Approximation Algorithms

Lecture 8: Approximation Schemes and the KNAPSACK Problem

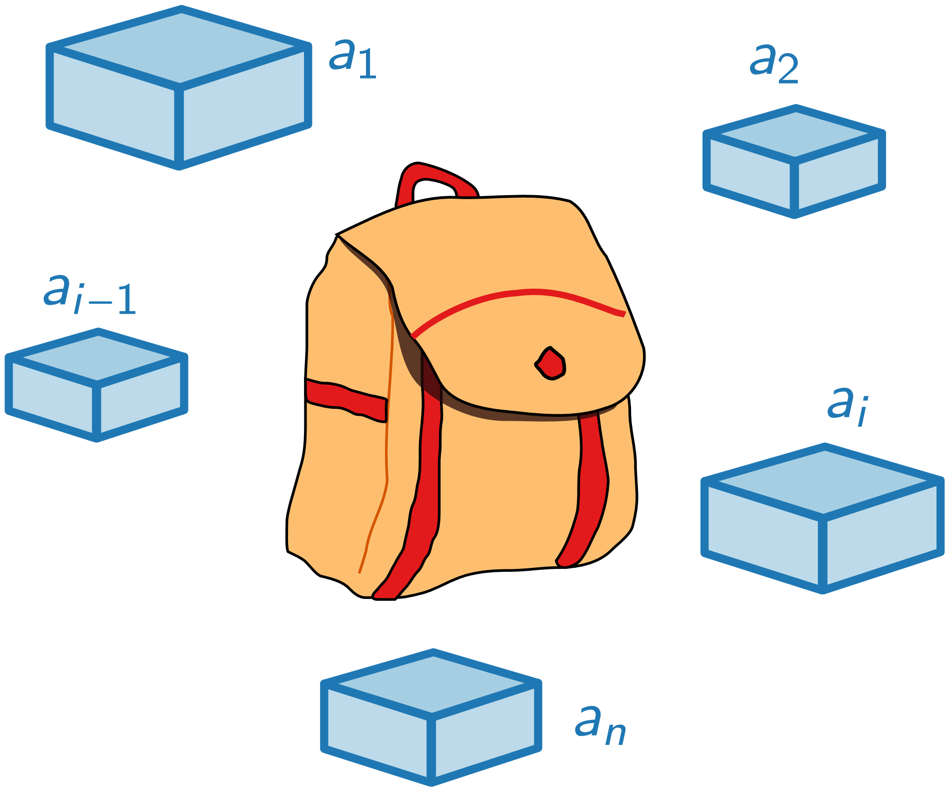
Part I: KNAPSACK

Alexander Wolff

Winter 2022/23

Knapsack

Given: ■ A set $S = \{a_1, \ldots, a_n\}$ of **objects**.



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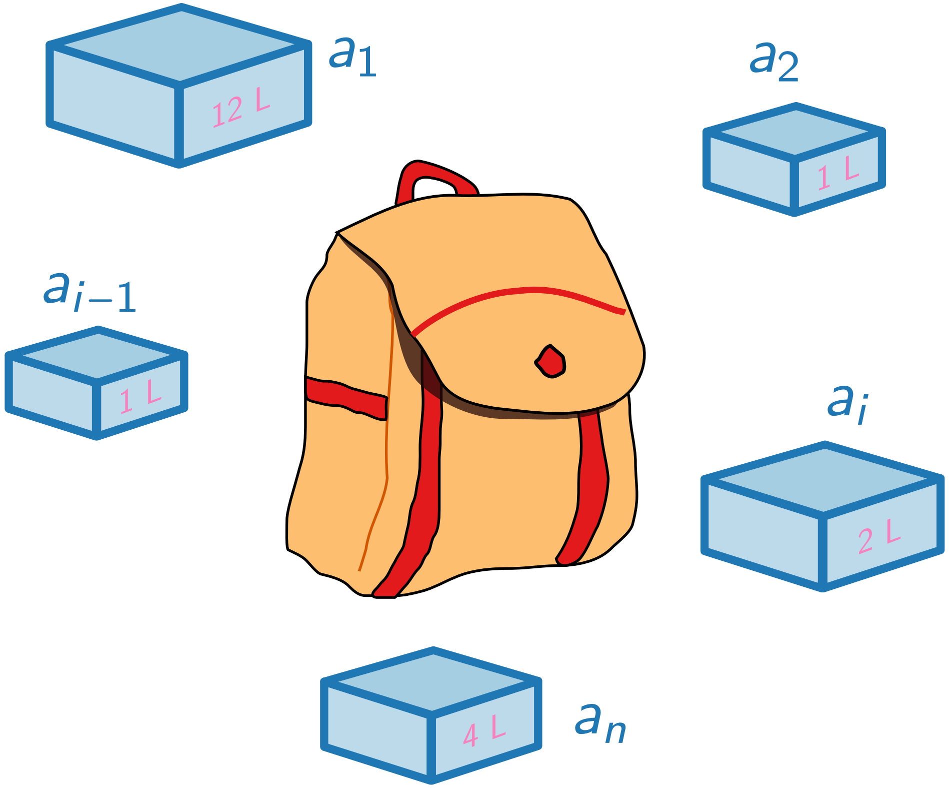

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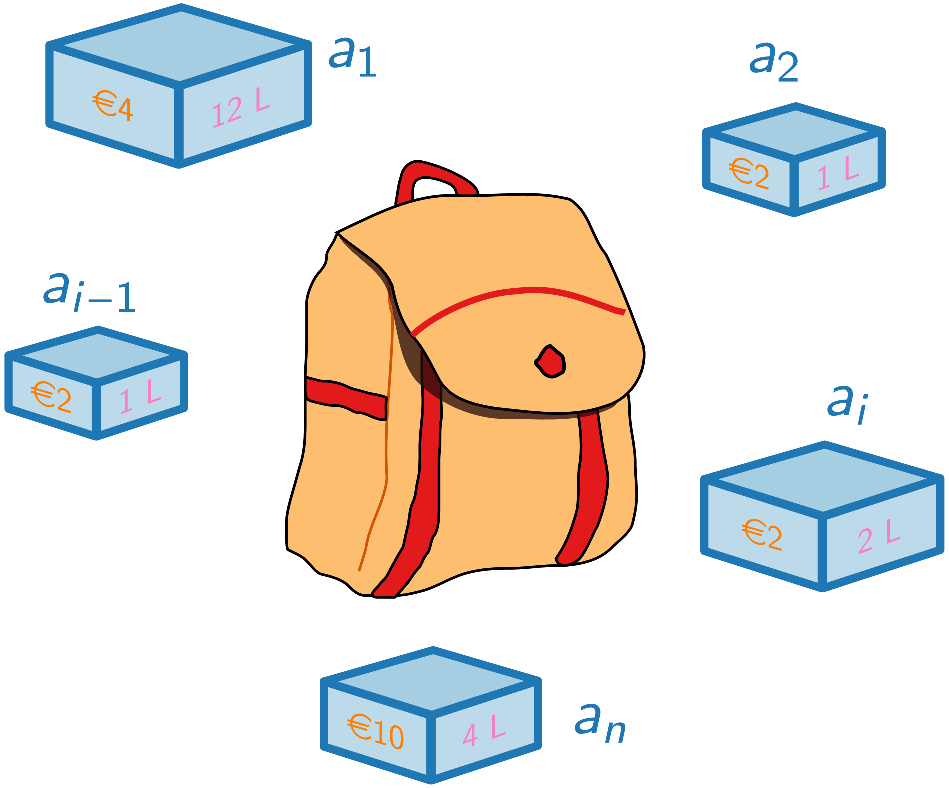

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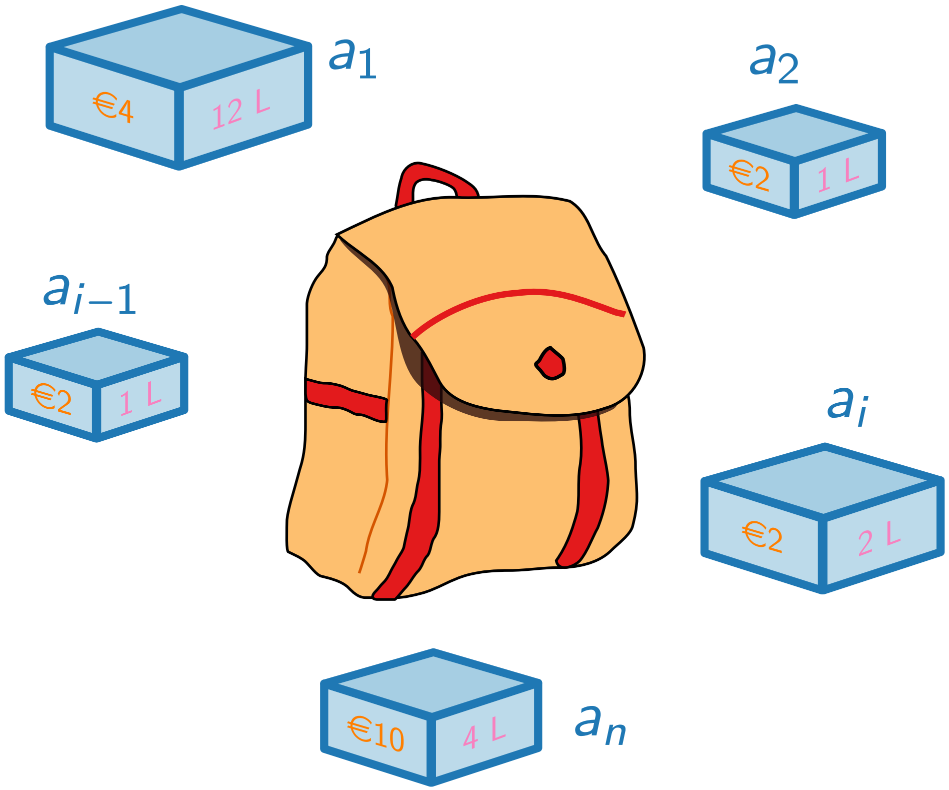

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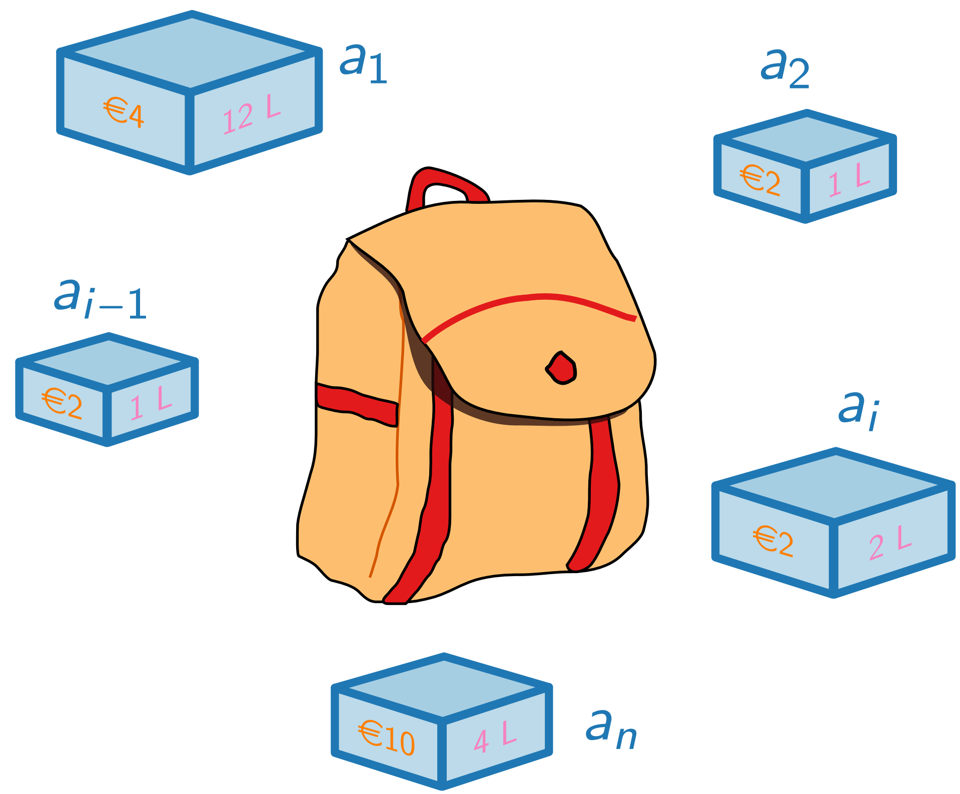

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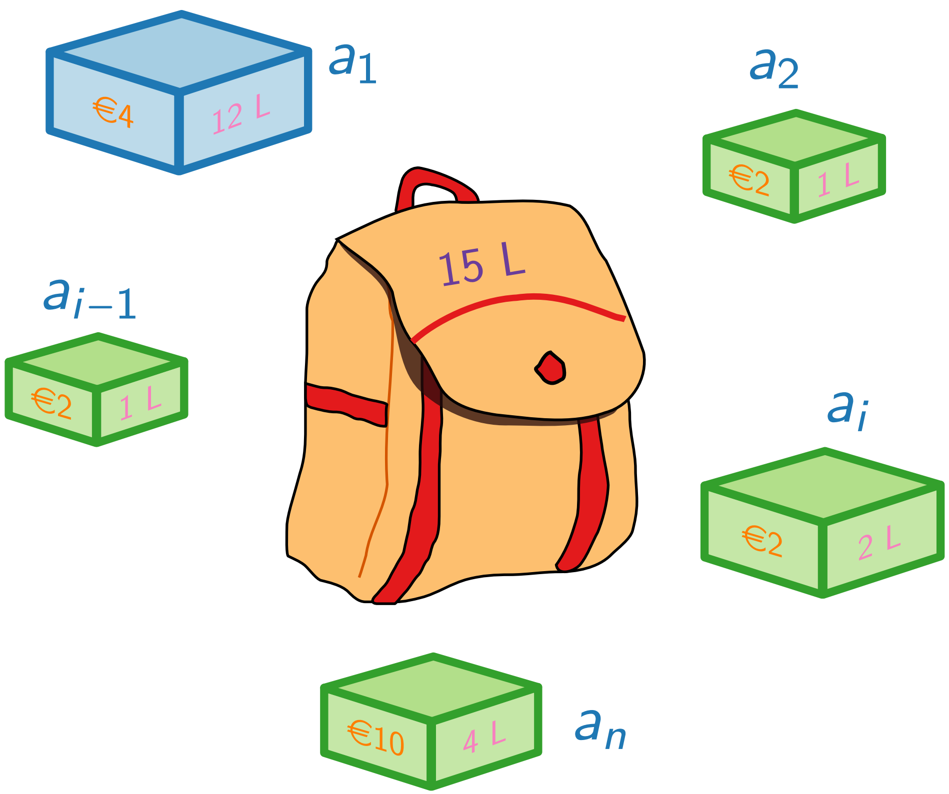

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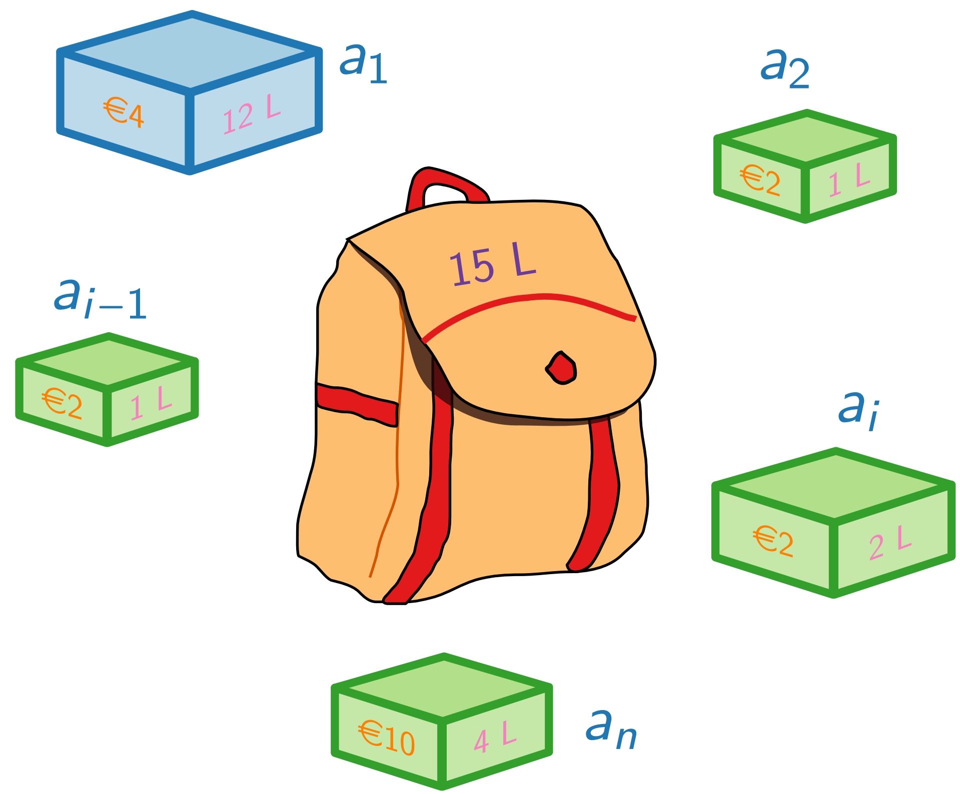

NP-hard

Approximation Algorithms

Lecture 8:
Approximation Schemes and the KNAPSACK Problem

Part II:
Pseudo-Polynomial Algorithms and Strong NP-Hardness

Pseudo-Polynomial Algorithms

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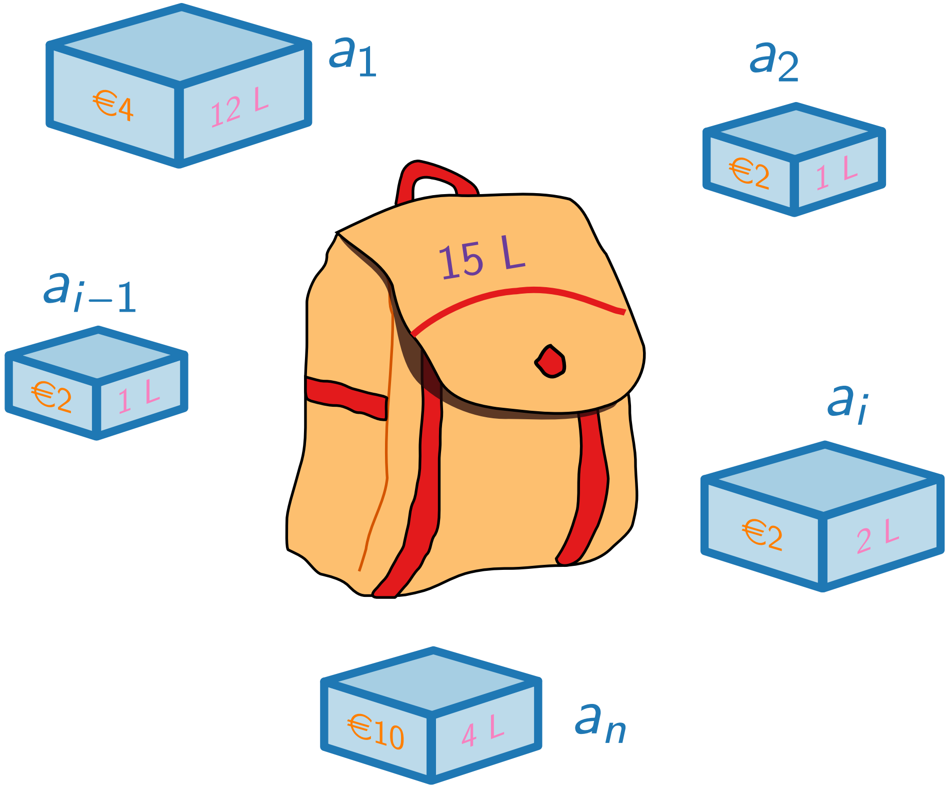

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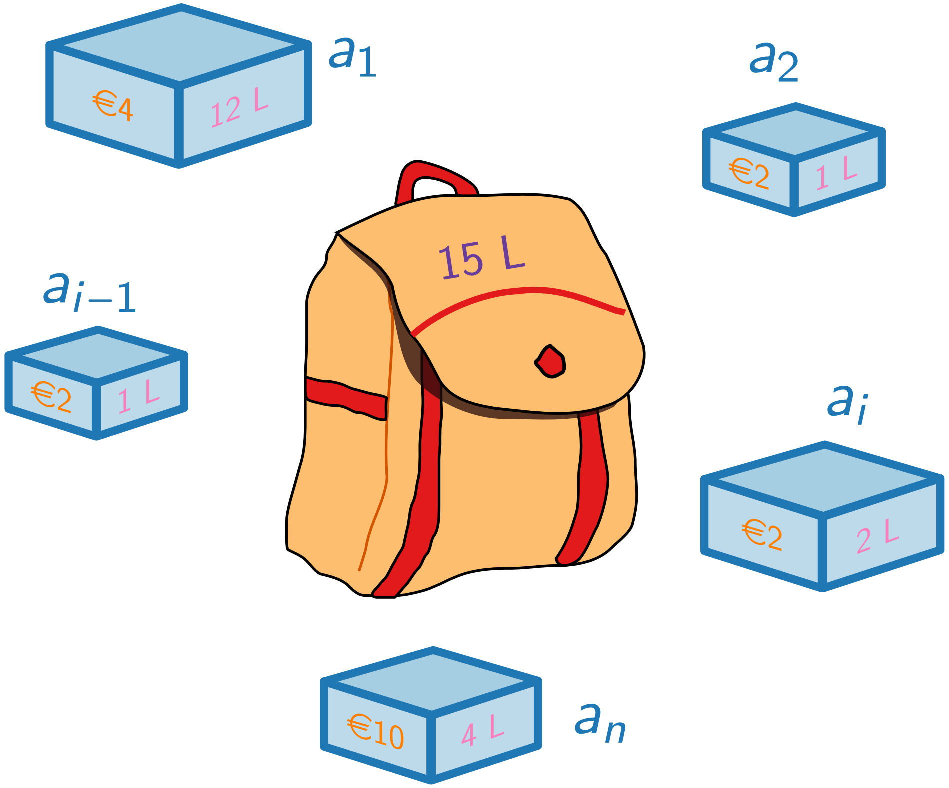

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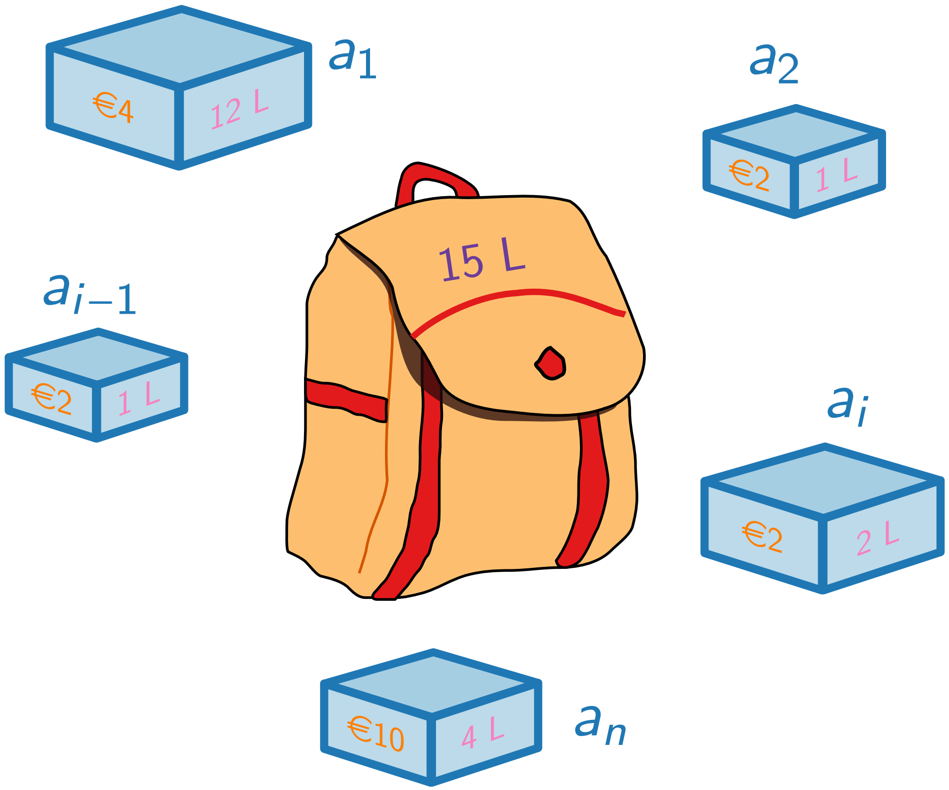

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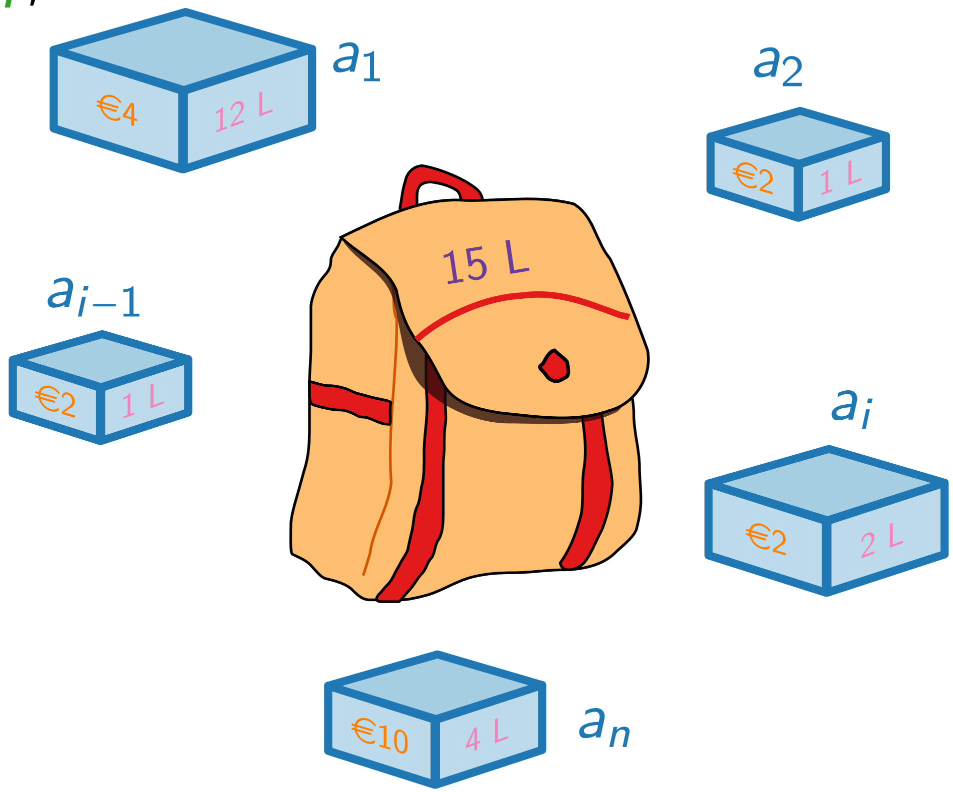

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The running time of a polynomial algorithm for Π is polynomial in $|I|$.

The running time of a **pseudo-polynomial algorithm** is polynomial in $|I|_u$.

The running time of a pseudo-polynomial algorithm may not be polynomial in $|I|$.

Strong NP-Hardness

An optimization problem is called **strongly NP-hard** if it remains NP-hard under unary encoding.

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Theorem. A strongly NP-hard problem has no pseudo-polynomial algorithm unless P = NP.

Approximation Algorithms

Lecture 8: Approximation Schemes and the Knapsack Problem

Part III: Pseudo-Polynomial Algorithm for Knapsack

Pseudo-Polynomial Alg. for KNAPSACK

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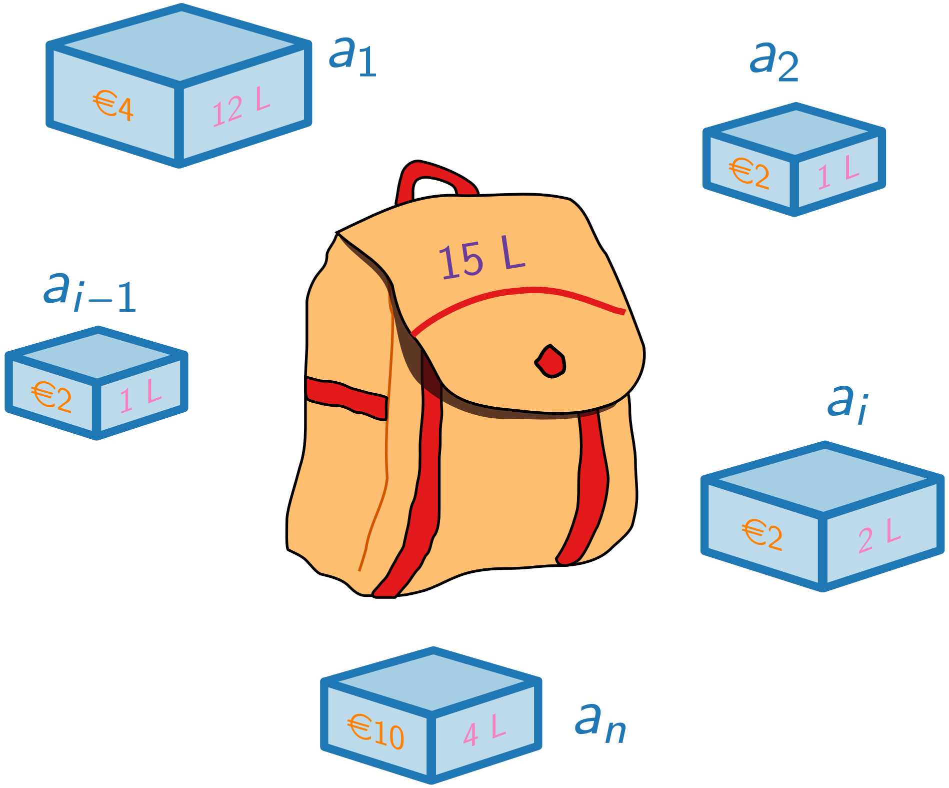

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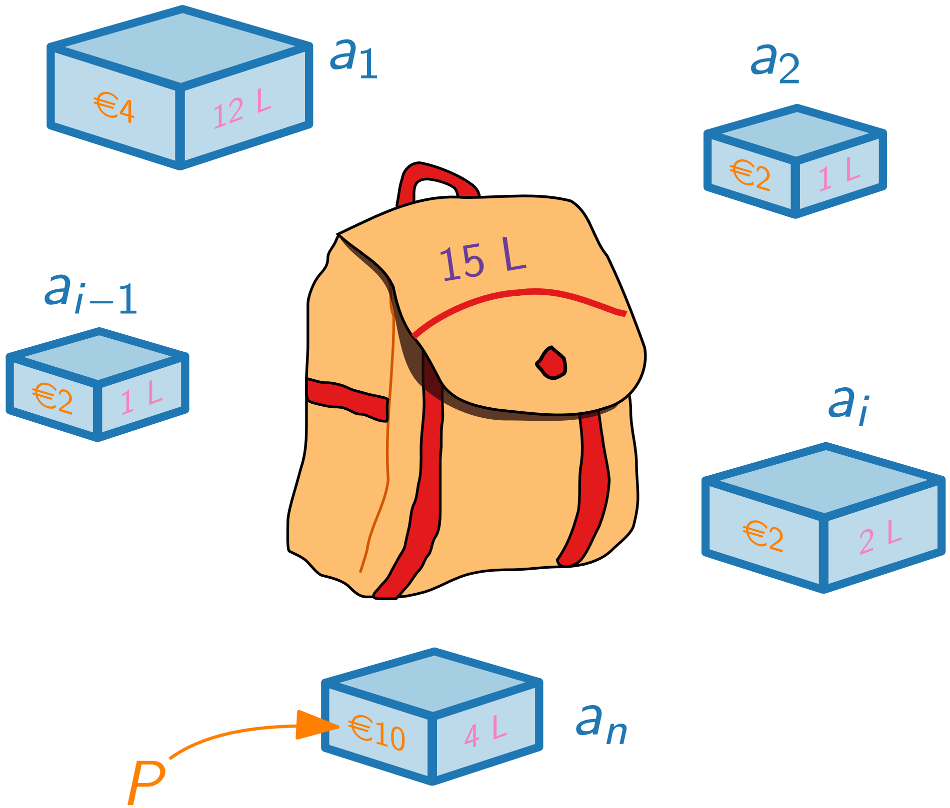

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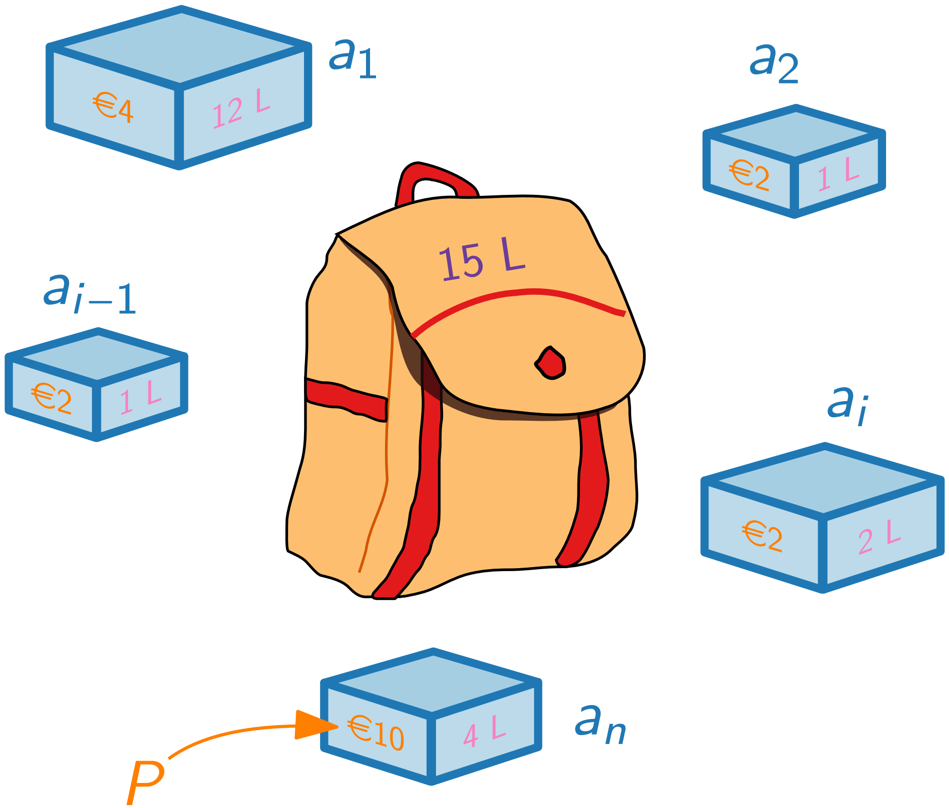

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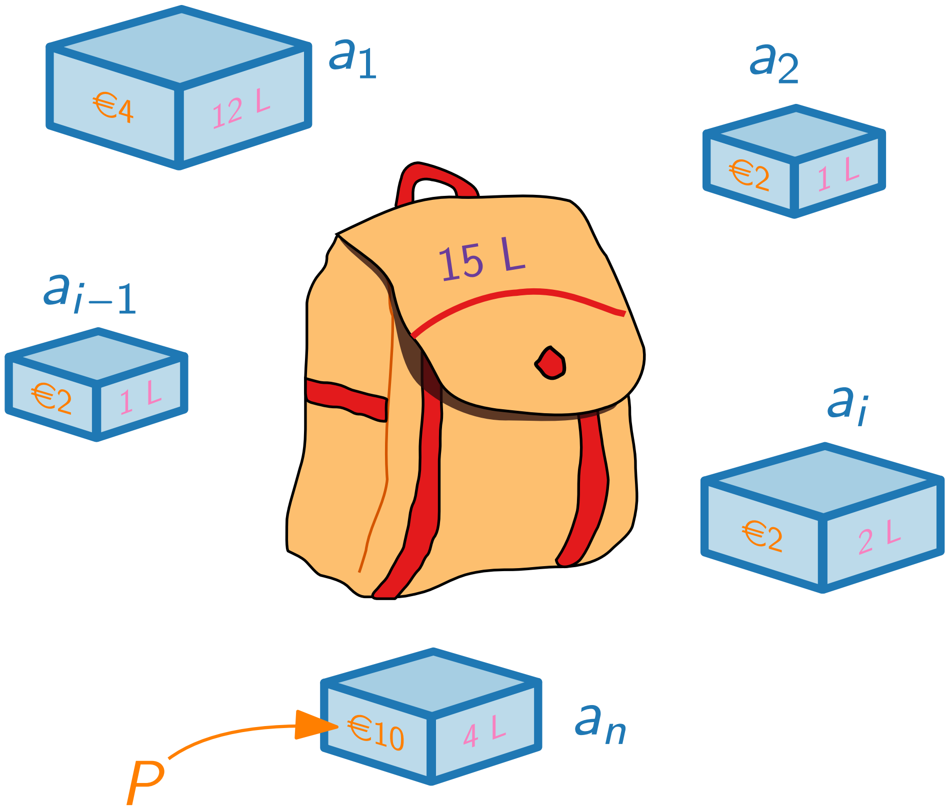

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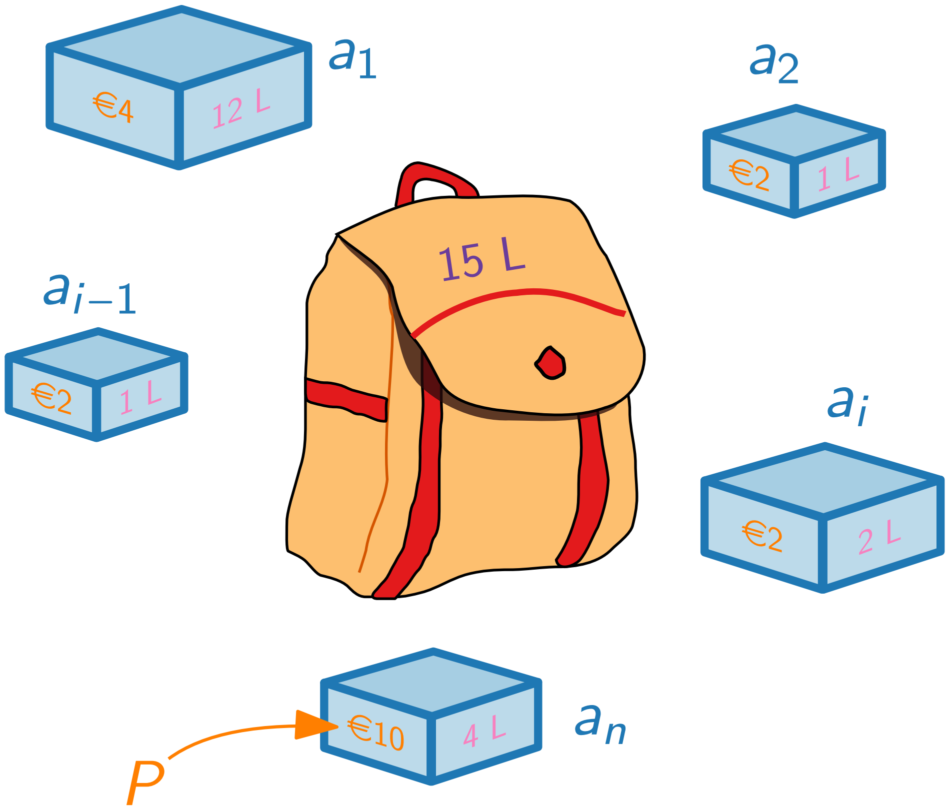

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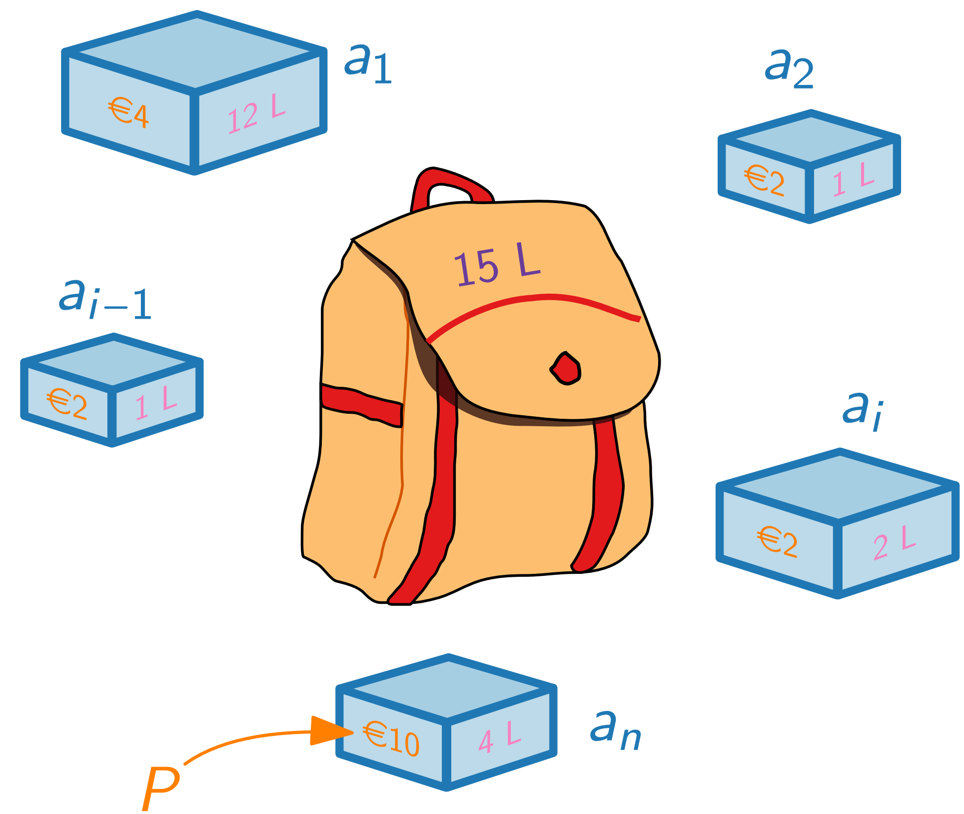

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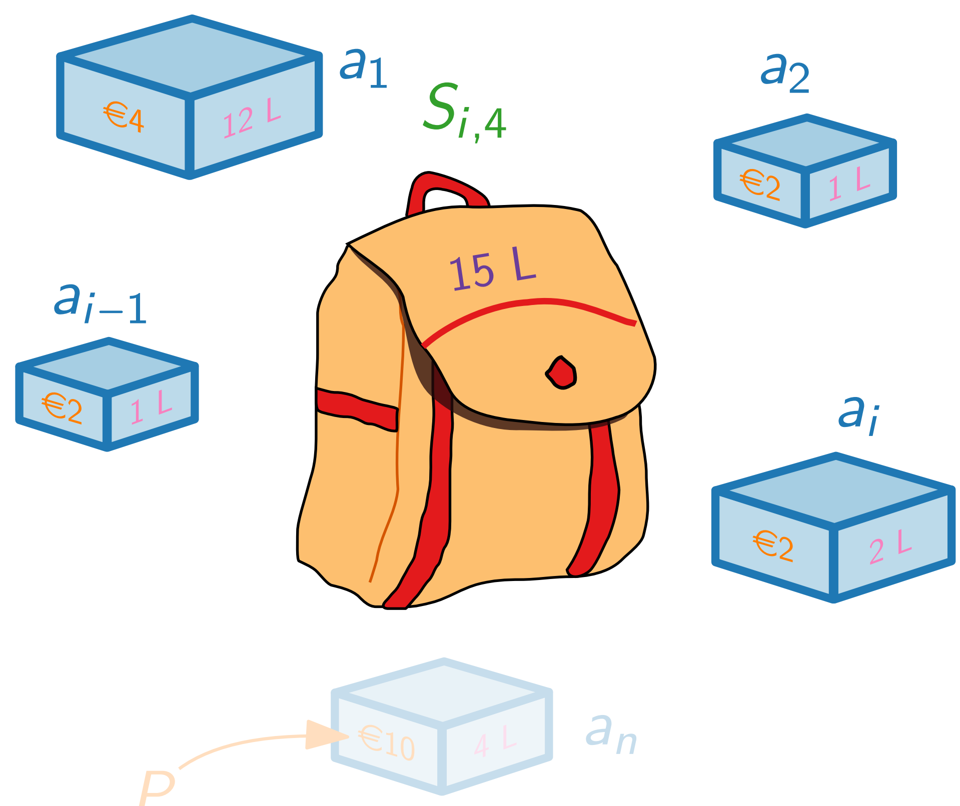

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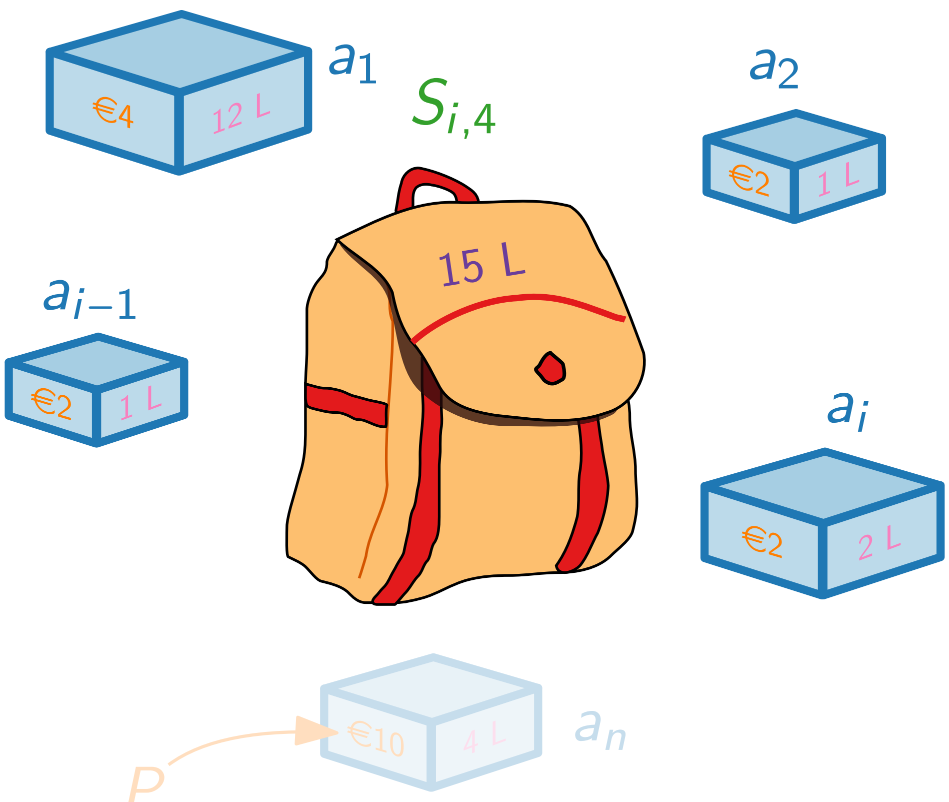

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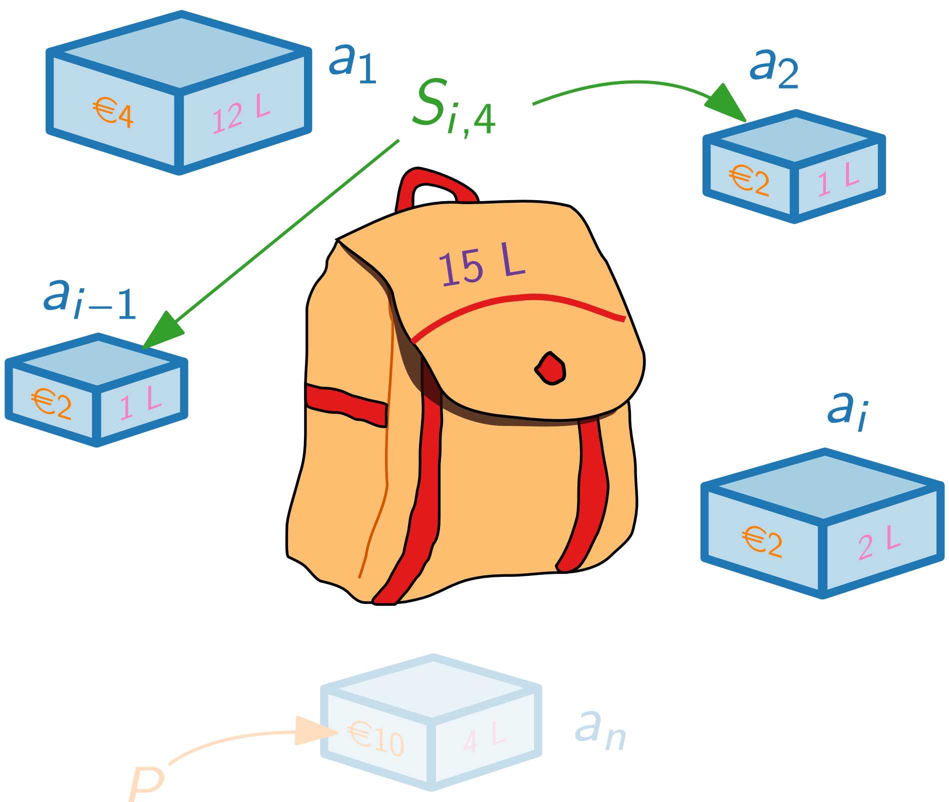

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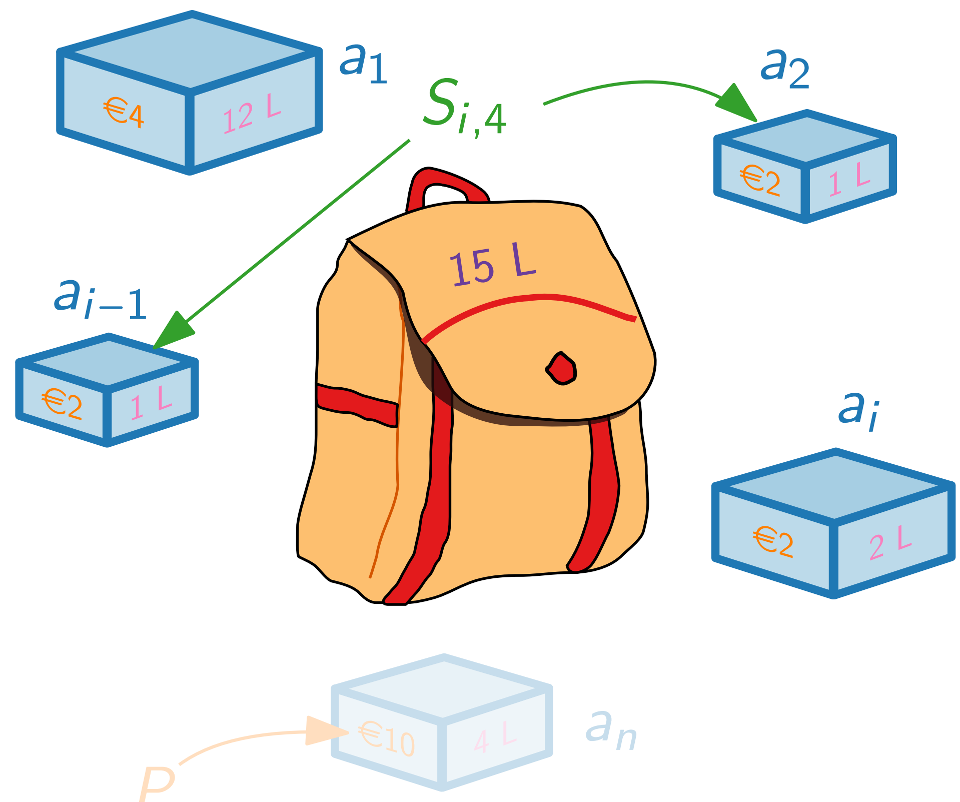

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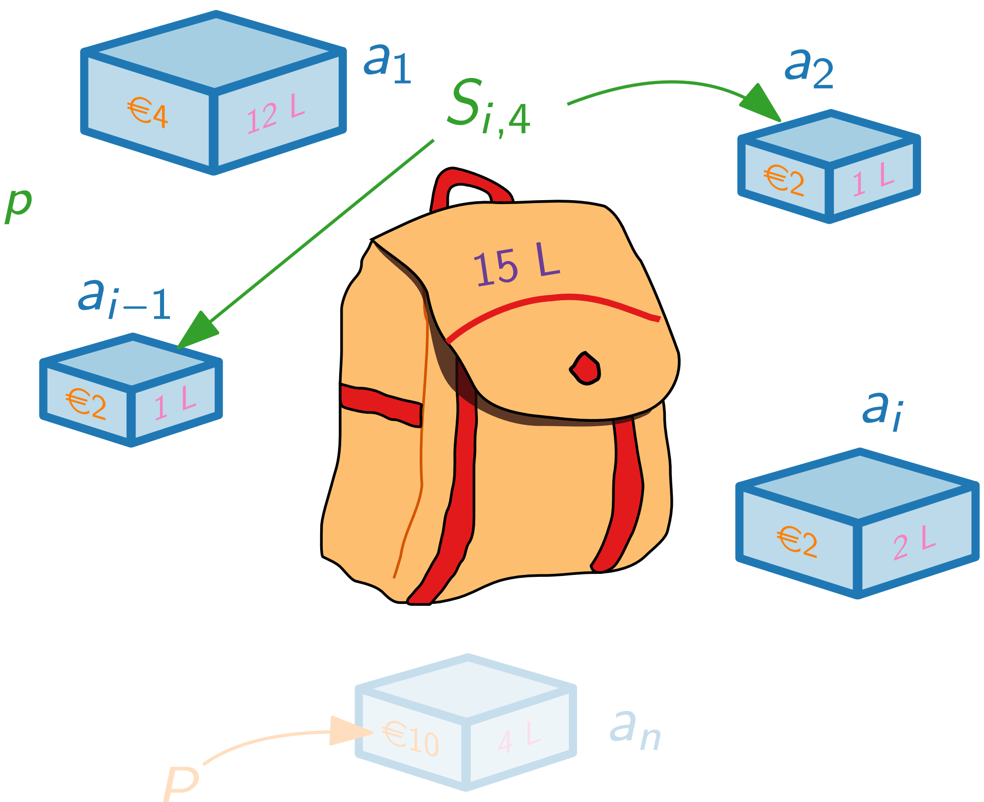

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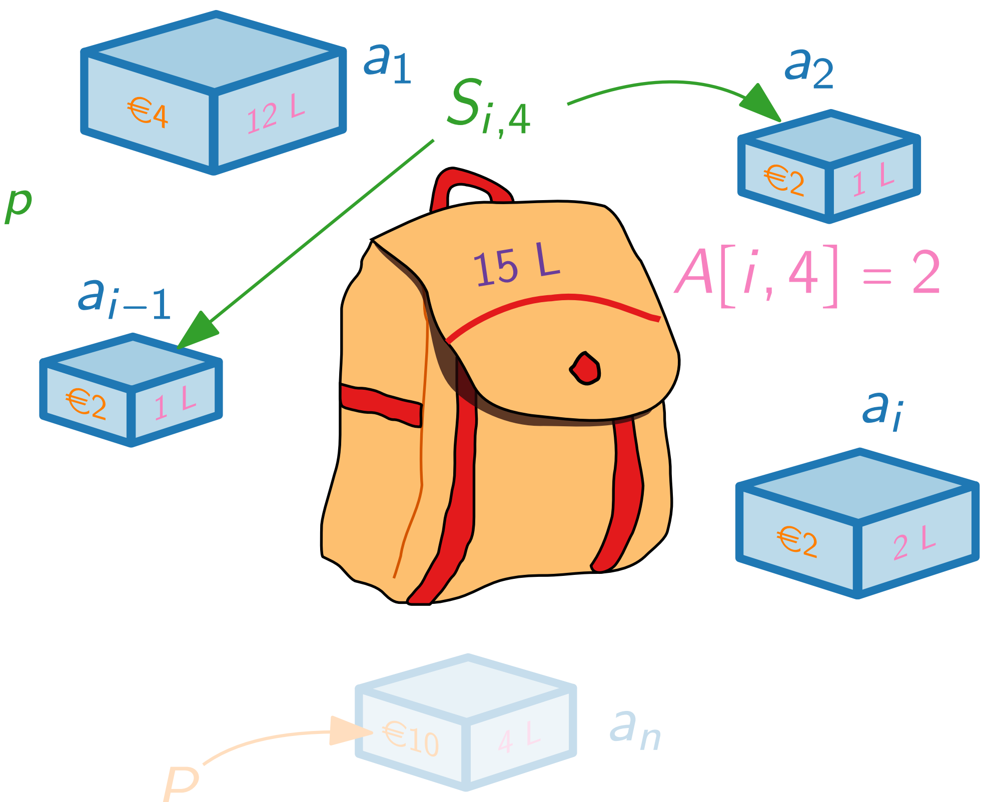

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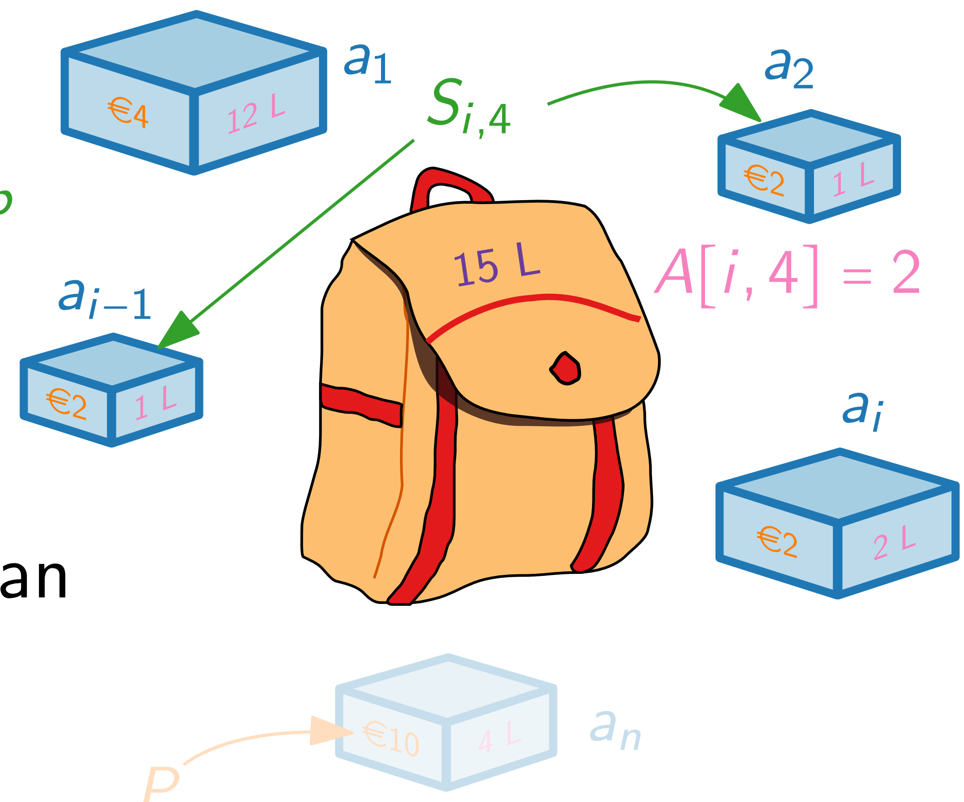

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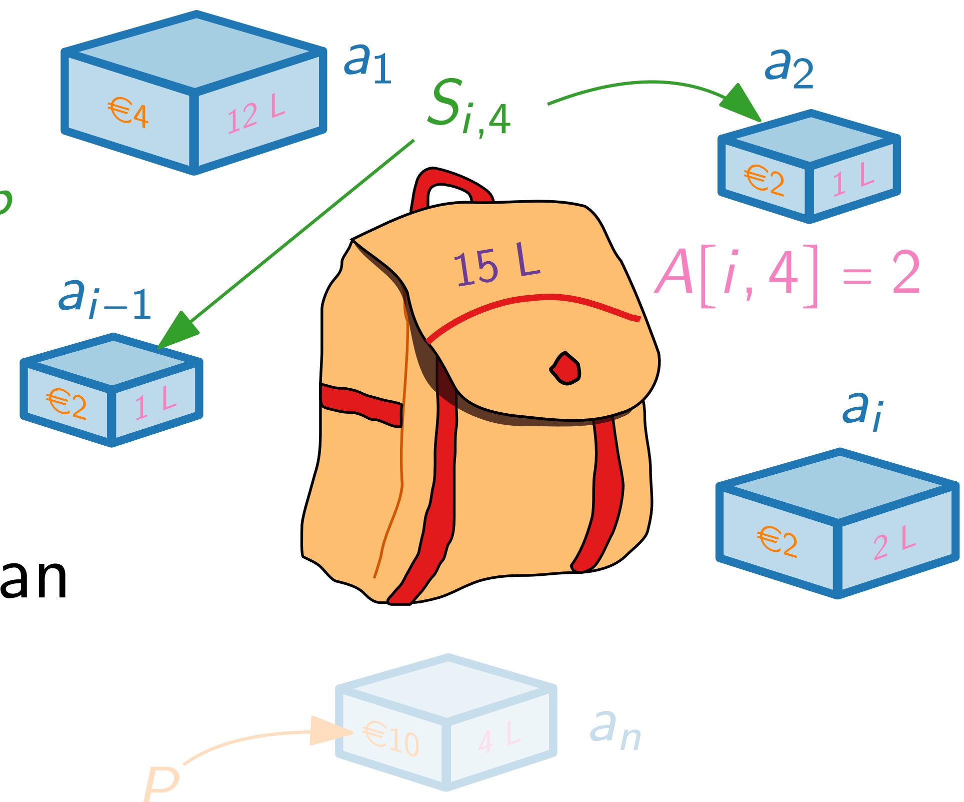

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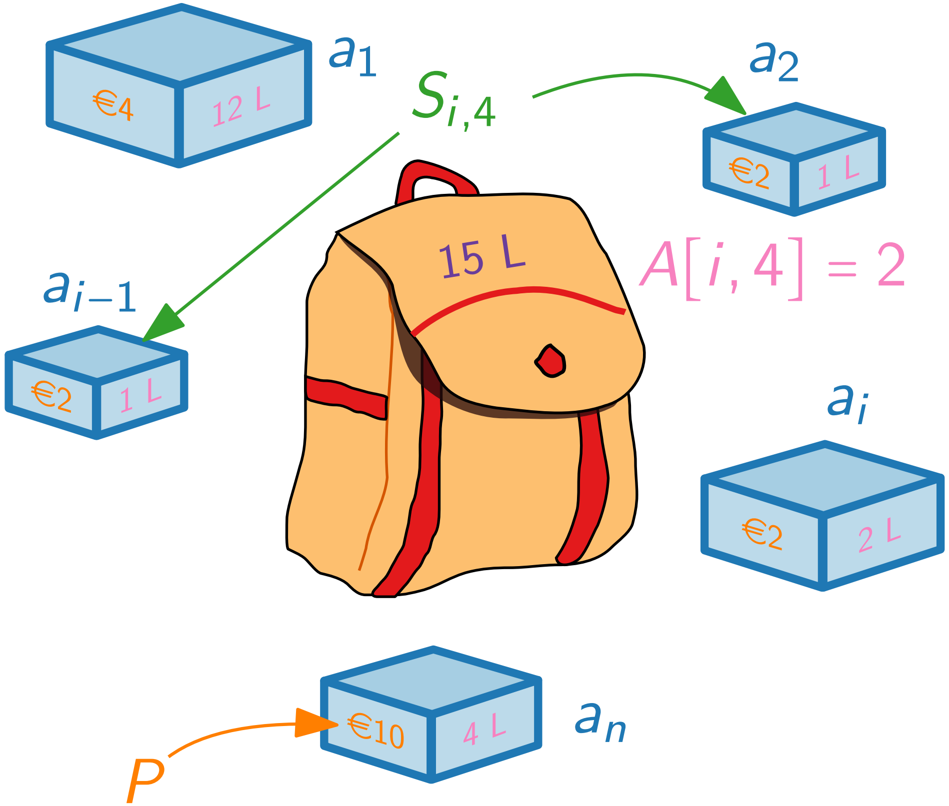
$A[1, p]$ can be computed for all $p \in \{0, \ldots, nP\}$.



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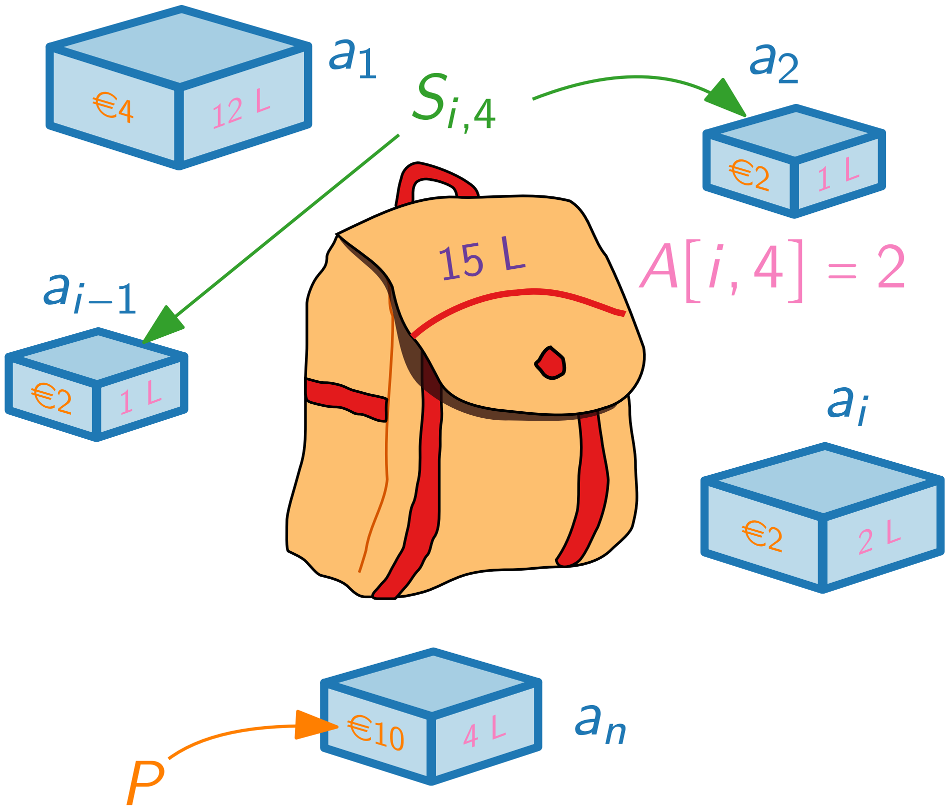

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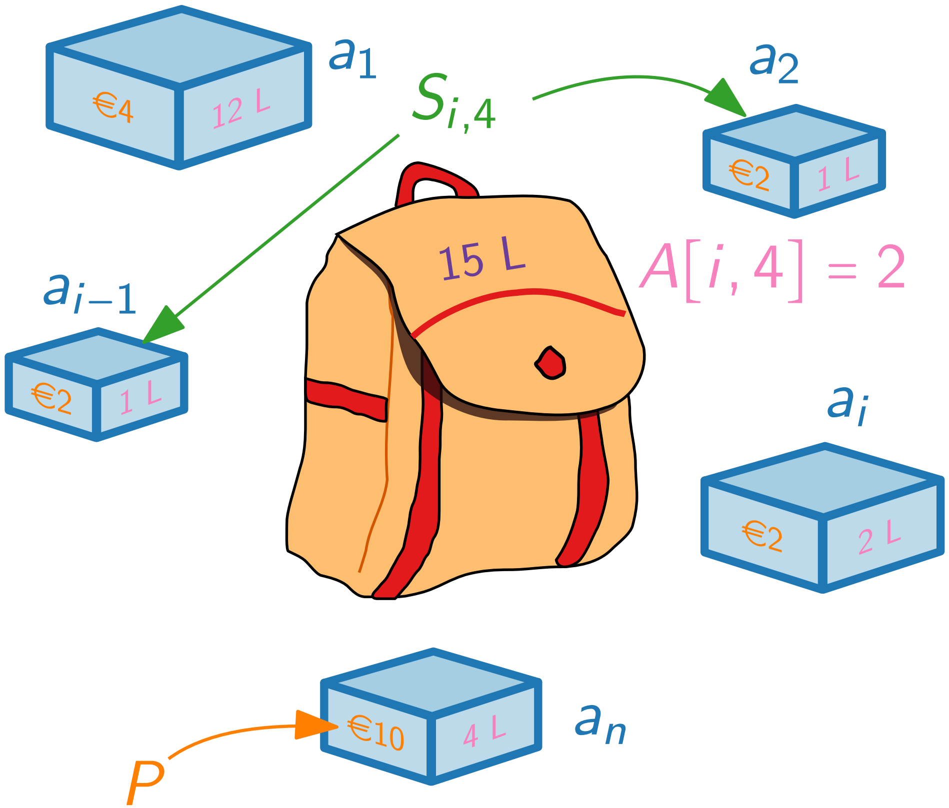

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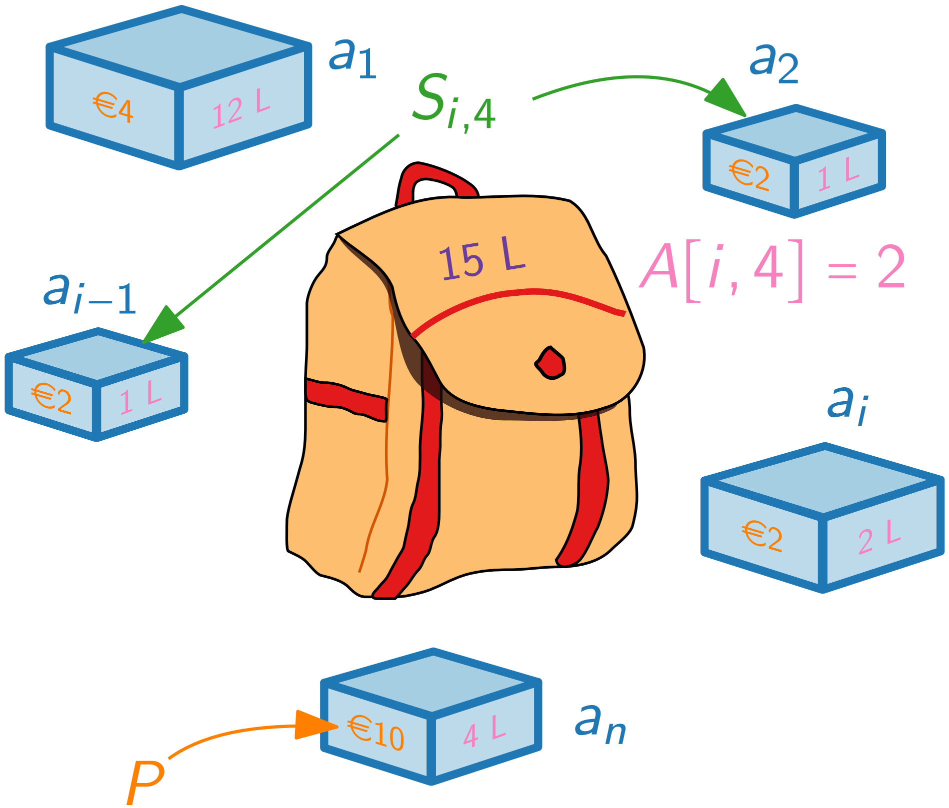

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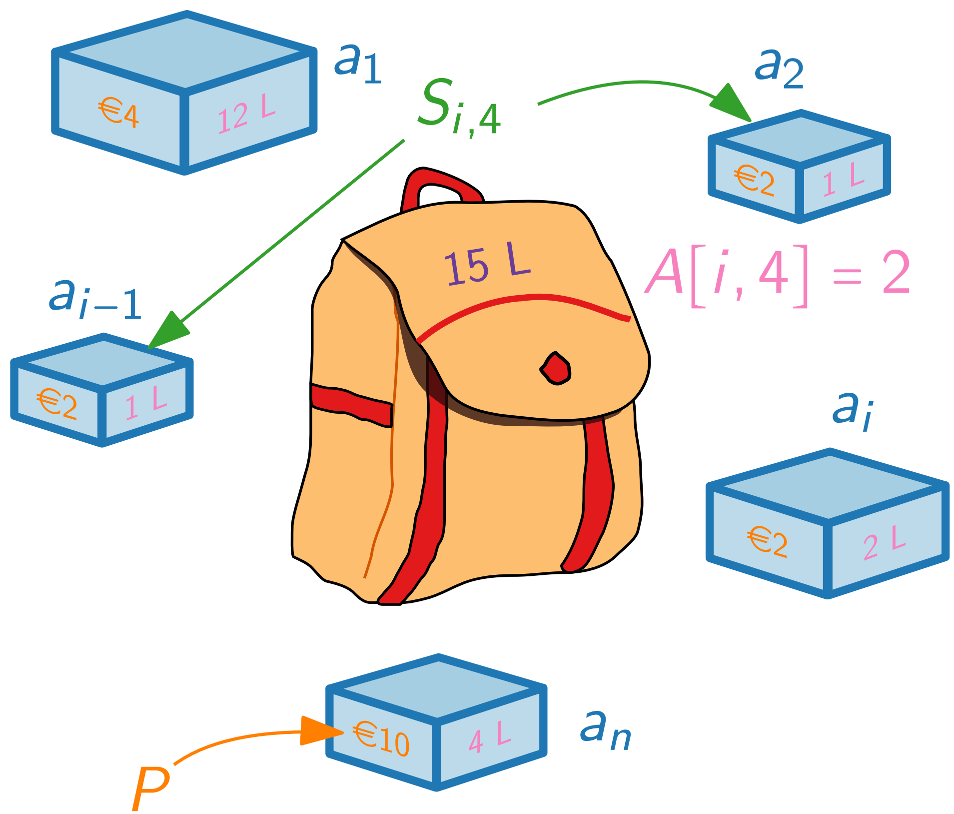

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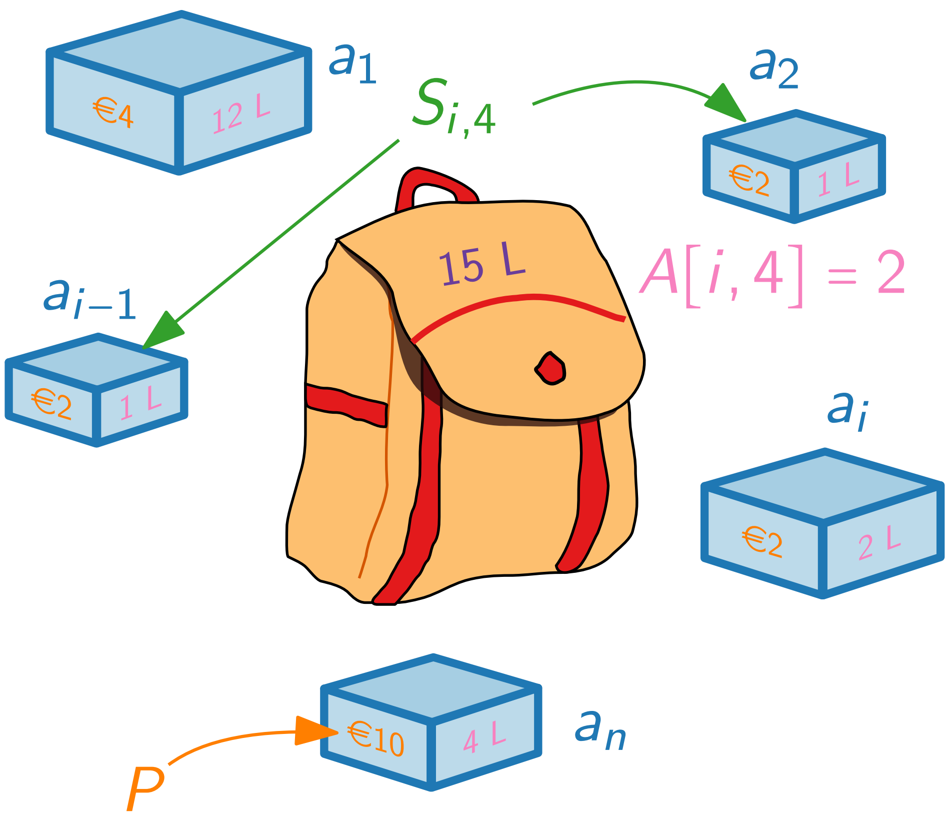

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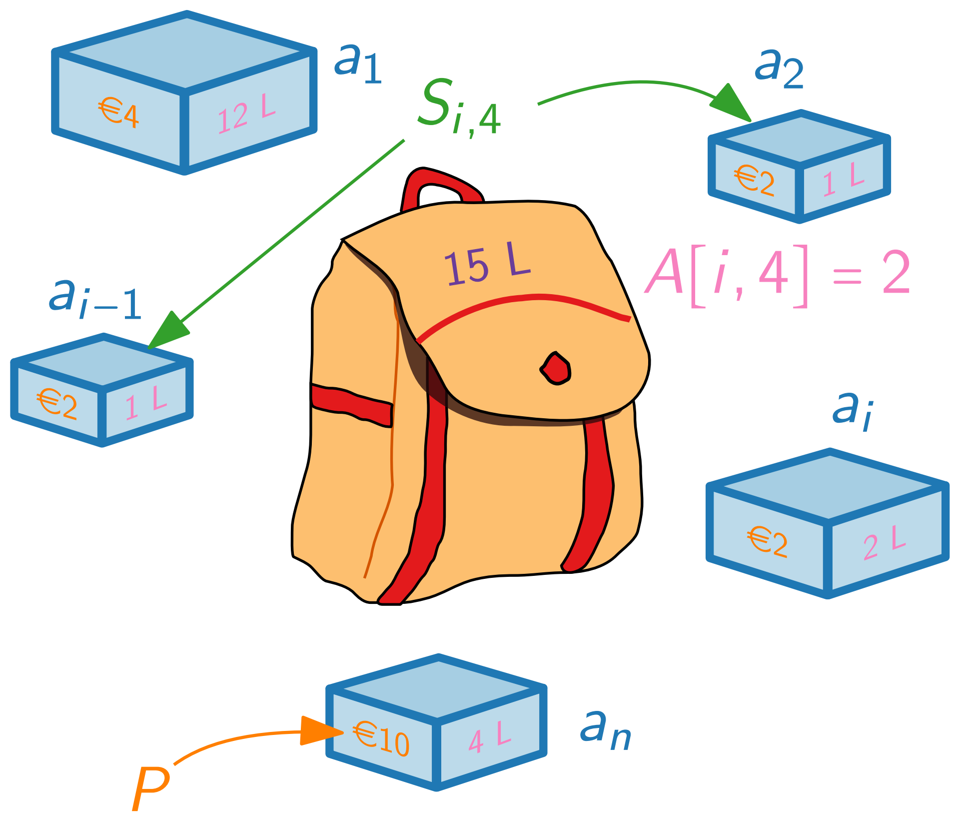

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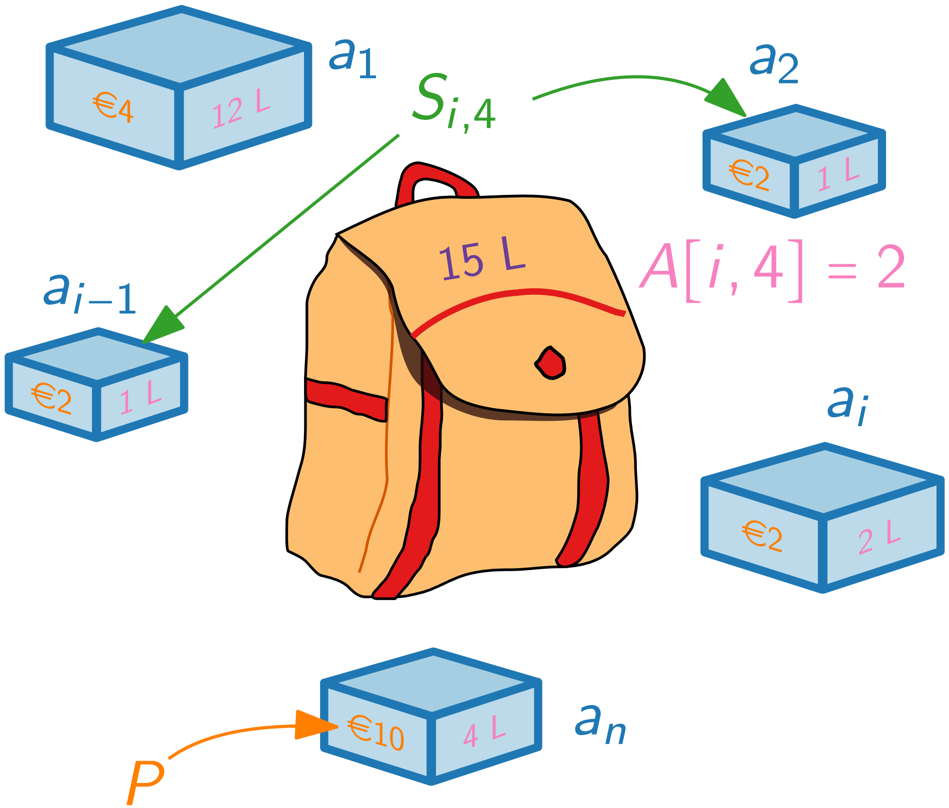

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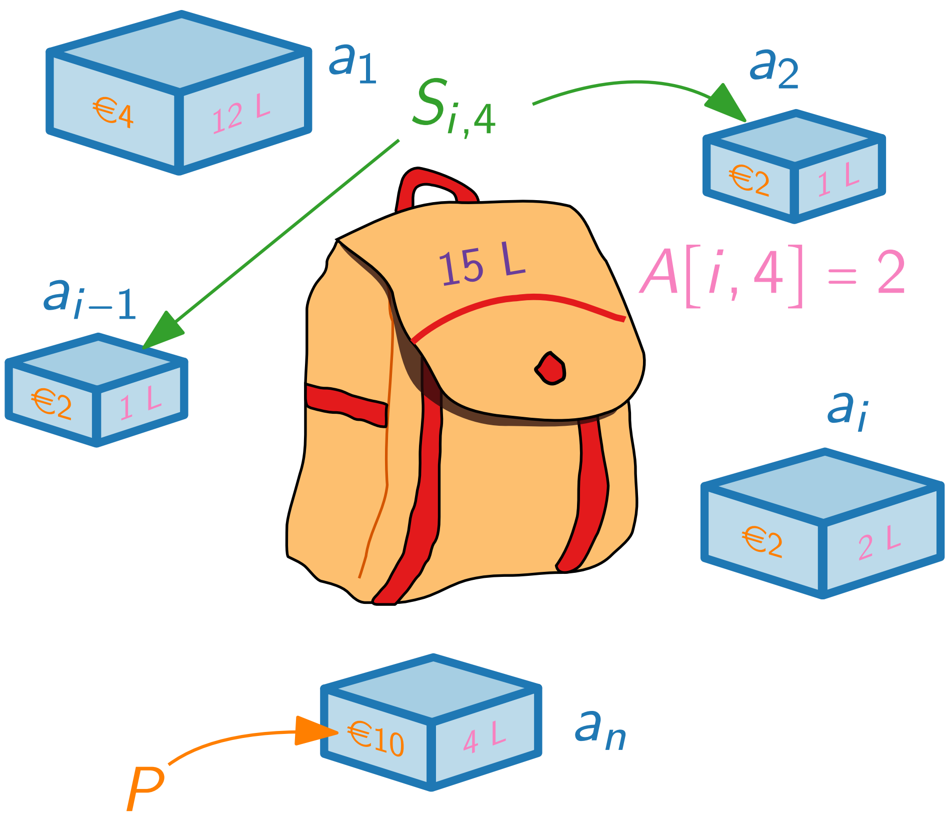
$\Rightarrow$ All values $A[i,p]$ can be computed in total time $O(n^2P)$.



Pseudo-Polynomial Alg. for KNAPSACK

$A[1, p]$ can be computed for all $p \in \{0, \dots, nP\}$.

Set $A[i, p] := \infty$ for $p < 0$ (for convenience).

$A[i+1, p] = \min\{A[i, p], \text{size}(a_{i+1}) + A[i, p - \text{profit}(a_{i+1})]\}$

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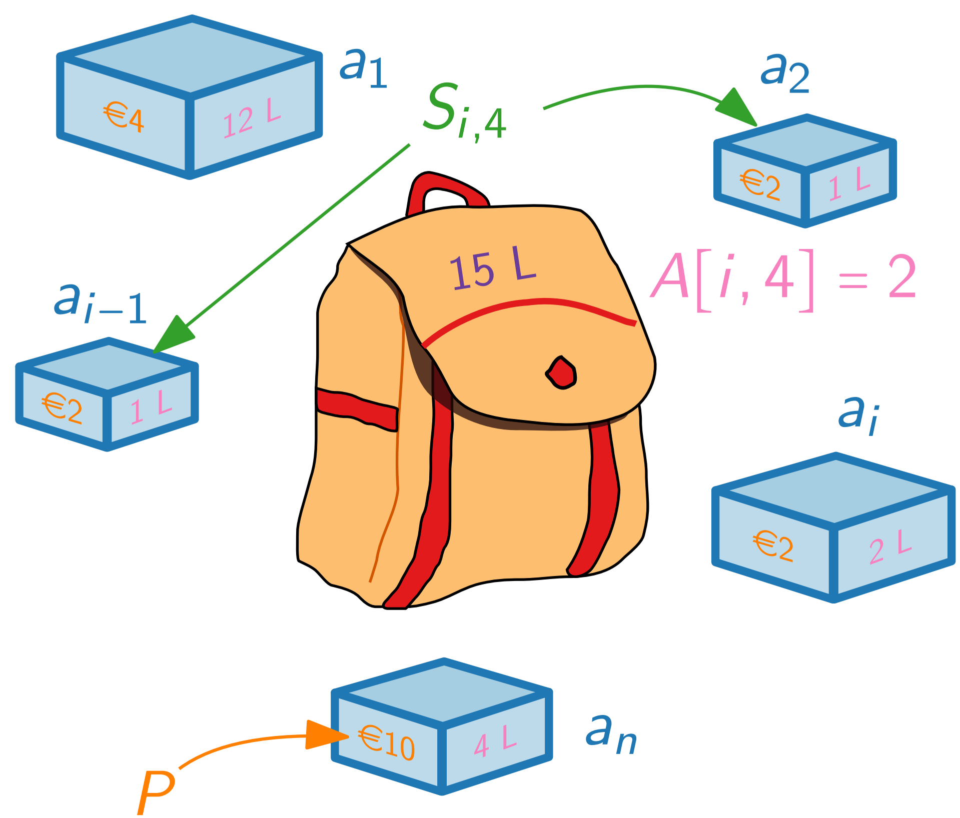

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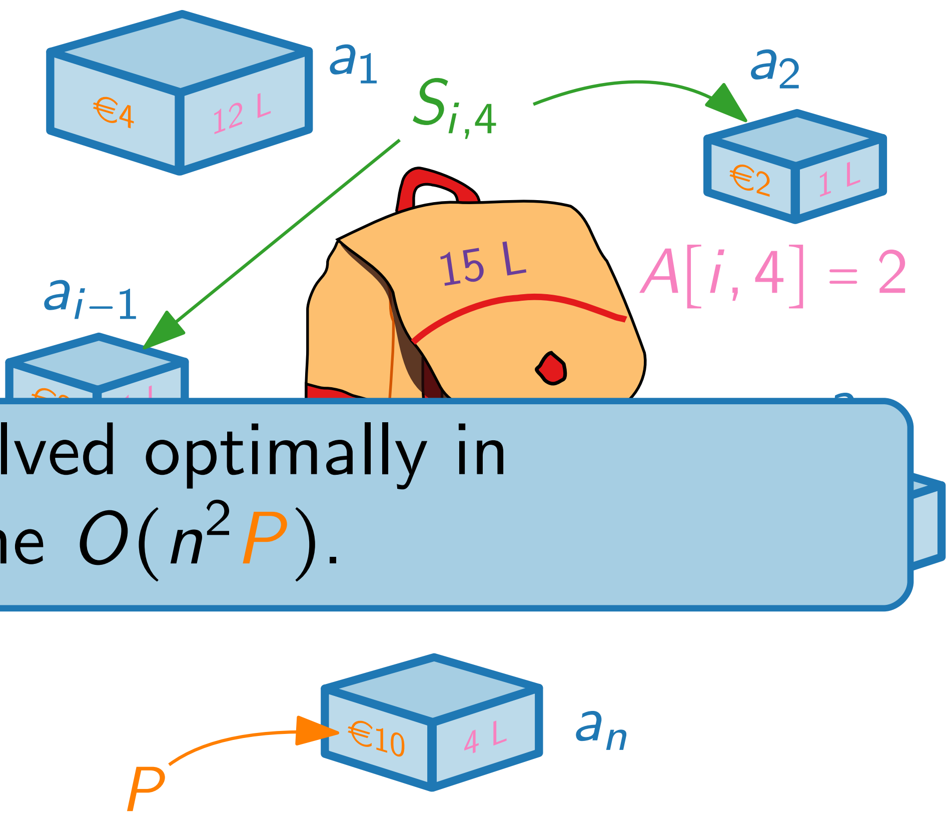

Theorem. KNAPSACK can be solved optimally in pseudo-polynomial time $O(n^2 P)$.

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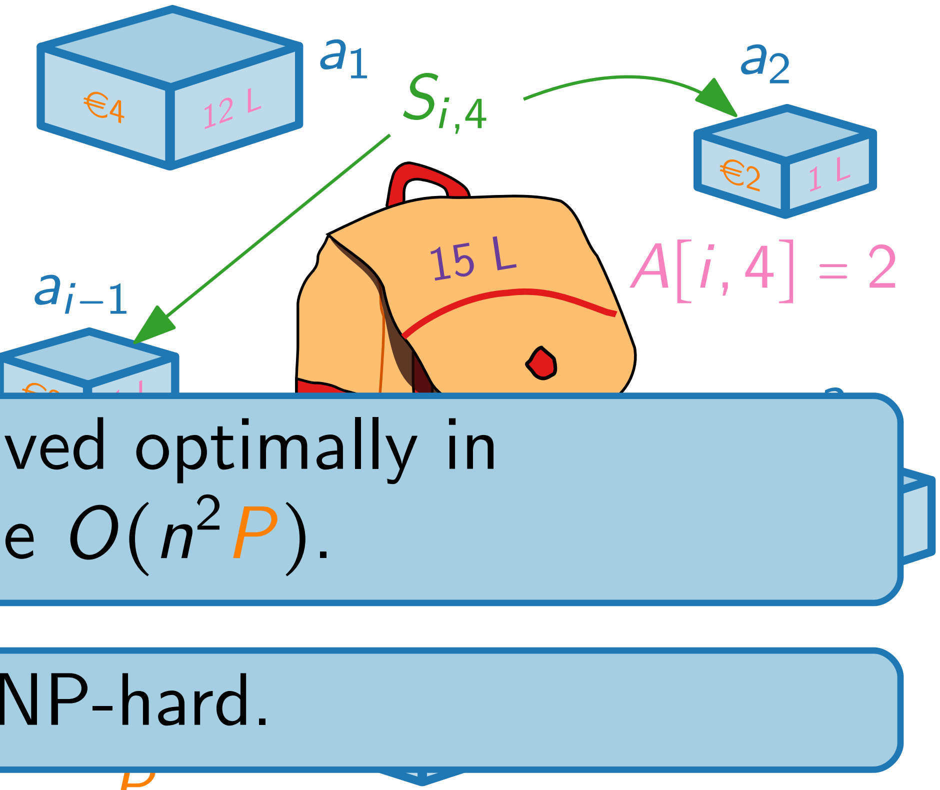

Theorem. Knapsack can be solved optimally in pseudo-polynomial time $O(n^2 P)$.

Corollary. Knapsack is weakly NP-hard.

Pseudo-Polynomial Alg. for KNAPSACK

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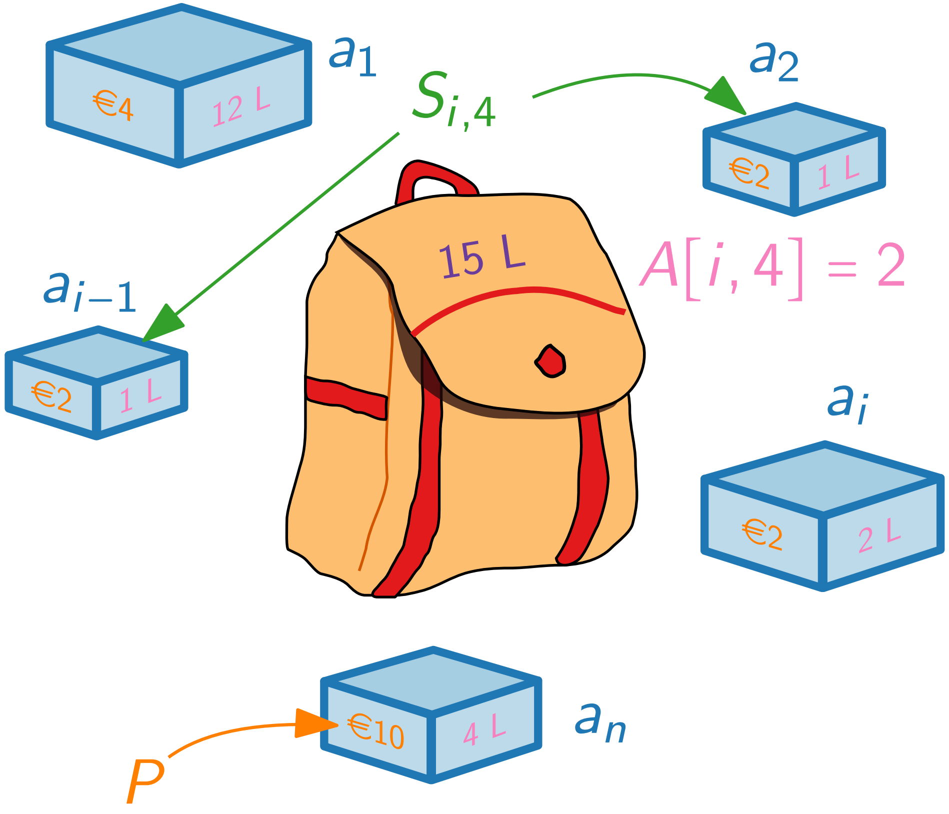
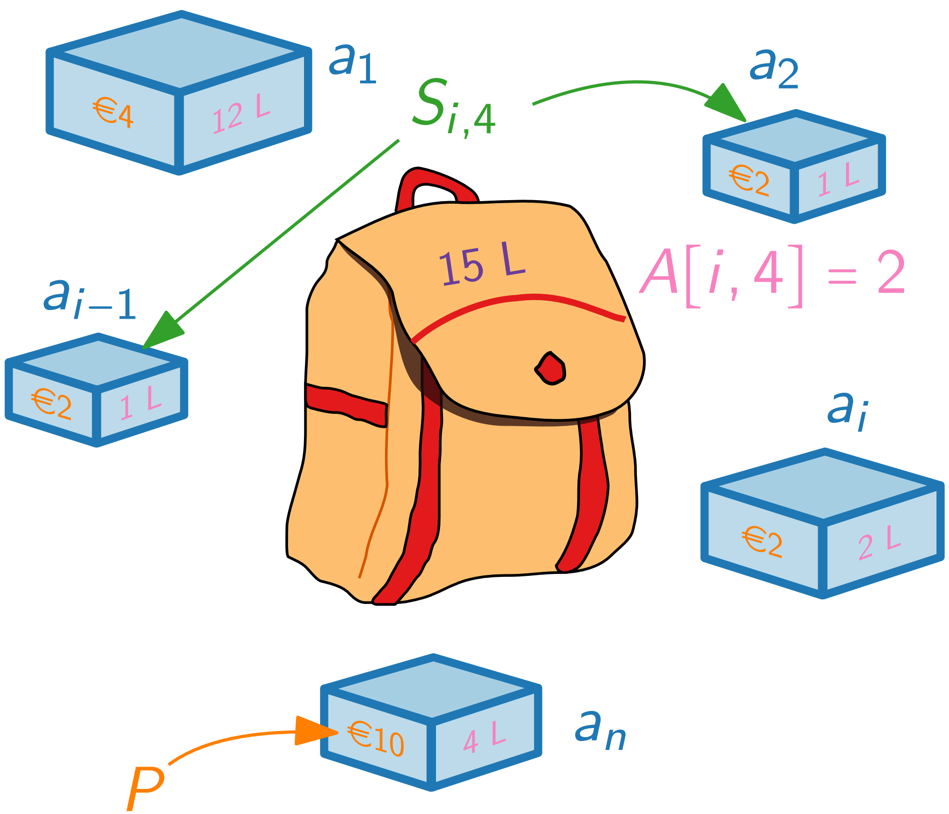


Pseudo-Polynomial Alg. for Knapsack

Theorem. Knapsack can be solved optimally in pseudo-polynomial time $O(n^2 P)$.

Observe. The running time $O(n^2 P)$ is polynomial in n if P is polynomial in n.



Approximation Algorithms

Lecture 8:
Approximation Schemes and the KNAPSACK Problem

Part IV:
Approximation Schemes

Approximation Schemes

Let Π be an optimization problem.

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Let Π be an optimization problem. An algorithm $\mathcal{A}$ is called a **polynomial-time approximation scheme** (PTAS) for Π if it outputs, for every input (I, ε) with $I \in D_\Pi$ and $\varepsilon > 0$, a solution $s \in S_\Pi(I)$ such that

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Approximation Algorithms

Lecture 8:
Approximation Schemes and the Knapsack Problem

Part V:
FPTAS for Knapsack

An FPTAS for KNAPSACK via Scaling

FPTAS idea: **Scale** profits to polynomial size (as required by the error parameter ε)...

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FPTAS idea: **Scale** profits to polynomial size (as required by the error parameter ε)...

An FPTAS for Knapsack via Scaling

KnapsackScaling (I, ε)

$K = \varepsilon P / n$ // scaling factor

$\text{profit}'(a_i) = \lfloor \text{profit}(a_i) / K \rfloor$

Compute optimal solution S' for I w.r.t. $\text{profit}'(\cdot)$.

return S'

Lemma. $\text{profit}(S') \geq (1 - \varepsilon) \cdot \text{OPT}$.

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Theorem. KnapsackScaling is an FPTAS for Knapsack with running time $O(n^3/\varepsilon)$

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Approximation Algorithms

Lecture 8:
Approximation Schemes and the KNAPSACK Problem

Part VI:
Connections Between the Concepts

FPTAS and Pseudo-Poly. Algorithms

Theorem. Let p be a polynomial and let Π be an NP-hard minimization problem

FPTAS and Pseudo-Poly. Algorithms

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FPTAS and Strong NP-Hardness

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Corollary. Let Π be an NP-hard optimization problem that fulfills the restrictions above.
If Π is strongly NP-hard, then there is no FPTAS for Π (unless P = NP).