Approximation Algorithms

Lecture 5: LP-based Approximation Algorithms for SetCover

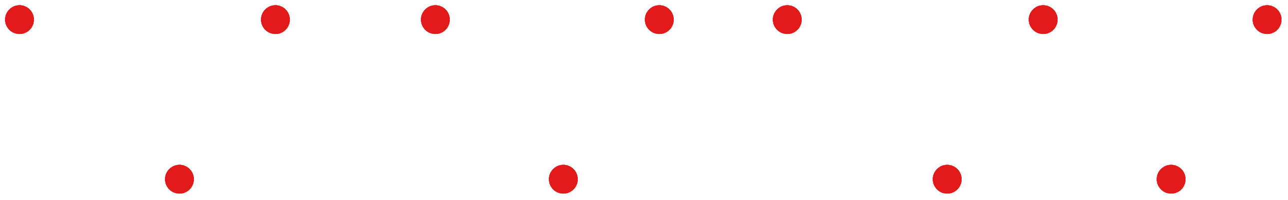
Part I: SetCover as an ILP

Alexander Wolff

Winter 2022/23

SetCover as an ILP

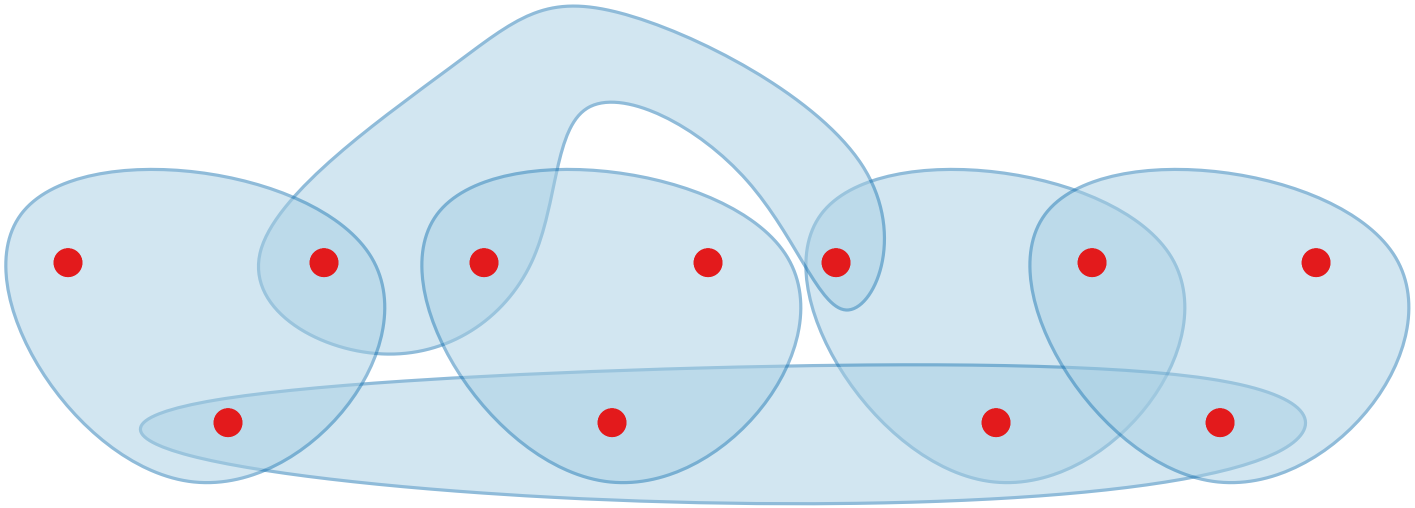
Ground set U



SetCover as an ILP

Ground set U

Family $\mathcal{S} \subseteq 2^U$ with $\bigcup \mathcal{S} = U$



SetCover as an ILP

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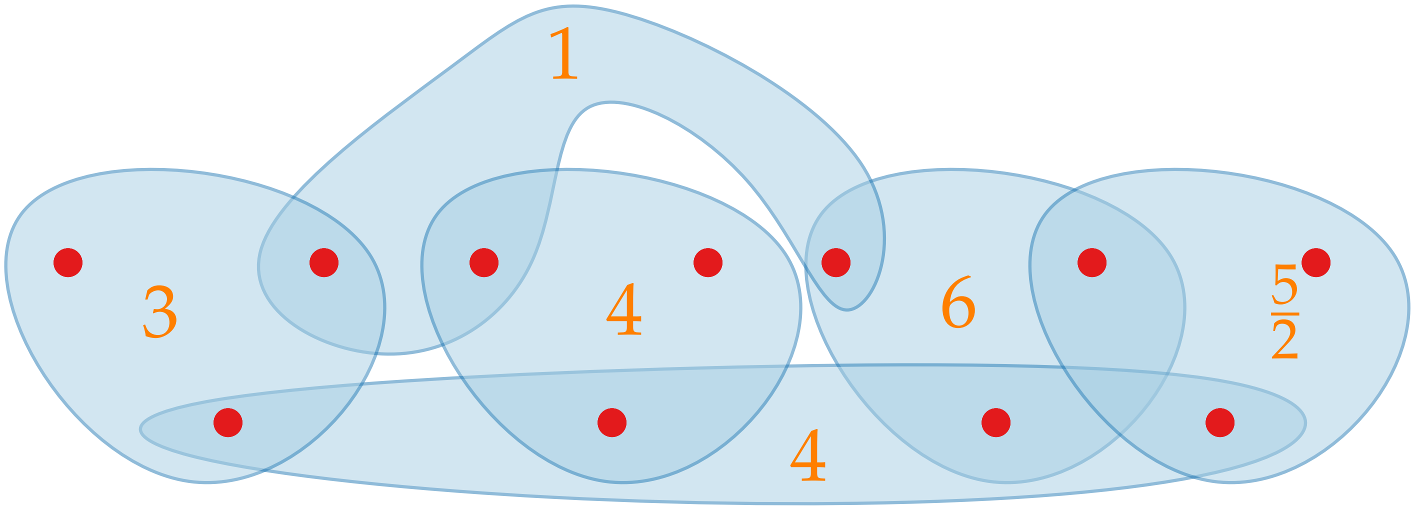
Family $\mathcal{S} \subseteq 2^U$ with $\bigcup \mathcal{S} = U$
Costs $c \colon \mathcal{S} \to \mathbb{Q}^+$



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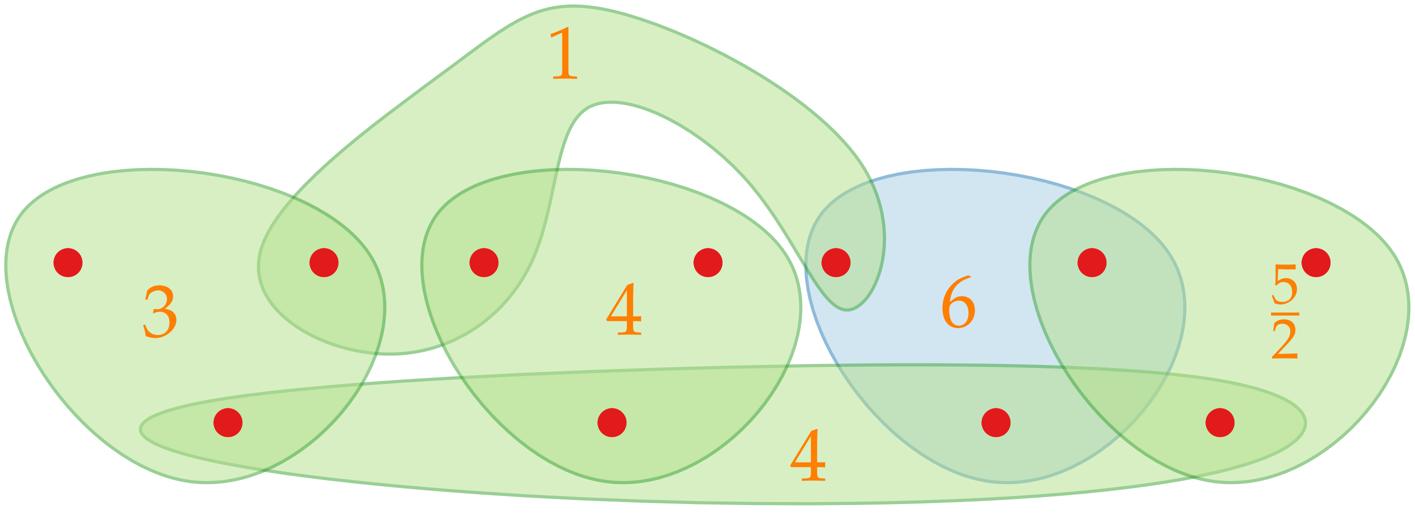

Find cover
$\mathcal{S}' \subseteq \mathcal{S}$ of U with
minimum cost.

SetCover as an ILP

minimize

subject to

Ground set U
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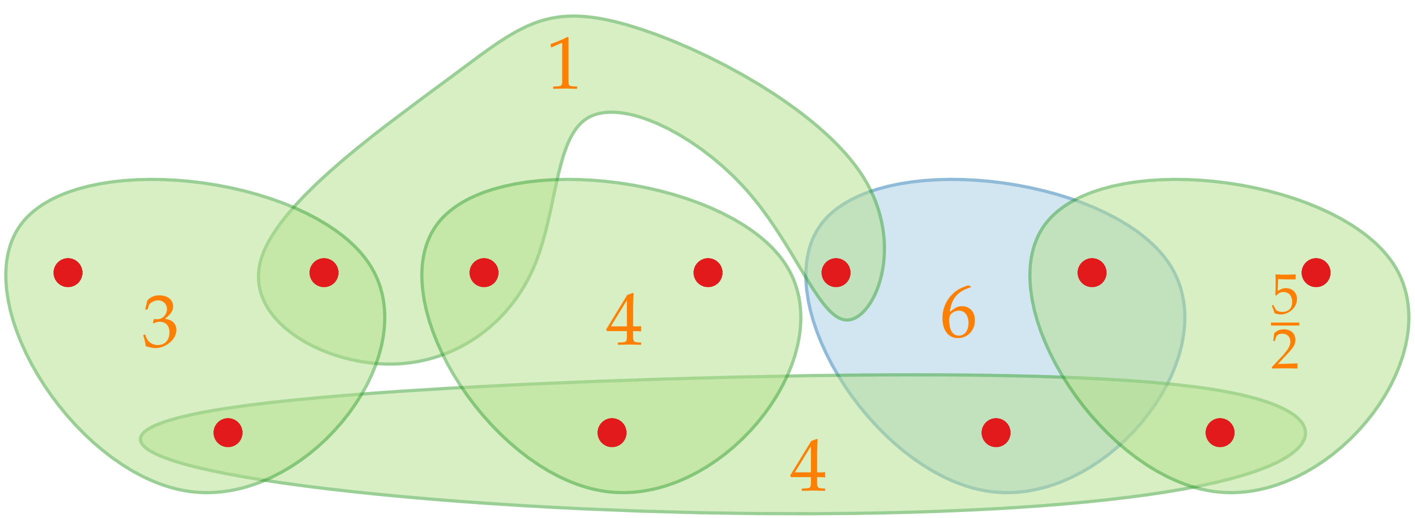
Costs $c \colon \mathcal{S} \to \mathbb{Q}^+$



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$x_S \qquad S \in \mathcal{S}$

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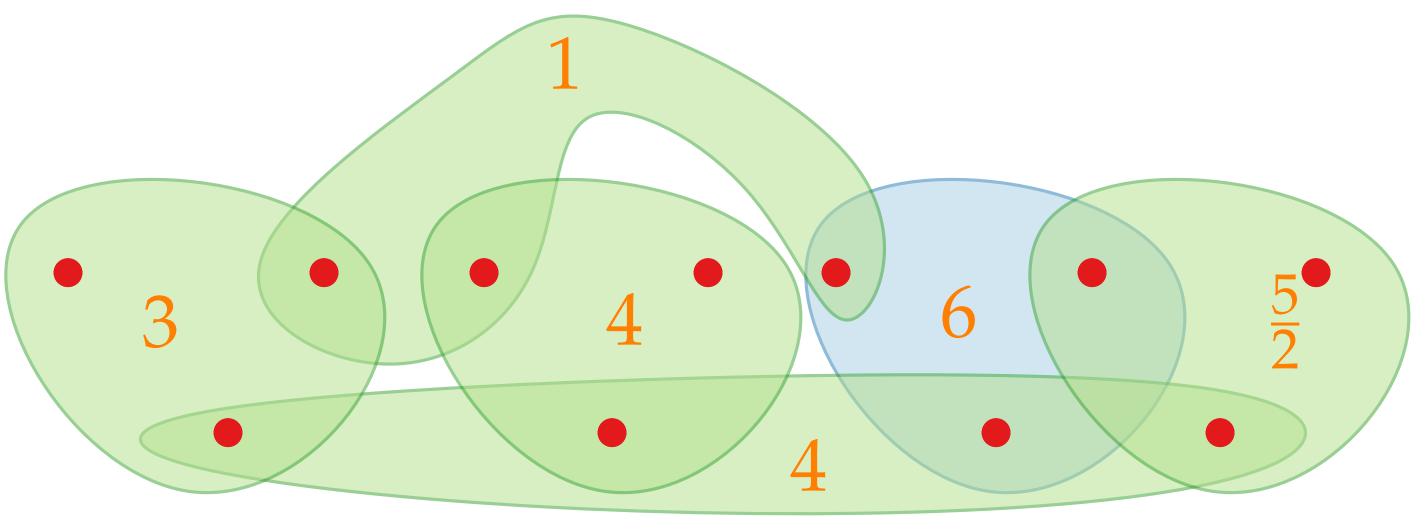

Find cover $\mathcal{S}' \subseteq \mathcal{S}$ of U with minimum cost.

SetCover as an ILP

minimize

subject to

$x_S \in \{0,1\} \quad S \in \mathcal{S}$

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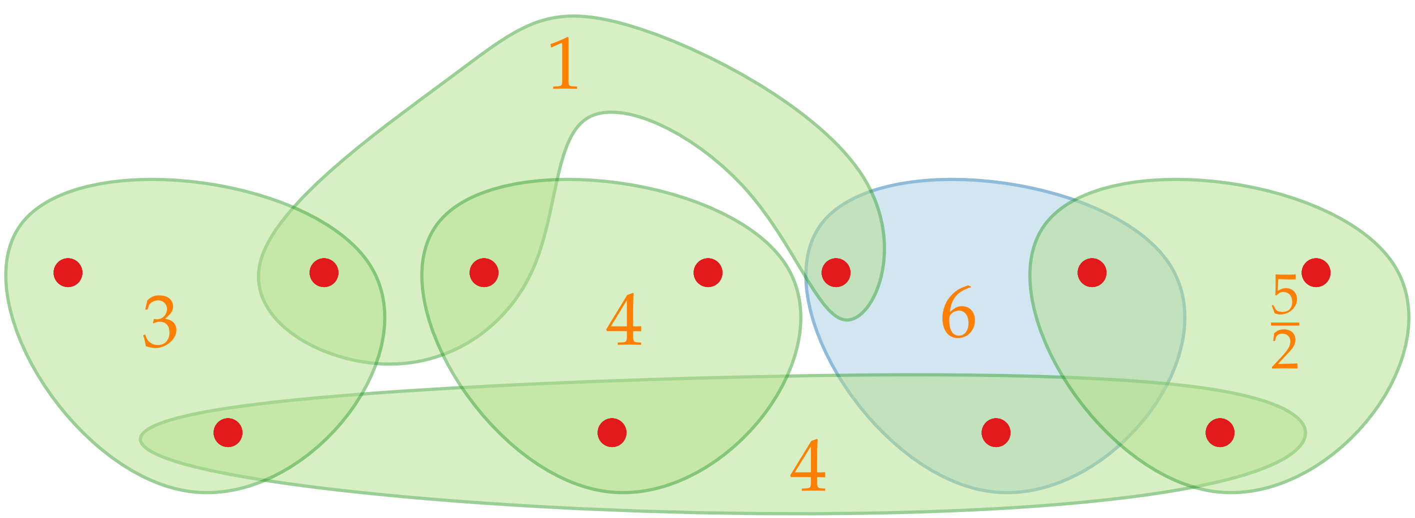

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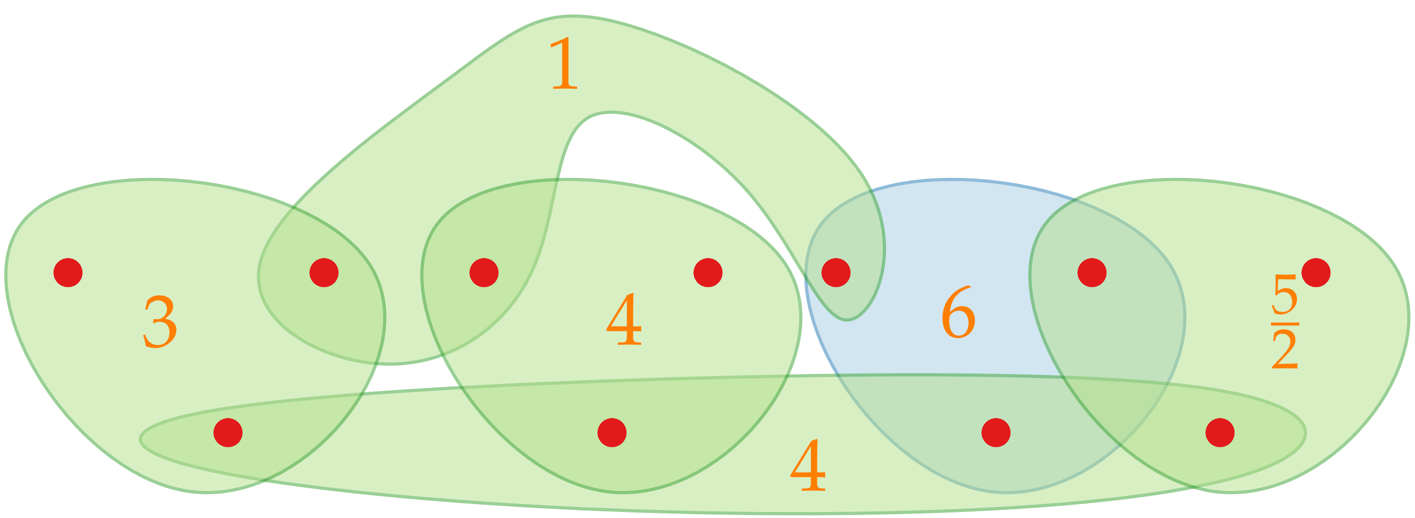
SetCover as an ILP

minimize	$\sum_{S \in \mathcal{S}} c_S x_S$	
subject to		
	$x_S \in \{0,1\}$	$S \in \mathcal{S}$

Ground set U

Family $\mathcal{S} \subseteq 2^U$ with $\bigcup \mathcal{S} = U$

Costs $c \colon \mathcal{S} \to \mathbb{Q}^+$



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SetCover as an ILP

$$
\begin{aligned}
\textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S \\
\textbf{subject to} \quad & \sum_{S \ni u} x_S \geq 1 \qquad u \in U \\
& x_S \in \{0,1\} \qquad S \in \mathcal{S}
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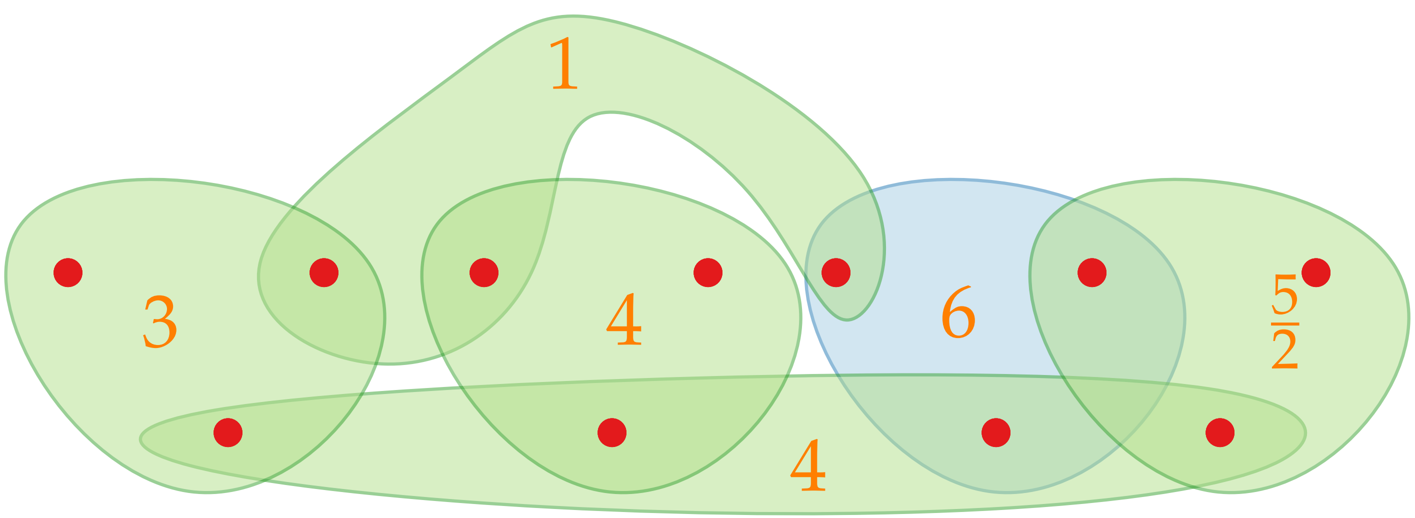

Find cover $\mathcal{S}' \subseteq \mathcal{S}$ of U with minimum cost.

Approximation Algorithms

Lecture 5:
LP-based Approximation Algorithms for SetCover

Part II:
LP-Rounding

Alexander Wolff

Winter 2022/23

Technique I) LP-Rounding

Consider a minimization problem Π in ILP form.

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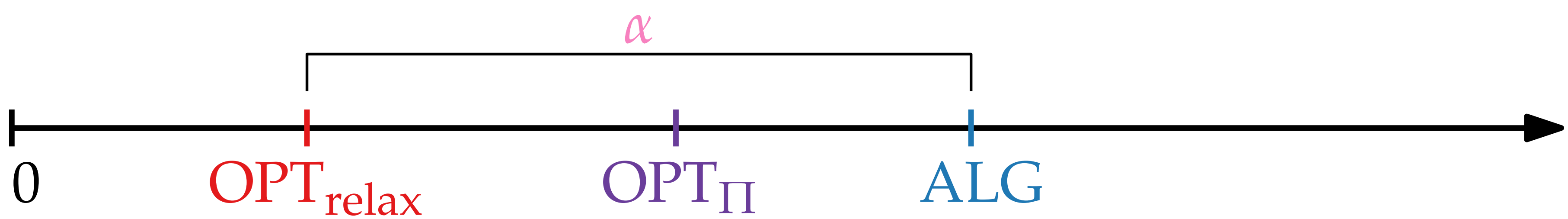
Difficulty: Ensure the **feasiblity** of the solution.

Technique I) LP-Rounding



Consider a minimization problem Π in ILP form.

Compute a solution for the LP-relaxation.

Round to obtain an integer solution for Π.

Difficulty: Ensure the **feasiblity** of the solution.

Approximation factor: $\text{ALG}/\text{OPT}_{\Pi} \leq \text{ALG}/\text{OPT}_{\text{relax}}$.

SetCover – LP-Relaxation

$$\begin{aligned}
\textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S & \\
\textbf{subject to} \quad & \sum_{S \ni u} x_S \geq 1 & u \in U \\
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\end{aligned}$$

SetCover – LP-Relaxation

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Optimal?

SetCover – LP-Relaxation

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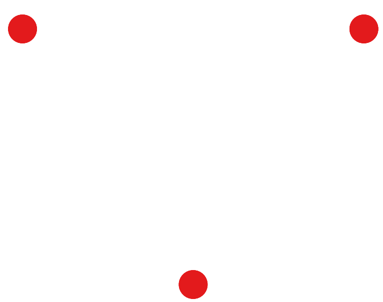

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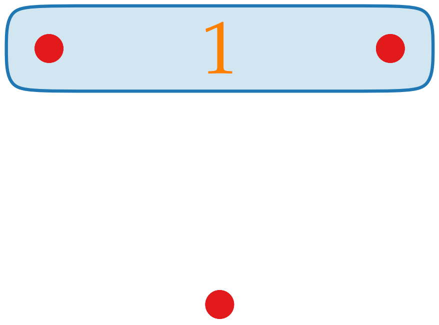

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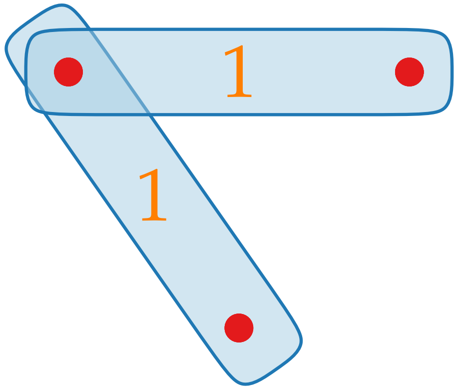

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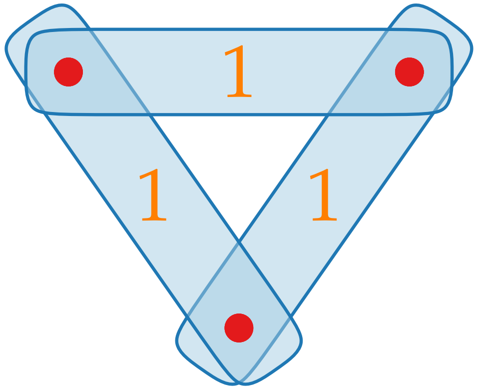

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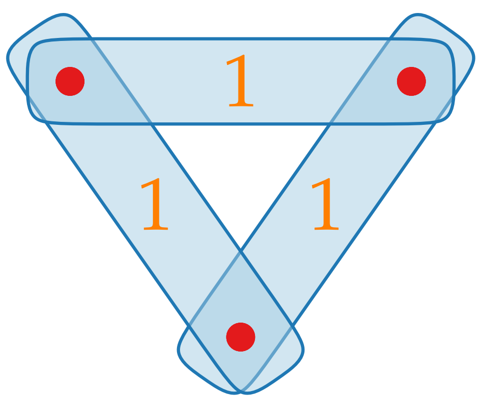

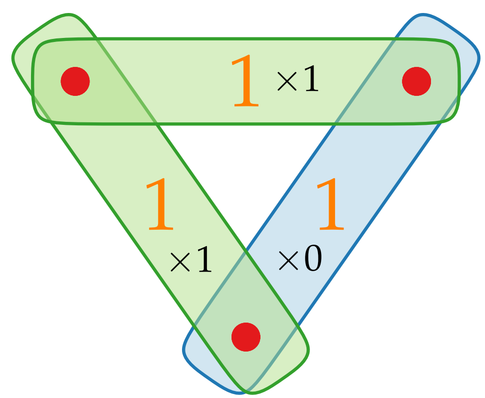

integer: 2

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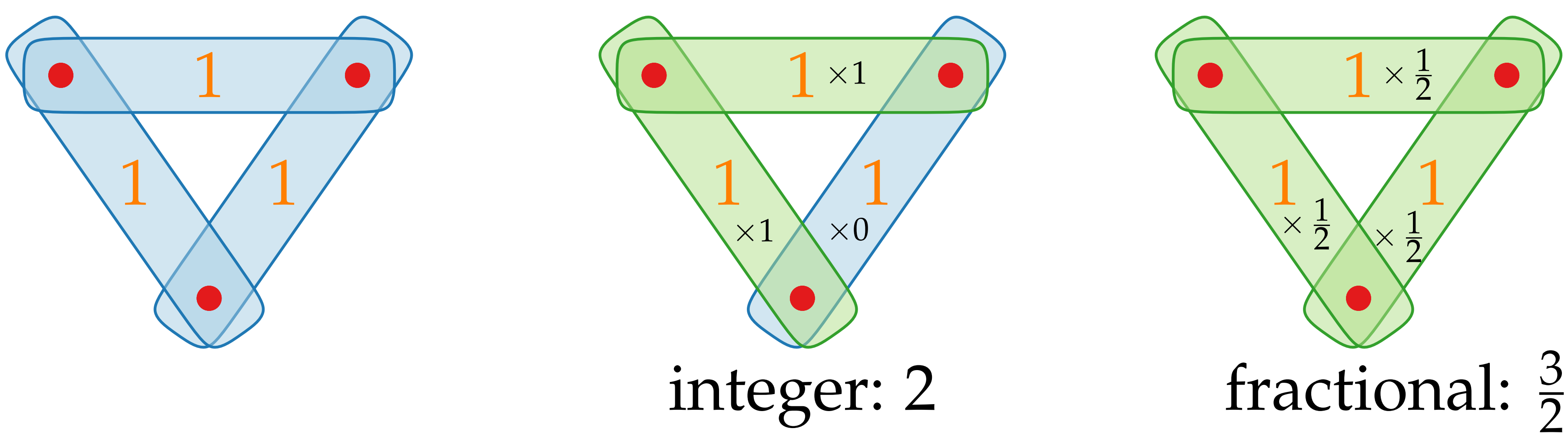

LP-Rounding: Approach I

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LP-Rounding-One$(U, \mathcal{S}, c)$

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LP-Rounding-One($U, \mathcal{S}, c$)

Compute optimal solution x for LP-relaxation.
Round each x_S with $x_S > 0$ to 1.

LP-Rounding: Approach I

$$\begin{aligned} \textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S \\ \textbf{subject to} \quad & \sum_{S \ni u} x_S \geq 1 && u \in U \\ & x_S \geq 0 && S \in \mathcal{S} \end{aligned}$$

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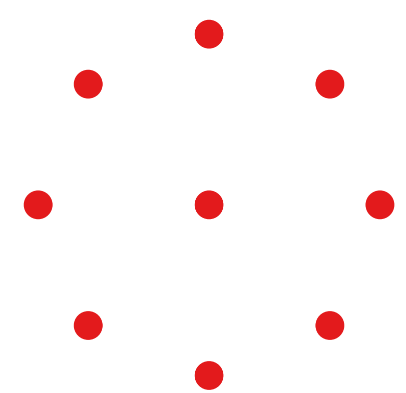

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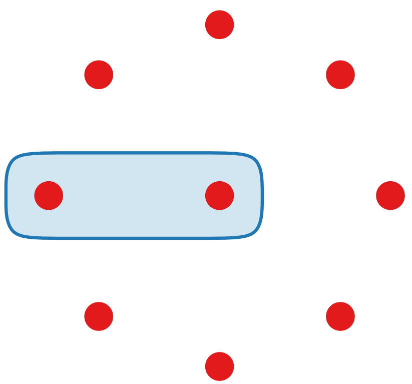

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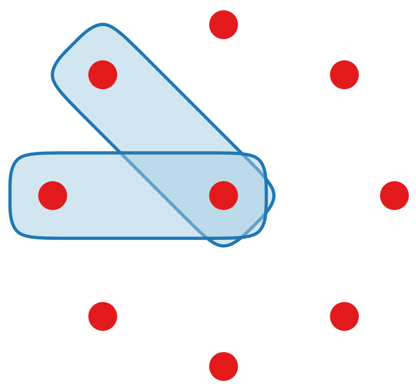

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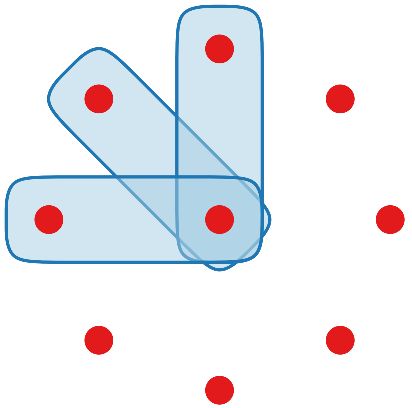

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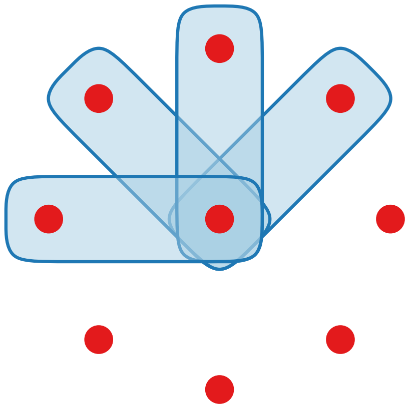

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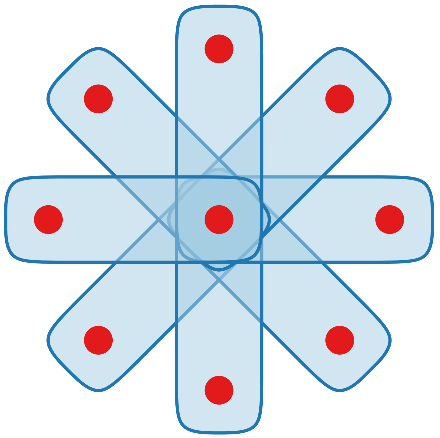

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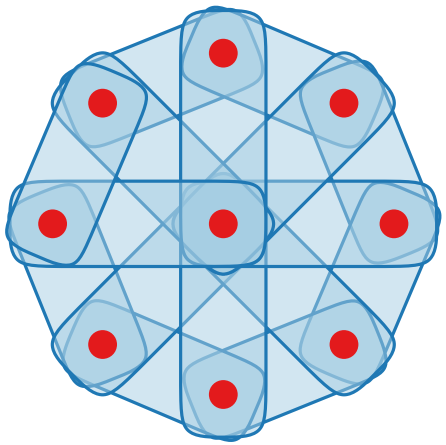

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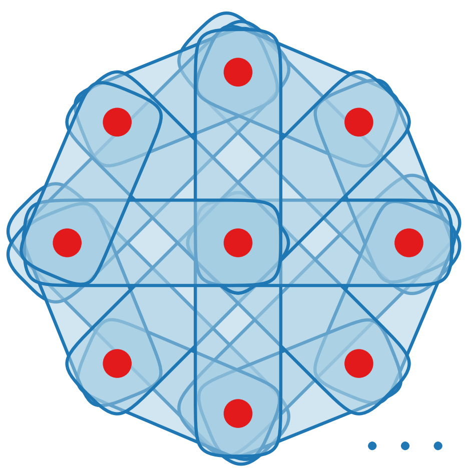

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Compute optimal solution x for LP-relaxation.

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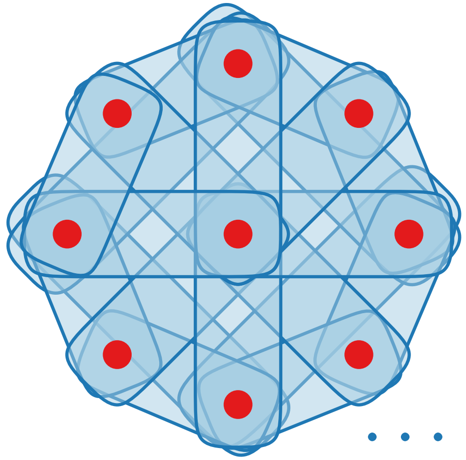
Use frequency f



LP-Rounding: Approach II

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LP-Rounding-Two($U, \mathcal{S}, c$)

Compute optimal solution x for LP-Relaxation.

Round each x_S with $x_S \geq 1/f$ to 1; remaining to 0.

Let f be the frequency of (i.e., the number of sets containing) the most frequent element.

LP-Rounding: Approach II

$$\begin{aligned} \textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S \\ \textbf{subject to} \quad & \sum_{S \ni u} x_S \geq 1 \qquad u \in U \\ & x_S \geq 0 \qquad S \in \mathcal{S} \end{aligned}$$

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Compute optimal solution x for LP-Relaxation.
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Theorem. LP-Rounding-Two is a factor-f approximation algorithm for SetCover.

Approximation Algorithms

Lecture 5: LP-based Approximation Algorithms for SetCover

Part III: The Primal-Dual Schema

Alexander Wolff

Winter 2022/23

Technique II) Primal–Dual Approach

Consider a minimization problem Π in ILP form.

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increase s_{d} according to CS and make s_{Π} "more feasible".

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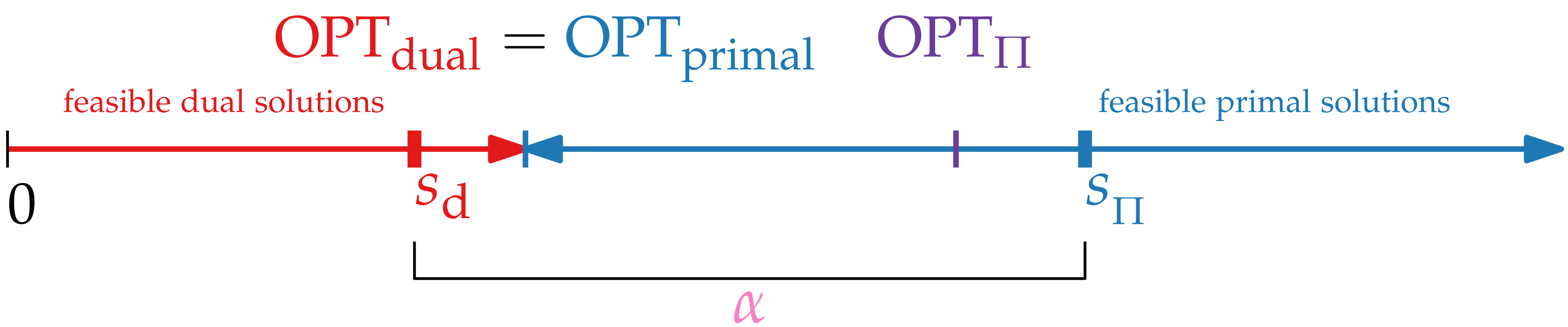

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Approximation factor $\leq \mathrm{obj}(s_{\Pi}) / \mathrm{obj}(s_{\mathrm{d}})$

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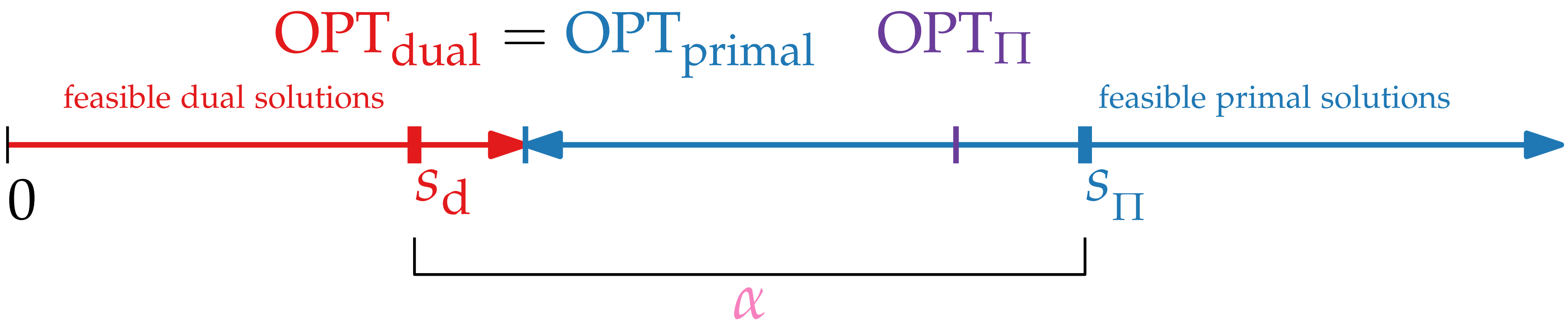

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increase s_{d} according to CS and make s_{Π} "more feasible".

Approximation factor $\leq \mathrm{obj}(s_{\Pi}) / \mathrm{obj}(s_{\mathrm{d}})$

Advantage: don't need LP-"machinery"; possibly faster, more flexible.

SetCover – Dual LP

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SetCover – Dual LP

$$\begin{aligned} \textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S \\ \textbf{subject to} \quad & \sum_{S \ni u} x_S \geq 1 \qquad u \in U \\ & x_S \geq 0 \qquad S \in \mathcal{S} \end{aligned}$$

maximize

subject to

SetCover – Dual LP

$$\begin{aligned}
\textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S \\
\textbf{subject to} \quad & \sum_{S \ni u} x_S \geq 1 && u \in U \\
& x_S \geq 0 && S \in \mathcal{S}
\end{aligned}$$

$$\begin{aligned}
\textbf{maximize} \quad & \\
\textbf{subject to} \quad & \\
& y_u \geq 0 && u \in U
\end{aligned}$$

SetCover – Dual LP

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$$\begin{aligned} \textbf{maximize} \quad & \sum_{u \in U} y_u \\ \textbf{subject to} \quad & \sum_{u \in S} y_u \leq c_S && S \in \mathcal{S} \\ & y_u \geq 0 && u \in U \end{aligned}$$

Complementary Slackness

minimize $c^\top x$
subject to $Ax \geq b$
$x \geq 0$

maximize $b^\top y$
subject to $A^\top y \leq c$
$y \geq 0$

Theorem. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be valid solutions for the primal and dual program (resp.). Then x and y are optimal if and only if the following conditions are met:

Primal CS:
For each $j = 1, \ldots, n$: $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
For each $i = 1, \ldots, m$: $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

Relaxing Complementary Slackness

minimize	$c^\top x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\top y$		
subject to	$A^\top y$	$\leq$	c
	y	$\geq$	0

Primal CS:

For each $j = 1, \ldots, n$: $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:

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$$\Leftrightarrow \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i y_i$$

Relaxing Complementary Slackness

minimize	$c^\top x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\top y$		
subject to	$A^\top y$	$\leq$	c
	y	$\geq$	0

~~**Primal CS**~~: Relaxed Primal CS

For each $j = 1, \ldots, n$: $x_j = 0$ or ~~$\sum_{i=1}^m a_{ij} y_i = c_j$~~

$$c_j/\alpha \leq \sum_{i=1}^m a_{ij} y_i \leq c_j$$

Dual CS:

For each $i = 1, \ldots, m$: $y_i = 0$ or $\sum_{j=1}^n a_{ij} x_j = b_i$

$$\Leftrightarrow \sum_{j=1}^n c_j x_j = \sum_{i=1}^m b_i y_i$$

Relaxing Complementary Slackness

$$\begin{array}{llll} \textbf{minimize} & c^\top x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{llll} \textbf{maximize} & b^\top y & & \\ \textbf{subject to} & A^\top y & \leq & c \\ & y & \geq & 0 \end{array}$$

~~Primal CS~~: Relaxed Primal CS

For each $j = 1, \ldots, n$: $x_j = 0$ or ~~$\sum_{i=1}^{m} a_{ij} y_i = c_j$~~

$$c_j / \alpha \leq \sum_{i=1}^{m} a_{ij} y_i \leq c_j$$

~~Dual CS~~: Relaxed Dual CS

For each $i = 1, \ldots, m$: $y_i = 0$ or ~~$\sum_{j=1}^{n} a_{ij} x_j = b_i$~~

$$b_i \leq \sum_{j=1}^{n} a_{ij} x_j \leq \beta \cdot b_i$$

$$\Leftrightarrow \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i y_i$$

Relaxing Complementary Slackness

minimize	$c^\top x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\top y$		
subject to	$A^\top y$	$\leq$	c
	y	$\geq$	0

~~**Primal CS**~~: Relaxed Primal CS

For each $j = 1, \ldots, n$: $x_j = 0$ or ~~$\sum_{i=1}^m a_{ij} y_i = c_j$~~

$$c_j/\alpha \leq \sum_{i=1}^m a_{ij} y_i \leq c_j$$

~~**Dual CS**~~: Relaxed Dual CS

For each $i = 1, \ldots, m$: $y_i = 0$ or ~~$\sum_{j=1}^n a_{ij} x_j = b_i$~~

$$b_i \leq \sum_{j=1}^n a_{ij} x_j \leq \beta \cdot b_i$$

~~$\Leftrightarrow \sum_{j=1}^n c_j x_j = \sum_{i=1}^m b_i y_i$~~ $\Rightarrow \sum_{j=1}^n c_j x_j \leq \alpha\beta \sum_{i=1}^m b_i y_i \leq \alpha\beta \cdot \mathrm{OPT}_{\mathrm{LP}}$

Primal–Dual Schema

Start with a feasible dual and infeasible primal solution (often trivial).

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Do so until the relaxed CS conditions are met.

Maintain that the primal solution is integer valued.

Primal–Dual Schema

Start with a feasible dual and infeasible primal solution (often trivial).

“Improve” the feasibility of the primal solution...

…and simultaneously the obj. value of the dual solution.

Do so until the relaxed CS conditions are met.

Maintain that the primal solution is integer valued.

The feasibility of the primal solution and relaxed CS condition provide an approximation ratio.

Relaxed CS for SetCover

minimize $\sum_{S \in \mathcal{S}} c_S x_S$

subject to $\sum_{S \ni u} x_S \geq 1 \quad u \in U$

$x_S \geq 0 \quad S \in \mathcal{S}$

maximize $\sum_{u \in U} y_u$

subject to $\sum_{u \in S} y_u \leq c_S \quad S \in \mathcal{S}$

$y_u \geq 0 \quad u \in U$

Relaxed CS for SetCover

minimize	$\sum_{S \in \mathcal{S}} c_S x_S$	
subject to	$\sum_{S \ni u} x_S \geq 1$	$u \in U$
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maximize	$\sum_{u \in U} y_u$	
subject to	$\sum_{u \in S} y_u \leq c_S$	$S \in \mathcal{S}$
	$y_u \geq 0$	$u \in U$

(Unrelaxed) primal CS:

Relaxed CS for SetCover

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subject to	$\sum_{S \ni u} x_S \geq 1$	$u \in U$
	$x_S \geq 0$	$S \in \mathcal{S}$

maximize	$\sum_{u \in U} y_u$	
subject to	$\sum_{u \in S} y_u \leq c_S$	$S \in \mathcal{S}$
	$y_u \geq 0$	$u \in U$

(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow$

Relaxed CS for SetCover

minimize	$\sum_{S \in \mathcal{S}} c_S x_S$	
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maximize	$\sum_{u \in U} y_u$	
subject to	$\sum_{u \in S} y_u \leq c_S$	$S \in \mathcal{S}$
	$y_u \geq 0$	$u \in U$

critical set ← c_S

(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

Relaxed CS for SetCover

minimize	$\sum_{S \in \mathcal{S}} c_S x_S$	
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critical set

(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

only chooses critical sets

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(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

only chooses critical sets

Relaxed dual CS:

Relaxed CS for SetCover

minimize	$\sum_{S \in \mathcal{S}} c_S x_S$	
subject to	$\sum_{S \ni u} x_S \geq 1$	$u \in U$
	$x_S \geq 0$	$S \in \mathcal{S}$

maximize	$\sum_{u \in U} y_u$	
subject to	$\sum_{u \in S} y_u \leq c_S$	$S \in \mathcal{S}$
	$y_u \geq 0$	$u \in U$

critical set

(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

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Relaxed dual CS: $y_u \neq 0 \Rightarrow$

Relaxed CS for SetCover

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(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

only chooses critical sets

Relaxed dual CS: $y_u \neq 0 \Rightarrow 1 \leq \sum_{S \ni u} x_S \leq f \cdot$

Relaxed CS for SetCover

minimize	$\sum_{S \in \mathcal{S}} c_S x_S$	
subject to	$\sum_{S \ni u} x_S \geq 1$	$u \in U$
	$x_S \geq 0$	$S \in \mathcal{S}$

maximize	$\sum_{u \in U} y_u$	
subject to	$\sum_{u \in S} y_u \leq c_S$	$S \in \mathcal{S}$
	$y_u \geq 0$	$u \in U$

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Relaxed dual CS: $y_u \neq 0 \Rightarrow 1 \leq \sum_{S \ni u} x_S \leq f \cdot 1$

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trivial for binary x

Relaxed dual CS: $y_u \neq 0 \Rightarrow 1 \leq \sum_{S \ni u} x_S \leq f \cdot 1$

Primal–Dual Schema for SetCover

PrimalDualSetCover($U, \mathcal{S}, c$)

$x \leftarrow 0, y \leftarrow 0$

repeat

until all elements are covered.

return x

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  Select an uncovered element u.

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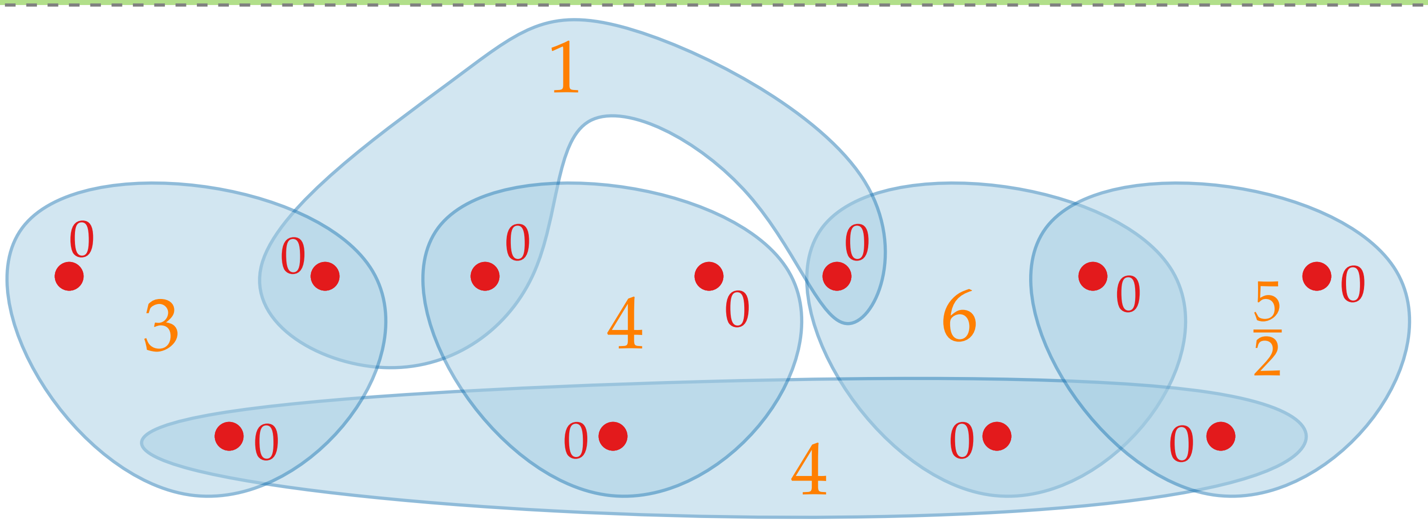

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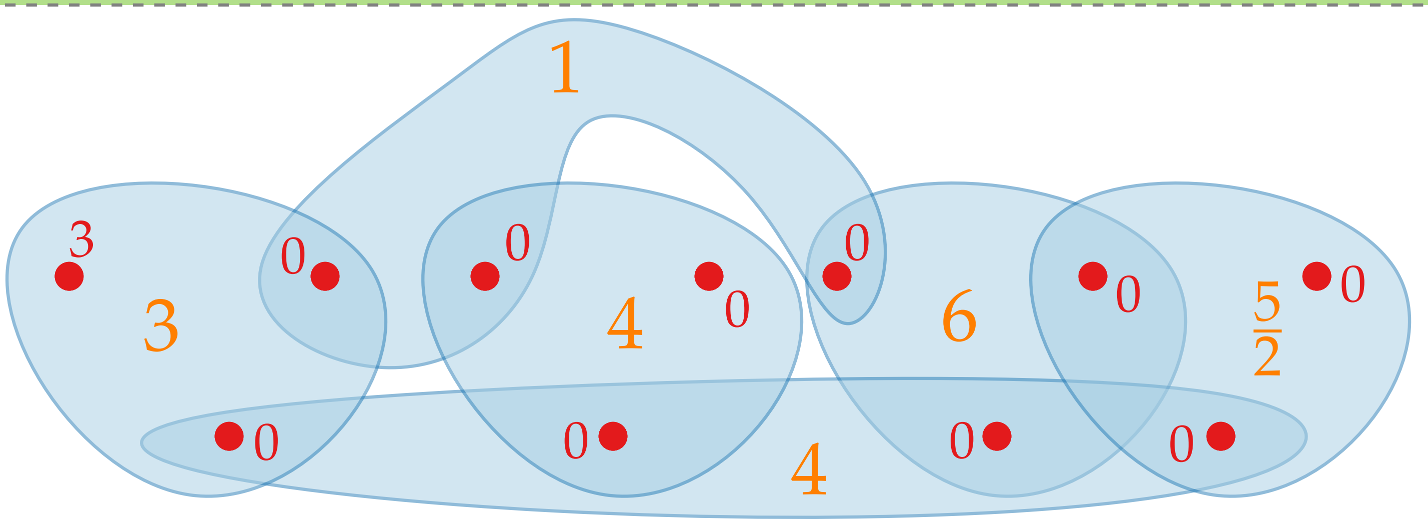

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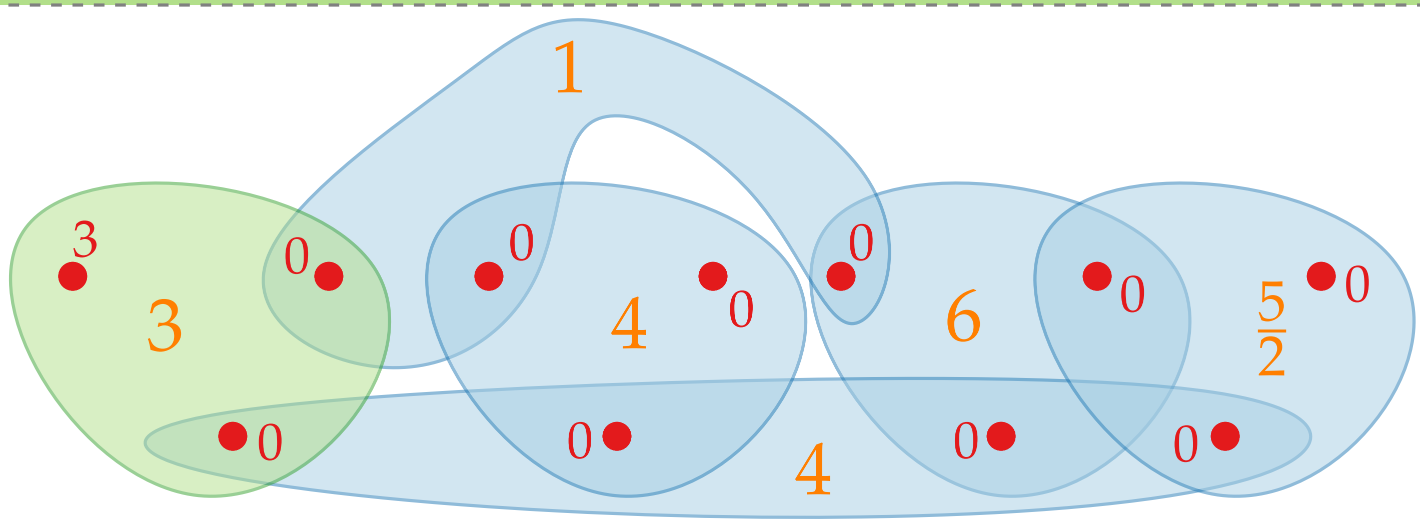

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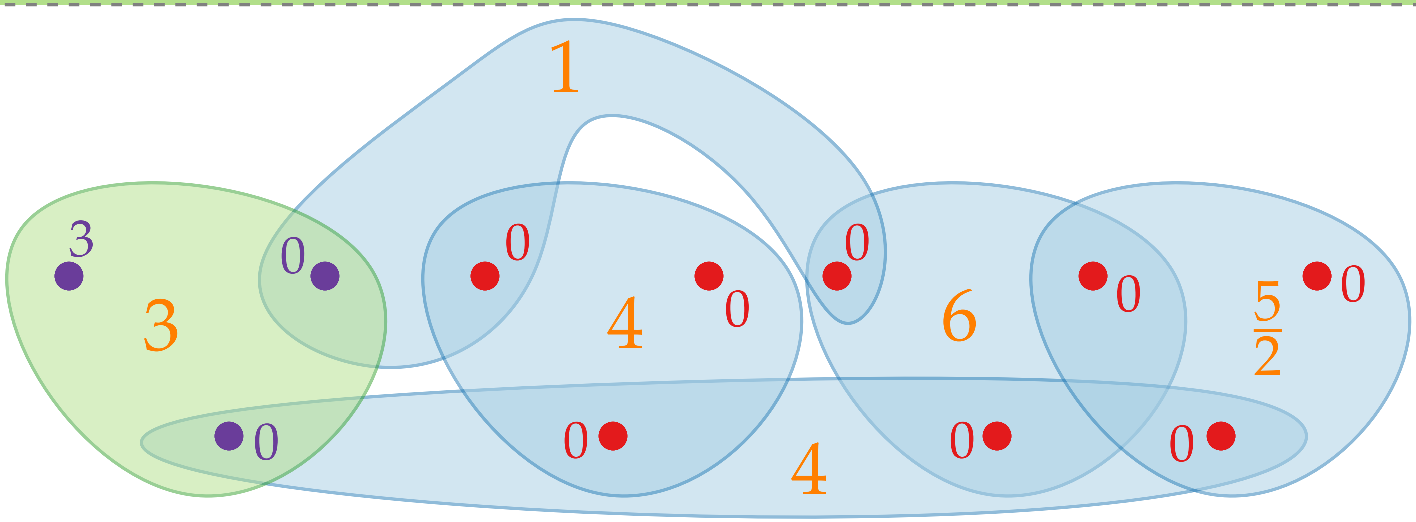

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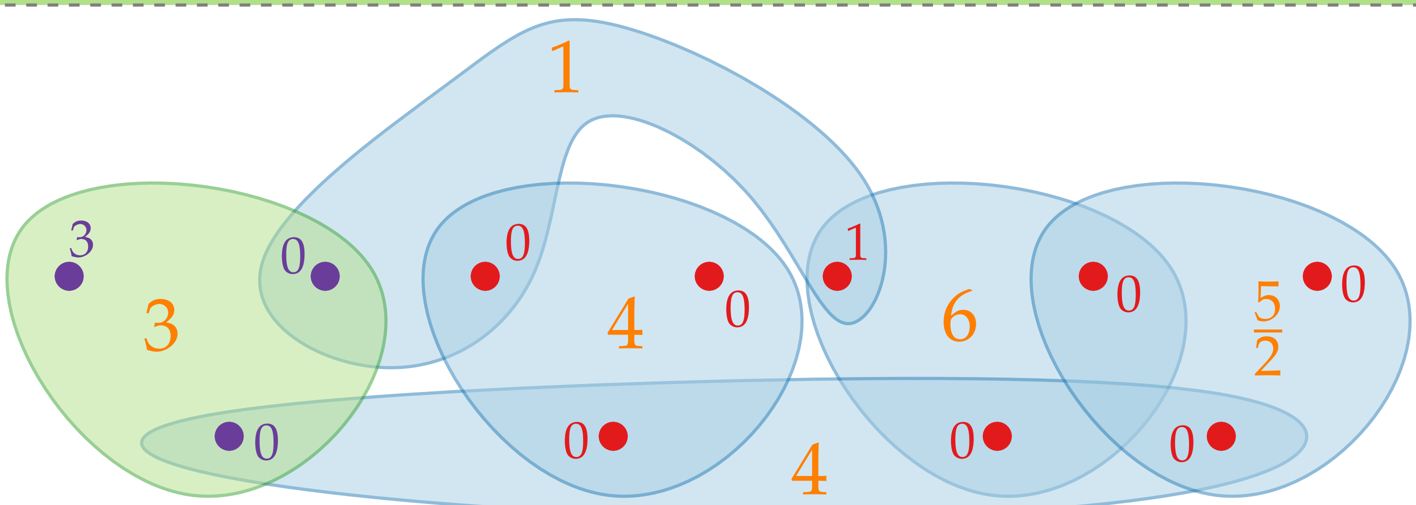

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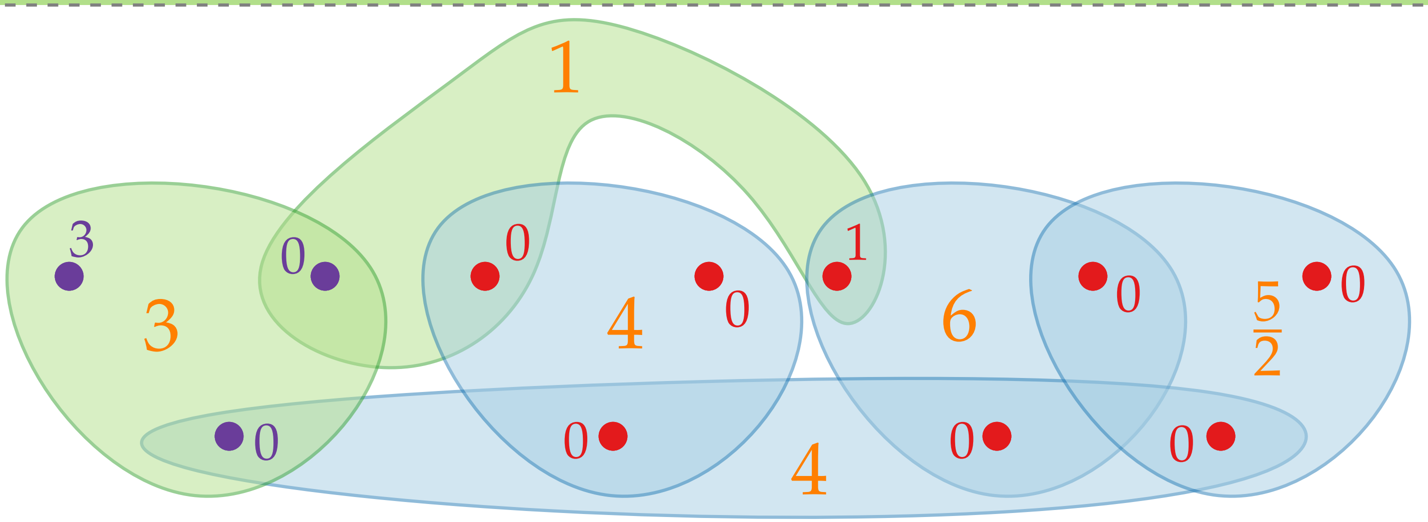

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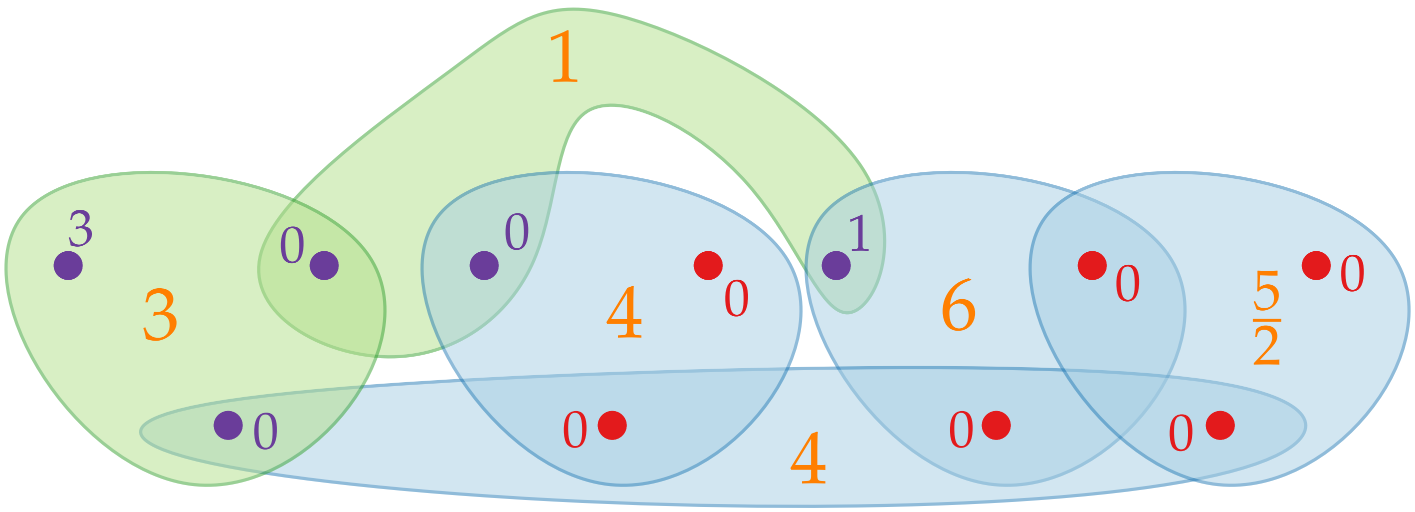

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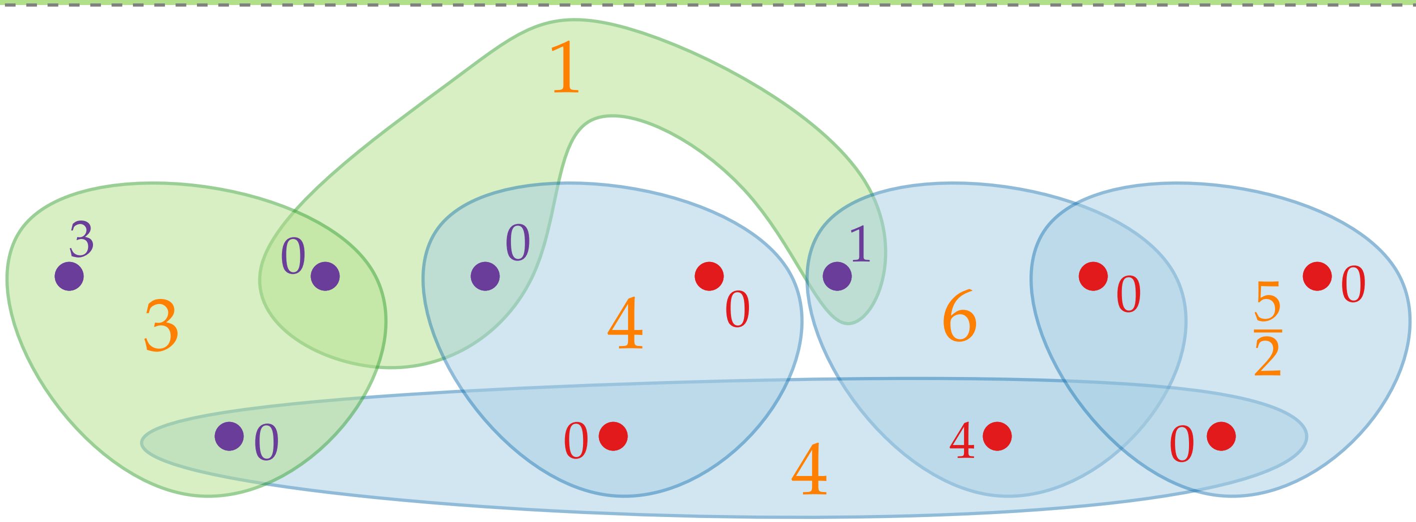

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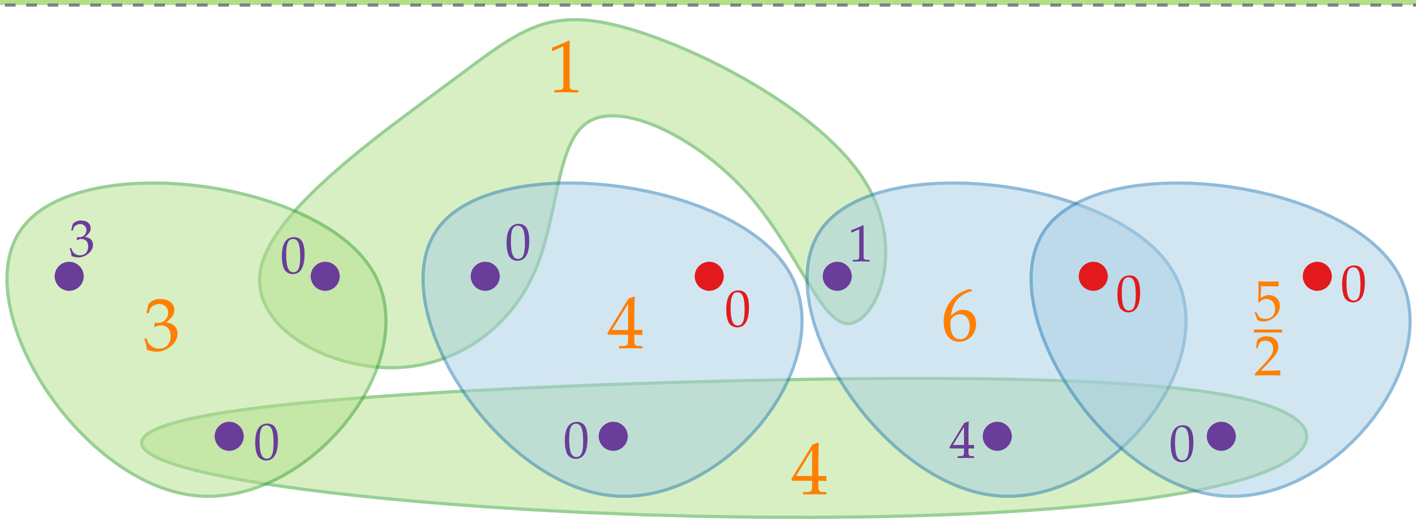

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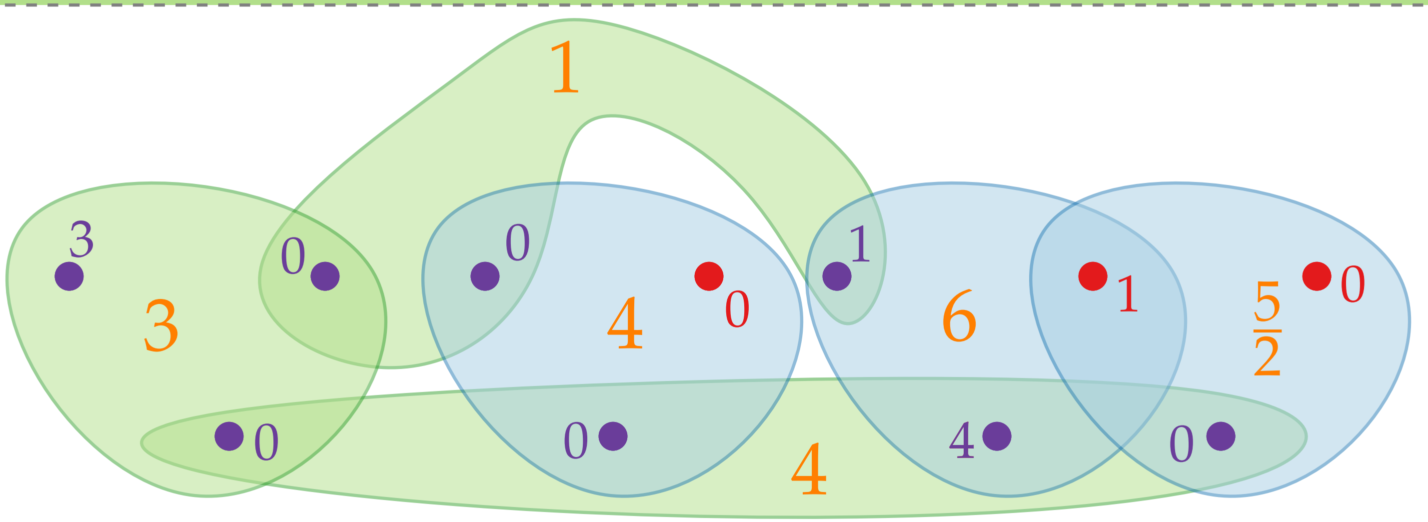

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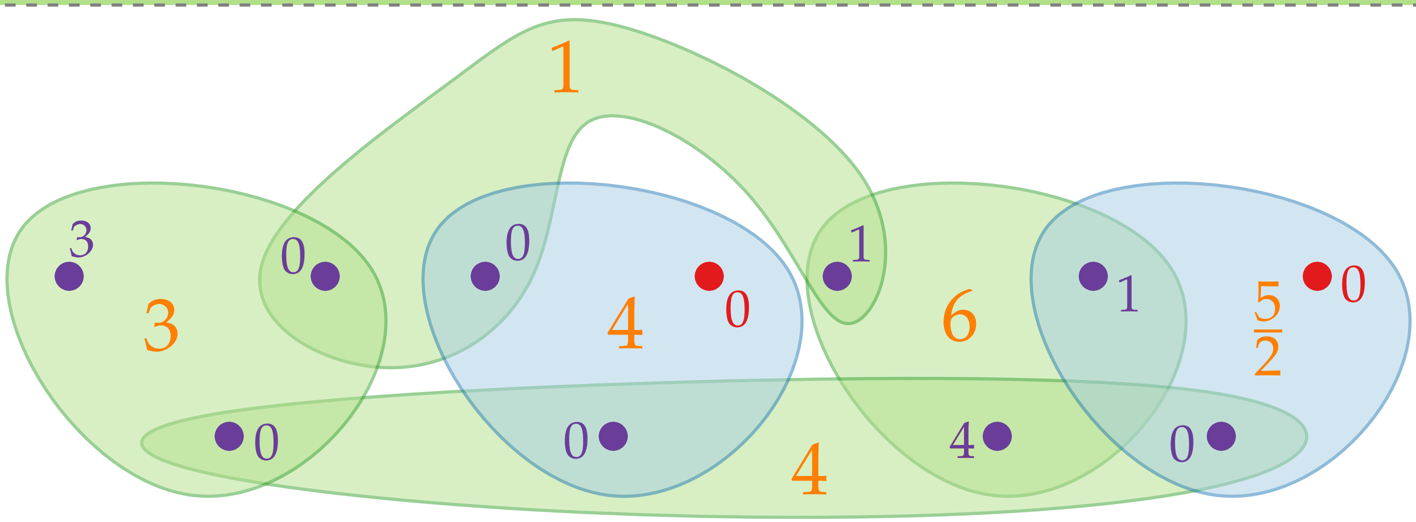

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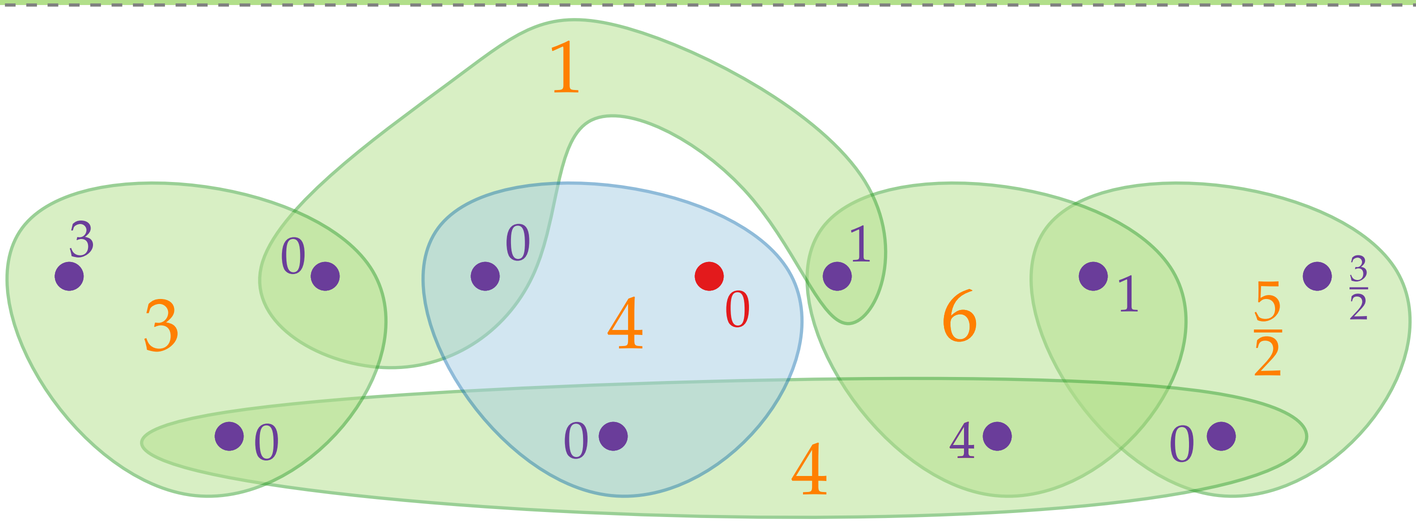

Primal–Dual Schema for SetCover

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$x \leftarrow 0, y \leftarrow 0$
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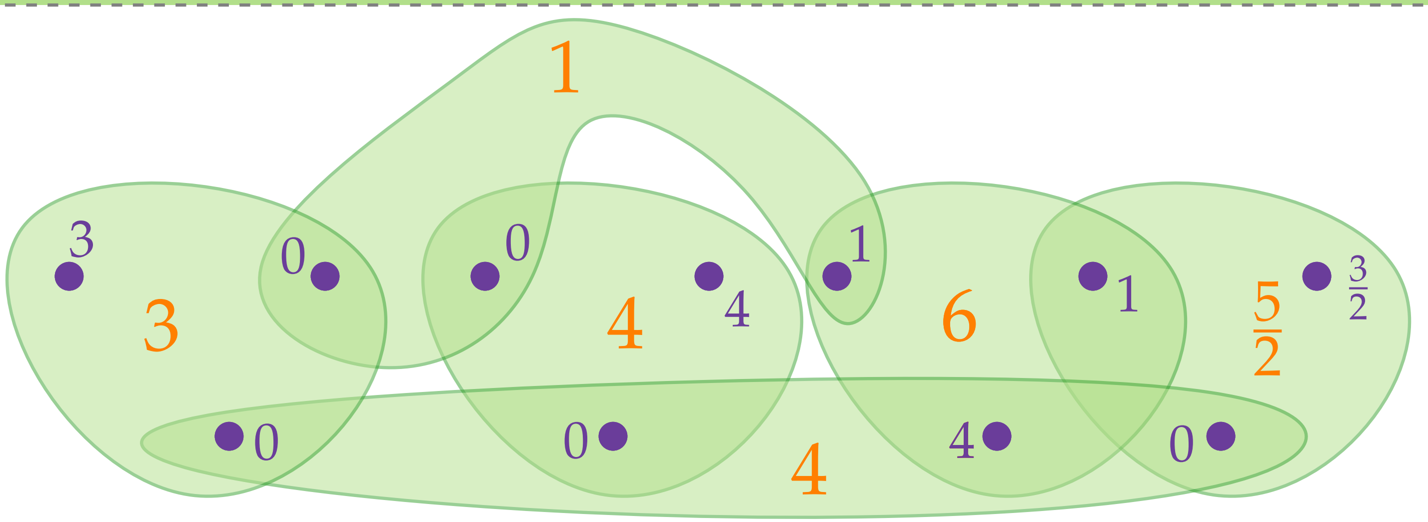

Primal–Dual Schema for SetCover

PrimalDualSetCover(U, $\mathcal{S}$, c)

$x \leftarrow 0, y \leftarrow 0$

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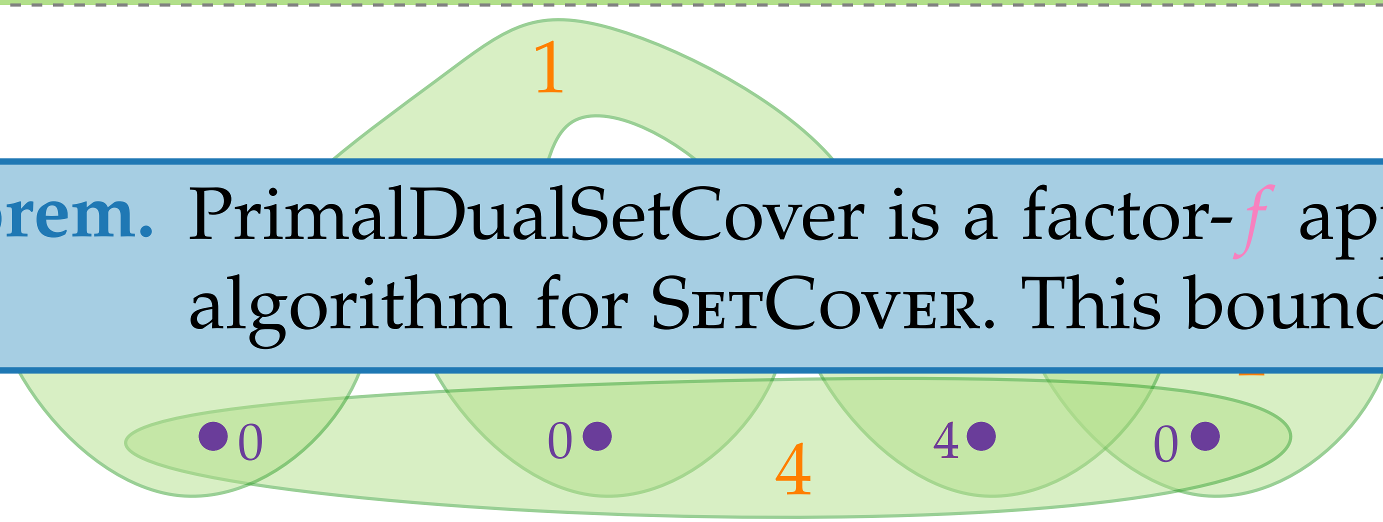

Theorem. PrimalDualSetCover is a factor-f approximation algorithm for SetCover. This bound is tight.

Tight Example

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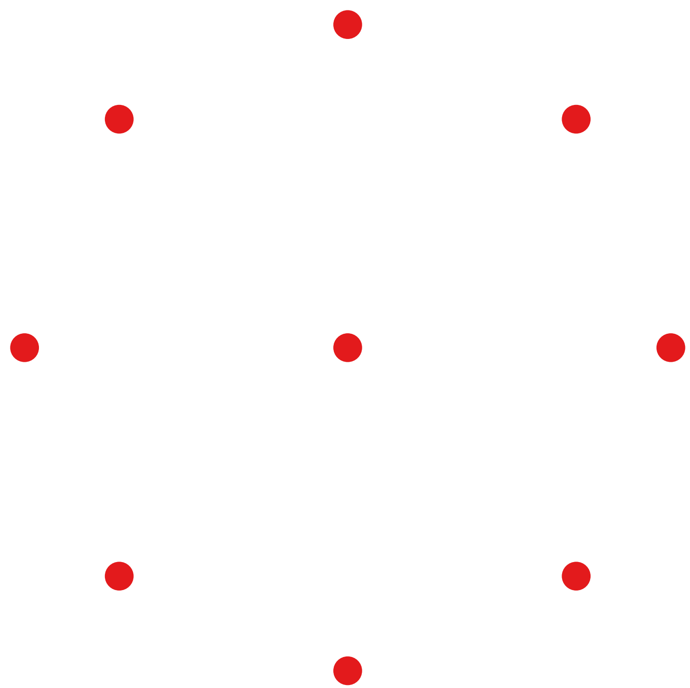

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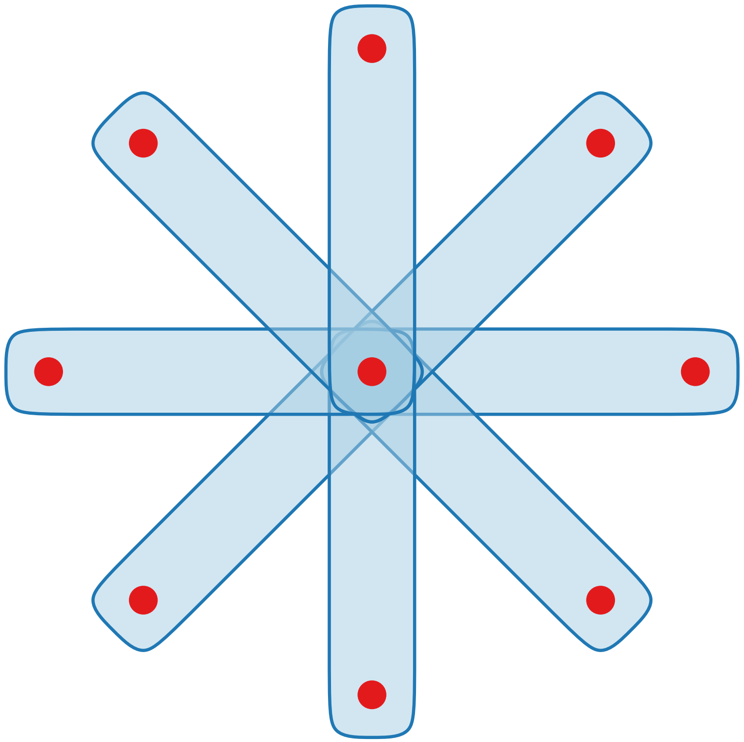

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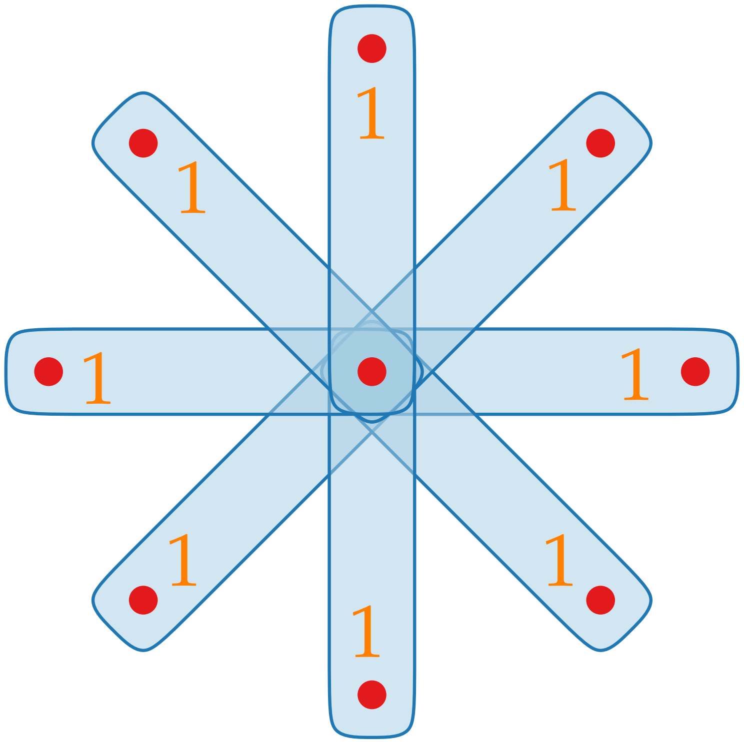

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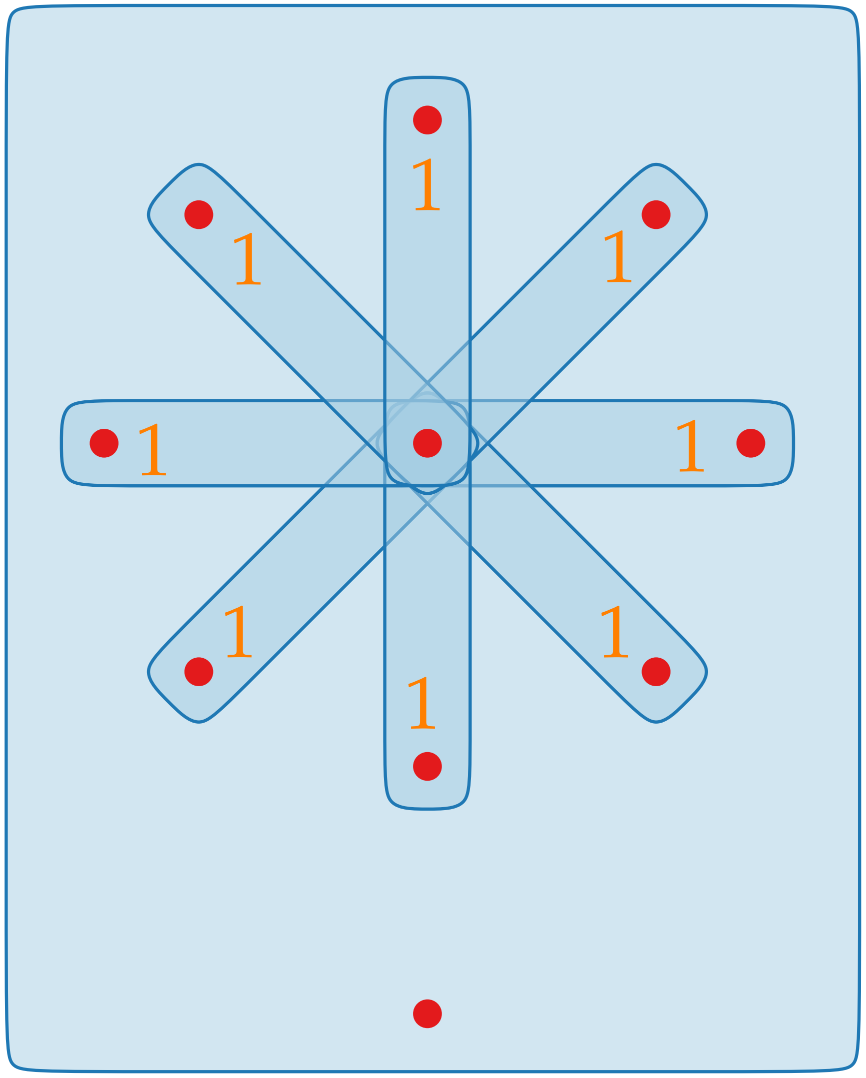

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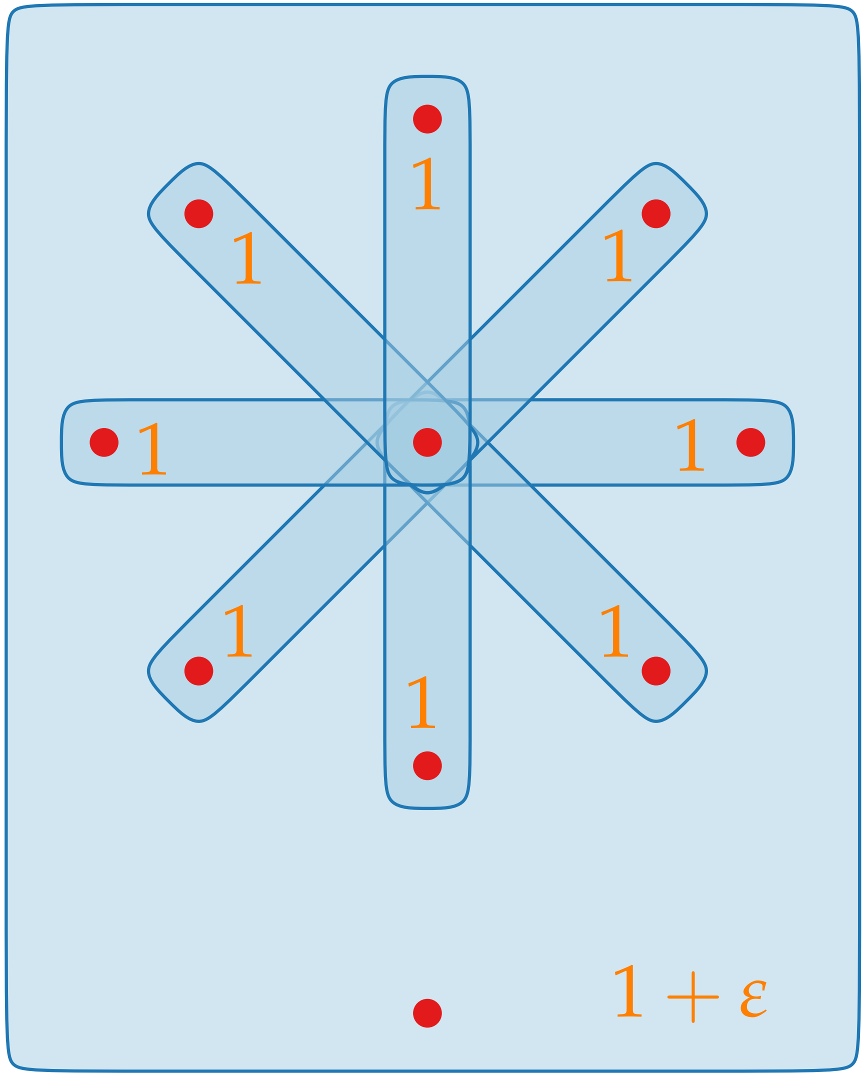
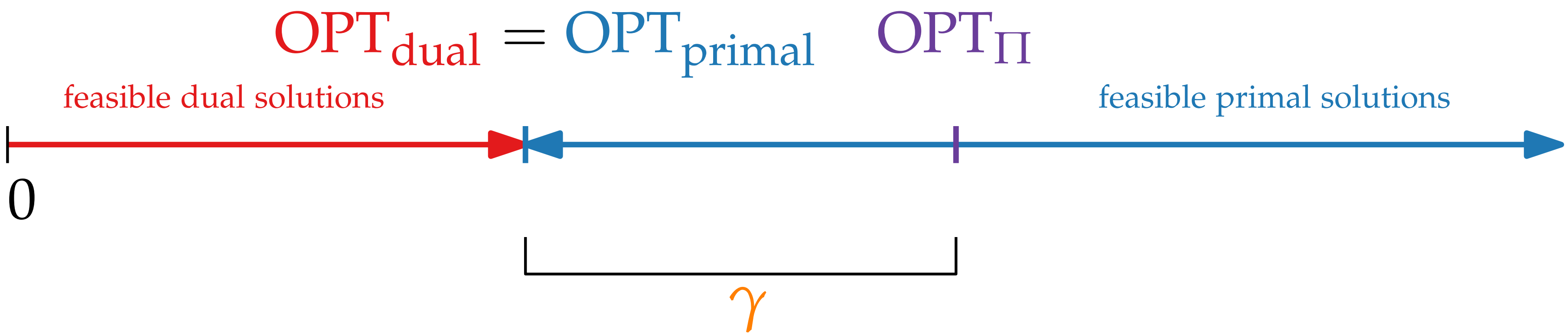


Integrality Gap



Consider a minimization problem Π in ILP form.

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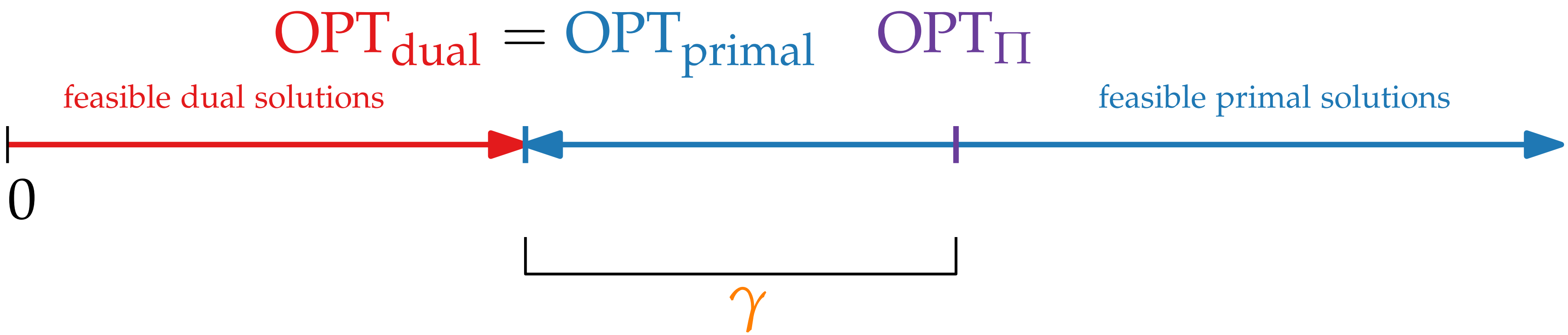

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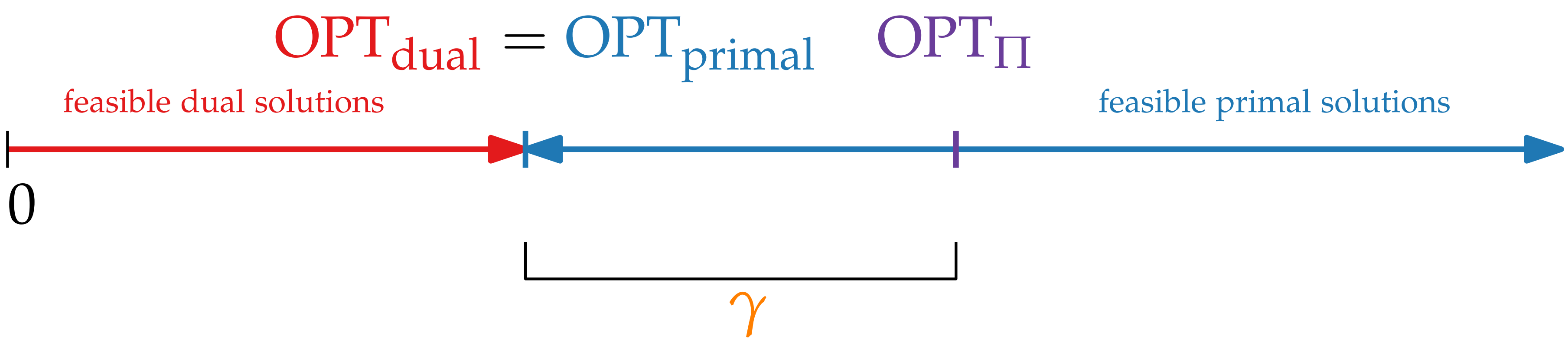

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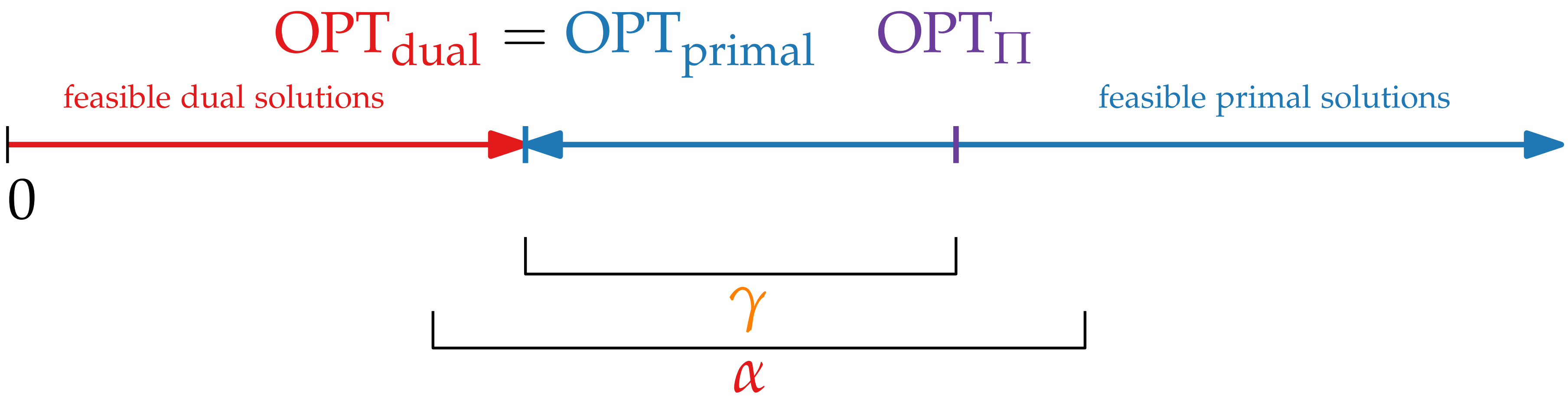

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$$\alpha \geq \gamma = \sup_I \frac{\mathrm{OPT}_\Pi(I)}{\mathrm{OPT}_{\mathrm{primal}}(I)}$$

Approximation Algorithms

Lecture 5:
LP-based Approximation Algorithms for SetCover

Part IV:
Dual Fitting

Alexander Wolff

Winter 2022/23

Technique III) Dual Fitting

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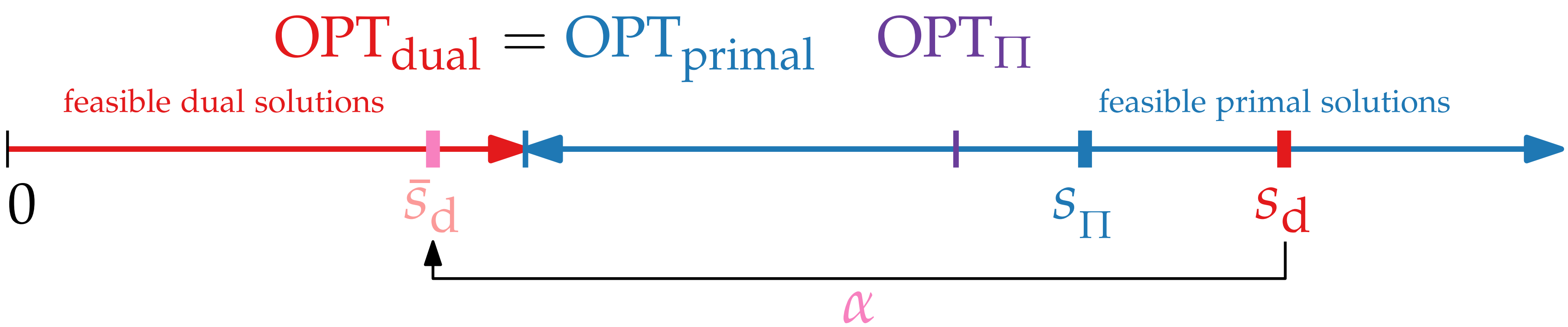

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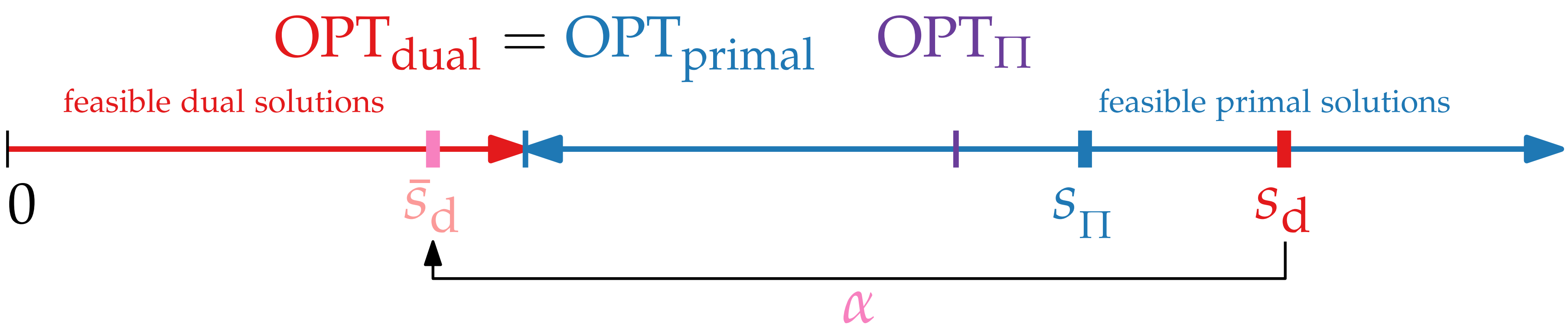

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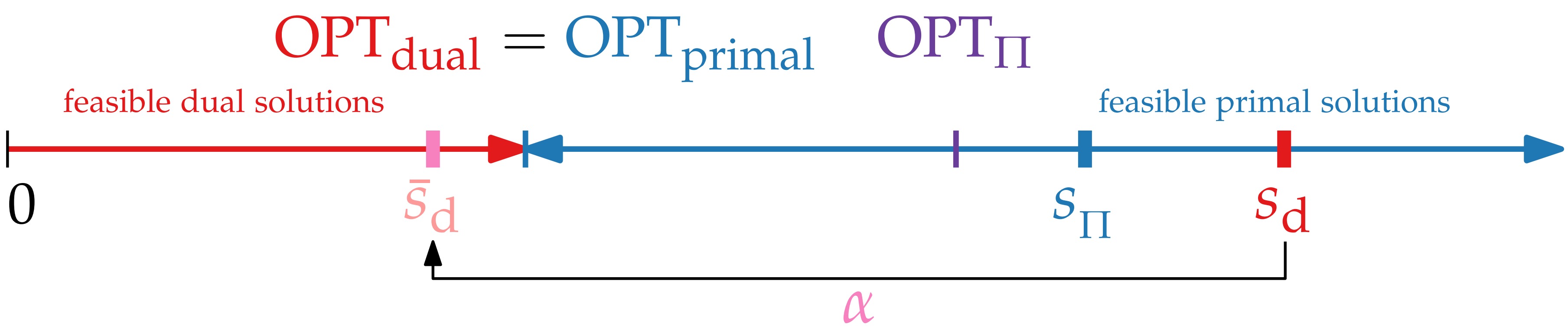

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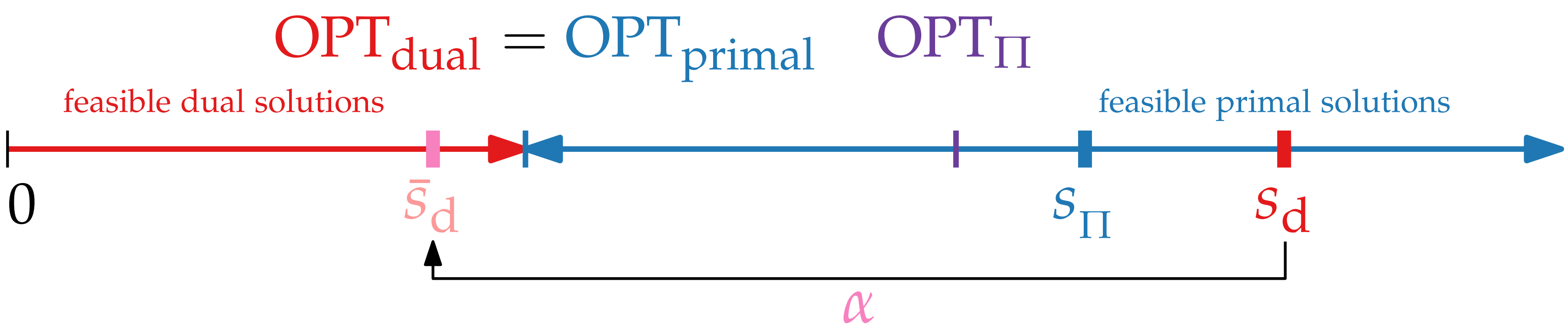

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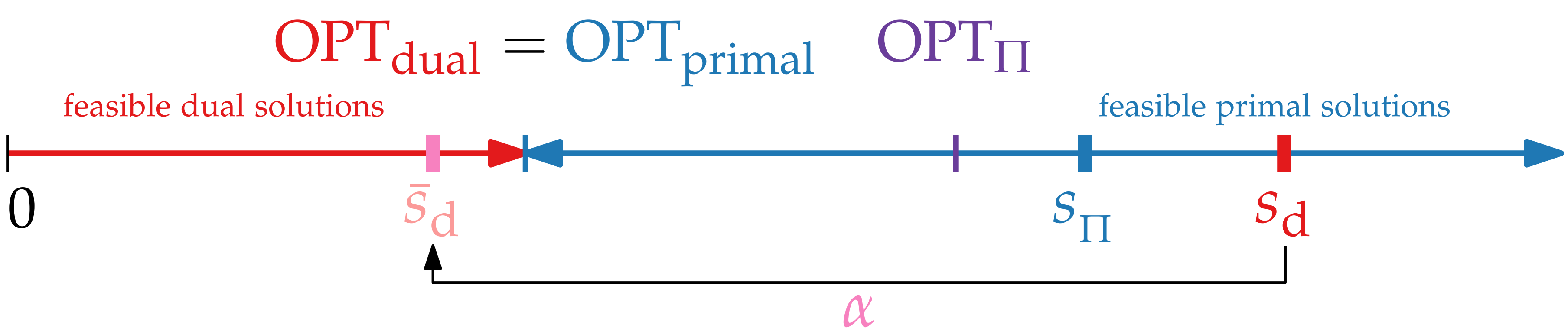

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$\Rightarrow$ Scaling factor α is approximation factor.

Dual Fitting for SetCover

Combinatorial (greedy) algorithm (see Lecture #2):

```
GreedySetCover(U, 𝒮, c)
  C ← ∅
  𝒮' ← ∅
  while C ≠ U do
     S ← Set from 𝒮 that minimizes c(S)/|S \ C|
     foreach u ∈ S \ C do
        price(u) ← c(S)/|S \ C|
     C ← C ∪ S
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Reminder: $\sum_{u \in U} \text{price}(u) \quad \ldots$

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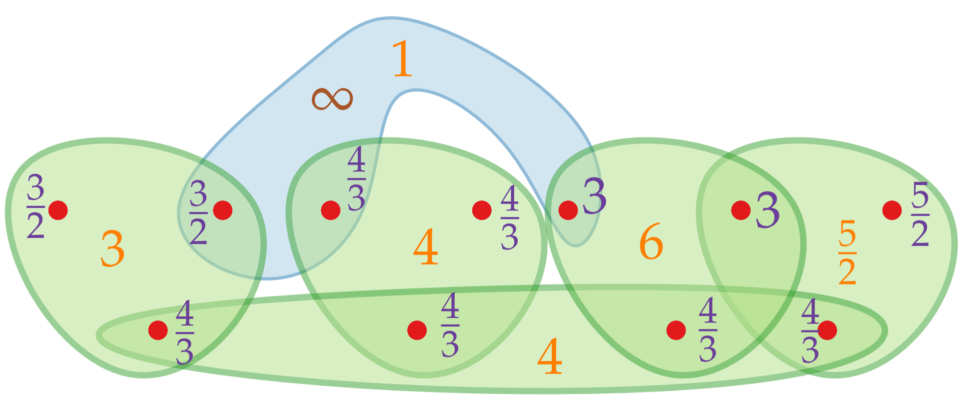

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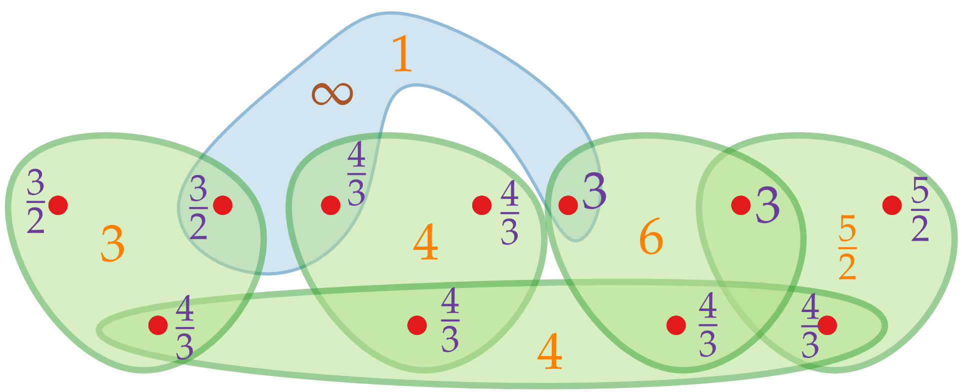

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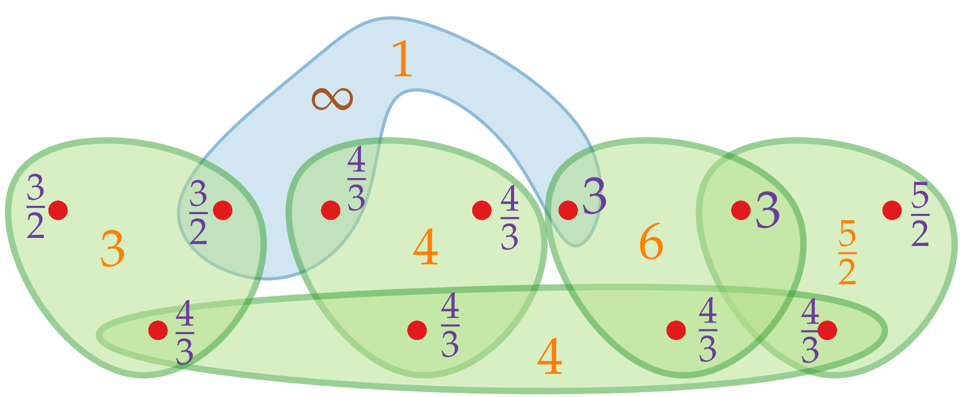

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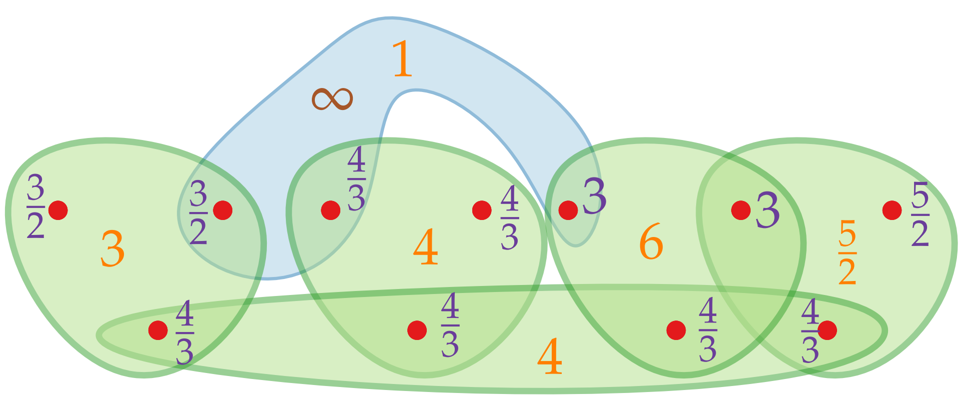

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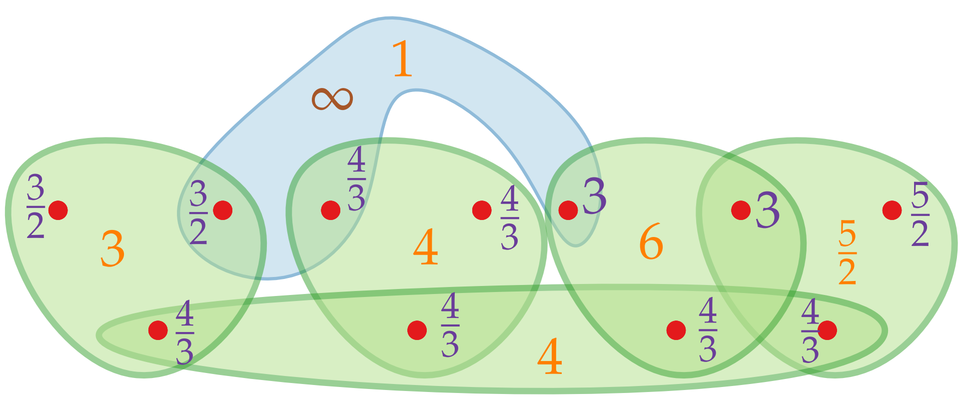

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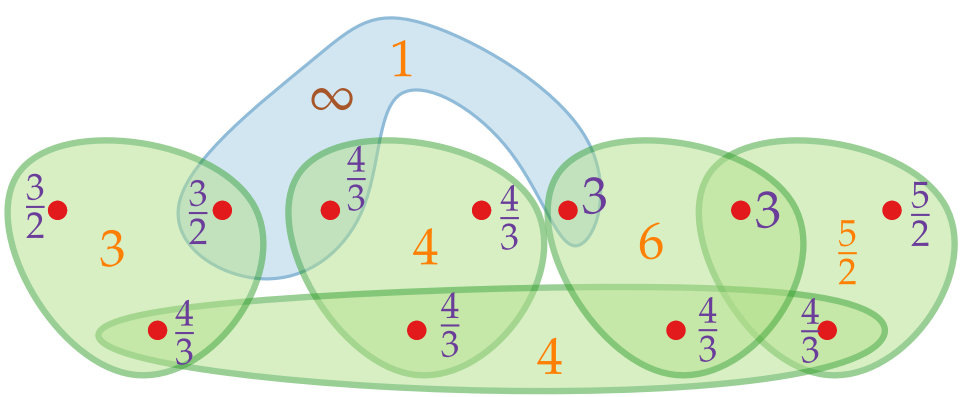

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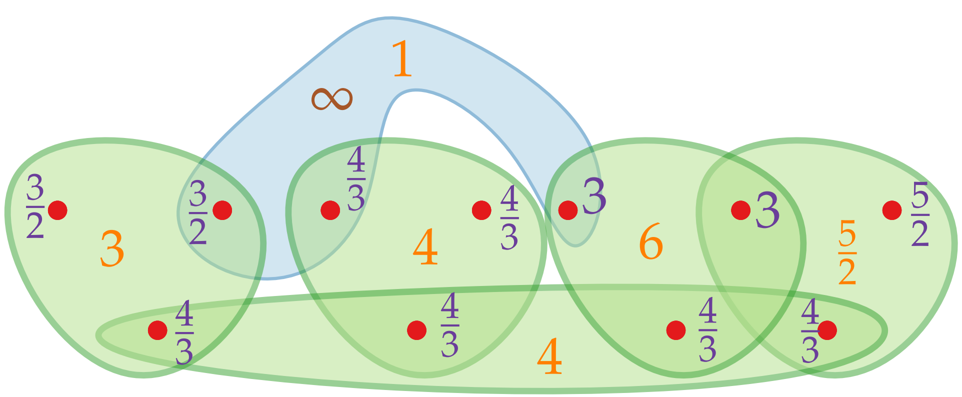

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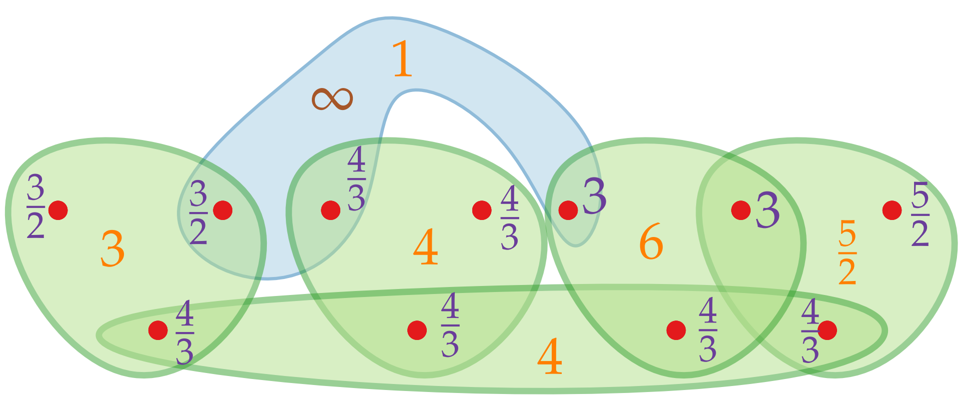

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Dual solution allows a *per-instance* estimation

…which may be stronger than worst-case bound $\mathcal{H}_k$.