

Approximation Algorithms

Lecture 4: Linear Programming and LP-Duality

Part I: Introduction to Linear Programming

Maximizing Profits

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Three machines M_A , M_B and M_C produce the required components A , B and C for the products. The components are used in different quantities for the products,

$$M_A: \quad 4x_1 \quad + \quad 11x_2$$

$$M_B: \quad x_1 \quad + \quad x_2$$

$$M_C: \quad \quad \quad x_2$$

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$$M_A: \quad 4x_1 + 11x_2 \leq 880$$

$$M_B: \quad x_1 + x_2 \leq 150$$

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Which choice of (x_1, x_2) maximizes the profit?

Solution

Linear constraints:

$$M_A: 4x_1 + 11x_2 \leq 880$$

$$M_B: x_1 + x_2 \leq 150$$

$$M_C: x_2 \leq 60$$

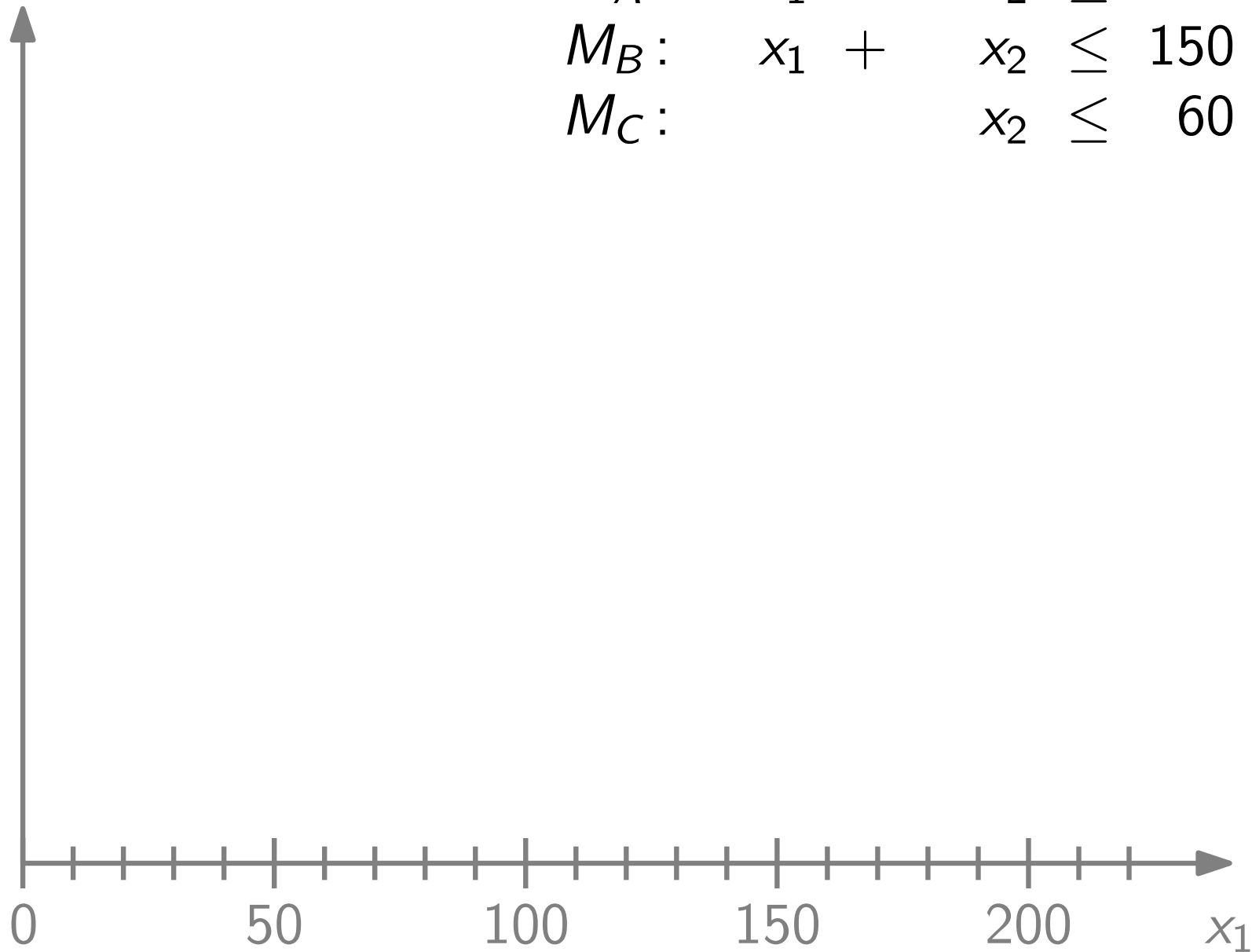
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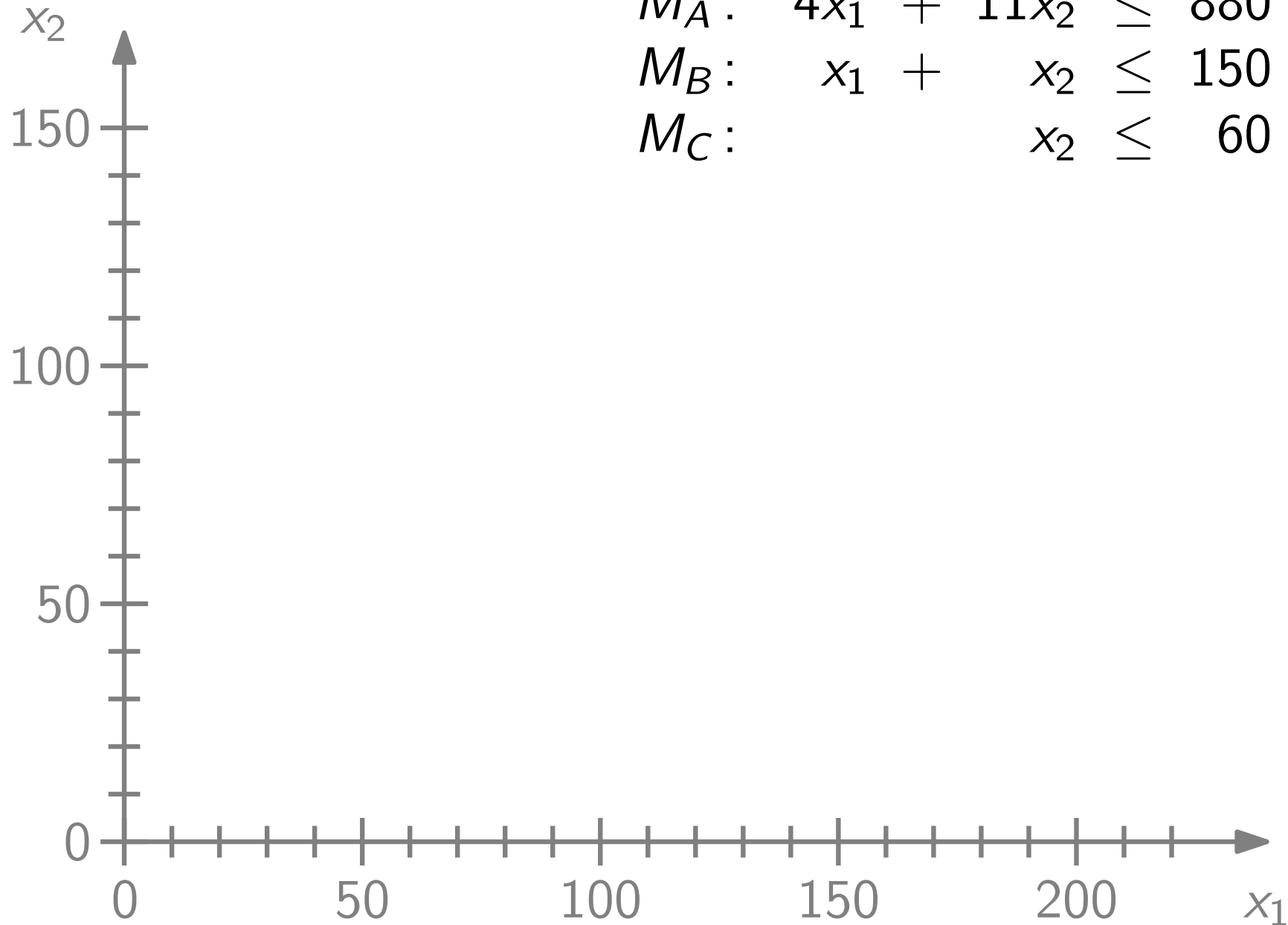
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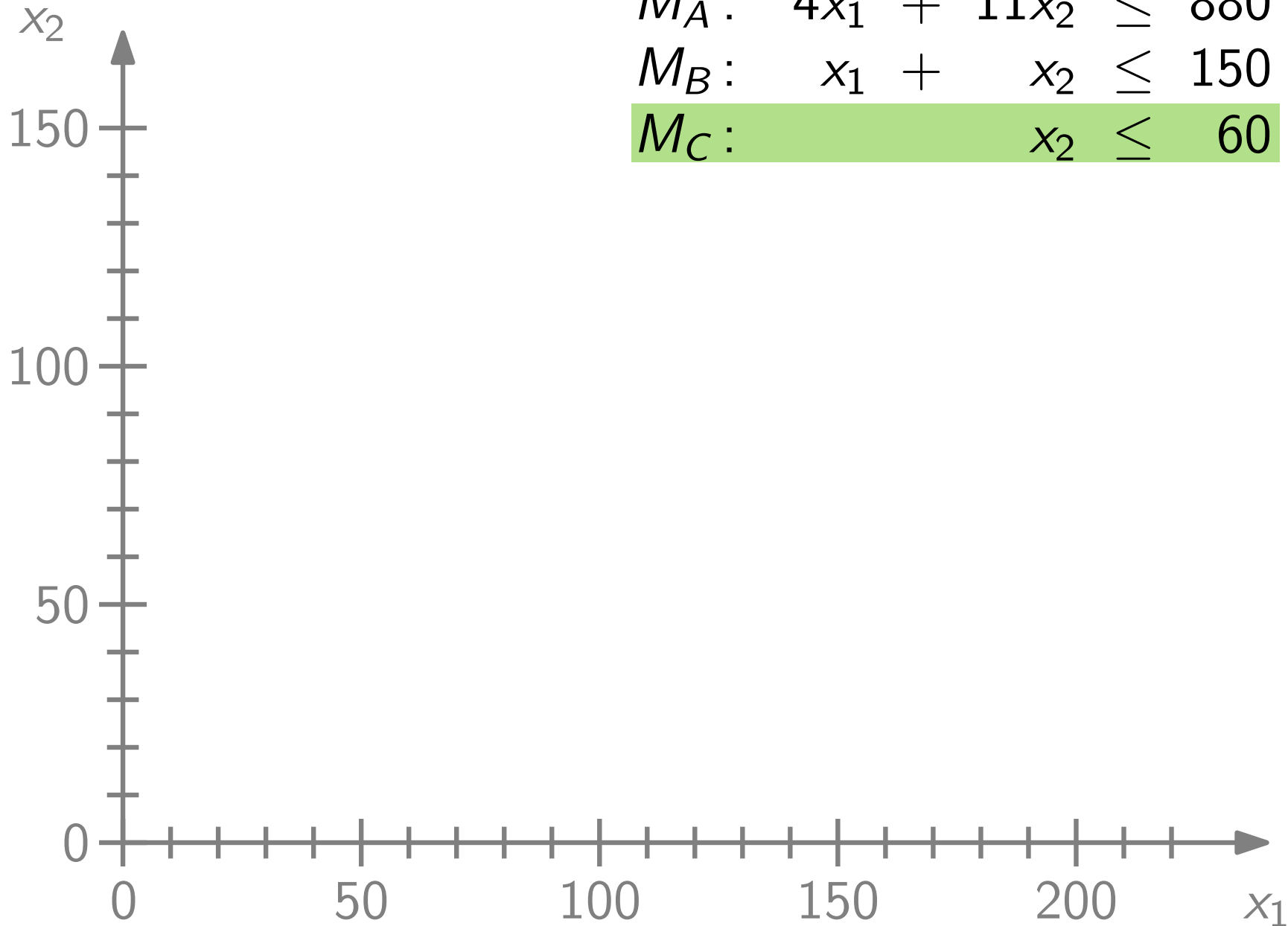
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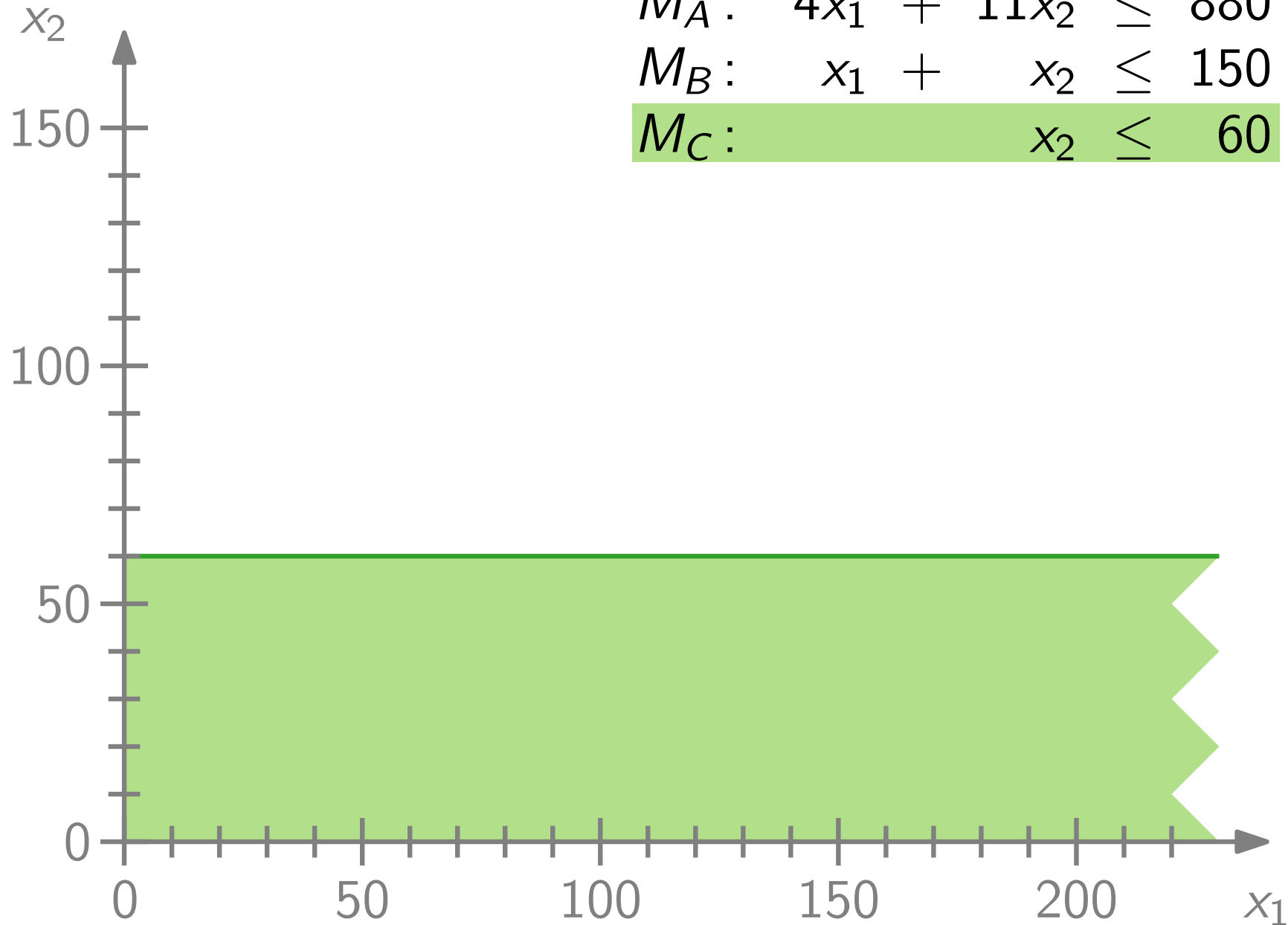
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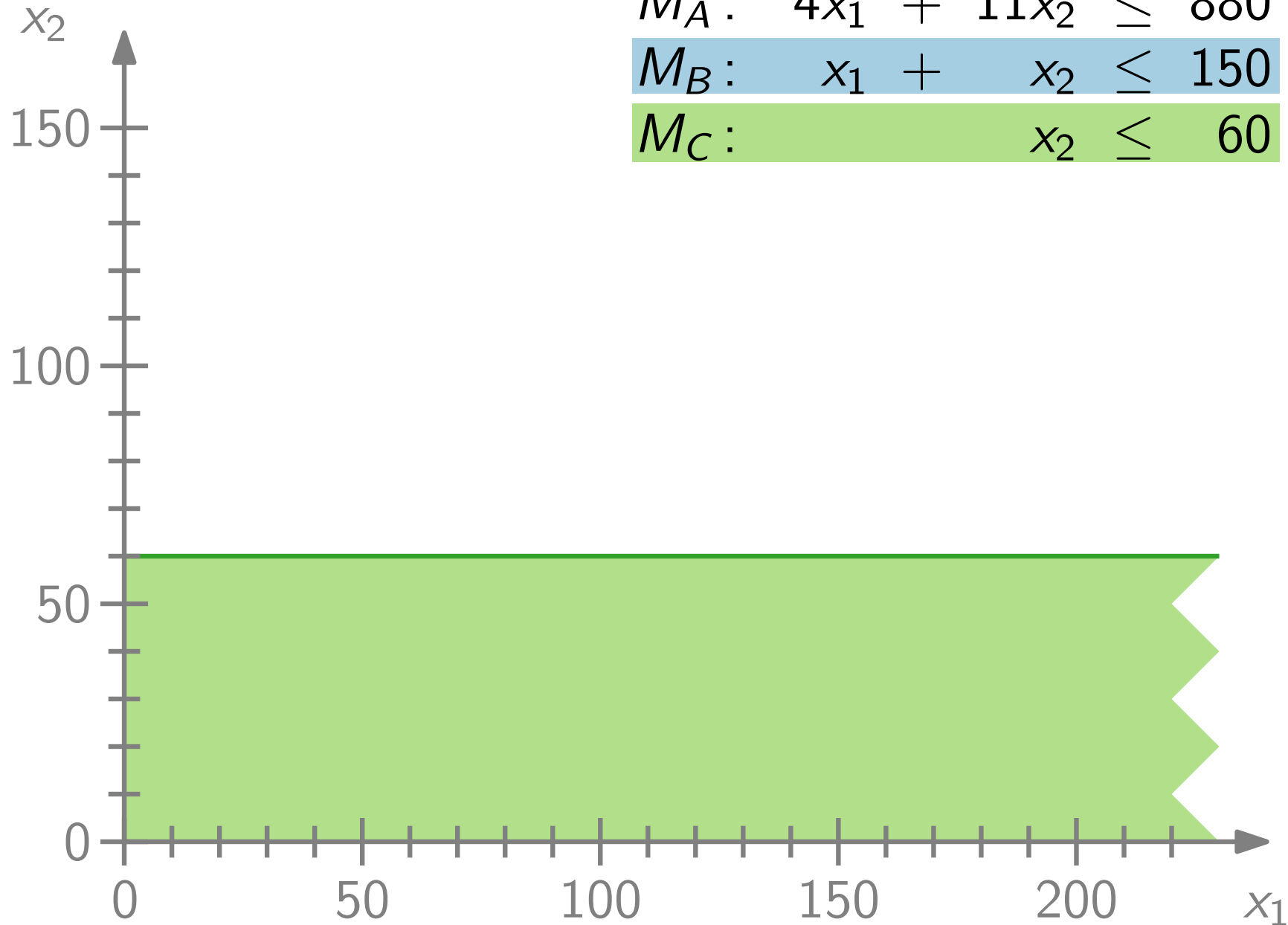
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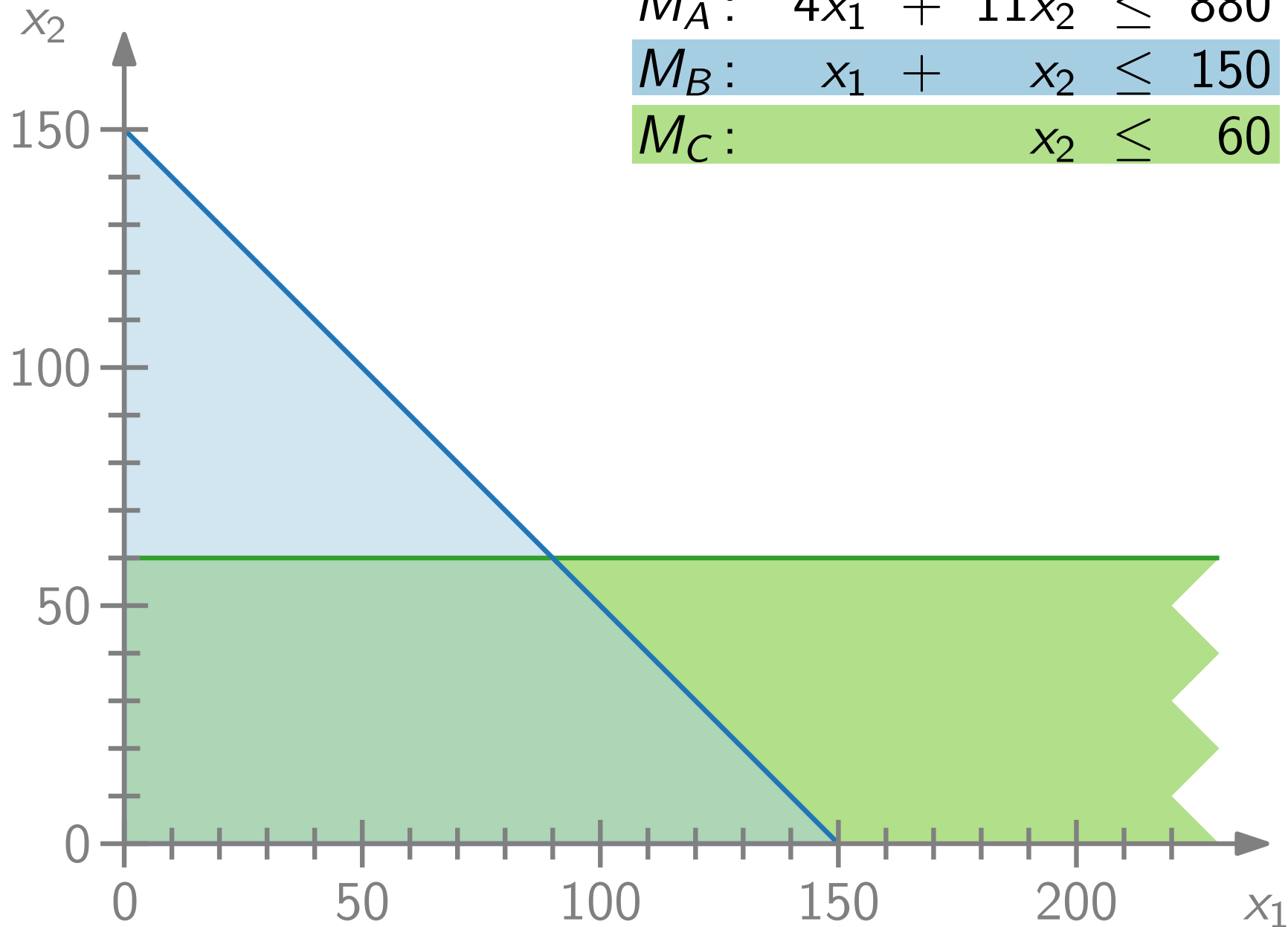
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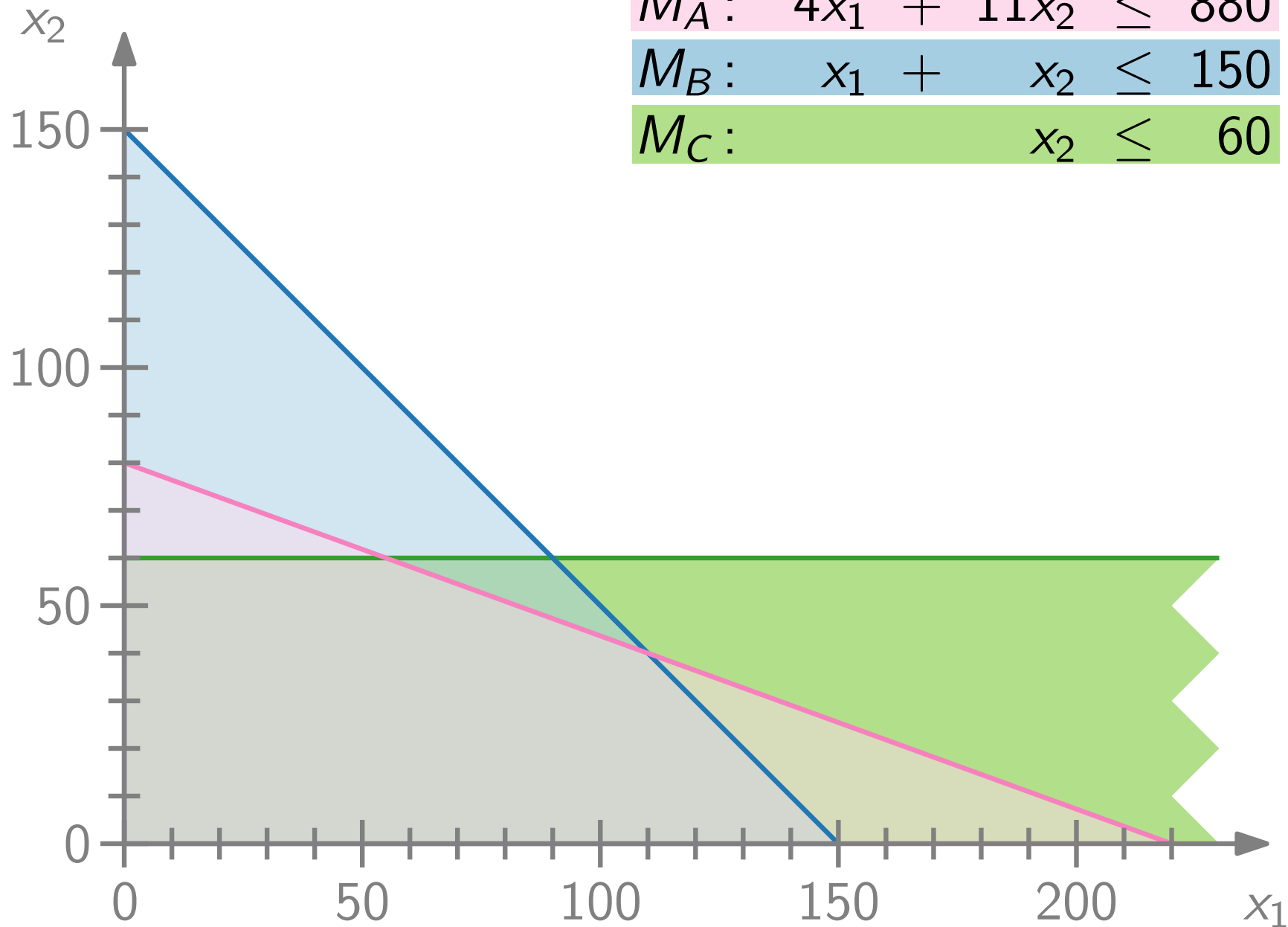
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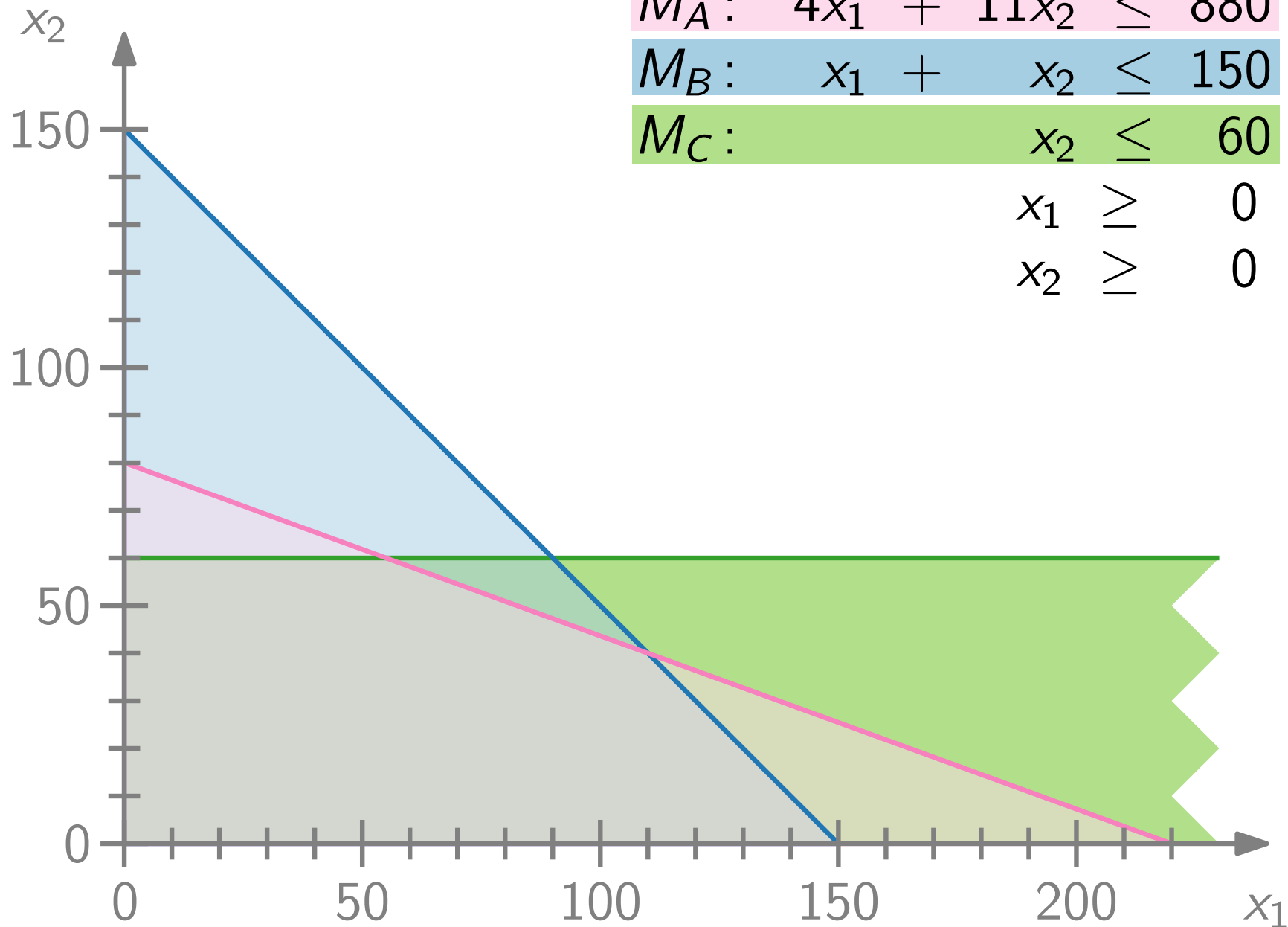
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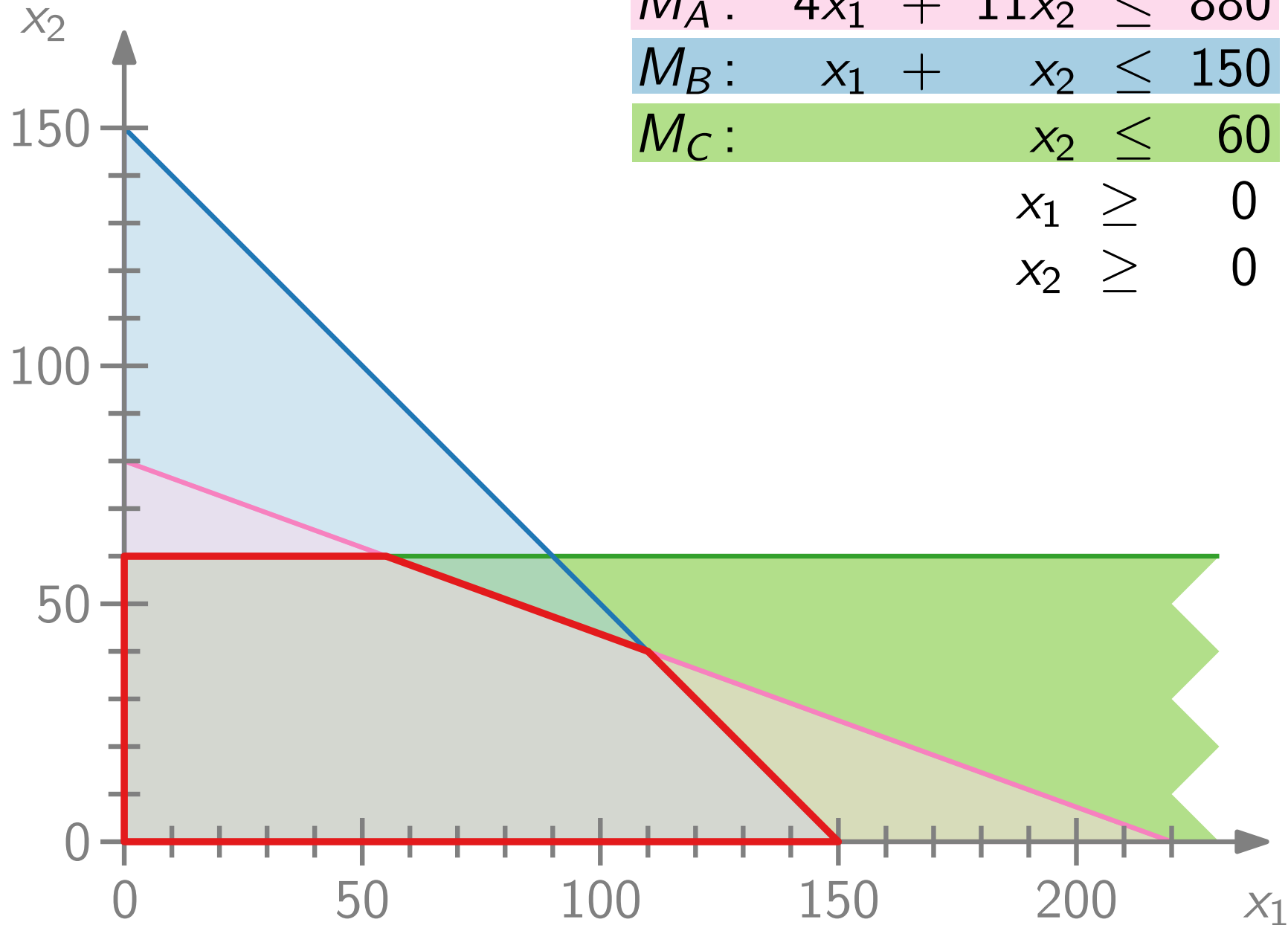
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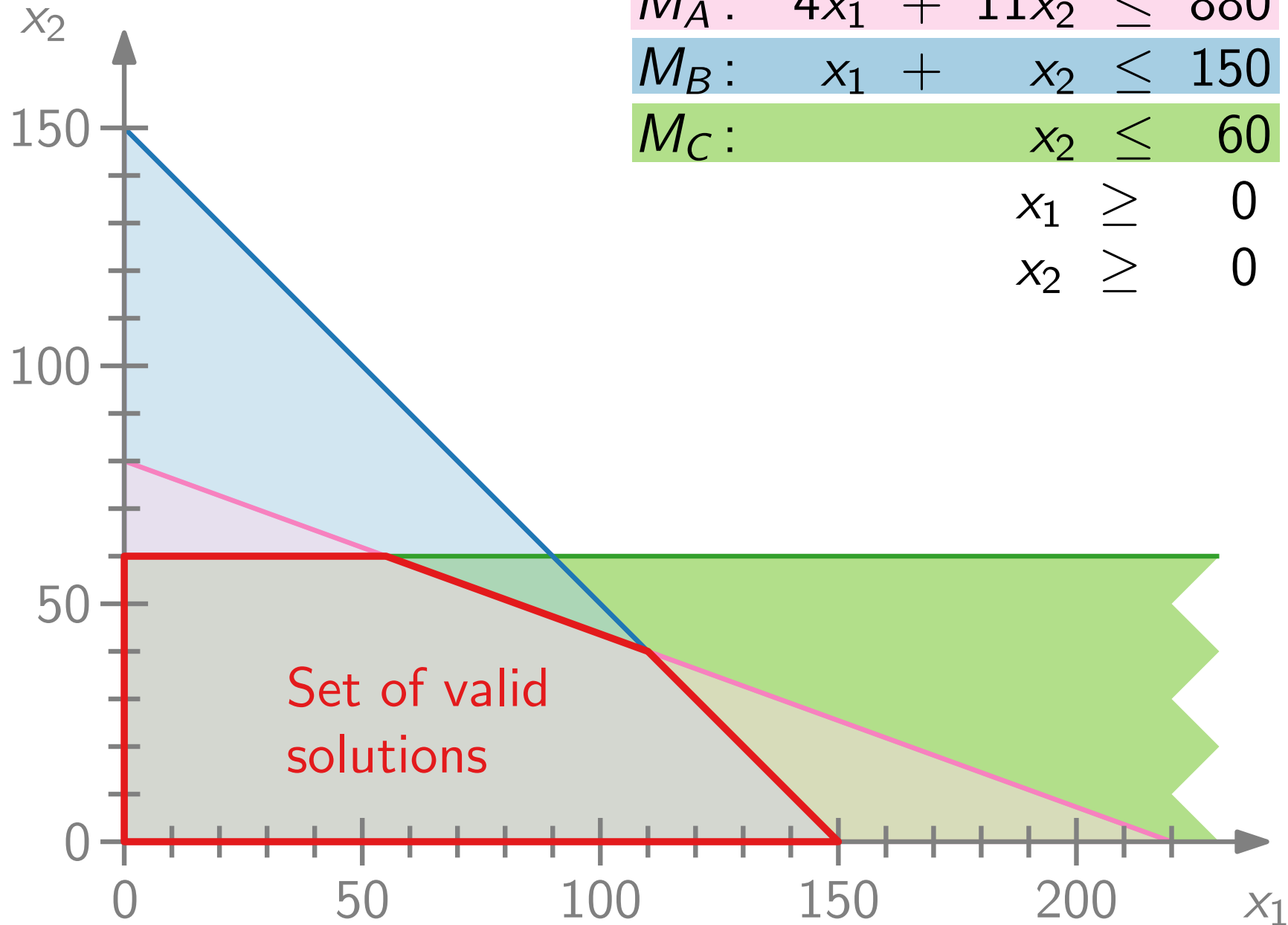
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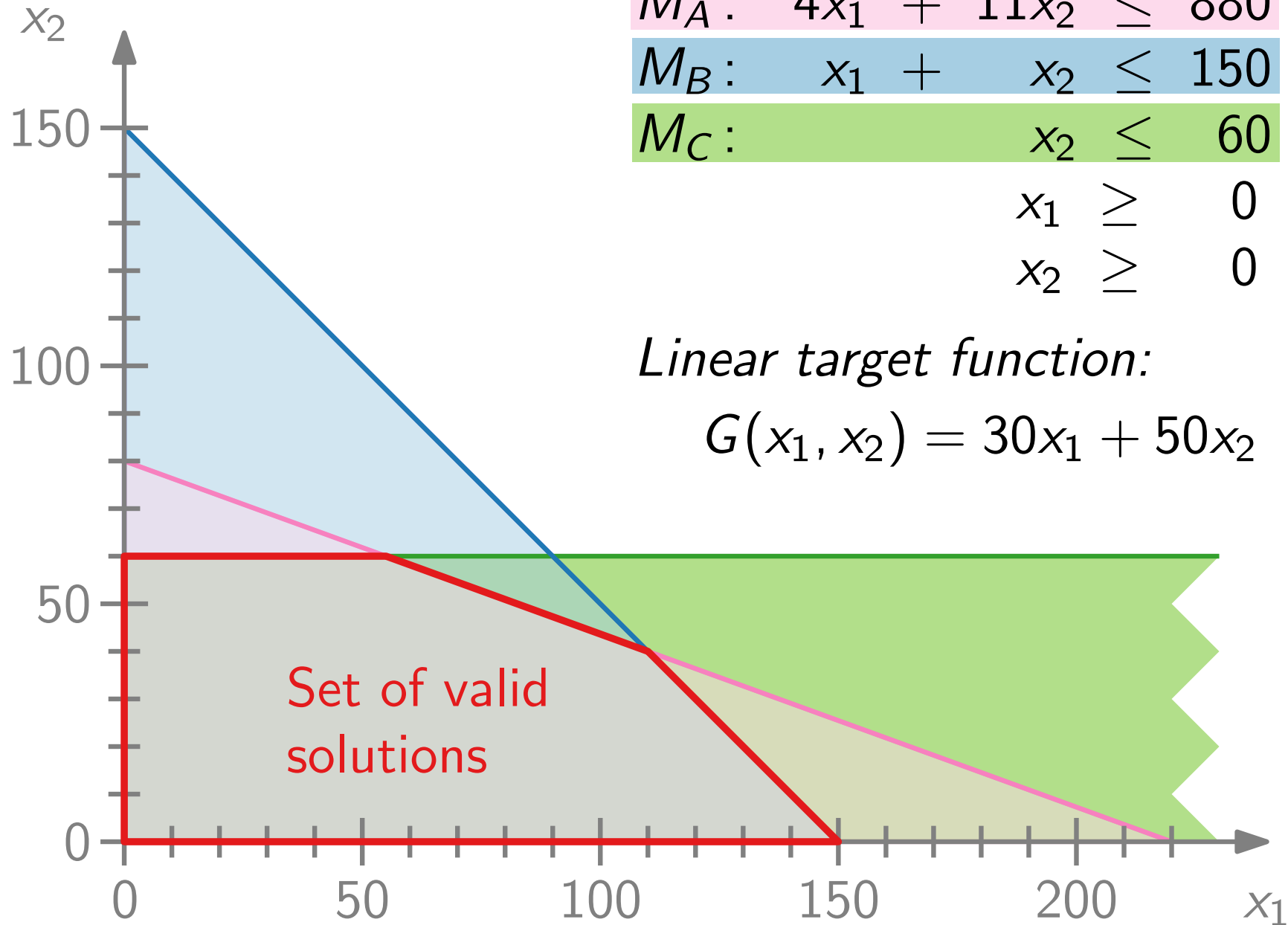
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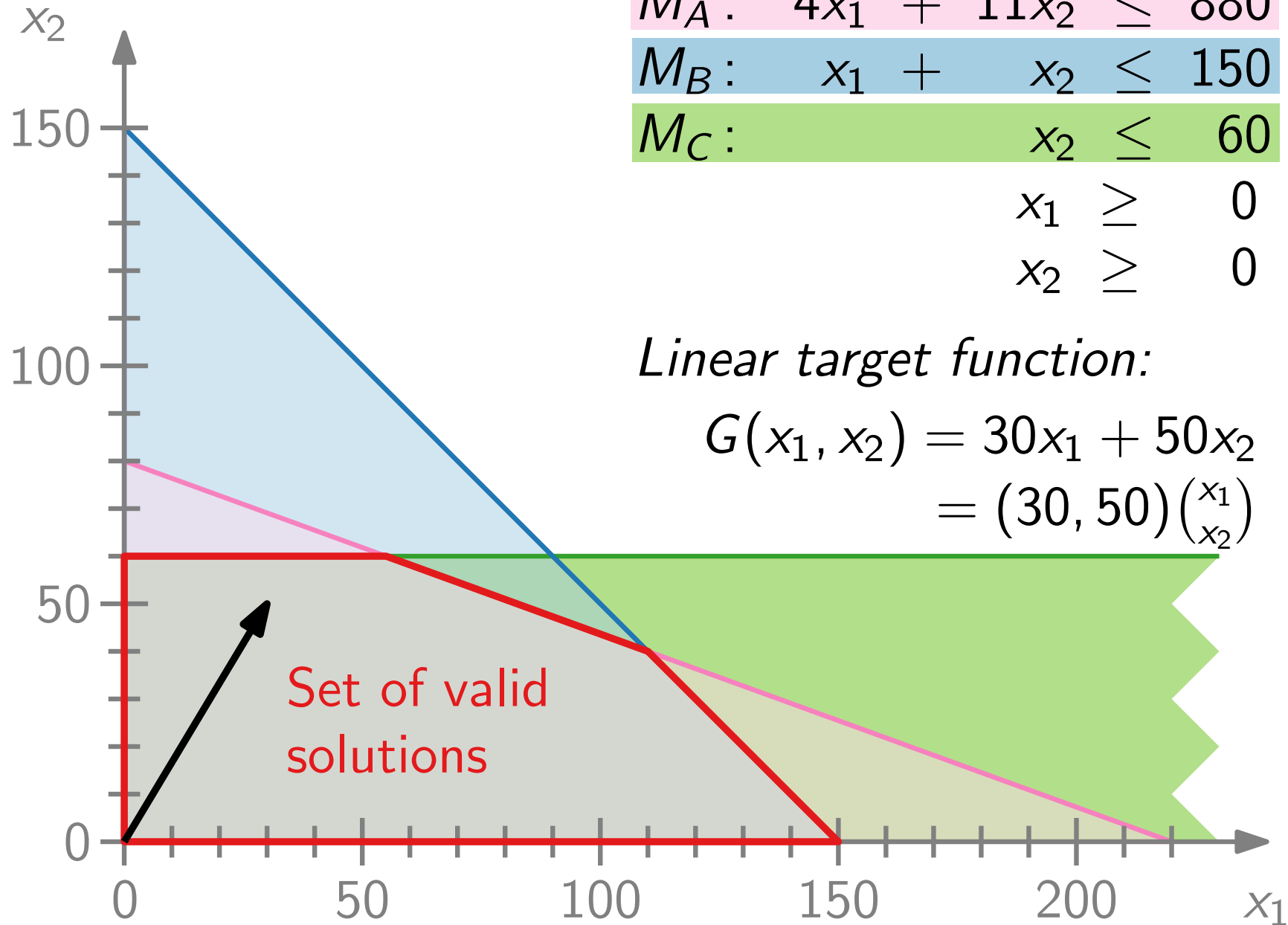
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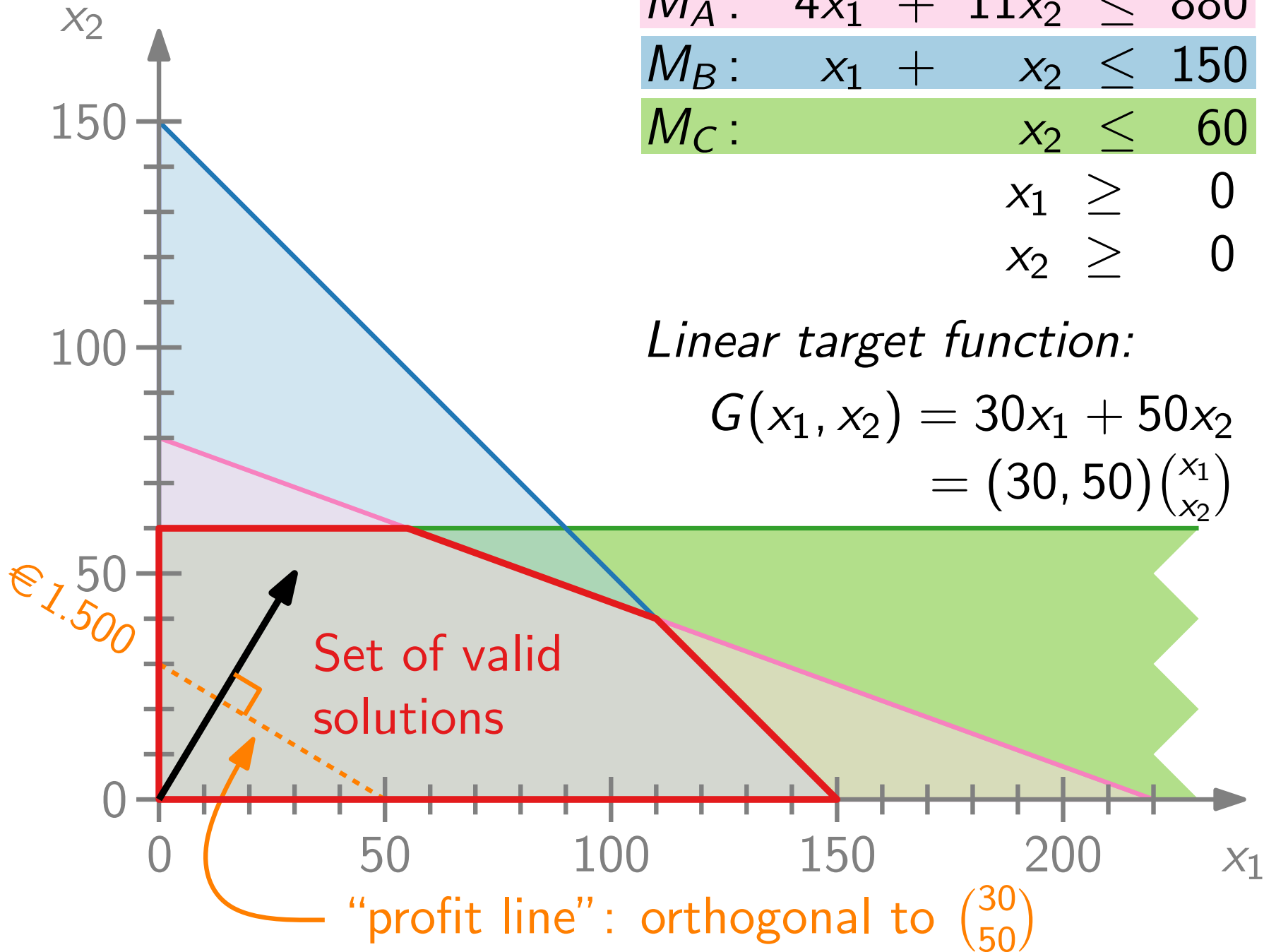
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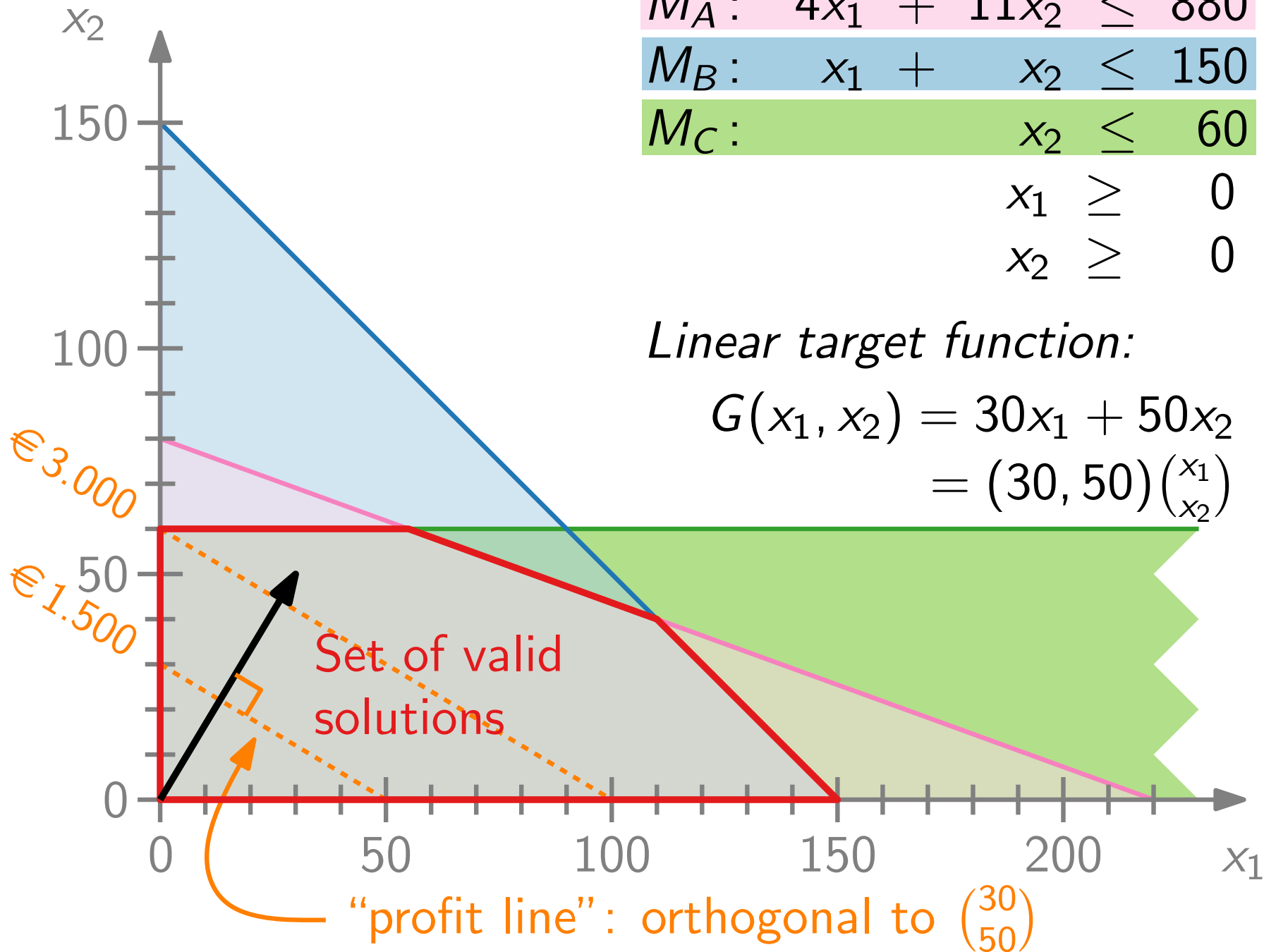
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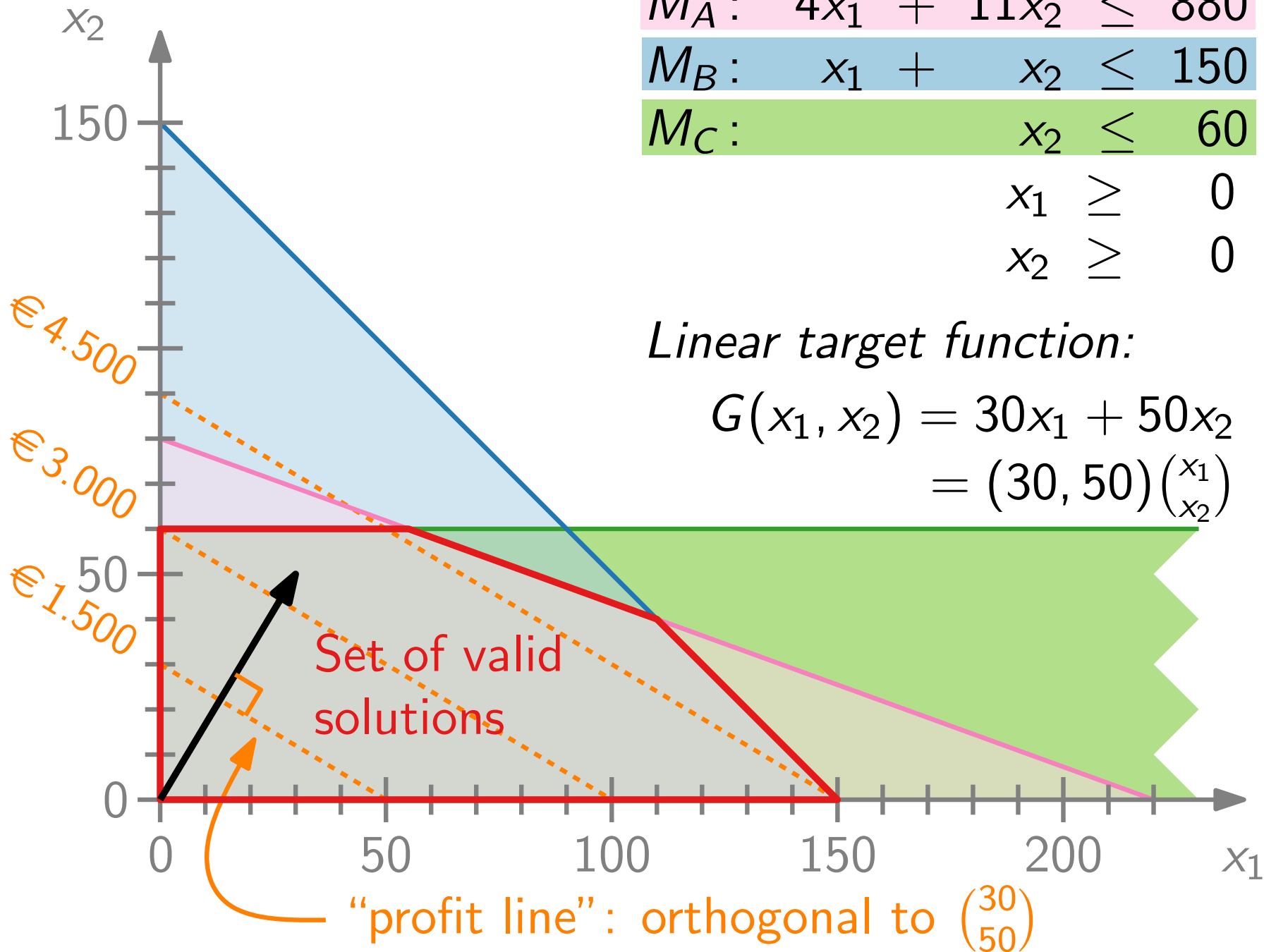
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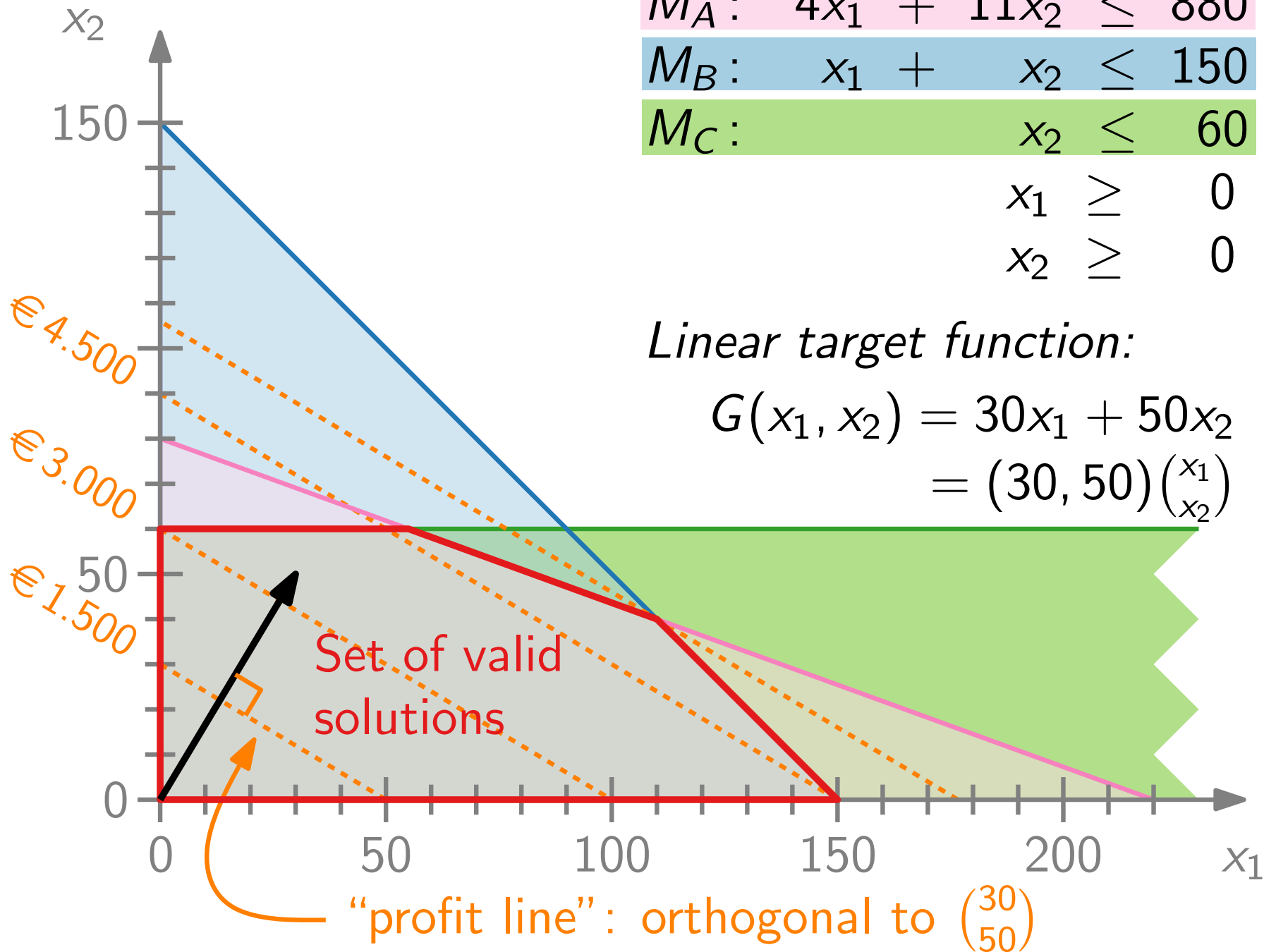
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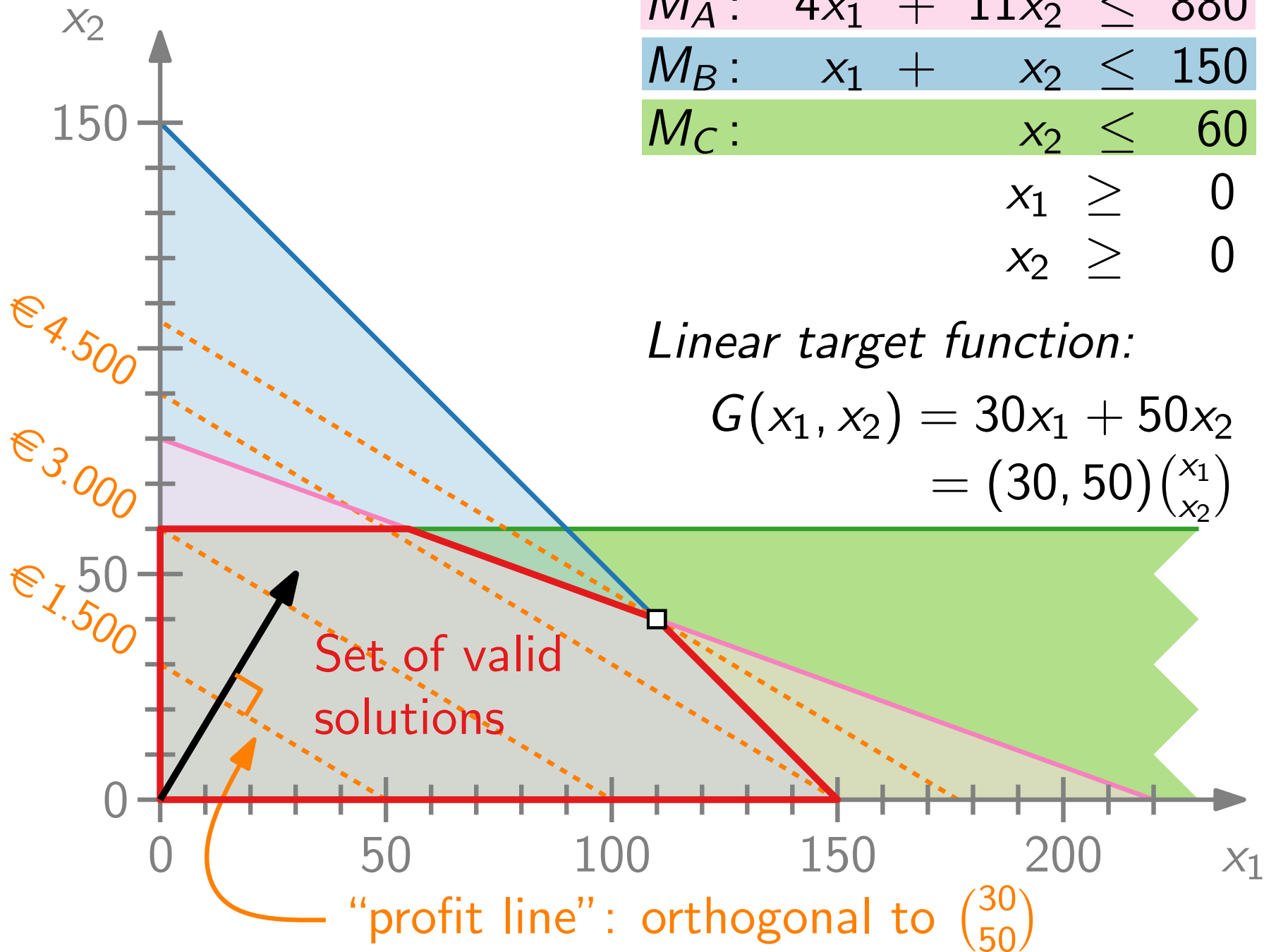
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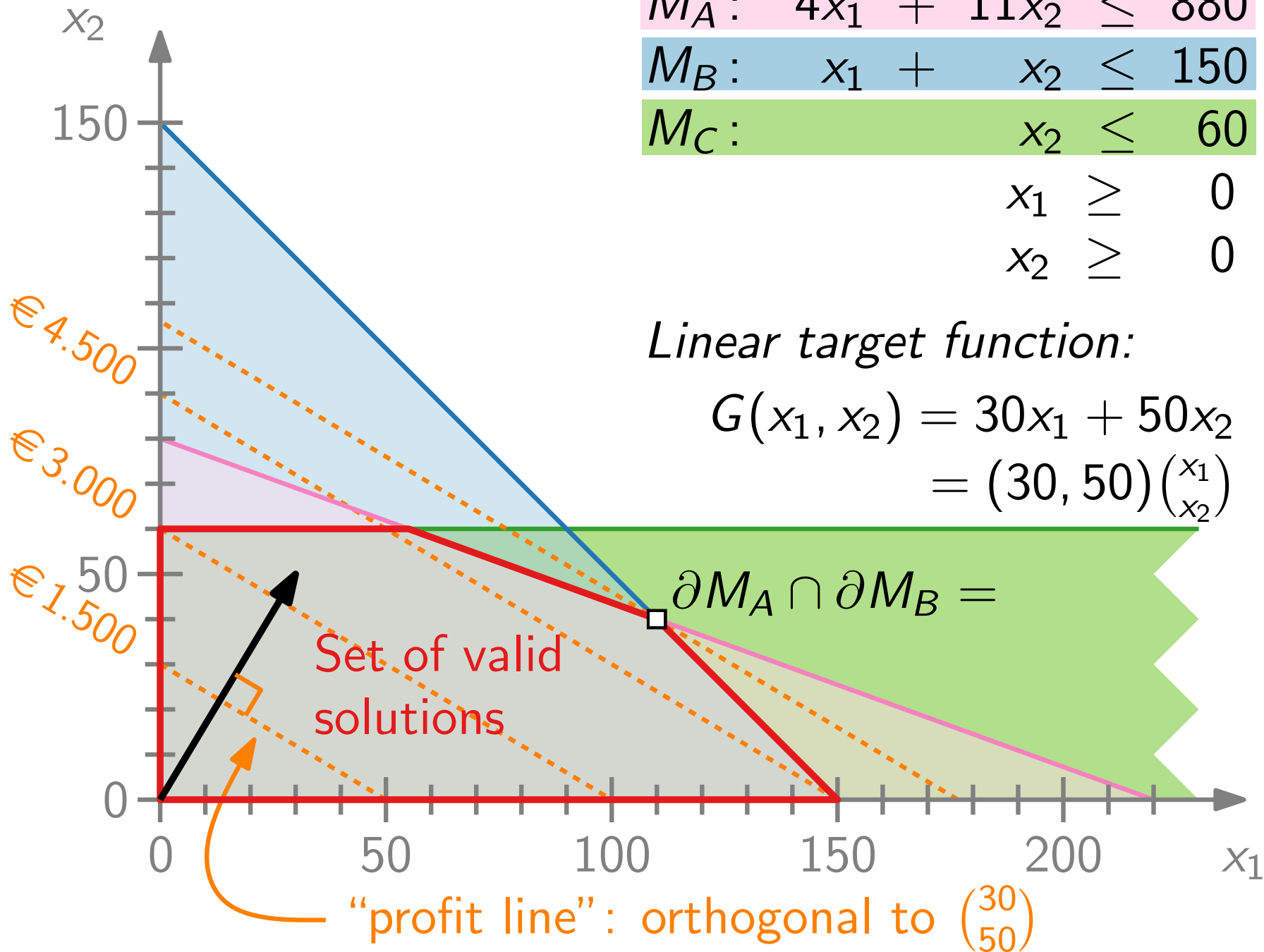
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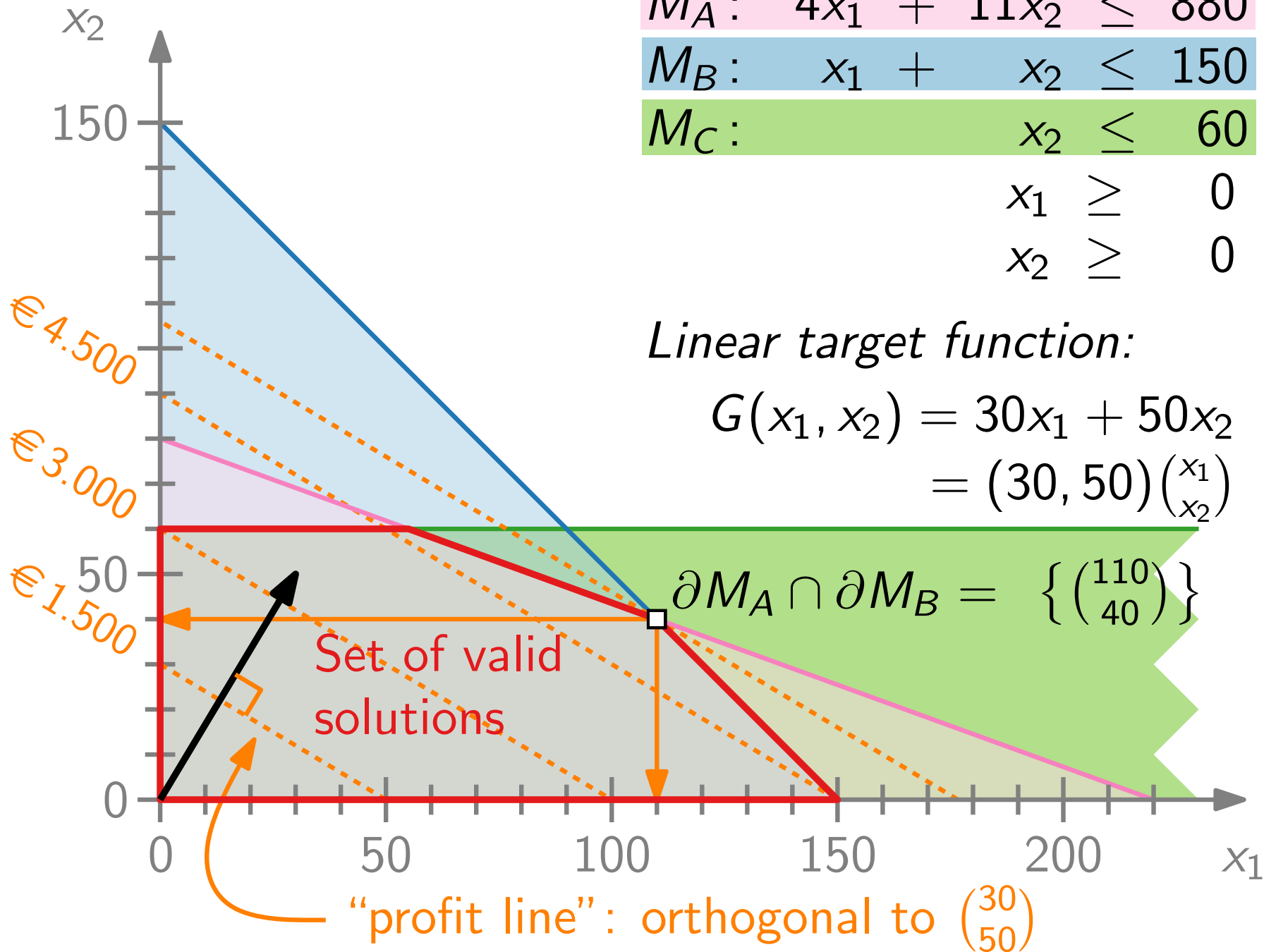
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$$\partial M_A \cap \partial M_B = \left\{ \begin{pmatrix} 110 \\ 40 \end{pmatrix} \right\}$$



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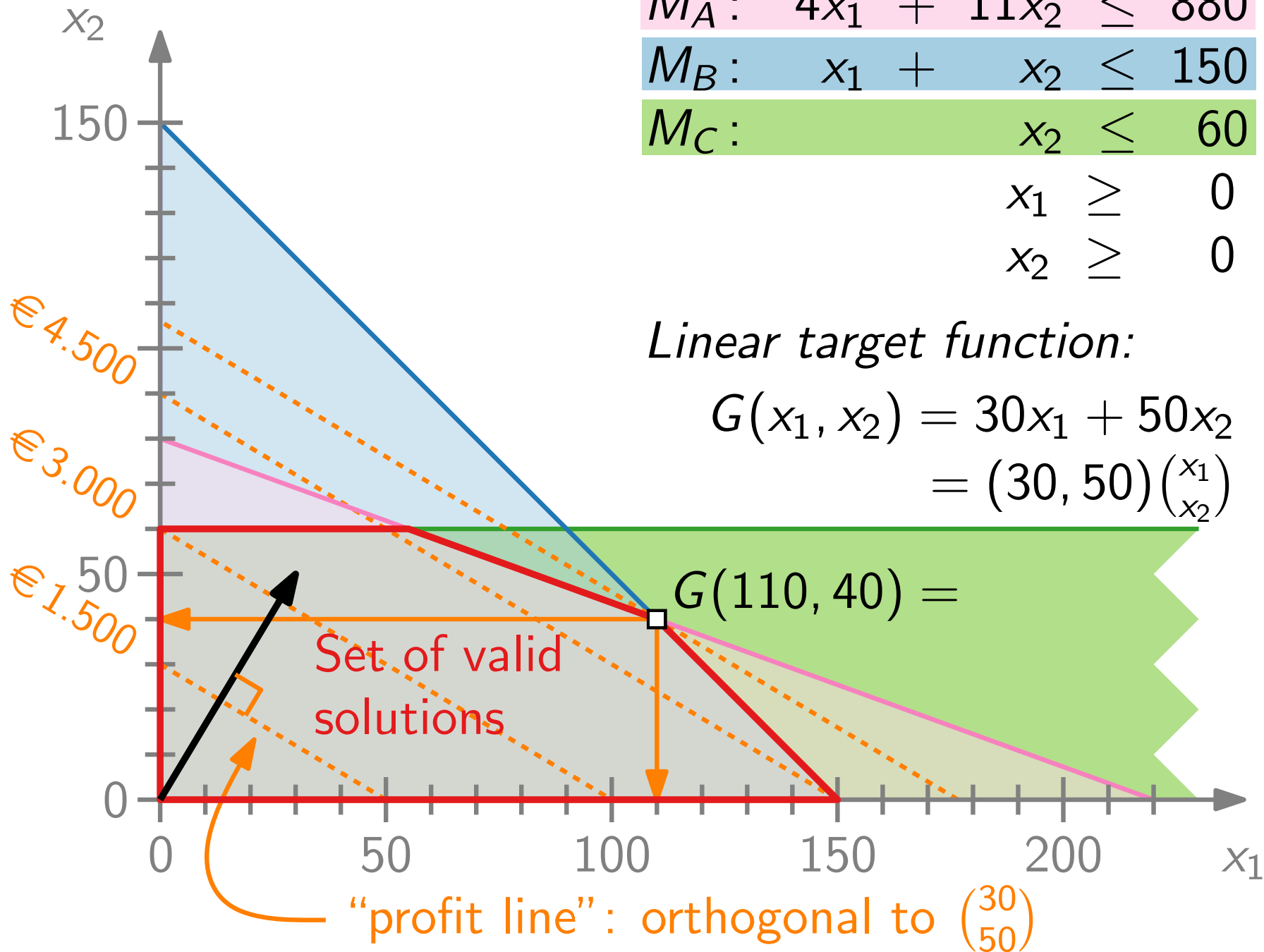
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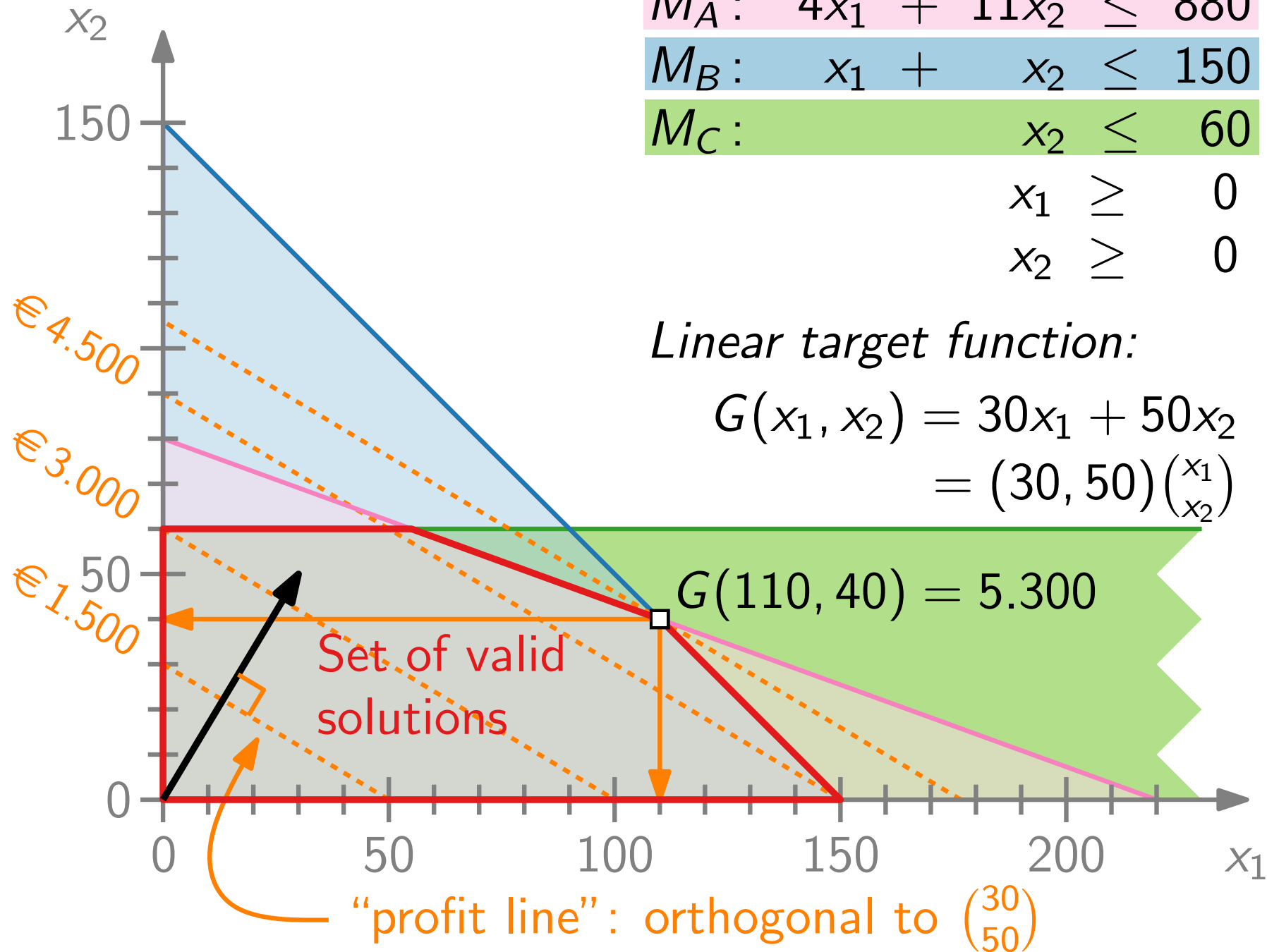
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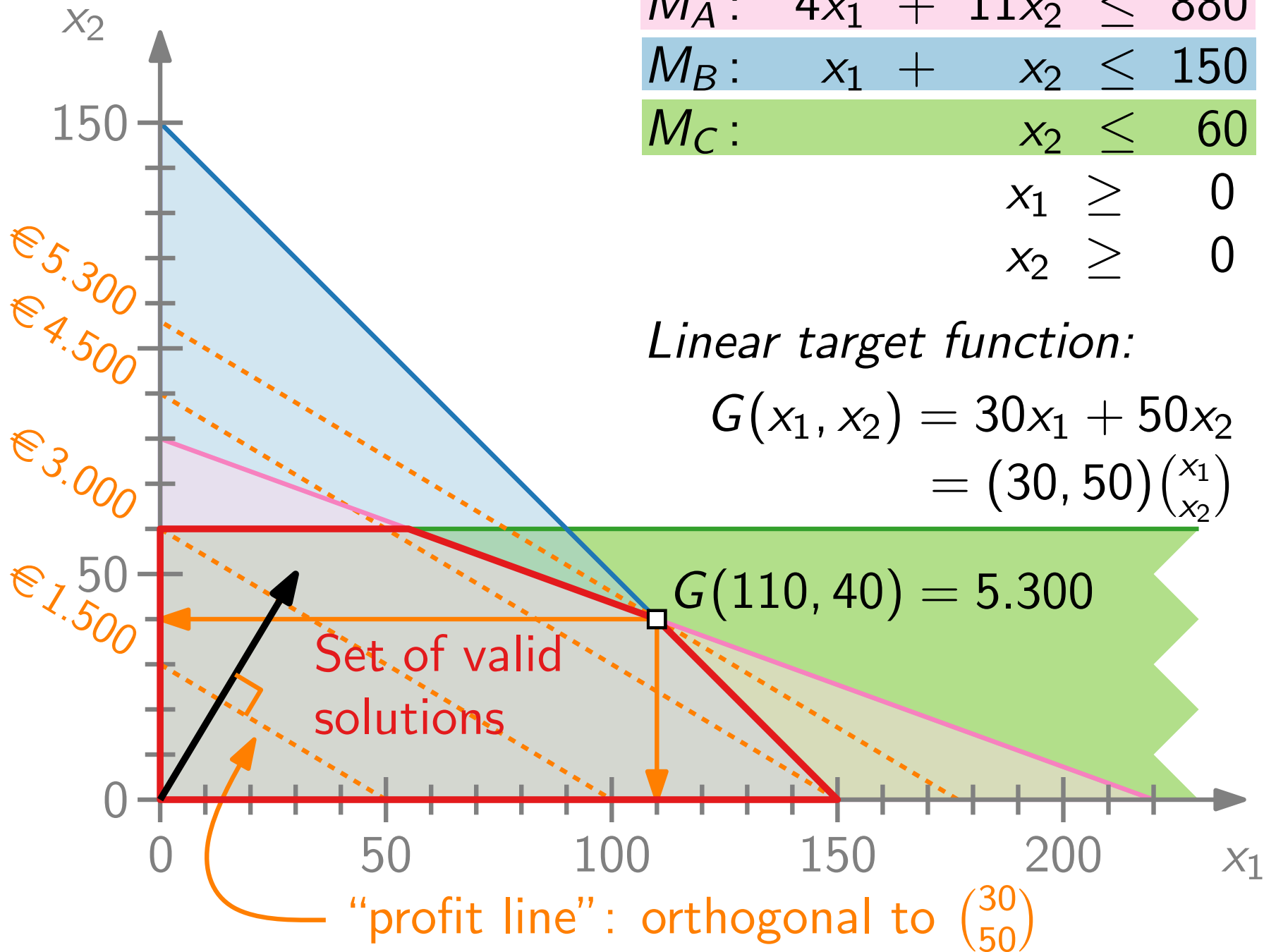
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Part II: Upper Bounds for LPs

Motivation: Upper and Lower Bounds

Consider an NP-hard minimization problem.

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\rightsquigarrow probably not! (conjecture: $NP \neq coNP$)

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$L \geq OPT/\alpha$ (i.e., an approximate “no”-certificate)!

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Examples:

- Vertex Cover: lower bound by matchings
- TSP: lower bound by MST or by cycle cover

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Example. $c = \begin{pmatrix} 7 \\ 1 \\ 5 \end{pmatrix}$ $A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & 2 & -1 \end{pmatrix}$ $b = \begin{pmatrix} 10 \\ 6 \end{pmatrix}$

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 \text{minimize} & 7x_1 + x_2 + 5x_3 \\
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	$5x_1$	+	$2x_2$	-	x_3	≥ 6

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minimize	$7x_1 + x_2 + 5x_3$	
subject to	$x_1 - 2x_2 + 3x_3 \geq 10$	10
	$5x_1 + 2x_2 - x_3 \geq 6$	9
	$x_1, x_2, x_3 \geq 0$	0

Valid solution?

$$x = (2, 1, 3)$$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$	$+ 14$	x_2	$+ 5x_3$		
subject to	x_1	$- 2$	x_2	$+ 1$	$+ 3x_3$	$9 \geq 10$
	$5x_1$	$+ 10$	$2x_2$	$- 2$	x_3	$3 \geq 6$
			x_1, x_2, x_3			≥ 0

Valid solution?

$$x = (2, 1, 3)$$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$	$+ 14$	x_2	$+ 1$	$5x_3$	
subject to	x_1	$- 2$	x_2	$+ 1$	$3x_3$	$9 \geq 10$
	$5x_1$	$+ 10$	$2x_2$	$- 2$	x_3	$3 \geq 6$
			x_1, x_2, x_3			≥ 0

Valid solution?

$$x = (2, 1, 3)$$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1 + 14x_2 + 5x_3$	
subject to	$x_1 - 2x_2 + 3x_3 \geq 10$	$10x_1 + 2x_2 + 3x_3 \geq 6$
		$x_1, x_2, x_3 \geq 0$

Valid solution?

$$x = (2, 1, 3)$$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$	$+ 14$	x_2	$+ 1$	$5x_3$	$+ 15$	$=$	30
subject to	x_1	$- 2$	x_2	$+ 1$	$3x_3$	$+ 9$	\geq	10
	$5x_1$	$+ 10$	$2x_2$	$- 2$	x_3	$+ 3$	\geq	6
			x_1, x_2, x_3	\geq			\geq	0

Valid solution?

$$x = (2, 1, 3)$$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$	$+ 14$	x_2	$+ 1$	$5x_3$	$+ 15$	$=$	30
subject to	x_1	$- 2$	x_2	$+ 1$	$3x_3$	$+ 9$	\geq	10
	$5x_1$	$+ 10$	$2x_2$	$- 2$	x_3	$+ 3$	\geq	6
			x_1, x_2, x_3	\geq			\geq	0

Valid solution?

$$x = (2, 1, 3)$$

$\Rightarrow \text{obj}(x) = 30$ is upper bound for **OPT**

Approximation Algorithms

Lecture 4: Linear Programming and LP-Duality

Part III: Lower Bounds for LPs

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$	+	x_2	+	$5x_3$		
subject to	x_1	−	x_2	+	$3x_3$	\geq	10
	$5x_1$	+	$2x_2$	−	x_3	\geq	6
					x_1, x_2, x_3	\geq	0

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$	+	x_2	+	$5x_3$		
subject to	x_1	-	x_2	+	$3x_3$	\geq	10
	$5x_1$	+	$2x_2$	-	x_3	\geq	6
					x_1, x_2, x_3	\geq	0

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$	+	x_2	+	$5x_3$		
subject to	x_1	-	x_2	+	$3x_3$	\geq	10
	$5x_1$	+	$2x_2$	-	x_3	\geq	6
					x_1, x_2, x_3	\geq	0

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$	+	x_2	+	$5x_3$		
subject to	x_1	-	x_2	+	$3x_3$	\geq	10
	$5x_1$	+	$2x_2$	-	x_3	\geq	6
					x_1, x_2, x_3	\geq	0

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$	+	x_2	+	$5x_3$		
subject to	x_1	-	x_2	+	$3x_3$	\geq	10
	$5x_1$	+	$2x_2$	-	x_3	\geq	6
					x_1, x_2, x_3	\geq	0

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$	+	x_2	+	$5x_3$		
subject to	x_1	-	x_2	+	$3x_3$	\geq	10
	$5x_1$	+	$2x_2$	-	x_3	\geq	6
					x_1, x_2, x_3	\geq	0

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \text{OPT} \geq 10$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$	+	x_2	+	$5x_3$	
subject to	x_1	-	x_2	+	$3x_3$	≥ 10
	$5x_1$	+	$2x_2$	-	x_3	≥ 6
					x_1, x_2, x_3	≥ 0

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \text{OPT} \geq 10$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$	+	x_2	+	$5x_3$		
subject to	x_1	-	x_2	+	$3x_3$	\geq	10
	$5x_1$	+	$2x_2$	-	x_3	\geq	6
					x_1, x_2, x_3	\geq	0

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \text{OPT} \geq 10$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$	+	x_2	+	$5x_3$		
subject to	x_1	-	x_2	+	$3x_3$	\geq	10
	$5x_1$	+	$2x_2$	-	x_3	\geq	6
					x_1, x_2, x_3	\geq	0

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \text{OPT} \geq 10$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$	+	x_2	+	$5x_3$		
subject to	x_1	-	x_2	+	$3x_3$	\geq	10
	$5x_1$	+	$2x_2$	-	x_3	\geq	6
					x_1, x_2, x_3	\geq	0

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \text{OPT} \geq 10$$

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\ &\geq 10 + 6 \end{aligned}$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$	+	x_2	+	$5x_3$		
subject to	x_1	-	x_2	+	$3x_3$	\geq	10
	$5x_1$	+	$2x_2$	-	x_3	\geq	6
					x_1, x_2, x_3	\geq	0

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \text{OPT} \geq 10$$

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\ &\geq 10 + 6 \quad \Rightarrow \text{OPT} \geq 16 \end{aligned}$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$	+	x_2	+	$5x_3$	
subject to	$2x_1$	-	x_2	+	$3x_3$	≥ 10
	$5x_1$	+	$2x_2$	-	x_3	≥ 6
					x_1, x_2, x_3	≥ 0

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \text{OPT} \geq 10$$

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\ &\geq 10 + 6 \quad \Rightarrow \text{OPT} \geq 16 \end{aligned}$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$	+	x_2	+	$5x_3$	
subject to	$2x_1$	-	x_2	+	$3x_3$	≥ 10
	$5x_1$	+	$2x_2$	-	x_3	≥ 6
					x_1, x_2, x_3	≥ 0

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \text{OPT} \geq 10$$

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\ &\geq 10 + 6 \quad \Rightarrow \text{OPT} \geq 16 \end{aligned}$$

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\geq 2 \cdot (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\ &\geq 2 \cdot 10 + 6 \end{aligned}$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

minimize	$7x_1$	$+$	x_2	$+$	$5x_3$	
	V		V		V	
subject to	2·		2·		2·3	≥ 2·10
	+		+		+	
	$5x_1$	$+$	$2x_2$	$-$	x_3	≥ 6
						≥ 0
					x_1, x_2, x_3	

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \text{OPT} \geq 10$$

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\ &\geq 10 + 6 \quad \Rightarrow \text{OPT} \geq 16 \end{aligned}$$

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\geq 2 \cdot (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\ &\geq 2 \cdot 10 + 6 \quad \Rightarrow \text{OPT} \geq 26 \end{aligned}$$

Linear Programming – Lower Bounds

minimize	$7x_1$	+	x_2	+	$5x_3$	
subject to	$2 \cdot x_1$	-	$2 \cdot x_2$	+	$2 \cdot 3x_3$	$\geq 2 \cdot 10$
	$5x_1$	+	$2x_2$	-	x_3	≥ 6
					x_1, x_2, x_3	≥ 0

$$\begin{aligned}
 7x_1 + x_2 + 5x_3 &\geq 2 \cdot (x_1 - x_2 + 3x_3) + 1 \cdot (5x_1 + 2x_2 - x_3) \\
 &\geq 2 \cdot 10 + 1 \cdot 6 \quad \Rightarrow \quad \text{OPT} \geq 2 \cdot 10 + 1 \cdot 6
 \end{aligned}$$

Linear Programming – Lower Bounds

$$\begin{array}{ll}
 \text{minimize} & 7x_1 + x_2 + 5x_3 \\
 \text{subject to} & y_1 (x_1 - x_2 + 3x_3) \geq 10 y_1 \\
 & 5x_1 + 2x_2 - x_3 \geq 6 \\
 & x_1, x_2, x_3 \geq 0
 \end{array}$$

$$\begin{aligned}
 7x_1 + x_2 + 5x_3 &\geq 2 \cdot (x_1 - x_2 + 3x_3) + 1 \cdot (5x_1 + 2x_2 - x_3) \\
 &\geq 2 \cdot 10 + 1 \cdot 6 \quad \Rightarrow \quad \text{OPT} \geq 2 \cdot 10 + 1 \cdot 6
 \end{aligned}$$

Linear Programming – Lower Bounds

$$\begin{array}{ll}
 \text{minimize} & 7x_1 + x_2 + 5x_3 \\
 \text{subject to} & y_1 (x_1 - x_2 + 3x_3) \geq 10 y_1 \\
 & y_2 (5x_1 + 2x_2 - x_3) \geq 6 y_2 \\
 & x_1, x_2, x_3 \geq 0
 \end{array}$$

$$\begin{aligned}
 7x_1 + x_2 + 5x_3 &\geq 2 \cdot (x_1 - x_2 + 3x_3) + 1 \cdot (5x_1 + 2x_2 - x_3) \\
 &\geq 2 \cdot 10 + 1 \cdot 6 \quad \Rightarrow \text{OPT} \geq 2 \cdot 10 + 1 \cdot 6
 \end{aligned}$$

Linear Programming – Lower Bounds

$$\begin{array}{ll}
 \text{minimize} & 7x_1 + x_2 + 5x_3 \\
 \text{subject to} & y_1(x_1 - x_2 + 3x_3) \geq 10y_1 \\
 & y_2(5x_1 + 2x_2 - x_3) \geq 6y_2 \\
 & x_1, x_2, x_3 \geq 0
 \end{array}$$

$$\begin{aligned}
 7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
 &\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
 \end{aligned}$$

Linear Programming – Lower Bounds

$$\begin{array}{ll}
 \text{minimize} & 7x_1 + x_2 + 5x_3 \\
 \text{subject to} & y_1 (x_1 - x_2 + 3x_3) \geq 10y_1 \\
 & y_2 (5x_1 + 2x_2 - x_3) \geq 6y_2 \\
 & x_1, x_2, x_3 \geq 0
 \end{array}$$

$$\begin{aligned}
 7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
 &\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
 \end{aligned}$$

$10y_1 + 6y_2$ is lower bound for **OPT**

Linear Programming – Lower Bounds

$$\begin{array}{ll}
 \text{minimize} & 7x_1 + x_2 + 5x_3 \\
 \text{subject to} & y_1 (x_1 - x_2 + 3x_3) \geq 10 y_1 \\
 & y_2 (5x_1 + 2x_2 - x_3) \geq 6 y_2 \\
 & x_1, x_2, x_3 \geq 0
 \end{array}$$

$$\begin{aligned}
 7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
 &\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
 \end{aligned}$$

Bounds for y_1, y_2 :

Linear Programming – Lower Bounds

$$\begin{array}{l}
 \text{minimize} \quad 7x_1 + x_2 + 5x_3 \\
 \text{subject to} \quad y_1 \left(\begin{array}{ccc} x_1 & -x_2 & +3x_3 \\ +x_1 & +x_2 & +3x_3 \end{array} \right) \geq 10 y_1 \\
 \quad \quad \quad y_2 \left(\begin{array}{ccc} 5x_1 & +2x_2 & -x_3 \end{array} \right) \geq 6 y_2 \\
 \quad \quad \quad x_1, x_2, x_3 \geq 0
 \end{array}$$

$$\begin{aligned}
 7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
 &\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
 \end{aligned}$$

Bounds for y_1, y_2 :

Linear Programming – Lower Bounds

$$\begin{array}{l}
 \text{minimize} \quad 7x_1 + x_2 + 5x_3 \\
 \text{subject to} \quad y_1 \left(\begin{array}{ccc} x_1 & -x_2 & +3x_3 \\ +x_1 & +x_2 & +3x_3 \end{array} \right) \geq 10 y_1 \\
 \quad \quad \quad y_2 \left(\begin{array}{ccc} 5x_1 & +2x_2 & -x_3 \end{array} \right) \geq 6 y_2 \\
 \quad \quad \quad x_1, x_2, x_3 \geq 0
 \end{array}$$

$$\begin{aligned}
 7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
 &\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
 \end{aligned}$$

$$y_1 + 5y_2 \leq 7$$

Bounds for y_1, y_2 :

Linear Programming – Lower Bounds

$$\begin{array}{l}
 \text{minimize} \quad 7x_1 + x_2 + 5x_3 \\
 \text{subject to} \quad y_1 \left(\begin{array}{ccc} x_1 & -x_2 & +3x_3 \\ +x_1 & +x_2 & +3x_3 \end{array} \right) \geq 10 y_1 \\
 \quad \quad \quad y_2 \left(\begin{array}{ccc} 5x_1 & +2x_2 & -x_3 \end{array} \right) \geq 6 y_2 \\
 \quad \quad \quad x_1, x_2, x_3 \geq 0
 \end{array}$$

$$\begin{aligned}
 7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
 &\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
 \end{aligned}$$

Bounds for y_1, y_2 :

$$\begin{array}{rcl}
 y_1 + 5y_2 &\leq & 7 \\
 -y_1 + 2y_2 &\leq & 1
 \end{array}$$

Linear Programming – Lower Bounds

$$\begin{array}{l}
 \text{minimize} \quad 7x_1 + x_2 + 5x_3 \\
 \text{subject to} \quad y_1 \left(\begin{array}{l} \overset{|}{\vee} x_1 - \overset{|}{\vee} x_2 + \overset{|}{\vee} 3x_3 \end{array} \right) \geq 10 y_1 \\
 \quad \quad \quad y_2 \left(\begin{array}{l} \overset{+}{+} 5x_1 + \overset{+}{+} 2x_2 - \overset{+}{+} x_3 \end{array} \right) \geq 6 y_2 \\
 \quad \quad \quad x_1, x_2, x_3 \geq 0
 \end{array}$$

$$\begin{aligned}
 7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
 &\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
 \end{aligned}$$

Bounds for y_1, y_2 :

$$\begin{array}{rcl}
 y_1 + 5y_2 &\leq & 7 \\
 -y_1 + 2y_2 &\leq & 1 \\
 3y_1 - y_2 &\leq & 5
 \end{array}$$

Linear Programming – Lower Bounds

$$\begin{array}{l}
 \text{minimize} \quad 7x_1 + x_2 + 5x_3 \\
 \text{subject to} \quad y_1 \left(\begin{array}{ccc} x_1 & -x_2 & +3x_3 \\ +x_1 & +x_2 & +3x_3 \end{array} \right) \geq 10 y_1 \\
 \quad \quad \quad y_2 \left(\begin{array}{ccc} 5x_1 & +2x_2 & -x_3 \end{array} \right) \geq 6 y_2 \\
 \quad \quad \quad x_1, x_2, x_3 \geq 0
 \end{array}$$

$$\begin{aligned}
 7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
 &\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
 \end{aligned}$$

Bounds for y_1, y_2 :

$$\begin{array}{rcl}
 y_1 + 5y_2 &\leq & 7 \\
 -y_1 + 2y_2 &\leq & 1 \\
 3y_1 - y_2 &\leq & 5 \\
 y_1, y_2 &\geq & 0
 \end{array}$$

Linear Programming – Lower Bounds

$$\begin{array}{ll}
 \text{minimize} & 7x_1 + x_2 + 5x_3 \\
 \text{subject to} & y_1 \left(\begin{array}{l} x_1 - x_2 + 3x_3 \\ 5x_1 + 2x_2 - x_3 \end{array} \right) \geq 10y_1 \\
 & y_2 \left(\begin{array}{l} x_1 - x_2 + 3x_3 \\ 5x_1 + 2x_2 - x_3 \end{array} \right) \geq 6y_2 \\
 & x_1, x_2, x_3 \geq 0
 \end{array}$$

$$\begin{aligned}
 7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
 &\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
 \end{aligned}$$

$$\begin{array}{ll}
 \text{maximize} & 10y_1 + 6y_2 \\
 \text{subject to} & y_1 + 5y_2 \leq 7 \\
 & -y_1 + 2y_2 \leq 1 \\
 & 3y_1 - y_2 \leq 5 \\
 & y_1, y_2 \geq 0
 \end{array}$$

Linear Programming – Lower Bounds

minimize	$7x_1$	+	x_2	+	$5x_3$		Primal			
subject to	y_1	(x_1	–	x_2	+	$3x_3$)	≥ 10	y_1
	y_2	($5x_1$	+	$2x_2$	–	x_3)	≥ 6	y_2
									≥ 0	x_1, x_2, x_3

$$\begin{aligned}
 7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
 &\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathbf{OPT} \geq 10y_1 + 6y_2
 \end{aligned}$$

maximize	$10y_1$	+	$6y_2$	
subject to	y_1	+	$5y_2$	≤ 7
	$-y_1$	+	$2y_2$	≤ 1
	$3y_1$	–	y_2	≤ 5
				≥ 0
			y_1, y_2	

Linear Programming – Lower Bounds

minimize	$7x_1$	+	x_2	+	$5x_3$		Primal			
subject to	y_1	$($	x_1	$-$	x_2	$+$	$3x_3)$	\geq	10	y_1
	y_2	$($	$5x_1$	$+$	$2x_2$	$-$	$x_3)$	\geq	6	y_2
								\geq	0	
					x_1, x_2, x_3			\geq	0	

$$7x_1 + x_2 + 5x_3 \geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3)$$

$$\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2$$

maximize	$10y_1$	+	$6y_2$		Dual
subject to	y_1	+	$5y_2$	\leq	7
	$-y_1$	+	$2y_2$	\leq	1
	$3y_1$	$-$	y_2	\leq	5
			y_1, y_2	\geq	0

Linear Programming – Lower Bounds

$$\begin{array}{ll}
 \text{minimize} & 7x_1 + x_2 + 5x_3 \\
 \text{subject to} & y_1 \begin{pmatrix} x_1 - x_2 + 3x_3 \\ 5x_1 + 2x_2 - x_3 \end{pmatrix} \geq \begin{pmatrix} 10 \\ 6 \end{pmatrix} \\
 & x_1, x_2, x_3 \geq 0
 \end{array}
 \quad \text{Primal}$$

$$\begin{aligned}
 7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
 &\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
 \end{aligned}$$

$$\begin{array}{ll}
 \text{maximize} & 10y_1 + 6y_2 \\
 \text{subject to} & y_1 + 5y_2 \leq 7 \\
 & -y_1 + 2y_2 \leq 1 \\
 & 3y_1 - y_2 \leq 5 \\
 & y_1, y_2 \geq 0
 \end{array}
 \quad \text{Dual}$$

Any feasible solution to the **dual** program provides a lower bound for the optimum of the **primal** program.

Linear Programming – Lower Bounds

$$\begin{array}{llllll}
 \text{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \text{Primal} \\
 \text{subject to} & y_1 \left(\begin{array}{l} x_1 \\ +x_1 \end{array} \right. & - & y_1 \left(\begin{array}{l} x_2 \\ +x_2 \end{array} \right. & + & y_1 \left(\begin{array}{l} 3x_3 \\ +x_3 \end{array} \right) & \geq & 10y_1 \\
 & y_2 \left(\begin{array}{l} 5x_1 \\ +5x_1 \end{array} \right. & + & y_2 \left(\begin{array}{l} 2x_2 \\ +2x_2 \end{array} \right. & - & y_2 \left(\begin{array}{l} x_3 \\ -x_3 \end{array} \right) & \geq & 6y_2 \\
 & & & & & & & x_1, x_2, x_3 \geq 0
 \end{array}$$

$$\begin{aligned}
 7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
 &\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
 \end{aligned}$$

$$\begin{array}{llllll}
 \text{maximize} & 10y_1 & + & 6y_2 & & \text{Dual} \\
 \text{subject to} & y_1 & + & 5y_2 & \leq & 7 \\
 & -y_1 & + & 2y_2 & \leq & 1 \\
 & 3y_1 & - & y_2 & \leq & 5 \\
 & & & & & y_1, y_2 \geq 0
 \end{array}$$

Any feasible solution to the **dual** program provides a lower bound for the optimum of the **primal** program.

$x = (7/4, 0, 11/4)$ both $y = (2, 1)$ provide objective value 26.

Linear Programming – Lower Bounds

$$\begin{array}{ll}
 \text{minimize} & 7x_1 + x_2 + 5x_3 \\
 \text{subject to} & y_1 \begin{pmatrix} x_1 - x_2 + 3x_3 \\ 5x_1 + 2x_2 - x_3 \end{pmatrix} \geq \begin{pmatrix} 10 \\ 6 \end{pmatrix} \\
 & x_1, x_2, x_3 \geq 0
 \end{array}
 \quad \text{Primal}$$

$$\begin{aligned}
 7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
 &\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
 \end{aligned}$$

$$\begin{array}{ll}
 \text{maximize} & 10y_1 + 6y_2 \\
 \text{subject to} & y_1 + 5y_2 \leq 7 \\
 & -y_1 + 2y_2 \leq 1 \\
 & 3y_1 - y_2 \leq 5 \\
 & y_1, y_2 \geq 0
 \end{array}
 \quad \text{Dual}$$

Any feasible solution to the **dual** program provides a lower bound for the optimum of the **primal** program.

$x = (7/4, 0, 11/4)$ both $y = (2, 1)$ provide objective value 26.

== OPT

Primal–Dual

primal program

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

Primal–Dual

primal program

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

dual program

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \\ & y \geq 0 \end{array}$$

Primal–Dual

primal program

$$\begin{array}{ll}
 \text{minimize} & c^T x \\
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 & x \geq 0
 \end{array}$$

dual program

$$\begin{array}{ll}
 \text{maximize} & b^T y \\
 \text{subject to} & A^T y \leq c \\
 & y \geq 0
 \end{array}$$

Dual Program of the Dual Program

Primal–Dual

primal program

$$\begin{array}{ll}
 \text{minimize} & c^T x \\
 \text{subject to} & Ax \geq b \\
 & x \geq 0
 \end{array}$$

dual program

$$\begin{array}{ll}
 \text{maximize} & b^T y \\
 \text{subject to} & A^T y \leq c \\
 & y \geq 0
 \end{array}$$

dual of the dual program

$$\begin{array}{ll}
 \text{minimize} & c^T x \\
 \text{subject to} & Ax \geq b \\
 & x \geq 0
 \end{array}$$

Approximation Algorithms

Lecture 4: Linear Programming and LP-Duality

Part IV: LP-Duality and Complementary Slackness

LP-Duality

minimize	$c^T x$			Primal
subject to	Ax	\geq	b	
	x	\geq	0	

maximize	$b^T y$			Dual
subject to	$A^T y$	\leq	c	
	y	\geq	0	

LP-Duality

minimize	$c^T x$	Primal
subject to	$Ax \geq b$	
	$x \geq 0$	

maximize	$b^T y$	Dual
subject to	$A^T y \leq c$	
	$y \geq 0$	

Theorem. The primal program has a finite optimum
 \Leftrightarrow the dual program has a finite optimum.

LP-Duality

minimize	$c^T x$		Primal
subject to	Ax	\geq	b
	x	\geq	0

maximize	$b^T y$		Dual
subject to	$A^T y$	\leq	c
	y	\geq	0

Theorem. The primal program has a finite optimum \Leftrightarrow the dual program has a finite optimum. Moreover, if $x^* = (x_1^*, \dots, x_n^*)$ and $y^* = (y_1^*, \dots, y_m^*)$ are optimal solutions for the primal and dual program (resp.), then

LP-Duality

minimize	$c^T x$		Primal
subject to	Ax	\geq	b
	x	\geq	0

maximize	$b^T y$		Dual
subject to	$A^T y$	\leq	c
	y	\geq	0

Theorem. The primal program has a finite optimum \Leftrightarrow the dual program has a finite optimum. Moreover, if $x^* = (x_1^*, \dots, x_n^*)$ and $y^* = (y_1^*, \dots, y_m^*)$ are optimal solutions for the primal and dual program (resp.), then

$$\sum_{j=1}^n c_j x_j^* = \sum_{i=1}^m b_i y_i^* .$$

Weak LP-Duality

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \\ & y \geq 0 \end{array}$$

Theorem. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ are *valid* solutions for the **primal** and **dual** program (resp.), then

Weak LP-Duality

$$\begin{array}{ll}
 \text{minimize} & c^T x \\
 \text{subject to} & Ax \geq b \\
 & x \geq 0
 \end{array}$$

$$\begin{array}{ll}
 \text{maximize} & b^T y \\
 \text{subject to} & A^T y \leq c \\
 & y \geq 0
 \end{array}$$

Theorem. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ are *valid* solutions for the **primal** and **dual** program (resp.), then

$$\sum_{j=1}^n c_j x_j \geq \sum_{i=1}^m b_i y_i .$$

Weak LP-Duality

$$\begin{array}{ll}
 \text{minimize} & c^T x \\
 \text{subject to} & Ax \geq b \\
 & x \geq 0
 \end{array}$$

$$\begin{array}{ll}
 \text{maximize} & b^T y \\
 \text{subject to} & A^T y \leq c \\
 & y \geq 0
 \end{array}$$

Theorem. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ are *valid* solutions for the **primal** and **dual** program (resp.), then

$$\sum_{j=1}^n c_j x_j \geq \sum_{i=1}^m b_i y_i .$$

Proof.

Weak LP-Duality

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Proof.

$$\sum_{j=1}^n c_j x_j \geq$$

$$\sum_{i=1}^m b_i y_i .$$

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Proof.

$$\sum_{j=1}^n c_j x_j \geq$$

$$\sum_{i=1}^m b_i y_i .$$

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Theorem. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ are valid solutions for the primal and dual program (resp.), then

$$\sum_{j=1}^n c_j x_j \geq \sum_{i=1}^m b_i y_i .$$

Proof.

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Theorem. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ are *valid* solutions for the **primal** and **dual** program (resp.), then

$$\sum_{j=1}^n c_j x_j \geq \sum_{i=1}^m b_i y_i .$$

Proof.

$$\sum_{j=1}^n c_j x_j \geq$$

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Weak LP-Duality

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Theorem. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ are valid solutions for the primal and dual program (resp.), then

$$\sum_{j=1}^n c_j x_j \geq \sum_{i=1}^m b_i y_i .$$

Proof.

$$\sum_{j=1}^n c_j x_j \geq \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} y_i \right) x_j = \sum_{i=1}^m b_i y_i .$$

Weak LP-Duality

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Proof.

$$\sum_{j=1}^n c_j x_j \geq \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} y_i \right) x_j = \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} y_i x_j \right) \geq \sum_{i=1}^m b_i y_i .$$

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Proof.

$$\sum_{j=1}^n c_j x_j \geq \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} y_i \right) x_j = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j \right) y_i \geq \sum_{i=1}^m b_i y_i .$$

Weak LP-Duality

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$$\sum_{j=1}^n c_j x_j \geq \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} y_i \right) x_j = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j \right) y_i \geq \sum_{i=1}^m b_i y_i .$$

Weak LP-Duality

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$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \\ & y \geq 0 \end{array}$$

Theorem. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ are valid solutions for the primal and dual program (resp.), then

$$\sum_{j=1}^n c_j x_j \geq \sum_{i=1}^m b_i y_i .$$

Proof.

$$\sum_{j=1}^n c_j x_j \geq \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} y_i \right) x_j = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j \right) y_i \geq \sum_{i=1}^m b_i y_i .$$

Complementary Slackness

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \\ & y \geq 0 \end{array}$$

Complementary Slackness

$$\begin{array}{ll}
 \text{minimize} & c^T x \\
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 & x \geq 0
 \end{array}$$

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 \text{maximize} & b^T y \\
 \text{subject to} & A^T y \leq c \\
 & y \geq 0
 \end{array}$$

Theorem. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ be valid solutions for the primal and dual program (resp.).

Complementary Slackness

$$\begin{array}{ll}
 \text{minimize} & c^T x \\
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 \end{array}$$

$$\begin{array}{ll}
 \text{maximize} & b^T y \\
 \text{subject to} & A^T y \leq c \\
 & y \geq 0
 \end{array}$$

Theorem. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ be valid solutions for the **primal** and **dual** program (resp.). Then x and y are optimal if and only if the following conditions are met:

Complementary Slackness

$$\begin{array}{ll}
 \text{minimize} & c^T x \\
 \text{subject to} & Ax \geq b \\
 & x \geq 0
 \end{array}$$

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Theorem. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ be valid solutions for the primal and dual program (resp.). Then x and y are optimal if and only if the following conditions are met:

Primal CS:

For each $j = 1, \dots, n$: $x_j = 0$ or $\sum_{i=1}^m a_{ij} y_i = c_j$

Complementary Slackness

$$\begin{array}{ll}
 \text{minimize} & c^T x \\
 \text{subject to} & Ax \geq b \\
 & x \geq 0
 \end{array}$$

$$\begin{array}{ll}
 \text{maximize} & b^T y \\
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 & y \geq 0
 \end{array}$$

Theorem. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ be valid solutions for the **primal** and **dual** program (resp.). Then x and y are optimal if and only if the following conditions are met:

Primal CS:

For each $j = 1, \dots, n$: $x_j = 0$ or $\sum_{i=1}^m a_{ij} y_i = c_j$

Dual CS:

For each $i = 1, \dots, m$: $y_i = 0$ or $\sum_{j=1}^n a_{ij} x_j = b_i$

Complementary Slackness

$$\begin{array}{ll}
 \text{minimize} & c^T x \\
 \text{subject to} & Ax \geq b \\
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 \text{maximize} & b^T y \\
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Theorem. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ be valid solutions for the primal and dual program (resp.). Then x and y are optimal if and only if the following conditions are met:

Primal CS:

For each $j = 1, \dots, n$: $x_j = 0$ or $\sum_{i=1}^m a_{ij} y_i = c_j$

Dual CS:

For each $i = 1, \dots, m$: $y_i = 0$ or $\sum_{j=1}^n a_{ij} x_j = b_i$

Proof.

Complementary Slackness

$$\begin{array}{ll}
 \text{minimize} & c^T x \\
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Theorem. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ be valid solutions for the primal and dual program (resp.). Then x and y are optimal if and only if the following conditions are met:

Primal CS:

For each $j = 1, \dots, n$: $x_j = 0$ or $\sum_{i=1}^m a_{ij} y_i = c_j$

Dual CS:

For each $i = 1, \dots, m$: $y_i = 0$ or $\sum_{j=1}^n a_{ij} x_j = b_i$

Proof. Follows from LP-duality:

Complementary Slackness

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

$$\begin{array}{ll} \text{maximize} & b^T y \\ \text{subject to} & A^T y \leq c \\ & y \geq 0 \end{array}$$

Theorem. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ be valid solutions for the primal and dual program (resp.). Then x and y are optimal if and only if the following conditions are met:

Primal CS:

For each $j = 1, \dots, n$: $x_j = 0$ or $\sum_{i=1}^m a_{ij} y_i = c_j$

Dual CS:

For each $i = 1, \dots, m$: $y_i = 0$ or $\sum_{j=1}^n a_{ij} x_j = b_i$

Proof. Follows from LP-duality:

$$\sum_{j=1}^n c_j x_j \geq \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} y_i \right) x_j = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j \right) y_i \geq \sum_{i=1}^m b_i y_i .$$

Complementary Slackness

$$\begin{array}{ll}
 \text{minimize} & c^T x \\
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 \text{maximize} & b^T y \\
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Theorem. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ be valid solutions for the primal and dual program (resp.). Then x and y are optimal if and only if the following conditions are met:

Primal CS:

For each $j = 1, \dots, n$: $x_j = 0$ or $\sum_{i=1}^m a_{ij} y_i = c_j$

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For each $i = 1, \dots, m$: $y_i = 0$ or $\sum_{j=1}^n a_{ij} x_j = b_i$

Proof. Follows from LP-duality:

$$\sum_{j=1}^n c_j x_j \geq \sum_{j=1}^n \left(\sum_{i=1}^m a_{ij} y_i \right) x_j = \sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} x_j \right) y_i \geq \sum_{i=1}^m b_i y_i .$$

Complementary Slackness

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

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LPs and Convex Polytopes

The feasible solutions of an LP with n variables form a **convex polytope** in \mathbb{R}^n (intersection of halfspaces).

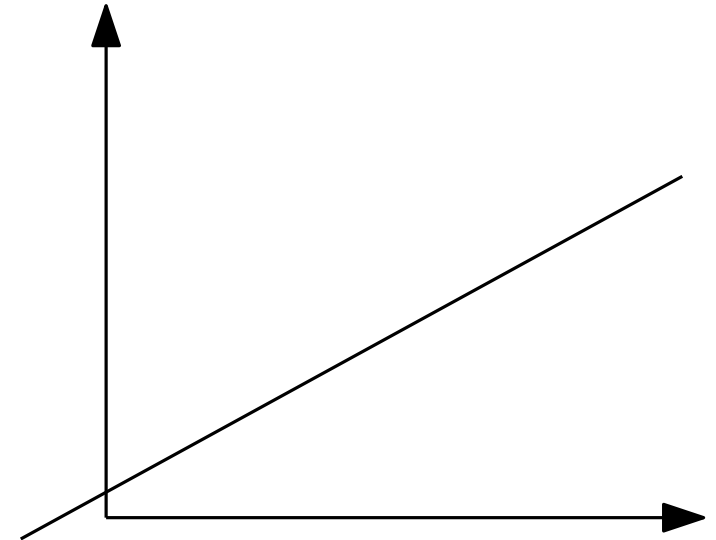
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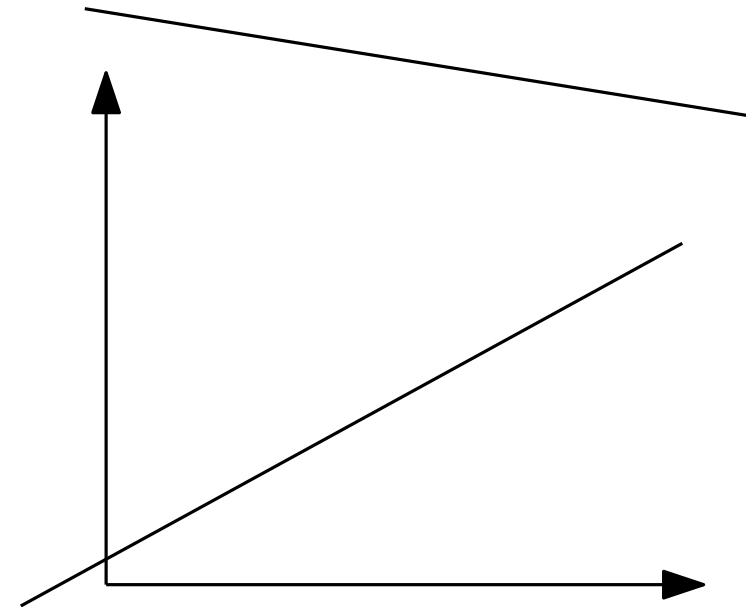
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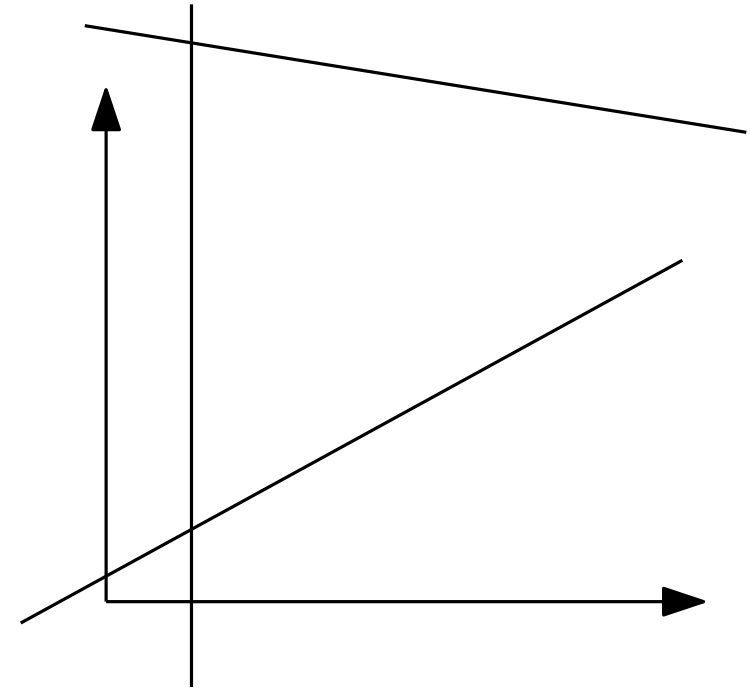
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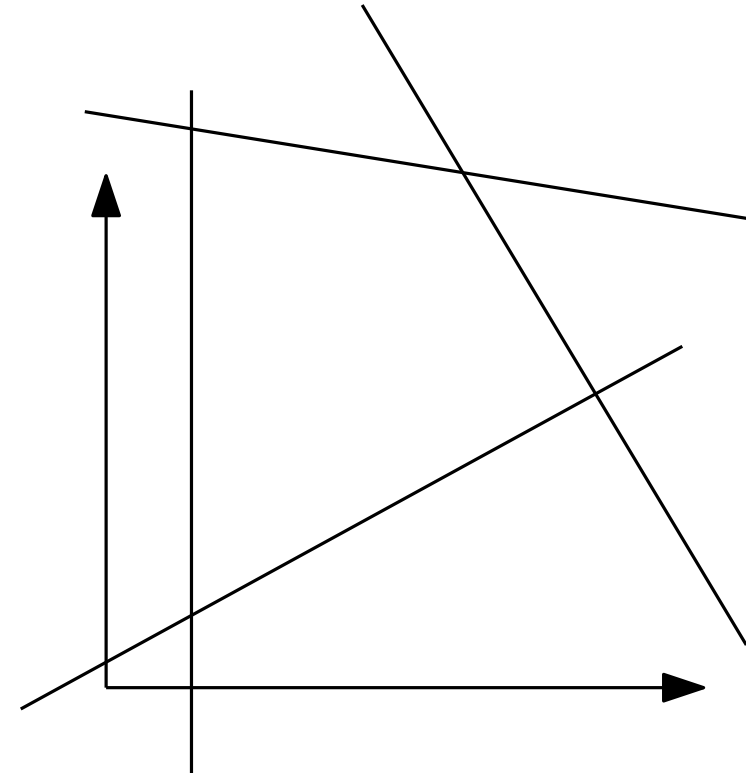
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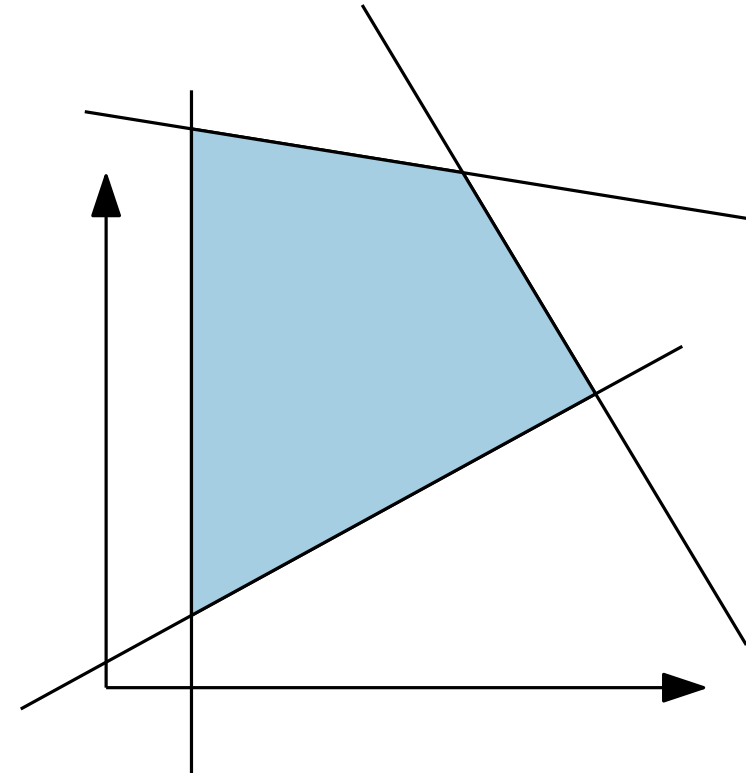
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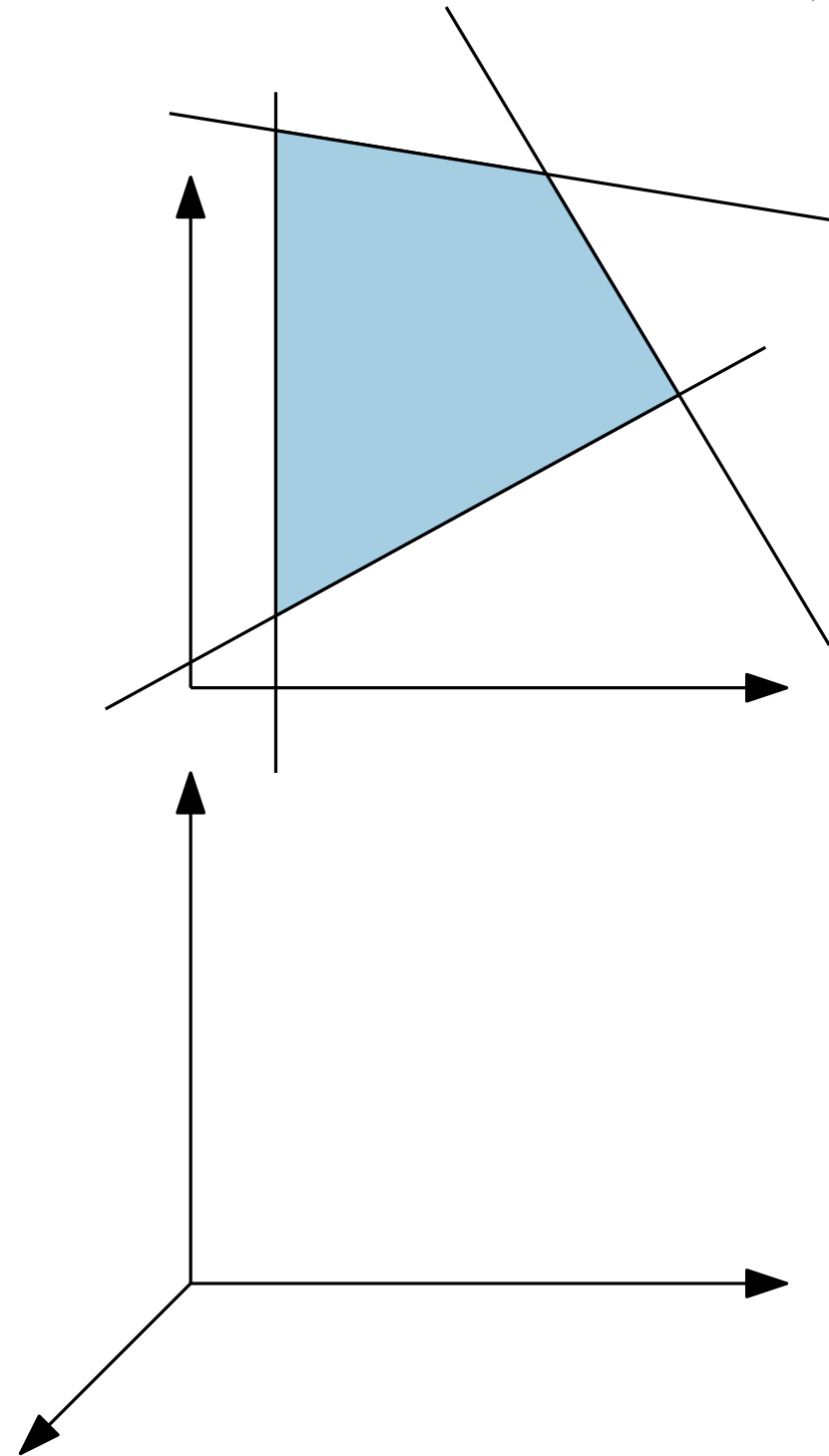
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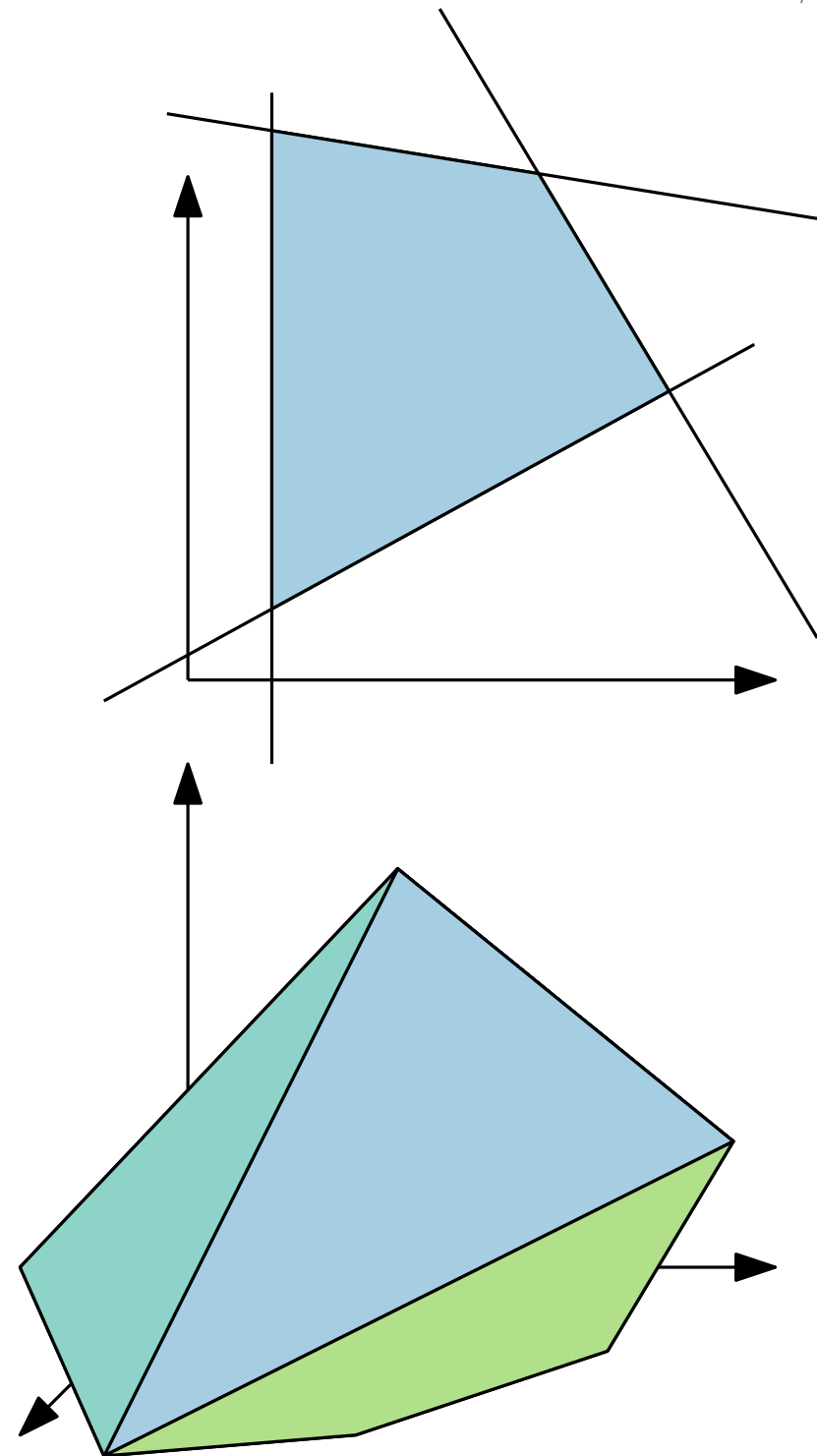
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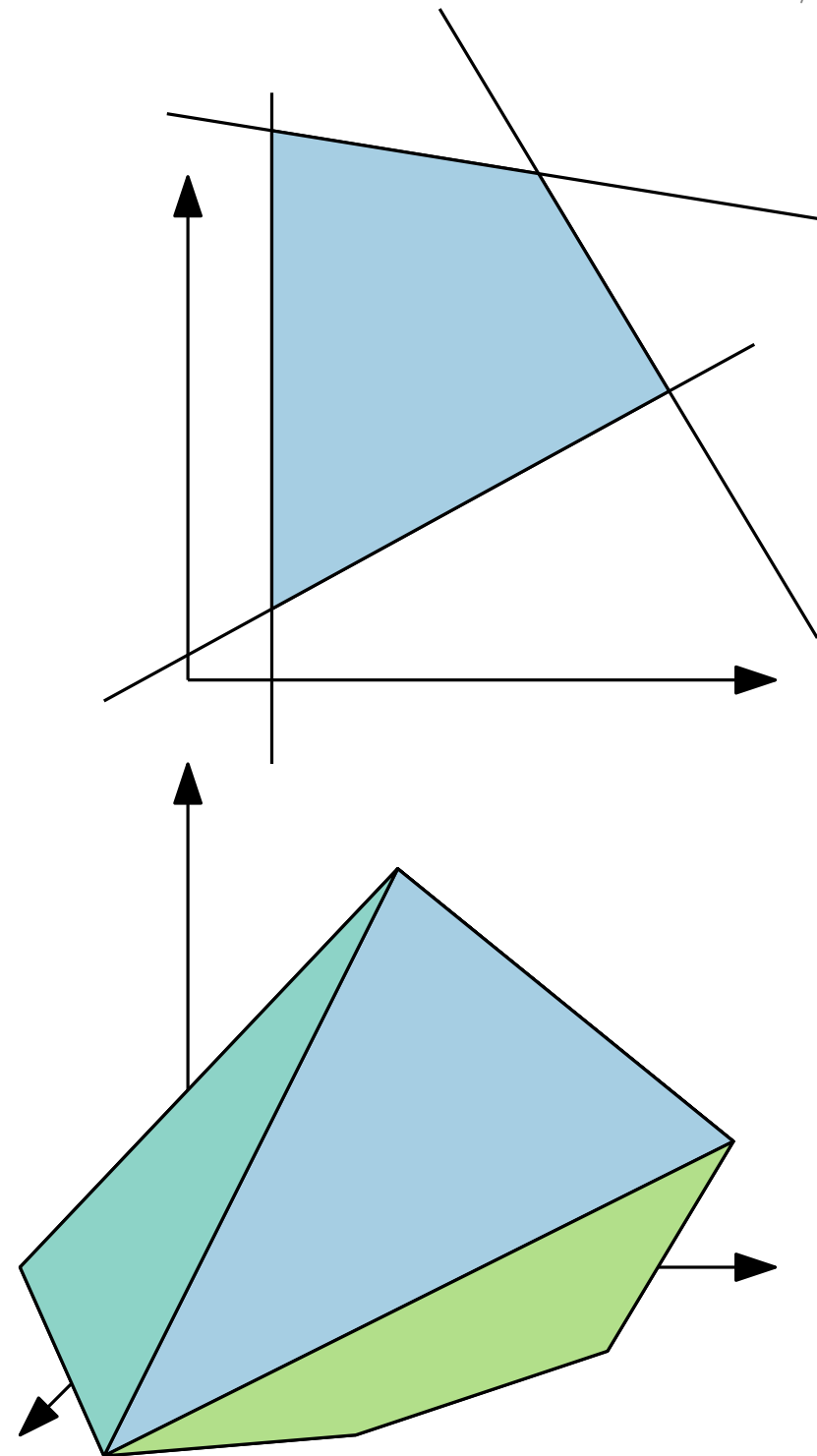
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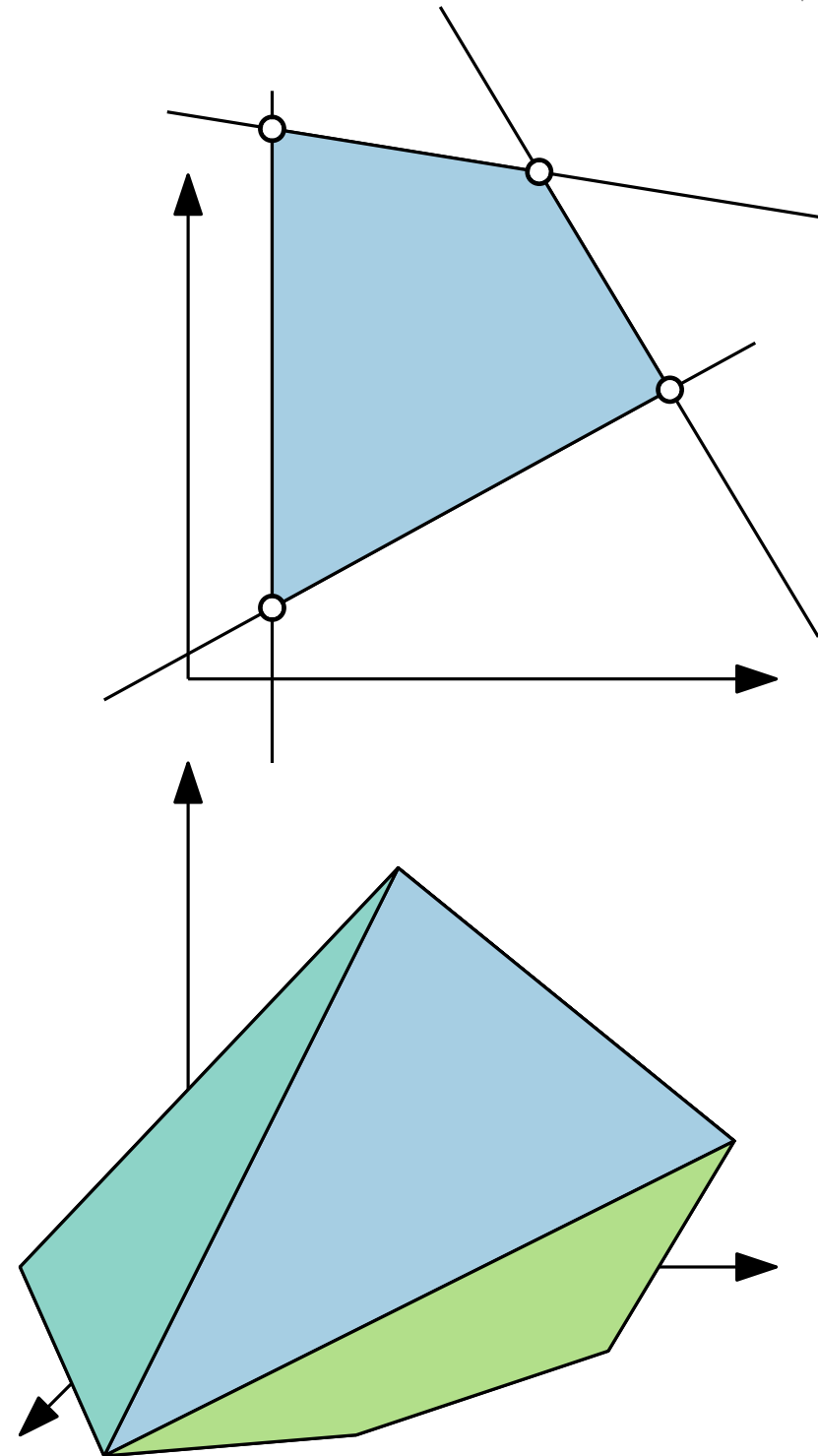
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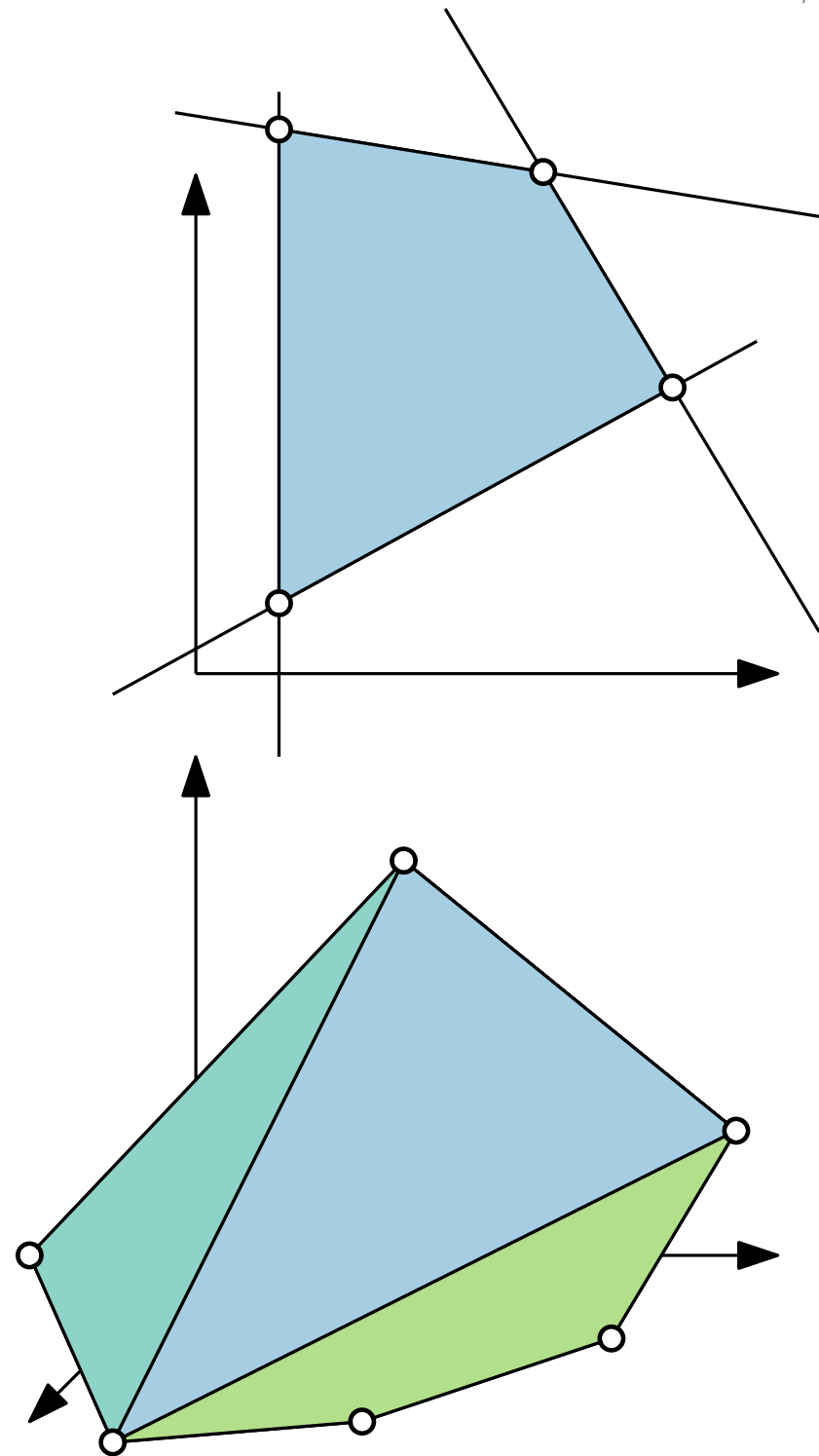
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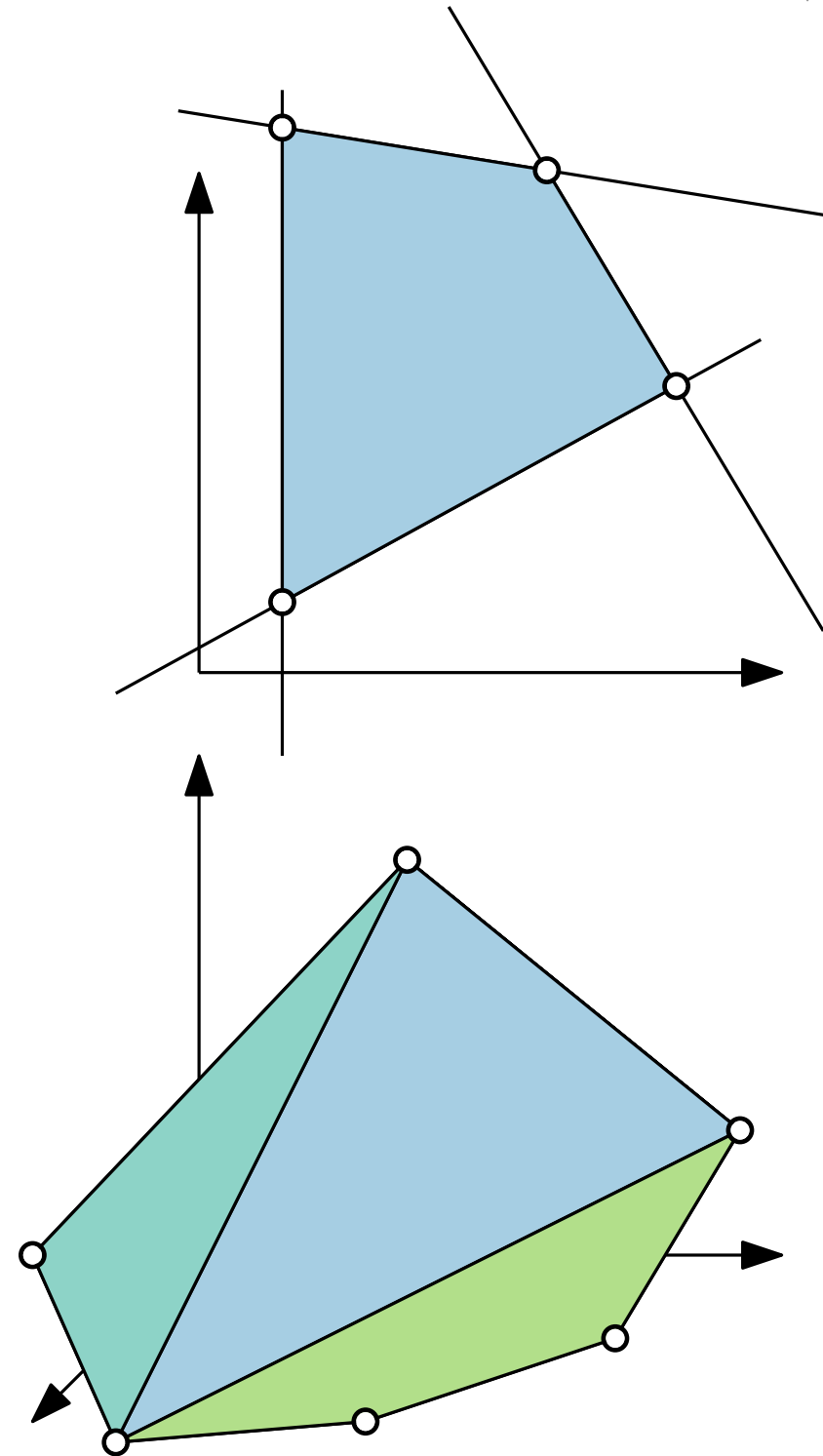


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If an optimal solution exists, some extreme point is also optimal.



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LP-relaxation provides a lower bound: $\text{OPT}_{\text{LP}} \leq \text{OPT}_{\text{ILP}}$

Approximation Algorithms

Lecture 4: Linear Programming and LP-Duality

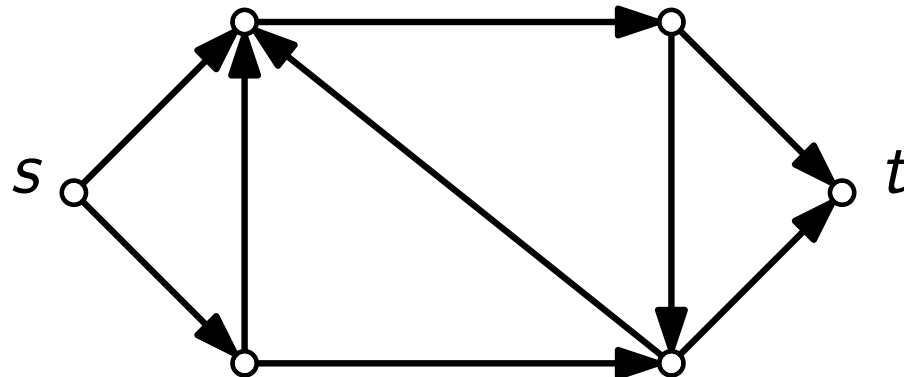
Part V: Min–Max Relationships

Max-Flow Problem

Given: A directed graph $G = (V, E)$ with edge capacities $c: E \rightarrow \mathbb{Q}_+$ and two special vertices: the source s and sink t .

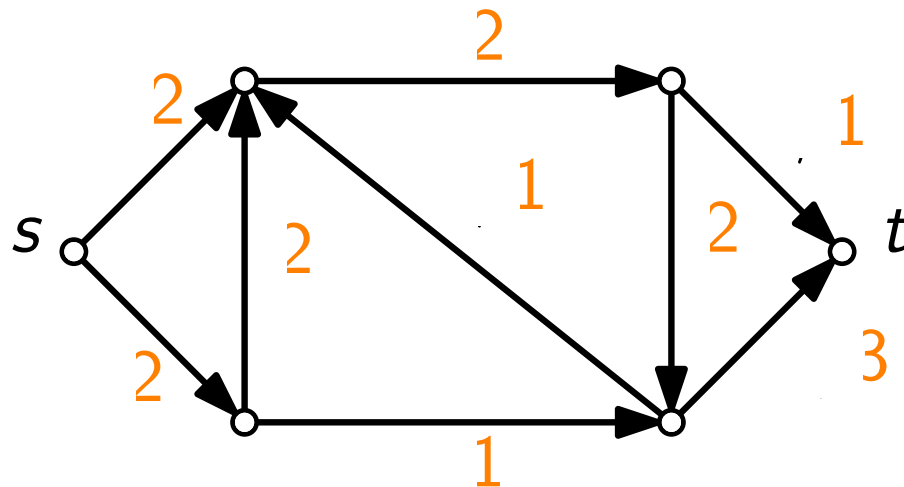
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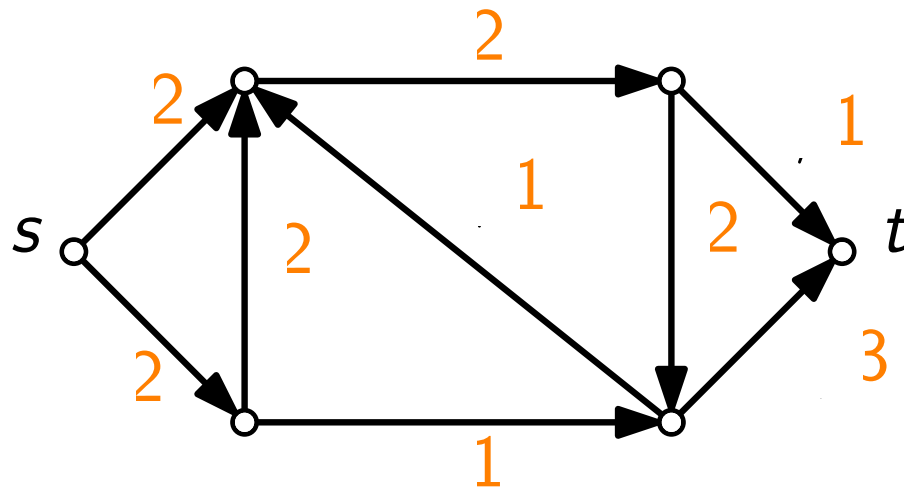


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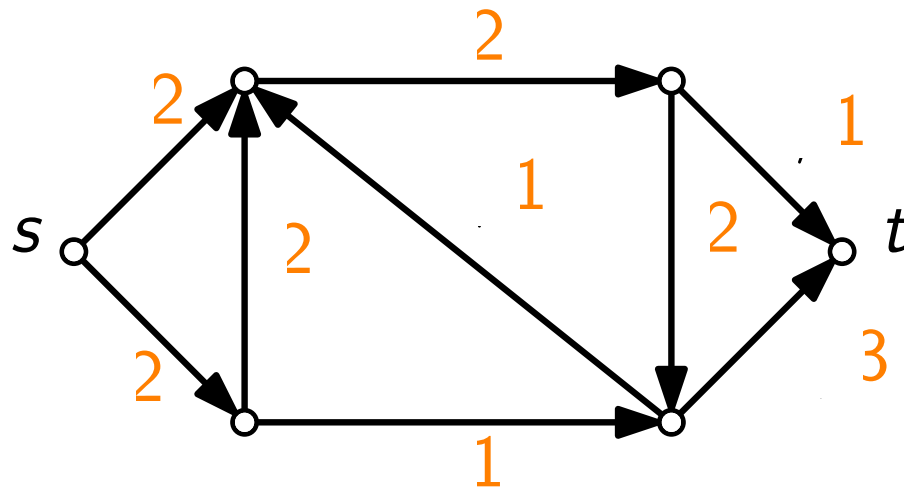
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The **flow value** is the inflow to t minus the outflow from t .



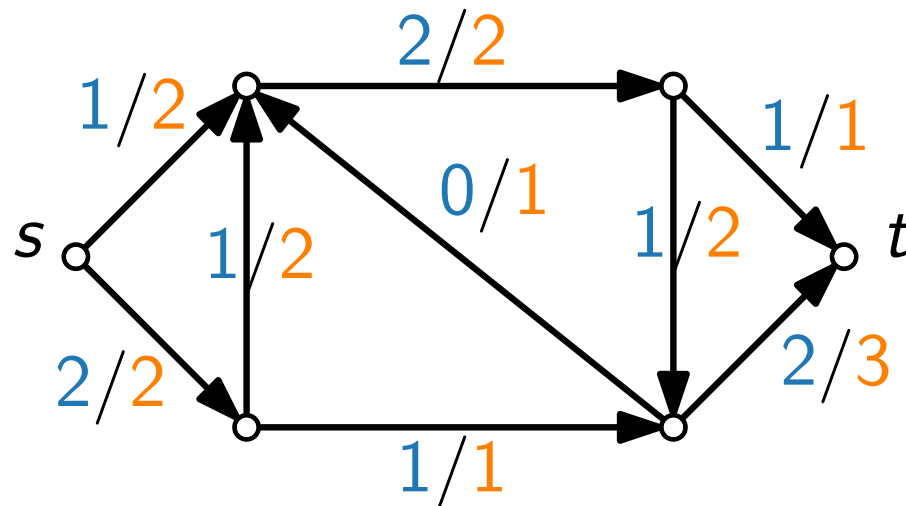
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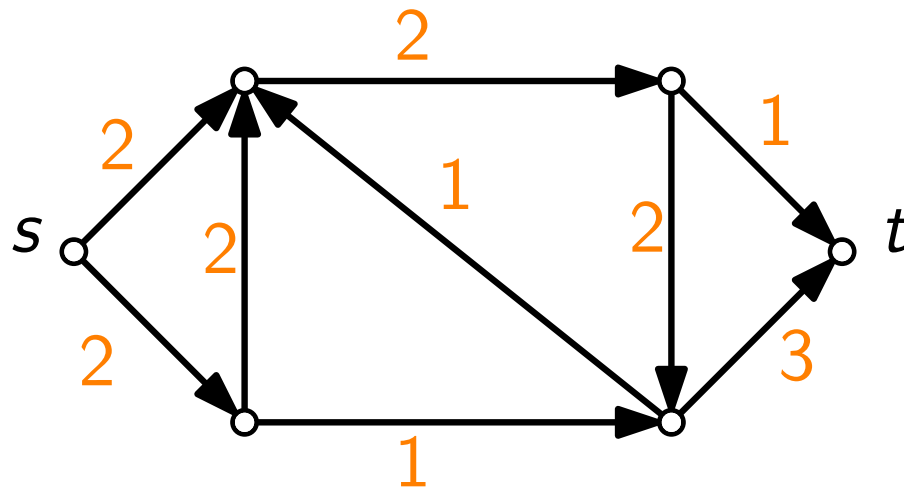
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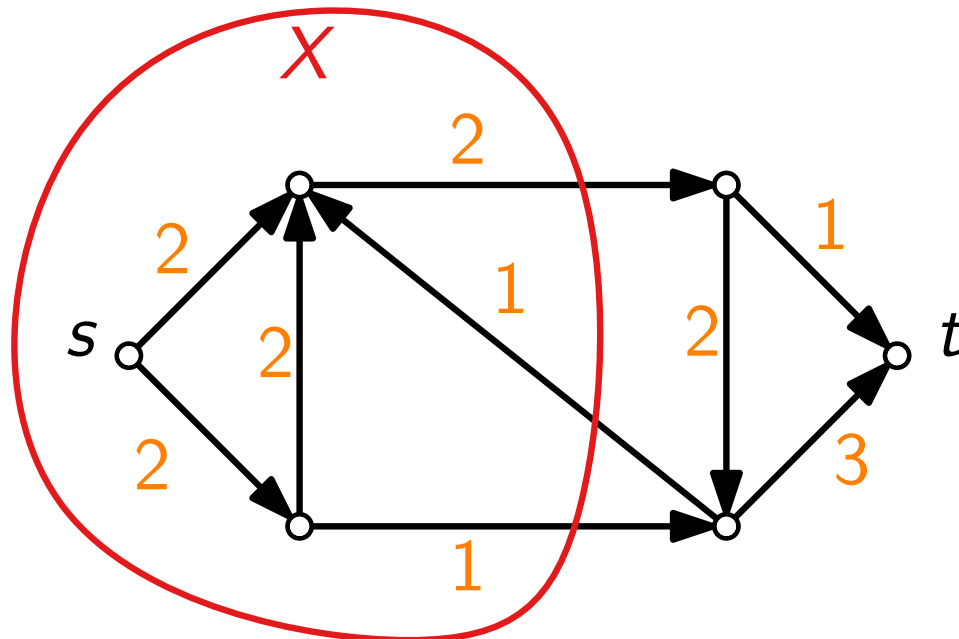
Find: An s - t cut, i.e., a vertex set X with $s \in X$ and $t \in \bar{X}$, such that the total weight $c(X, \bar{X})$ of the edges from X to \bar{X} is minimum.



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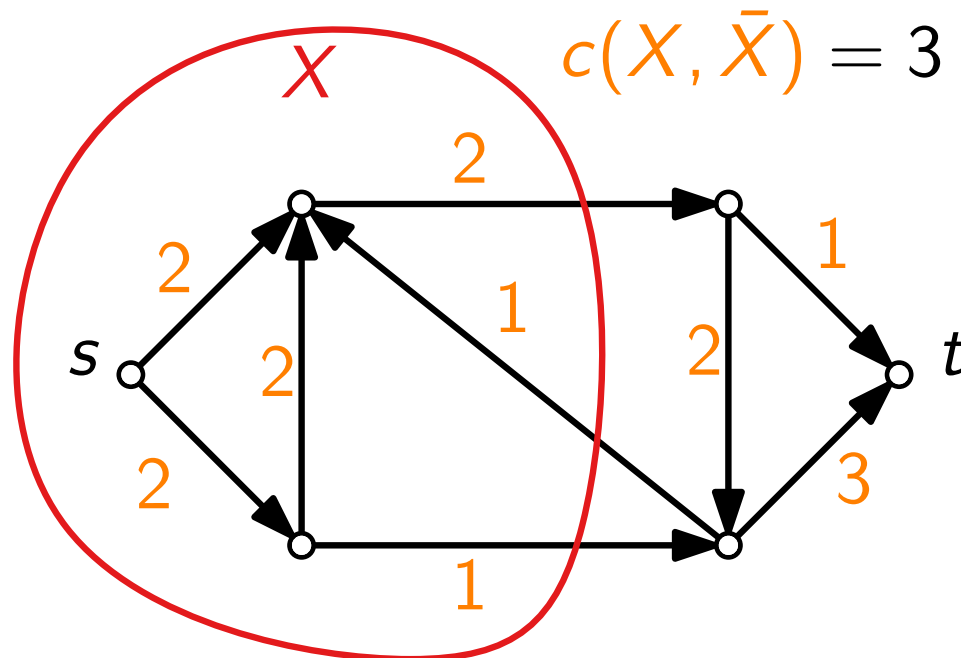
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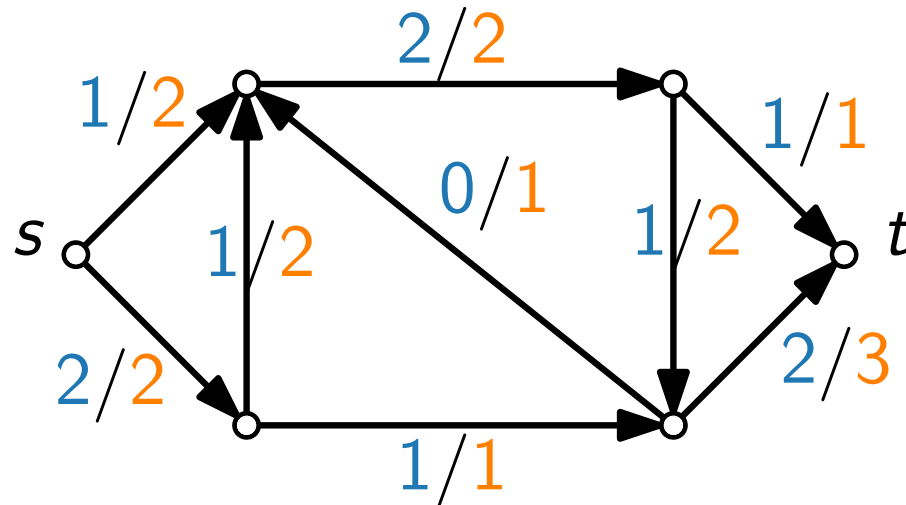
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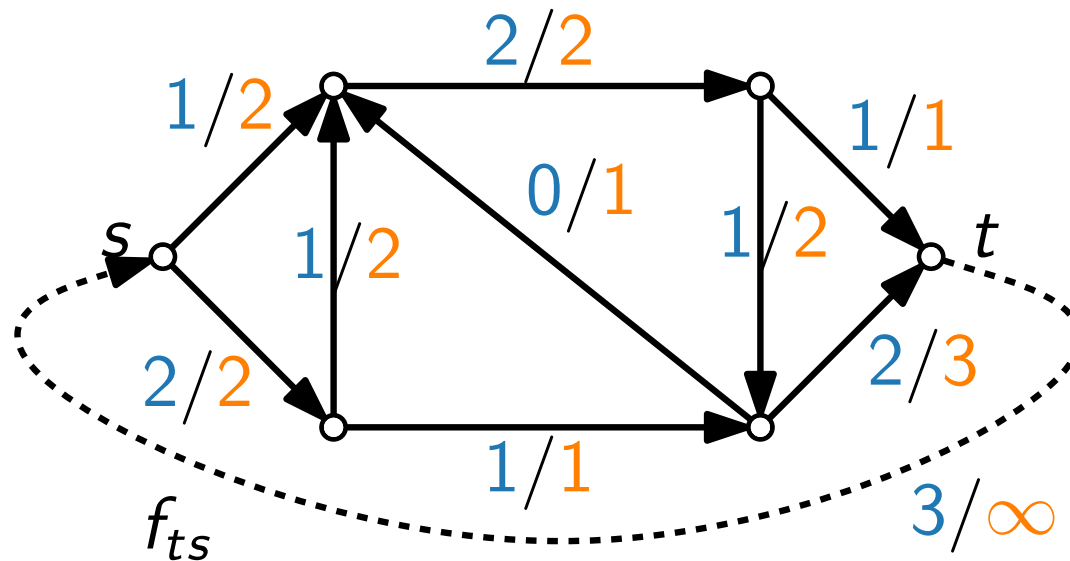


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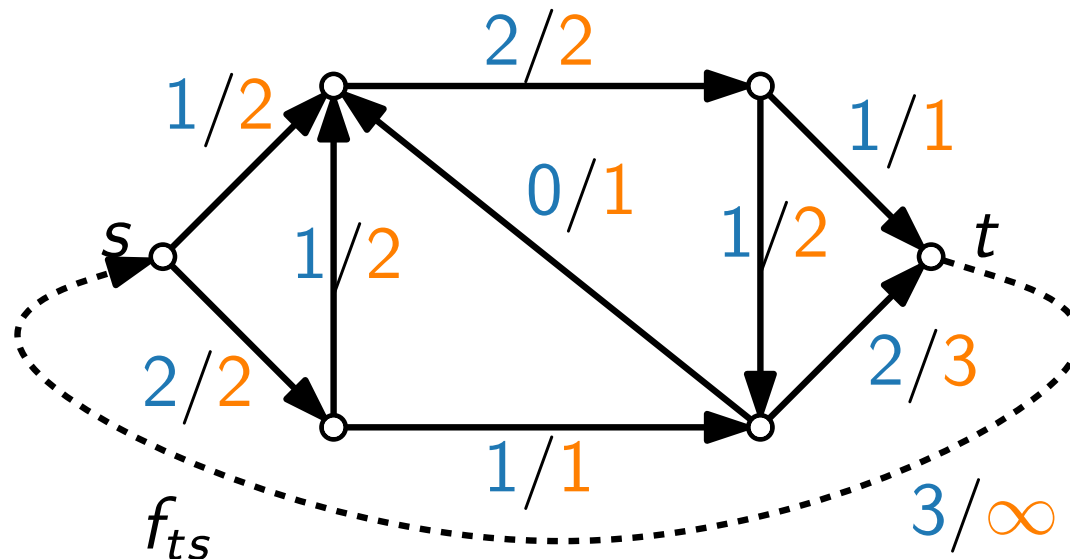


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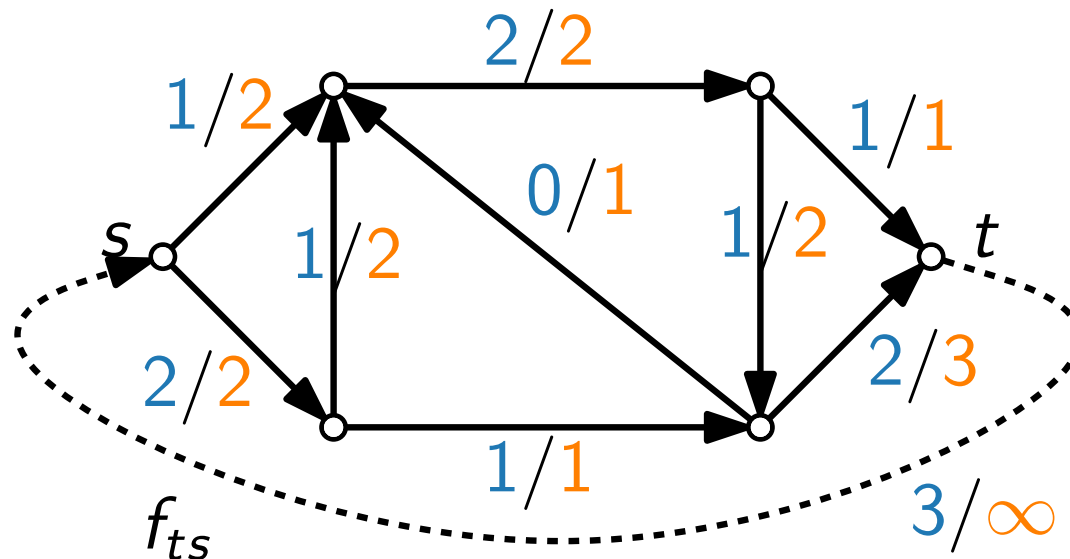


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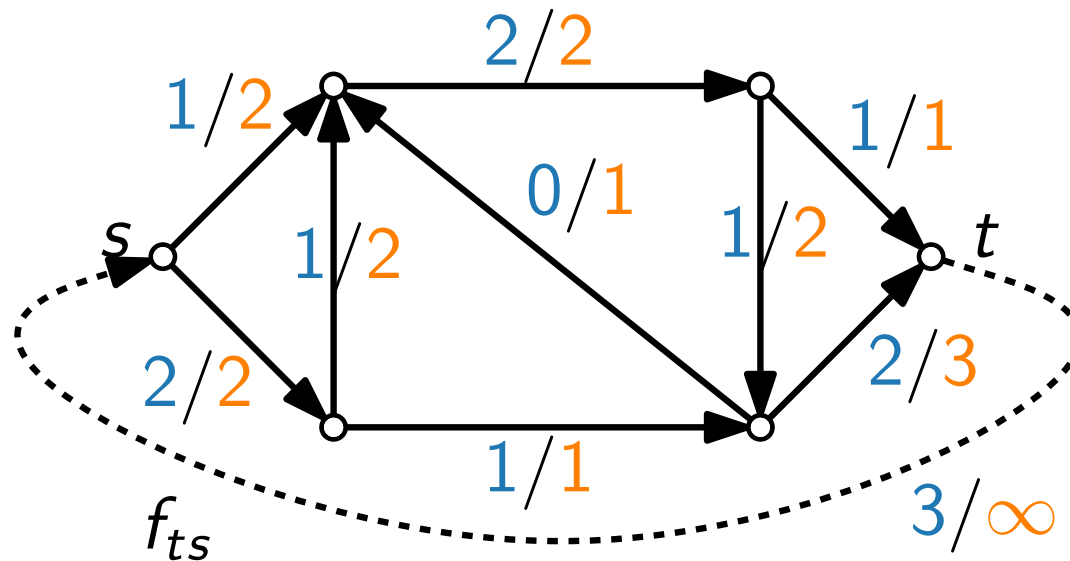


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 \end{array}$$

$$\text{maximize } c^T x = \sum_{(u,v) \in E} (0 \cdot f_{uv}) + 1 \cdot f_{ts} \Rightarrow c = (0, \dots, 0, 1)$$

Which constraints contain $f_{uv} \neq f_{ts}$? d_{uv}, p_u, p_v

Max-Flow-Min-Cut Theorem

Theorem. The value of a **maximum $s-t$ flow** and the weight of a **minimum $s-t$ cut** are the same.

Proof. Special case of LP-Duality ...

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$$\Rightarrow p_s - p_t \geq 1$$

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Approximation Algorithms

Lecture 4: Linear Programming and LP-Duality

Part VI: Dual LP of Max Flow

Dual LP – Interpretation as ILP

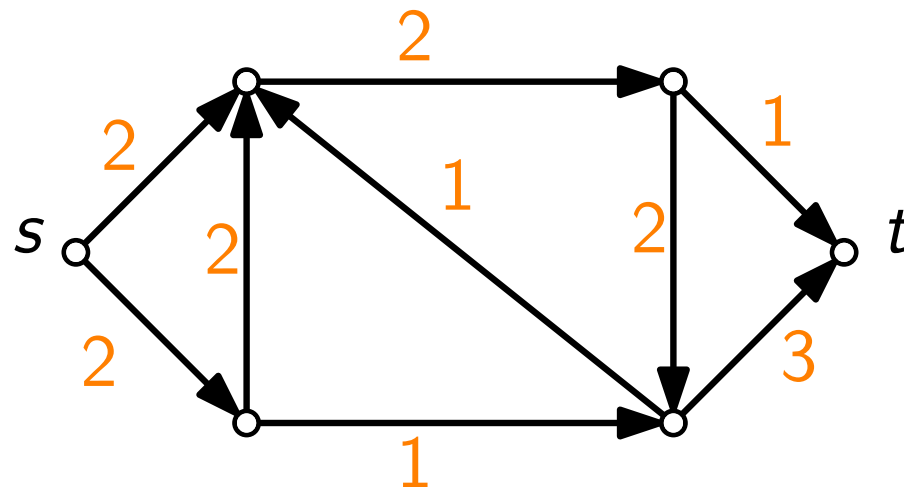
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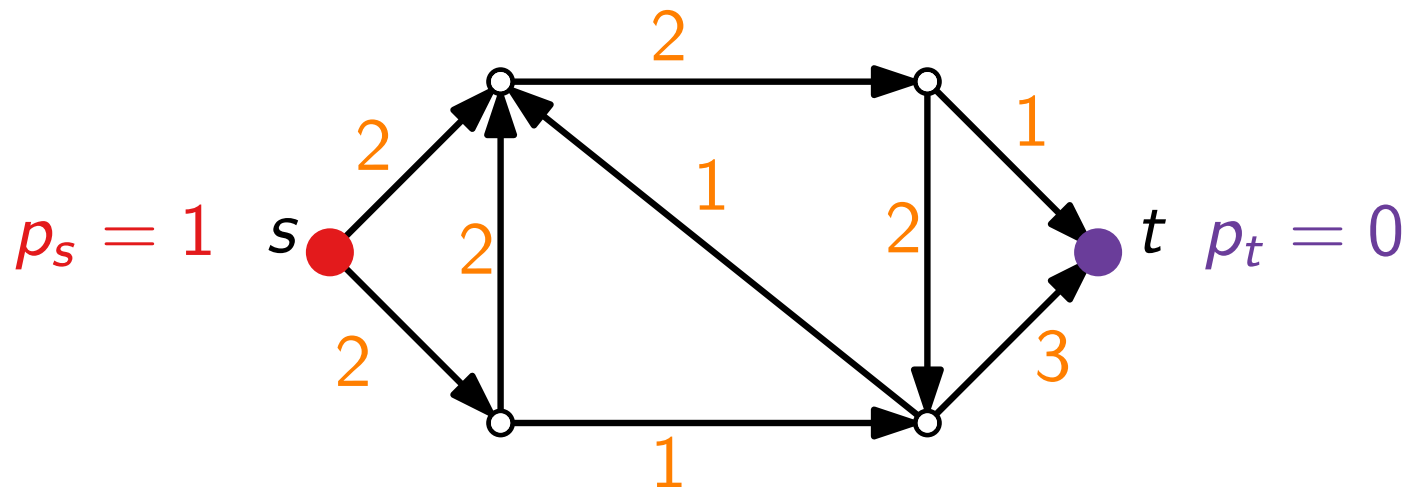
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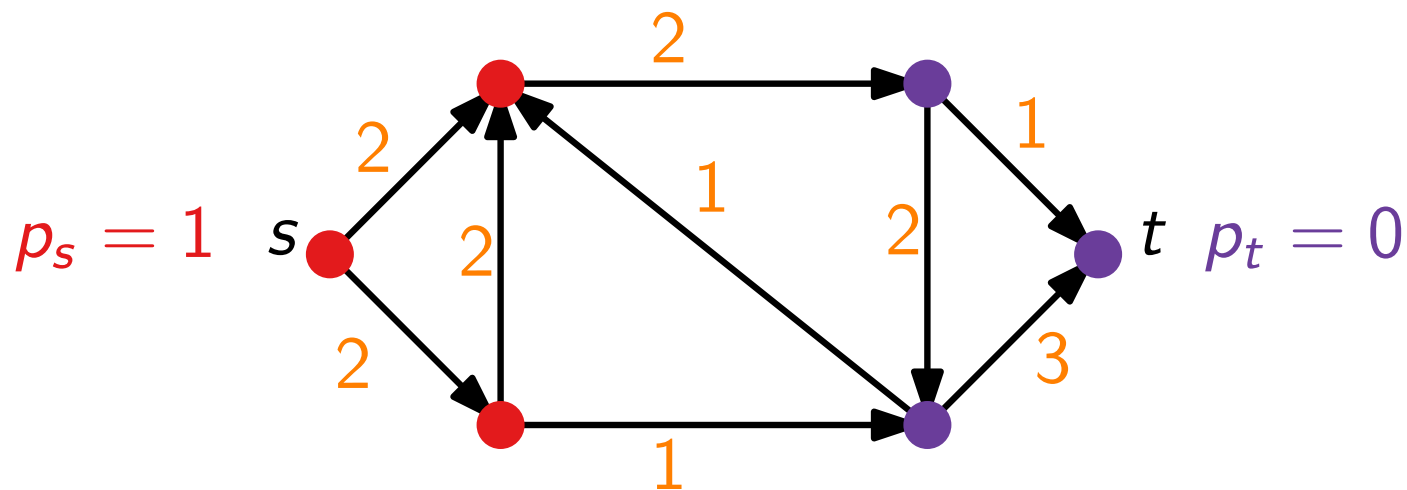
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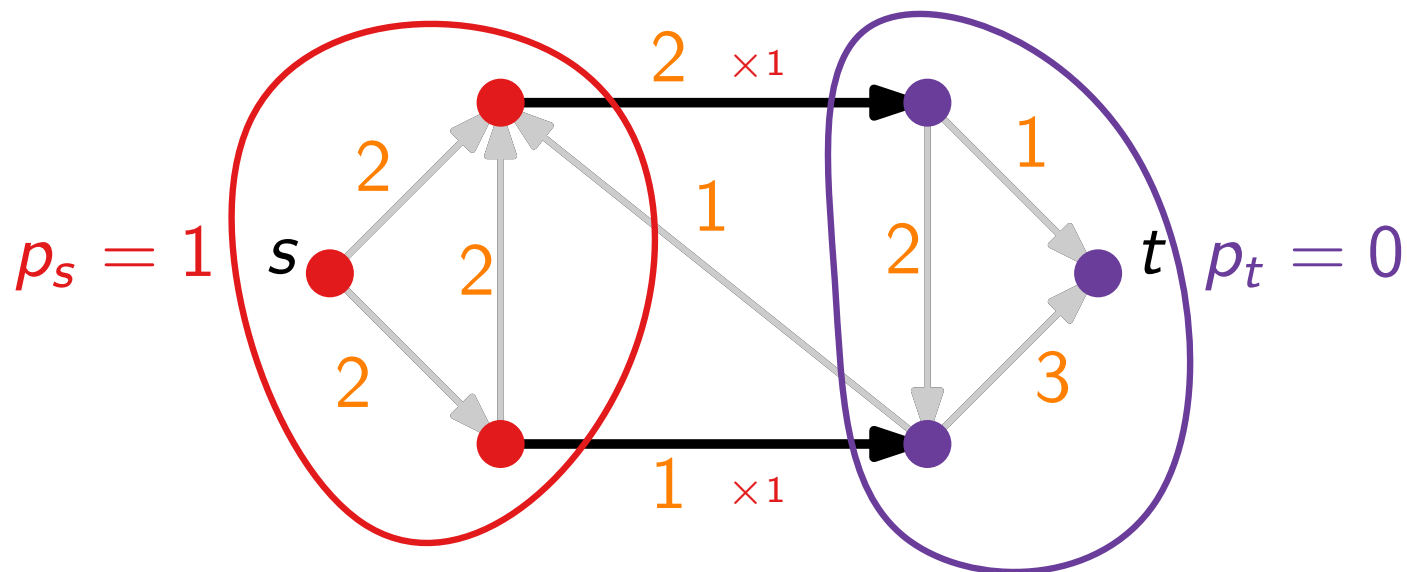
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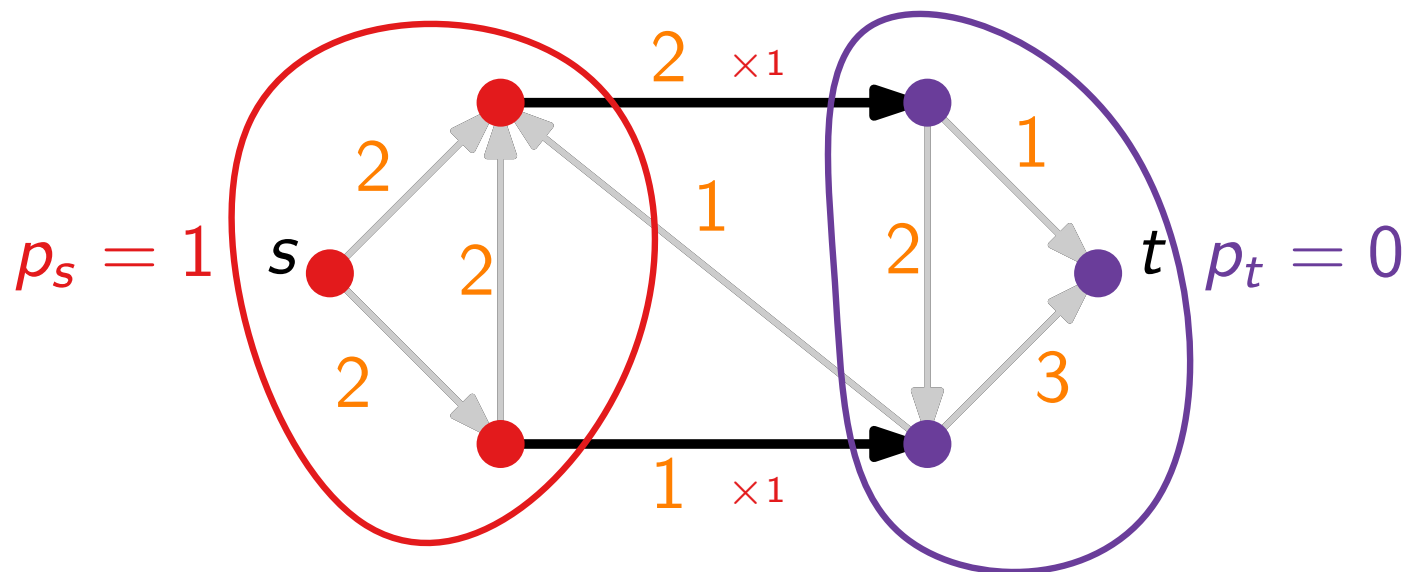
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equivalent to Min-Cut!

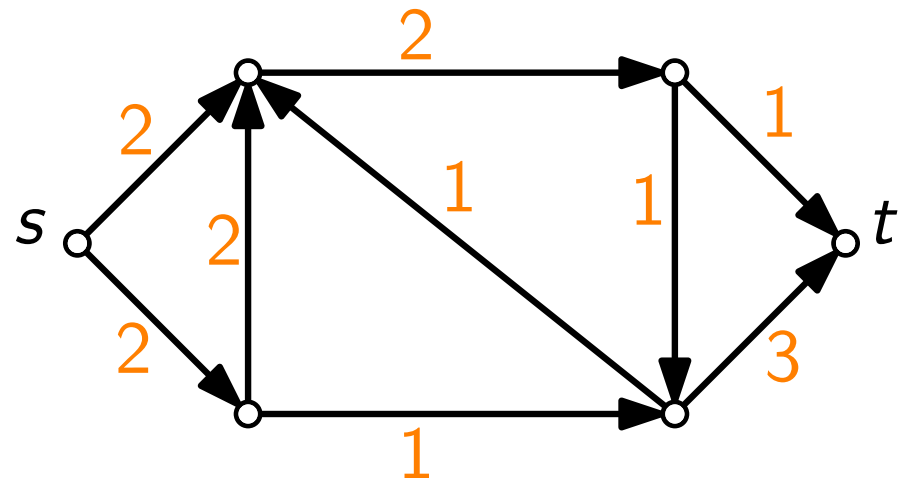


Dual LP – Fractional Cuts

minimize	$\sum_{(u,v) \in E \setminus \{(t,s)\}} c_{uv} \cdot d_{uv}$	\equiv LP-relaxation of the ILP
subject to	$d_{uv} - p_u + p_v \geq 0$	$\forall (u, v) \in E \setminus \{(t, s)\}$
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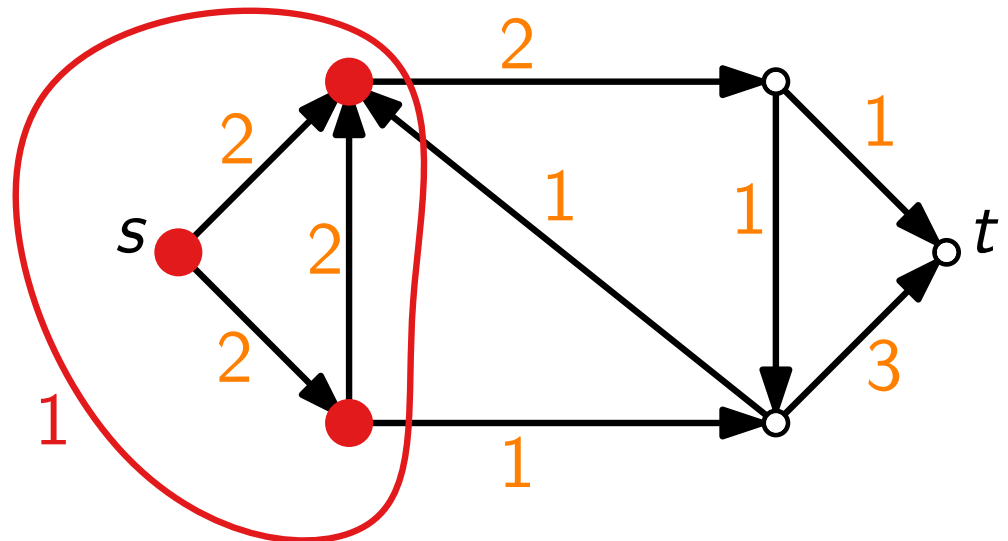
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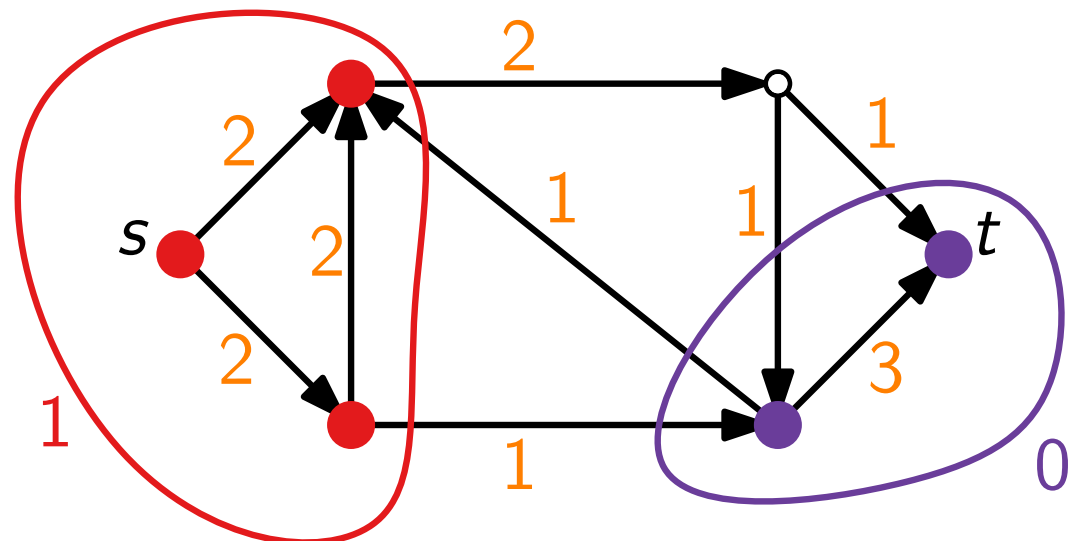
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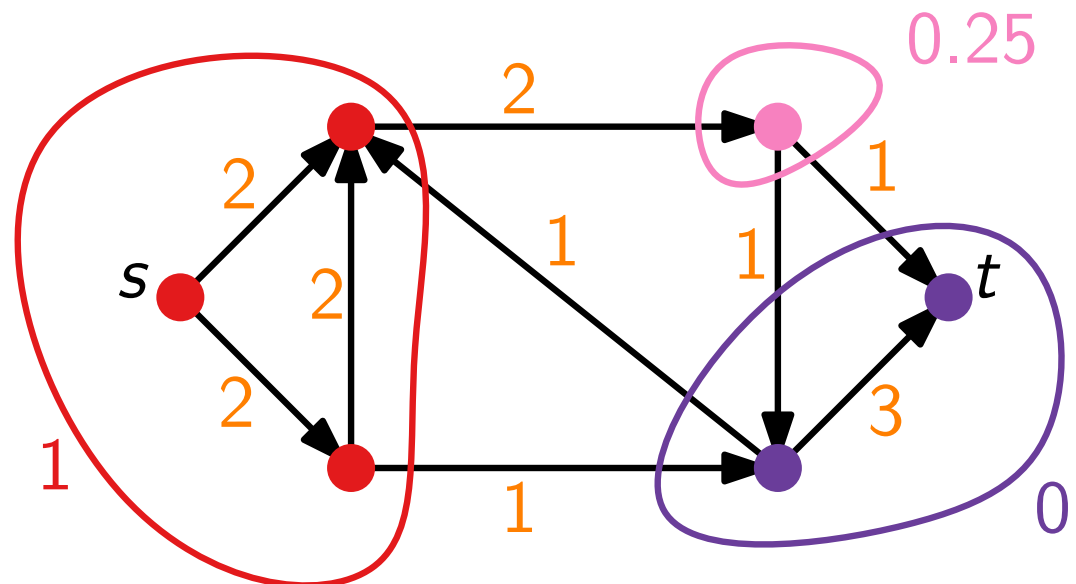
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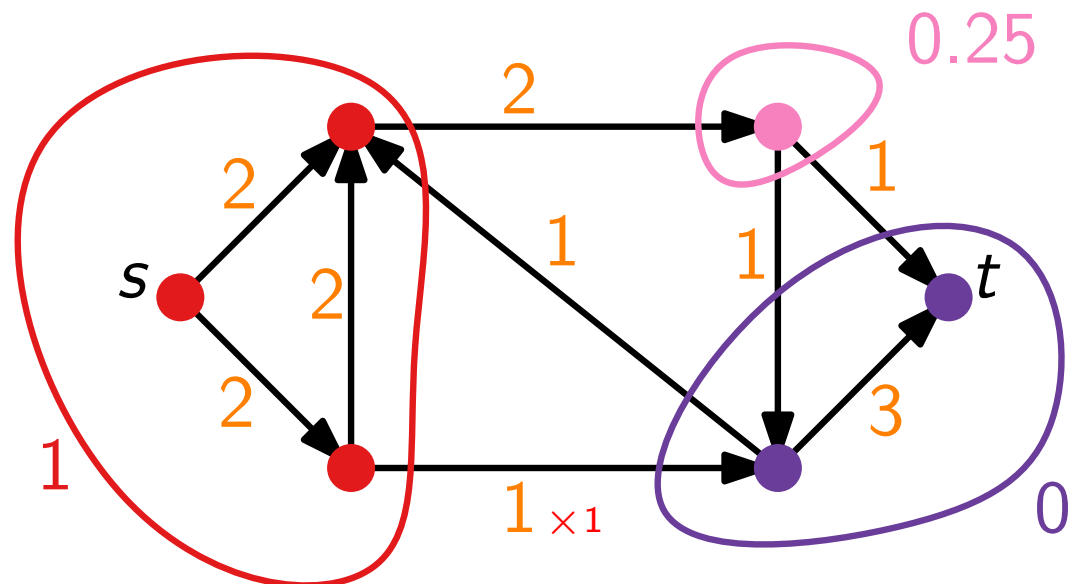
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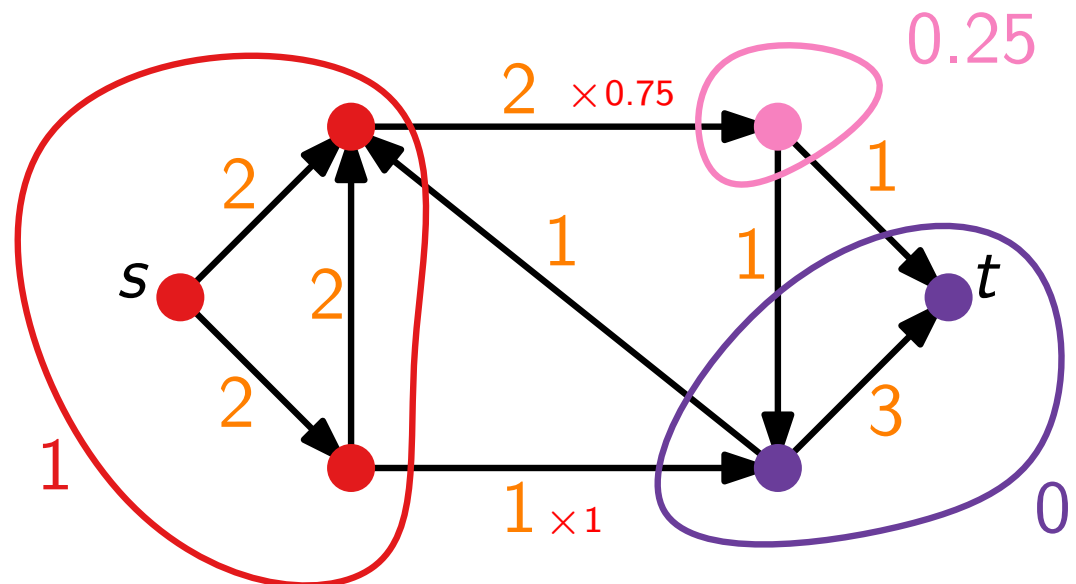
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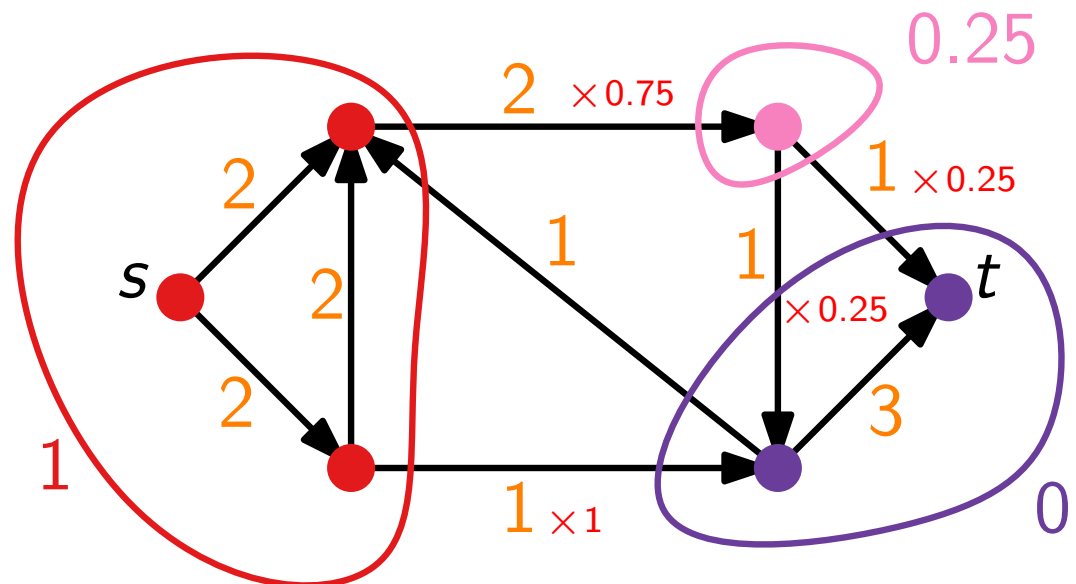
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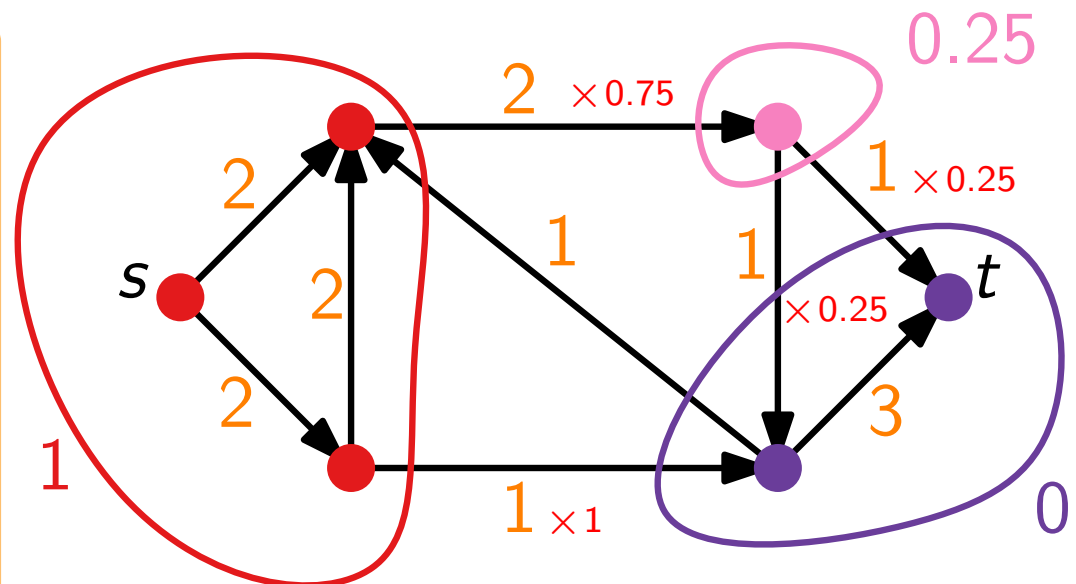


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 \end{array}$$

Each s – t -path

$s = v_0, \dots, v_k = t$ has
length ≥ 1 w.r.t. d :



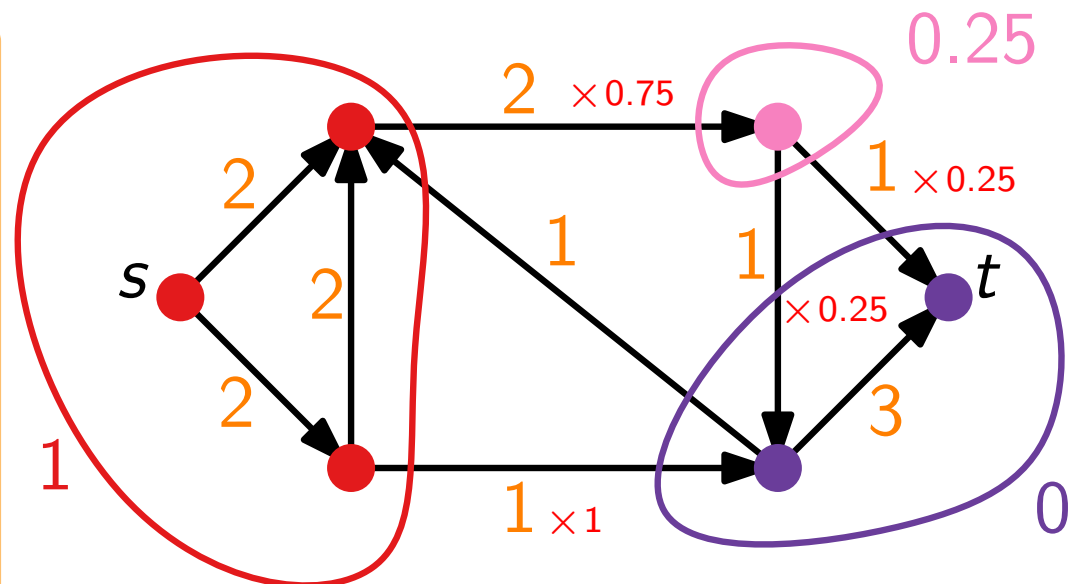
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$s = v_0, \dots, v_k = t$ has length ≥ 1 w.r.t. d :

$$\begin{aligned}
 \sum_{i=0}^{k-1} d_{i,i+1} &\geq \sum_{i=0}^{k-1} (p_i - p_{i+1}) \\
 &= p_s - p_t
 \end{aligned}$$



Dual LP – Fractional Cuts

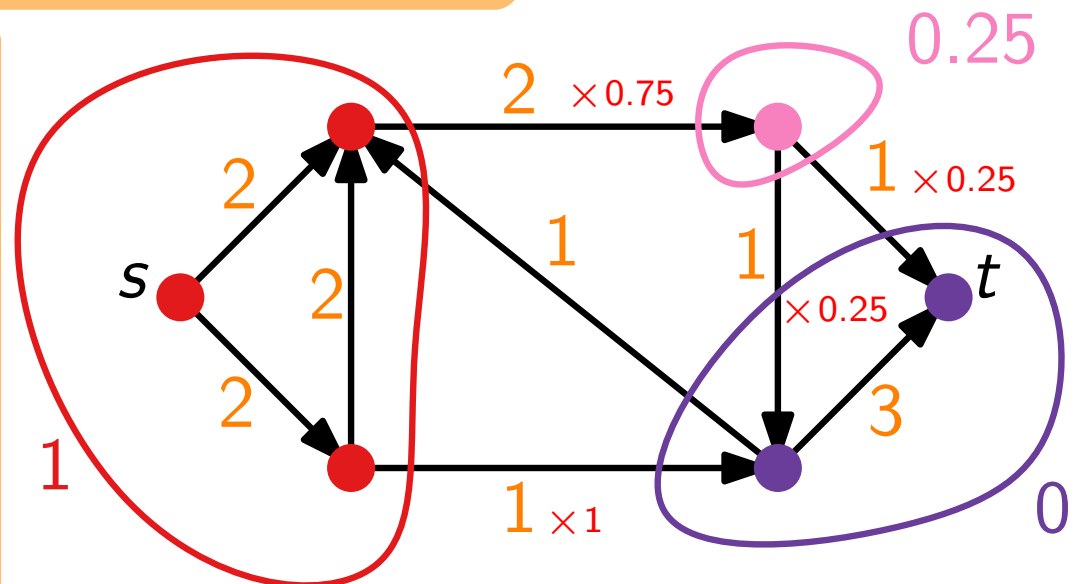
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 \end{array}$$

Each extreme-point solution is **integral!** (HW)

Each s – t -path

$s = v_0, \dots, v_k = t$ has length ≥ 1 w.r.t. d :

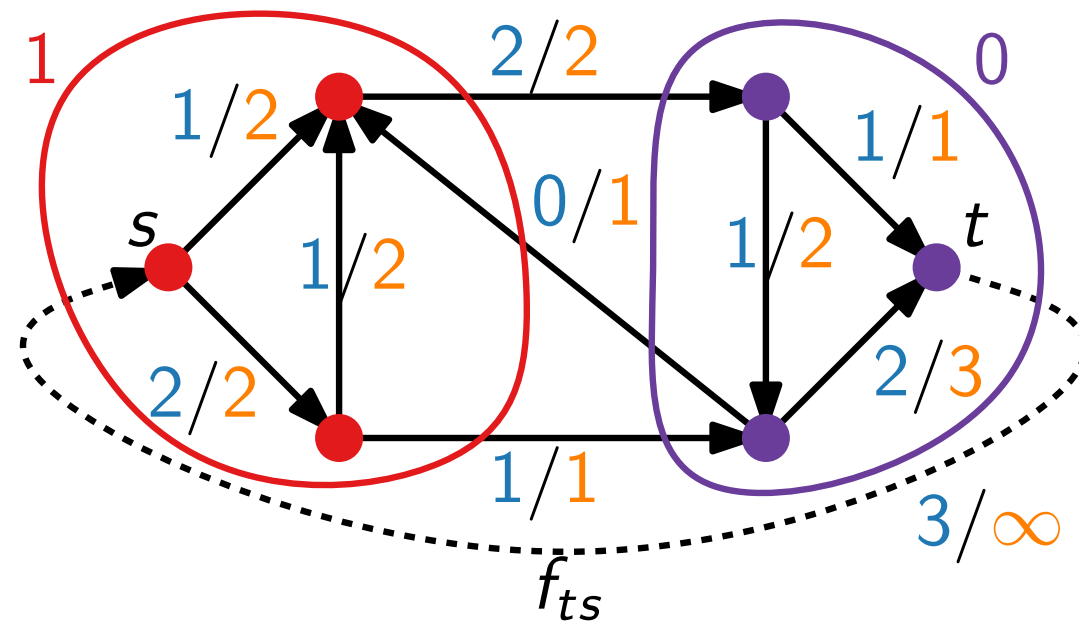
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Dual LP – Complementary Slackness

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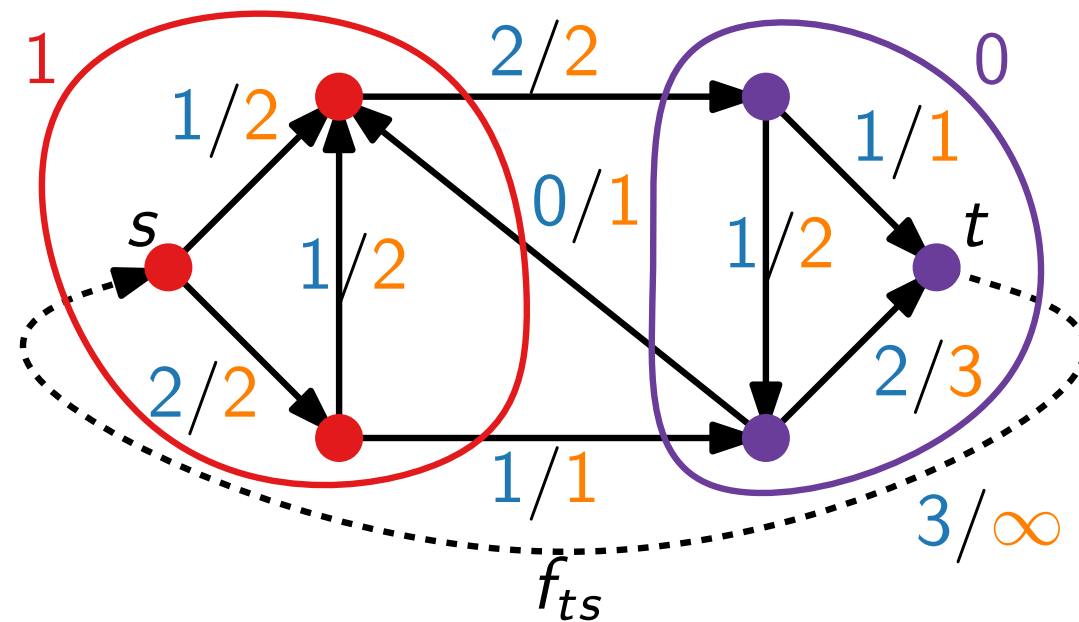


Dual LP – Complementary Slackness

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For a max flow and min cut:



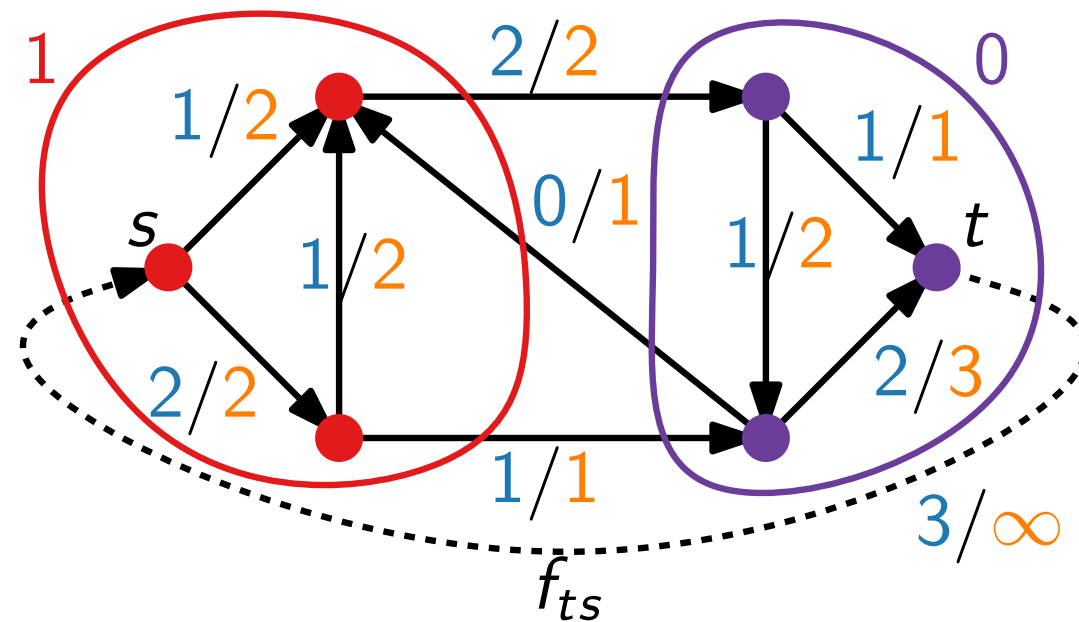
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For a max flow and min cut:

- For each forward edge (u, v) of the cut: $f_{uv} = c_{uv}$.



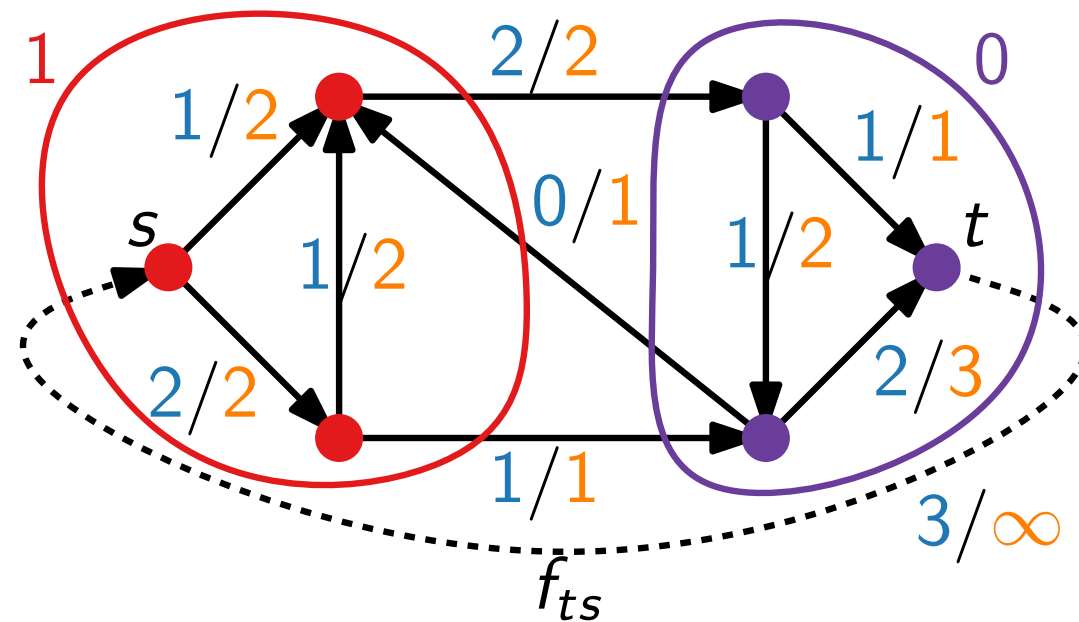
Dual LP – Complementary Slackness

$$\begin{array}{ll}
 \text{maximize} & f_{ts} \\
 \text{subject to} & f_{uv} \leq c_{uv} \quad \forall (u, v) \in E \setminus \{(t, s)\} \\
 & \sum_{u: (u,v) \in E} f_{uv} - \sum_{z: (v,z) \in E} f_{vz} \leq 0 \quad \forall v \in V \\
 & f_{uv} \geq 0 \quad \forall (u, v) \in E
 \end{array}$$

$$\begin{array}{ll}
 \text{minimize} & \sum_{(u,v) \in E \setminus \{(t,s)\}} c_{uv} \cdot d_{uv} \\
 \text{subject to} & d_{uv} - p_u + p_v \geq 0 \quad \forall (u, v) \in E \setminus \{(t, s)\} \\
 & p_s - p_t \geq 1 \\
 & d_{uv} \geq 0 \quad \forall (u, v) \in E \\
 & p_u \geq 0 \quad \forall u \in V
 \end{array}$$

For a max flow and min cut:

- For each forward edge (u, v) of the cut: $f_{uv} = c_{uv}$.
($d_{uv} = 1$, so by dual CS: $f_{uv} = c_{uv}$.)



Dual LP – Complementary Slackness

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 & d_{uv} \geq 0 \\
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 \end{array}$$

Primal CS:

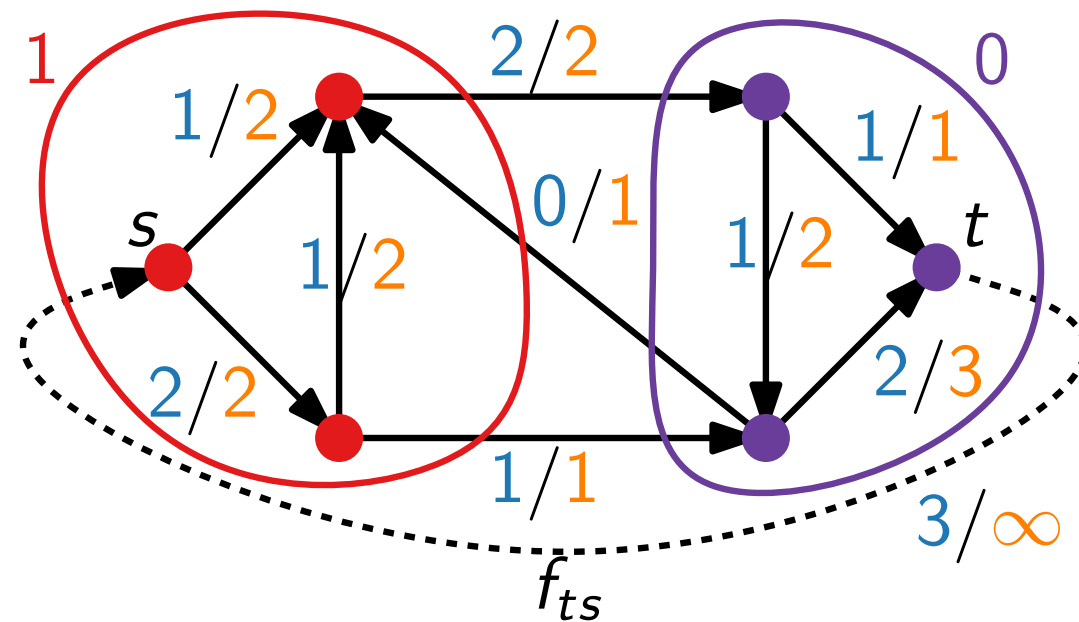
$$\forall j: \quad x_j = 0 \quad \text{or} \quad \sum_{i=1}^m a_{ij} y_i = c_j$$

Dual CS:

$$\forall i: \quad y_i = 0 \quad \text{or} \quad \sum_{j=1}^n a_{ij} x_j = b_i$$

For a max flow and min cut:

- For each forward edge (u, v) of the cut: $f_{uv} = c_{uv}$.
($d_{uv} = 1$, so by dual CS: $f_{uv} = c_{uv}$.)



Dual LP – Complementary Slackness

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Primal CS:

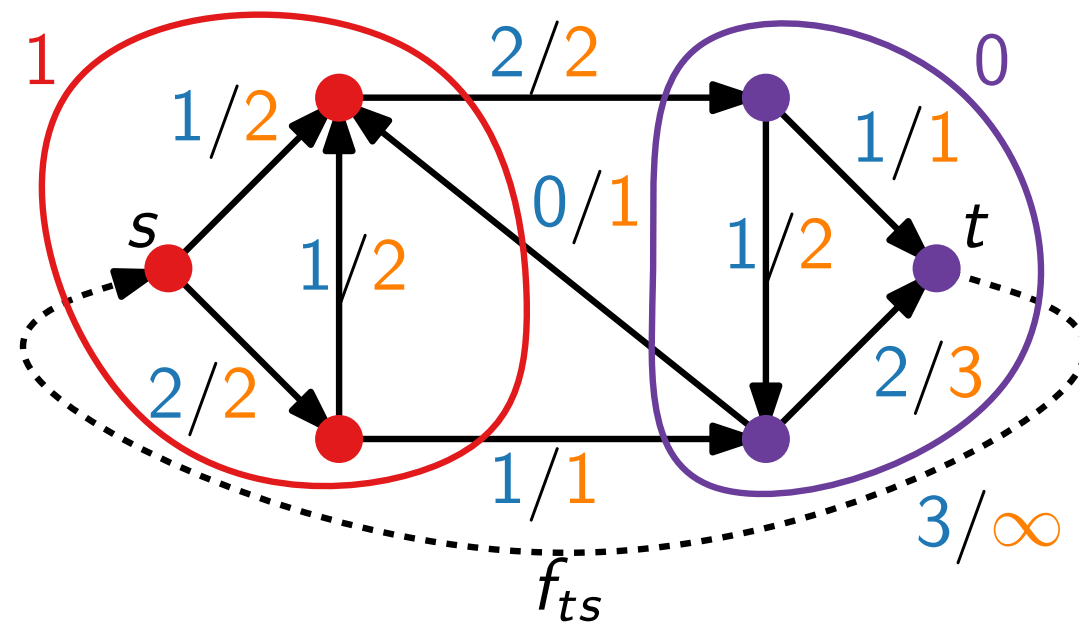
$$\forall j: \quad x_j = 0 \quad \text{or} \quad \sum_{i=1}^m a_{ij} y_i = c_j$$

Dual CS:

$$\forall i: \quad y_i = 0 \quad \text{or} \quad \sum_{j=1}^n a_{ij} x_j = b_i$$

For a max flow and min cut:

- For each forward edge (u, v) of the cut: $f_{uv} = c_{uv}$.
($d_{uv} = 1$, so by dual CS: $f_{uv} = c_{uv}$.)
- For each backward edge (u, v) of the cut: $f_{uv} = 0$.



Dual LP – Complementary Slackness

$$\begin{array}{ll}
 \text{maximize} & f_{ts} \\
 \text{subject to} & f_{uv} \leq c_{uv} \quad \forall (u, v) \in E \setminus \{(t, s)\} \\
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Primal CS:

$$\forall j: \quad x_j = 0 \quad \text{or} \quad \sum_{i=1}^m a_{ij} y_i = c_j$$

Dual CS:

$$\forall i: \quad y_i = 0 \quad \text{or} \quad \sum_{j=1}^n a_{ij} x_j = b_i$$

For a max flow and min cut:

- For each forward edge (u, v) of the cut: $f_{uv} = c_{uv}$.
($d_{uv} = 1$, so by dual CS: $f_{uv} = c_{uv}$.)
- For each backward edge (u, v) of the cut: $f_{uv} = 0$.
(Otherwise, by primal CS: $d_{uv} - 0 + 1 = 0$.)

