

Approximation Algorithms

Lecture 6:

k -CENTER via Parametric Pruning

Part I:

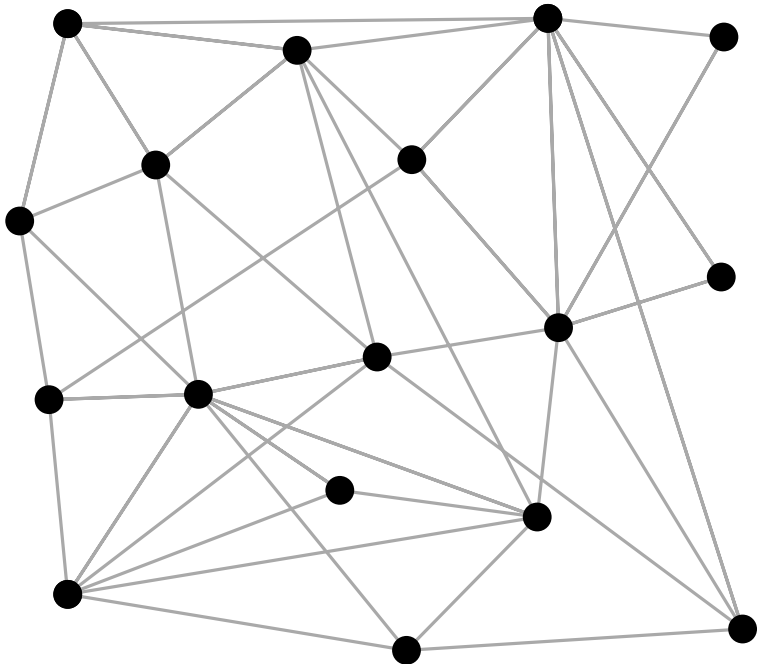
Metric k -CENTER

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Given: A graph $G = (V, E)$

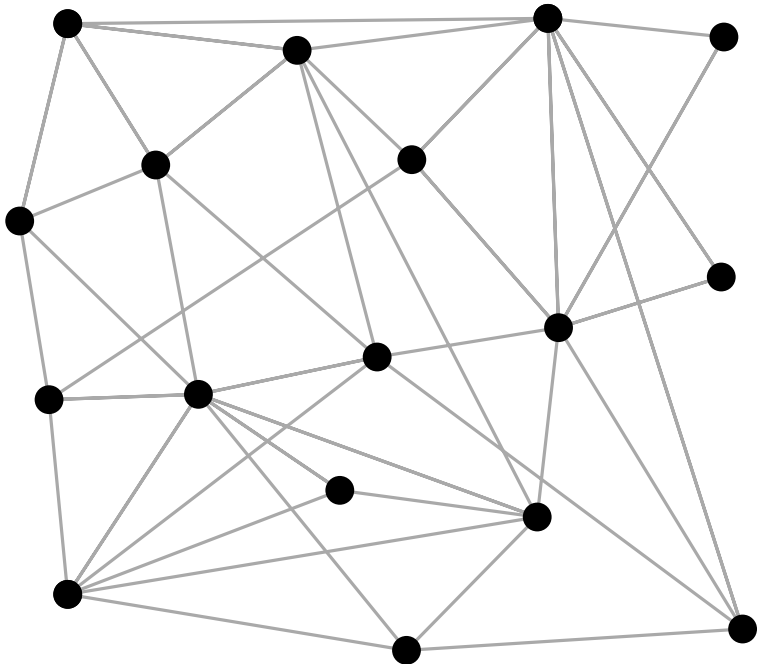
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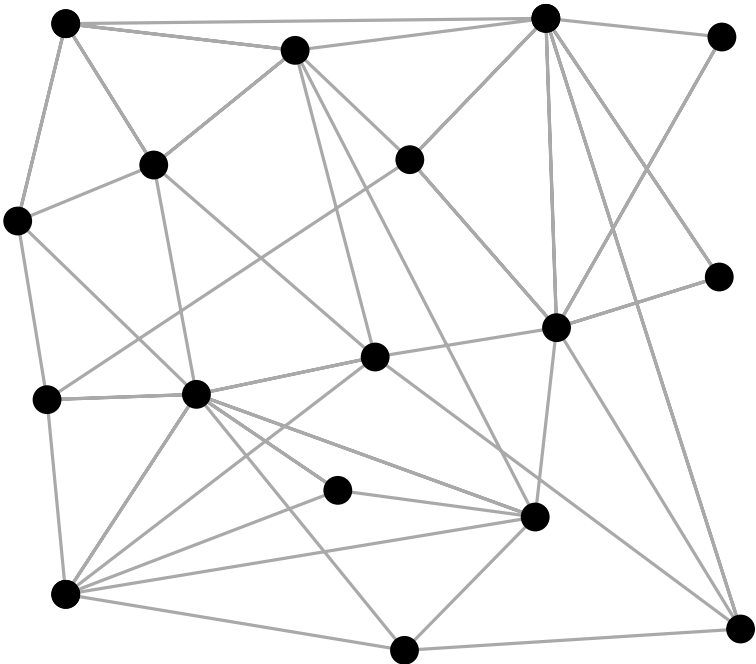
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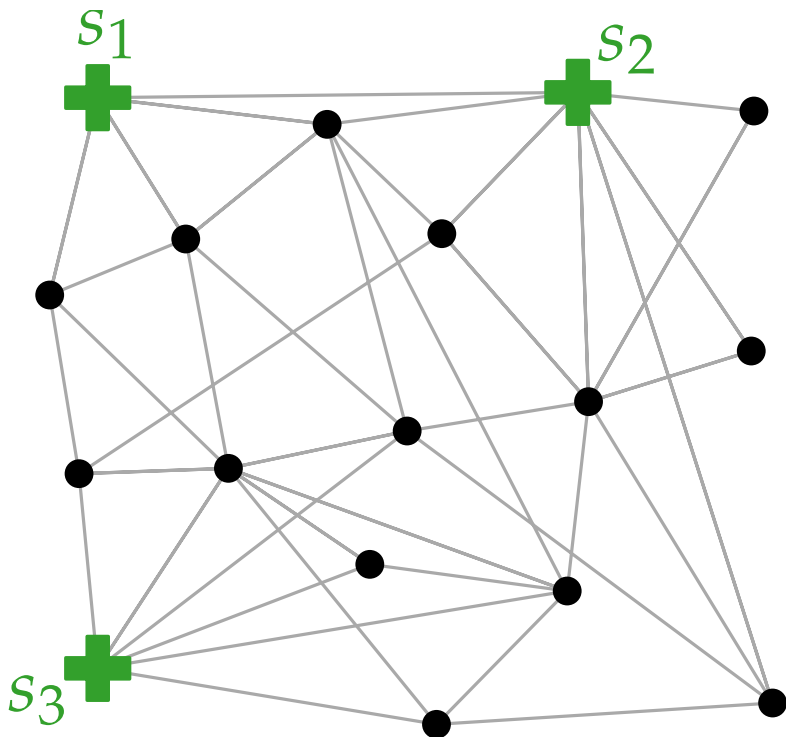
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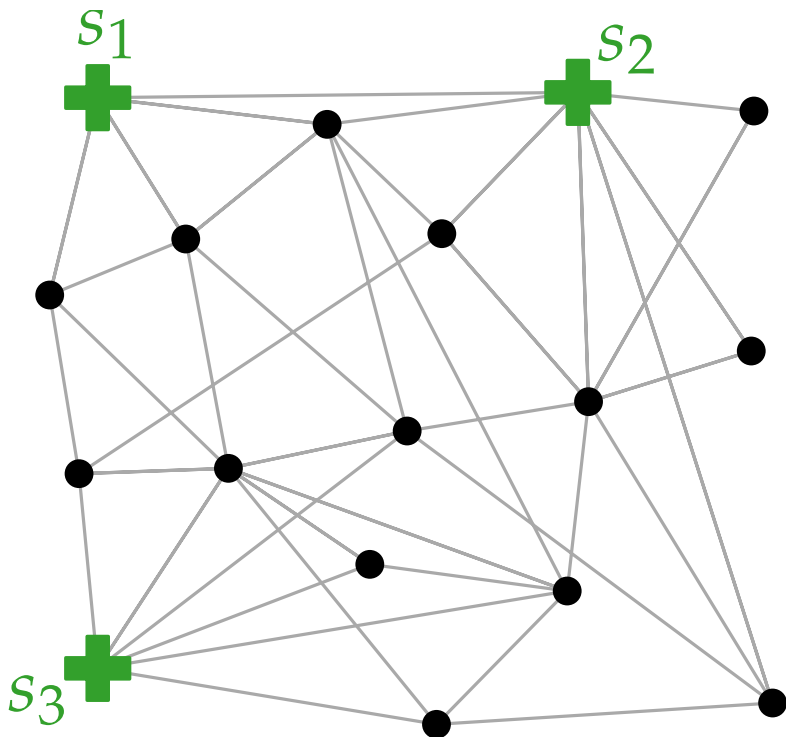
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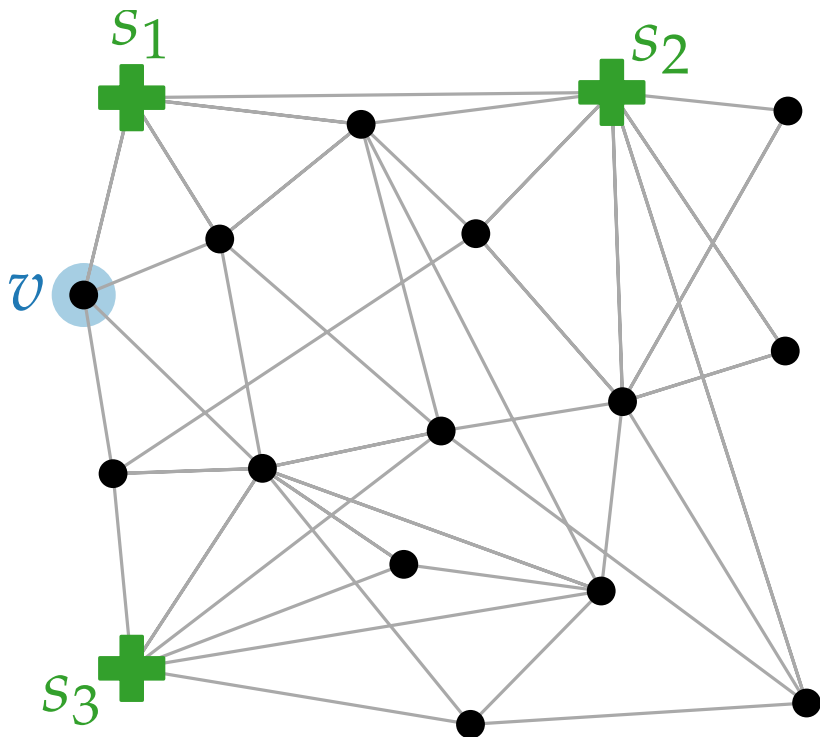
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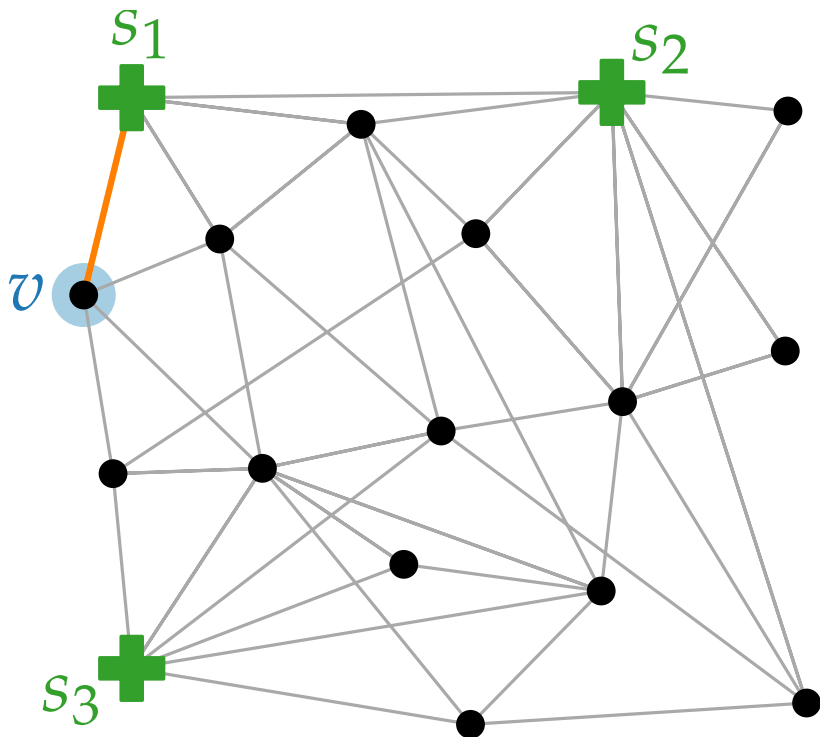
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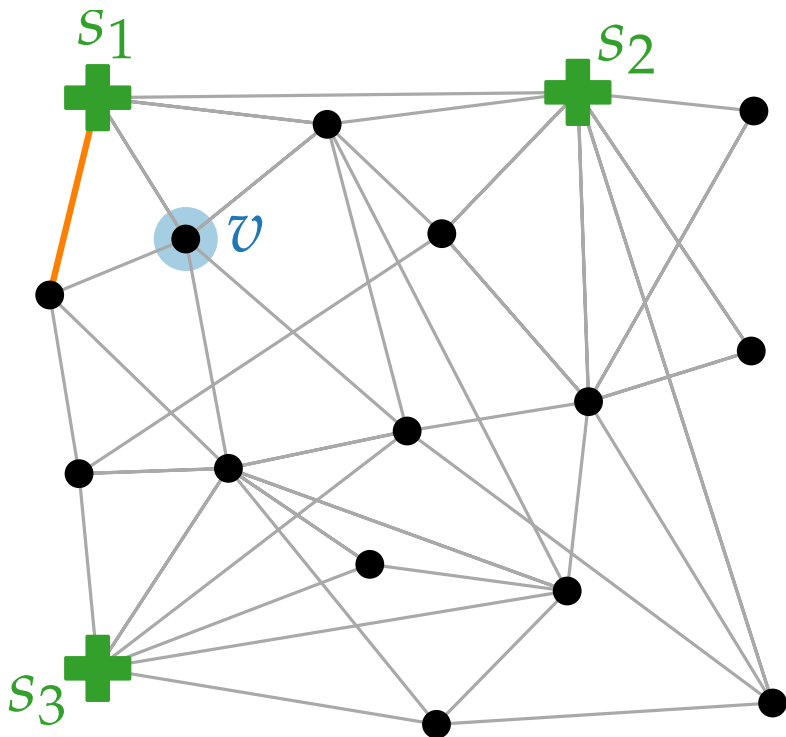
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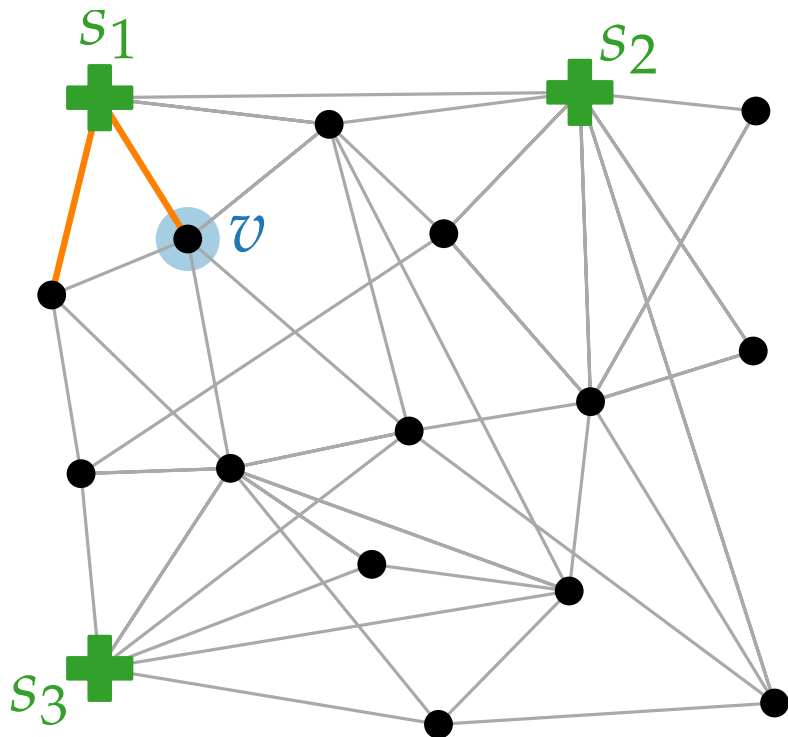
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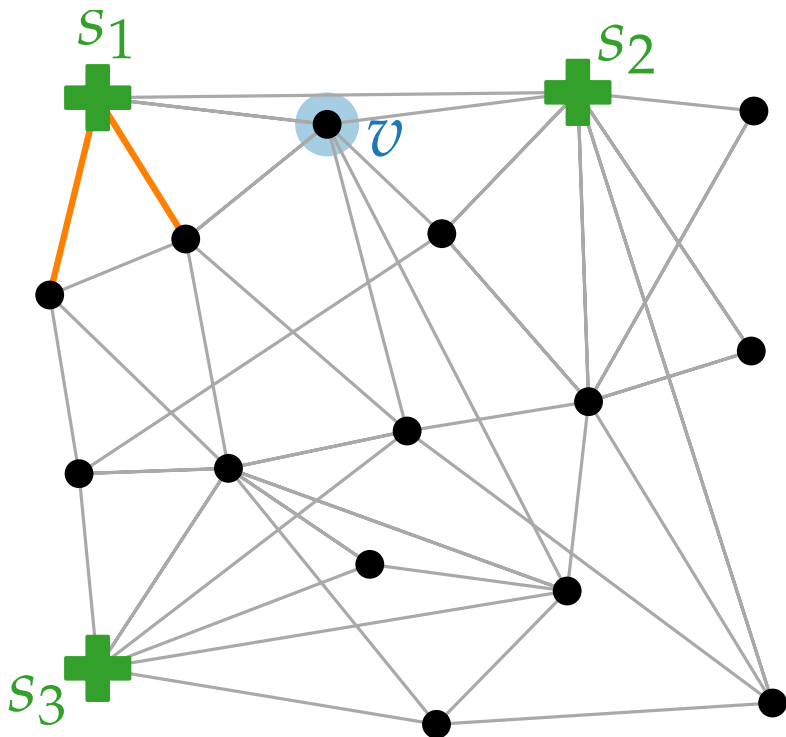
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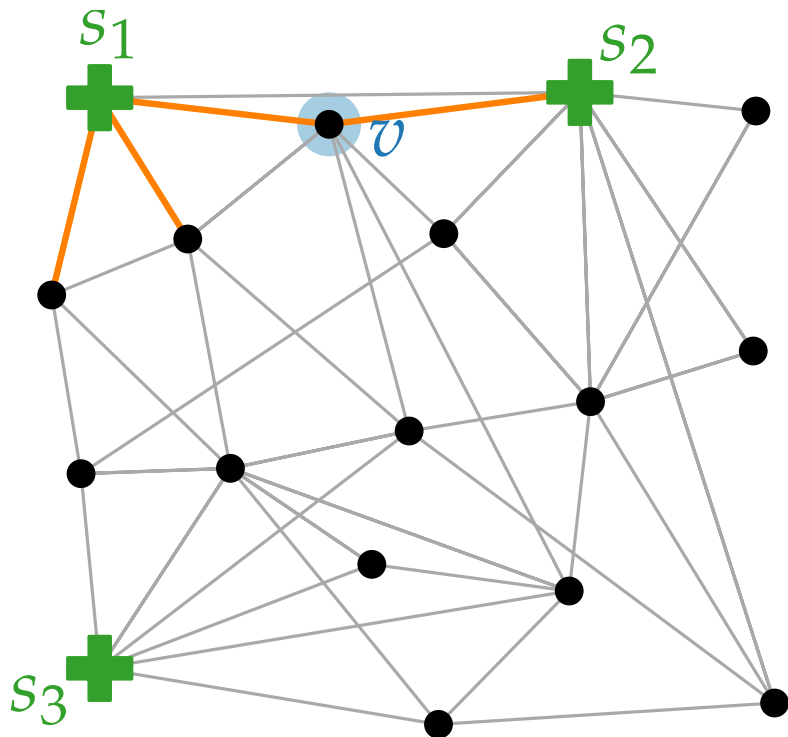
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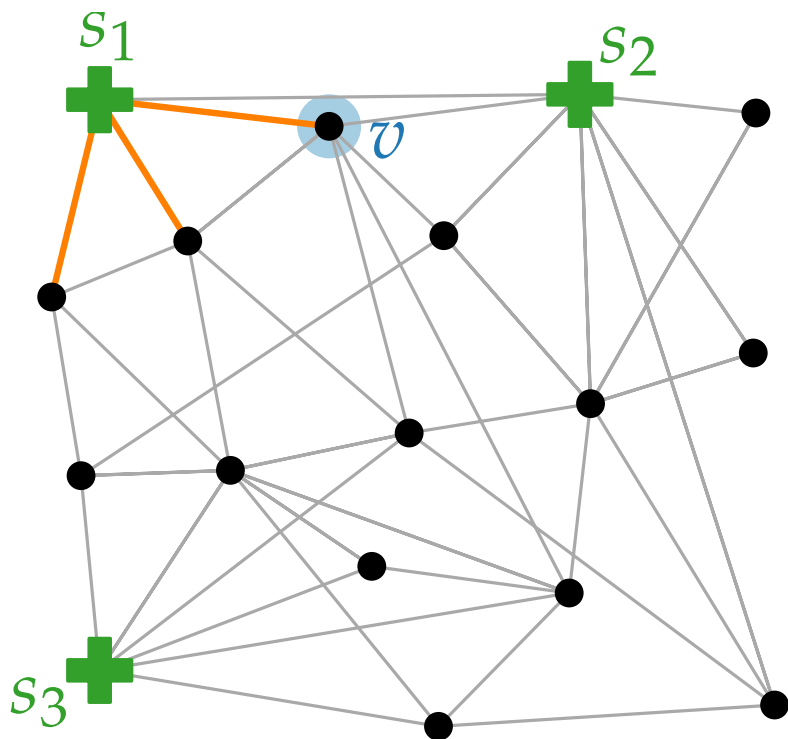
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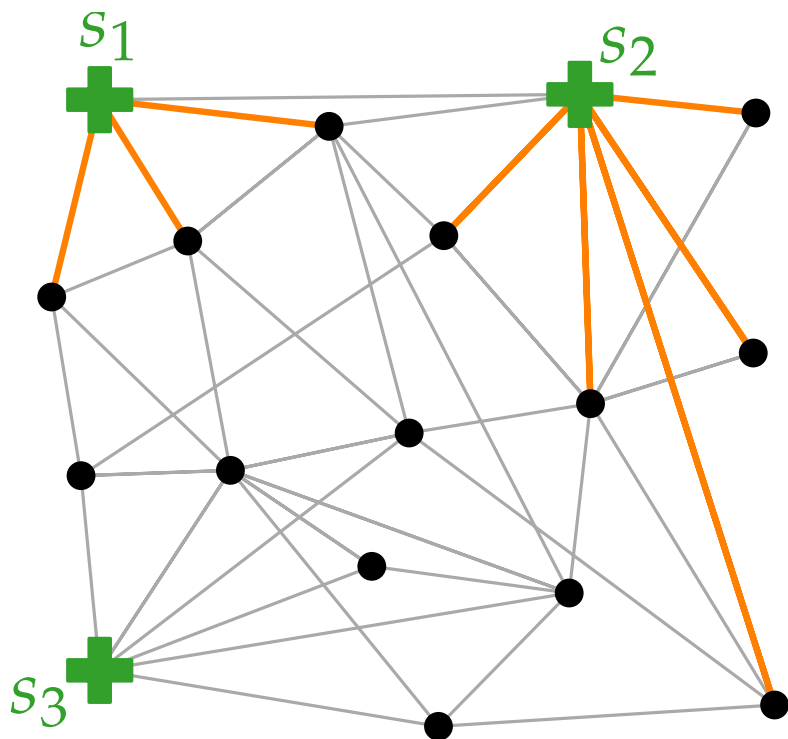
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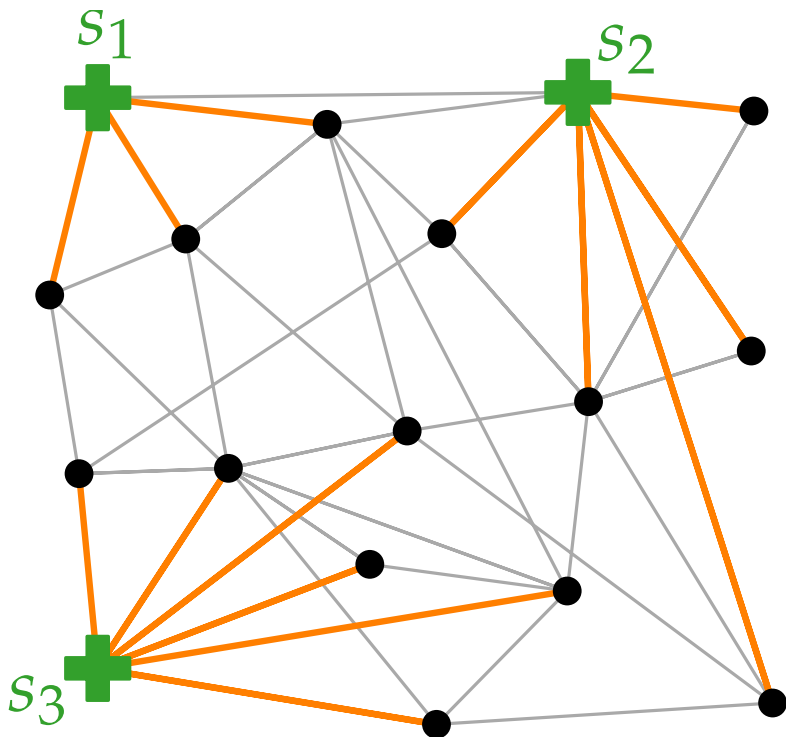
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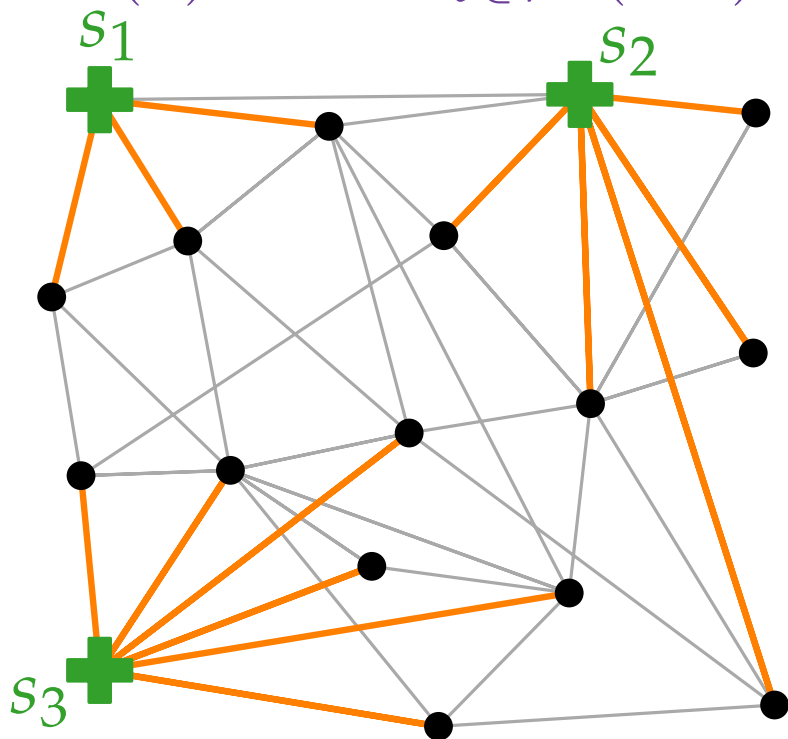


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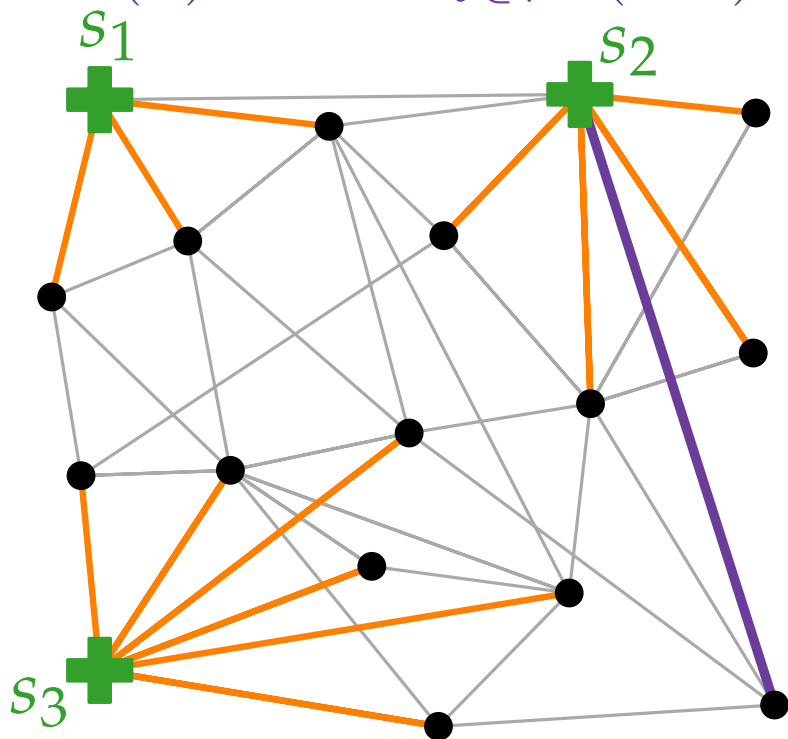


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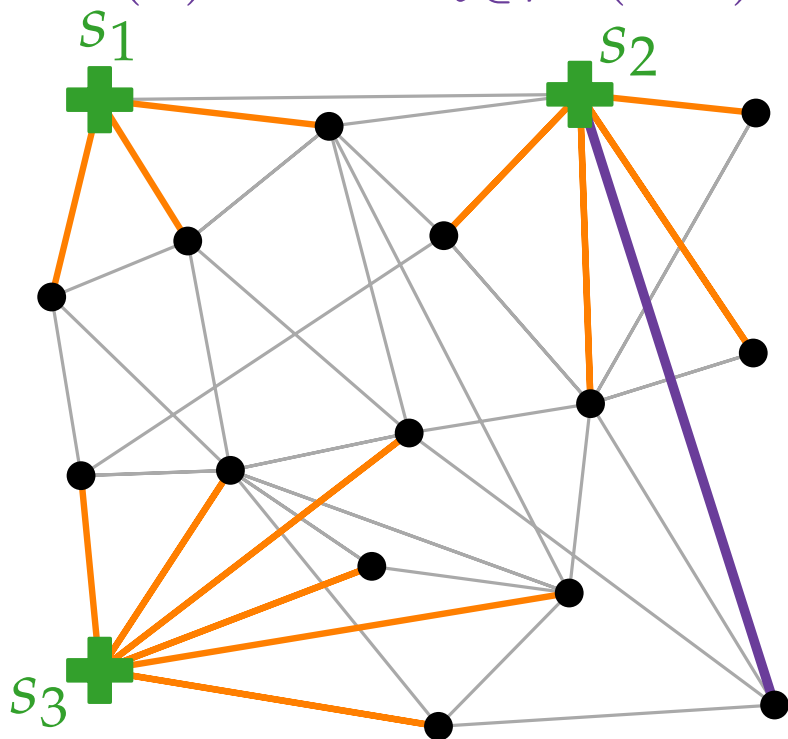


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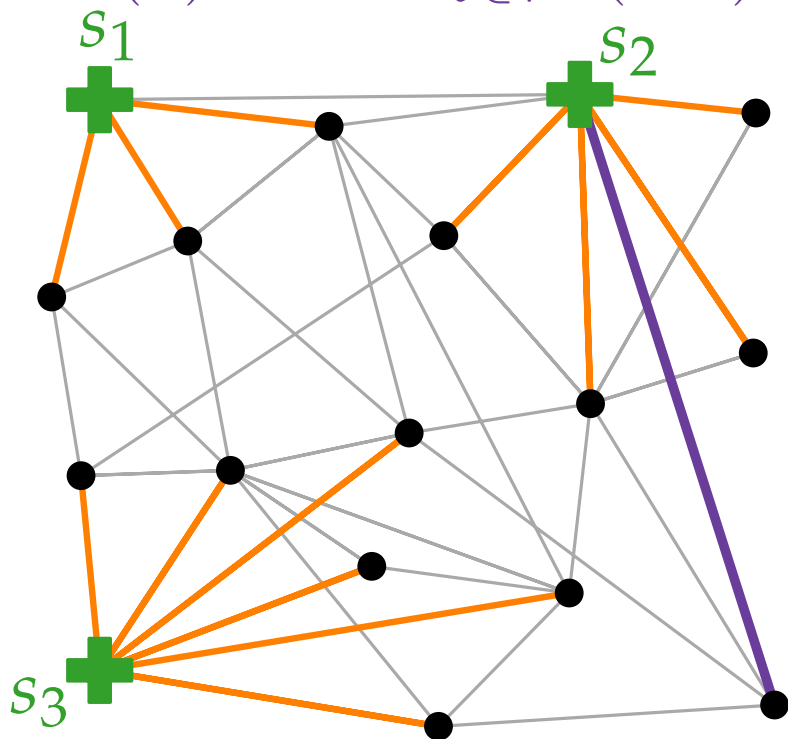


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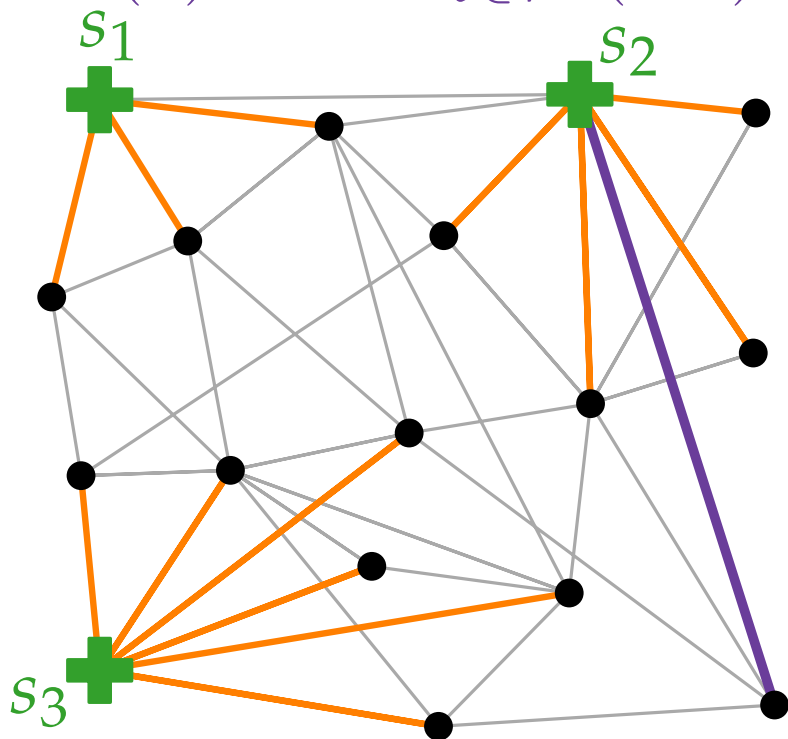


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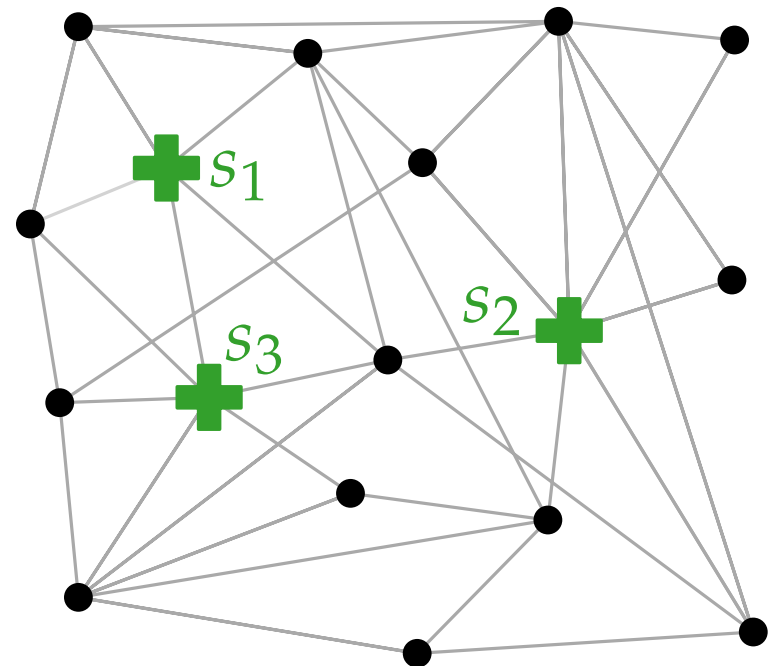
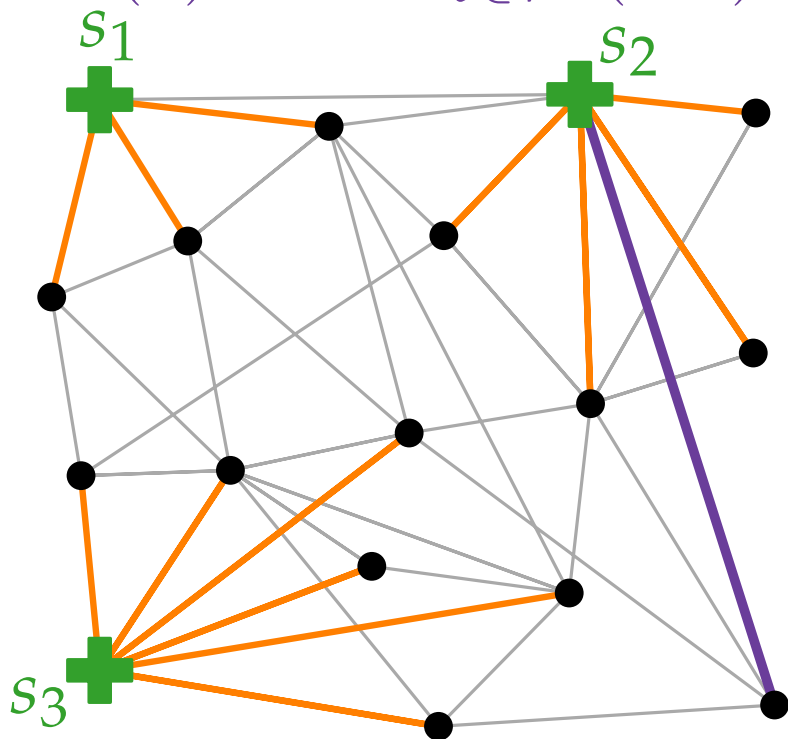


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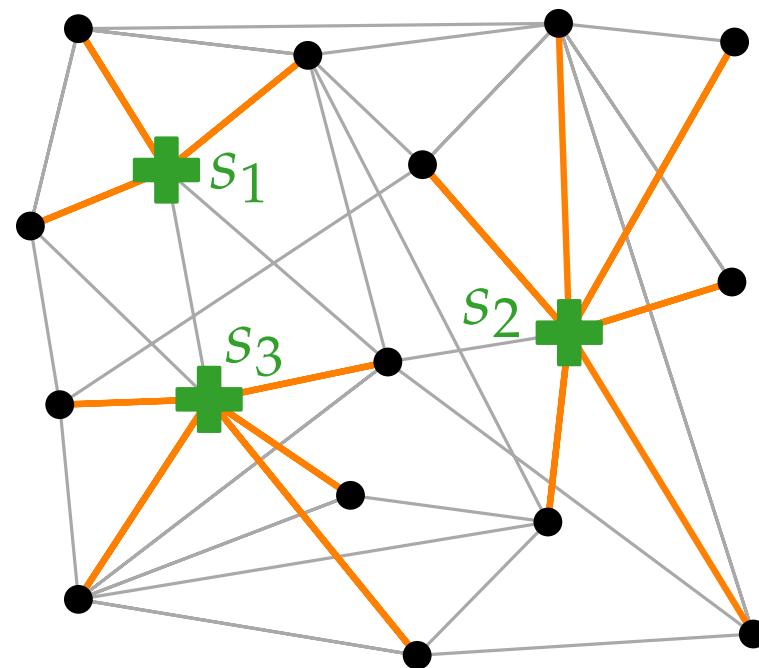
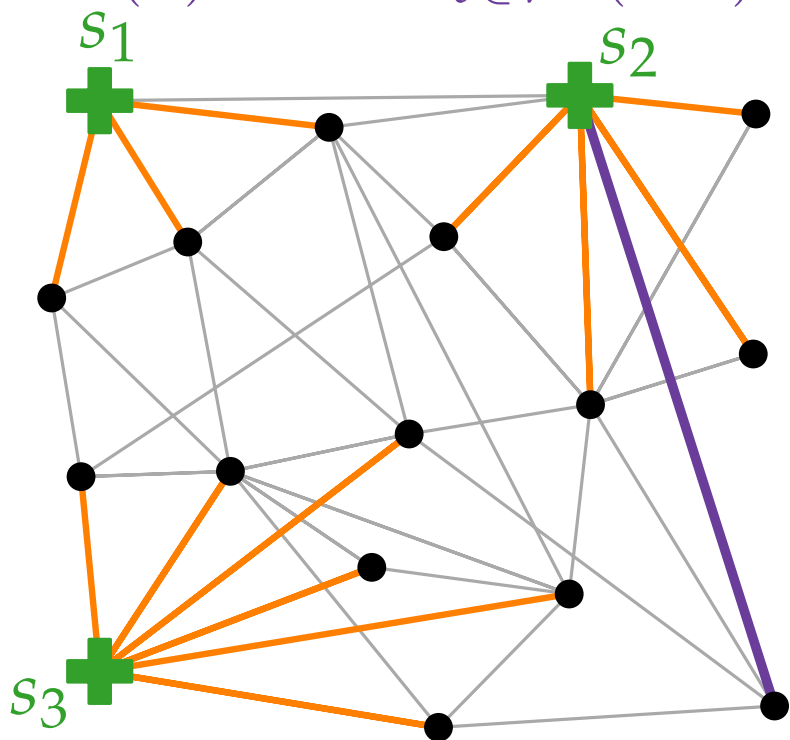


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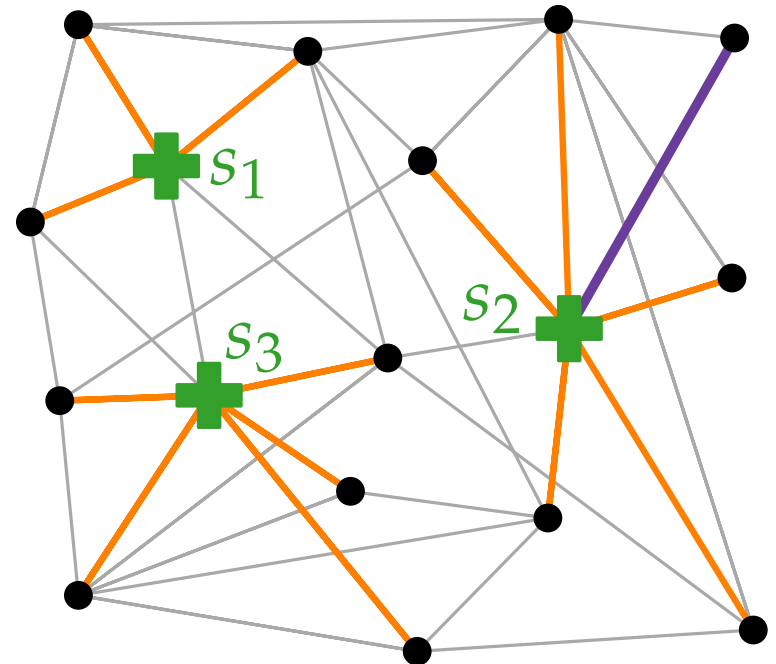
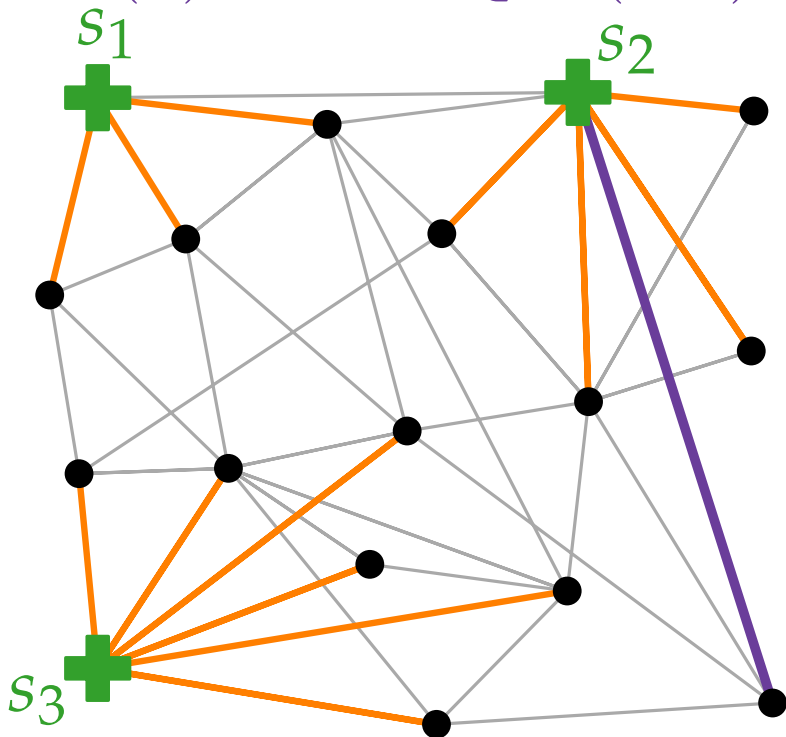


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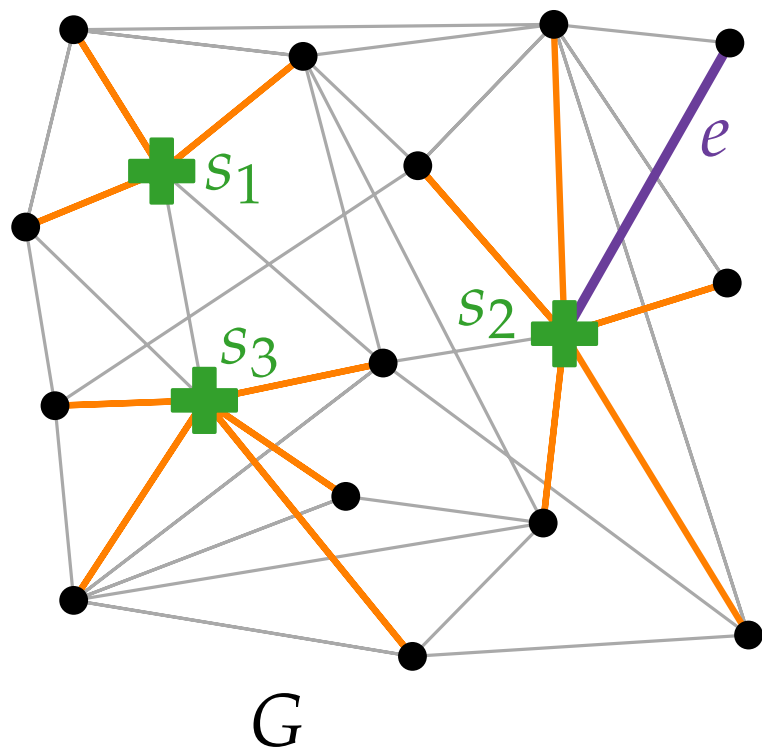
Lecture 6:

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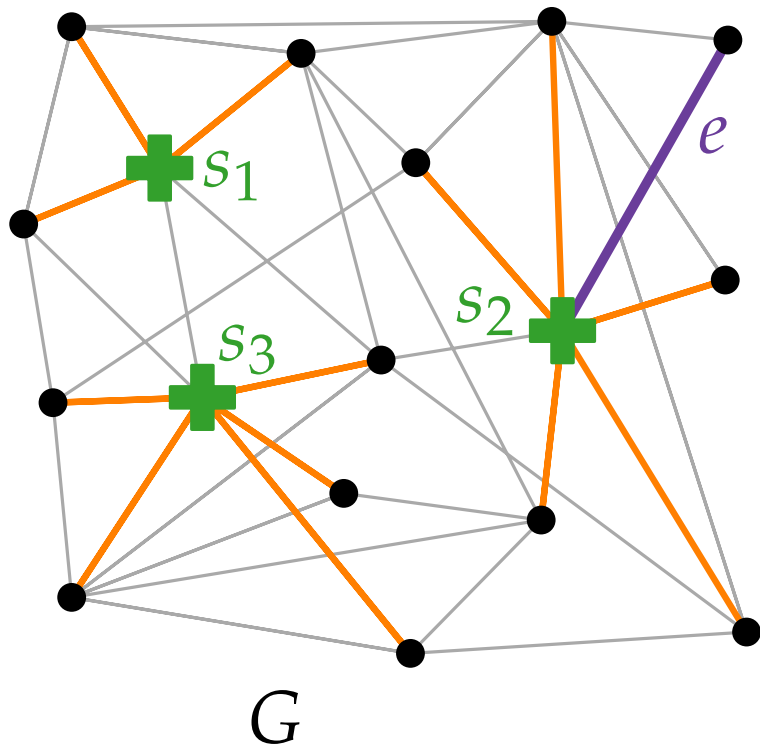
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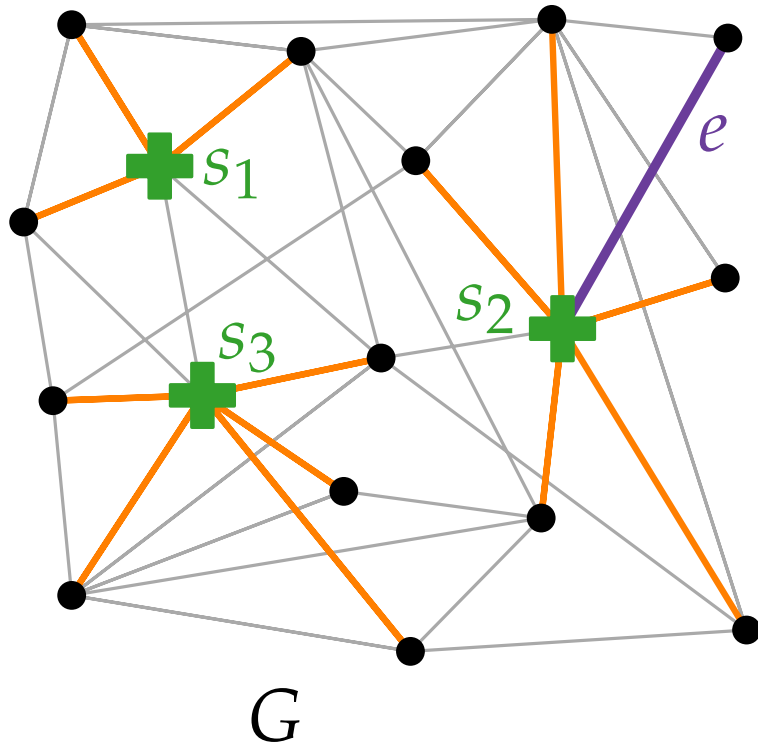
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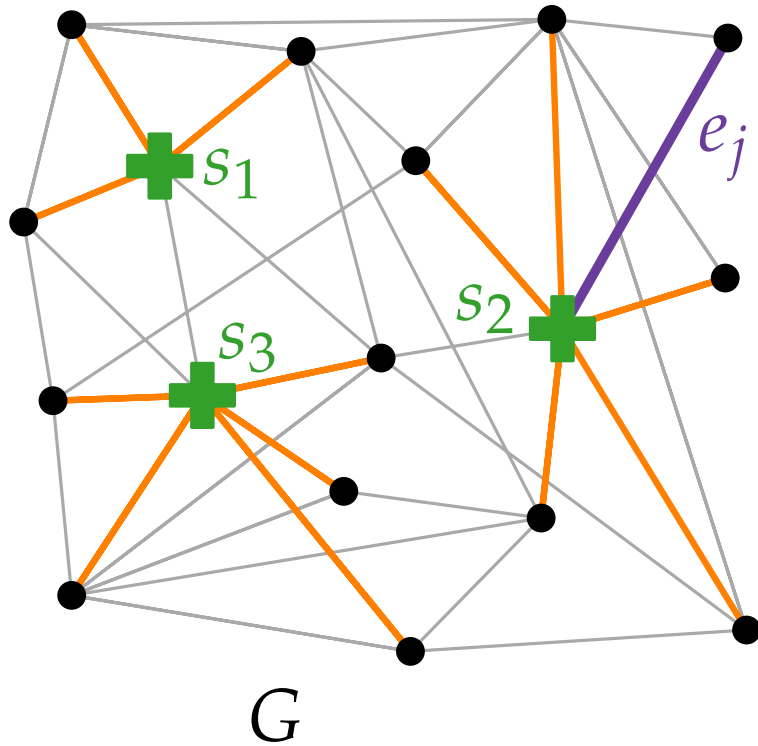
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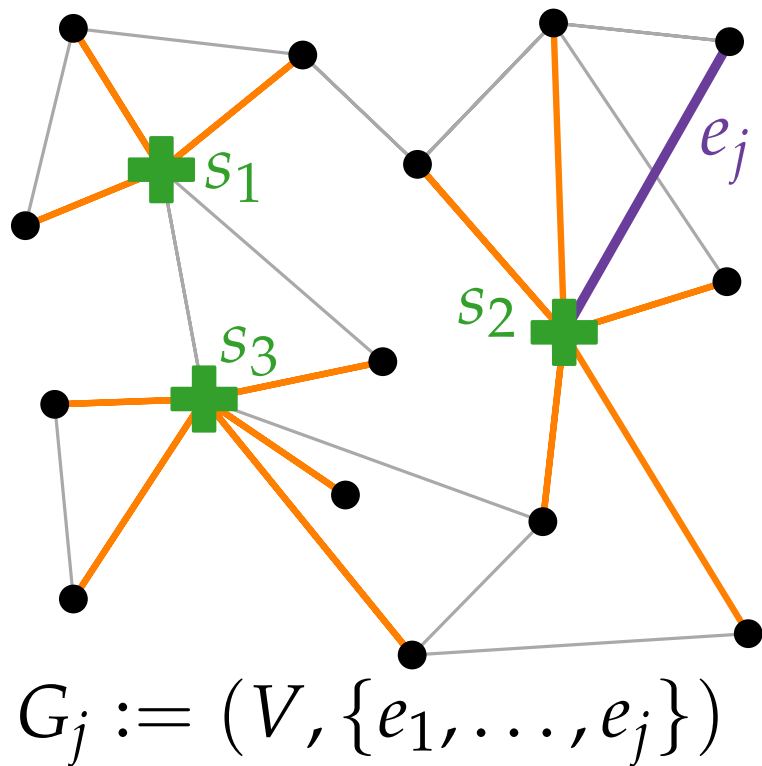
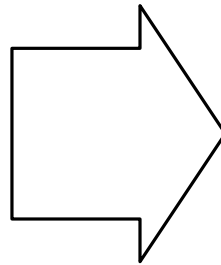
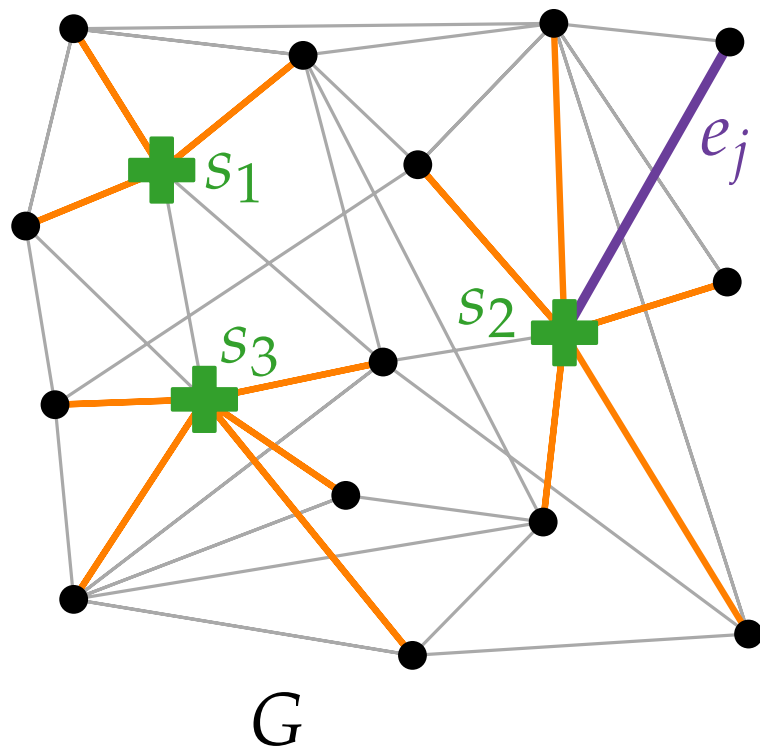
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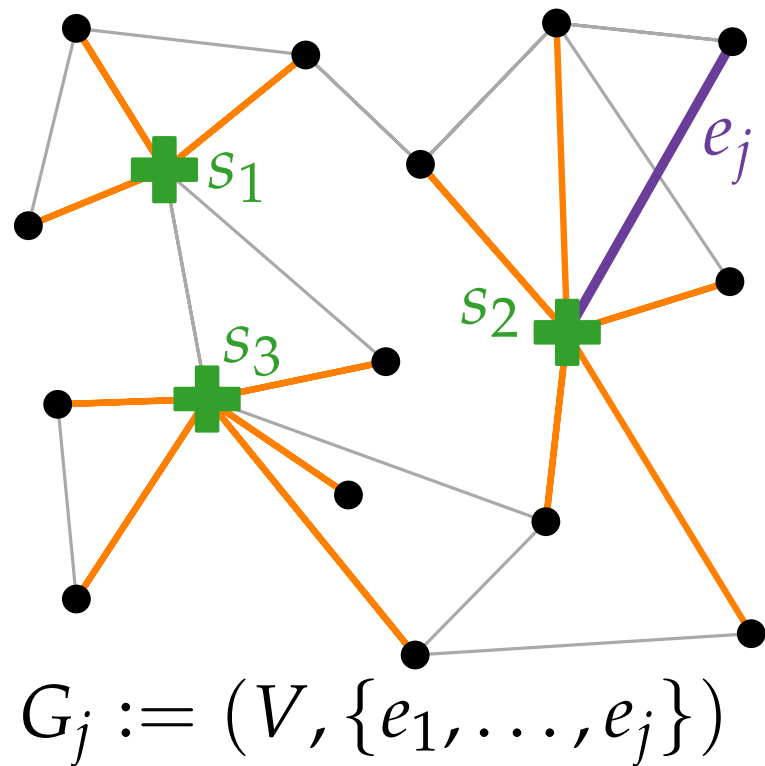
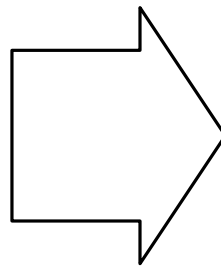
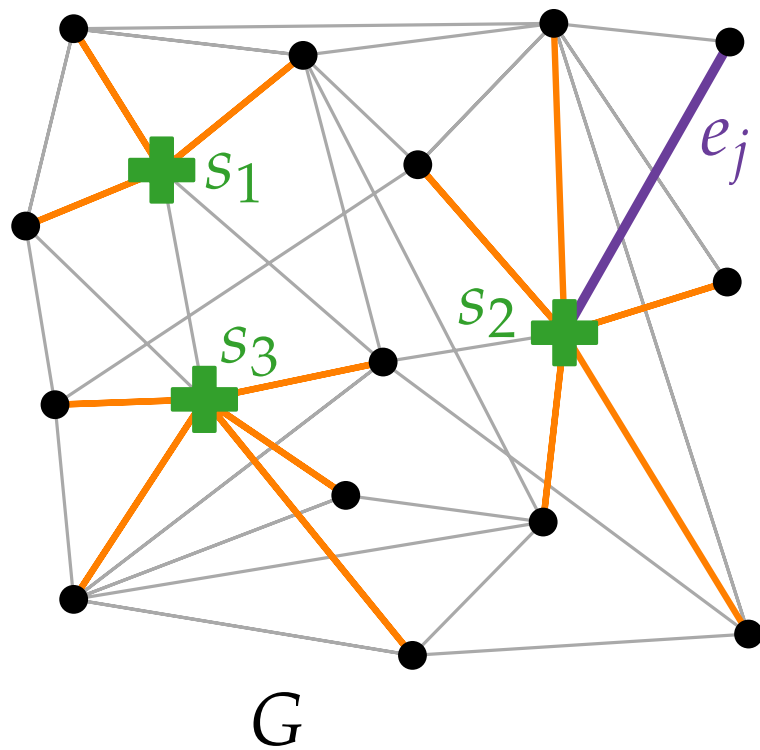
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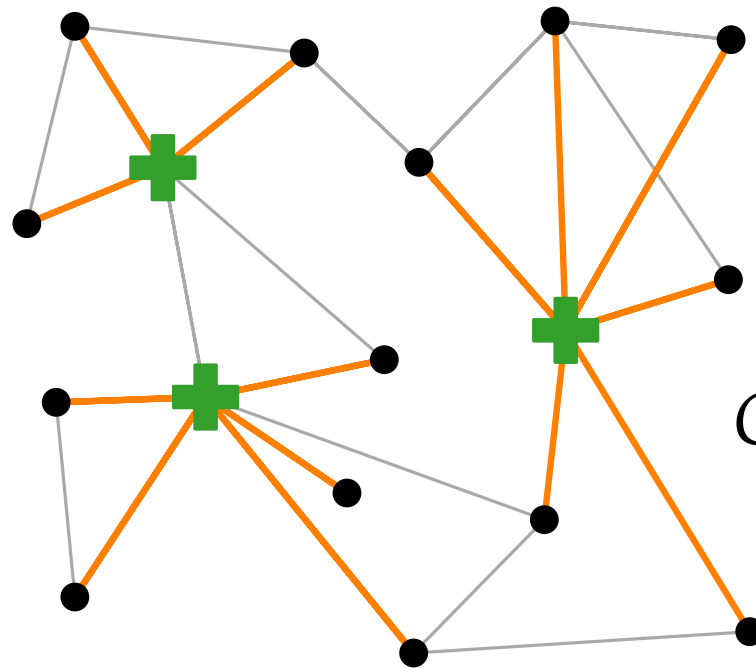
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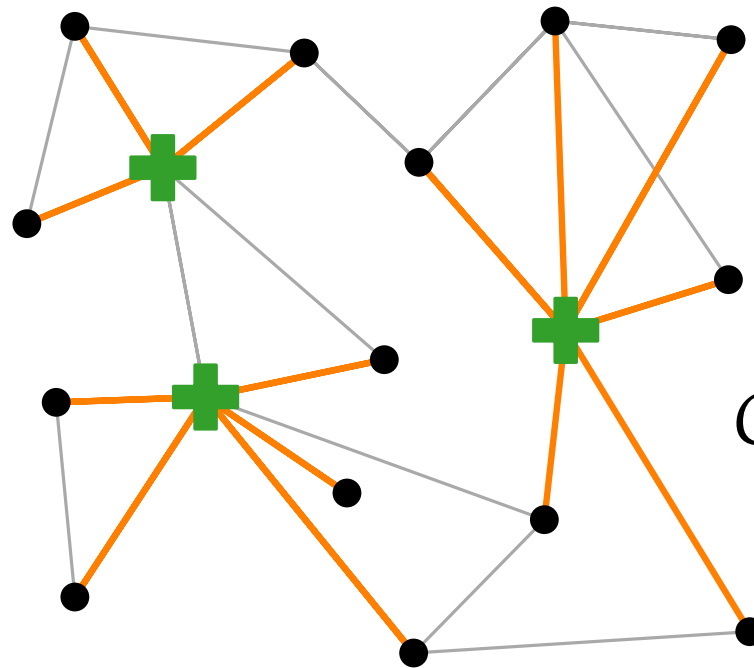
Def.



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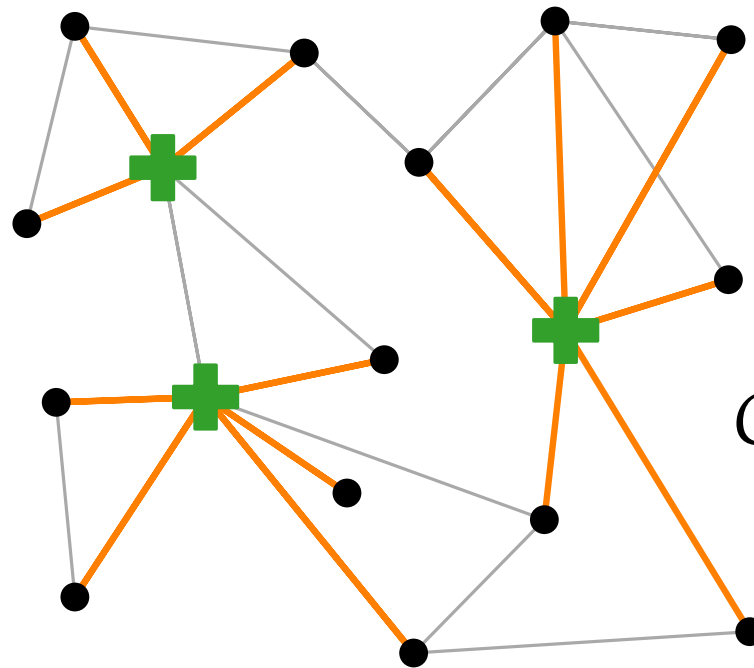
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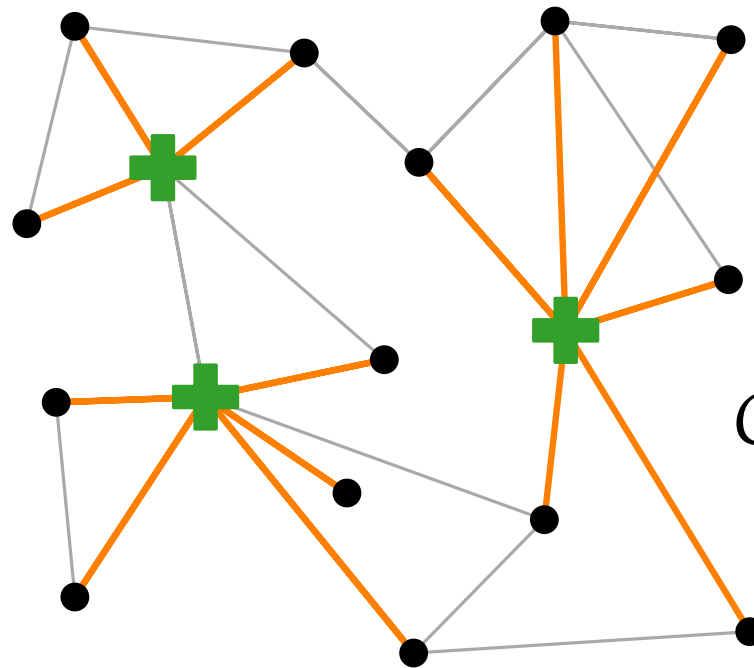
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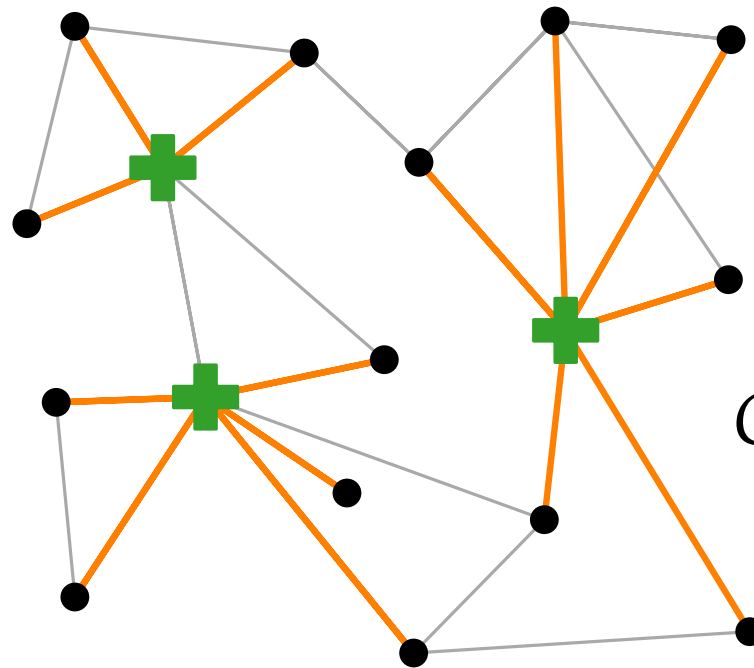


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... but computing $\text{dom}(H)$ is NP-hard.



Approximation Algorithms

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Part III:

Square of a Graph

Square of a Graph

Idea: Find a small dominating set in a “coarsened” G_j .

Square of a Graph

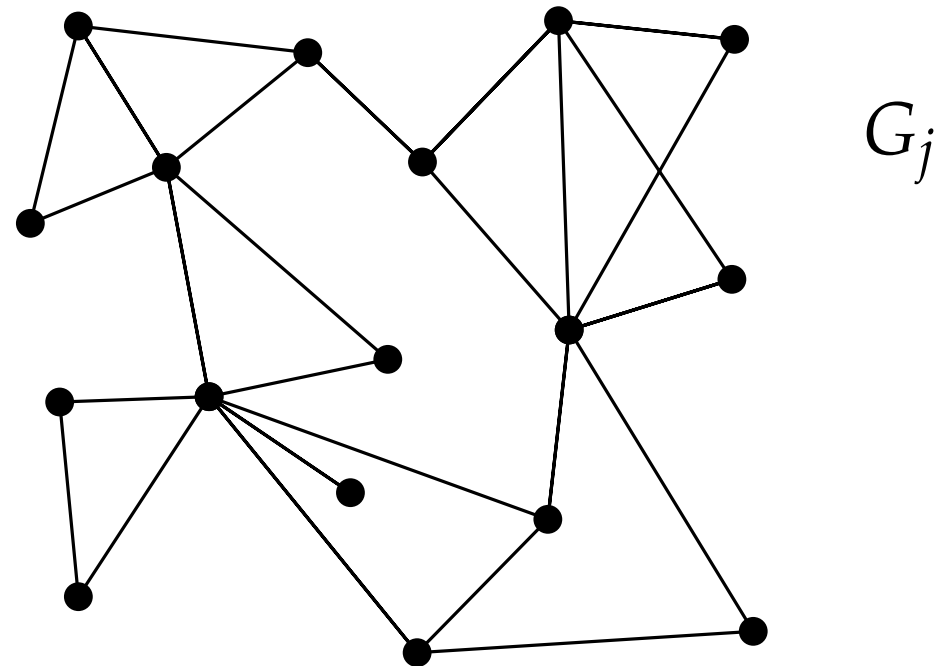
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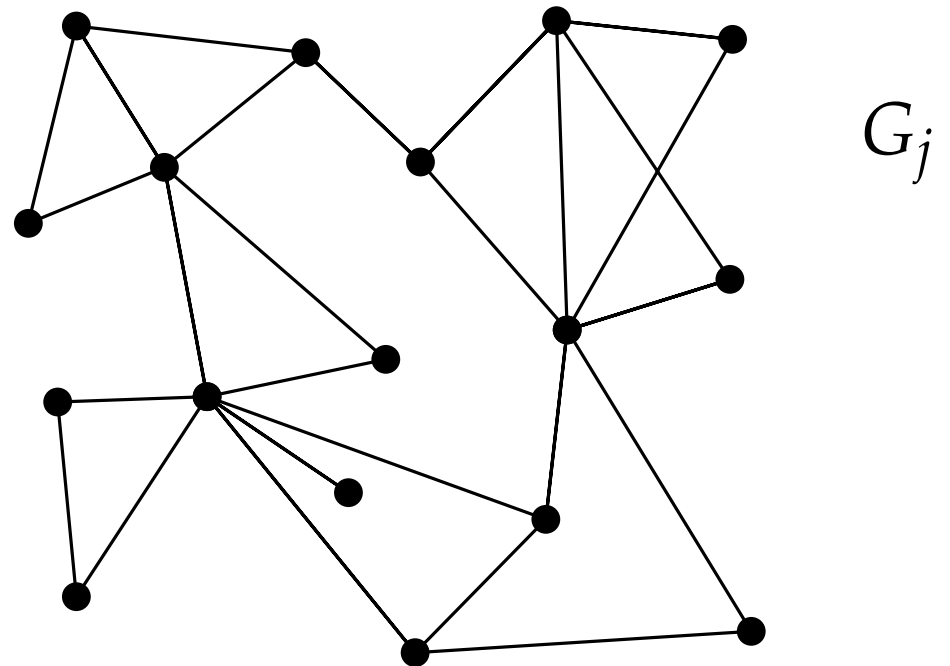
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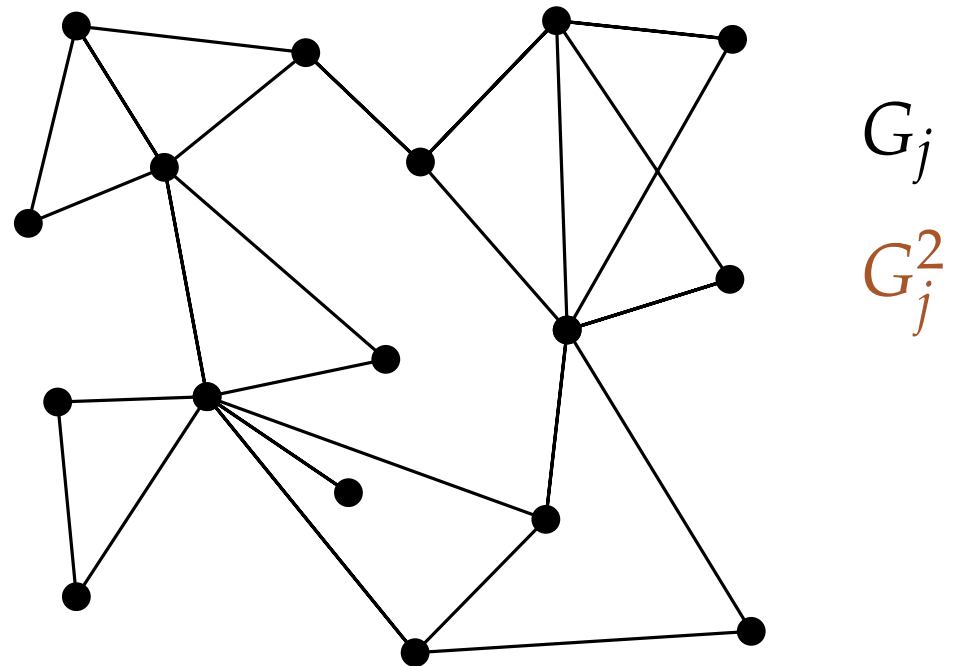
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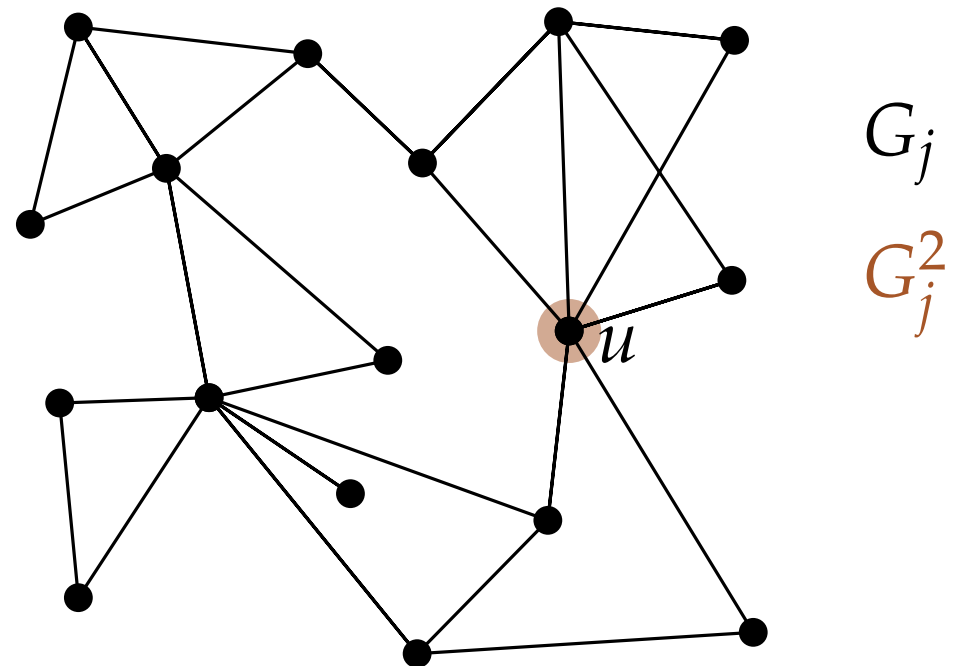
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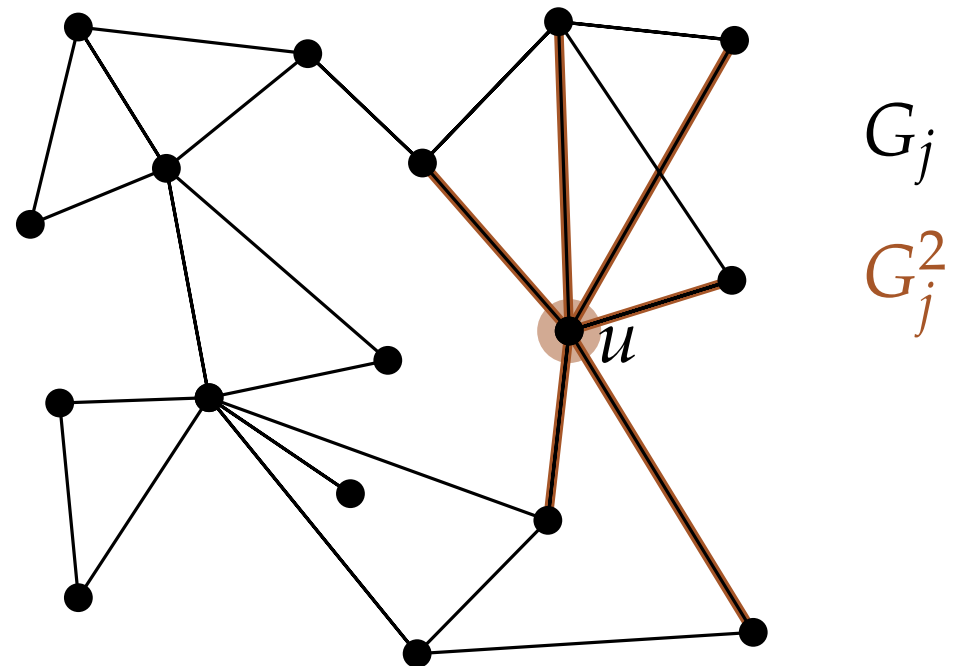
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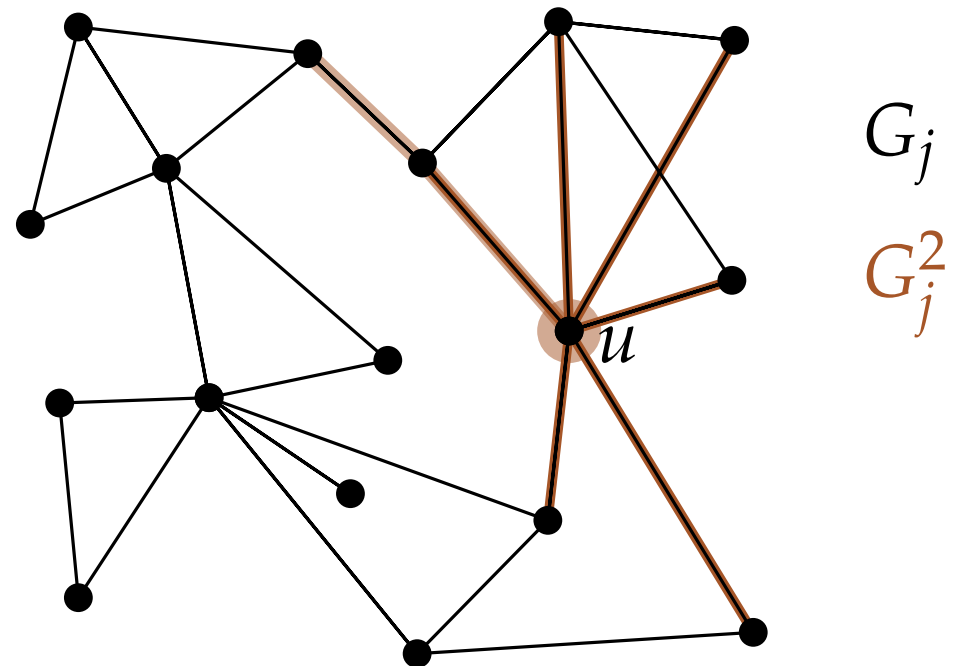
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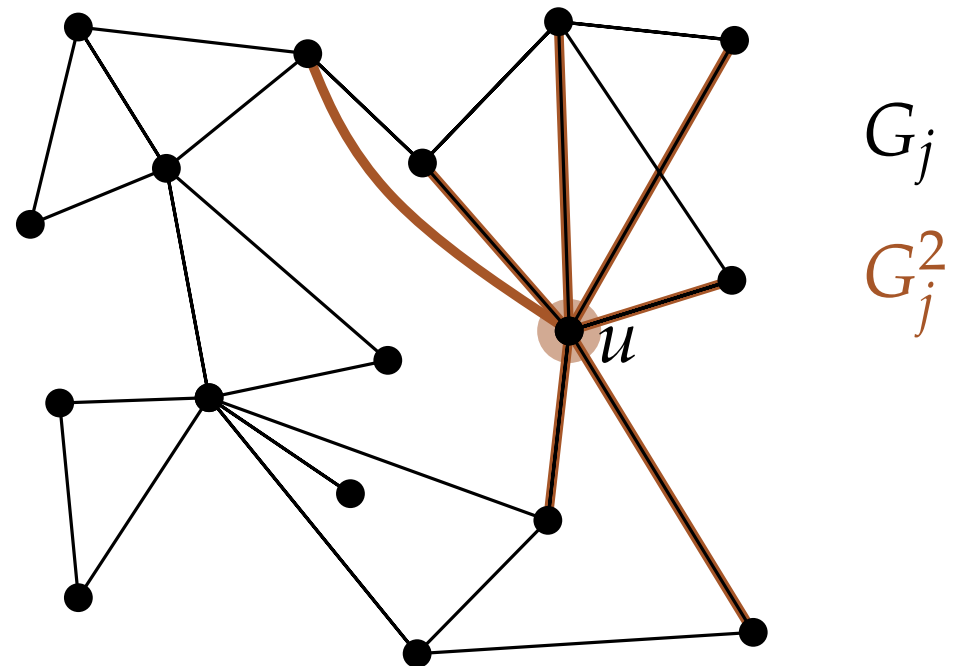
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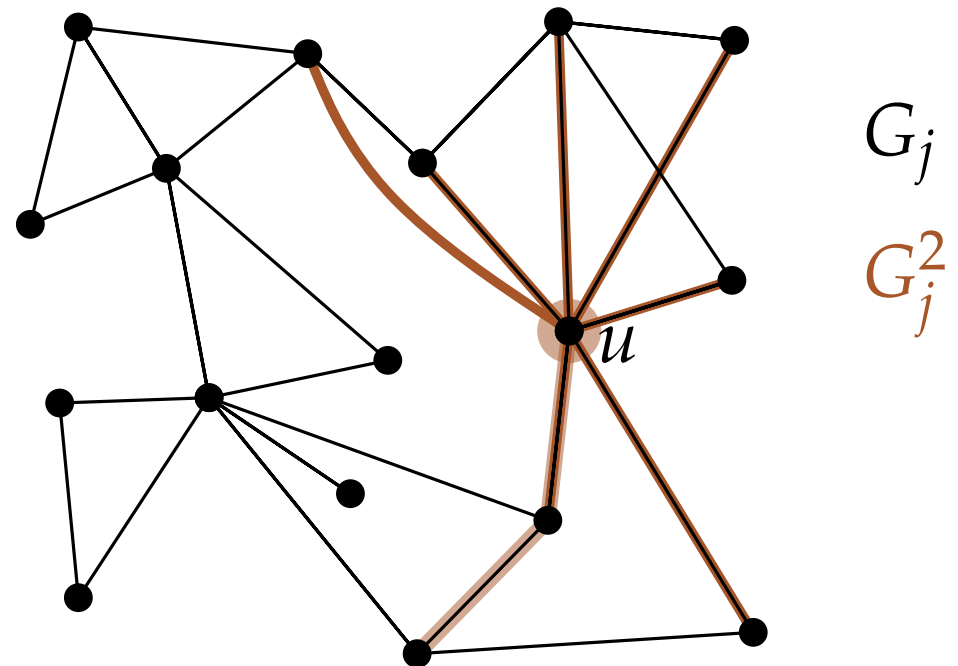
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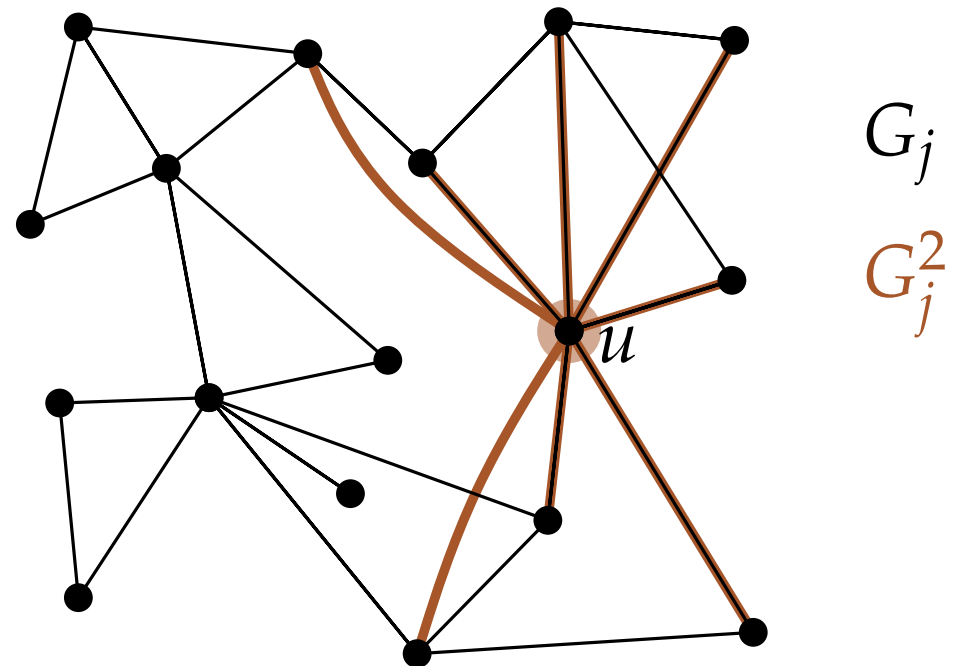
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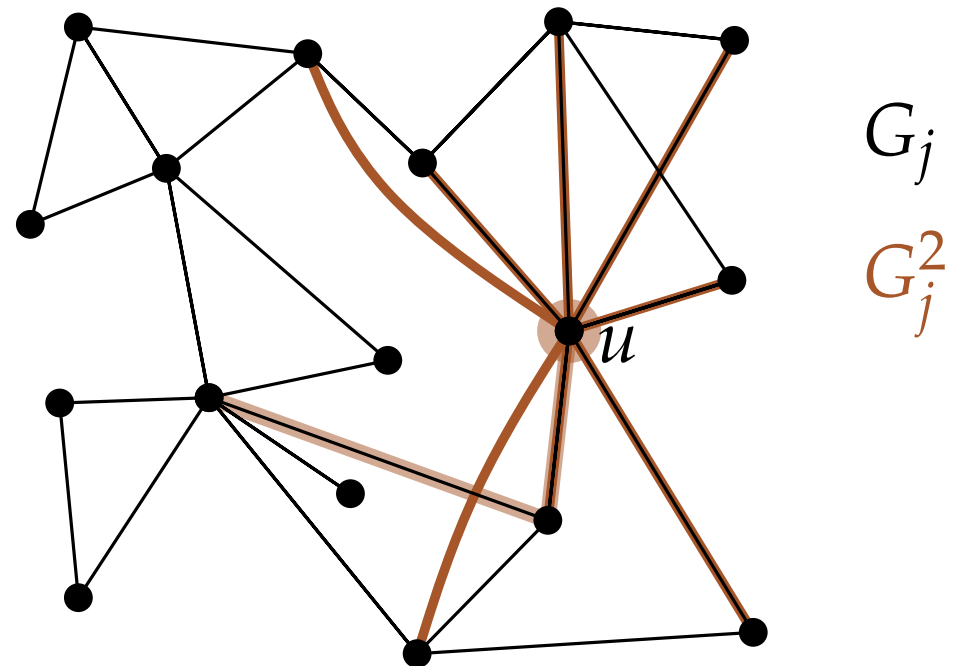
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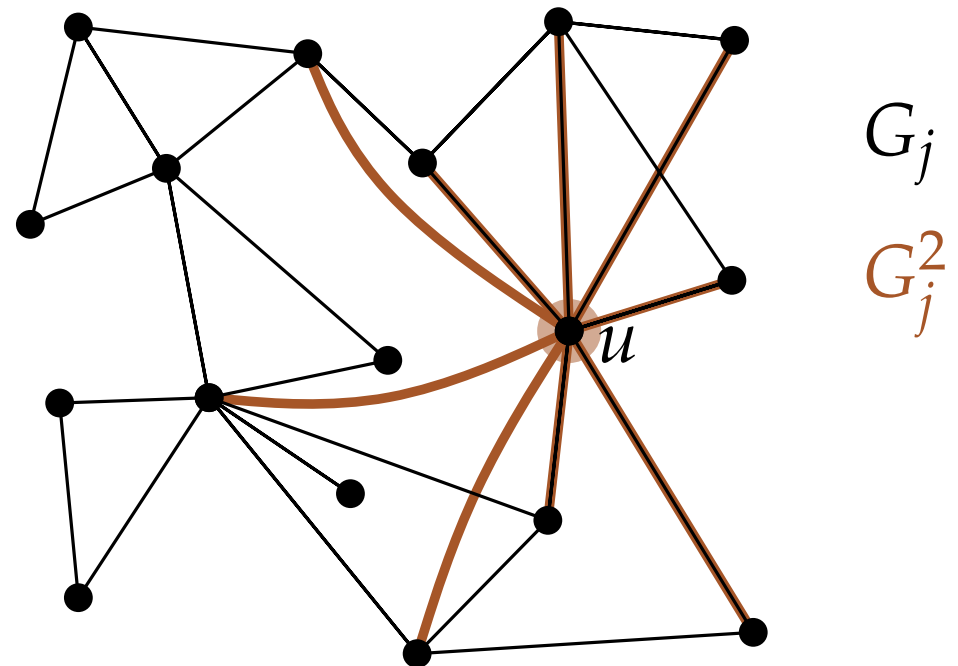
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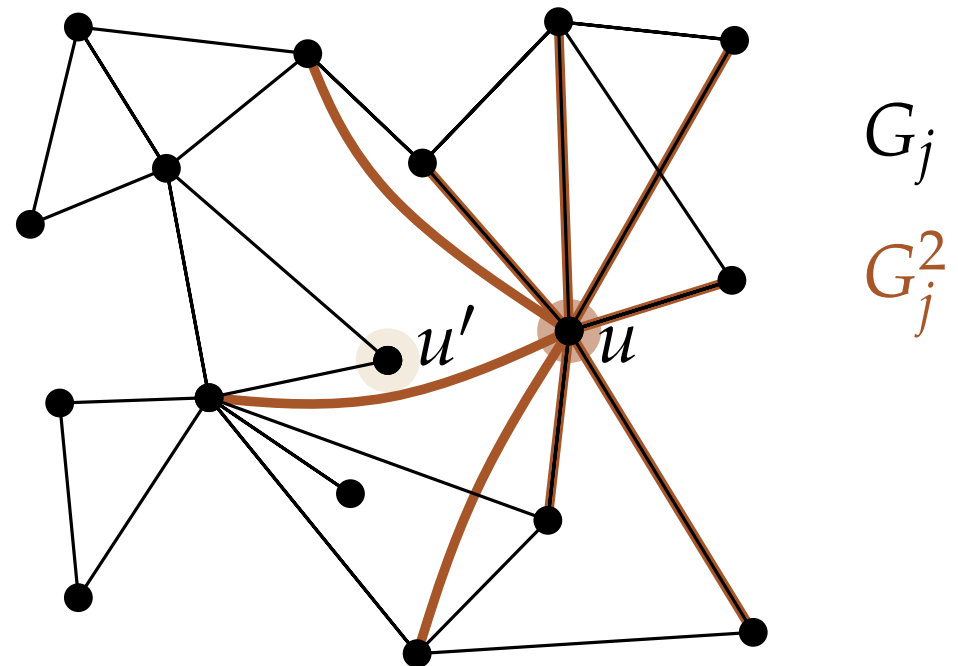
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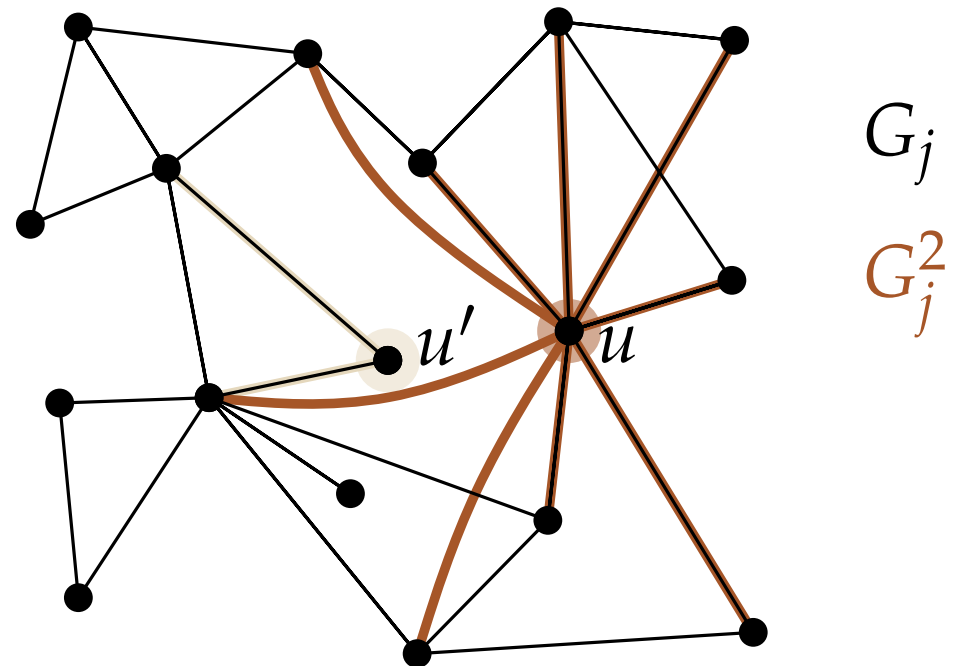
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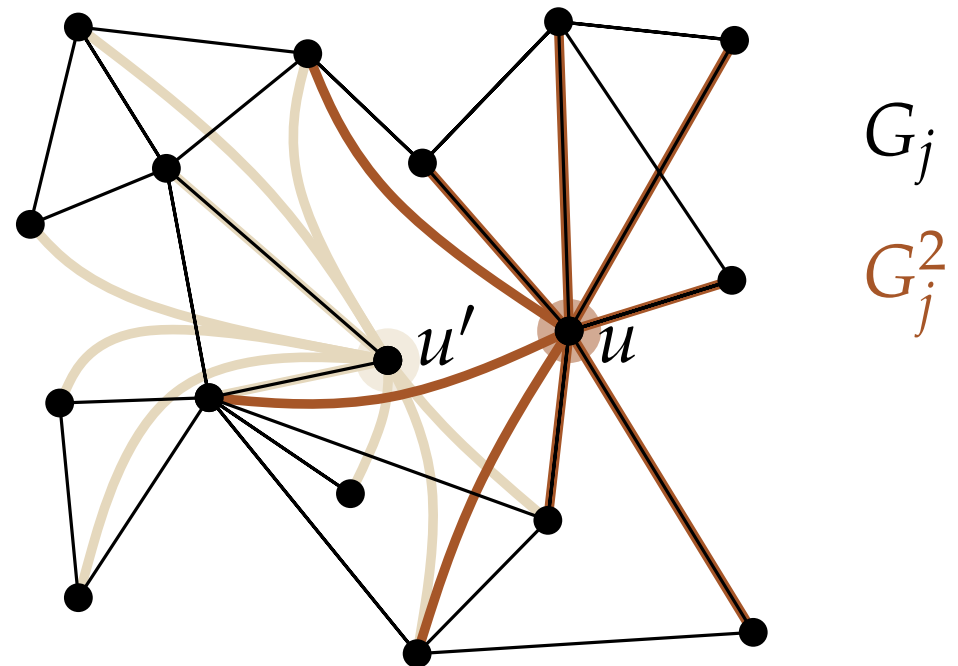
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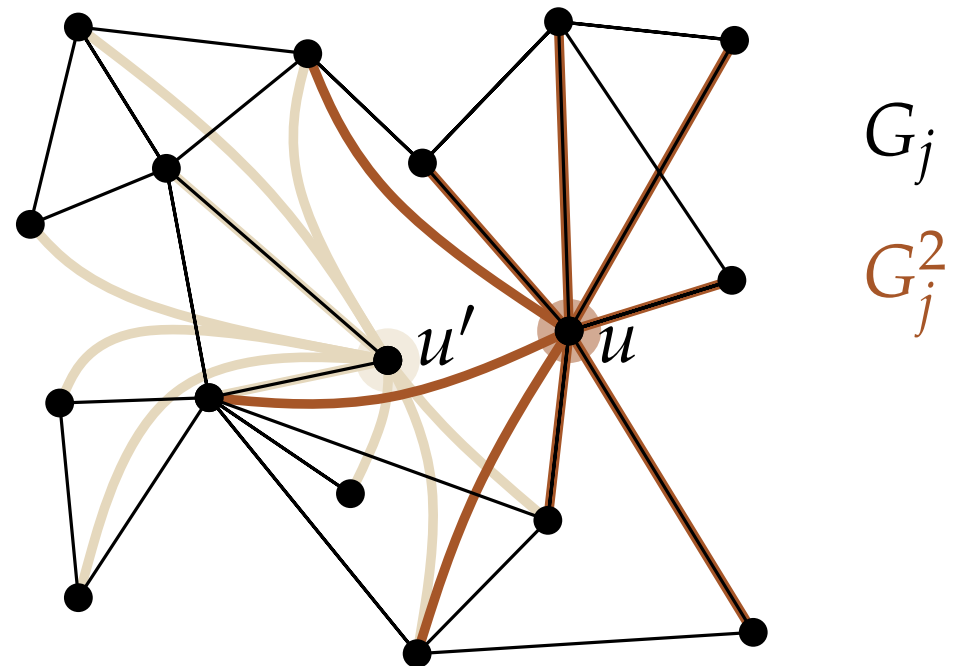


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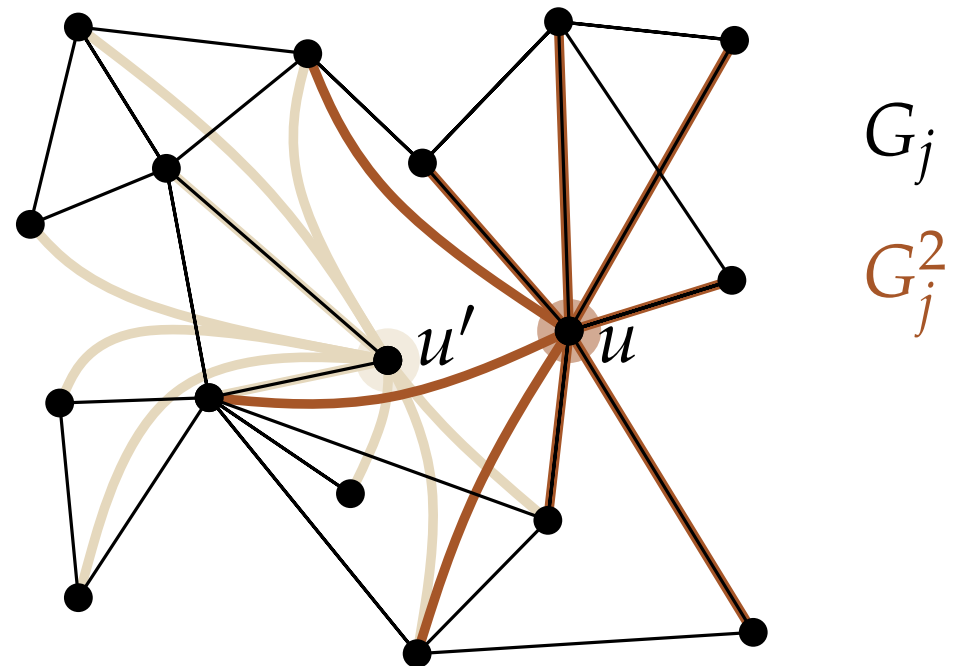
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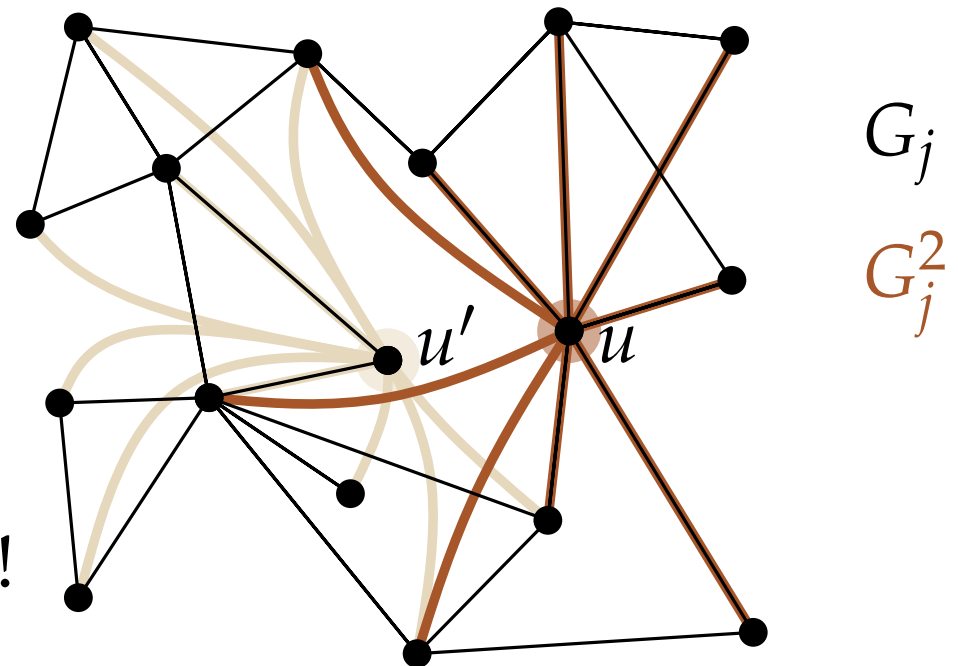
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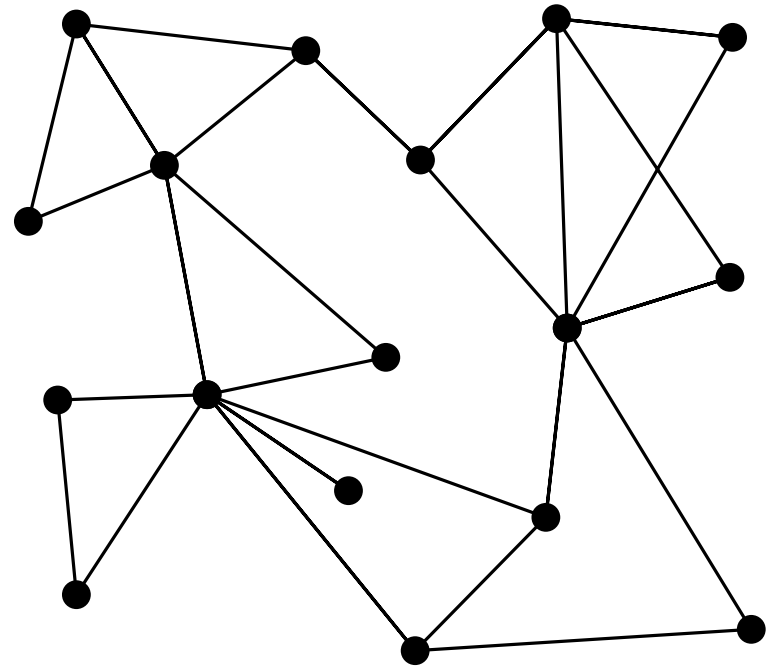
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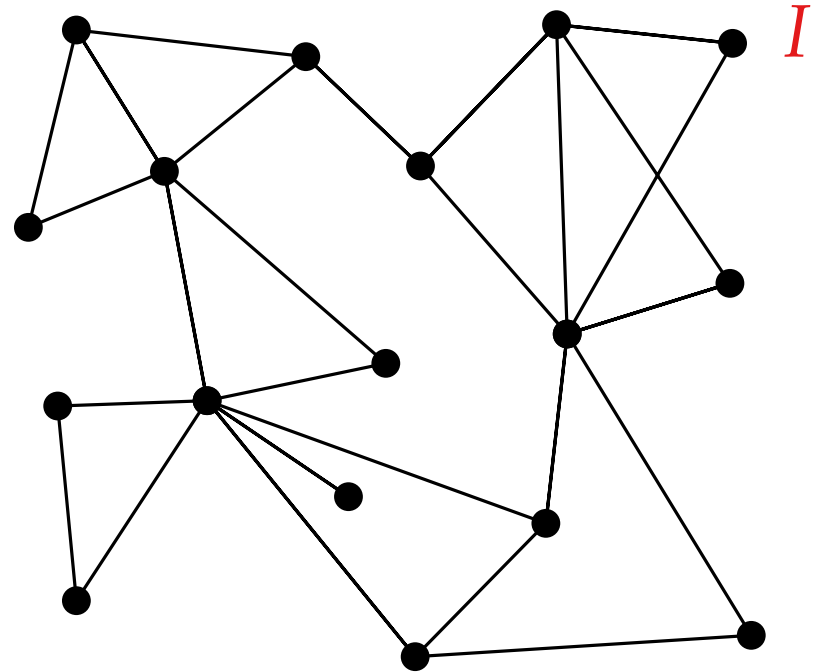
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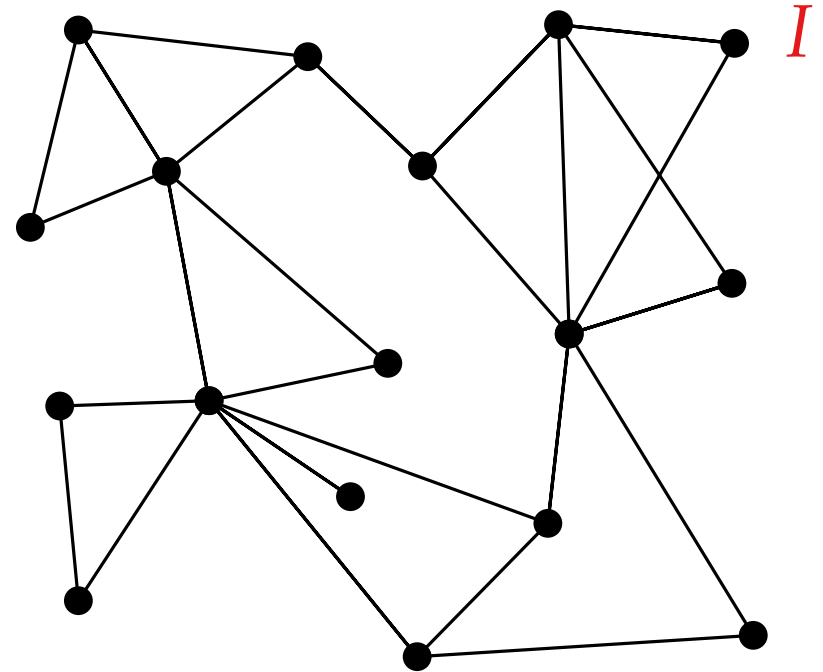
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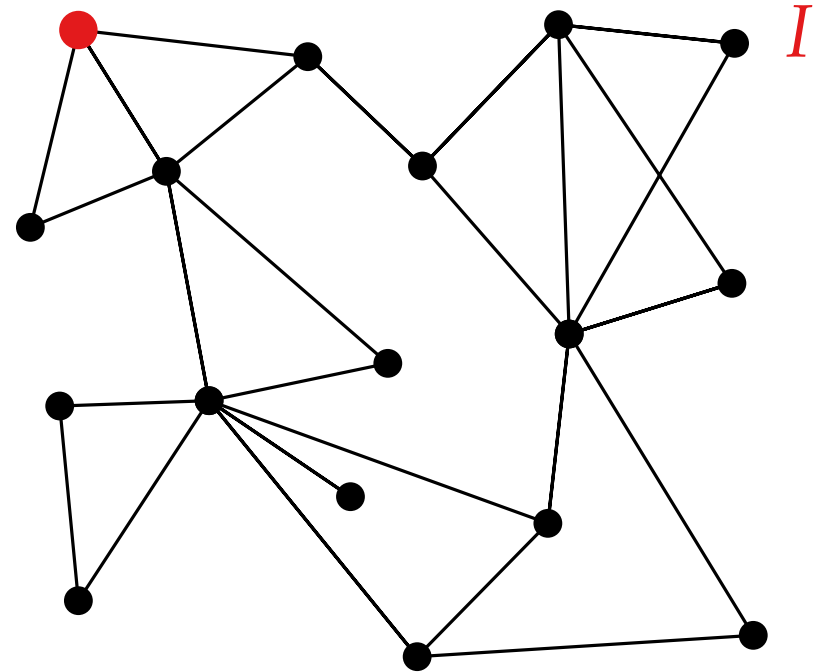
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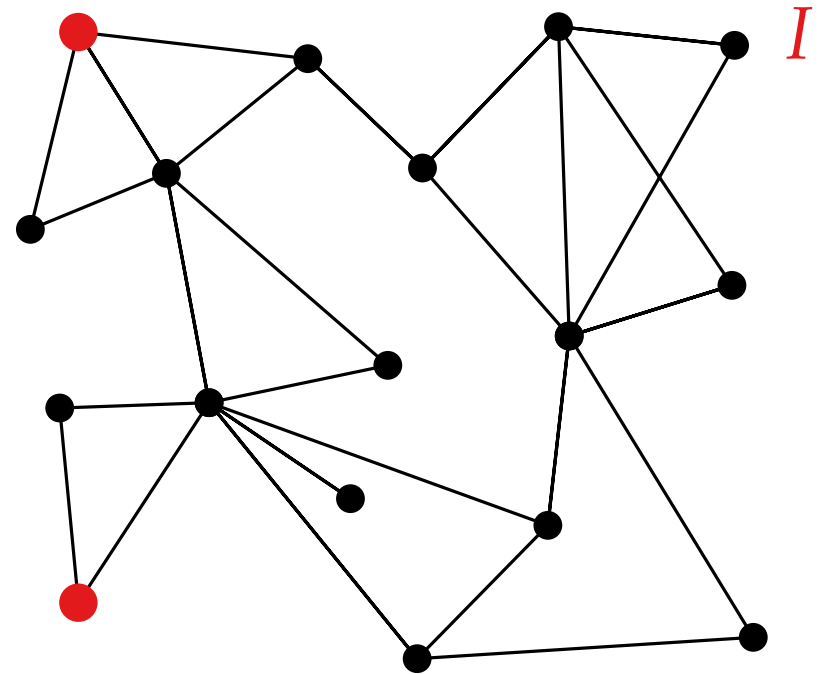
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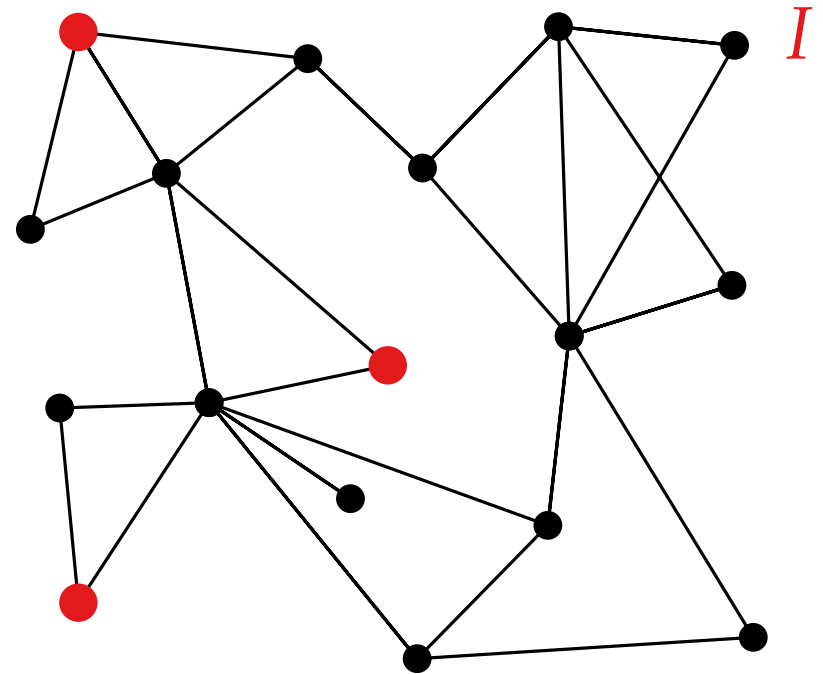
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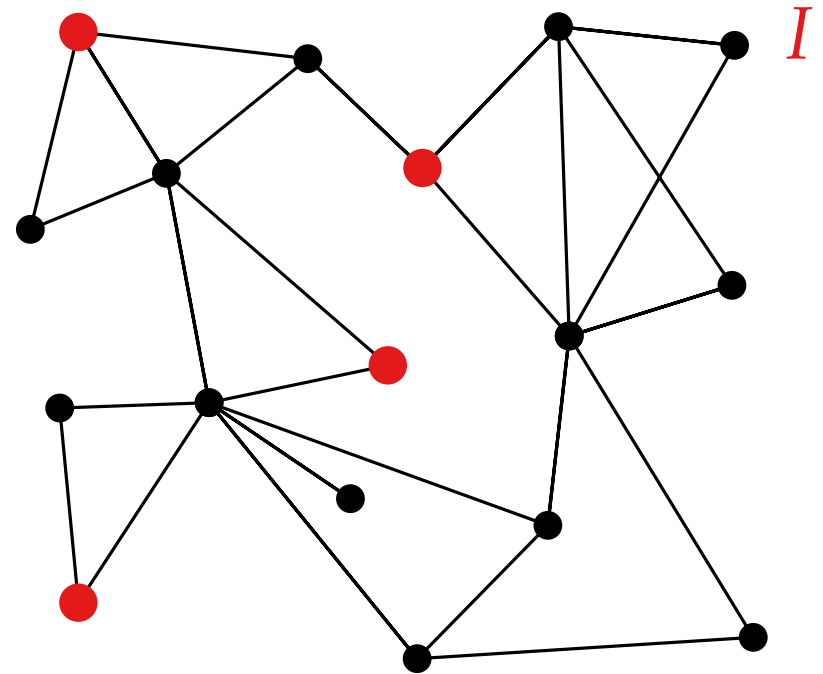
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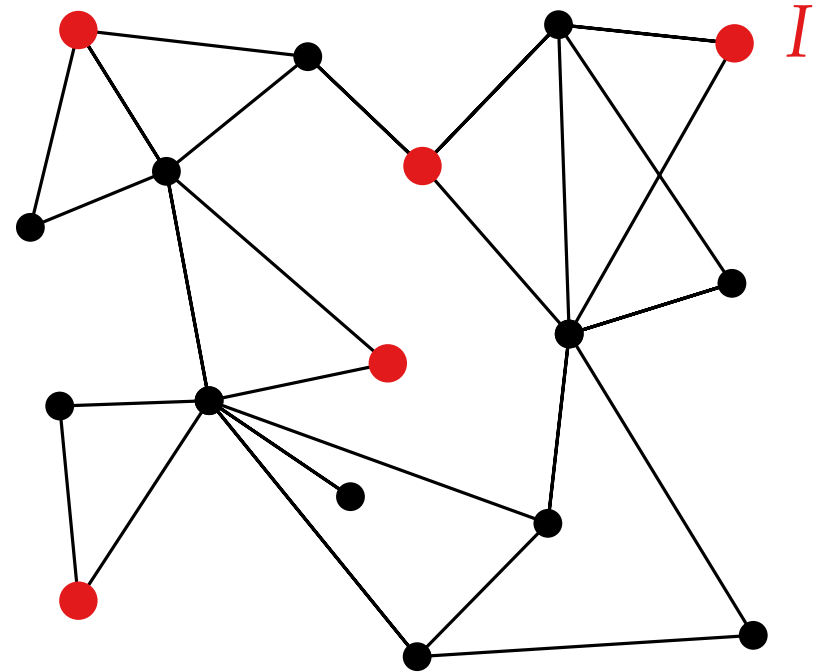
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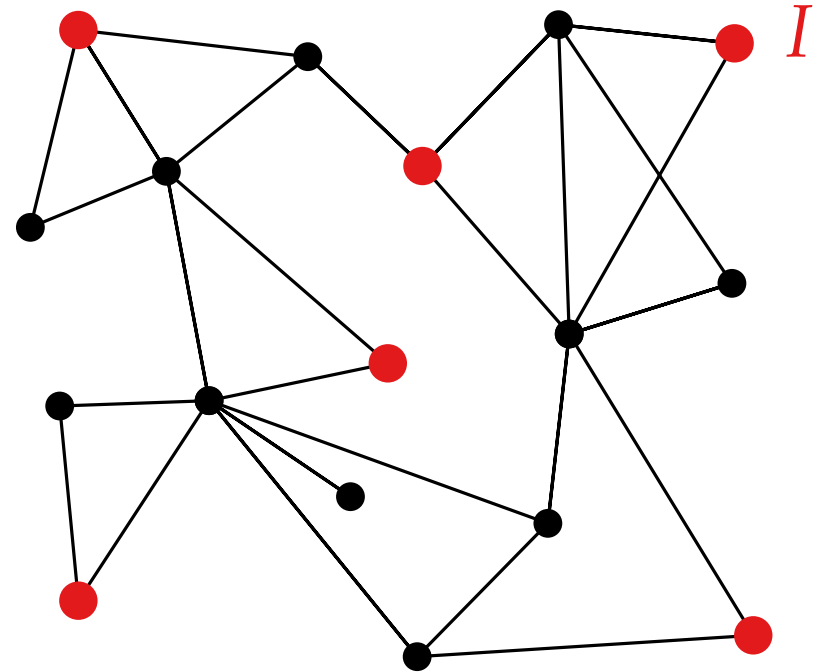
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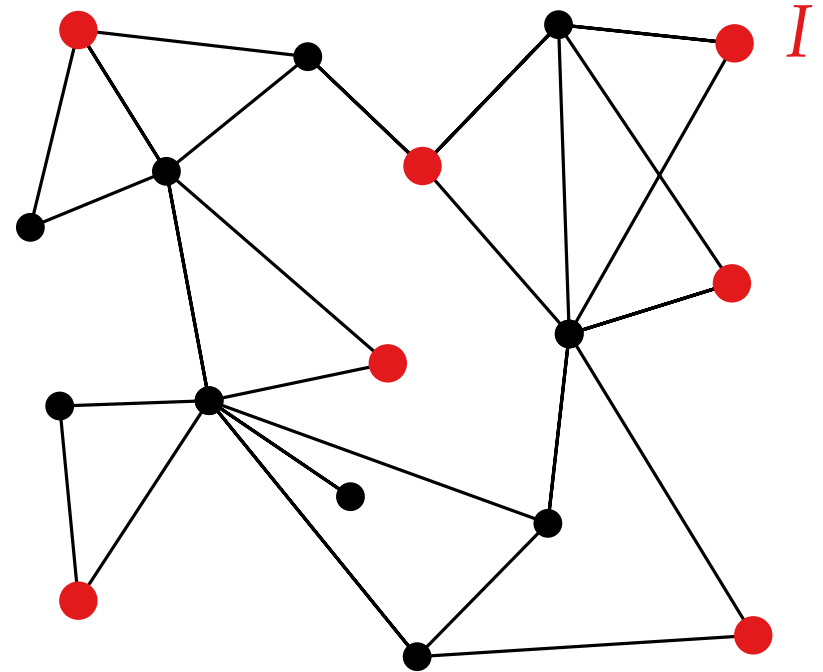
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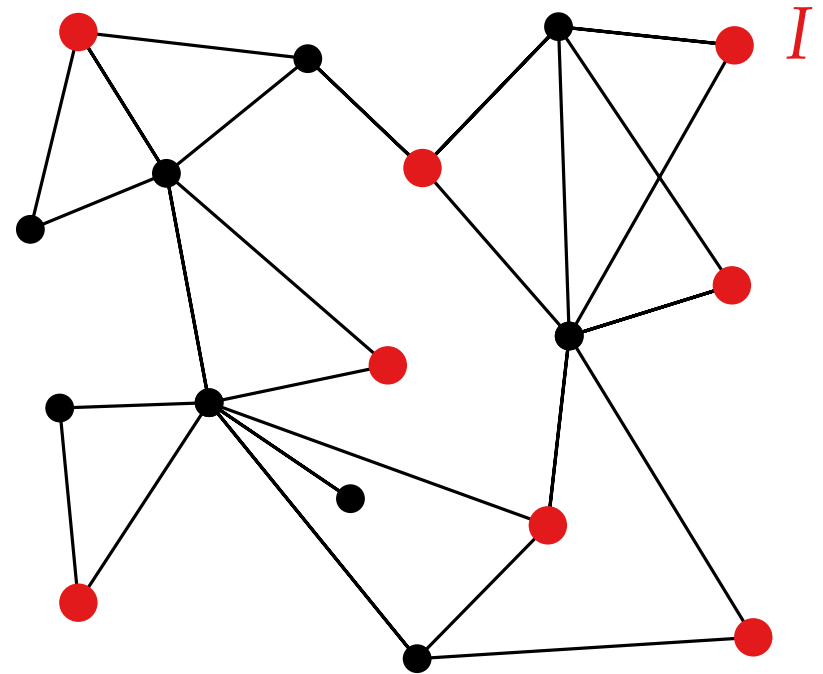
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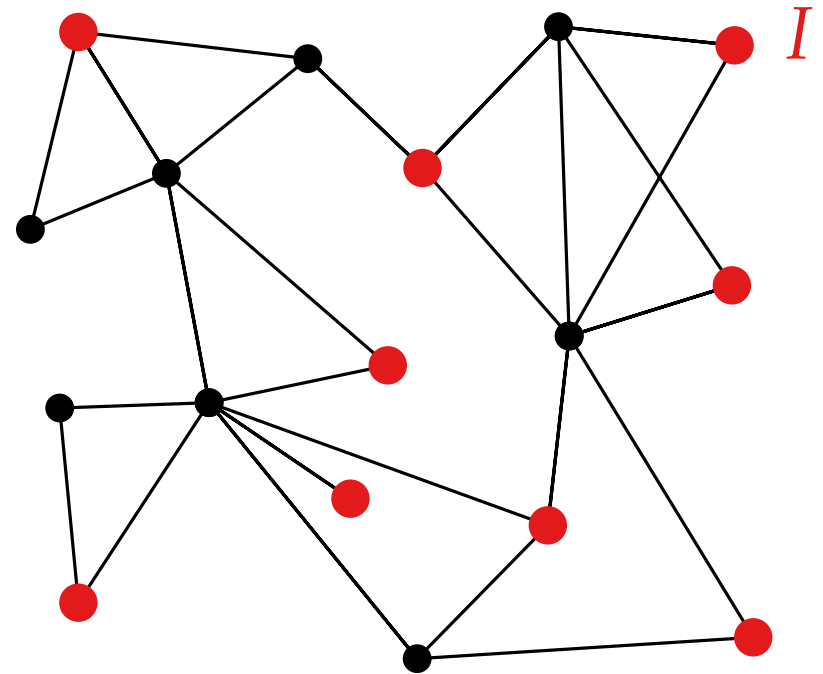
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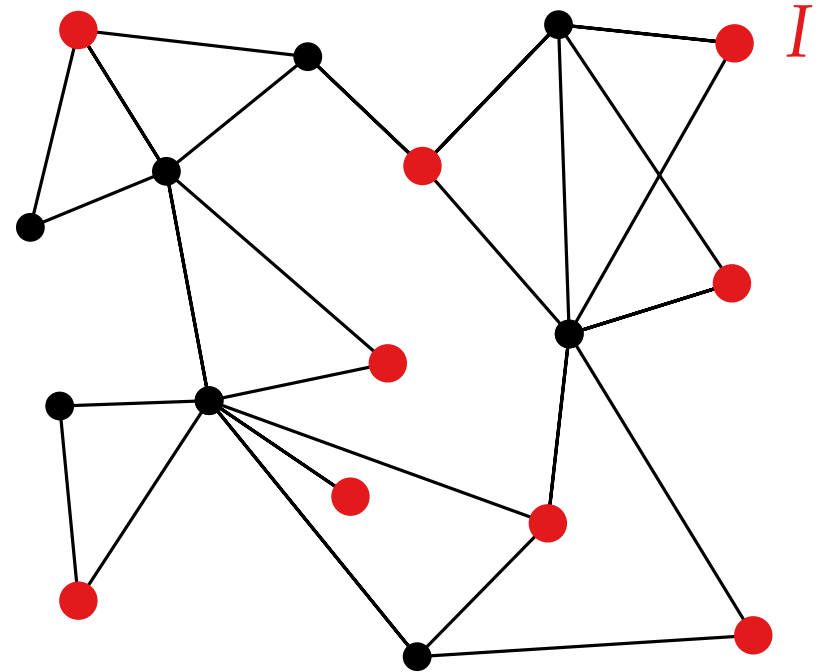
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Obs. Maximal independent sets are dominating sets :-)



Independent Sets in H^2

Lemma. For a graph H and an independent set I in H^2 ,
 $|I| \leq \text{dom}(H)$.

Independent Sets in H^2

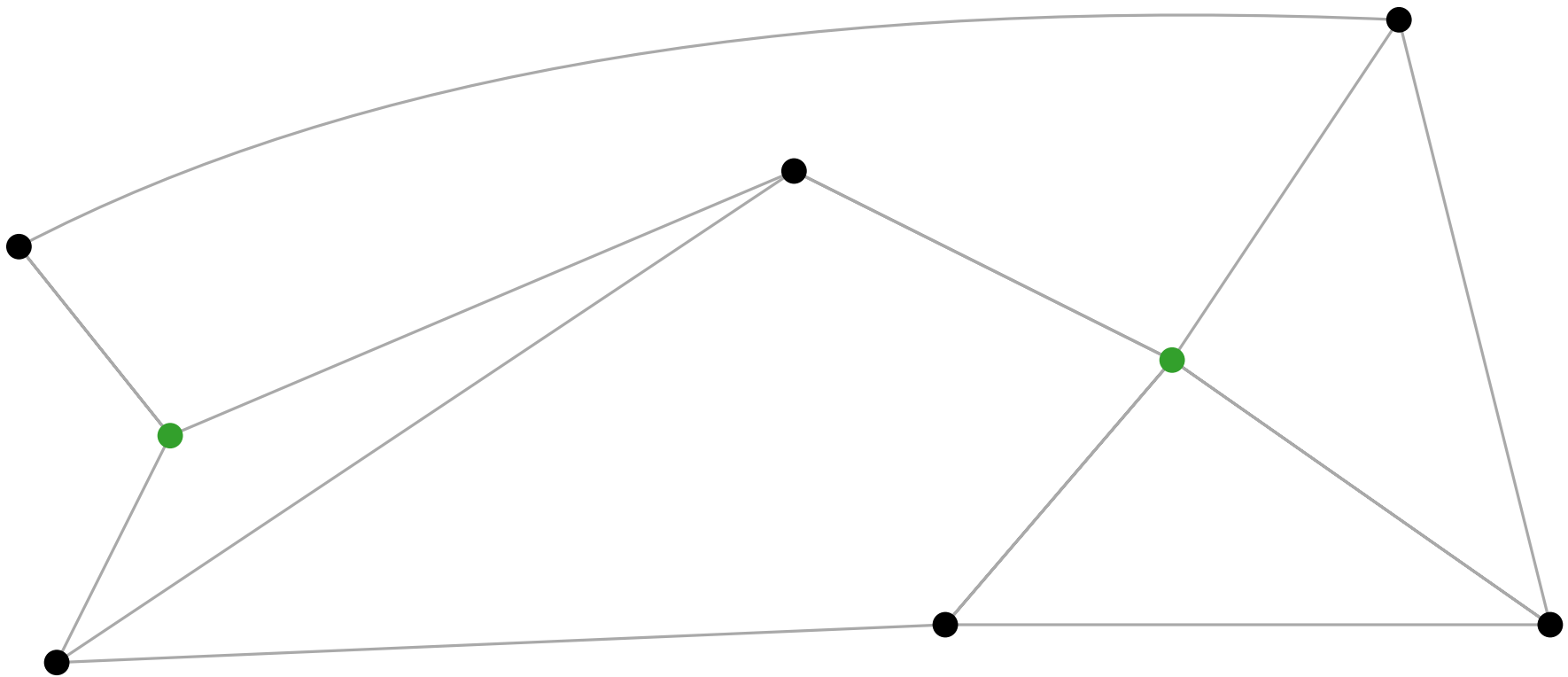
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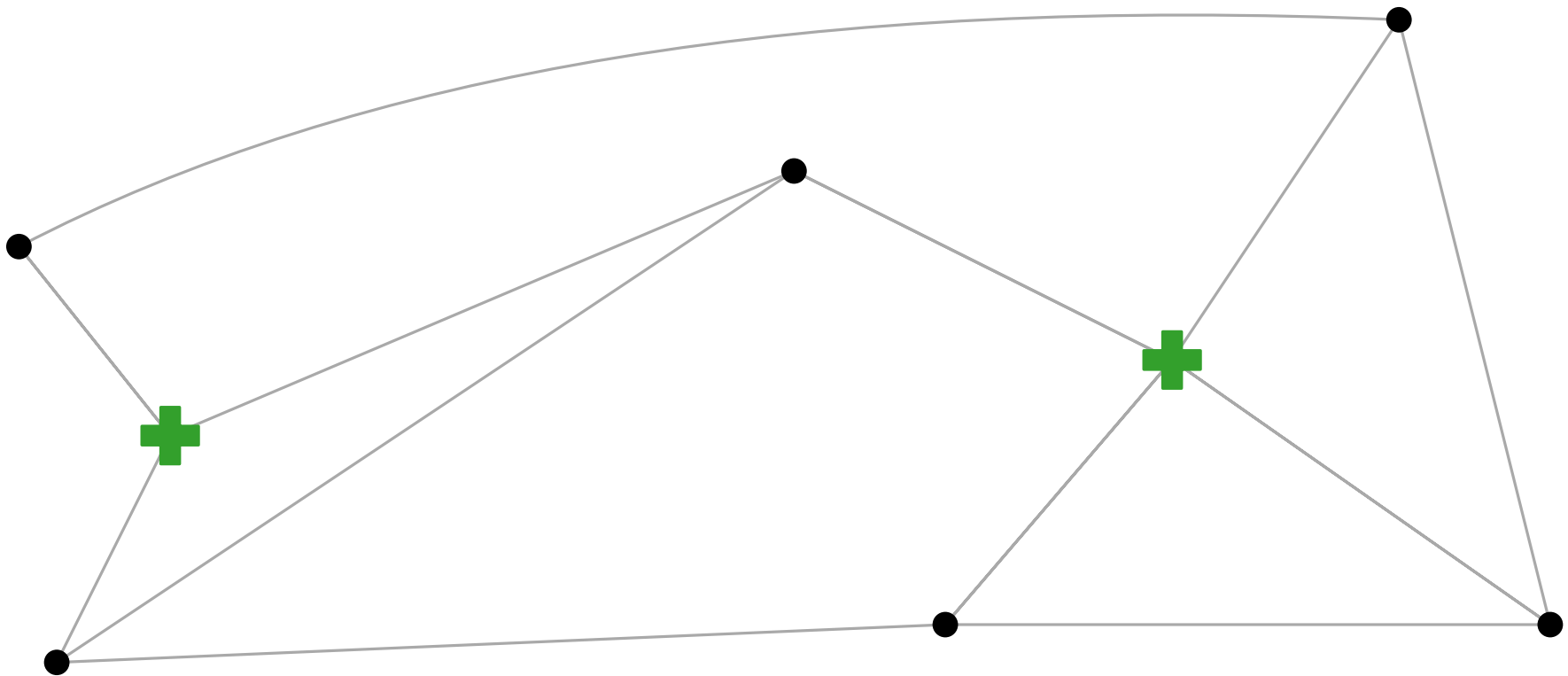
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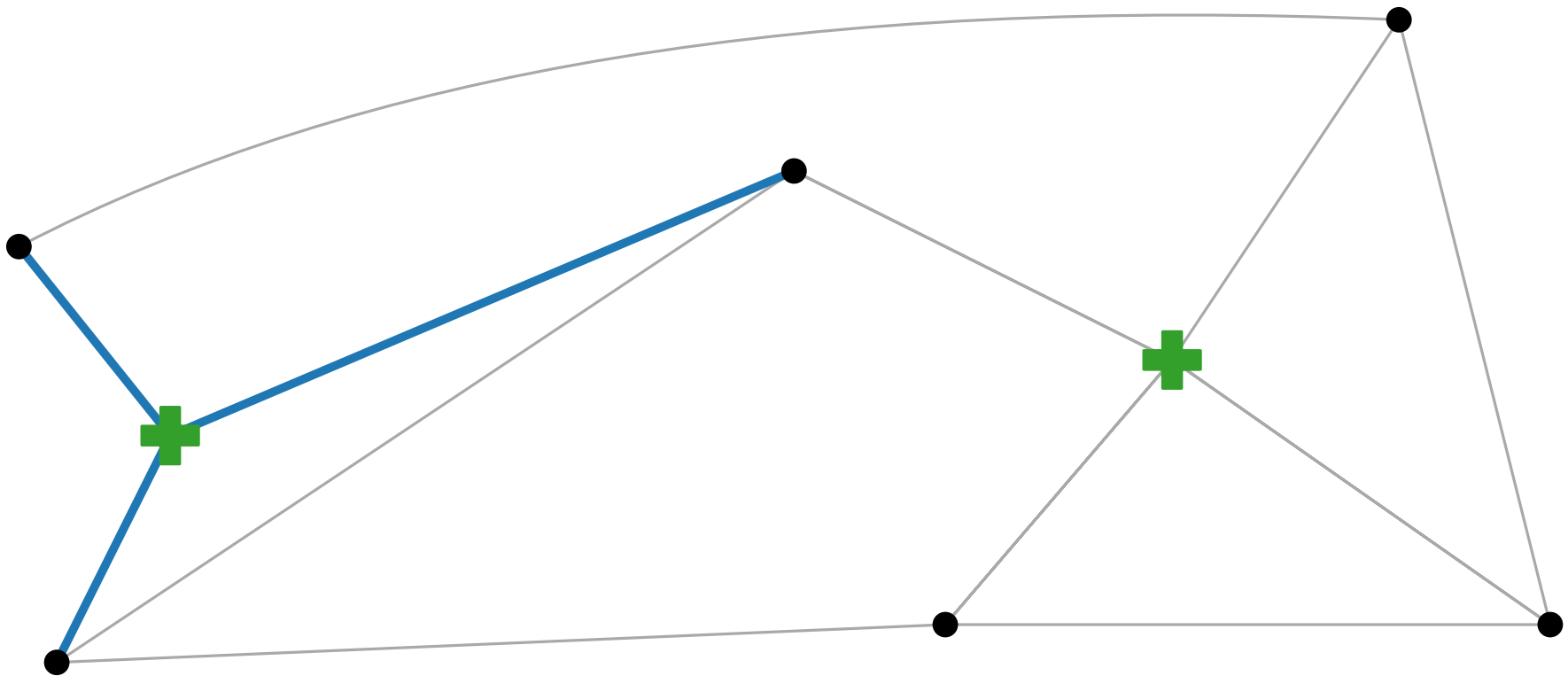
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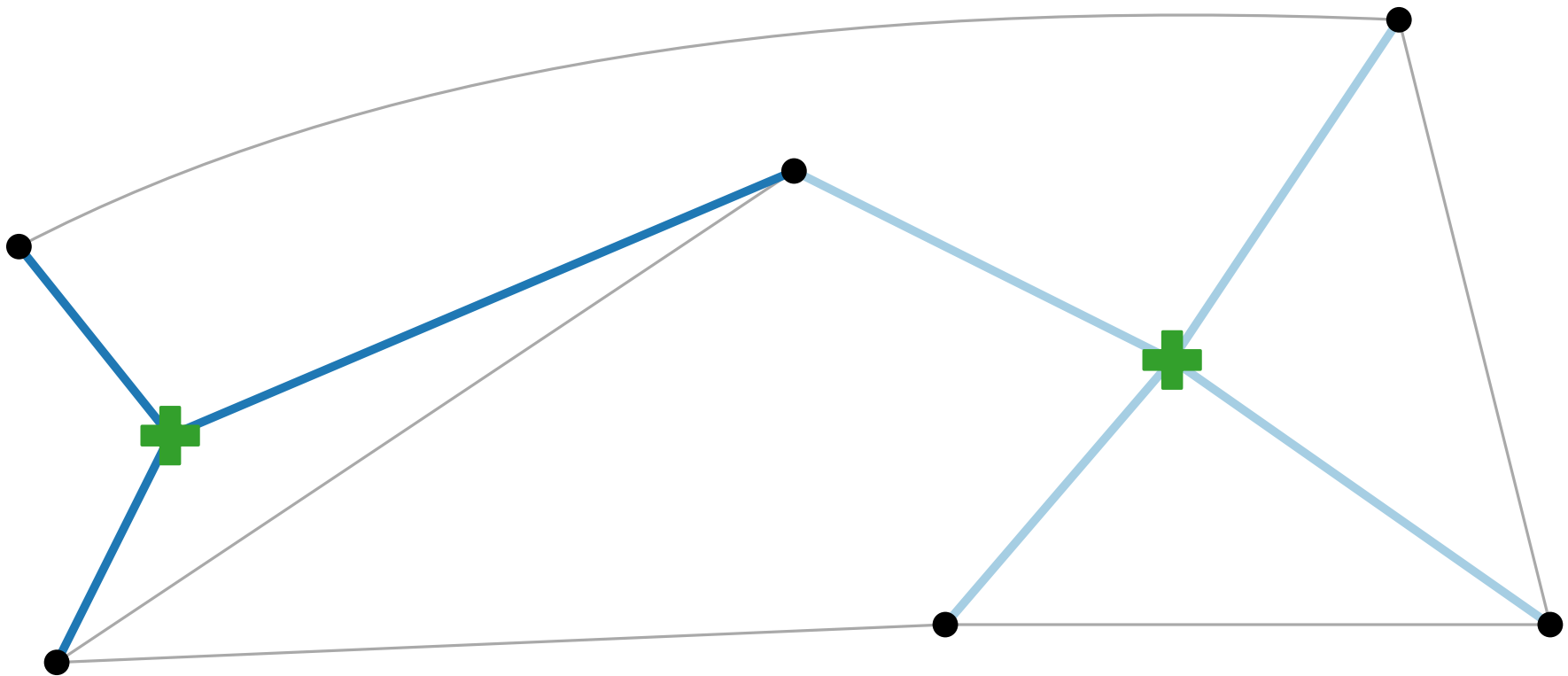
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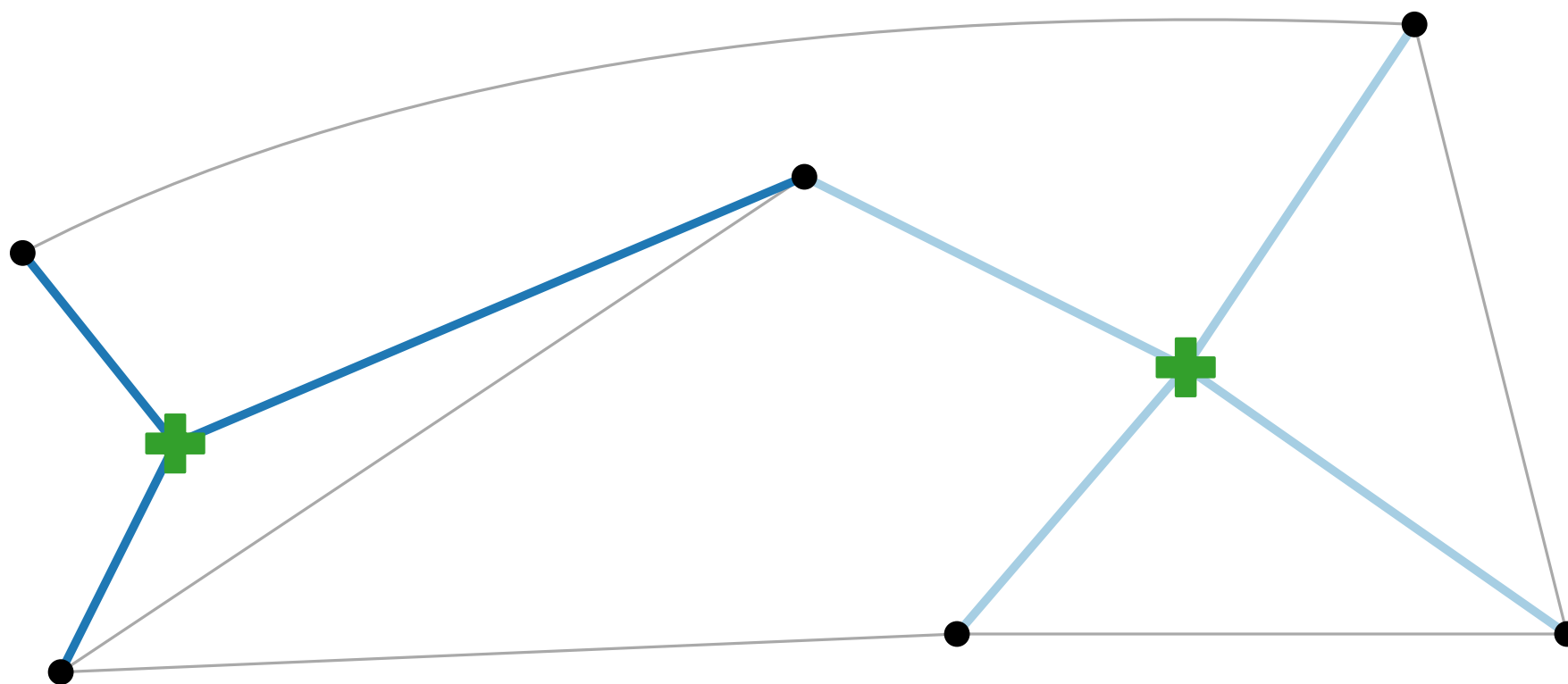
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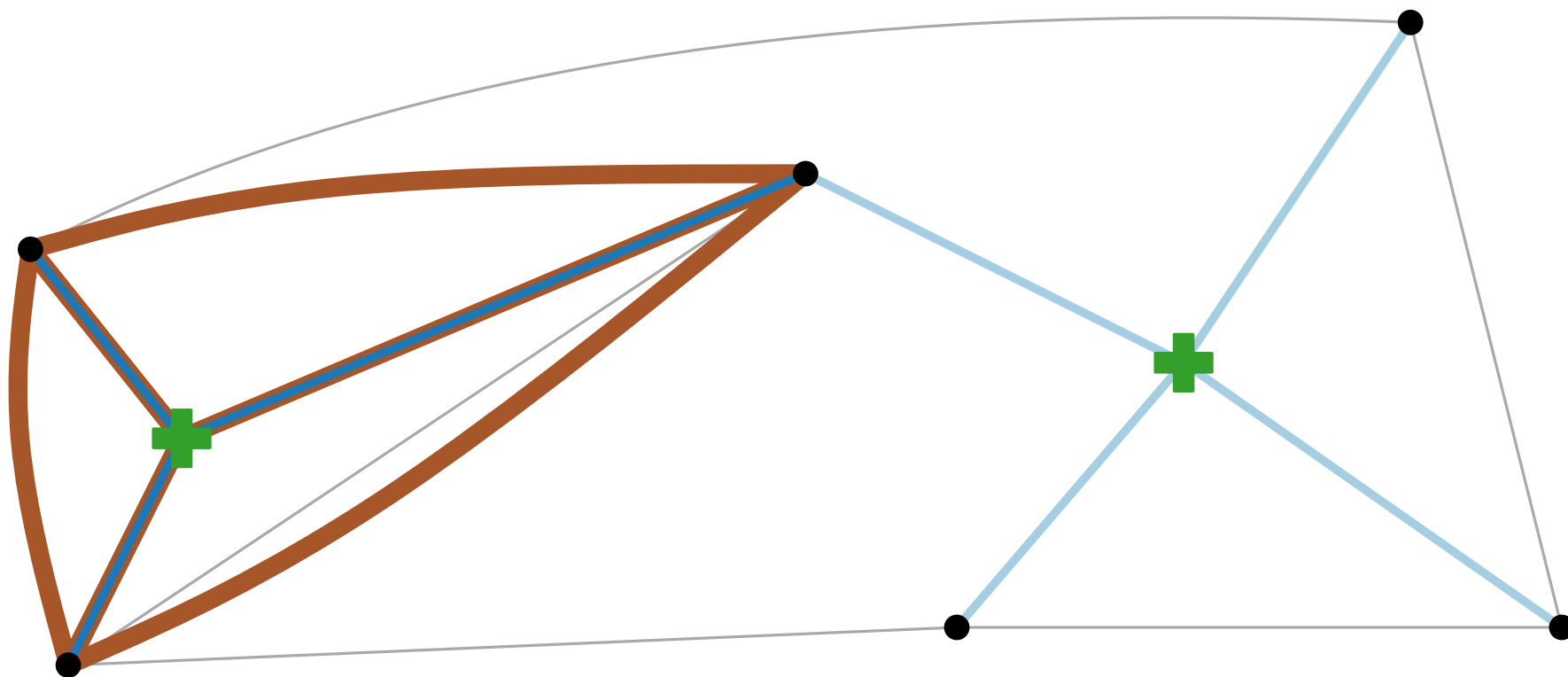


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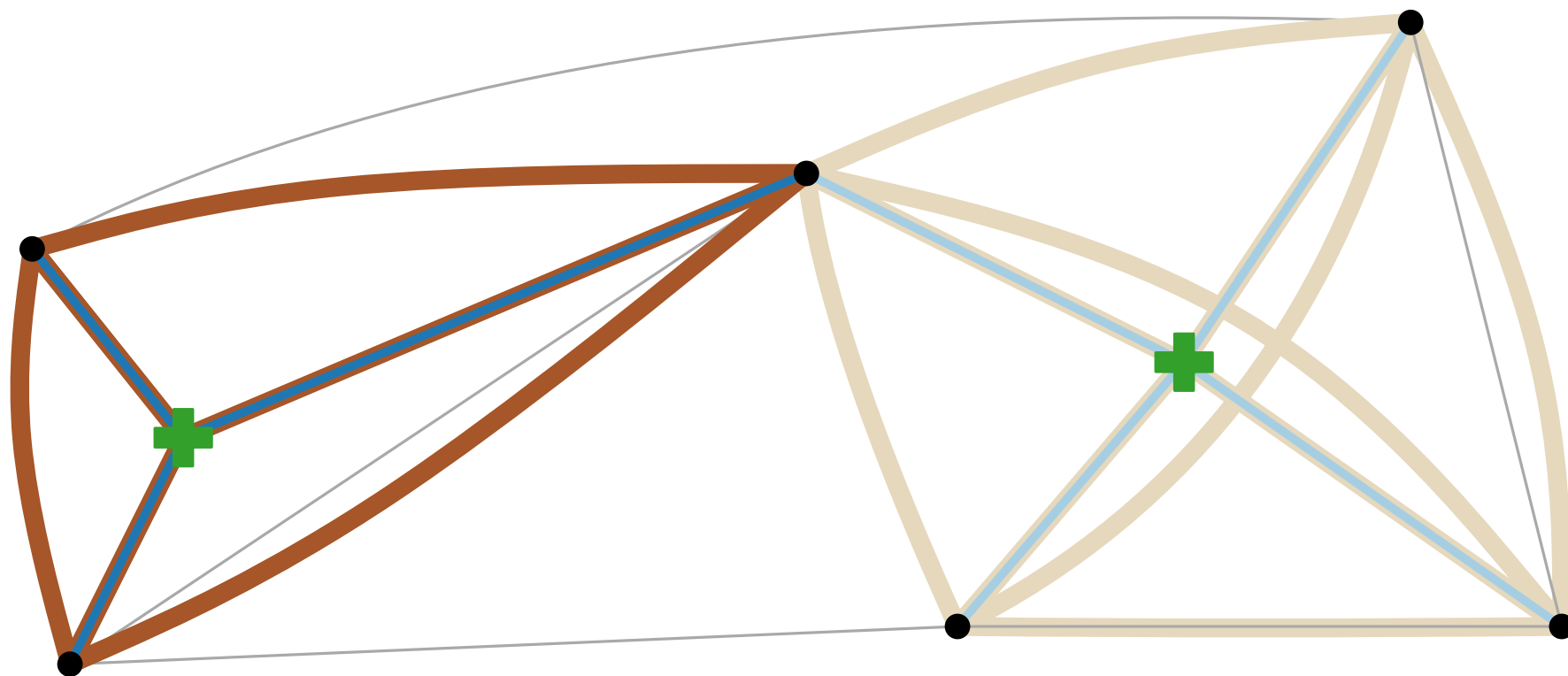


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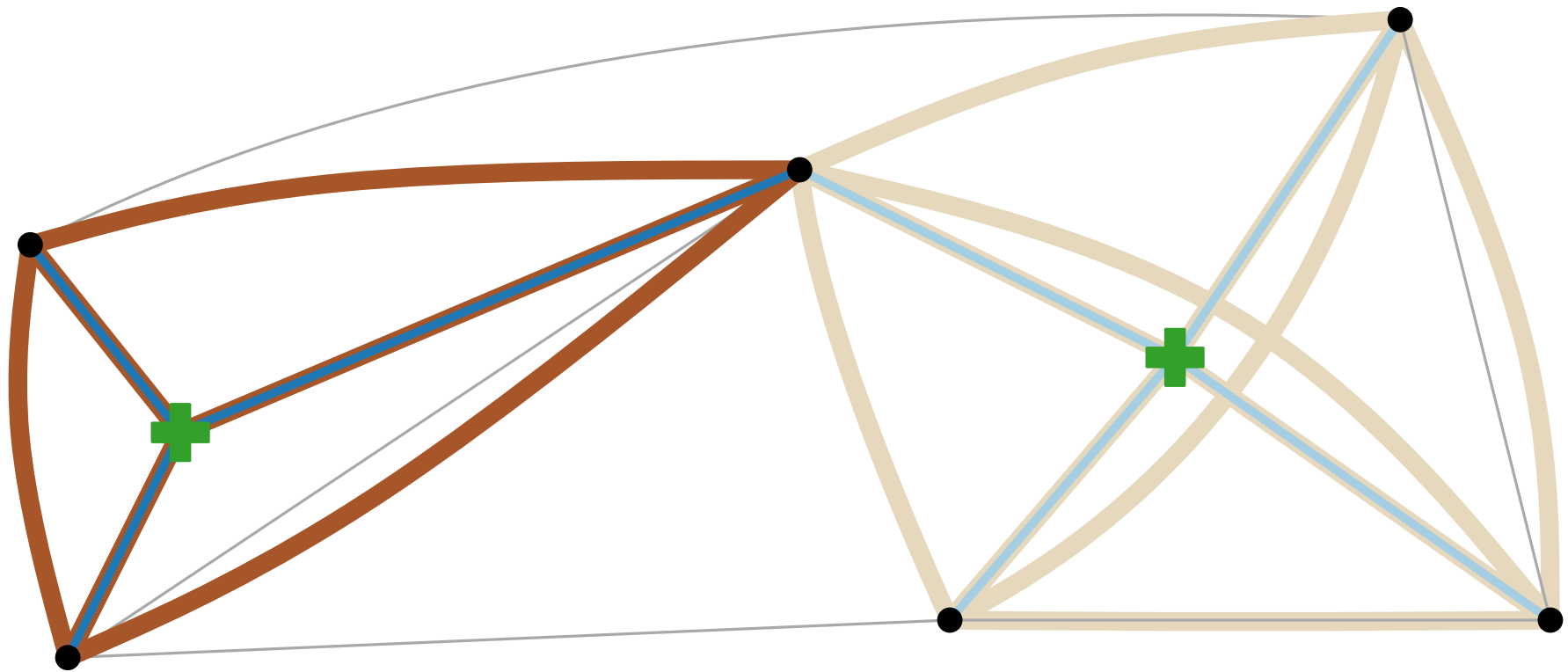


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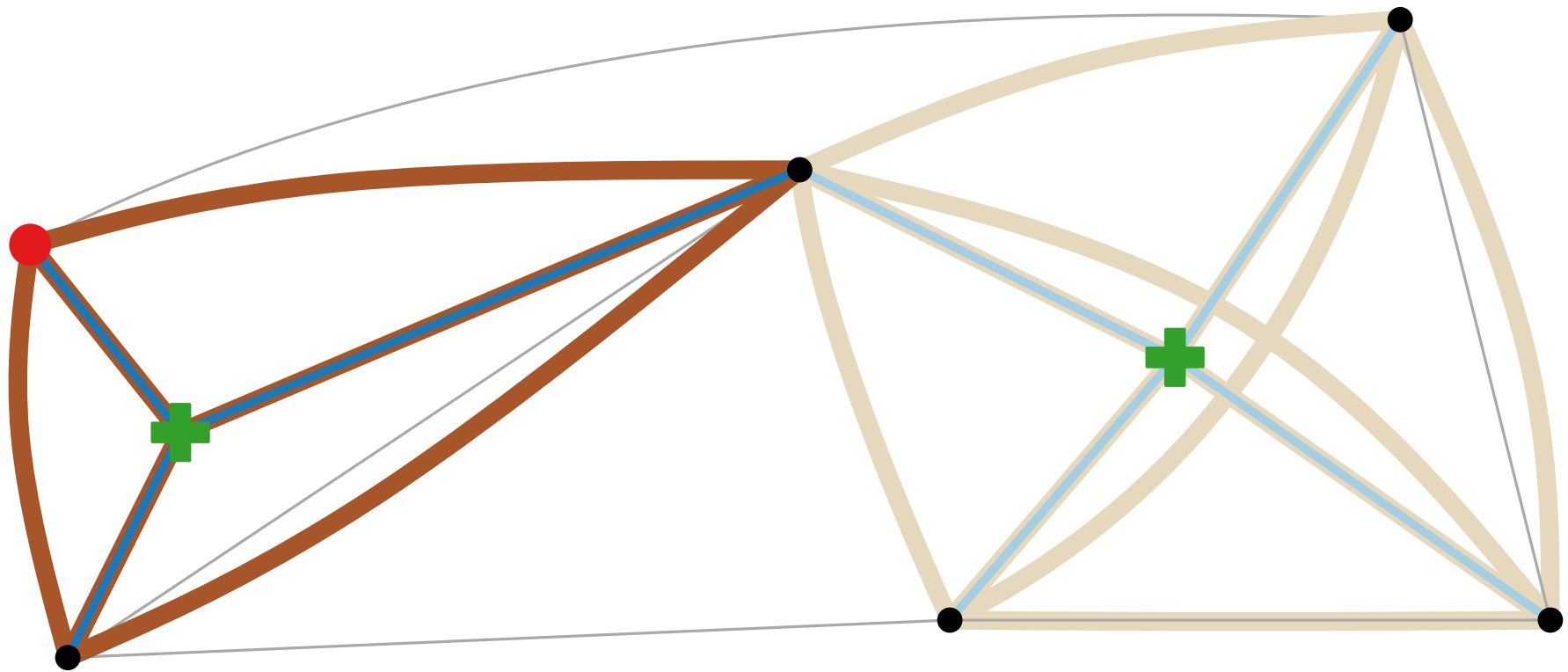
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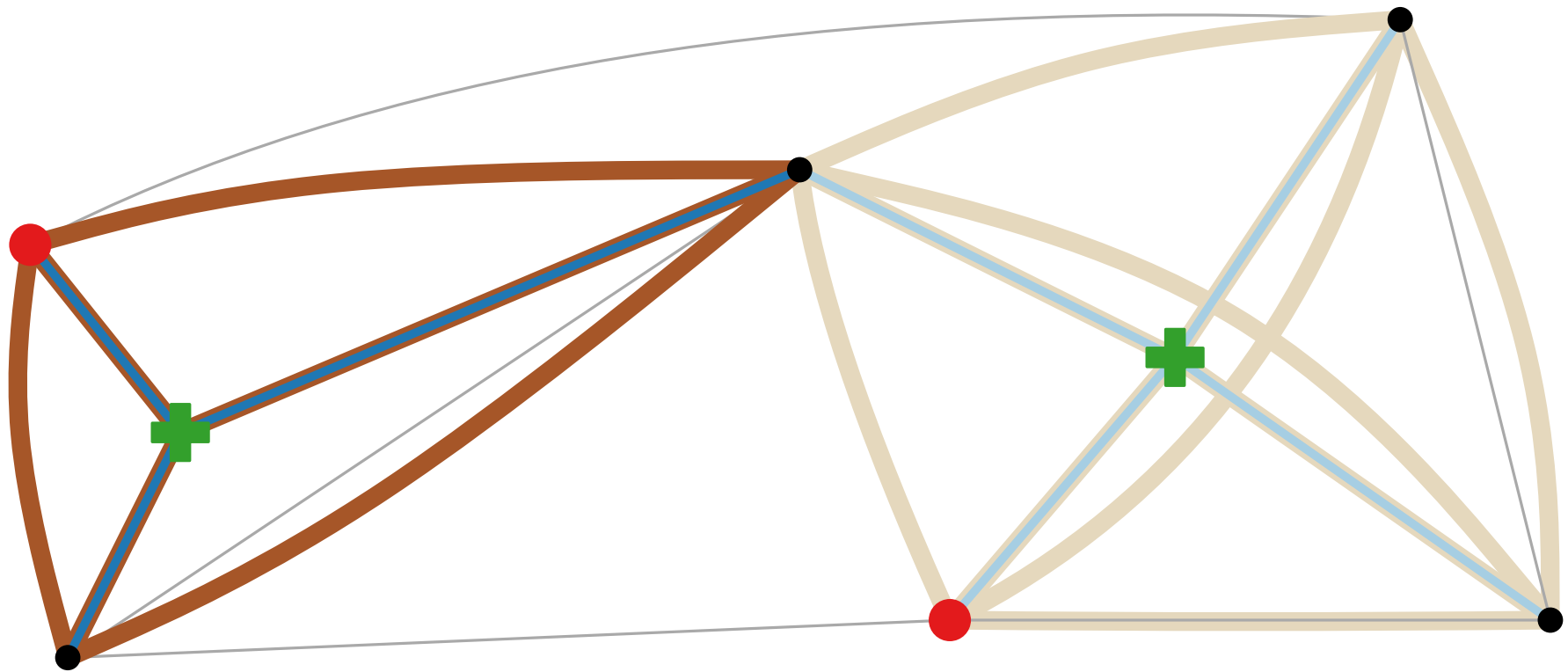
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Approximation Algorithms

Lecture 6:

k -CENTER via Parametric Pruning

Part IV:

Factor-2 Approximation for METRIC- k -CENTER

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Metric- k -CENTER($G = (V, E; c), k$)

Sort the edges of G by cost: $c(e_1) \leq \dots \leq c(e_m)$

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Theorem. The above algorithm is a factor-2 approximation algorithm for the metric k -CENTER problem.

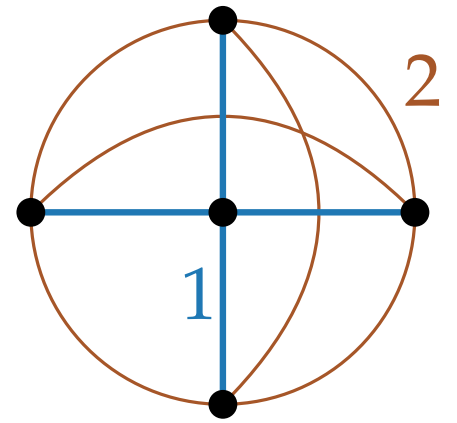
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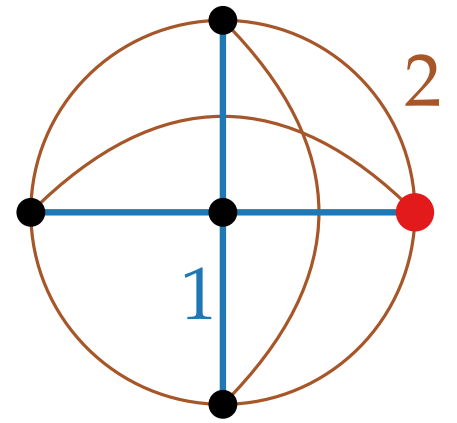
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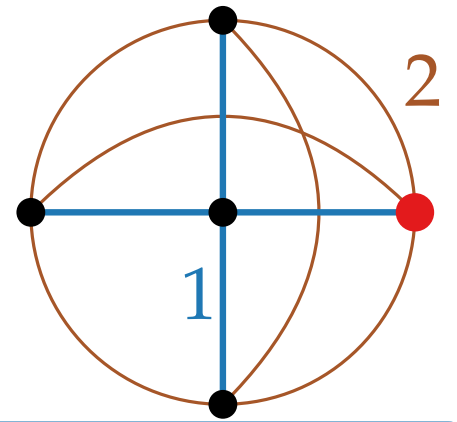
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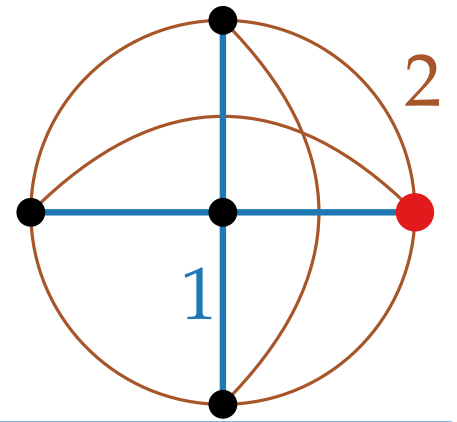
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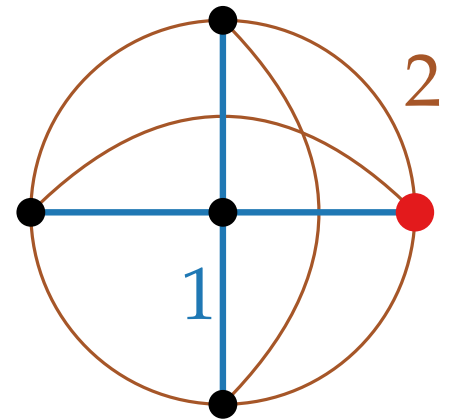


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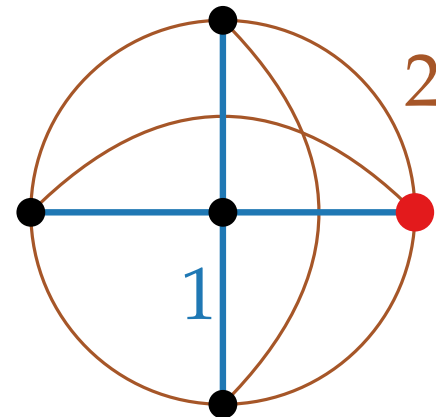


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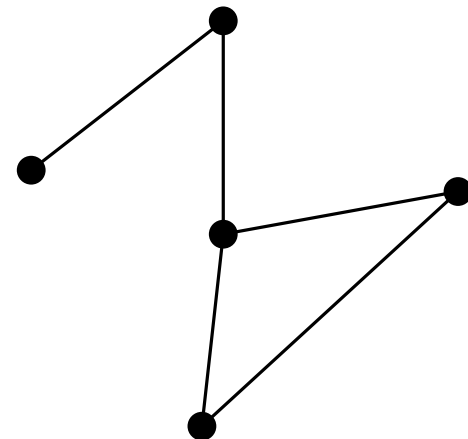
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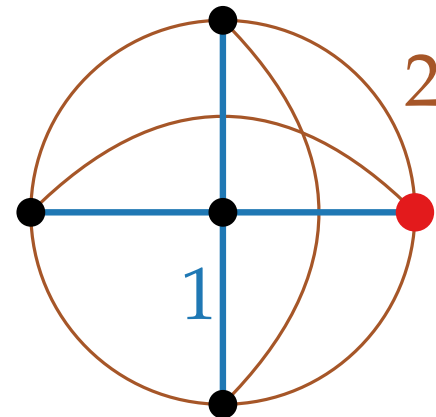
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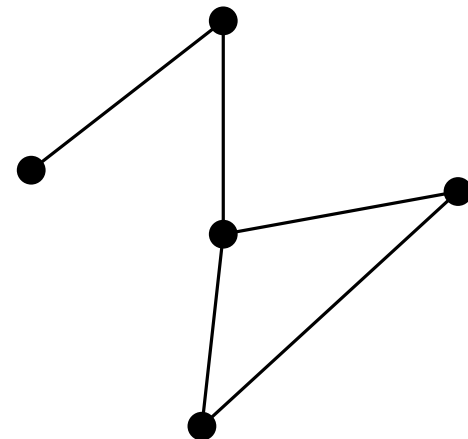
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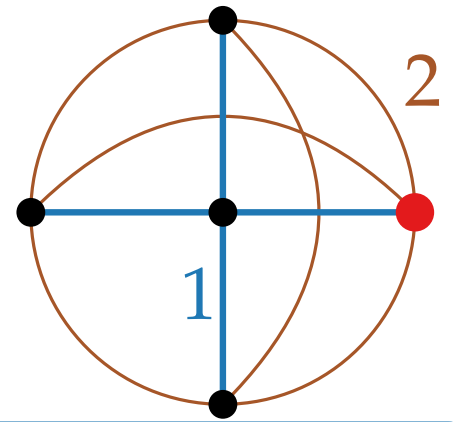
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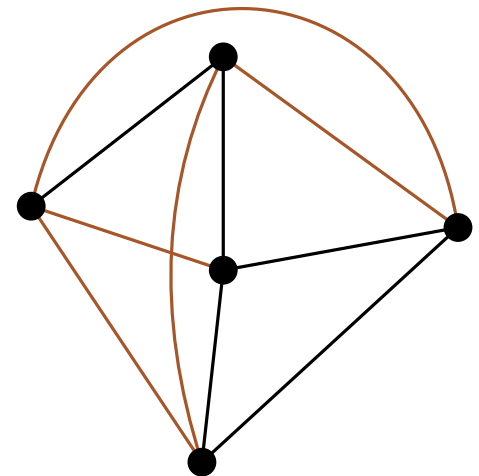
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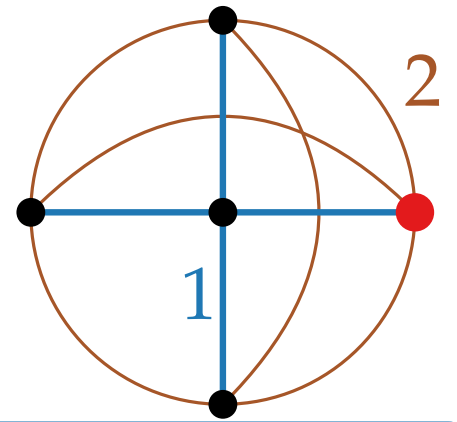
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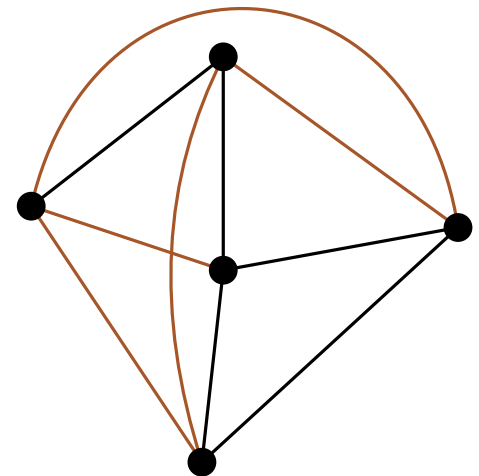
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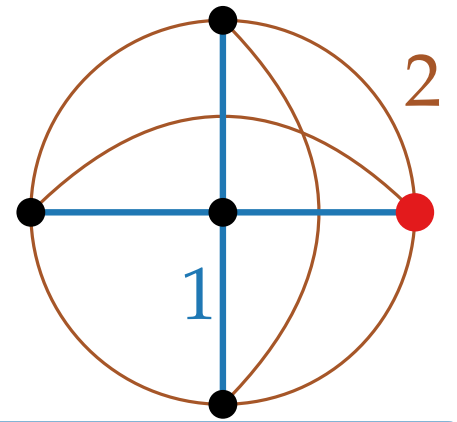
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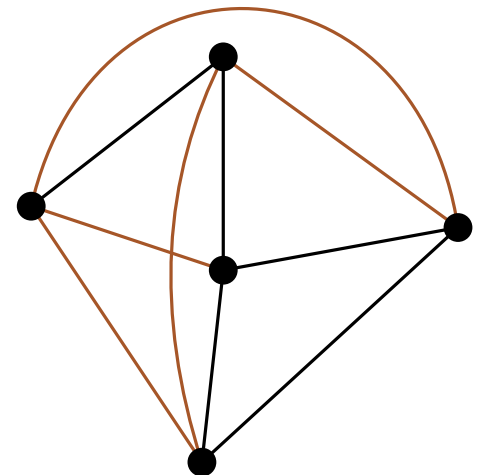


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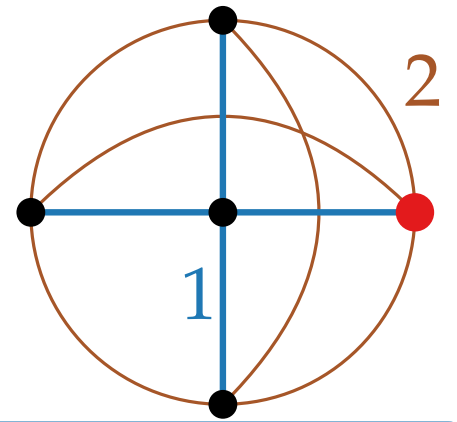
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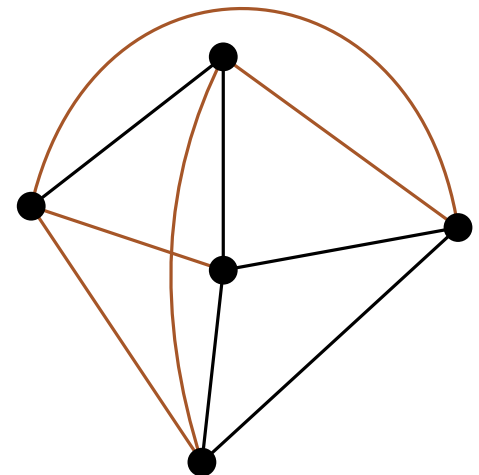
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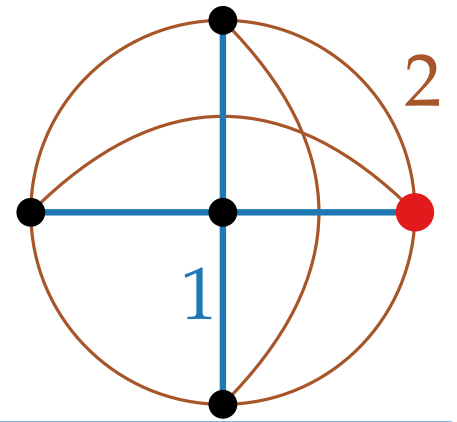
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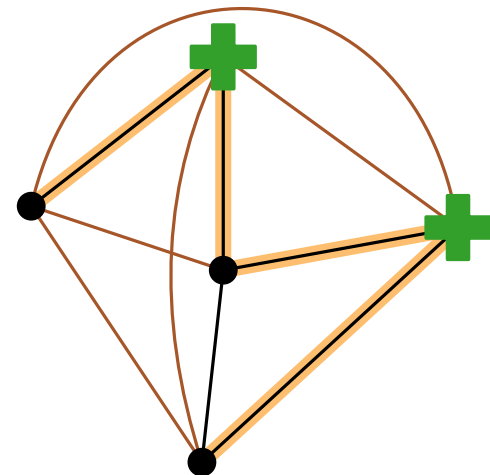


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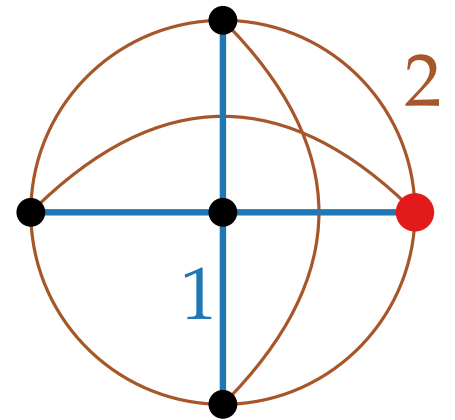
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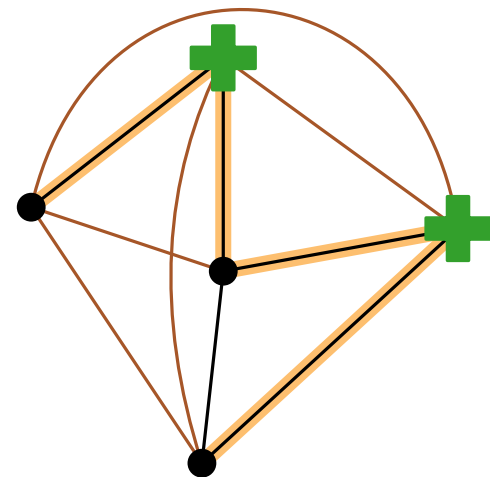
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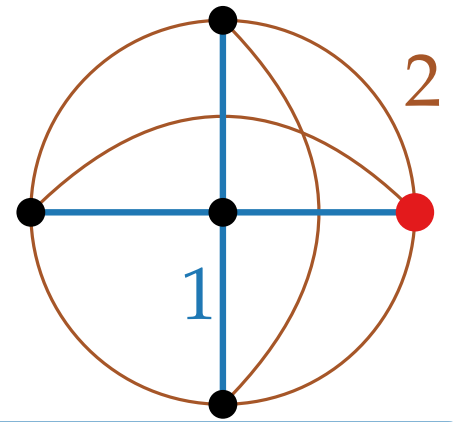
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If $\text{dom}(G) \leq k$, then $\text{cost}(S) = 1$.
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Can we do better ... ?

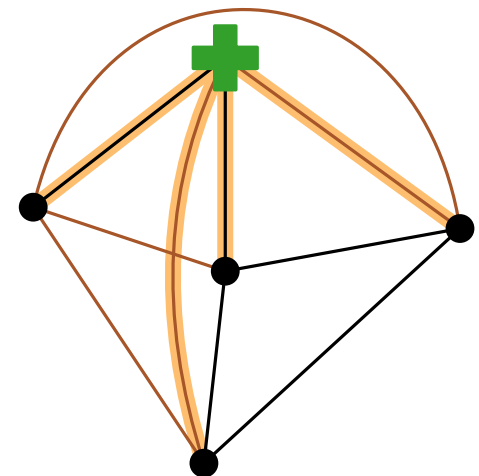
What about a tight example?



Theorem. Assuming $P \neq NP$, there is no factor- $(2 - \varepsilon)$ approximation algorithm for the metric k -CENTER problem, for any $\varepsilon > 0$.

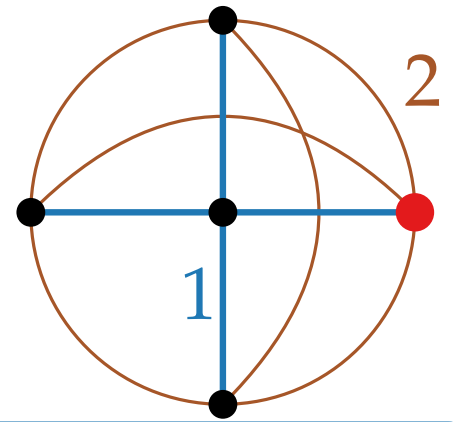
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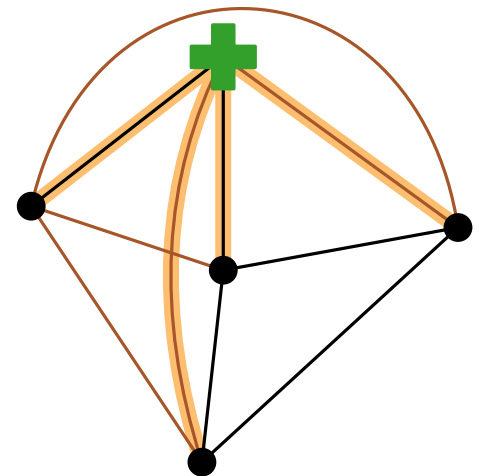
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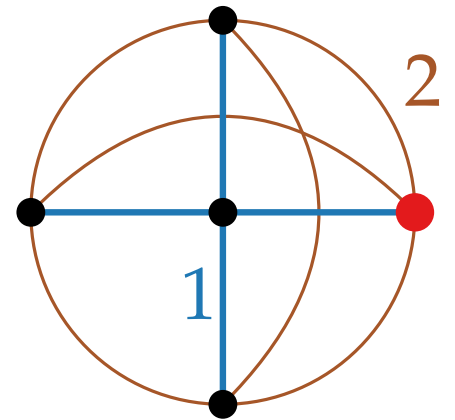
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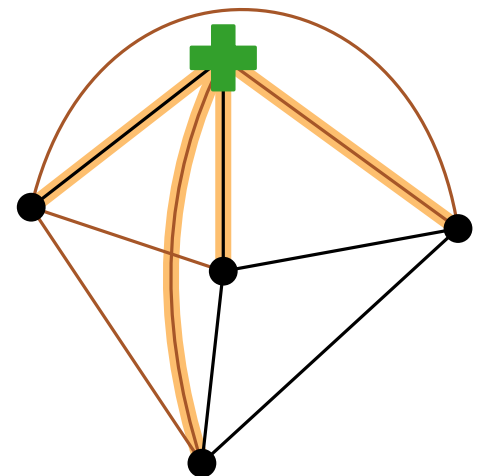
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Approximation Algorithms

Lecture 6:

k -CENTER via Parametric Pruning

Part V:

METRIC-WEIGHTED-CENTER

METRIC- k -CENTER

Given: A complete graph $G = (V, E)$ with metric edge costs $c: E \rightarrow \mathbb{Q}_{\geq 0}$ and a natural number $k \leq |V|$.

For $S \subseteq V$, $c(v, S)$ is the cost of the cheapest edge from v to a vertex in S .

Find: A k -element vertex set S such that $\text{cost}(S) := \max_{v \in V} c(v, S)$ is minimized.

METRIC-~~k~~-CENTER

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vertex set S of weight at most W

Find: A ~~k -element vertex set S~~ such that $\text{cost}(S) := \max_{v \in V} c(v, S)$ is minimized.

Algorithm for the Weighted Version

Algorithm Metric- -CENTER

Sort the edges of G by cost : $c(e_1) \leq \dots \leq c(e_m)$

for $j = 1, \dots, m$ **do**

 Construct G_j^2

 Find a maximal independent set I_j in G_j^2

if $|I_j| \leq k$ **then**

return I_j

Algorithm for the Weighted Version

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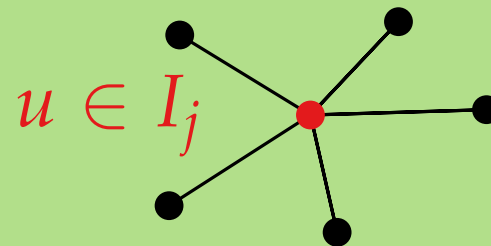
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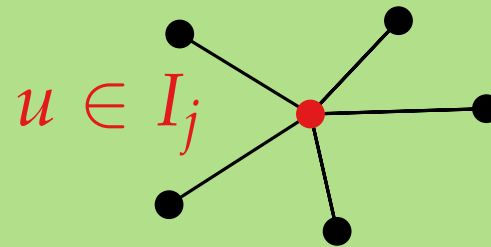
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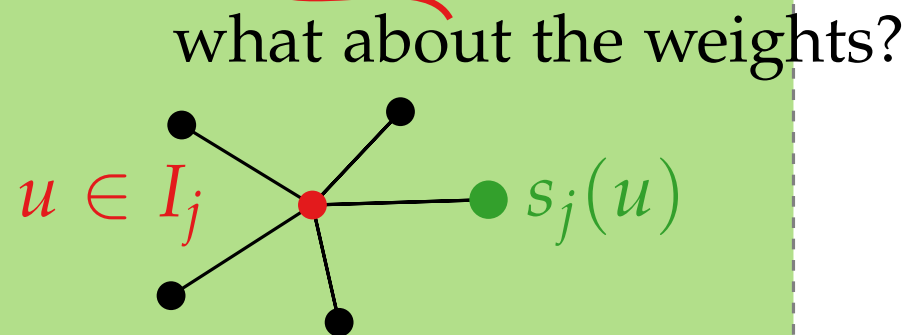
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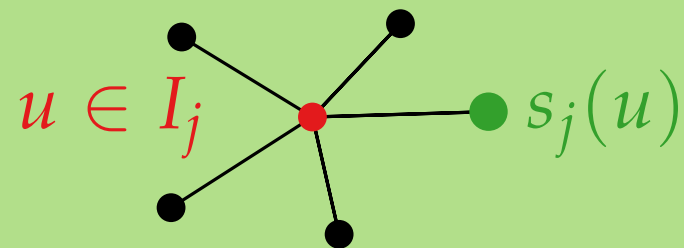
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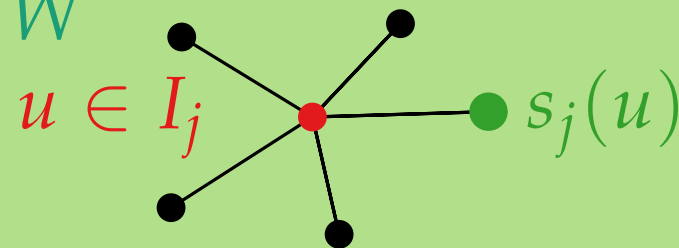
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if $|I_j| \leq k$ **then** $w(S_j) \leq W$

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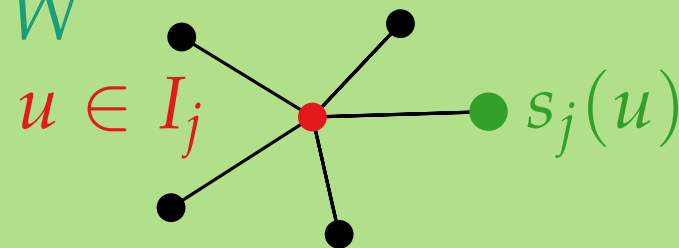
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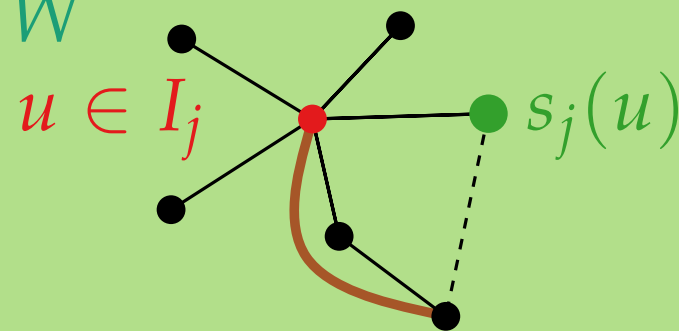
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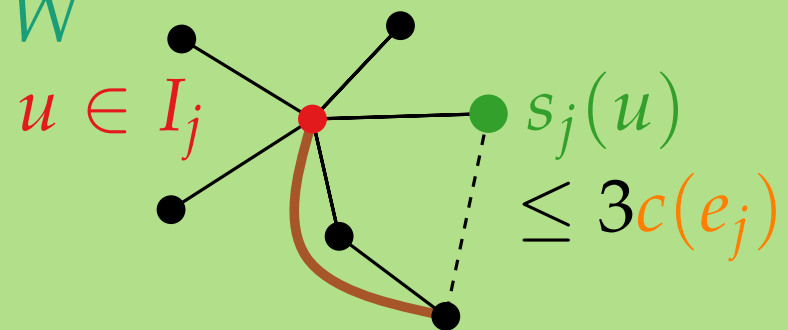
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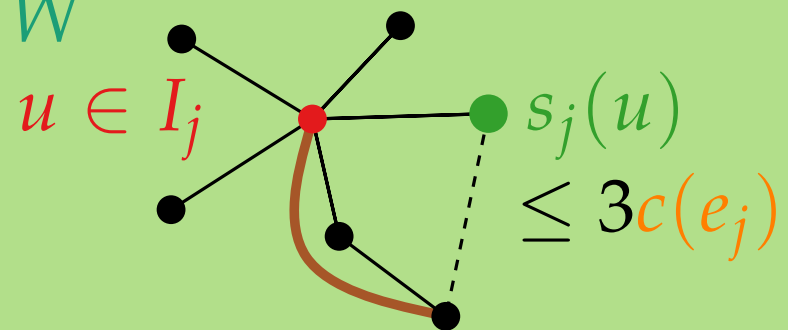
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Theorem. The above is a factor-3 approximation algorithm for METRIC-WEIGHTED-CENTER.

Tight Example... ?

Here, we need to have a budget W ,
and edge costs satisfying the triangle inequality.

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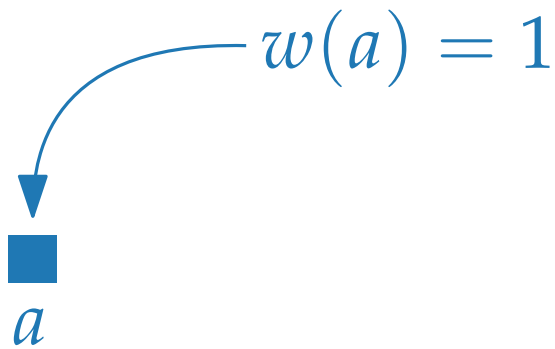
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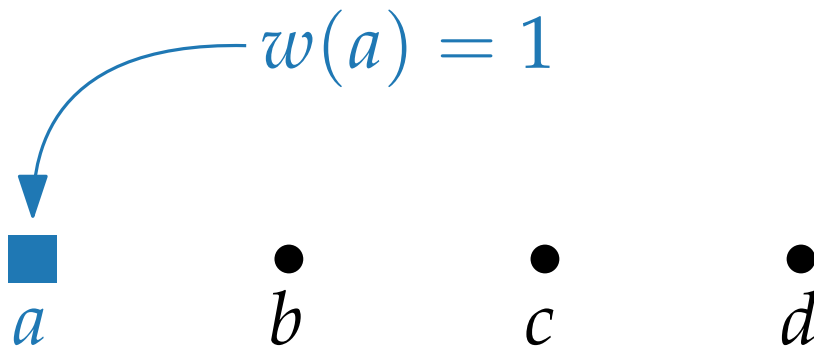
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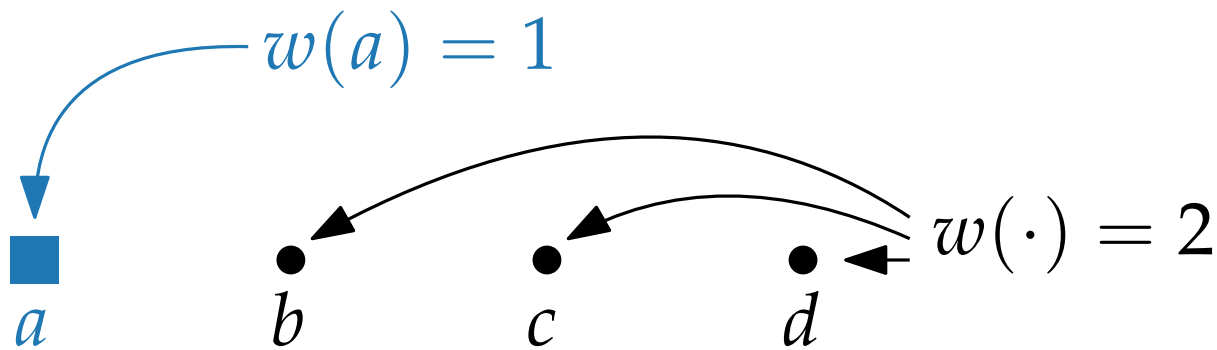
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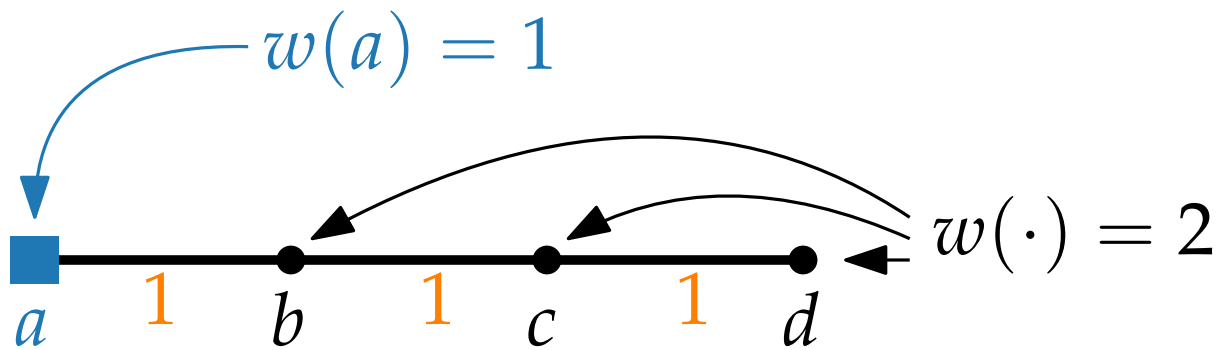
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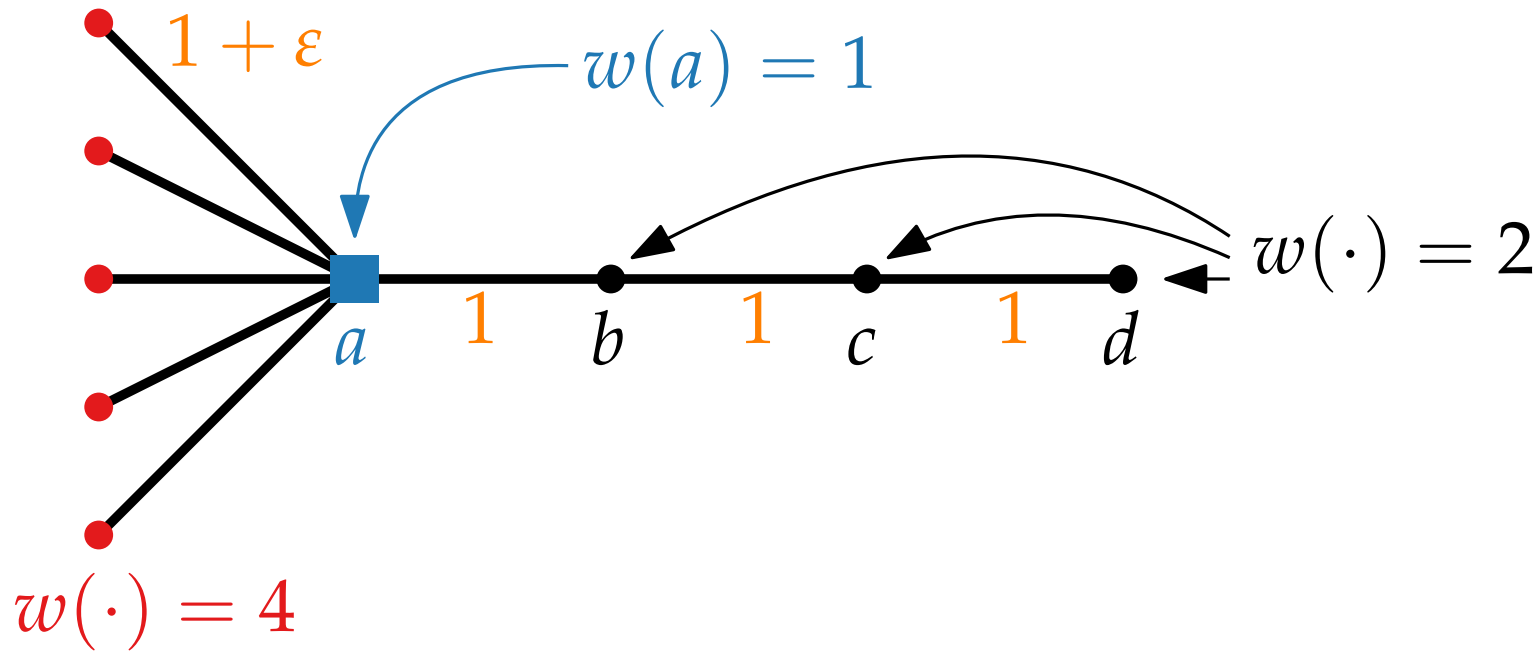
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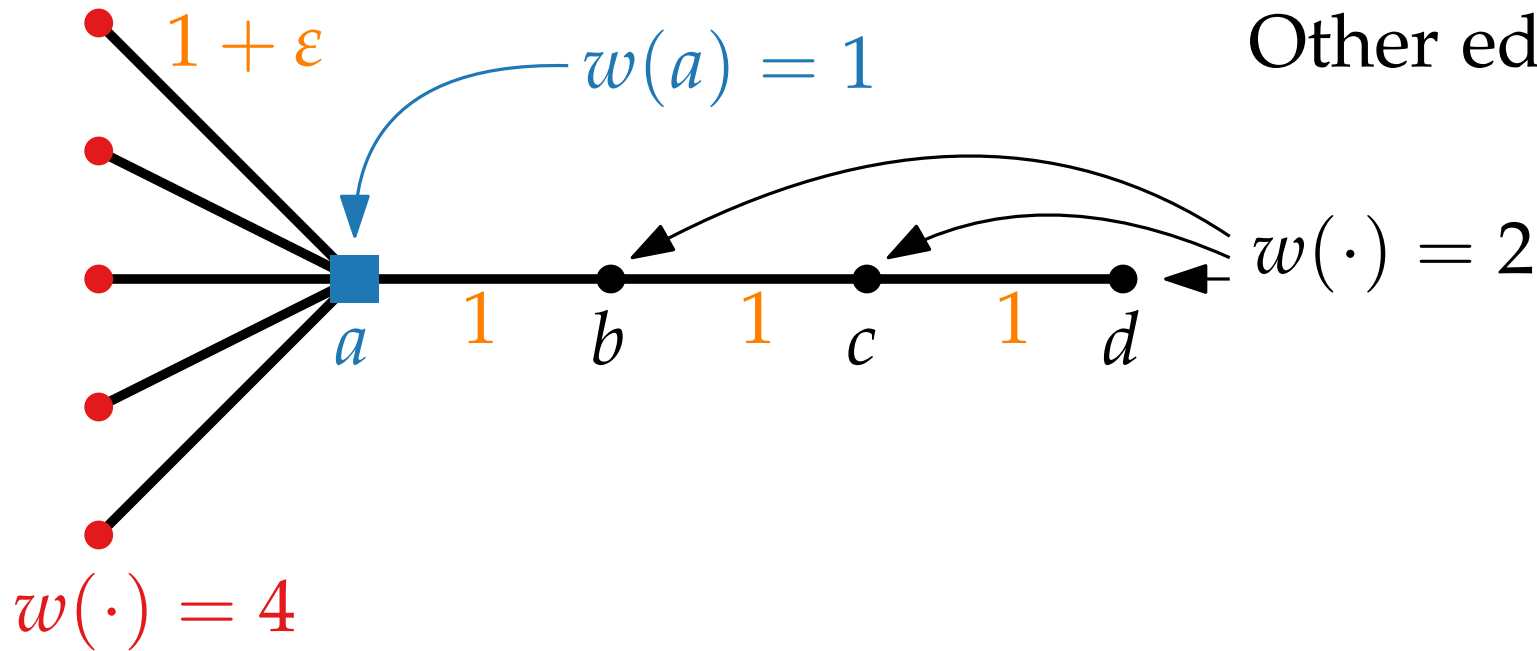


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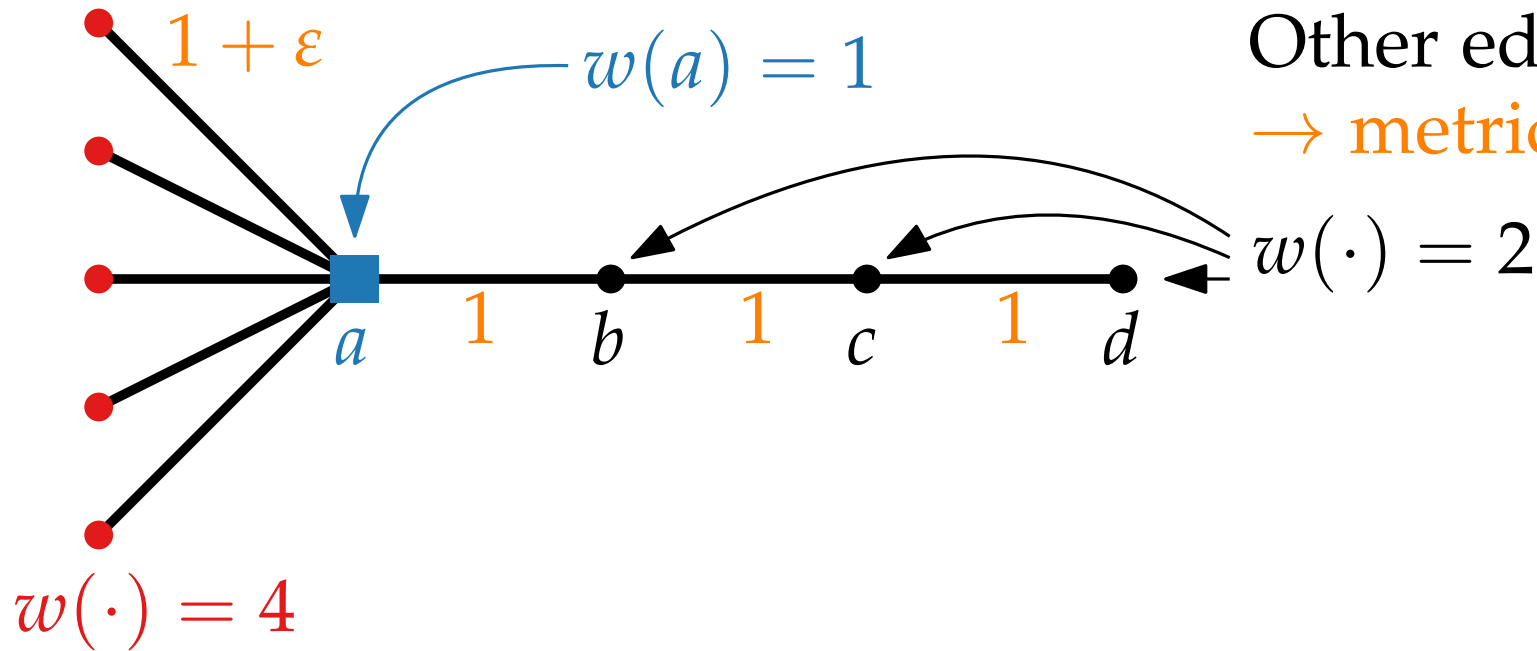


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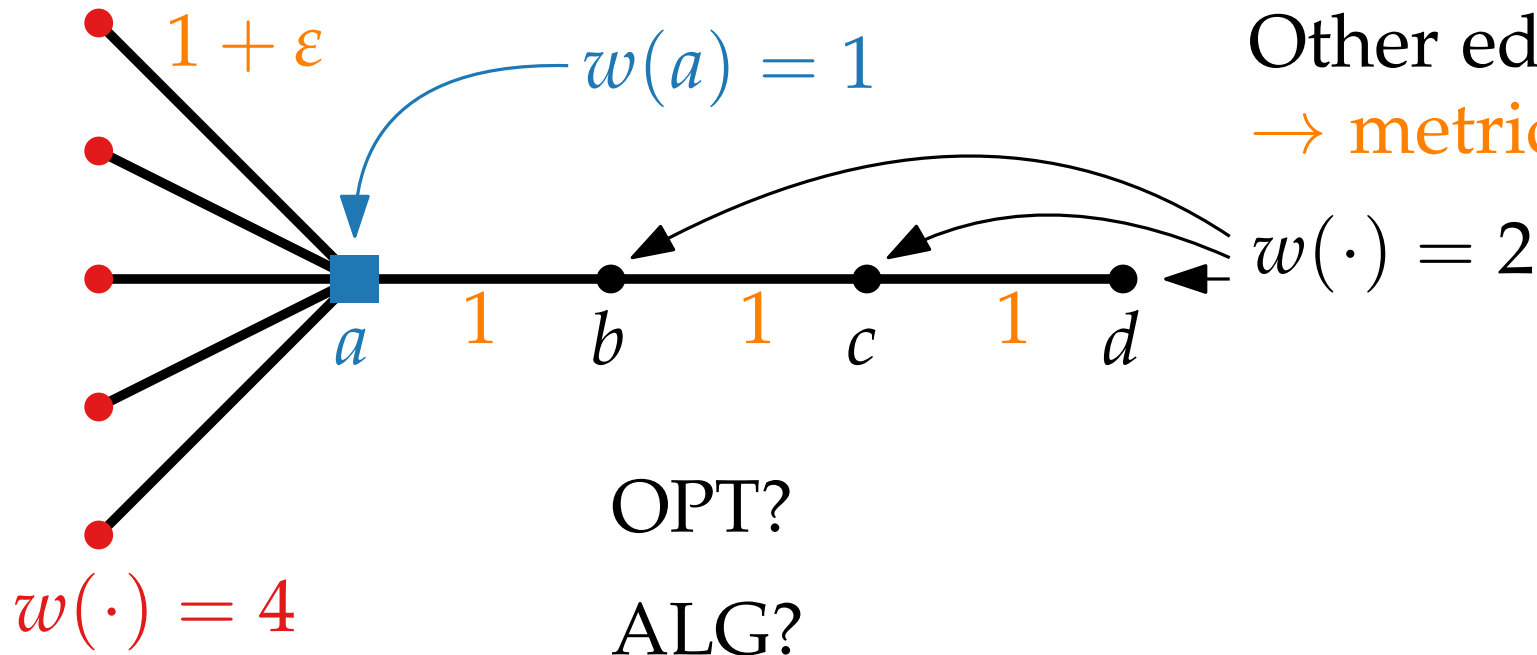


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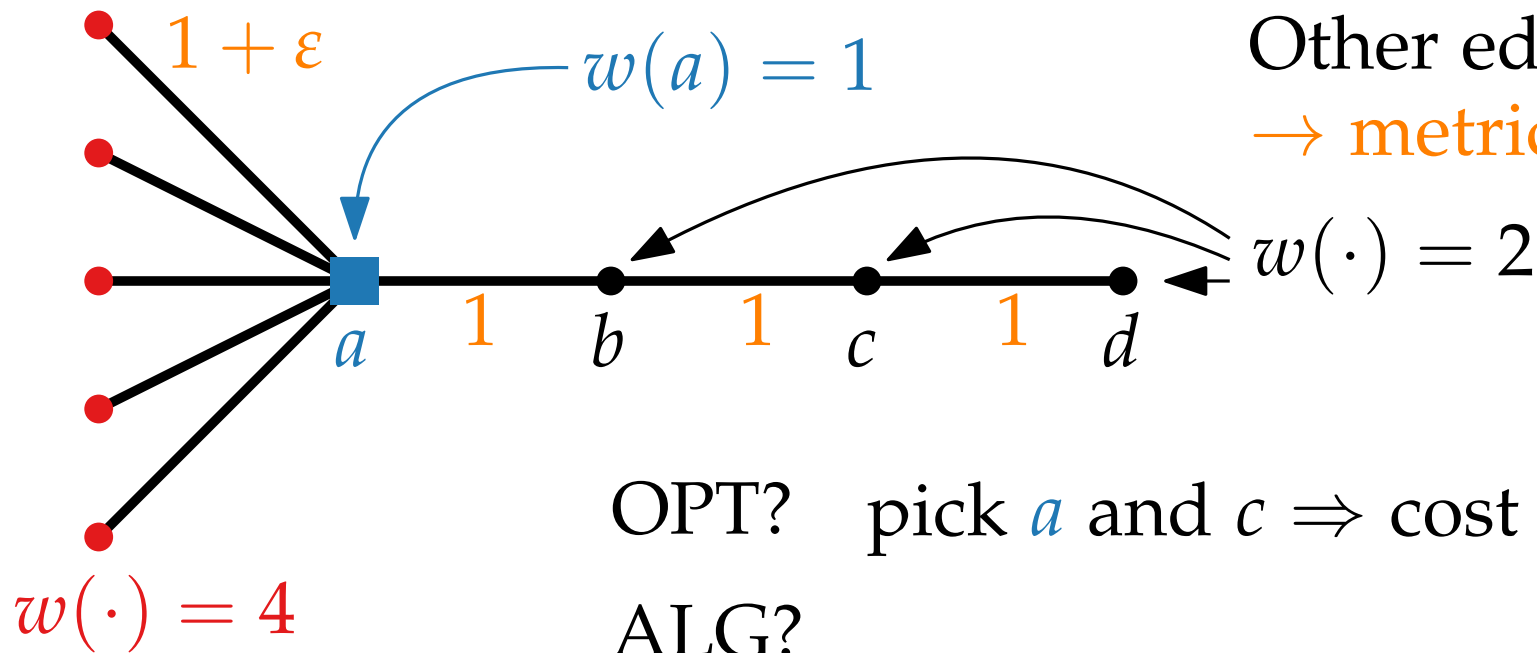


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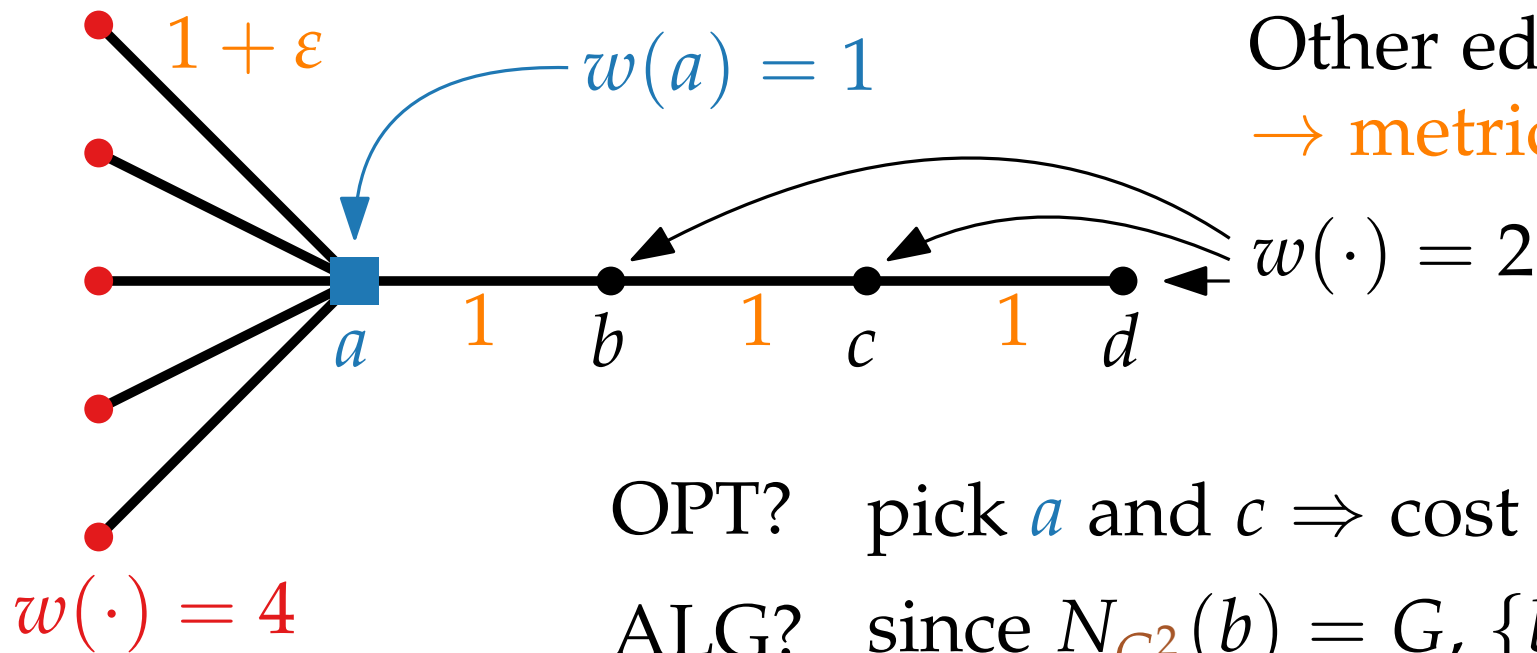


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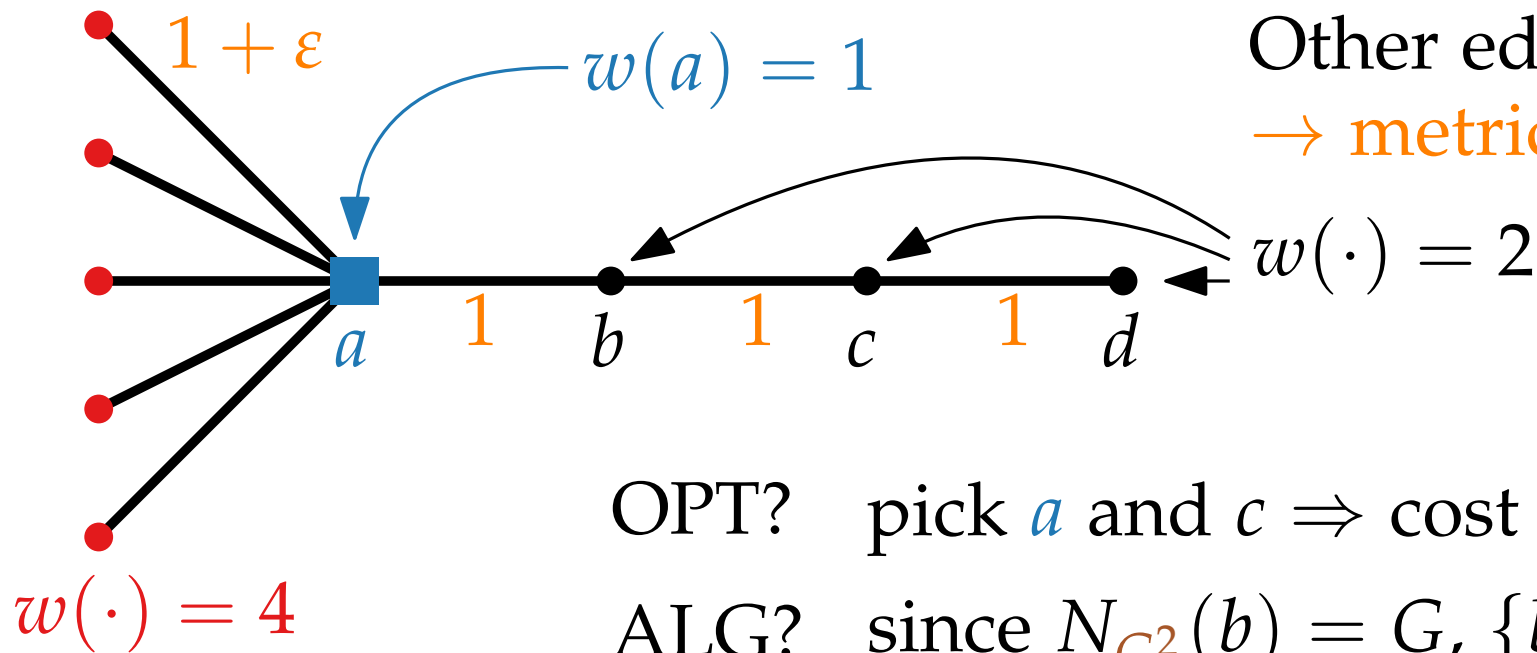
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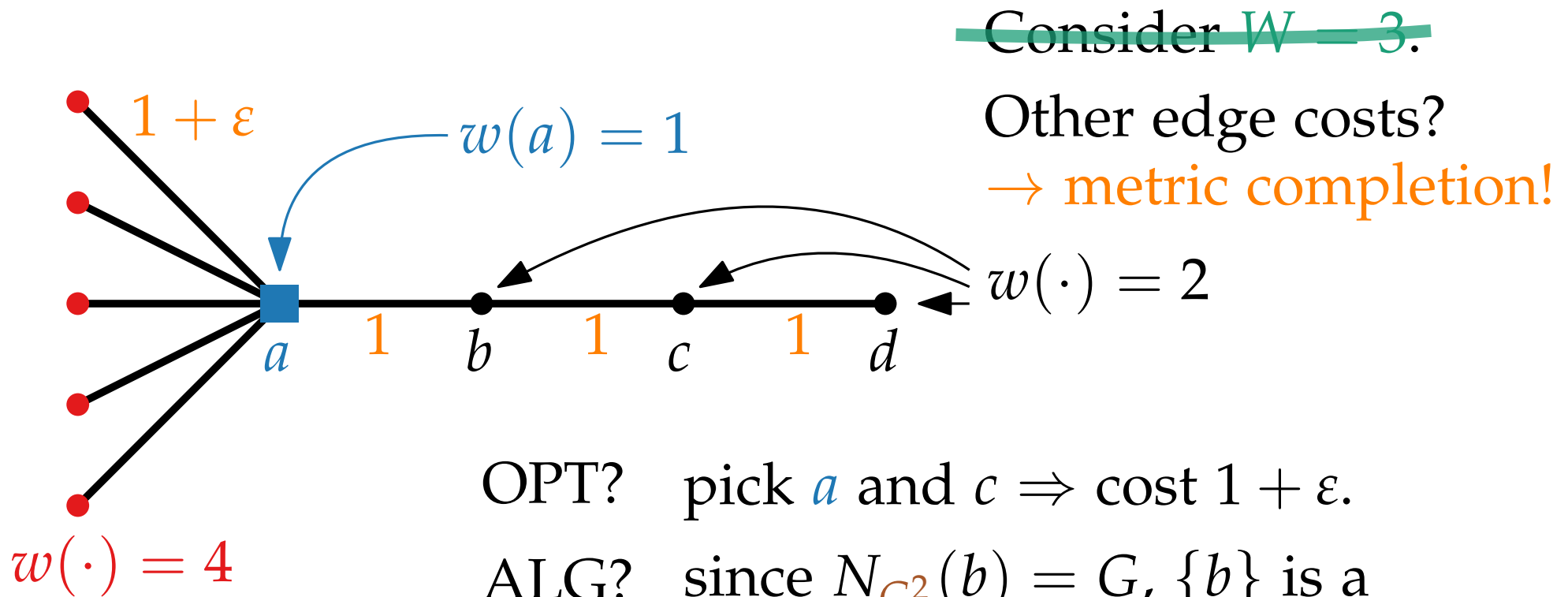


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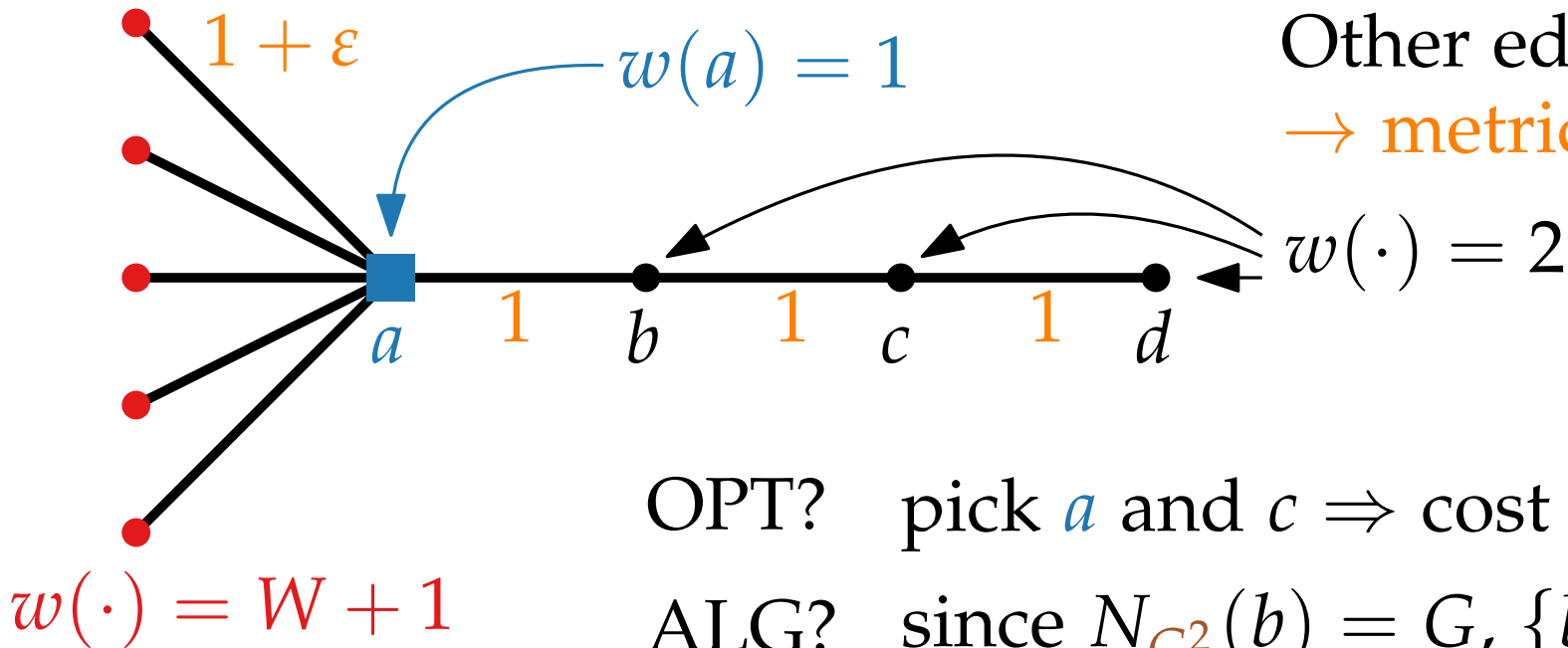
How can we generalize this to larger W ?

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