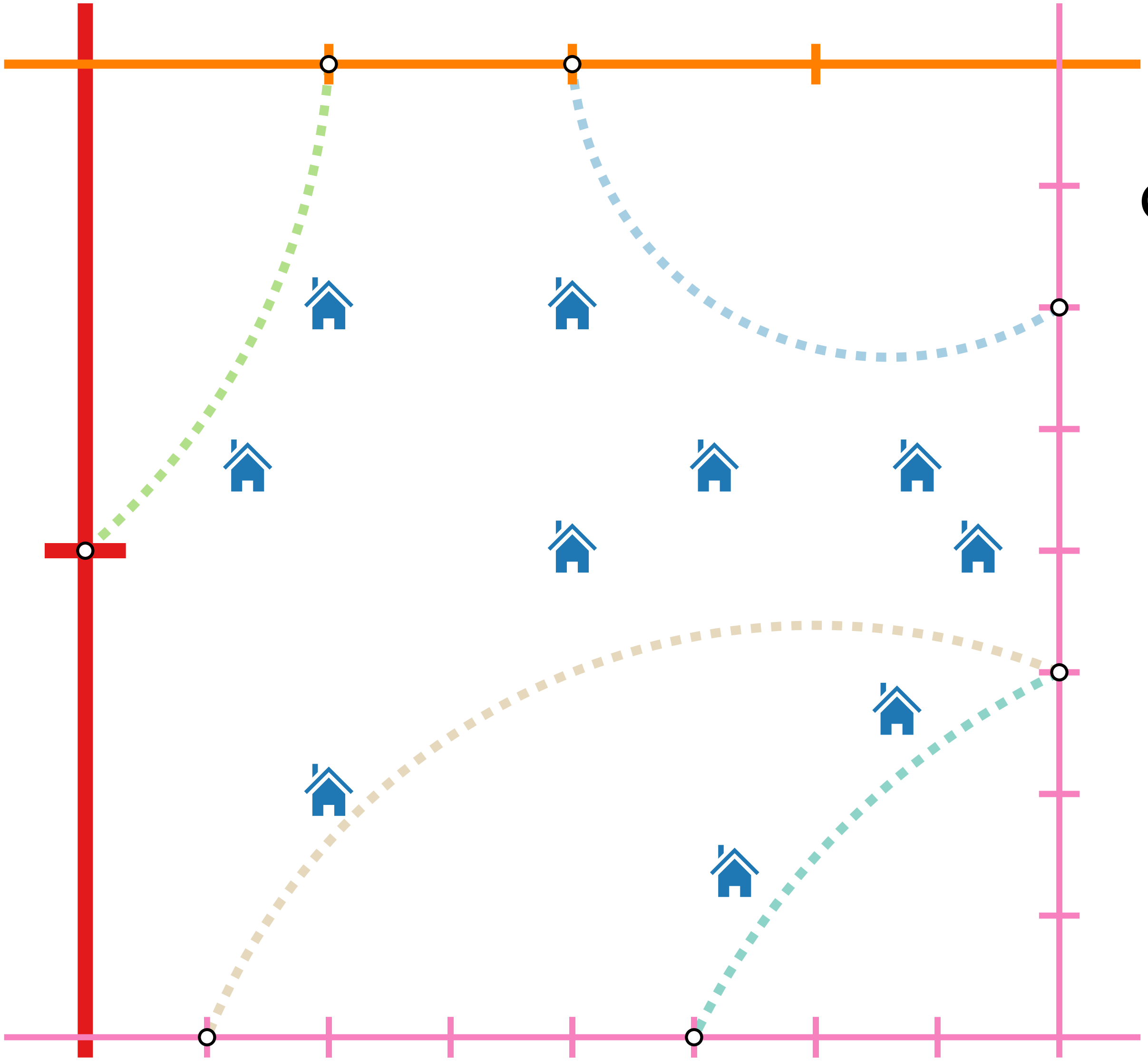
Dynamic Program (II)



Compute

- for each square Q in the dissection and

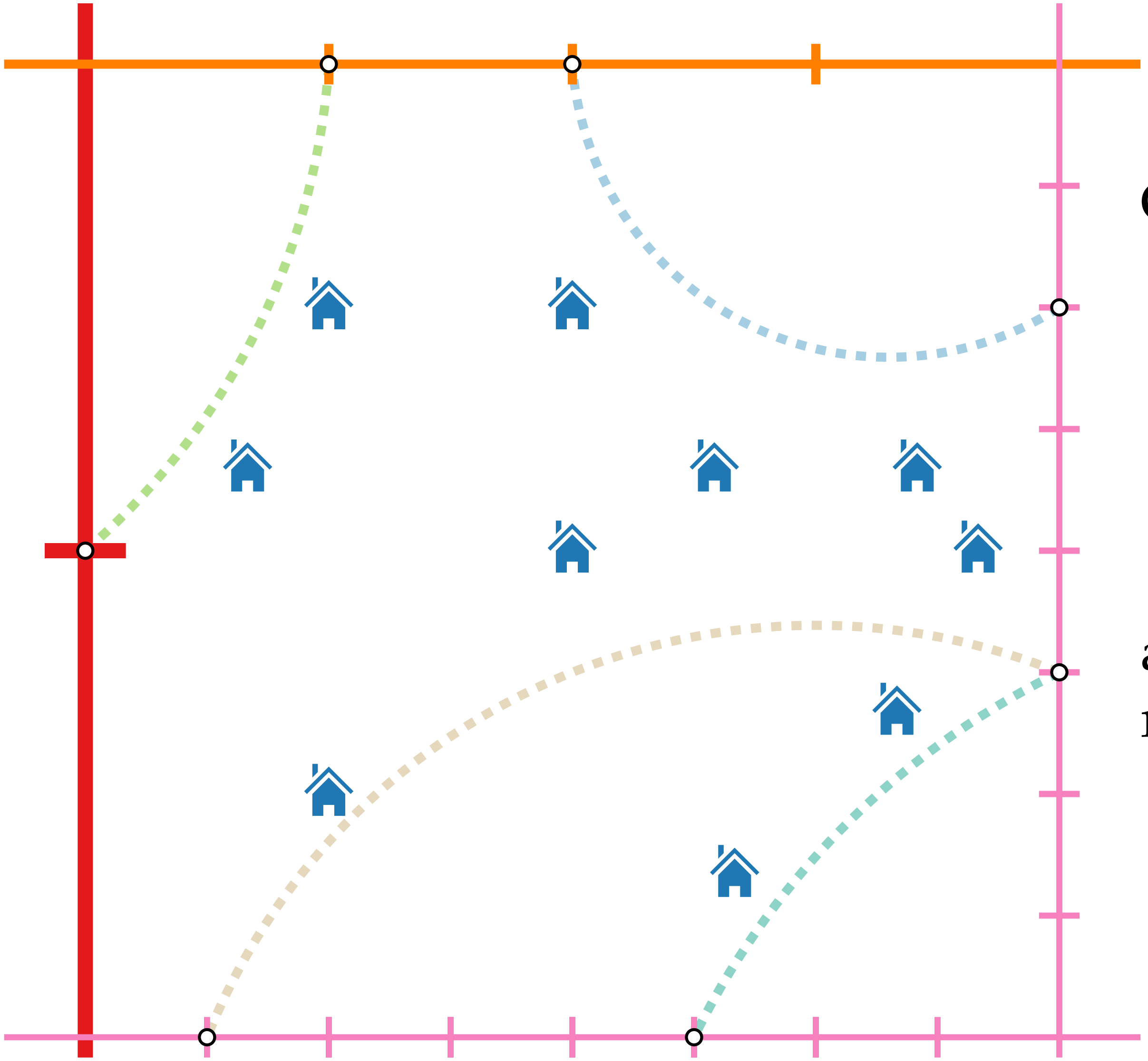
Dynamic Program (II)



Compute

- for each square Q in the dissection and
- for each crossing-free pairing P of Q,

Dynamic Program (II)



Compute

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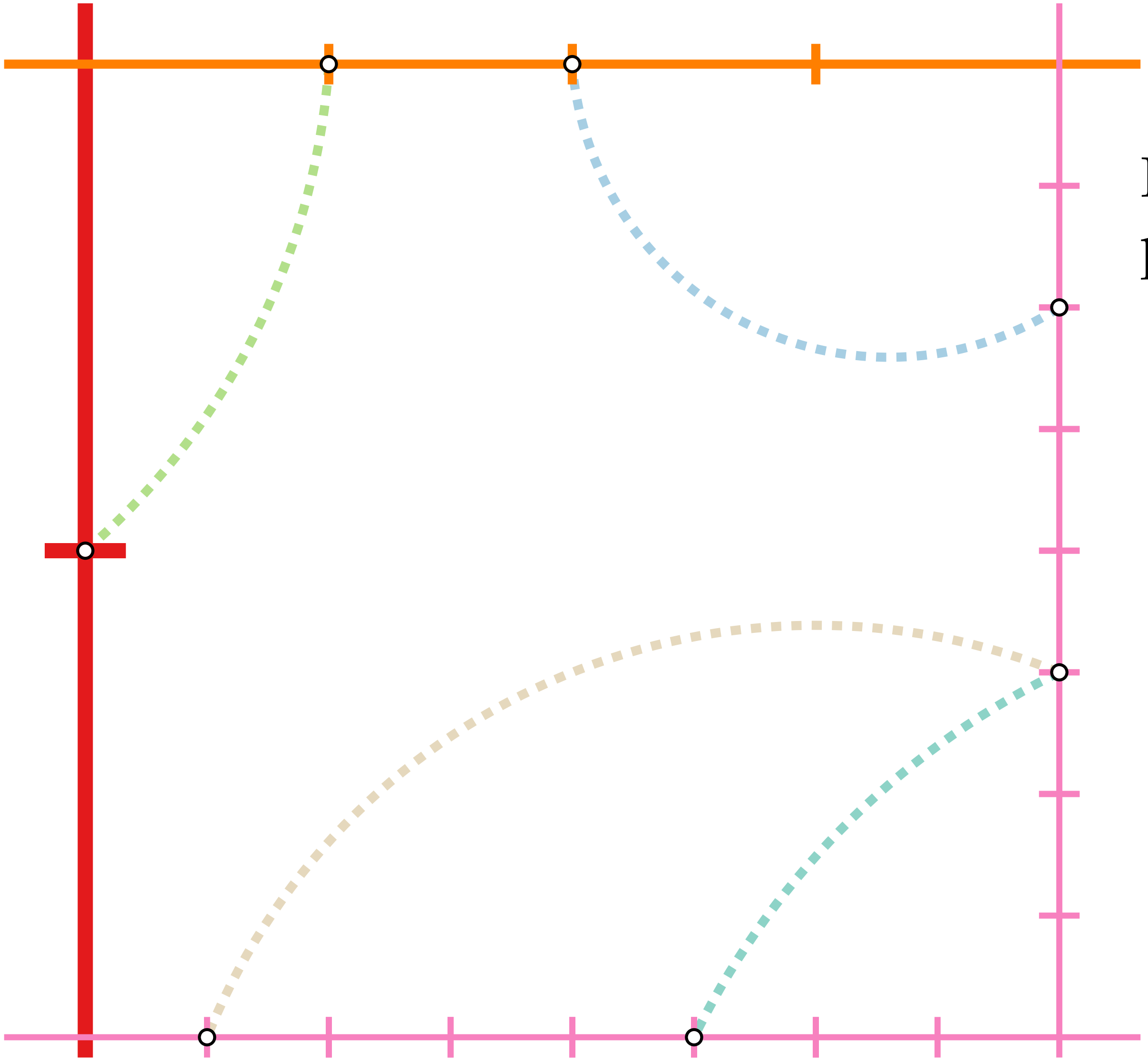
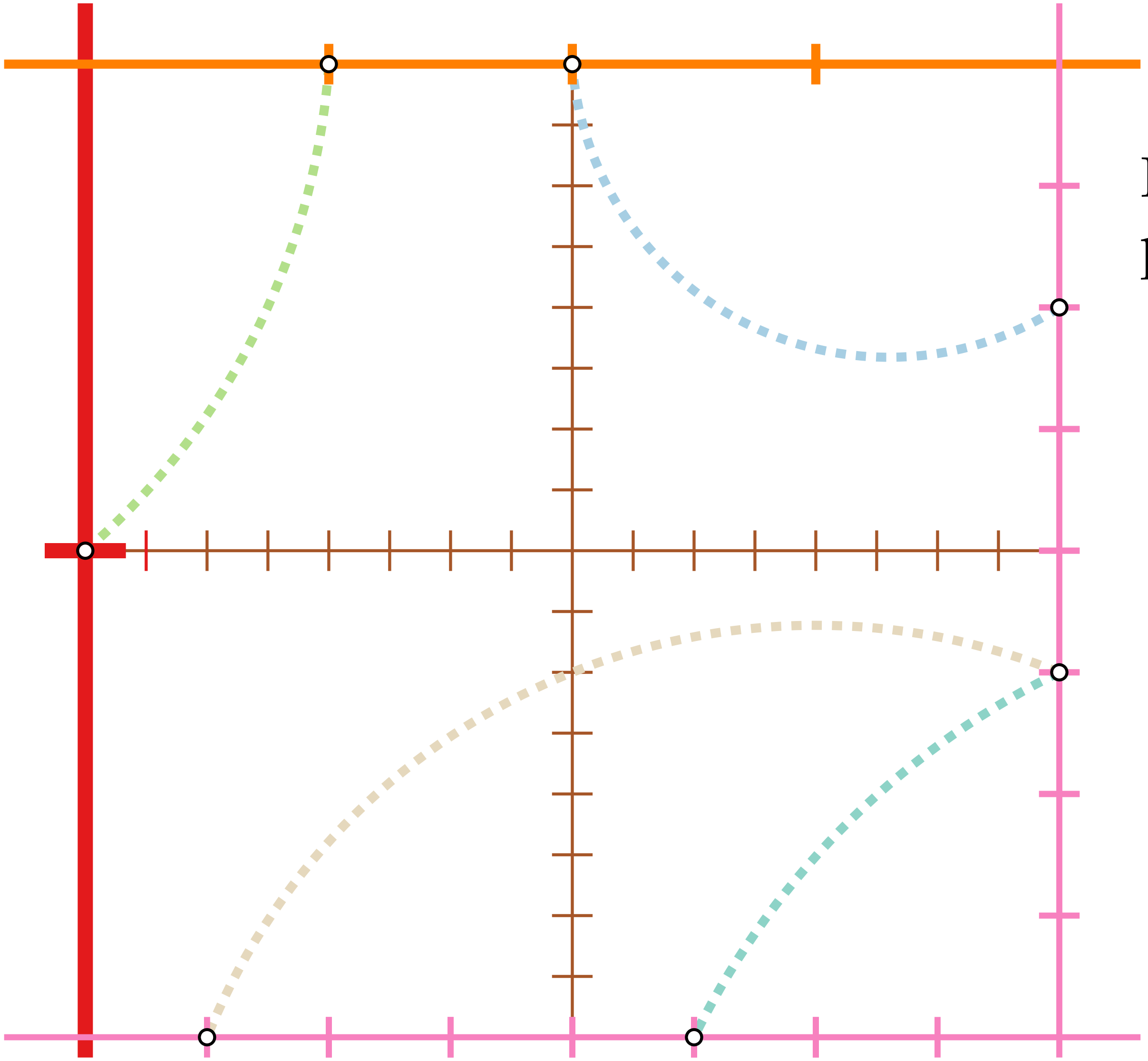
an optimal path cover that respects P.

Dynamic Program (III)



For a given square Q and pairing P:

Dynamic Program (III)



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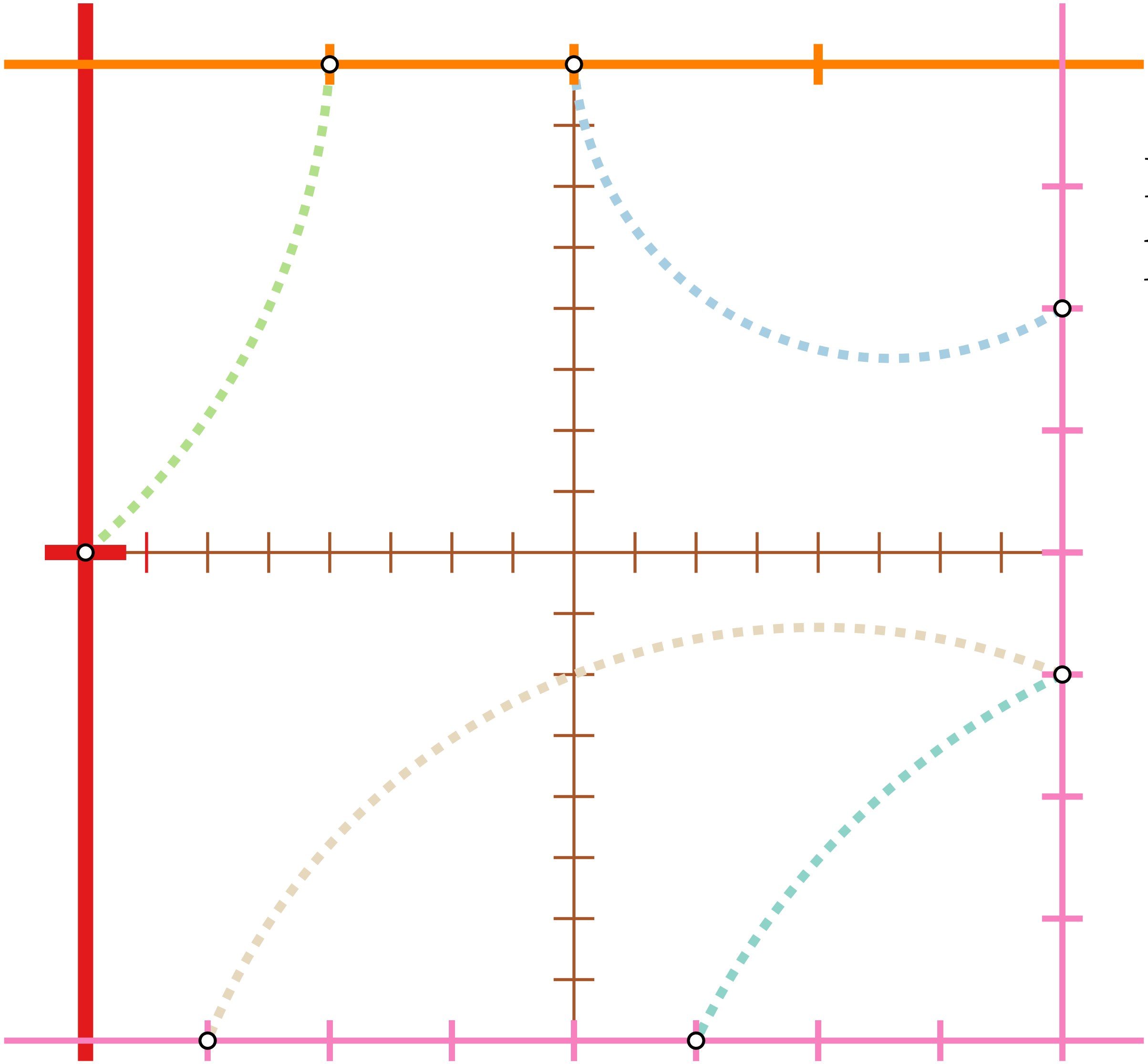

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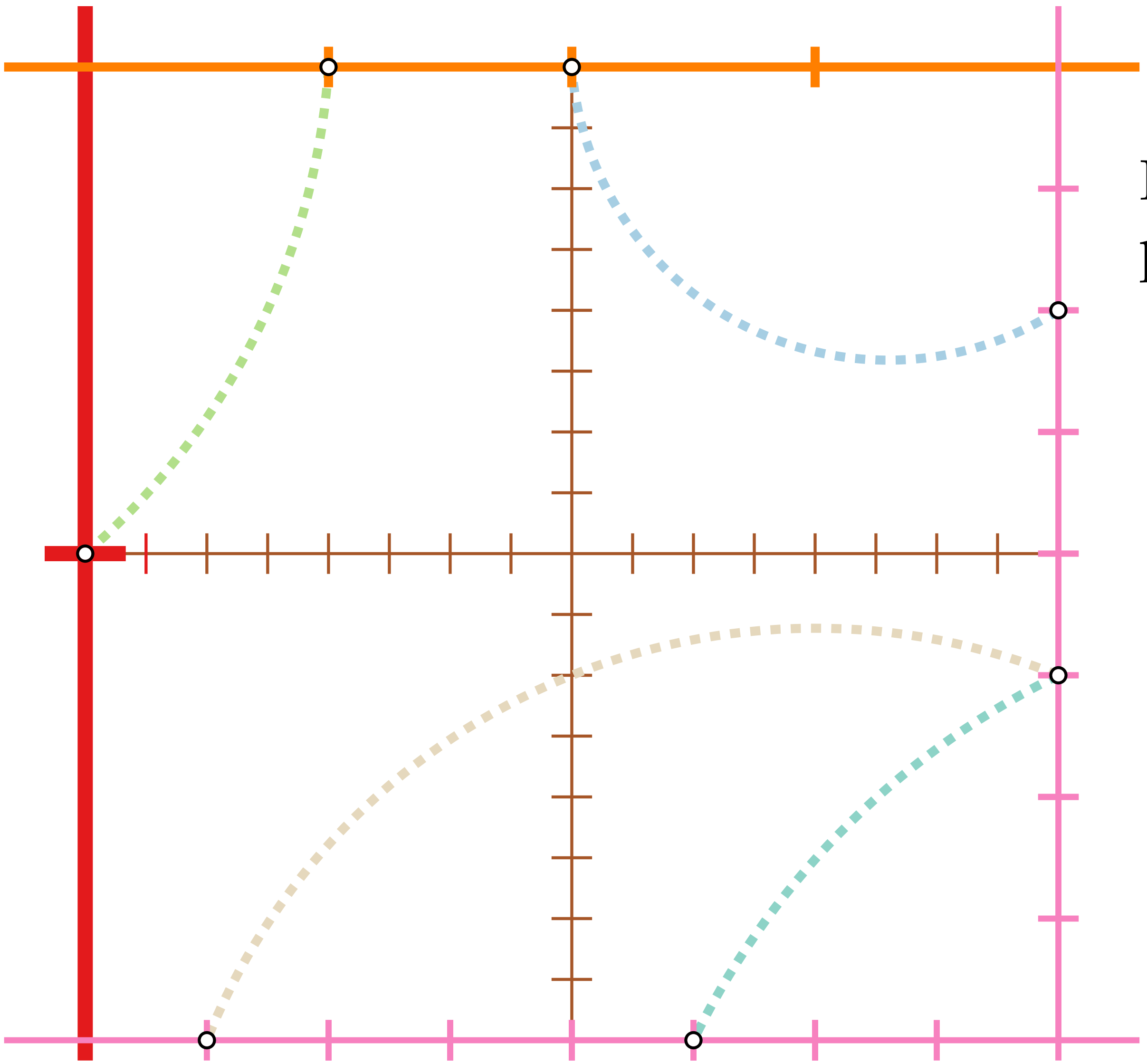
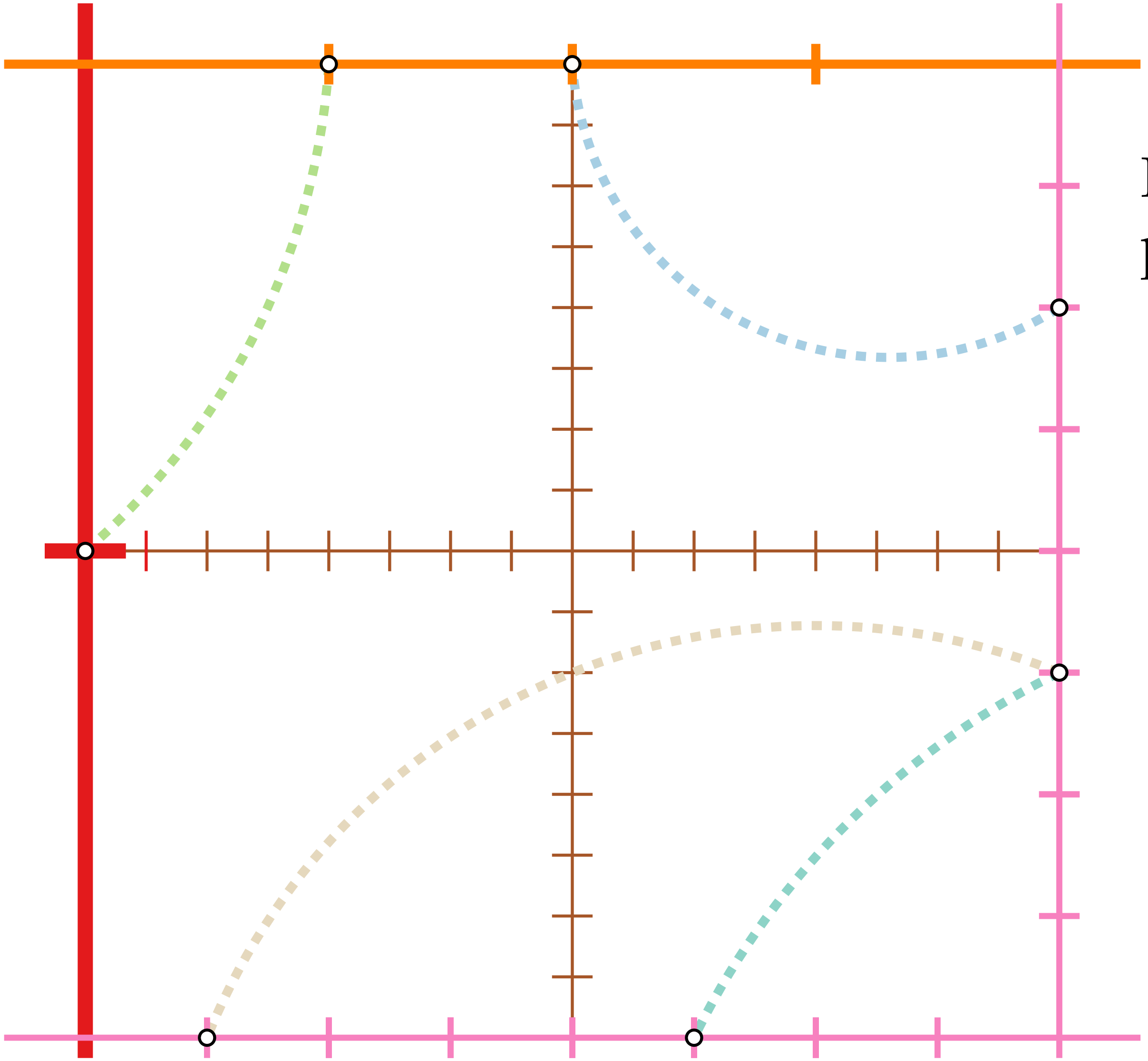


For a given square Q and pairing P:

- Iterate over all $(n^{O(1/\varepsilon)})^4 =$ crossing-free pairings of the child squares.

Dynamic Program (III)



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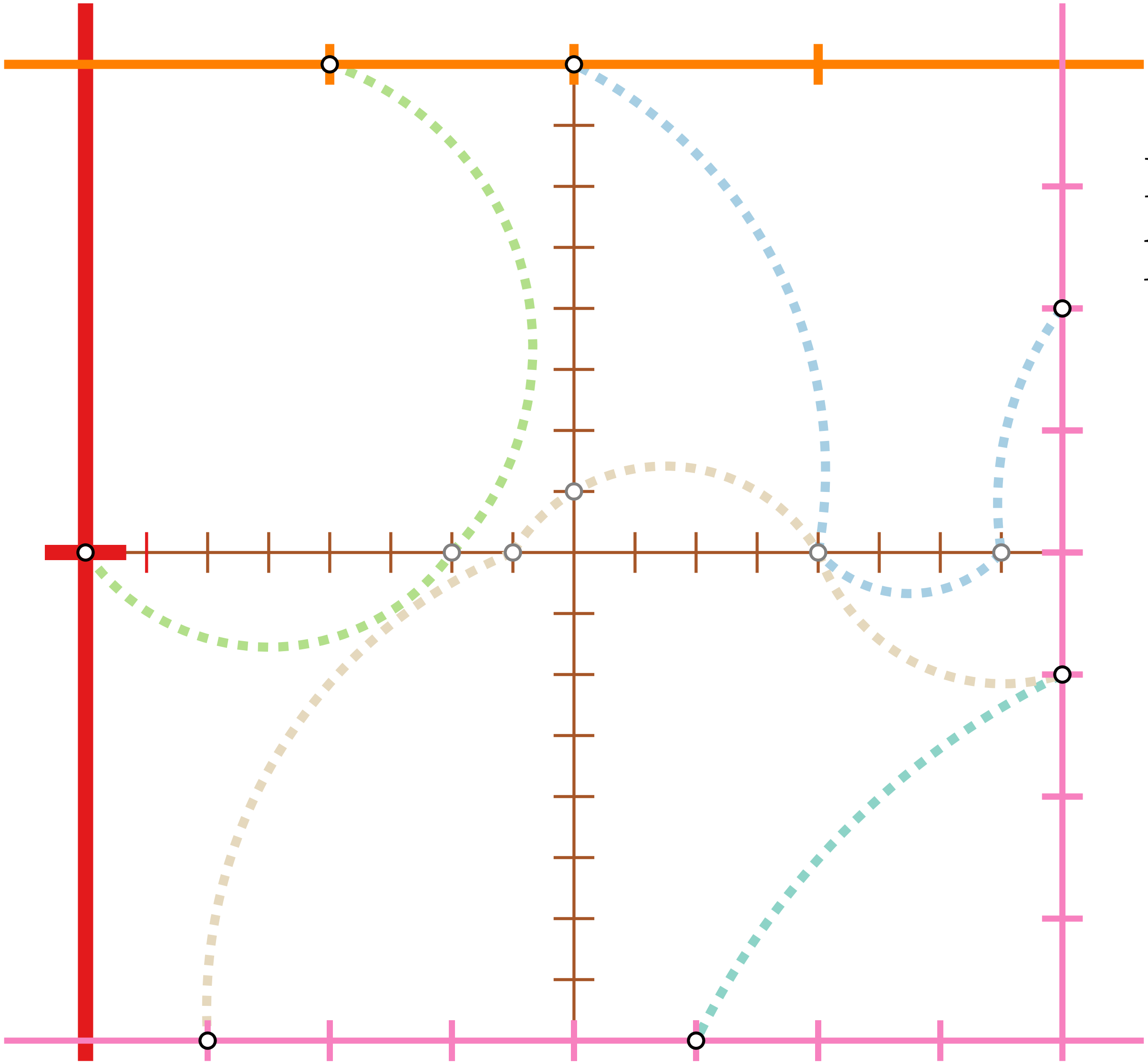

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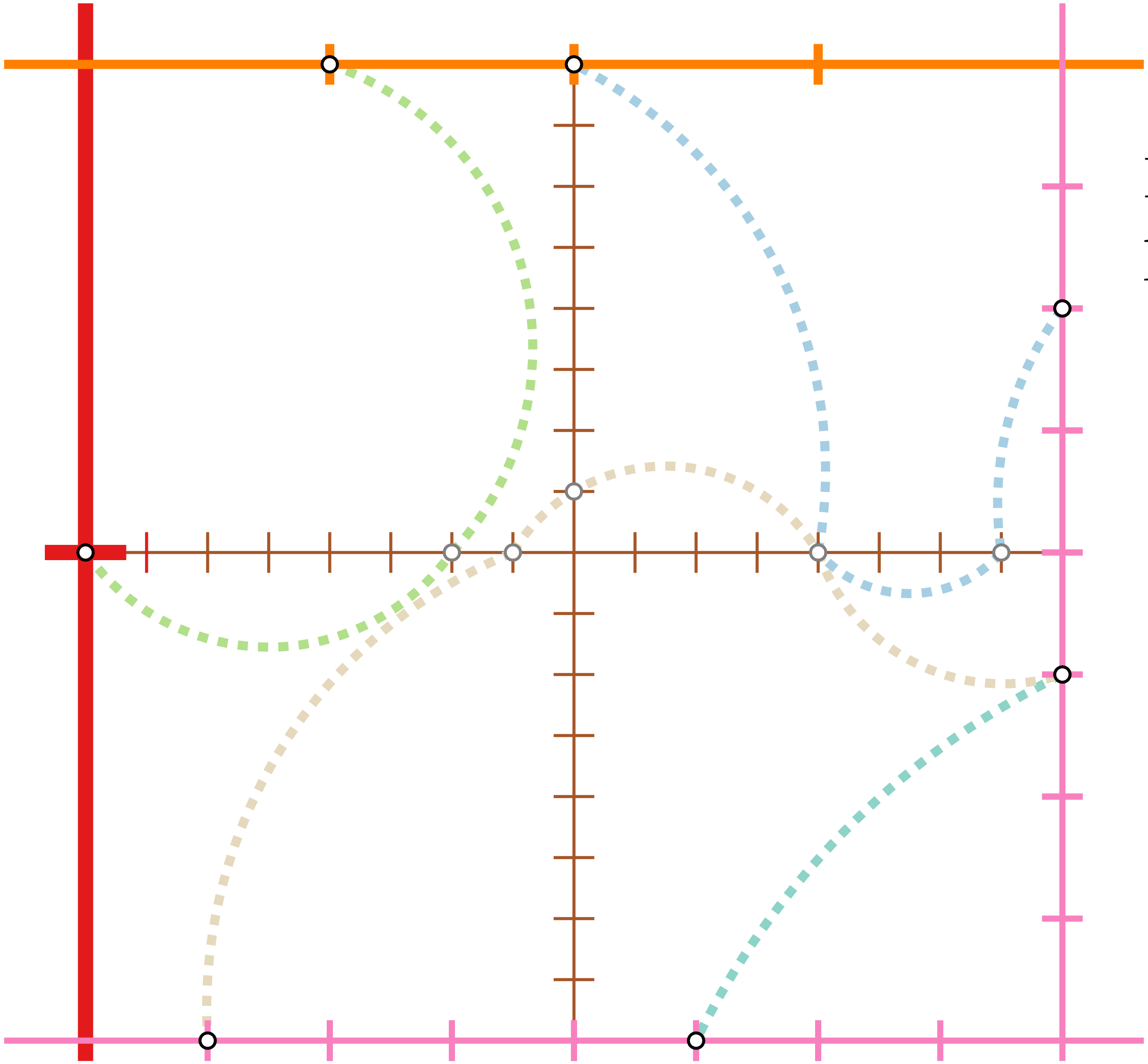

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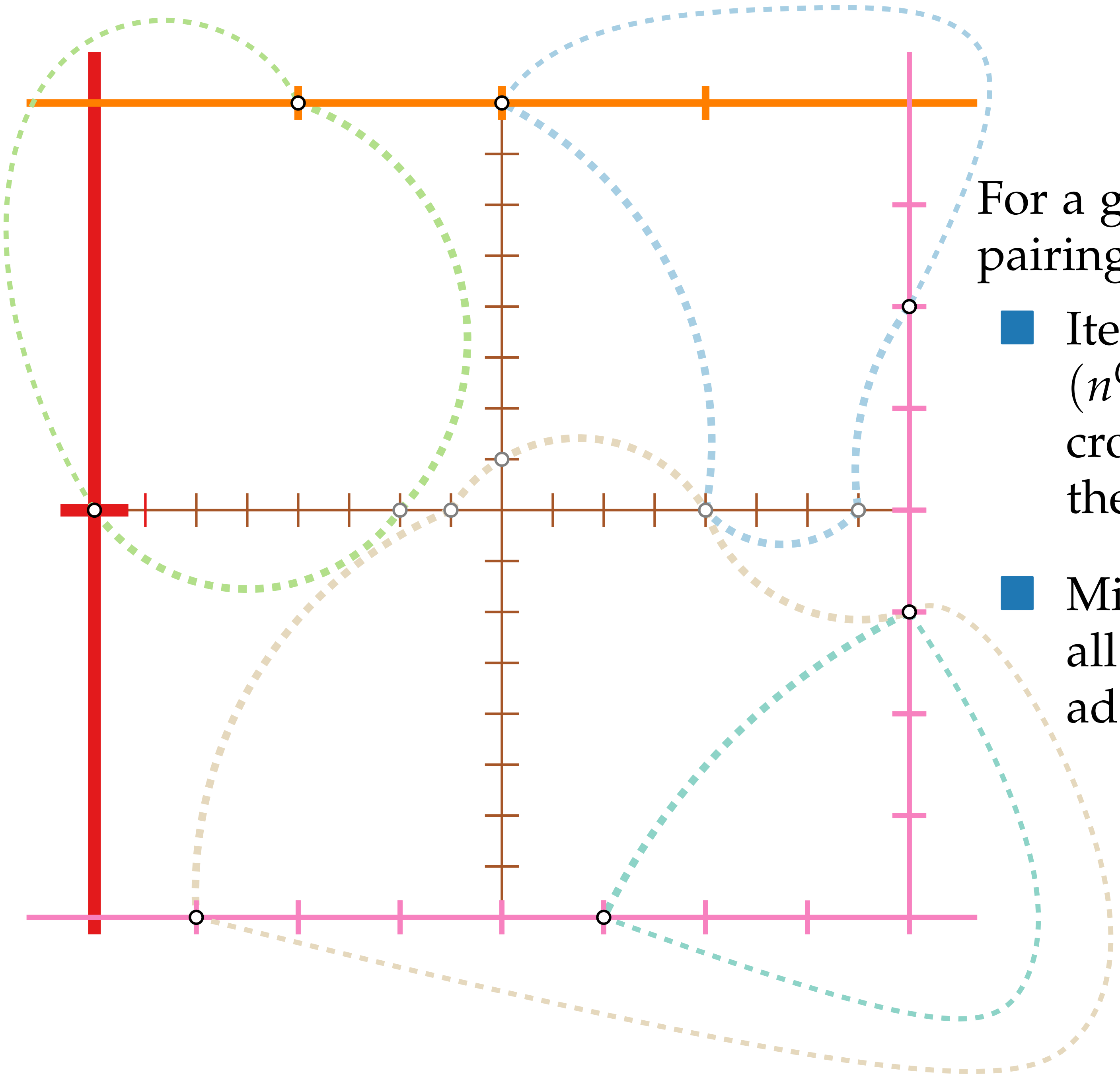

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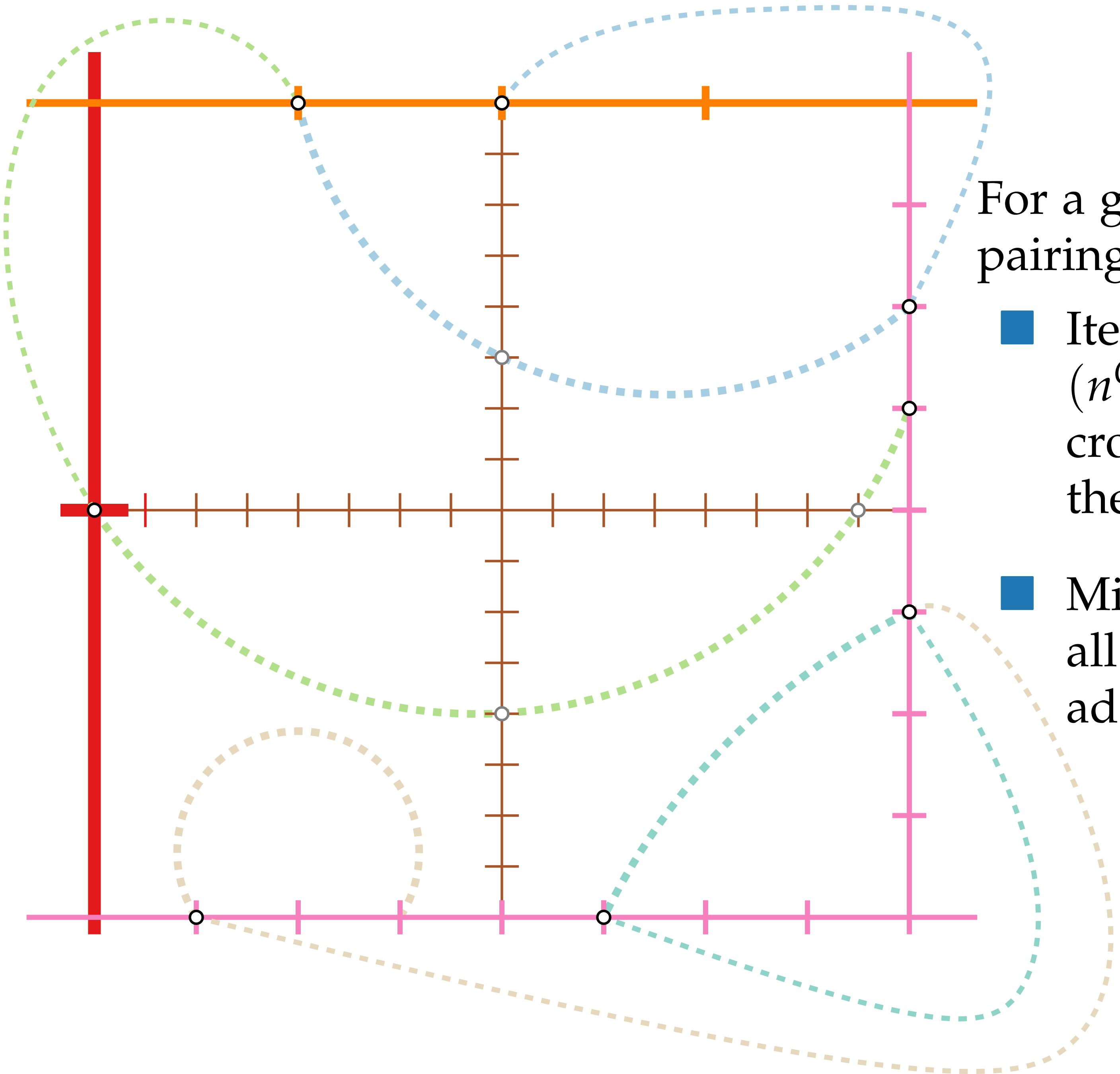

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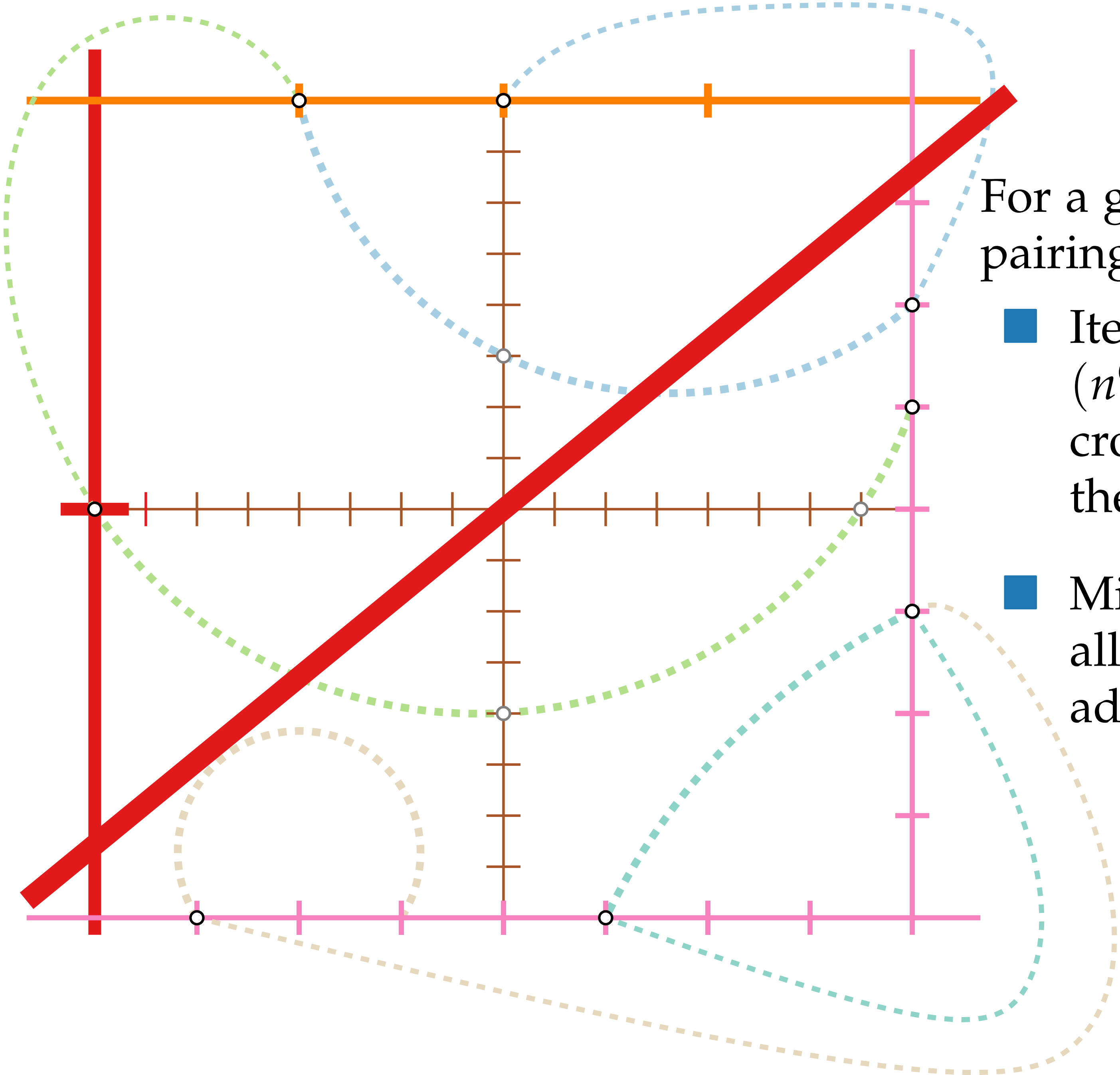

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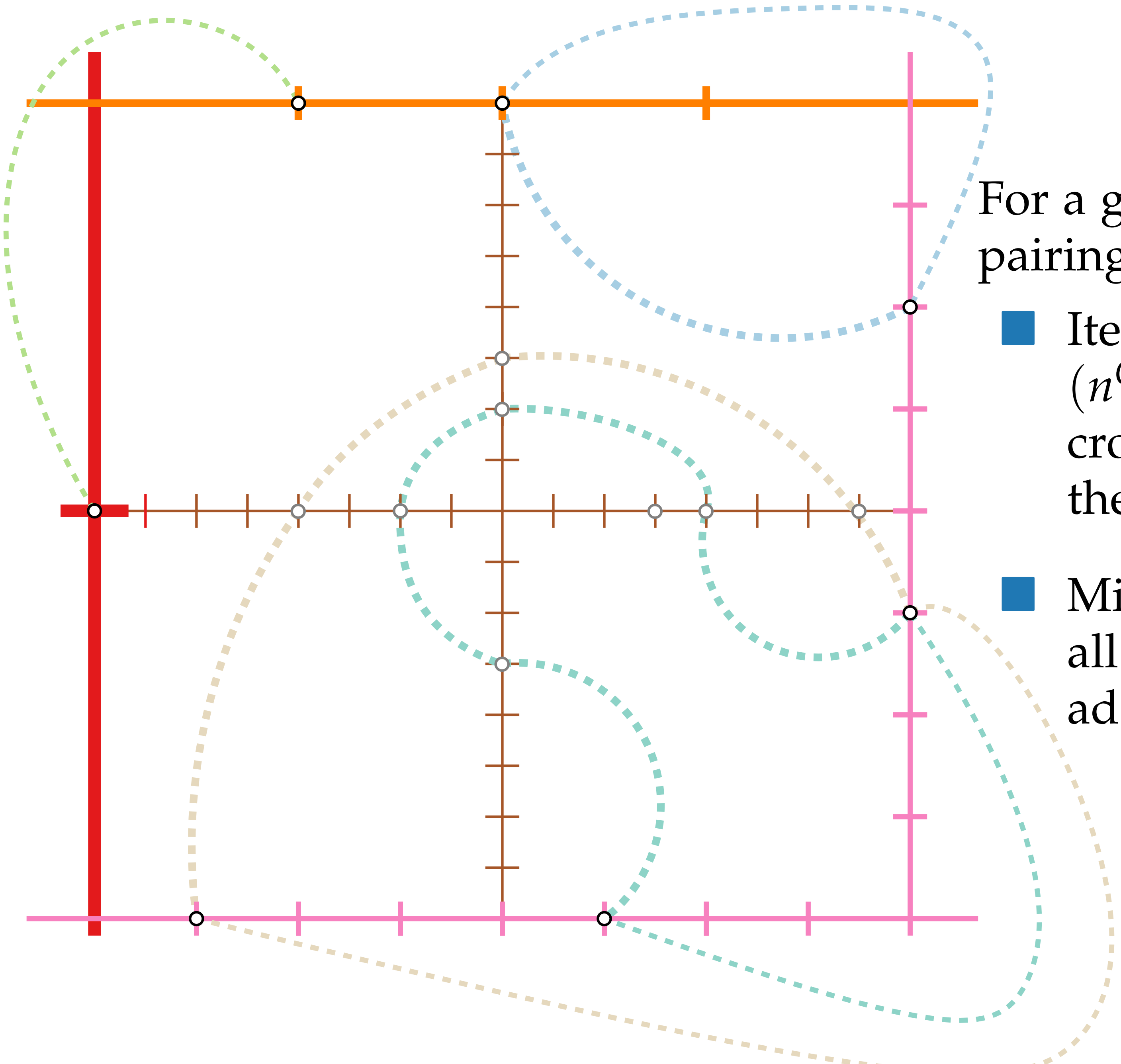

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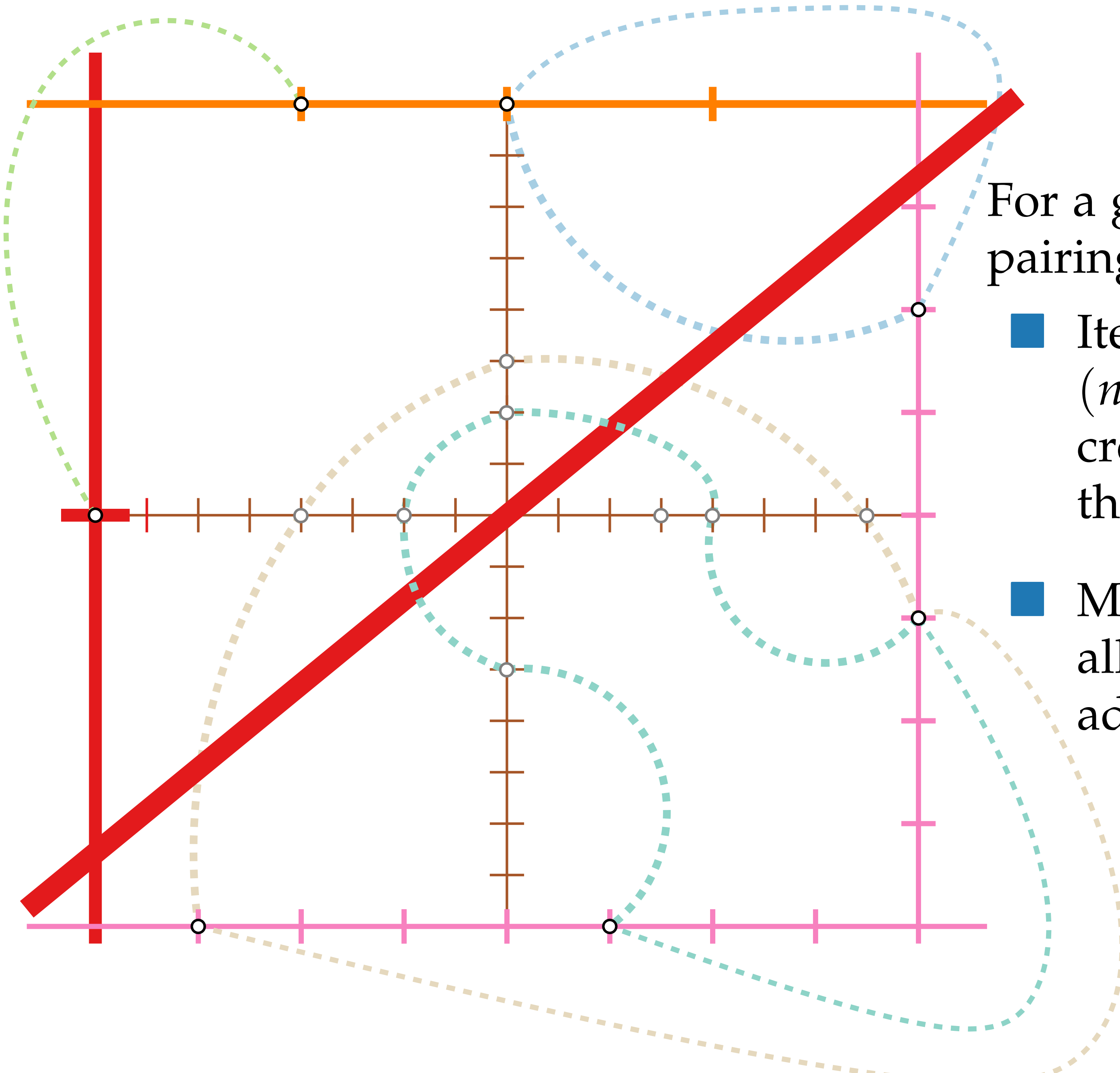

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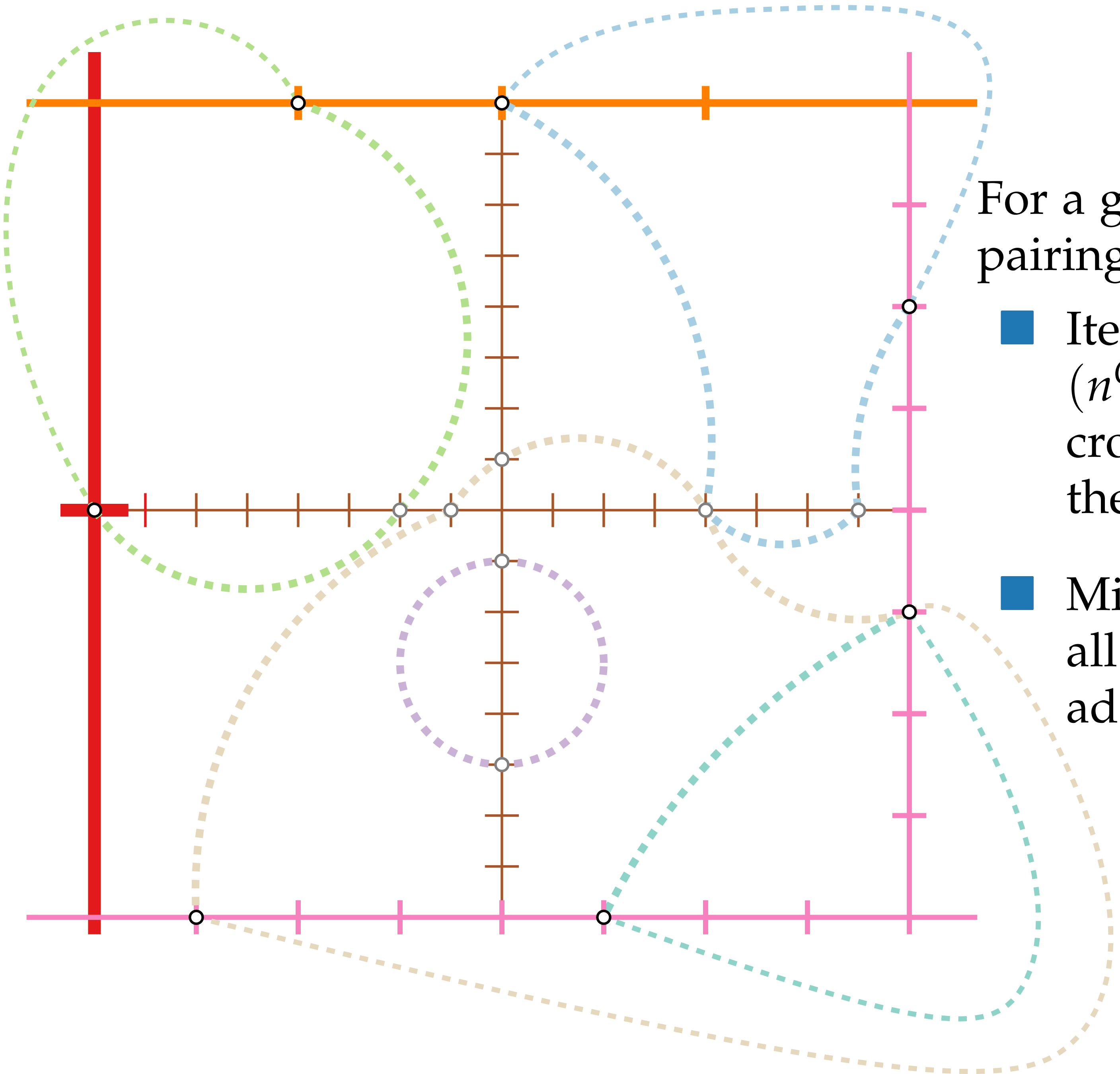

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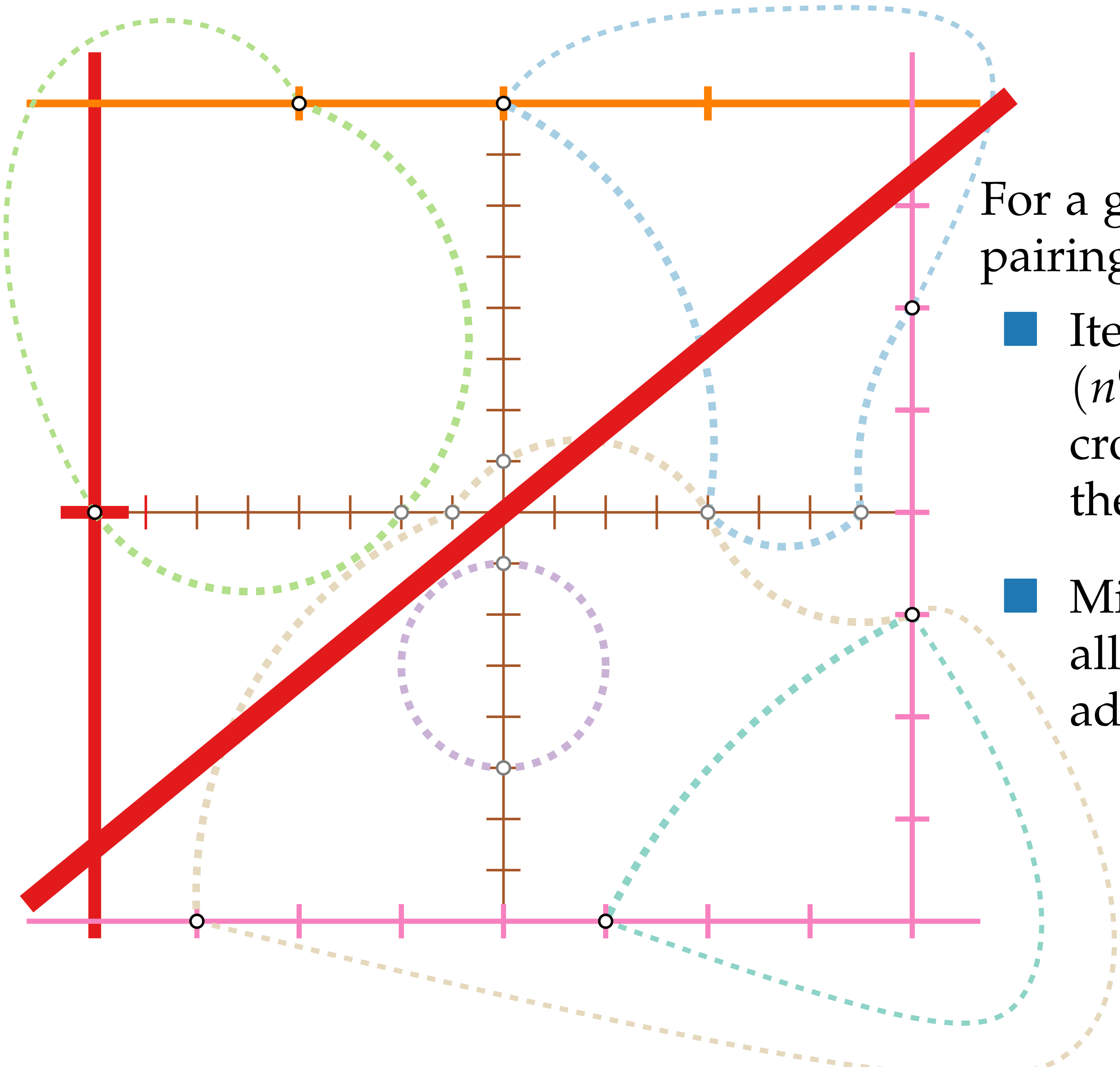

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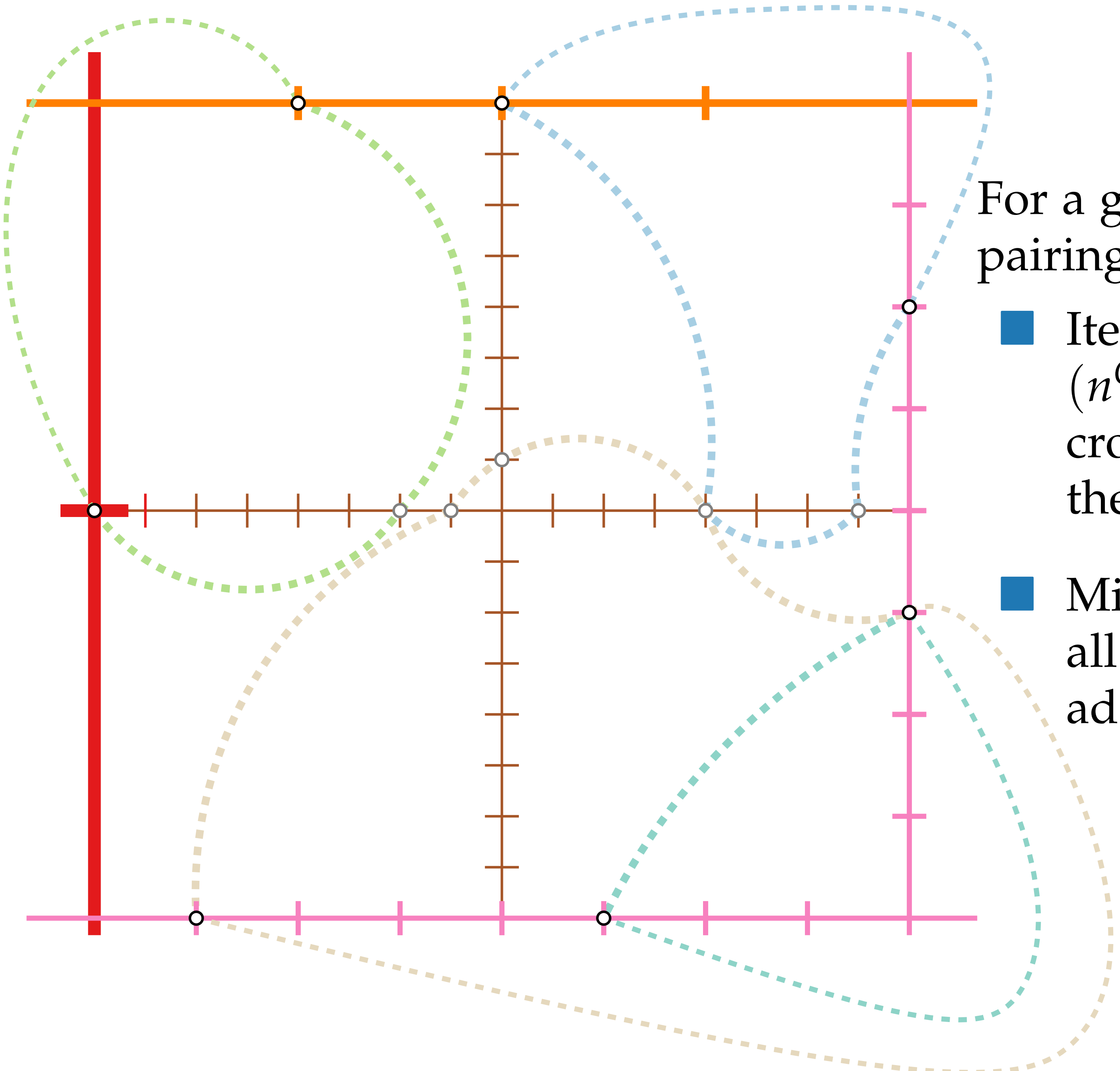

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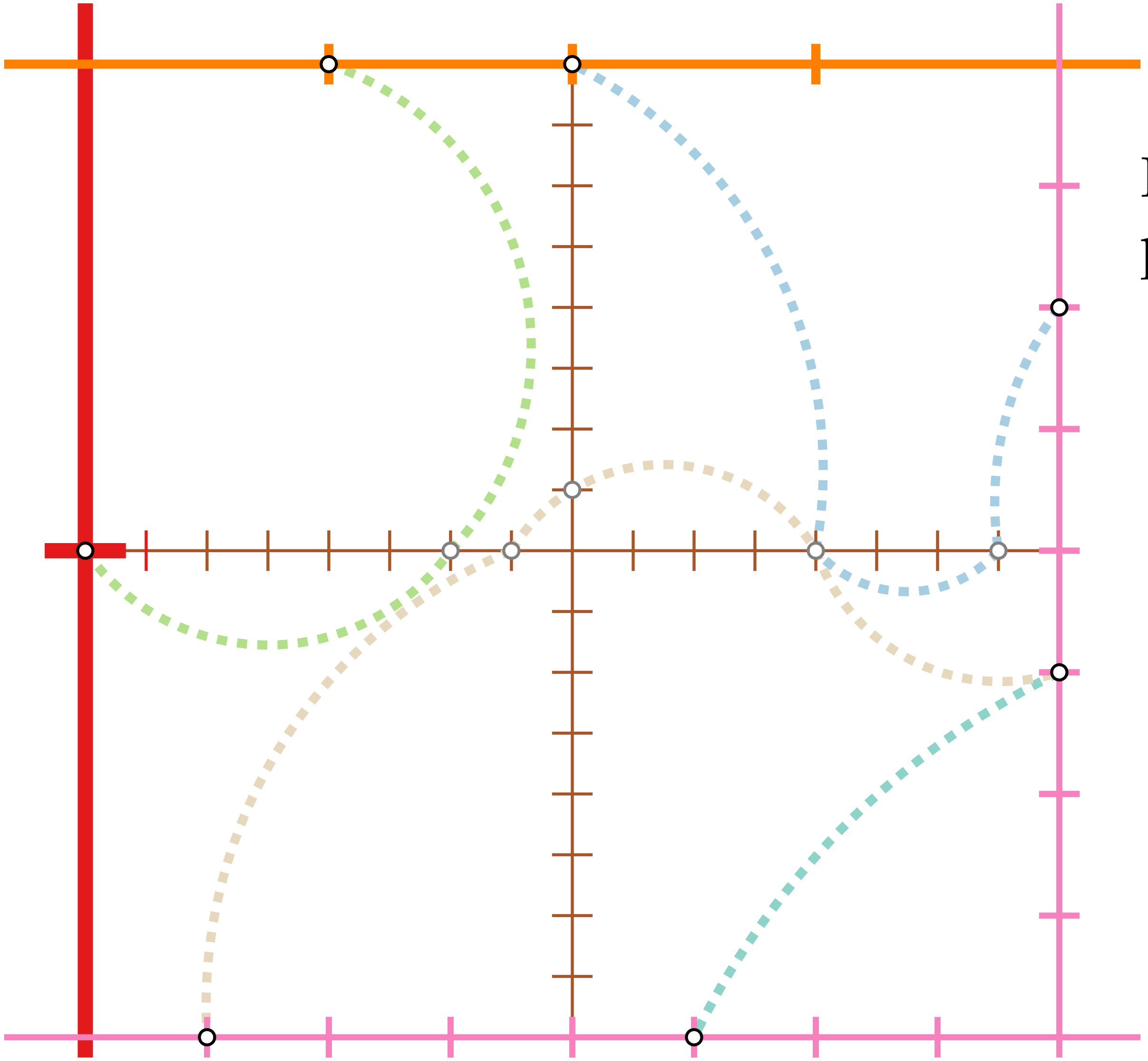

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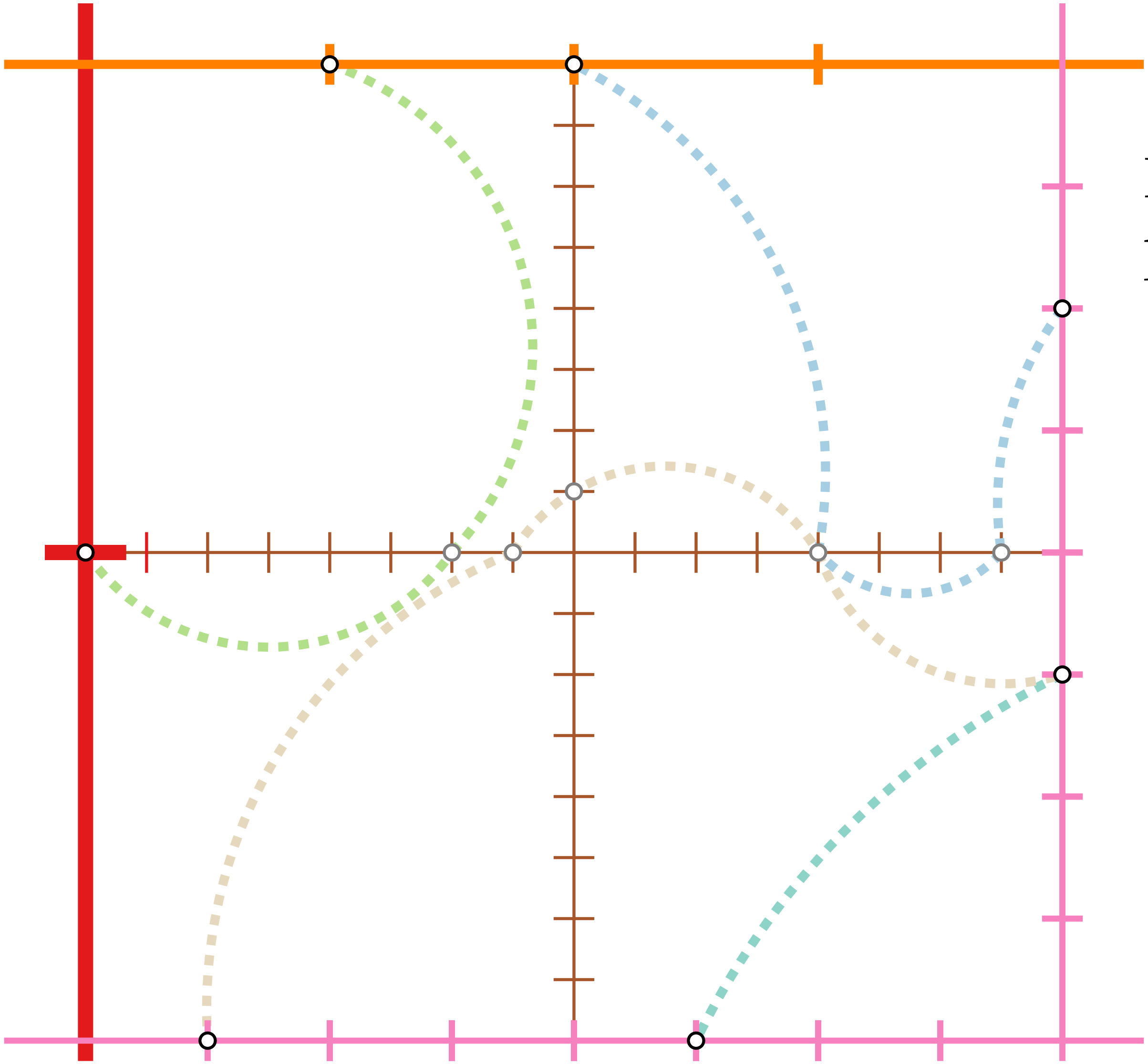

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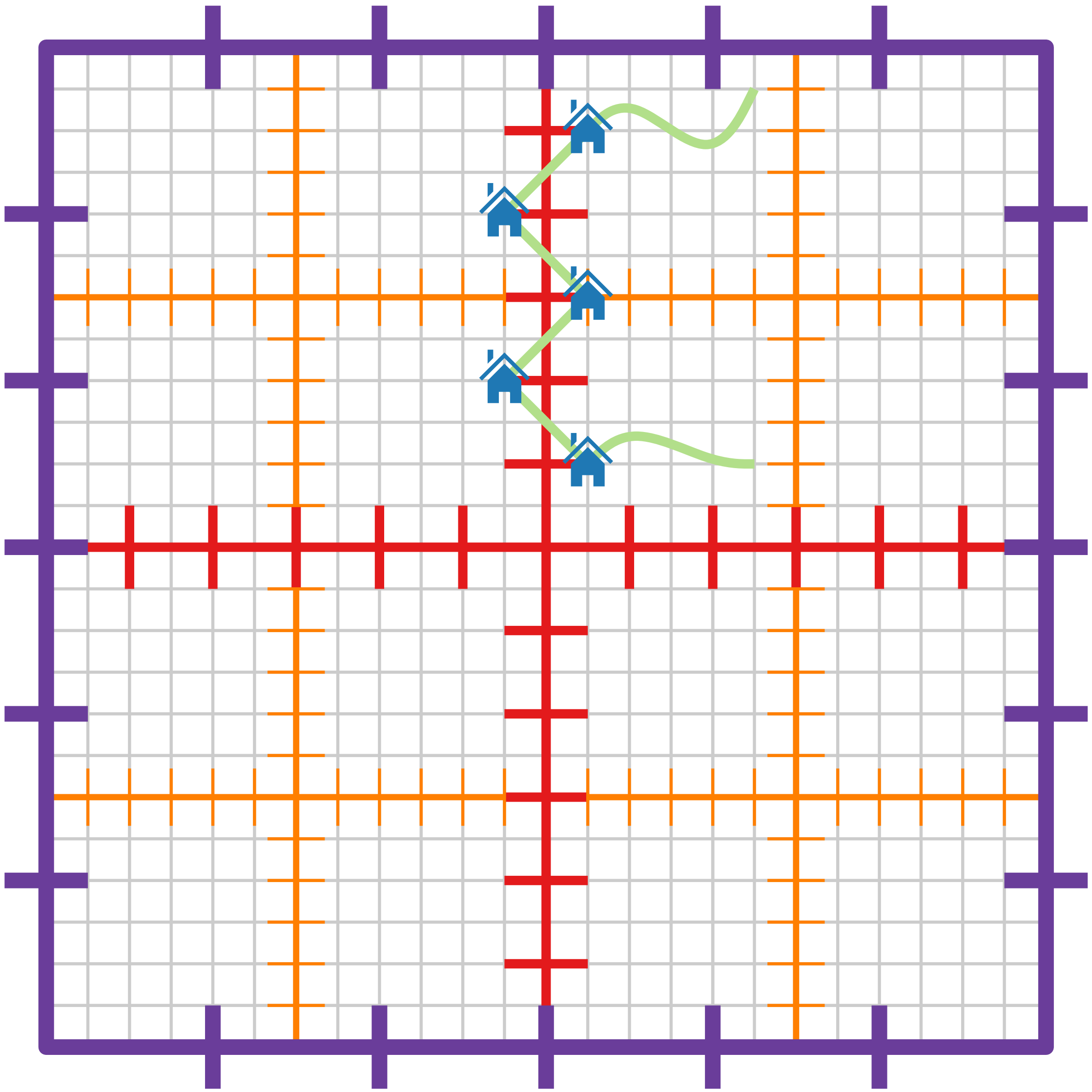
Lemma. An optimal well-behaved tour can be computed in $2^{O(m)} = n^{O(1/\varepsilon)}$ time.

Approximation Algorithms

Lecture 9:
A PTAS for Euclidean TSP

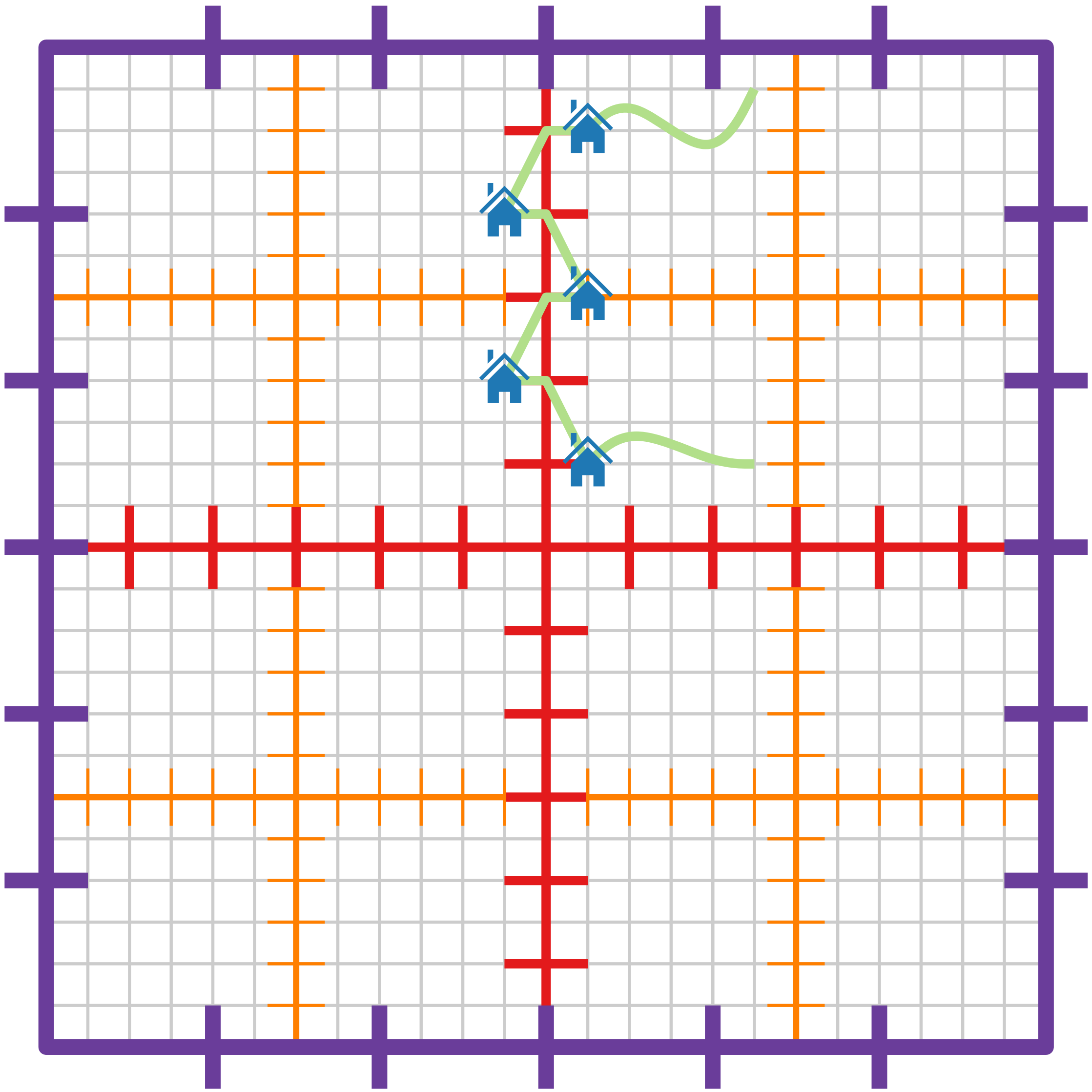
Part V:
Shifted Dissections

Shifted Dissections



- The best well-behaved tour can be a bad approximation.

Shifted Dissections



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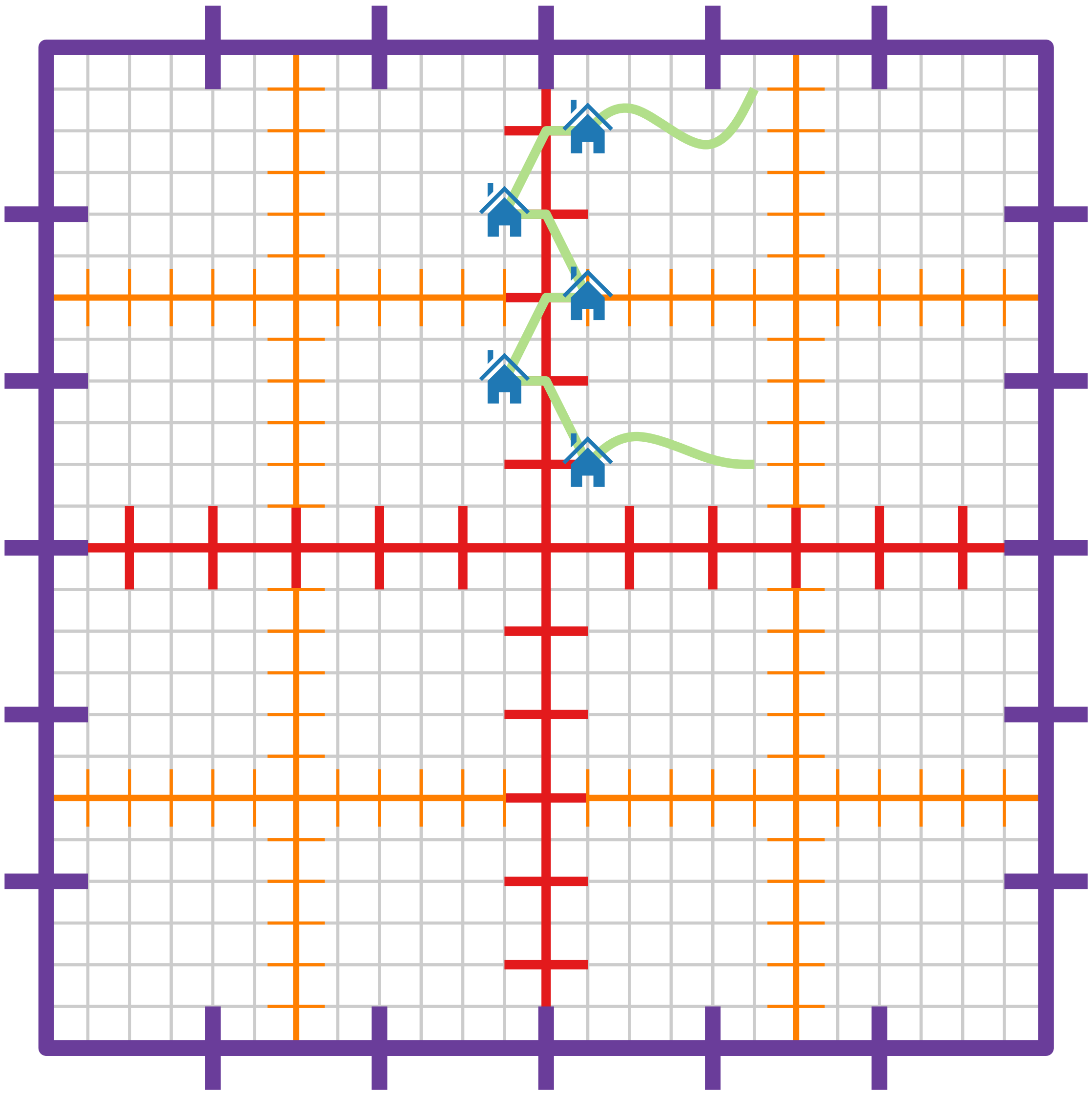
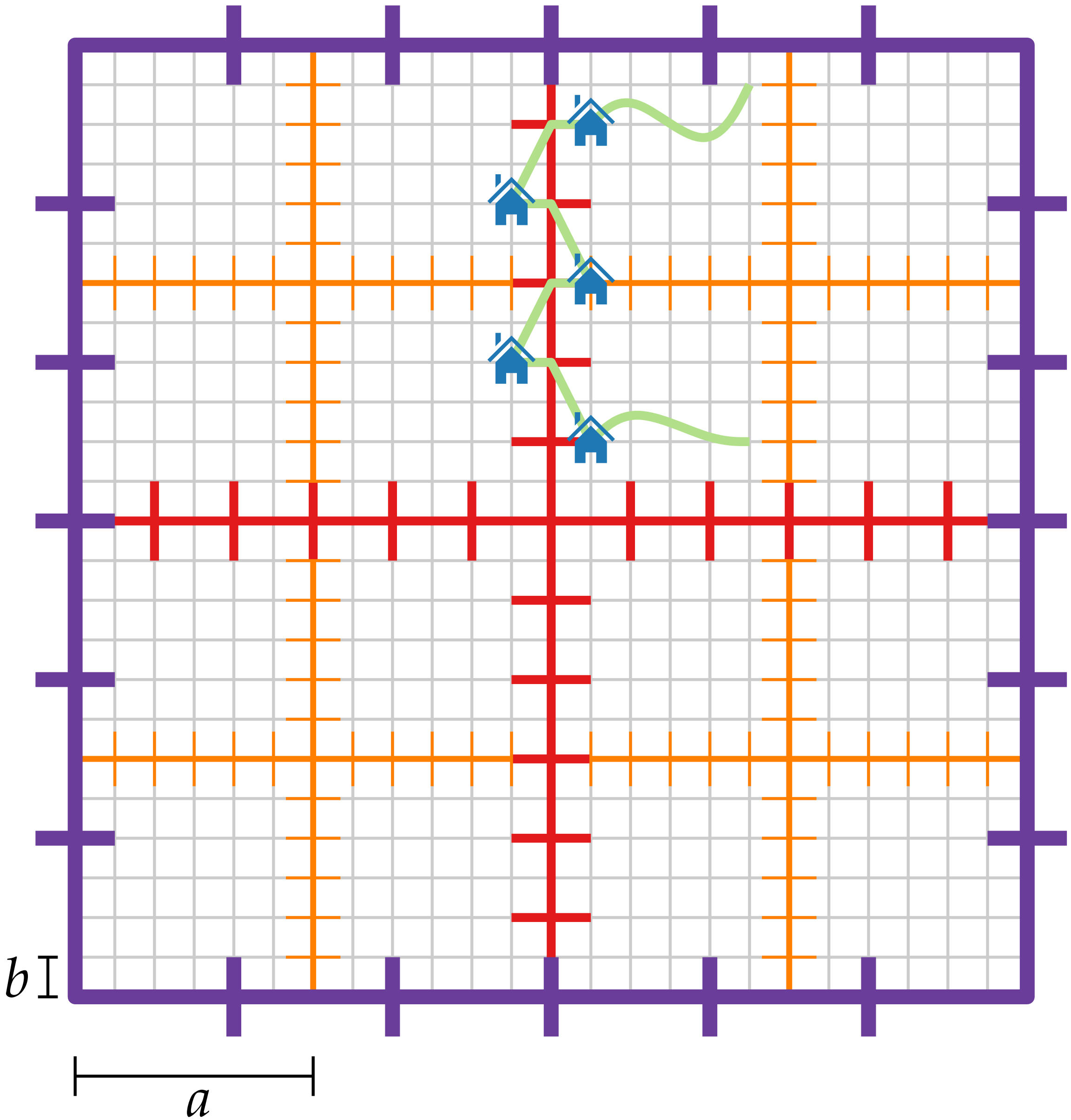


- The best well-behaved tour can be a bad approximation.
- Consider an (a, b)-shifted dissection:

$$x \mapsto (x + a) \bmod L$$
$$y \mapsto (y + b) \bmod L$$

Shifted Dissections



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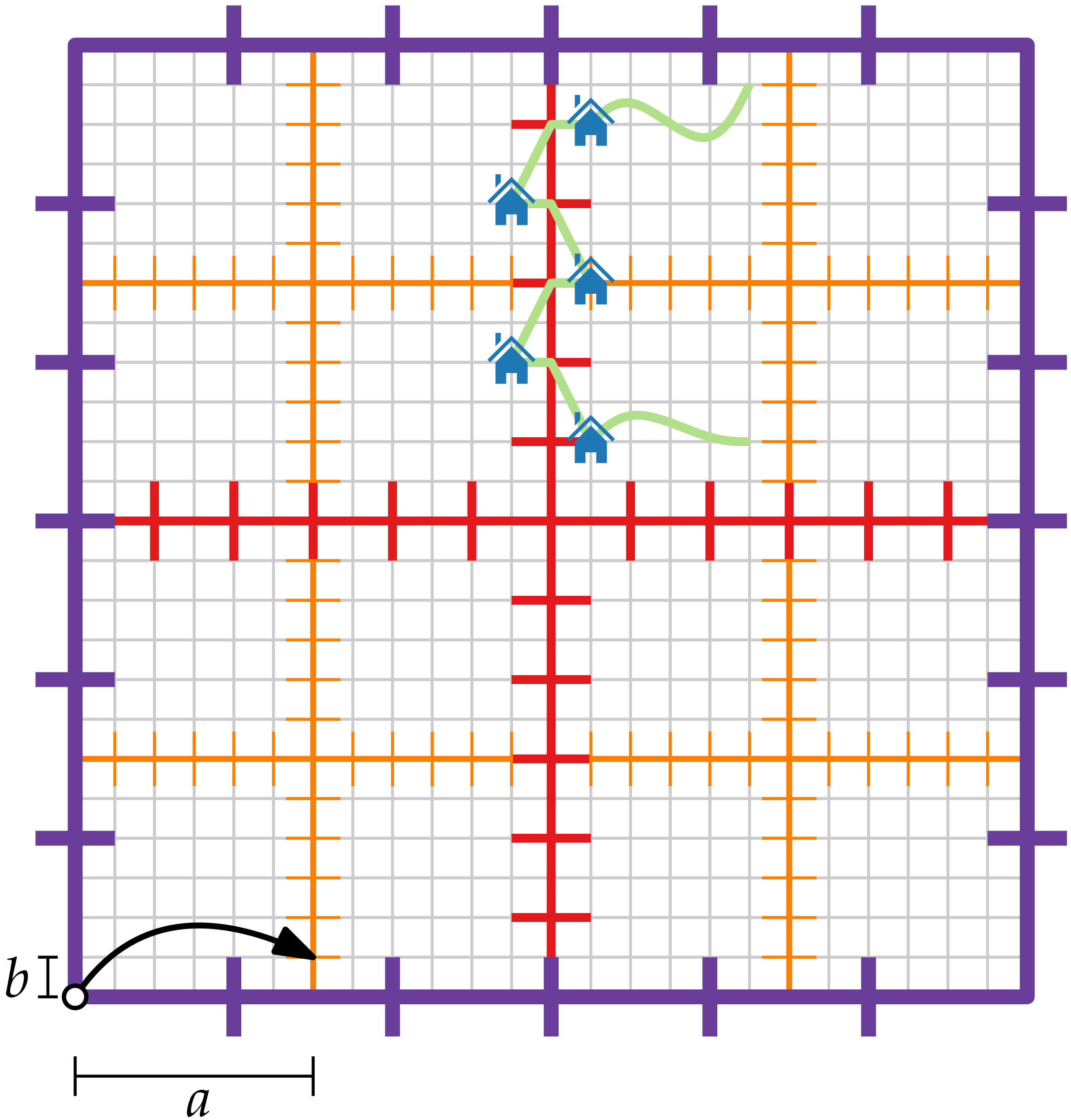

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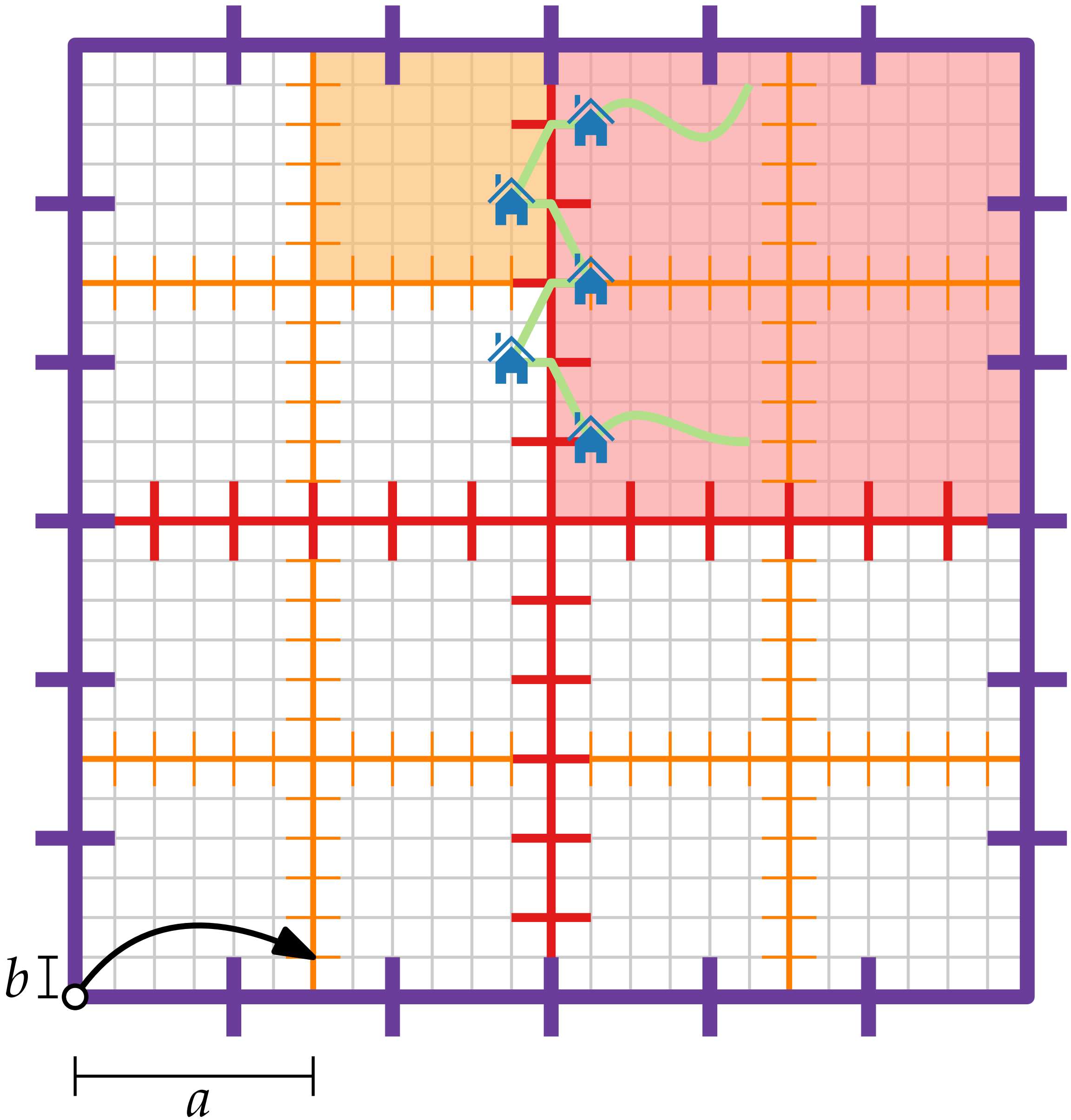

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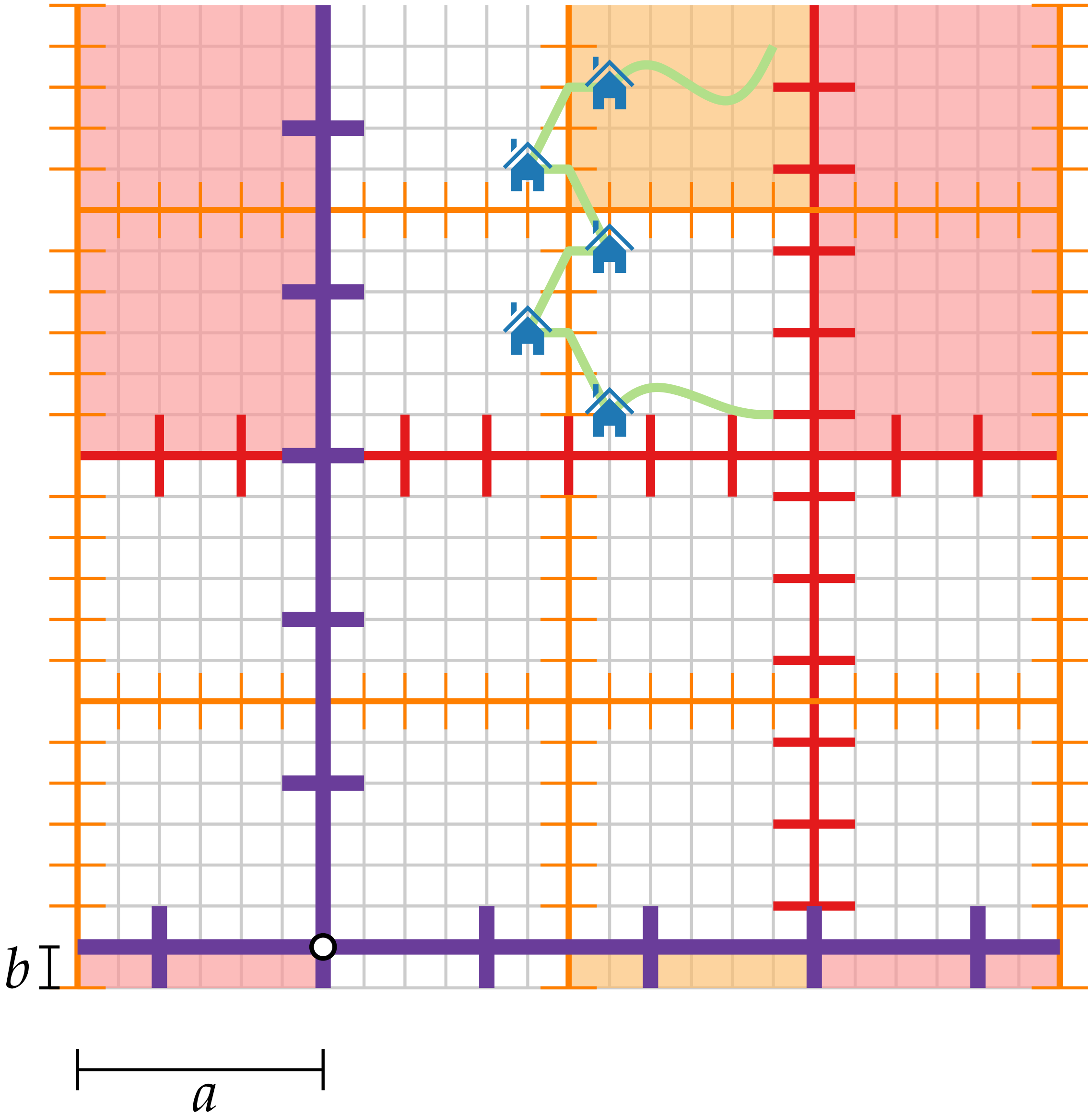

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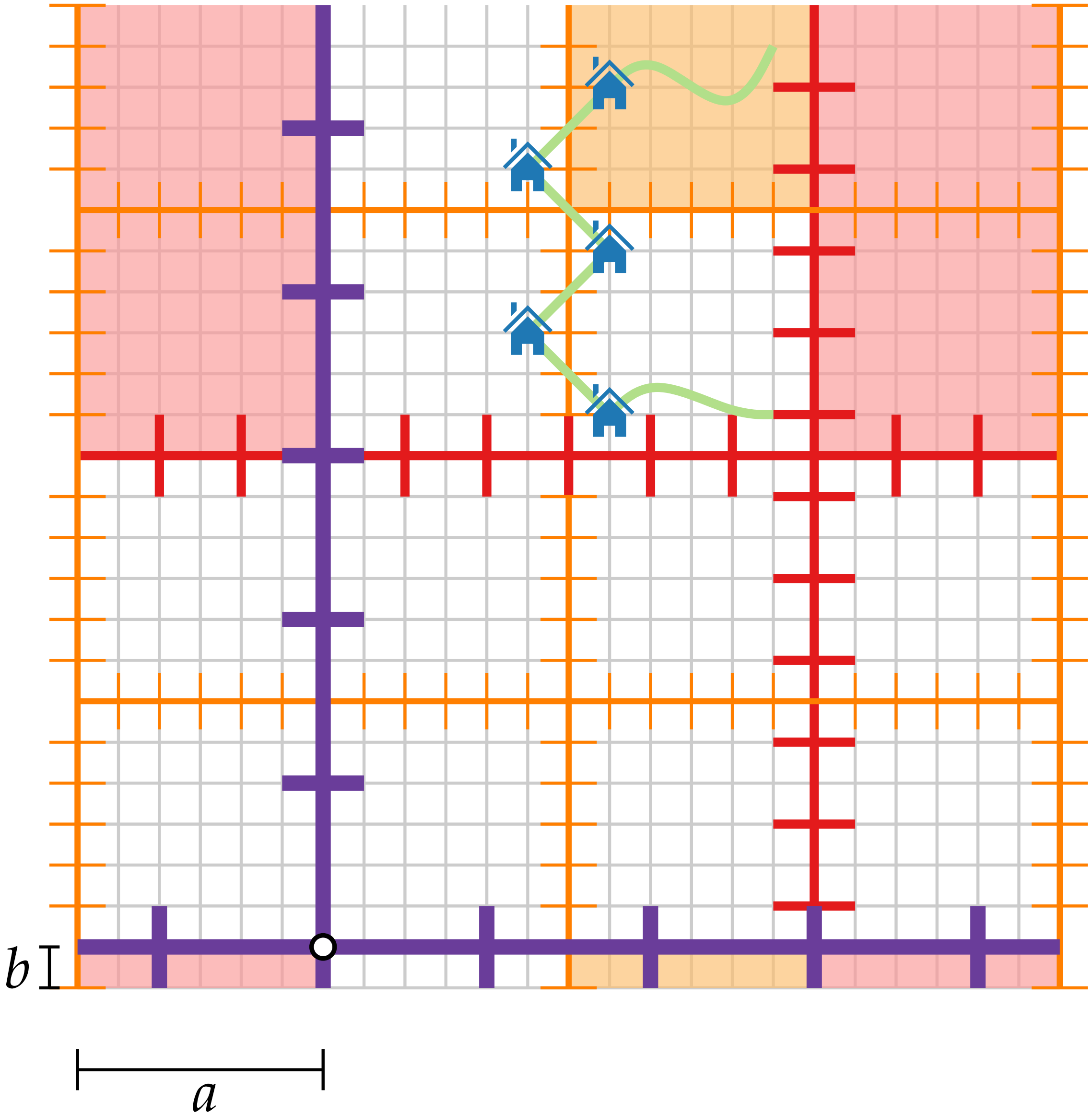

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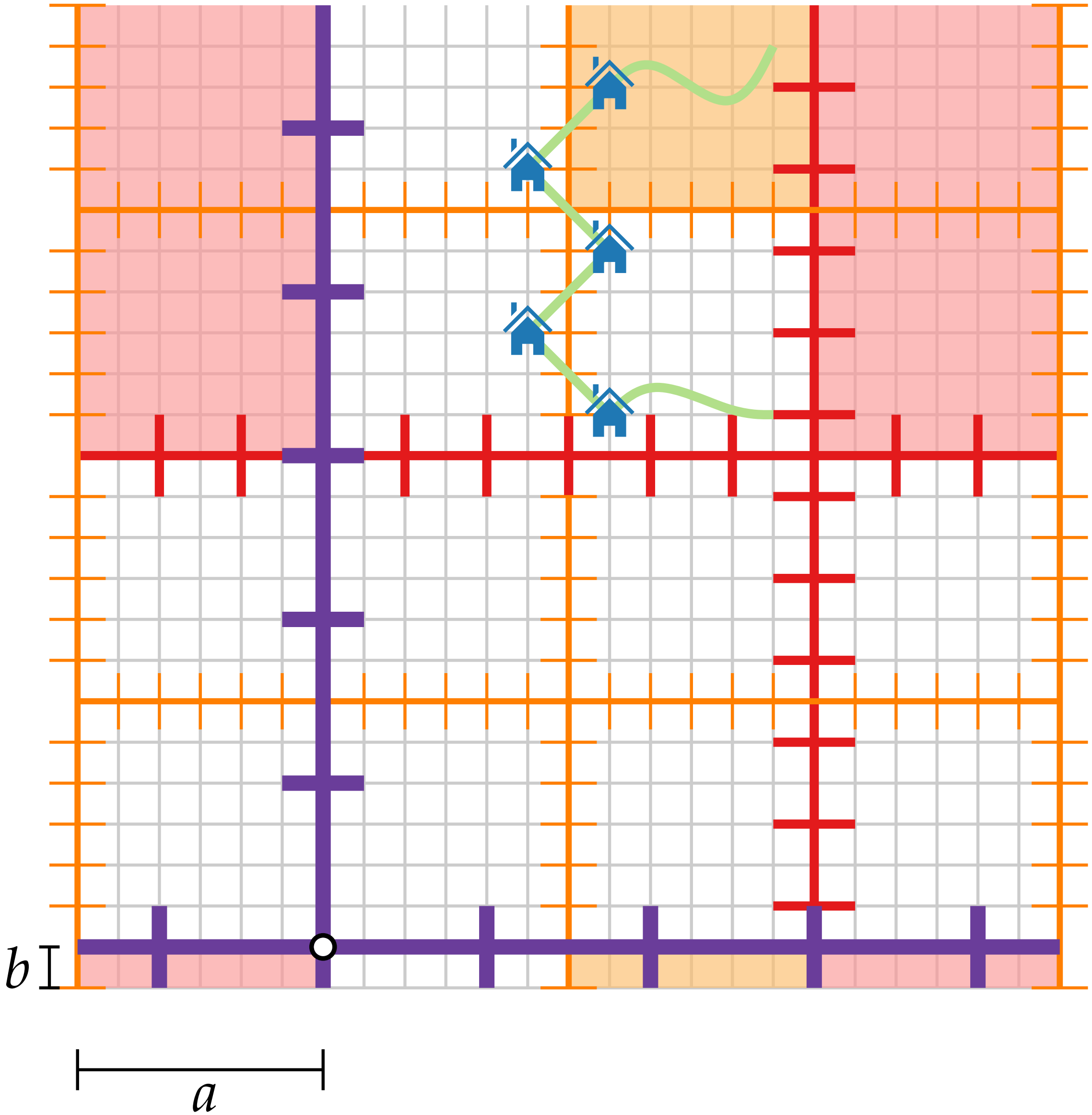

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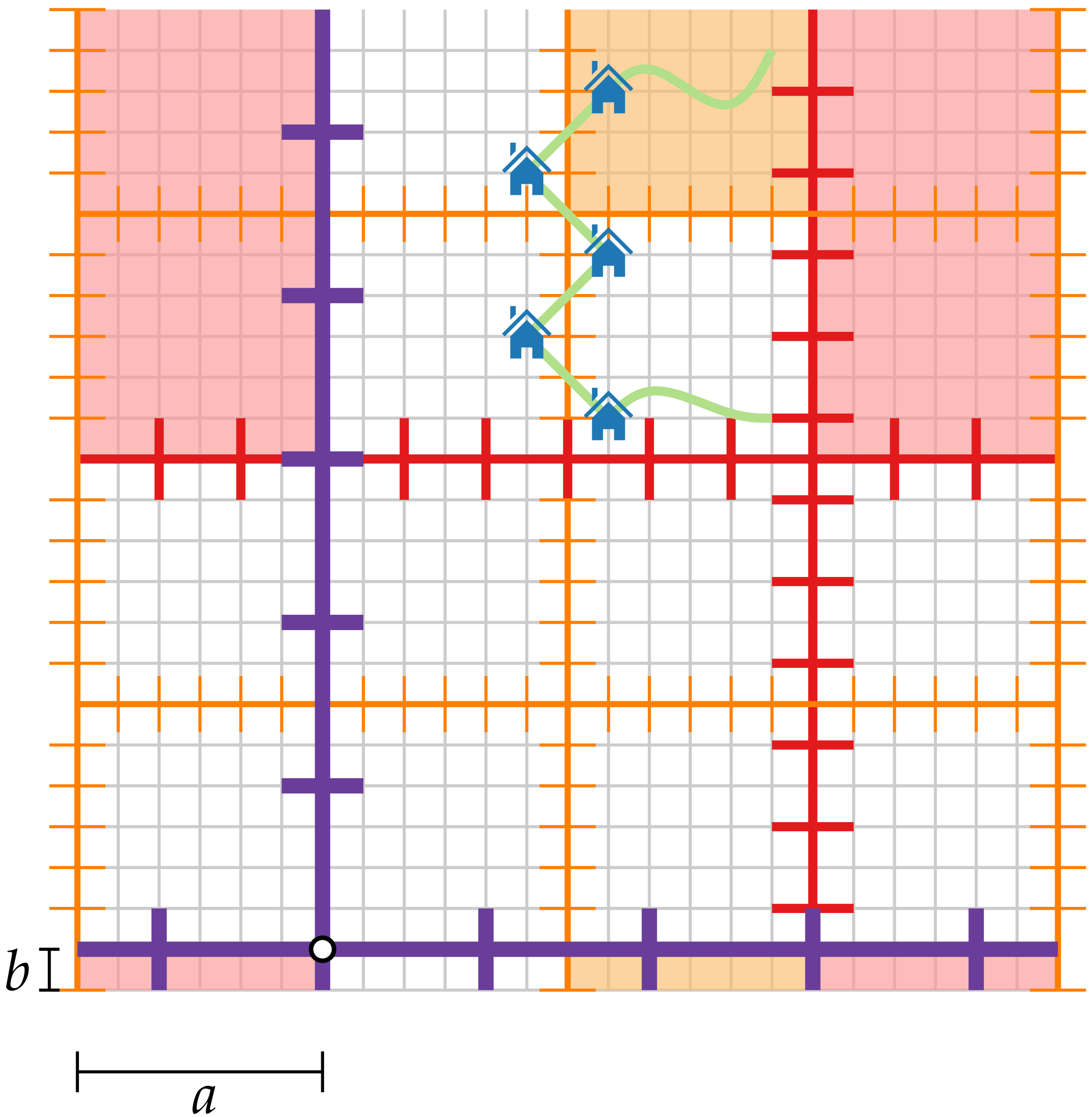

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- Squares in the dissection tree are "wrapped around".
- Dynamic program must be modified accordingly.

Shifted Dissections (II)

Lemma. Let π be an optimal tour, and let $N(\pi)$ be the number of crossings of π with the lines of the $(L \times L)$-grid.

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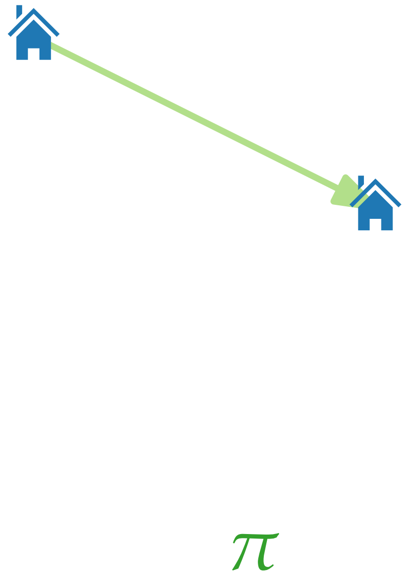

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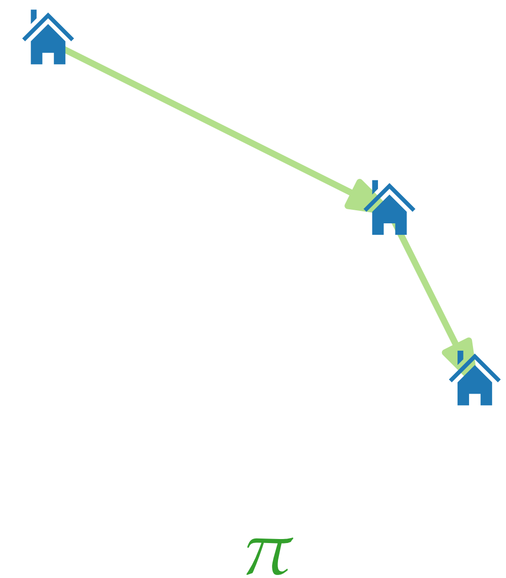

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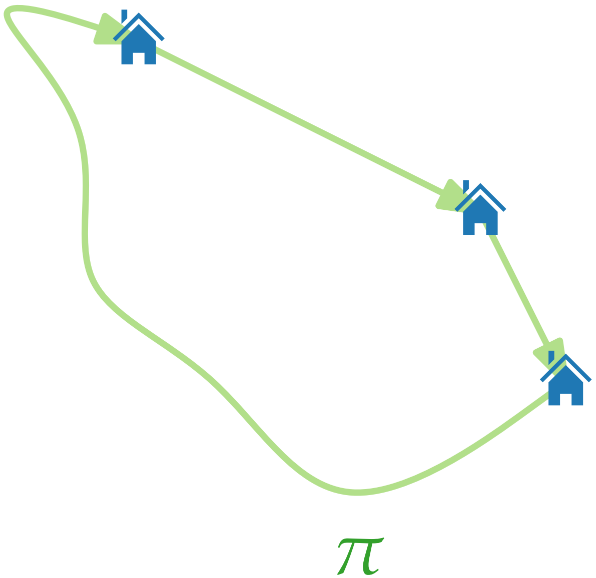

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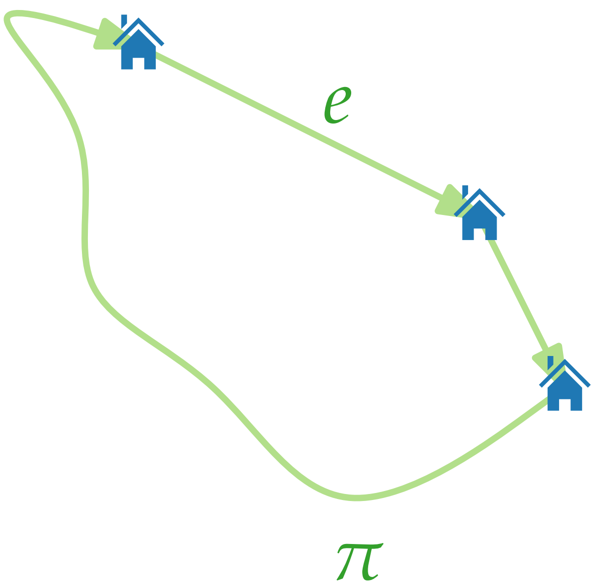

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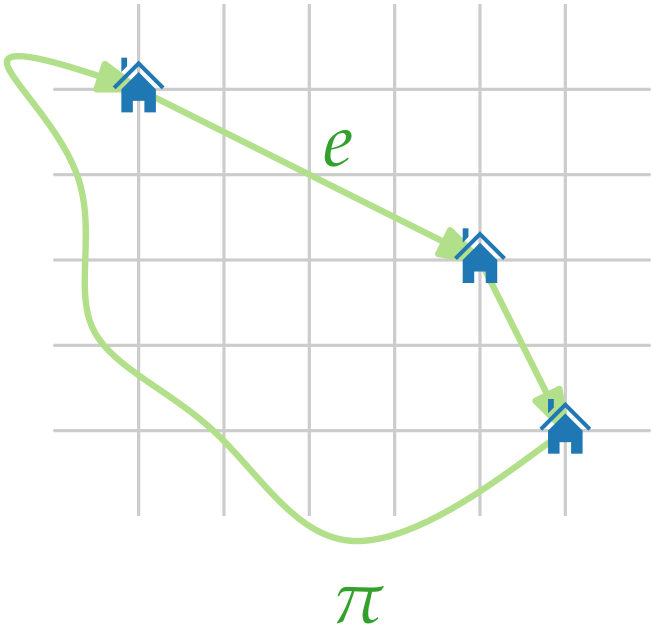

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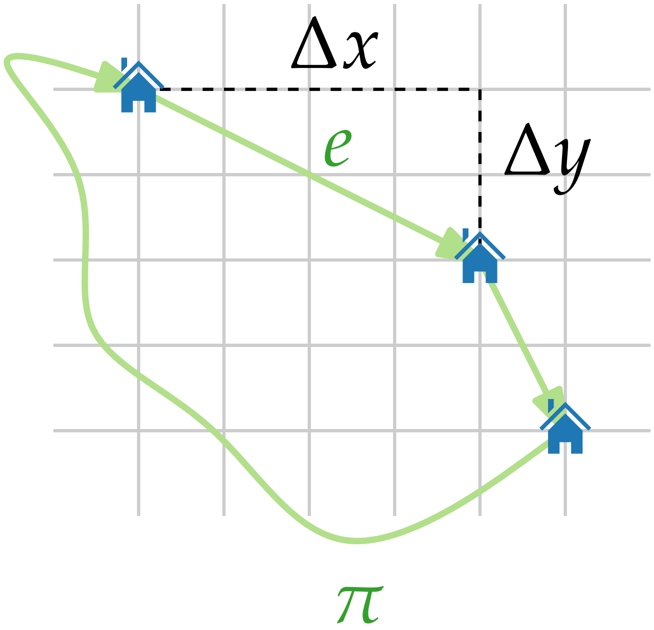

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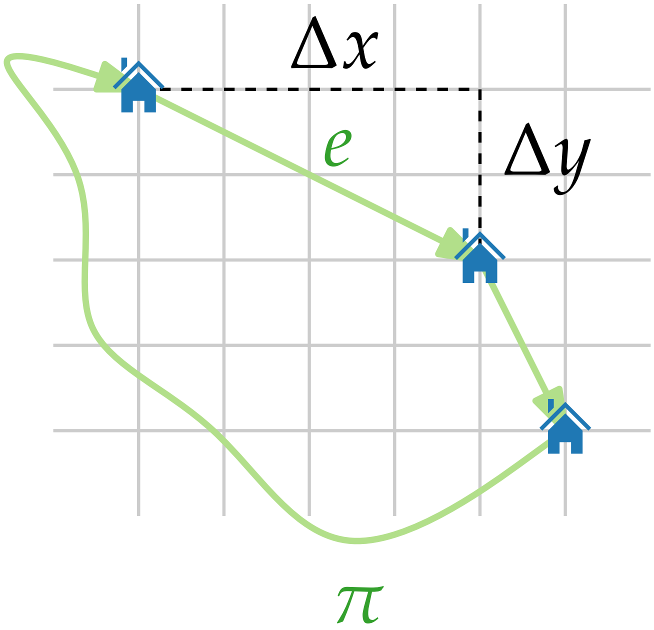

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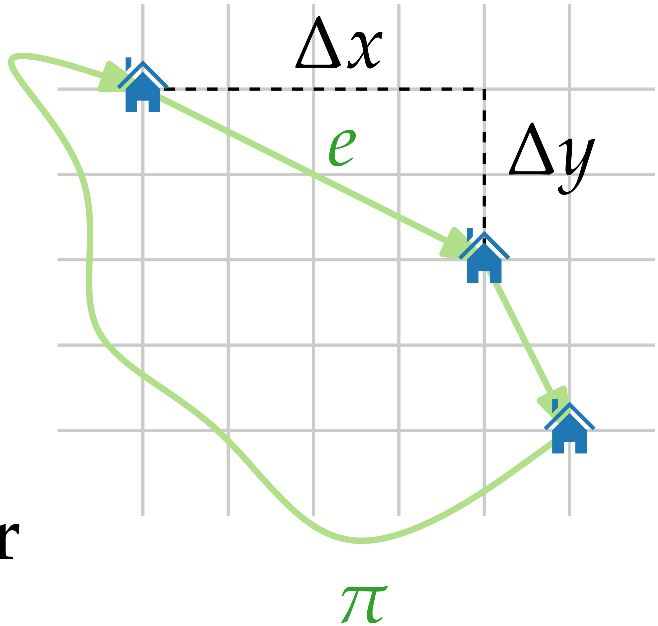

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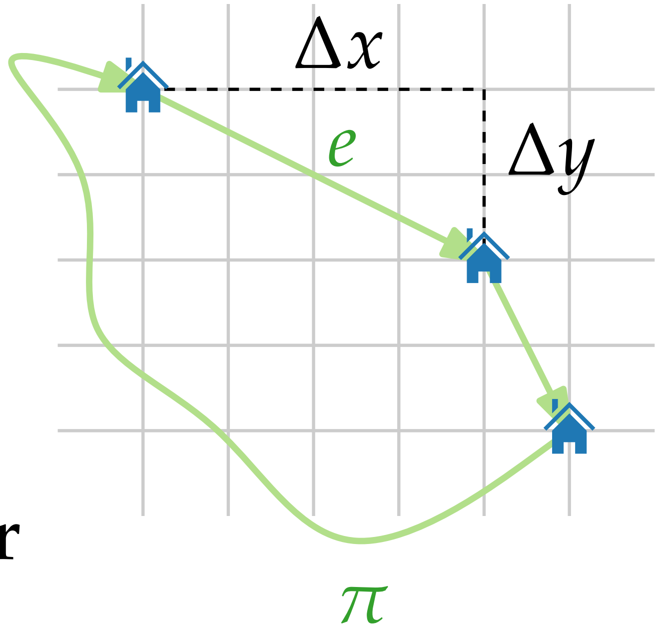

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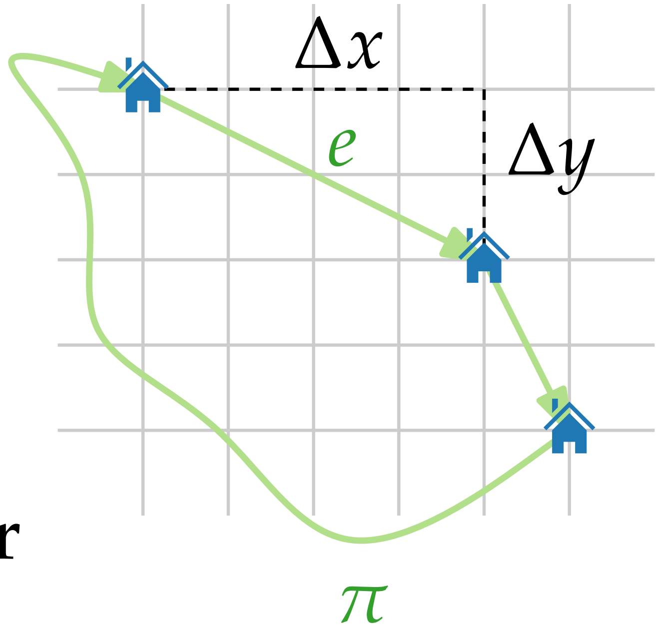
- $N_e{}^2 \leq (\Delta x + \Delta y)^2 \leq$



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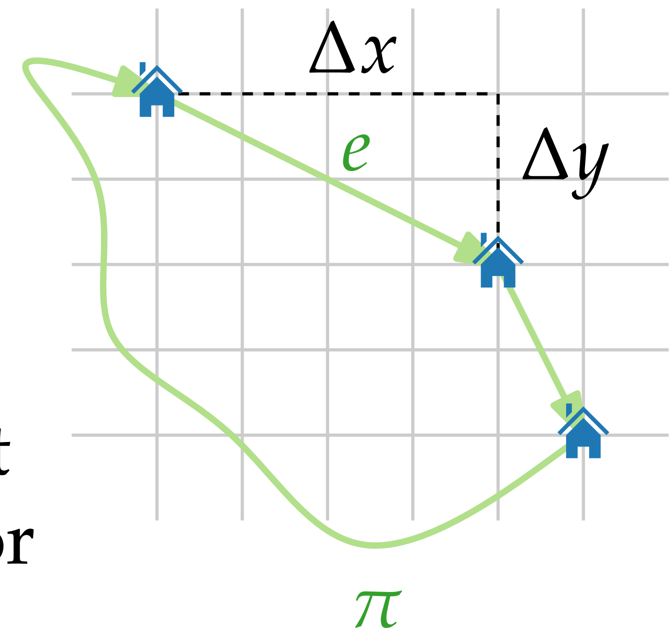

$0 \leq (\Delta x - \Delta y)^2$

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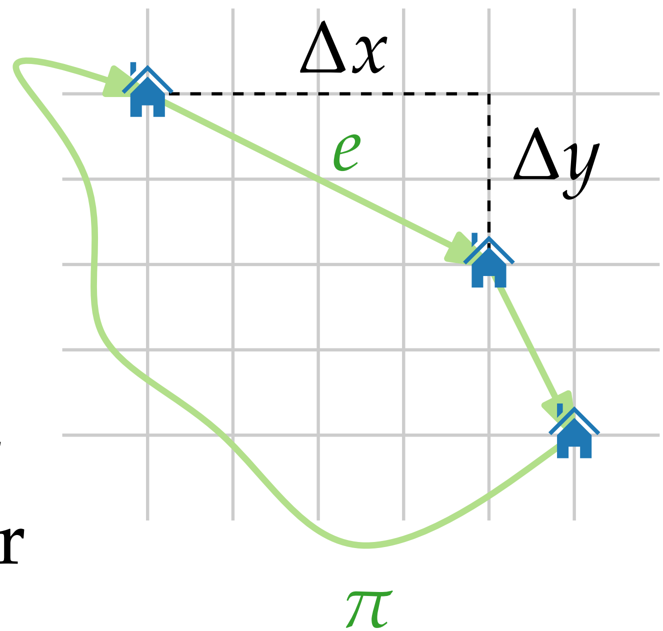

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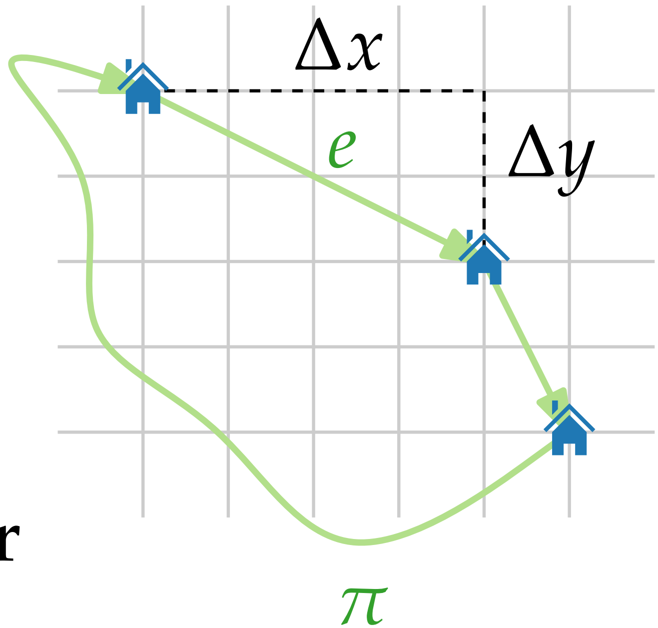

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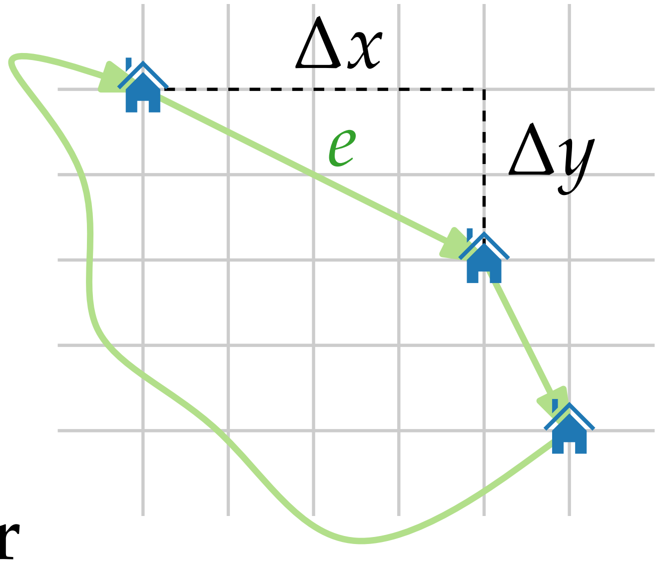

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Lemma. Let π be an optimal tour, and let $N(\pi)$ be the number of crossings of π with the lines of the $(L \times L)$-grid. Then we have $N(\pi) \leq \sqrt{2} \cdot \mathrm{OPT}$.

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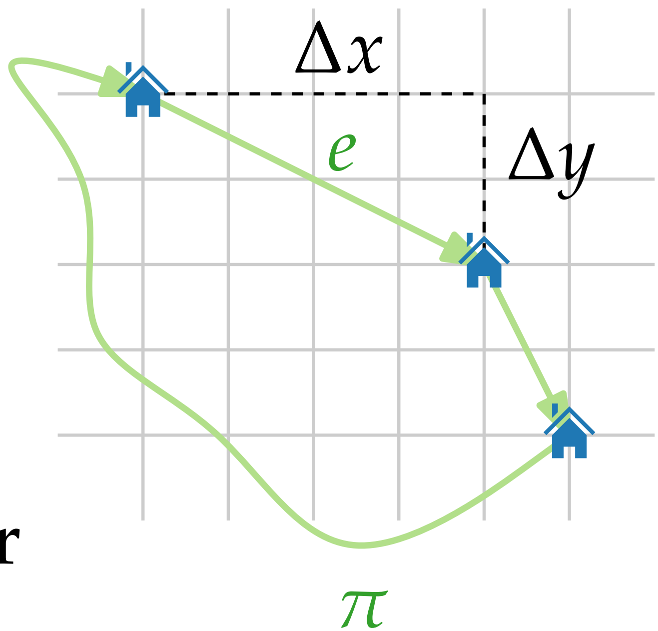

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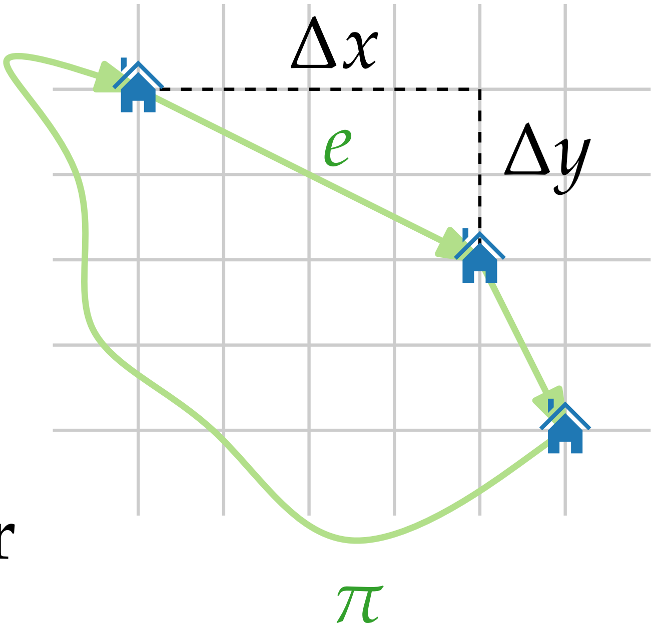

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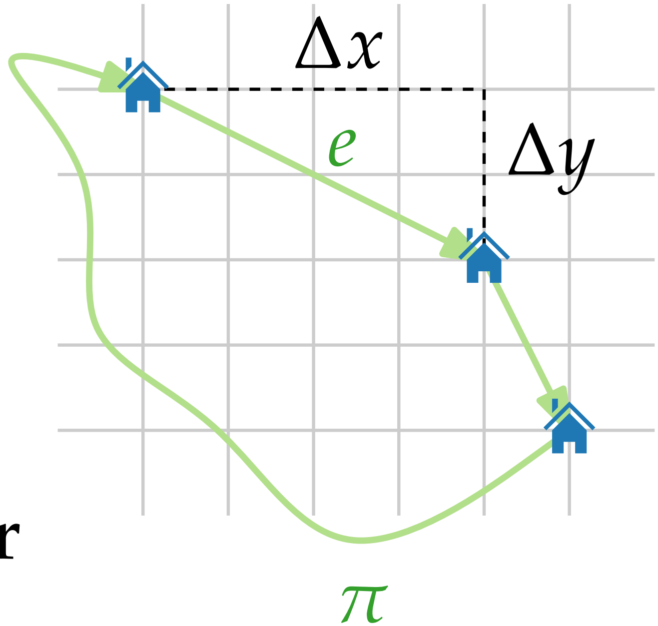

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Approximation Algorithms

Lecture 9:
A PTAS for Euclidean TSP

Part VI:
Approximation Factor

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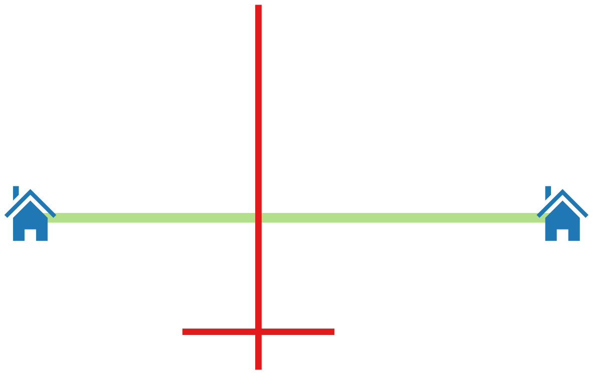

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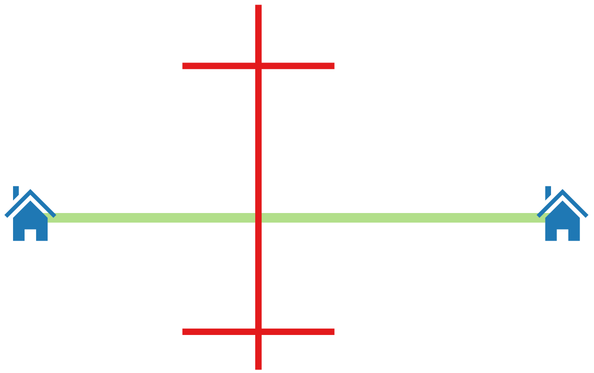

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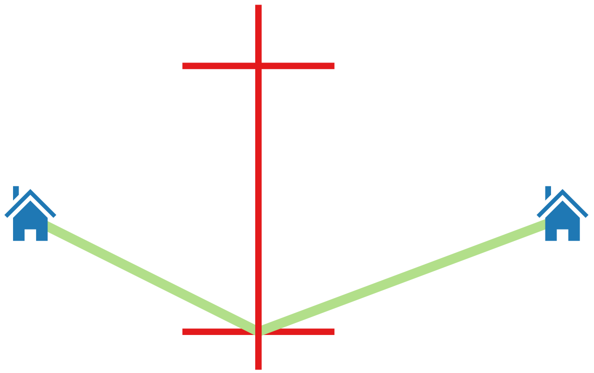

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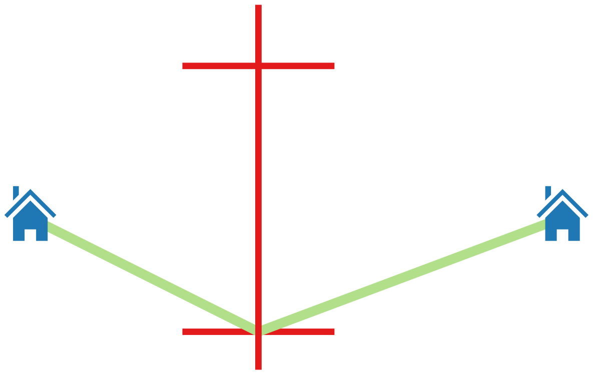

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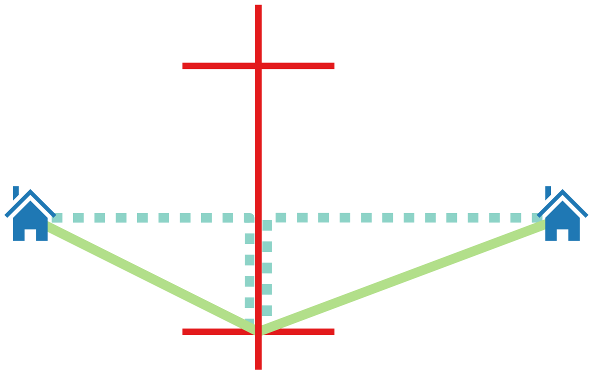

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- Summing over all $N(\pi) \leq \sqrt{2} \cdot \mathrm{OPT}$ intersection points and applying linearity of expectation yields the claim.

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Theorem. Let $a, b \in [0, L-1]$ be chosen independently and uniformly at random. Then the expected cost of an optimal well-behaved tour with respect to the (a, b)-shifted dissection is $\leq (1 + 2\sqrt{2}\varepsilon)\mathrm{OPT}$.

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Literature

- William J. Cook: Pursuit of the Traveling Salesman: Mathematics at the Limits of Computation.
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Randomized, $O\left(n(\log n)^{O(1/\varepsilon)}\right)$ time.

Literature (cont'd)

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