

Approximation Algorithms

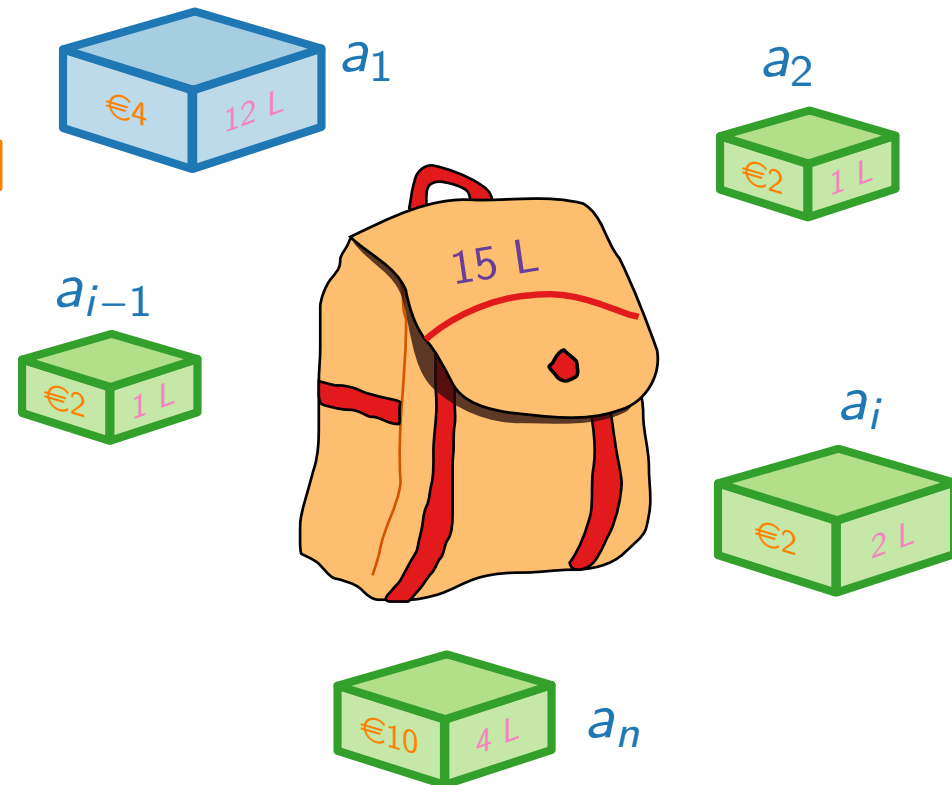
Lecture 8: Approximation Schemes and the KNAPSACK Problem

Part I: KNAPSACK

KNAPSACK

- Given:**
- A set $S = \{a_1, \dots, a_n\}$ of **objects**.
 - For every object a_i a **size** $\text{size}(a_i) \in \mathbb{N}^+$
 - For every object a_i a **profit** $\text{profit}(a_i) \in \mathbb{N}^+$
 - A knapsack **capacity** $B \in \mathbb{N}^+$

Task: Find a subset of objects whose **total size** is at most B and whose **total profit** is maximum.



NP-hard

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Part II:

Pseudo-Polynomial Algorithms and
Strong NP-Hardness

Pseudo-Polynomial Algorithms

Let Π be an optimization problem whose instances can be represented by **objects** (such as sets, elements, edges, nodes) and **numbers** (such as costs, weights, profits).

$|I|$: The size of an instance $I \in D_\Pi$, where all numbers in I are encoded in **binary**.
 (5 $\hat{=}$ 101_b \Rightarrow $|I| = 3$)

$|I|_u$: The size of an instance $I \in D_\Pi$, where all numbers in I are encoded in **unary**.
 (5 $\hat{=}$ 11111_u \Rightarrow $|I|_u = 5$)

The running time of a polynomial algorithm for Π is polynomial in $|I|$.

The running time of a **pseudo-polynomial algorithm** is polynomial in $|I|_u$.

The running time of a pseudo-polynomial algorithm may not be polynomial in $|I|$.

Strong NP-Hardness

An optimization problem is called **strongly NP-hard** if it remains NP-hard under unary encoding.

An optimization problem is called **weakly NP-hard** if it is NP-hard under binary encoding but has a pseudo-polynomial algorithm.

Theorem. A strongly NP-hard problem has no pseudo-polynomial algorithm unless $P = NP$.

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Part III:

Pseudo-Polynomial Algorithm for **KNAPSACK**

Pseudo-Polynomial Alg. for KNAPSACK

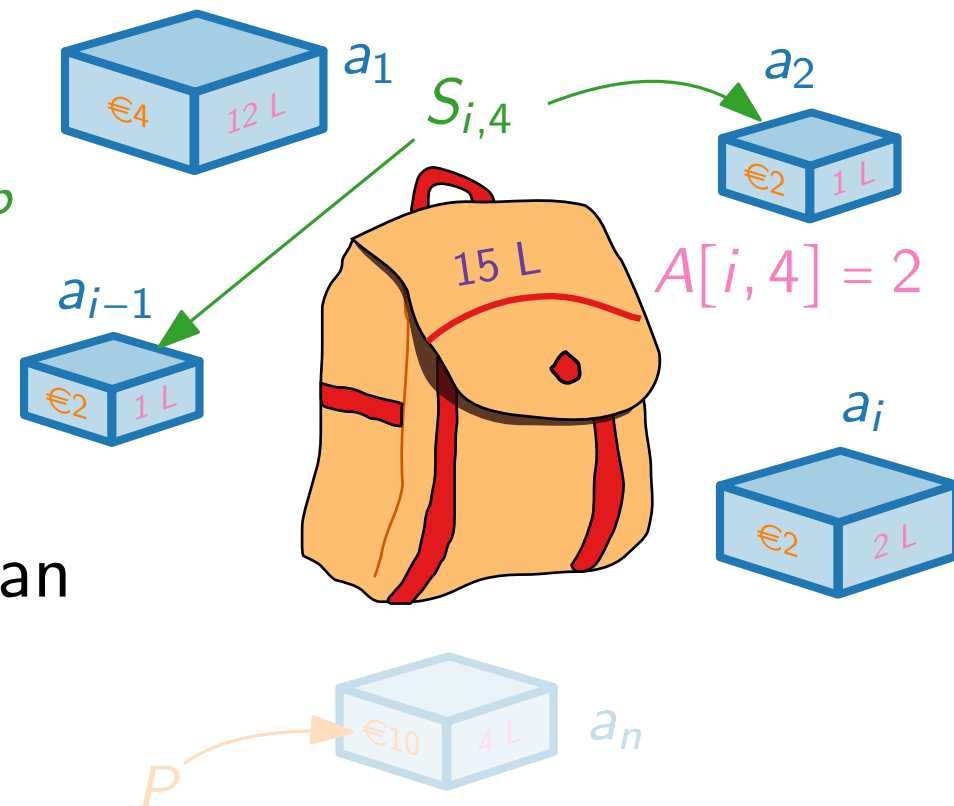
Let $P := \max_i \text{profit}(a_i) \Rightarrow P \leq \text{OPT} \leq nP$

For every $i = 1, \dots, n$ and every $p \in \{1, \dots, nP\}$, let $S_{i,p}$ be a subset of $\{a_1, \dots, a_i\}$ whose total profit is precisely p and whose total size is minimum among all subsets with these properties. Such a set may not exist.

Let $A[i, p]$ be the total size of $S_{i,p}$ (set $A[i, p] = \infty$ if no such set exists).

If all $A[i, p]$ are known, then we can compute

$\text{OPT} = \max\{p \mid A[n, p] \leq B\}$.



Pseudo-Polynomial Alg. for KNAPSACK

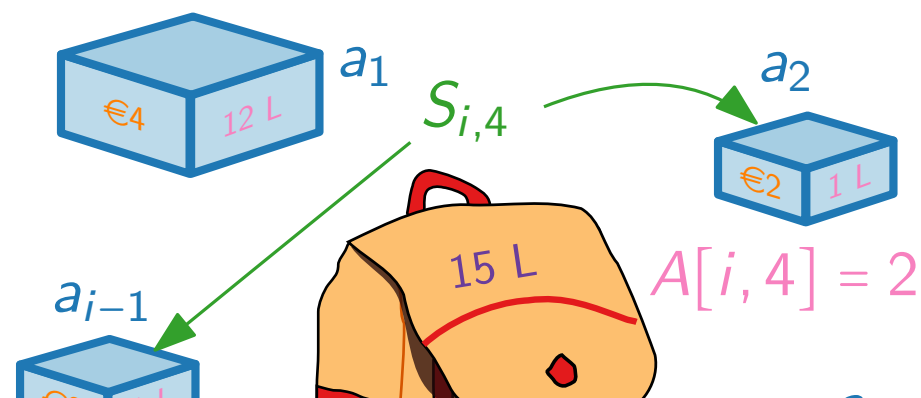
$A[1, p]$ can be computed for all $p \in \{0, \dots, nP\}$.

Set $A[i, p] := \infty$ for $p < 0$ (for convenience).

$A[i+1, p] = \min\{A[i, p], \text{size}(a_{i+1}) + A[i, p - \text{profit}(a_{i+1})]\}$

\Rightarrow All values $A[i, p]$ can be computed in total time $O(n^2 P)$.

\Rightarrow OPT can be computed in $O(n^2 P)$ total time.



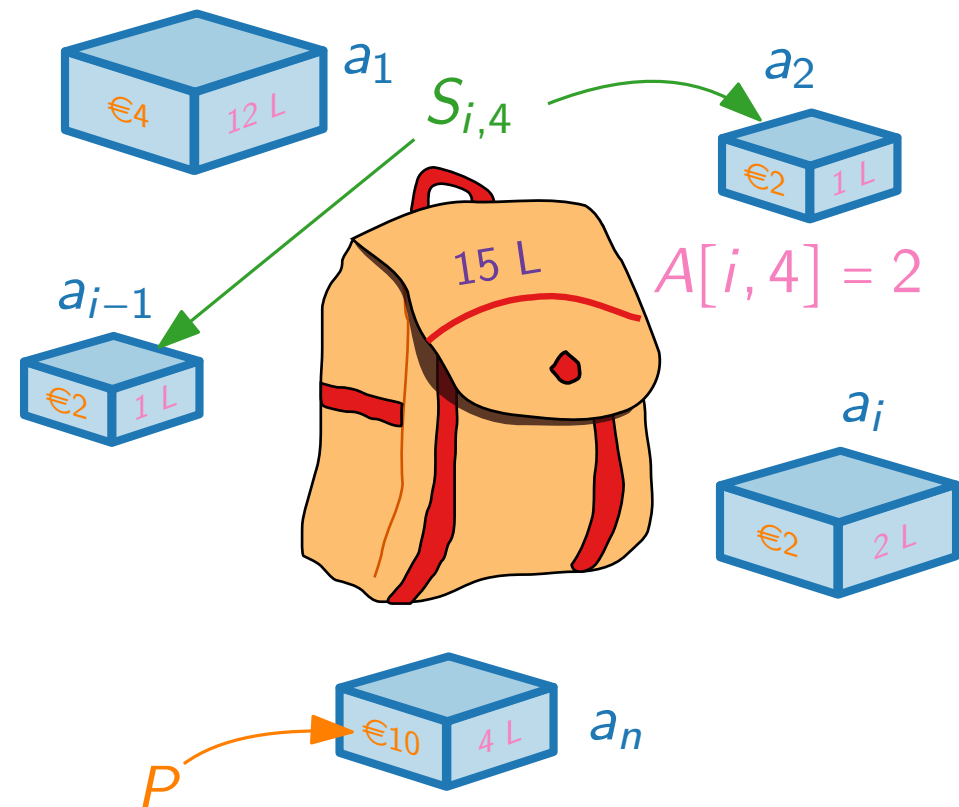
Theorem. KNAPSACK can be solved optimally in pseudo-polynomial time $O(n^2 P)$.

Corollary. KNAPSACK is weakly NP-hard.

Pseudo-Polynomial Alg. for KNAPSACK

Theorem. KNAPSACK can be solved optimally in pseudo-polynomial time $O(n^2 P)$.

Observe. The running time $O(n^2 P)$ is polynomial in n if P is polynomial in n .



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Part IV: Approximation Schemes

Approximation Schemes

Let Π be an optimization problem. An algorithm \mathcal{A} is called a **polynomial-time approximation scheme (PTAS)** for Π if it outputs, for every input (I, ε) with $I \in D_\Pi$ and $\varepsilon > 0$, a solution $s \in S_\Pi(I)$ such that

- $\text{obj}_\Pi(I, s) \leq (1 + \varepsilon) \cdot \text{OPT}$ if Π is a minimization problem,
- $\text{obj}_\Pi(I, s) \geq (1 - \varepsilon) \cdot \text{OPT}$ if Π is a maximization problem,

and the runtime of \mathcal{A} is polynomial in $|I|$ for **every fixed** $\varepsilon > 0$.

\mathcal{A} is called **fully polynomial-time approximation scheme (FPTAS)** if its running time is polynomial in $|I|$ and $1/\varepsilon$.

Example running times

- $O(n^{1/\varepsilon}) \rightsquigarrow \text{PTAS}$
- $O(n^3/\varepsilon^2) \rightsquigarrow \text{FPTAS}$
- $O(2^{1/\varepsilon} n^4) \rightsquigarrow \text{PTAS}$

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Part V: FPTAS for KNAPSACK

An FPTAS for KNAPSACK via Scaling

KnapsackScaling (I, ϵ)

$K = \epsilon P / n$ // scaling factor

$\text{profit}'(a_i) = \lfloor \text{profit}(a_i) / K \rfloor$

Compute optimal solution S' for I w.r.t. $\text{profit}'(\cdot)$.

return S'

Lemma. $\text{profit}(S') \geq (1 - \epsilon) \cdot \text{OPT}$.

Proof. Let $\text{OPT} = \{o_1, \dots, o_\ell\}$.

Obs. 1. For $i = 1, \dots, \ell$, $\text{profit}(o_i) - K \leq K \cdot \text{profit}'(o_i) \leq \text{profit}(o_i)$
 $\Rightarrow K \cdot \sum_i \text{profit}'(o_i) \geq \text{OPT} - \ell K \geq \text{OPT} - nK = \text{OPT} - \epsilon P$.

Obs. 2. $\text{profit}(S') \geq K \cdot \text{profit}'(S') \geq K \cdot \sum_i \text{profit}'(o_i)$
 $\Rightarrow \text{profit}(S') \geq \text{OPT} - \epsilon P \geq \text{OPT} - \epsilon \text{OPT} = (1 - \epsilon) \cdot \text{OPT} \quad \square$

Theorem. KnapsackScaling is an FPTAS for KNAPSACK with running time $O(n^3 / \epsilon) = O\left(n^2 \cdot \frac{P}{\epsilon P / n}\right)$.

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Part VI:

Connections Between the Concepts

FPTAS and Pseudo-Poly. Algorithms

Theorem. Let p be a polynomial and let Π be an NP-hard minimization problem with integral objective function and $\text{OPT}(I) < p(|I|_u)$ for all instances I of Π . If Π has an FPTAS, then there is a pseudo-polynomial algorithm for Π .

Proof.

Assuming there is an FPTAS for Π (in $q(|I|, 1/\varepsilon)$ time).

Set $\varepsilon = 1/p(|I|_u)$.

$\Rightarrow \text{ALG} \leq (1 + \varepsilon)\text{OPT} < \text{OPT} + \varepsilon p(|I|_u) = \text{OPT} + 1.$

$\Rightarrow \text{ALG} = \text{OPT}.$

Running time: $q(|I|, p(|I|_u))$, so $\text{poly}(|I|_u)$.

FPTAS and Strong NP-Hardness

Theorem. A strongly NP-hard problem has no pseudo-polynomial algorithm unless $P = NP$.

Theorem. Let p be a polynomial and let Π be an NP-hard minimization problem with integral objective function and $\text{OPT}(I) < p(\|I\|_u)$ for all instances I of Π . If Π has an FPTAS, then there is a pseudo-polynomial algorithm for Π .

Corollary. Let Π be an NP-hard optimization problem that fulfills the restrictions above.
If Π is strongly NP-hard, then there is no FPTAS for Π (unless $P = NP$).