

Approximation Algorithms

Lecture 6:

k -CENTER via Parametric Pruning

Part I:

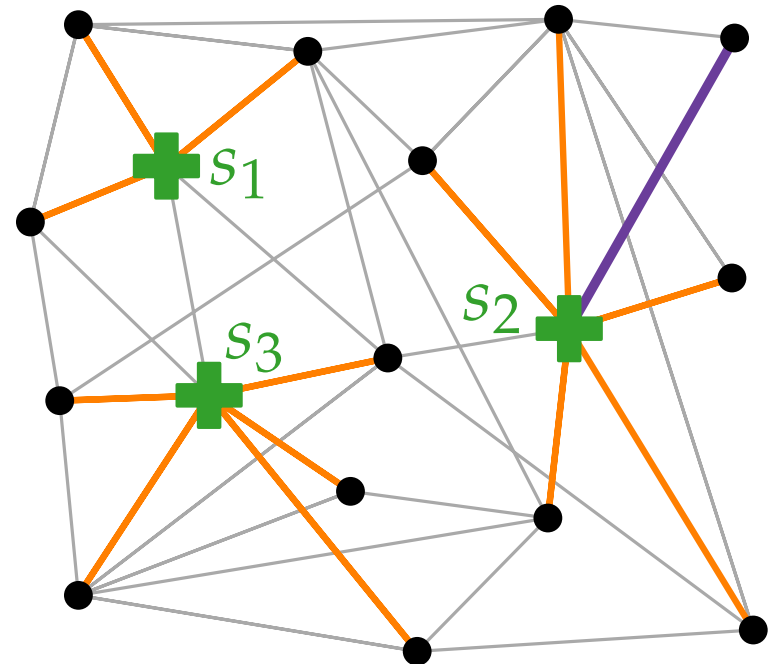
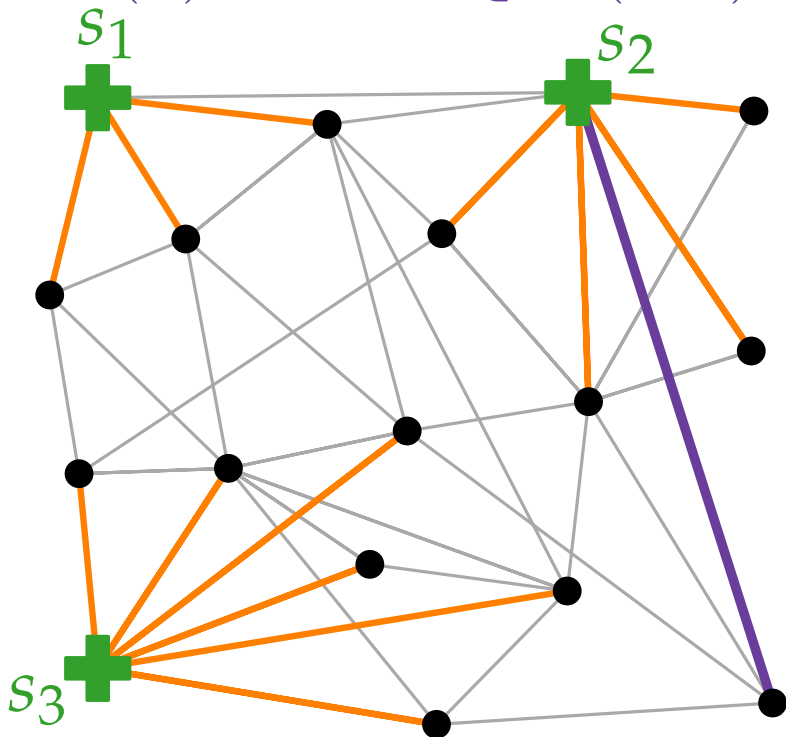
Metric k -CENTER

Metric k -CENTER

Given: A complete graph $G = (V, E)$ with **edge costs** $c: E \rightarrow \mathbb{Q}_{\geq 0}$ satisfying the triangle inequality and a natural number $k \leq |V|$.

For each vertex set $S \subseteq V$, $c(v, S)$ is the cost of the cheapest edge from v to a vertex in S .

Find: A k -element vertex set S such that $\text{cost}(S) := \max_{v \in V} c(v, S)$ is minimized.



Approximation Algorithms

Lecture 6:

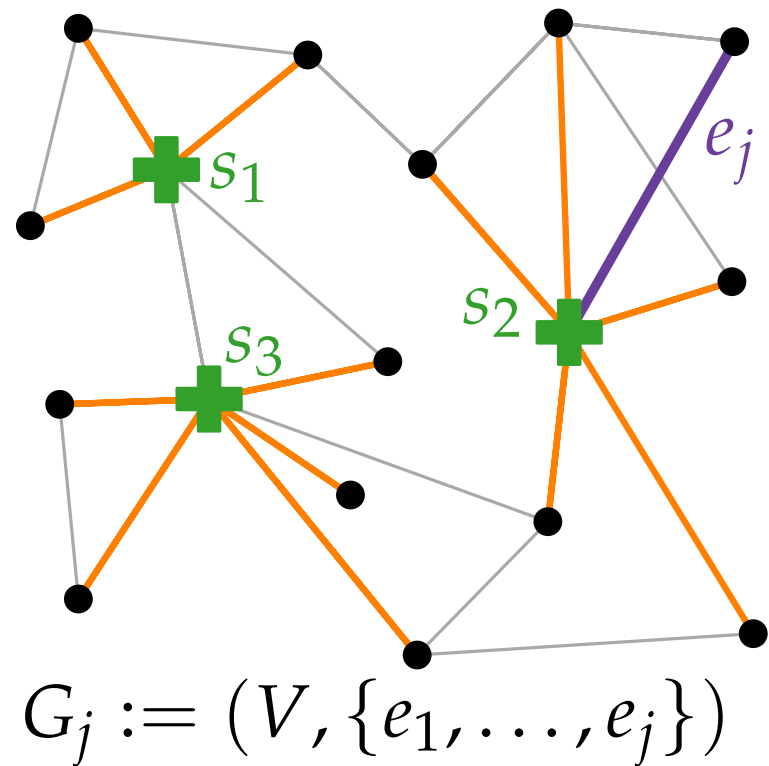
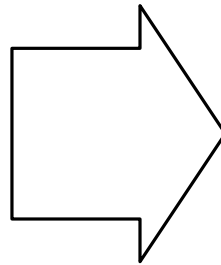
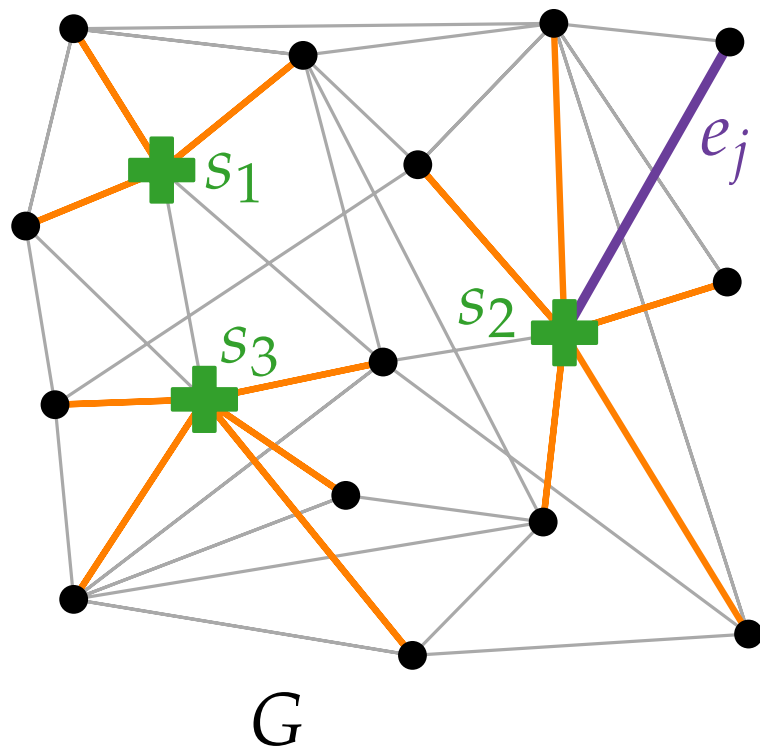
k -CENTER via Parametric Pruning

Part II:

Parametric Pruning

Parametric Pruning

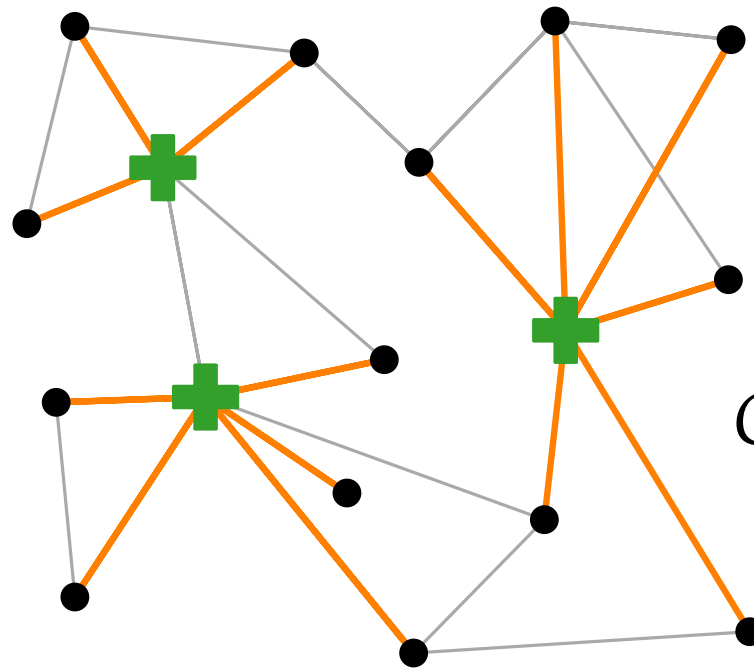
Let $E = \{e_1, \dots, e_m\}$ with $c(e_1) \leq \dots \leq c(e_m)$.
Suppose we know that $\text{OPT} = c(e_j)$.



... try each G_j .

... try each G_j .

Def. A vertex set D of a graph H is **dominating** if each vertex is either in D or adjacent to a vertex in D . The cardinality of a smallest dominating set in H is denoted by $\text{dom}(H)$.



$$\text{dom}(G_j) \leq k$$

$$G_j := (V, \{e_1, \dots, e_j\})$$

... but computing $\text{dom}(H)$ is NP-hard.



Approximation Algorithms

Lecture 6:

k -CENTER via Parametric Pruning

Part III:

Square of a Graph

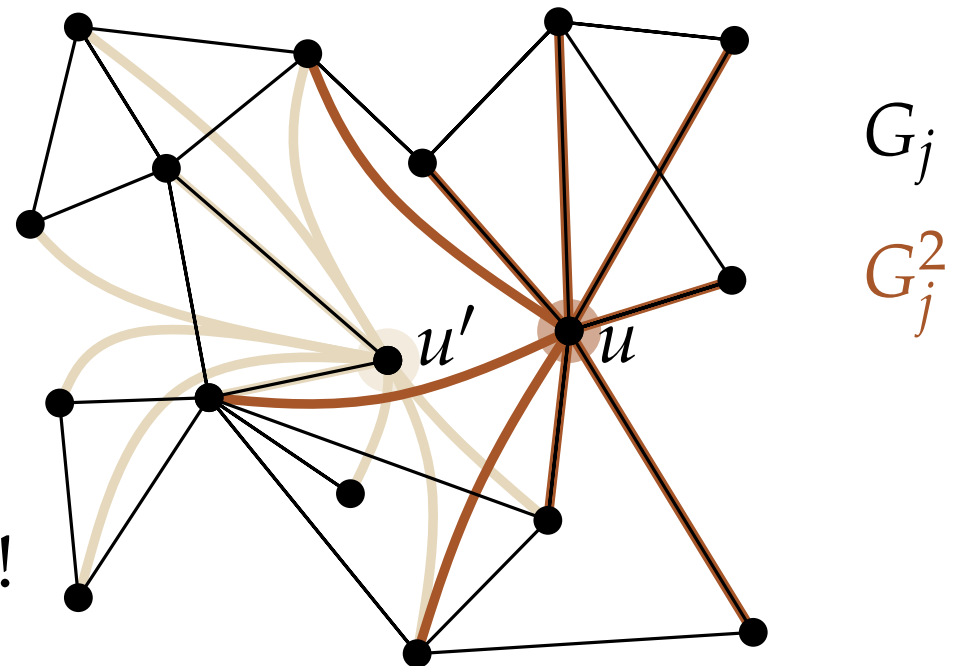
Square of a Graph

Idea: Find a small dominating set in a “coarsened” G_j .

Def. The **square** H^2 of a graph H has the same vertex set as H . Two vertices $u \neq v$ are adjacent in H^2 iff they are within distance at most **two** in H .

Obs. A dominating set with at most k elements in G_j^2 is a 2-approximation for metric k -CENTER.

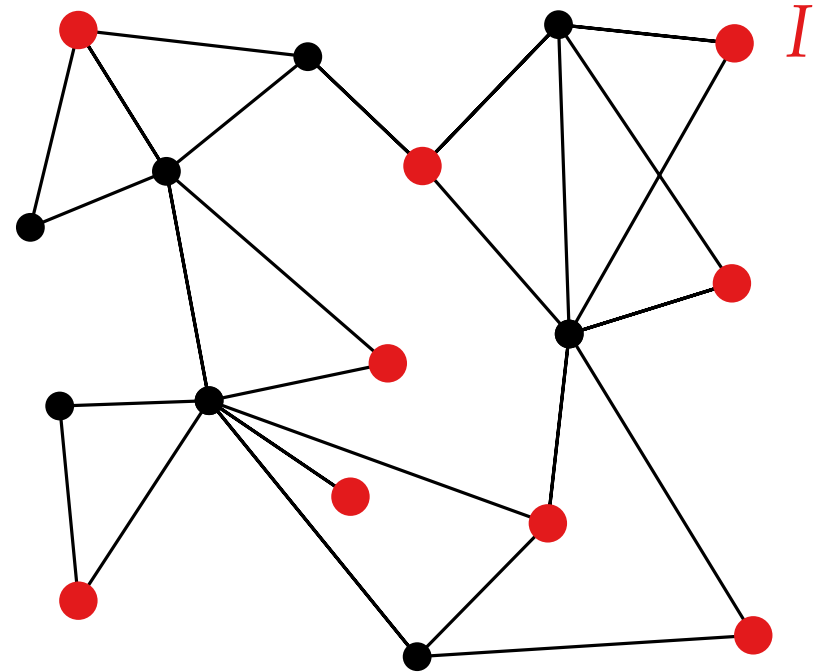
Why? $\max_{e \in E(G_j)} c(e) = \text{OPT} !$



Independent Sets

Def. A vertex set I in a graph is called **independent** (or **stable**) if no pair of vertices in I forms an edge. An independent set is called **maximal** if no superset of it is independent.

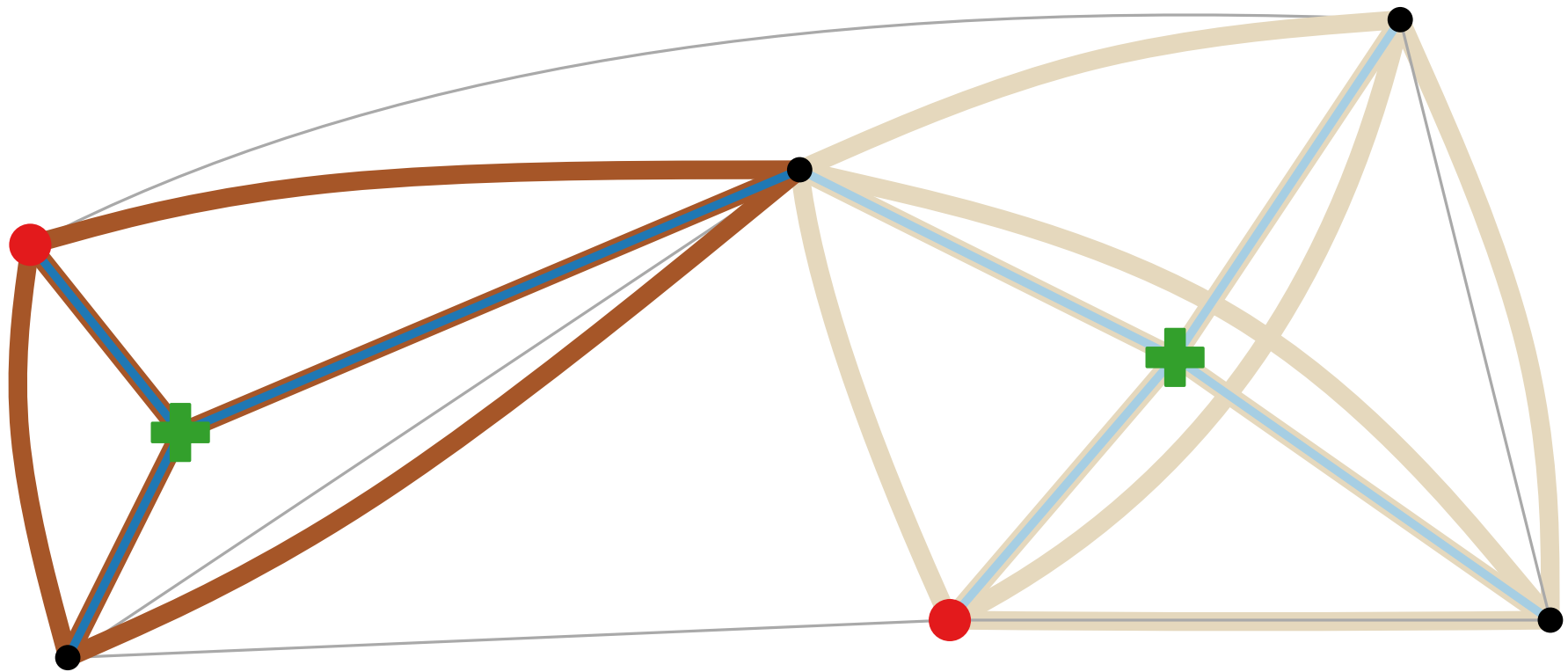
Obs. Maximal independent sets are dominating sets :-)



Independent Sets in H^2

Lemma. For a graph H and an independent set I in H^2 ,
 $|I| \leq \text{dom}(H)$.

What does a dominating set of H look like in H^2 ?



Star in H

Clique in H^2

Approximation Algorithms

Lecture 6:

k -CENTER via Parametric Pruning

Part IV:

Factor-2 Approximation for METRIC- k -CENTER

Factor-2 Approx. for Metric k -CENTER

Metric- k -CENTER($G = (V, E; c), k$)

Sort the edges of G by cost: $c(e_1) \leq \dots \leq c(e_m)$

for $j = 1$ **to** m **do**

 Construct G_j^2

 Find a maximal independent set I_j in G_j^2

if $|I_j| \leq k$ **then**

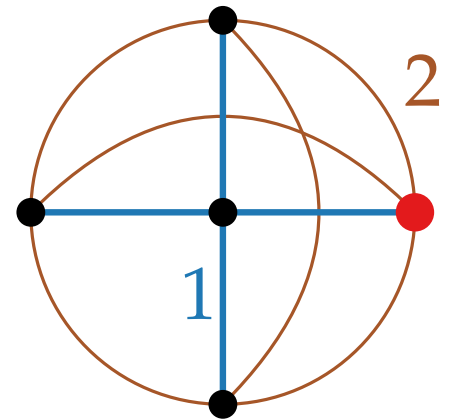
return I_j

Lemma. For j provided by the algorithm, it holds that $c(e_j) \leq \text{OPT}$.

Theorem. The above algorithm is a factor-2 approximation algorithm for the metric k -CENTER problem.

Can we do better ... ?

What about a tight example?



Theorem. Assuming $P \neq NP$, there is no factor- $(2 - \epsilon)$ approximation algorithm for the metric k -CENTER problem, for any $\epsilon > 0$.

Proof. Reduce from dominating set to metric k -CENTER. Given graph $G = (V, E)$ and integer k , construct complete graph $G' = (V, E \cup E')$

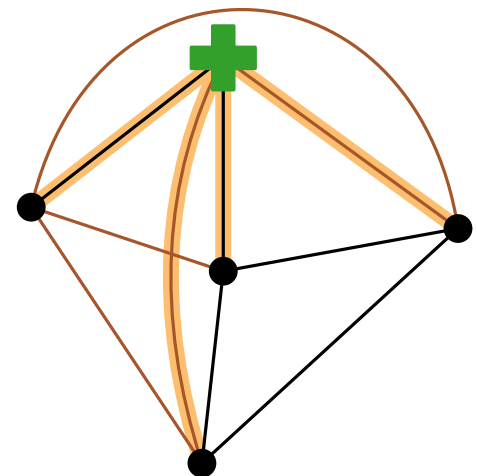
$$\text{with } c(e) = \begin{cases} 1, & \text{if } e \in E \\ 2, & \text{if } e \in E' \end{cases}$$

Δ -inequality holds

Let S be a metric k -center of G' .

If $\text{dom}(G) \leq k$, then $\text{cost}(S) = 1$.

If $\text{dom}(G) > k$, then $\text{cost}(S) = 2$.



Approximation Algorithms

Lecture 6:

k -CENTER via Parametric Pruning

Part V:

METRIC-WEIGHTED-CENTER

METRIC-~~k~~-CENTER

WEIGHTED

Given: A complete graph $G = (V, E)$ with metric edge costs $c: E \rightarrow \mathbb{Q}_{\geq 0}$ and ~~a natural number $k \leq |V|$~~ , vertex weights $w: V \rightarrow \mathbb{Q}_{\geq 0}$ and a budget $W \in \mathbb{Q}_+$

For $S \subseteq V$, $c(v, S)$ is the cost of the cheapest edge from v to the a vertex in S .

vertex set S of weight at most W

Find: A ~~k -element vertex set S~~ such that $\text{cost}(S) := \max_{v \in V} c(v, S)$ is minimized.

Algorithm for the Weighted Version

Algorithm Metric-Weighted-CENTER

Sort the edges of G by cost : $c(e_1) \leq \dots \leq c(e_m)$

for $j = 1, \dots, m$ **do**

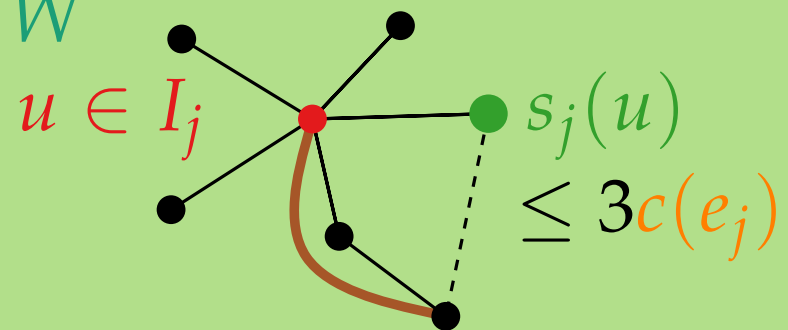
Construct G_j^2

Find a maximal independent set I_j in G_j^2

Compute $S_j := \{ s_j(u) \mid u \in I_j \}$

if $|I_j| \leq k$ **then** $w(S_j) \leq W$

return I_j S_j



$s_j(u) :=$ lightest node in $N_{G_j}(u) \cup \{u\}$

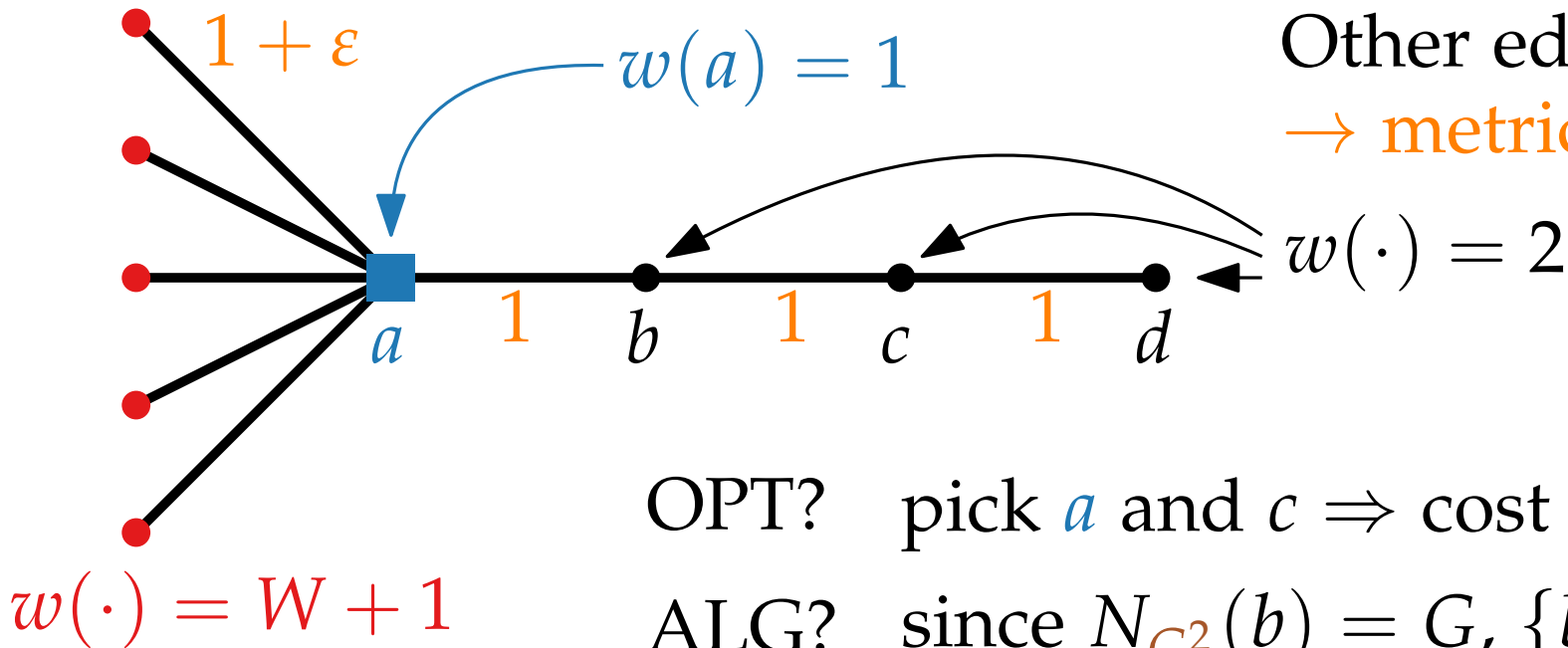
Theorem. The above is a factor-3 approximation algorithm for METRIC-WEIGHTED-CENTER.

Tight Example... ?

Here, we need to have a budget W ,
and edge costs satisfying the triangle inequality.

~~Consider $W = 3$.~~

Other edge costs?
→ metric completion!



OPT? pick a and $c \Rightarrow$ cost $1 + \varepsilon$.

ALG? since $N_{G^2}(b) = G$, $\{b\}$ is a
maximal independent set in G^2
Thus, alg. picks only $a \Rightarrow$ cost 3.

How can we generalize this to larger W ?