Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part I:
Introduction to Linear Programming

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Maximizing Profits

You're the boss of a small company that produces two products P_1 and P_2. For the production of x_1 units of P_1 and x_2 units of P_2, your profit in € is:

$$G(x_1, x_2) = 30x_1 + 50x_2$$

Three machines M_A, M_B and M_C produce the required components A, B and C for the products. The components are used in different quantities for the products, and each machine requires some time for the production.

$$\begin{array}{lrcrcr} M_A: & 4x_1 & + & 11x_2 & \leq & 880 \\ M_B: & x_1 & + & x_2 & \leq & 150 \\ M_C: & & & x_2 & \leq & 60 \end{array}$$

Which choice of (x_1, x_2) maximizes the profit?

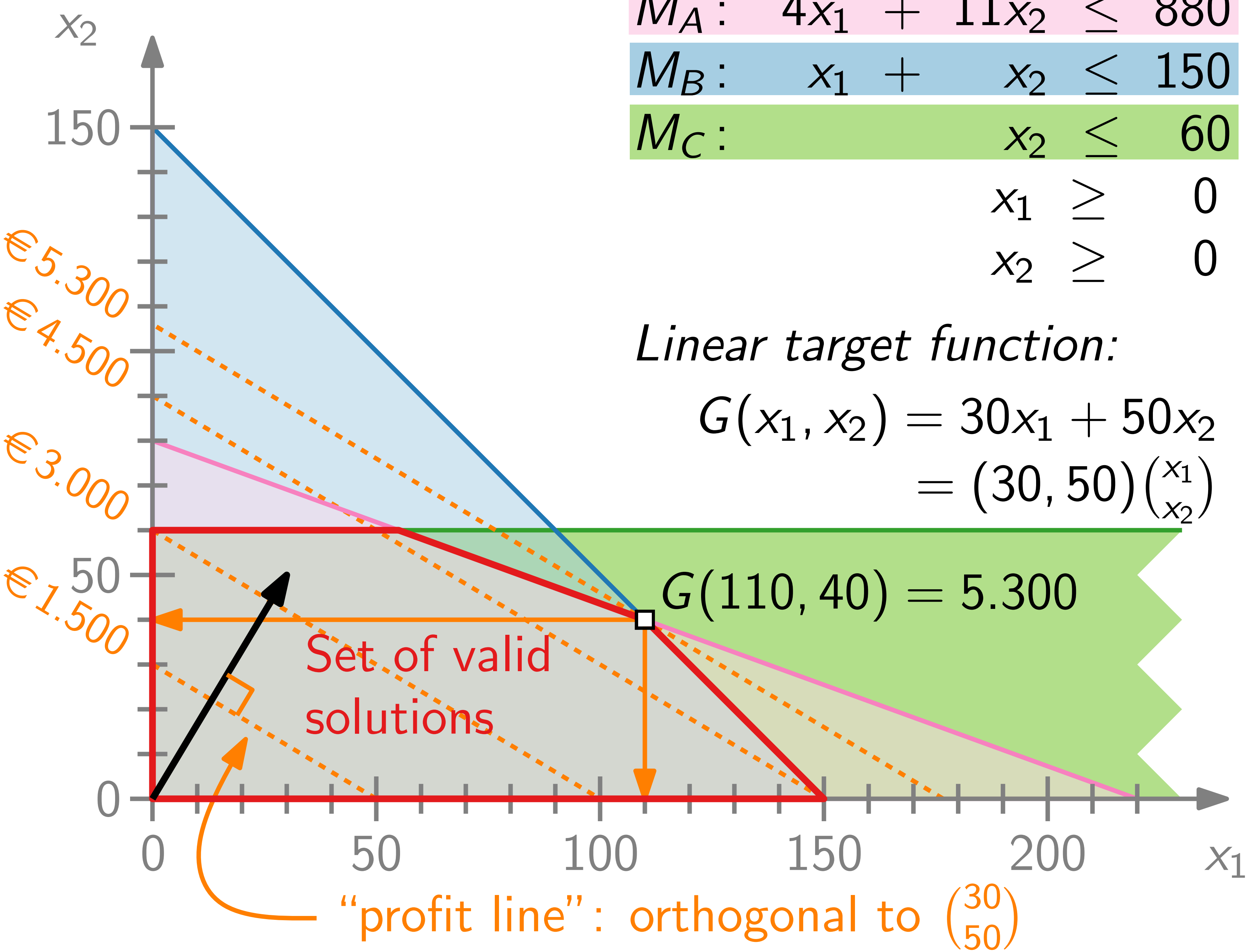
Solution

Linear constraints:

M_A :	$4x_1$	$+$	$11x_2 \leq$	880
M_B :	x_1	$+$	$x_2 \leq$	150
M_C :			$x_2 \leq$	60
			$x_1 \geq$	0
			$x_2 \geq$	0

Linear target function:

$$G(x_1, x_2) = 30x_1 + 50x_2 = (30, 50)\binom{x_1}{x_2}$$



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Part II:
Upper Bounds for LPs

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Motivation: Upper and Lower Bounds

Consider an NP-hard minimization problem.

Decision Problem:
Is a given U an **upper bound** on OPT?

A feasible sol. S provides efficiently verifiable "yes"-certificate.

Lower bounds / "no"-certificates?
$\rightsquigarrow$ probably not! (conjecture: NP $\neq$ coNP)

For an approximation algorithm, we need a lower bound
$L \geq \mathrm{OPT}/\alpha$ (i.e., an approximate "no"-certificate)!

Examples:

- Vertex Cover: lower bound by matchings
- TSP: lower bound by MST or by cycle cover

Linear Programming

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Standard form (HW)

Example. $c = \begin{pmatrix} 7 \\ 1 \\ 5 \end{pmatrix}$ $A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & 2 & -1 \end{pmatrix}$ $b = \begin{pmatrix} 10 \\ 6 \end{pmatrix}$

$$\begin{array}{lrcrcrcr} \textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\ & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$\begin{array}{lrcrcrcrr}
\textbf{minimize} & 7x_1\;14 & + & x_2\;1 & + & 5x_3\;15 & = & & 30 \\
\textbf{subject to} & x_1\;2 & - & x_2\;1 & + & 3x_3\;9 & \geq & 10 & 10 \\
 & 5x_1\;10 & + & 2x_2\;2 & - & x_3\;3 & \geq & 6 & 9 \\
 & & & & & x_1, x_2, x_3 & \geq & 0 &
\end{array}$$

Valid solution?

$x = (2, 1, 3)$

$\Rightarrow$ obj$(x) = 30$ is upper bound for OPT

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Part III:
Lower Bounds for LPs

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Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \text{IV} & & \text{IV} & & \text{IV} & & \\
\textbf{subject to} & 2\cdot\, x_1 & - & 2\cdot\, x_2 & + & 2\cdot 3x_3 & \geq & 2\cdot 10 \\
 & + & & + & & + & & \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \quad \Rightarrow \text{OPT} \geq 10$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 10 + 6 \quad \Rightarrow \text{OPT} \geq 16
\end{aligned}$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq 2 \cdot (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 2\cdot 10 + 6 \quad \Rightarrow \text{OPT} \geq 26
\end{aligned}$$

Linear Programming – Lower Bounds

Primal

$$
\begin{array}{llcccccl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
& \text{\scriptsize IV} & & \text{\scriptsize IV} & & \text{\scriptsize IV} & & \\
\textbf{subject to}\ y_1(& x_1 & - & x_2 & + & 3x_3\ ) & \geq & 10\ y_1 \\
& + & & + & & + & & \\
y_2(& 5x_1 & + & 2x_2 & - & x_3\ ) & \geq & 6\ y_2 \\
& & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
\end{aligned}
$$

Dual

$$
\begin{array}{lcccccl}
\textbf{maximize} & 10y_1 & + & 6y_2 & & \\
\textbf{subject to} & y_1 & + & 5y_2 & \leq & 7 \\
& -y_1 & + & 2y_2 & \leq & 1 \\
& 3y_1 & - & y_2 & \leq & 5 \\
& & & y_1, y_2 & \geq & 0
\end{array}
$$

Any feasible solution to the **dual** program provides a lower bound for the optimum of the **primal** program.

$x = (7/4, 0, 11/4)$ both $y = (2, 1)$ provide objective value 26. $= \text{OPT}$

Primal–Dual

primal program

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

dual program

$$\begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

dual of the dual program

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

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Part IV:
LP-Duality and Complementary Slackness

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LP-Duality

Primal

$$\begin{array}{lrcl} \textbf{minimize} & c^{\mathsf{T}} x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Dual

$$\begin{array}{lrcl} \textbf{maximize} & b^{\mathsf{T}} y & & \\ \textbf{subject to} & A^{\mathsf{T}} y & \leq & c \\ & y & \geq & 0 \end{array}$$

Theorem. The primal program has a finite optimum $\Leftrightarrow$ the dual program has a finite optimum. Moreover, if $x^* = (x_1^*, \ldots, x_n^*)$ and $y^* = (y_1^*, \ldots, y_m^*)$ are *optimal* solutions for the primal and dual program (resp.), then

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^* .$$

Weak LP-Duality

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\intercal y$		
subject to	$A^\intercal y$	$\leq$	c
	y	$\geq$	0

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program (resp.), then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \geq \sum_{i=1}^{m} b_i y_i \,.$$

Complementary Slackness

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\intercal y$		
subject to	$A^\intercal y$	$\leq$	c
	y	$\geq$	0

Theorem. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be valid solutions for the primal and dual program (resp.). Then x and y are optimal if and only if the following conditions are met:

Primal CS:
For each $j = 1, \ldots, n$: $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
For each $i = 1, \ldots, m$: $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

Proof. Follows from LP-duality:

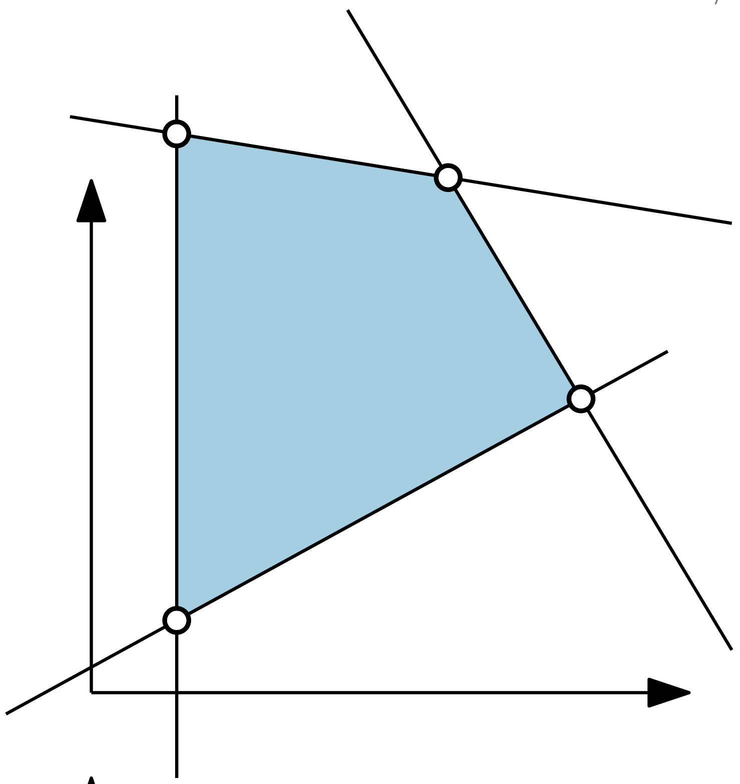
$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \geq \sum_{i=1}^{m} b_i y_i \,.$$

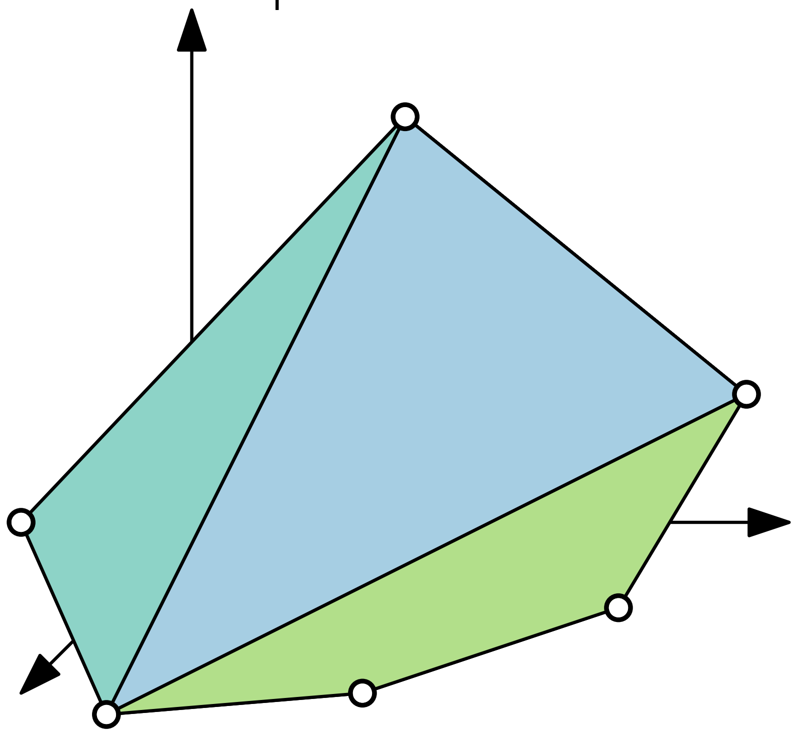
LPs and Convex Polytopes

The feasible solutions of an LP with n variables form a **convex polytope** in $\mathbb{R}^n$ (intersection of halfspaces).

Corners of the polytope are called **extreme point solutions** $\Leftrightarrow$ n linearly independent inequalities (constraints) are satisfied with equality.

If an optimal solution exists, some extreme point is also optimal.





Integer Linear Programs (ILPs)

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \in & \mathbb{N} \end{array}$$

Many NP-optimization problems can be formulated as ILPs; thus ILPs are NP-hard to solve.

LP-relaxation provides a lower bound: $\mathrm{OPT}_{\mathrm{LP}} \leq \mathrm{OPT}_{\mathrm{ILP}}$

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Part V:
Min–Max Relationships

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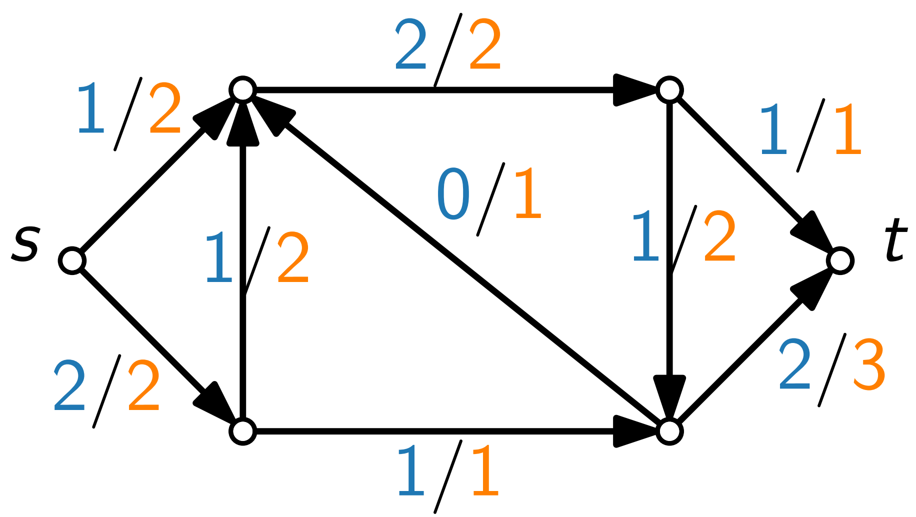
Max-Flow Problem

Given: A directed graph $G = (V, E)$ with edge capacities $c \colon E \to \mathbb{Q}_+$ and two special vertices: the source s and sink t.

Find: A maximum s–t flow (i.e., non-negative edge weights f), such that

- $f(u, v) \leq c(u, v)$ for each edge $(u, v) \in E$
- $\sum_{(u,v)\in E} f(u, v) = \sum_{(v,z)\in E} f(v, z)$ for each vertex $v \in V \setminus \{s, t\}$

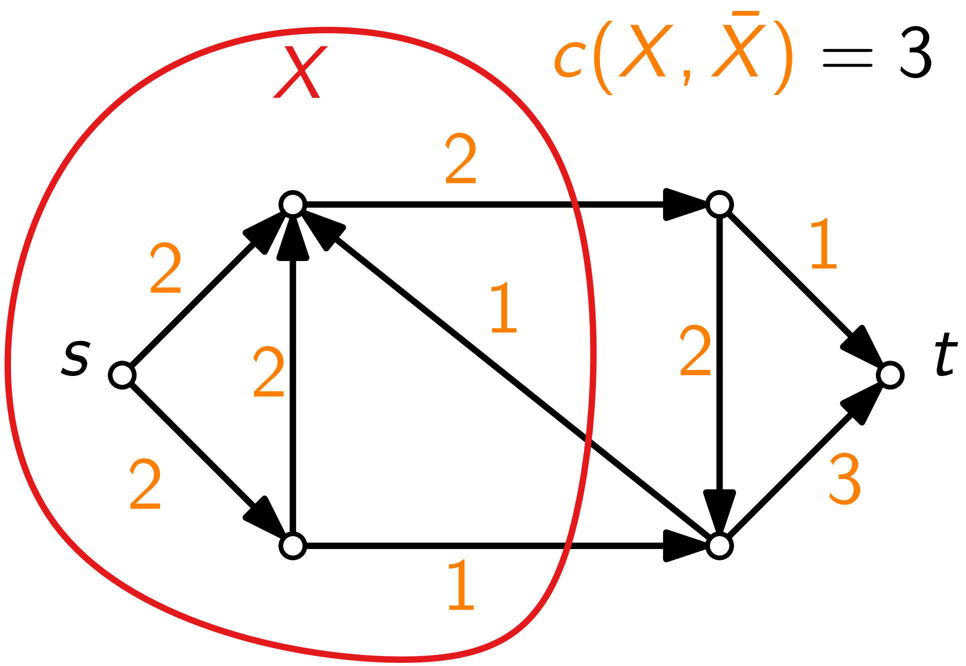
The **flow value** is the inflow to t minus the outflow from t.



Min-Cut Problem

Given: A directed graph $G = (V, E)$ with edge capacities $c \colon E \to \mathbb{Q}_+$ and two special vertices: the source s and sink t.

Find: An s–t cut, i.e., a vertex set X with $s \in X$ and $t \in \overline{X}$, such that the total weight $c(X, \overline{X})$ of the edges from X to $\overline{X}$ is minimum.



Max-Flow-Min-Cut Theorem

Theorem. The value of a maximum s–t flow and the weight of a minimum s–t cut are the same.

Proof. Special case of LP-Duality . . .

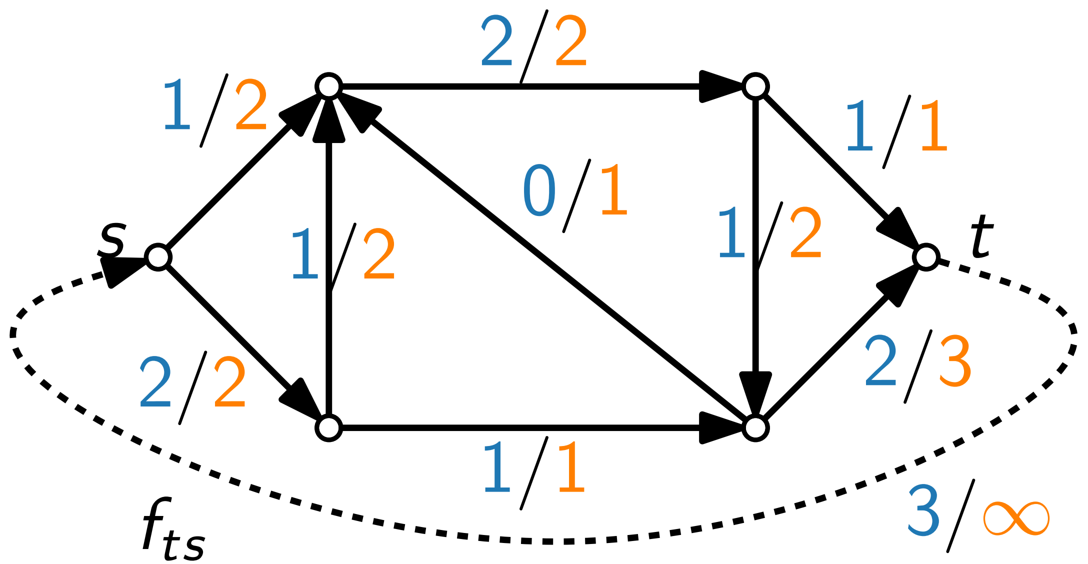
maximize f_{ts}

subject to

$$f_{uv} \leq c_{uv} \qquad \forall (u,v) \in E \setminus \{(t,s)\}$$

$$\sum_{u\colon (u,v) \in E} f_{uv} - \sum_{z\colon (v,z) \in E} f_{vz} \leq 0 \qquad \forall v \in V$$

$$f_{uv} \geq 0 \qquad \forall (u,v) \in E$$



Max-Flow-Min-Cut Theorem

Theorem. The value of a maximum s–t flow and the weight of a minimum s–t cut are the same.

Proof. Special case of LP-Duality ...

$$
\begin{array}{llll}
\textbf{maximize} & f_{ts} & & \\
\textbf{subject to} & f_{uv} \le c_{uv} & \forall (u,v) \in E \setminus \{(t,s)\} & d_{uv} \\
& \sum_{u\colon (u,v)\in E} f_{uv} - \sum_{z\colon (v,z)\in E} f_{vz} \le 0 & \forall v \in V & p_v \\
& f_{uv} \ge 0 & \forall (u,v) \in E &
\end{array}
$$

maximize $c^\mathsf{T} x \quad = \sum_{(u,v)\in E} (0 \cdot f_{uv}) \quad +1 \cdot f_{ts} \quad \Rightarrow c = (0, \dots, 0, 1)$

Which constraints contain $f_{uv} \neq f_{ts}$? d_{uv}, p_u, p_v

$$\Rightarrow d_{uv} - p_u + p_v \ge 0$$

Which constraints contain f_{ts}? p_s, p_t

$$\Rightarrow p_s - p_t \ge 1$$

Max-Flow-Min-Cut Theorem

Theorem. The value of a maximum s–t flow and the weight of a minimum s–t cut are the same.

Proof. Special case of LP-Duality . . .

$$\begin{array}{llll}
\textbf{maximize} & f_{ts} & & \\
\textbf{subject to} & f_{uv} \leq c_{uv} & \forall (u,v) \in E \setminus \{(t,s)\} & d_{uv} \\
 & \displaystyle\sum_{u:\,(u,v)\in E} f_{uv} - \sum_{z:\,(v,z)\in E} f_{vz} \leq 0 & \forall v \in V & p_v \\
 & f_{uv} \geq 0 & \forall (u,v) \in E &
\end{array}$$

$$\begin{array}{lll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} & \\
\textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v) \in E \setminus \{(t,s)\} \\
 & p_s - p_t \geq 1 & \\
 & d_{uv} \geq 0 & \forall (u,v) \in E \\
 & p_u \geq 0 & \forall u \in V
\end{array}$$

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Part VI:
Dual LP of Max Flow

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Dual LP – Interpretation as ILP

minimize $\displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$

subject to

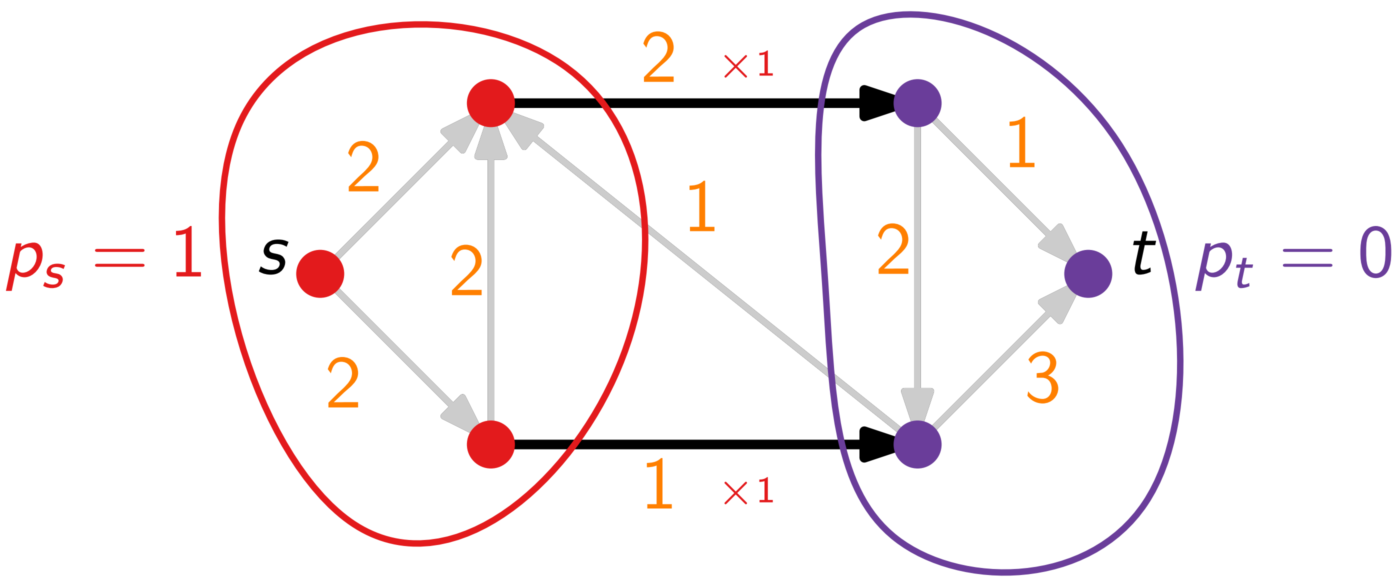
$d_{uv} - p_u + p_v \geq 0 \qquad \forall (u,v) \in E \setminus \{(t,s)\}$

$p_s - p_t \geq 1$

d_{uv} ~~≥ 0~~ $\in \{0,1\} \qquad \forall (u,v) \in E$

p_u ~~≥ 0~~ $\in \{0,1\} \qquad \forall u \in V$

equivalent to Min-Cut!



Dual LP – Fractional Cuts

minimize $\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$ $\equiv$ LP-relaxation of the ILP

subject to $d_{uv} - p_u + p_v \geq 0 \quad \forall (u,v) \in E \setminus \{(t,s)\}$

$p_s - p_t \geq 1$

$d_{uv} \geq 0 \quad \forall (u, v) \in E$

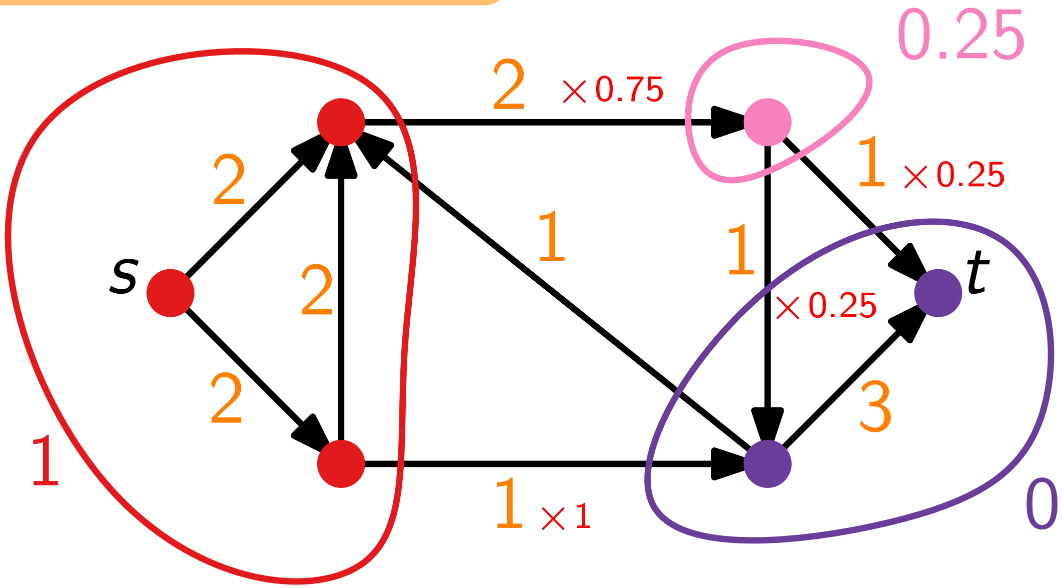
$p_u \geq 0 \quad \forall u \in V$

Each extreme-point solution is **integral**! (HW)

Each s–t-path $s = v_0, \ldots, v_k = t$ has length ≥ 1 w.r.t. d:

$$\sum_{i=0}^{k-1} d_{i,i+1} \geq \sum_{i=0}^{k-1} (p_i - p_{i+1})$$

$$= p_s - p_t$$



Dual LP – Complementary Slackness

maximize f_{ts}

subject to

$$f_{uv} \leq c_{uv} \quad \forall (u,v) \in E \setminus \{(t,s)\}$$

$$\sum_{u:\,(u,v)\in E} f_{uv} - \sum_{z:\,(v,z)\in E} f_{vz} \leq 0 \quad \forall v \in V$$

$$f_{uv} \geq 0 \quad \forall (u,v) \in E$$

minimize $\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$

subject to

$$d_{uv} - p_u + p_v \geq 0$$

$$p_s - p_t \geq 1$$

$$d_{uv} \geq 0$$

$$p_u \geq 0$$

Primal CS:

$\forall j$: $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:

$\forall i$: $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

For a max flow and min cut:

- For each forward edge (u,v) of the cut: $f_{uv} = c_{uv}$.
  ($d_{uv} = 1$, so by dual CS: $f_{uv} = c_{uv}$.)
- For each backward edge (u,v) of the cut: $f_{uv} = 0$.
  (Otherwise, by primal CS: $d_{uv} - 0 + 1 = 0$.)