# Approximation Algorithms

## Lecture 7:
## Scheduling Jobs on Parallel Machines

### Part I:
### ILP & Parametric Pruning

Alexander Wolff

Winter 2022/23

# Scheduling on Parallel Machines

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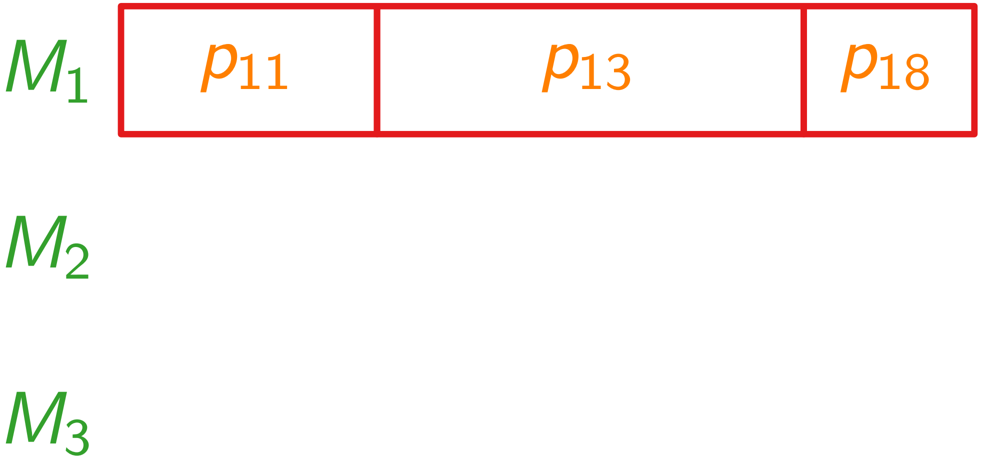

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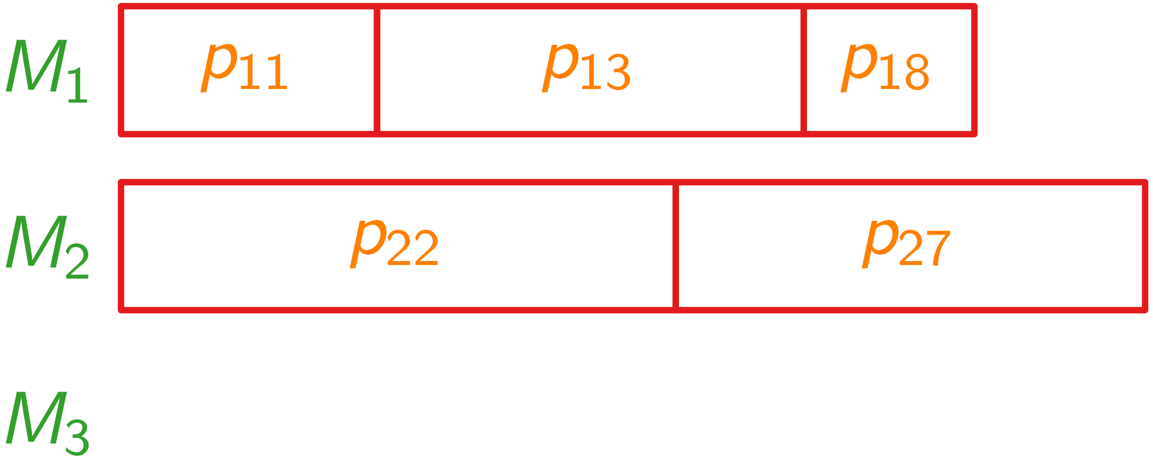

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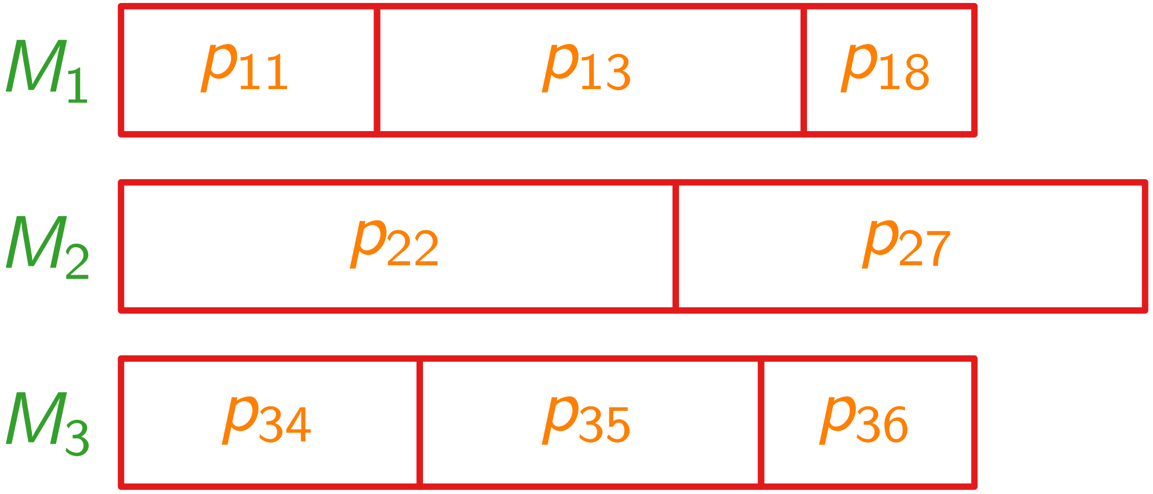
$M_1$: $p_{11}$, $p_{13}$, $p_{18}$

$M_2$: $p_{22}$, $p_{27}$

$M_3$: $p_{34}$, $p_{35}$, $p_{36}$

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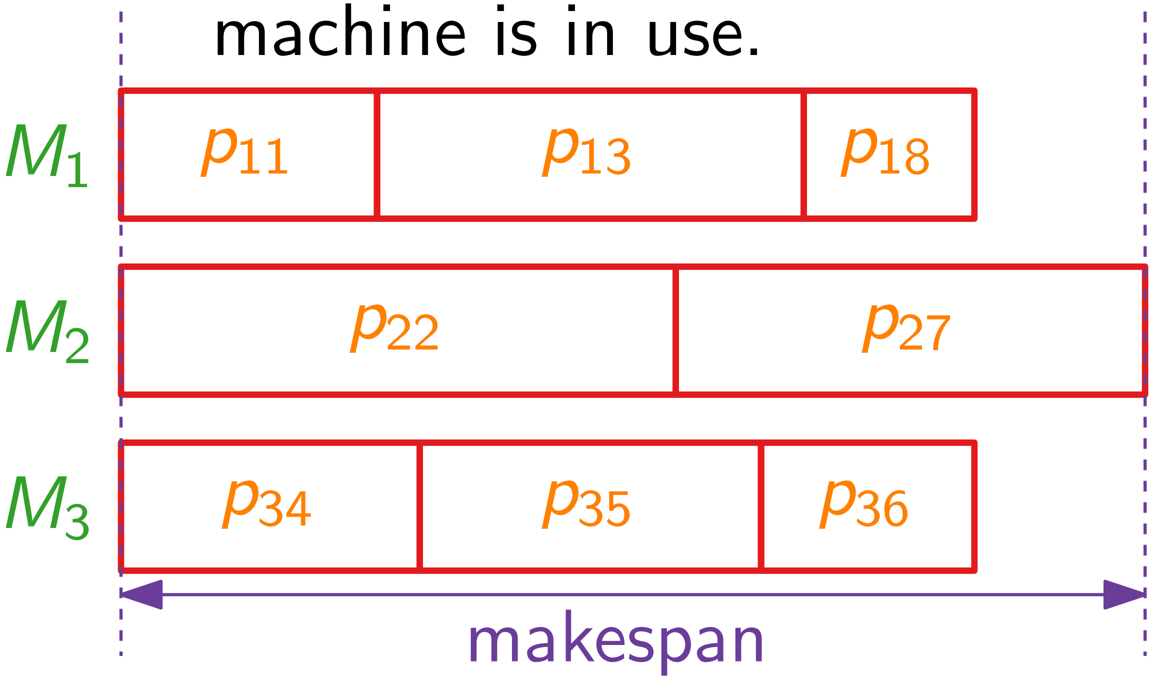

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**Task:** Prove that the integrality gap is unbounded!

**Solution:** $m$ machines and one job with processing time $m$
$\Rightarrow \mathrm{OPT} = m$ and $\mathrm{OPT}_{\mathrm{frac}} = 1$.

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If $p_{ij} > t$, then set $x_{ij} = 0$.

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But why does this LP give a good integrality gap?

# Approximation Algorithms

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### Part II:
### Properties of Extreme-Point Solutions

# Properties of Extreme Point Solutions

Use binary search to find the smallest $T$ so that LP($T$) has a solution.

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Every extreme-point solution of LP($T$) has at most $|\mathcal{M}| + |\mathcal{J}|$ positive variables.

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**Lemma 2.**
Every extreme-point solution of LP($T$) sets at least $|\mathcal{J}| - |\mathcal{M}|$ jobs integrally.

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# Approximation Algorithms

## Lecture 7:
## Scheduling Jobs on Parallel Machines

### Part III:
### An Algorithm

# Extreme Point Solutions of LP($T$)

**Definition:** Bipartite graph $G = (\mathcal{M} \cup \mathcal{J}, E)$ with $(i, j) \in E \Leftrightarrow x_{ij} \neq 0$ (in extreme-point sol.).

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And why is this useful . . . ?

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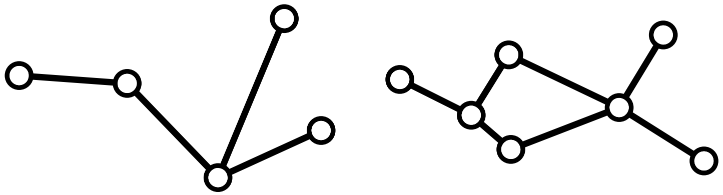
$\Rightarrow$ total makespan $\le 2T^* \le 2\mathrm{OPT}$ □

# Approximation Algorithms

## Lecture 7:
## Scheduling Jobs on Parallel Machines

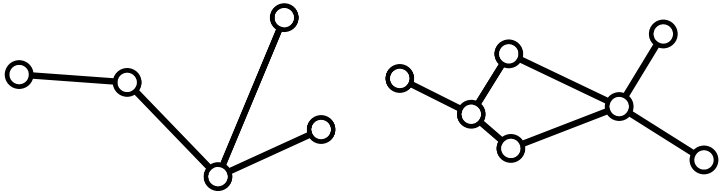
### Part IV:
### Pseudo-Trees and -Forests

# Pseudo-Trees and -Forests



**Pseudo-tree**: a connected graph with at most as many edges as vertices.

# Pseudo-Trees and -Forests



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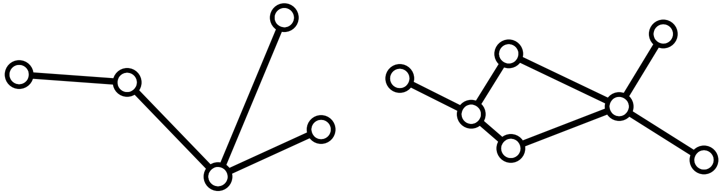

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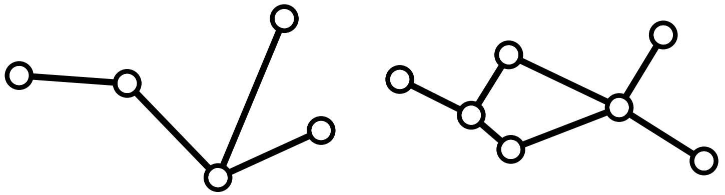

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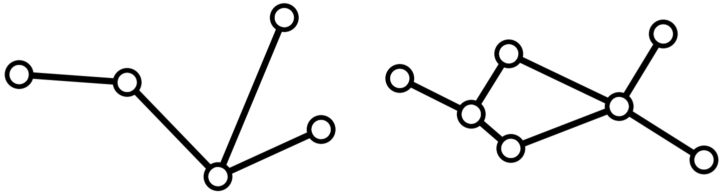

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**Lemma 3.**
The bipartite graph $G = (\mathcal{M} \cup \mathcal{J}, E)$ is a pseudo-forest.

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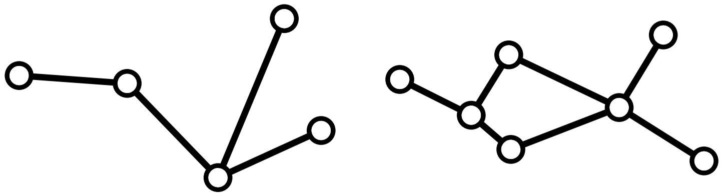

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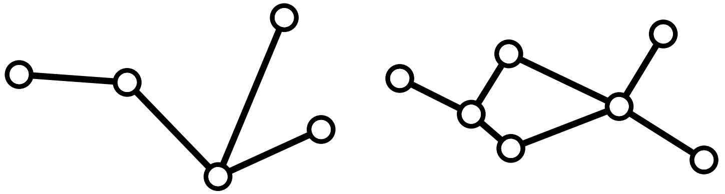

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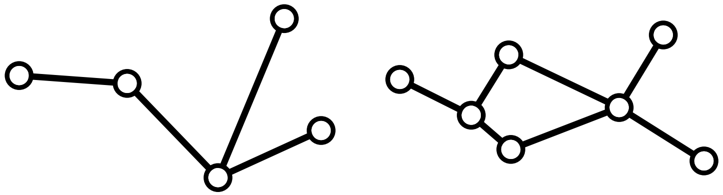

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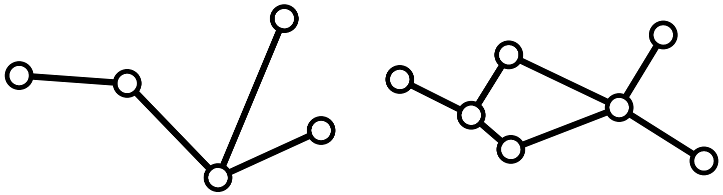

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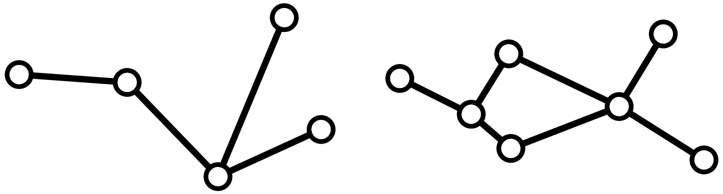

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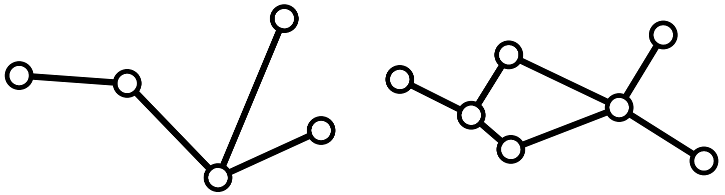

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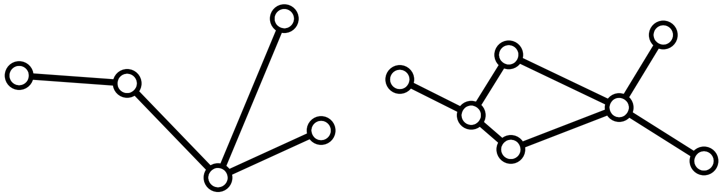

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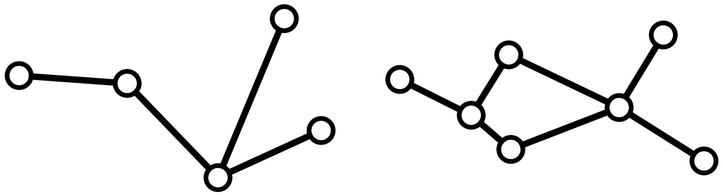

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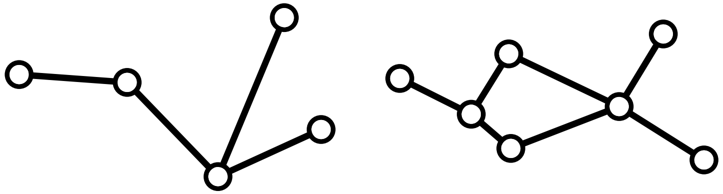

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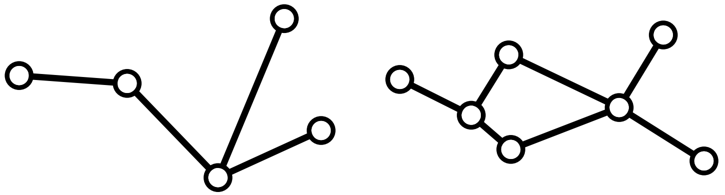

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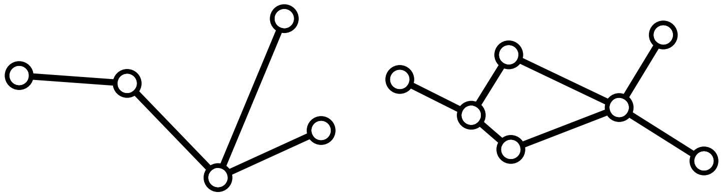

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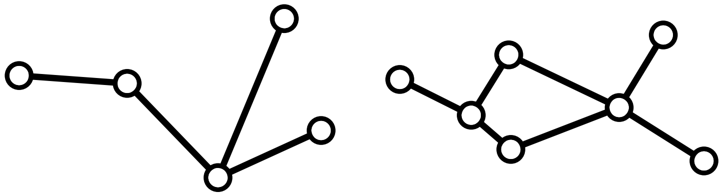

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