Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

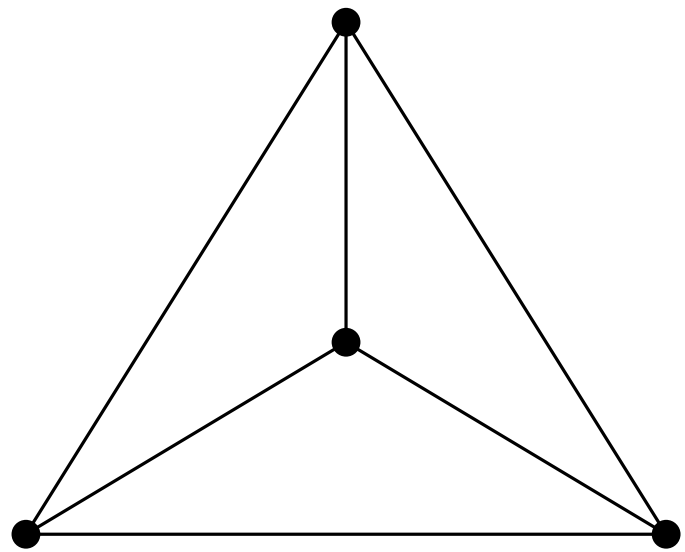
Part I:
SteinerTree

Alexander Wolff

Winter 2022/23

SteinerTree

Given: A graph G



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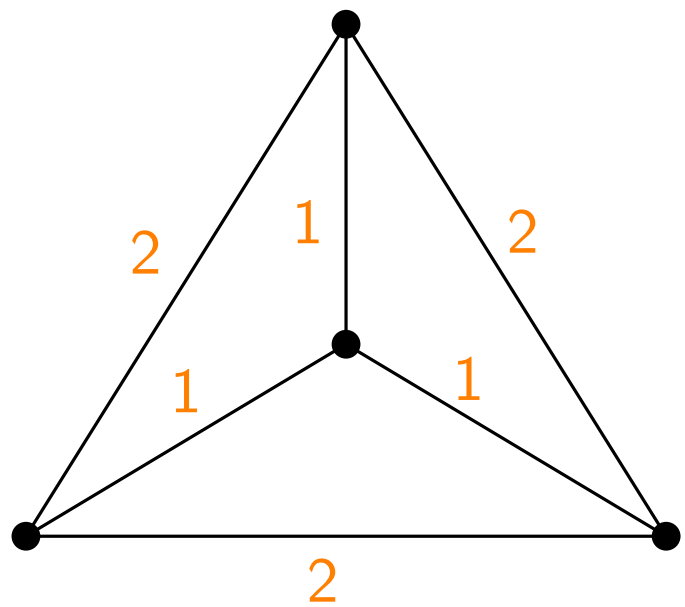
Given: A graph G with edge weights $c\colon E(G) \to \mathbb{Q}^+$



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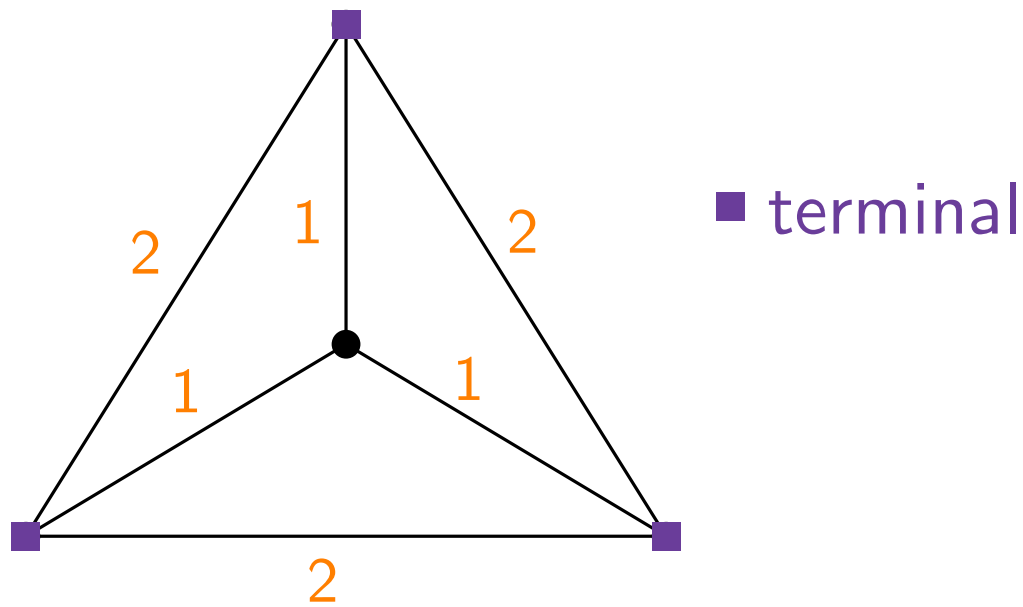
and a partition of $V(G)$ into a set T of **terminals**



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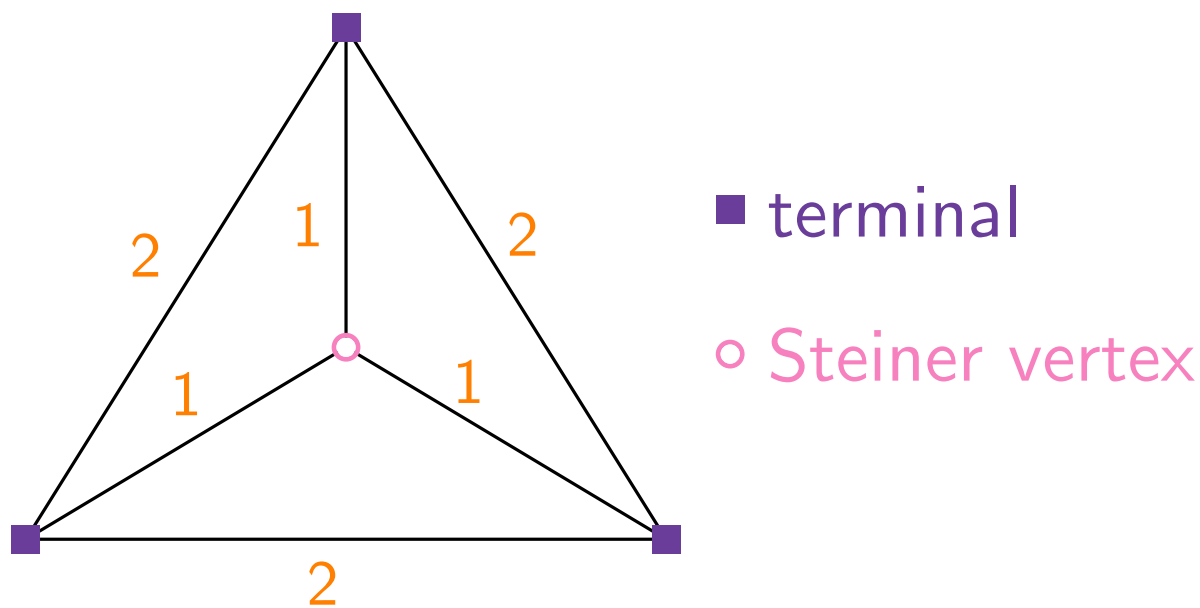

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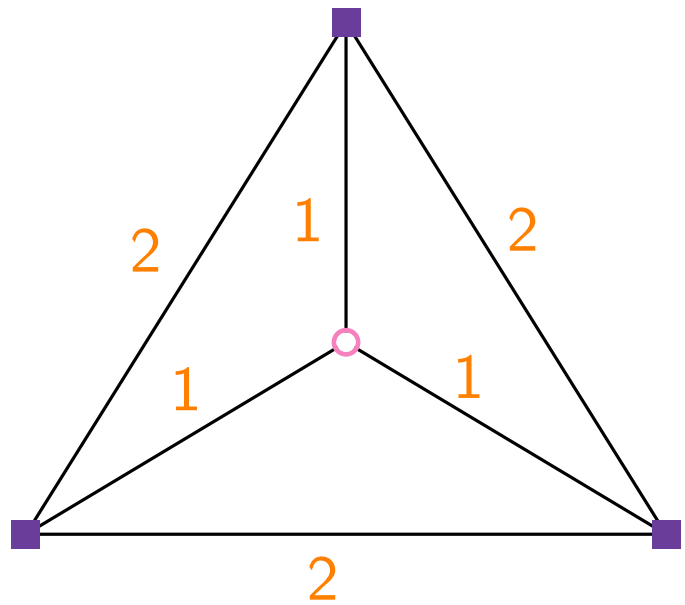

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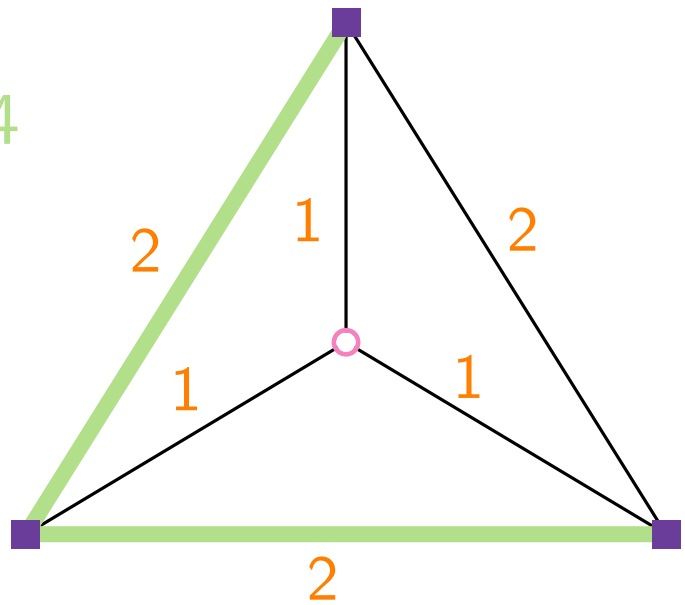

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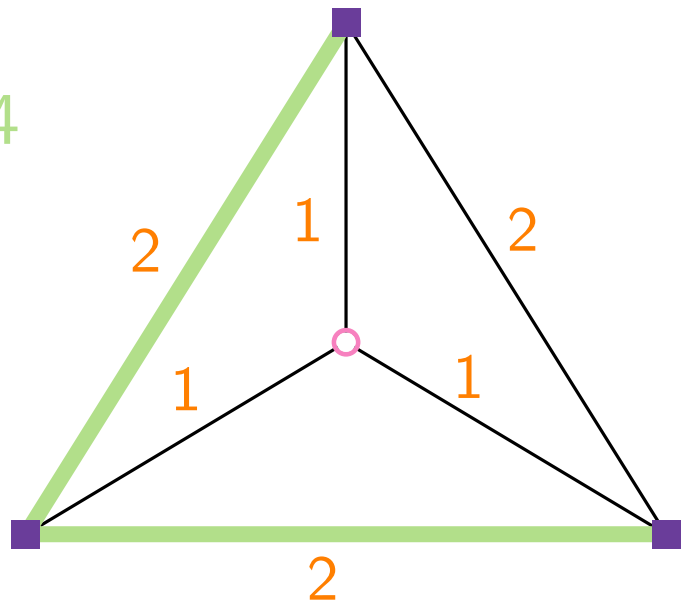

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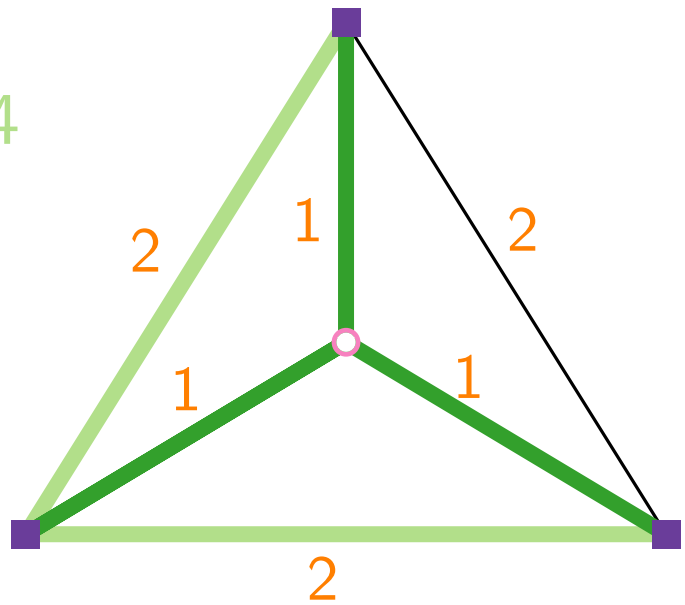

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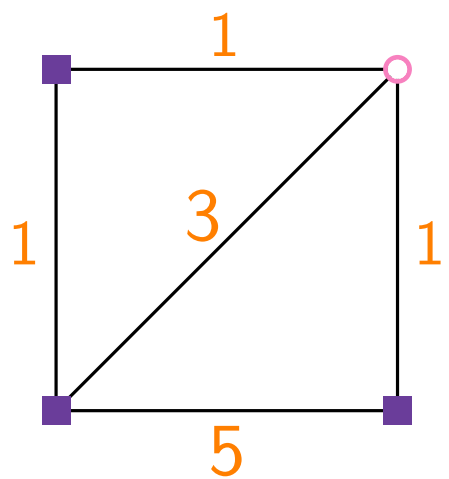

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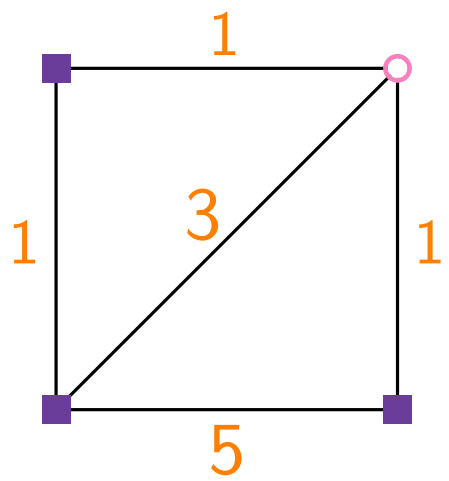

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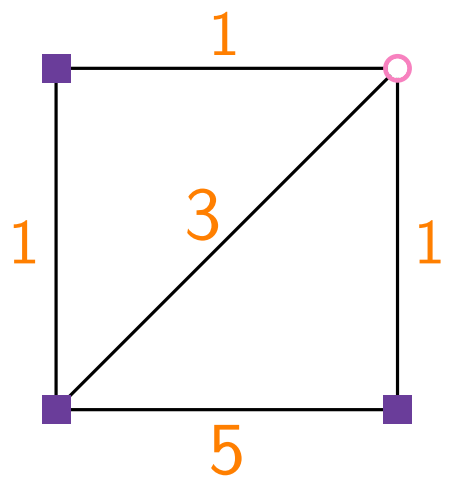

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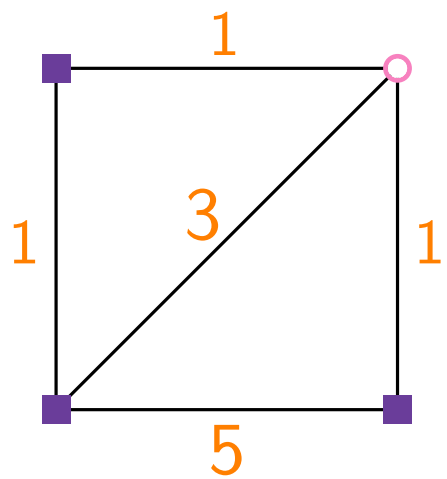

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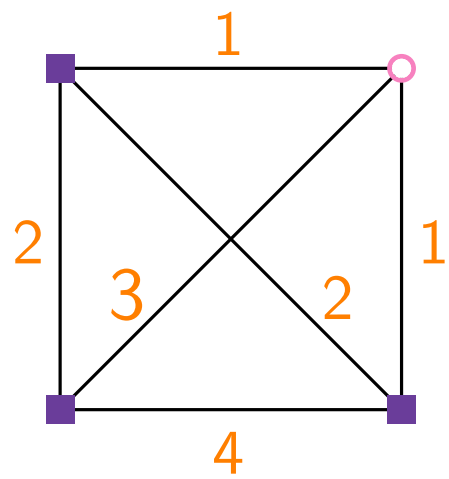

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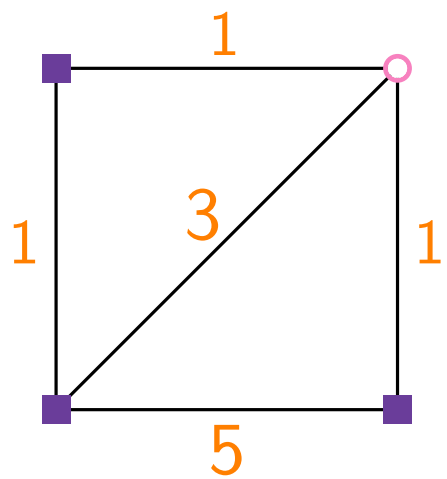

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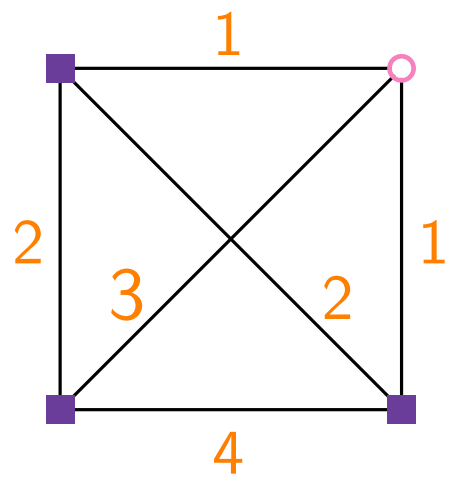

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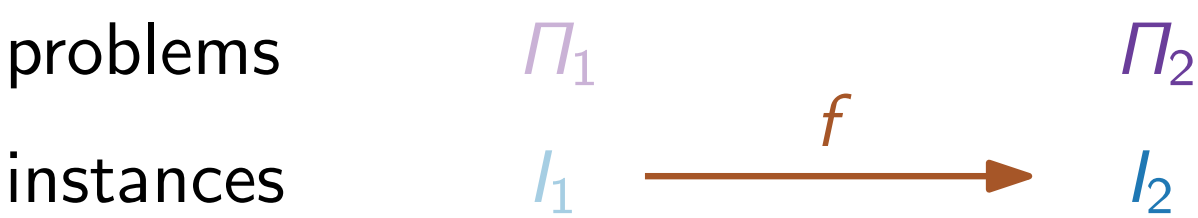

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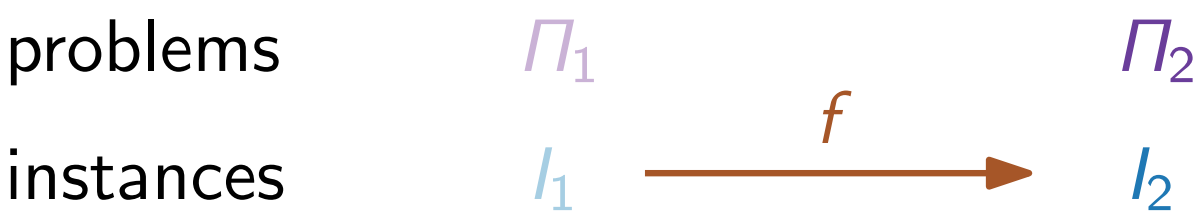

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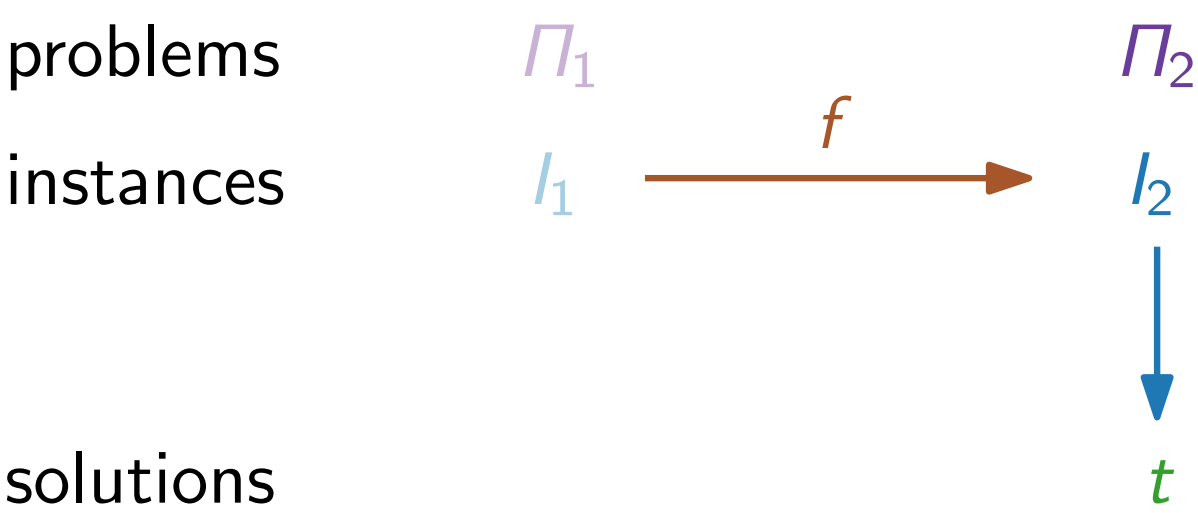

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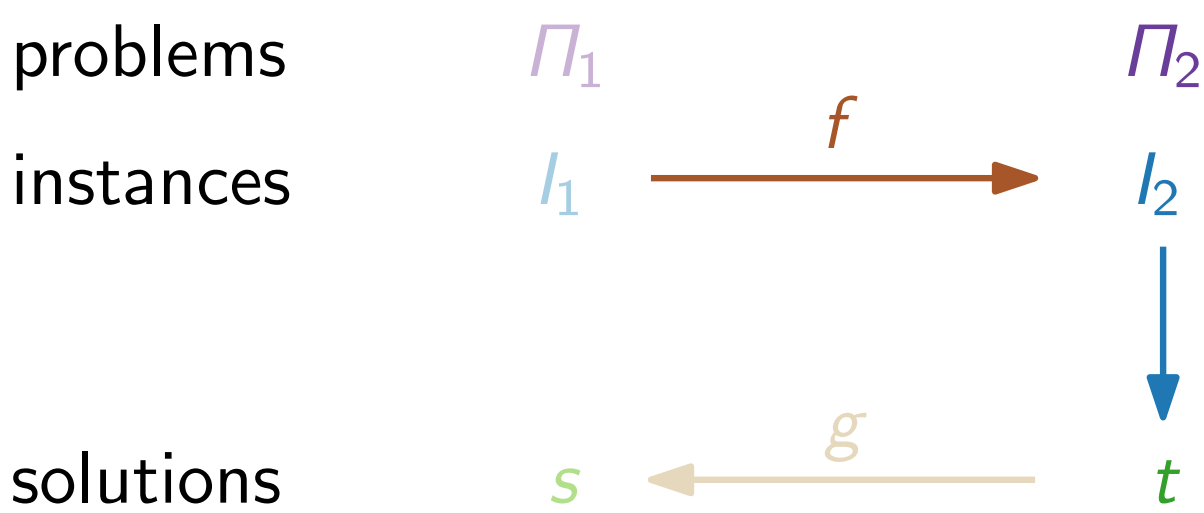

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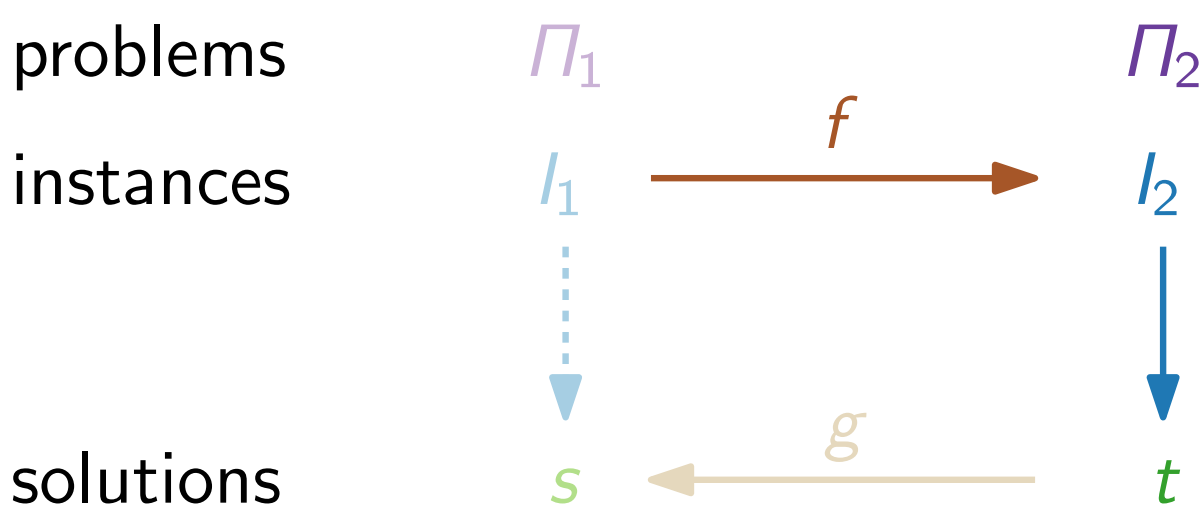

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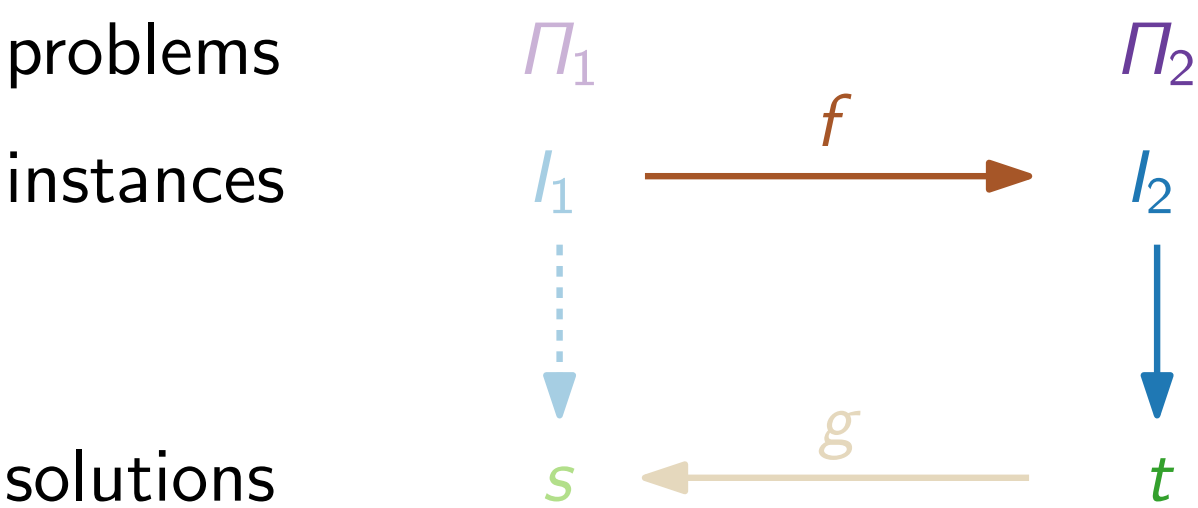

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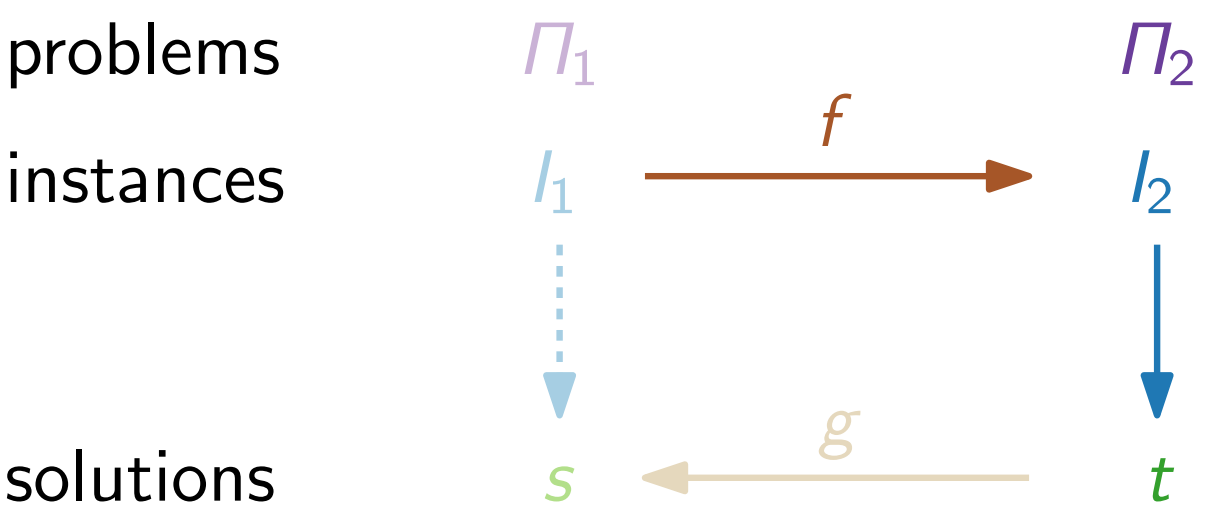

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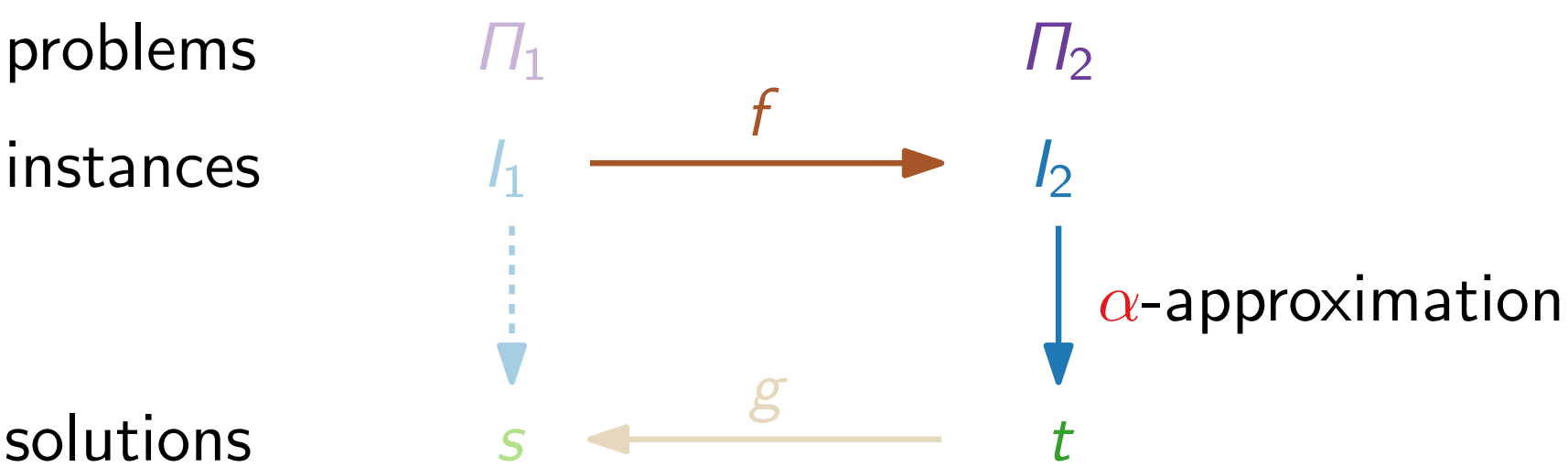

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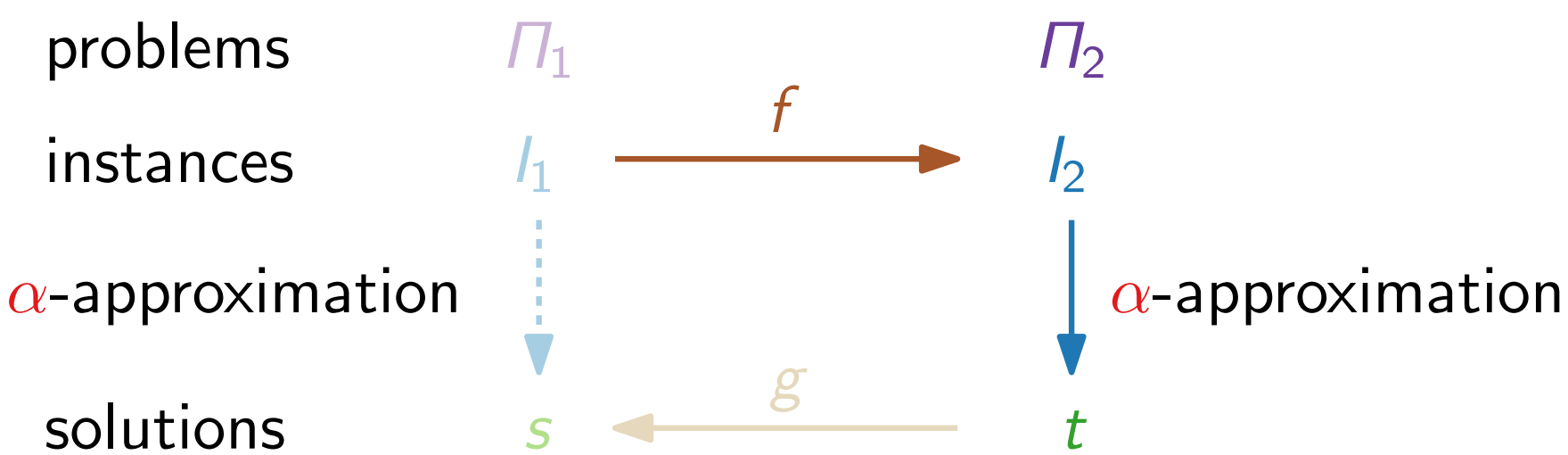

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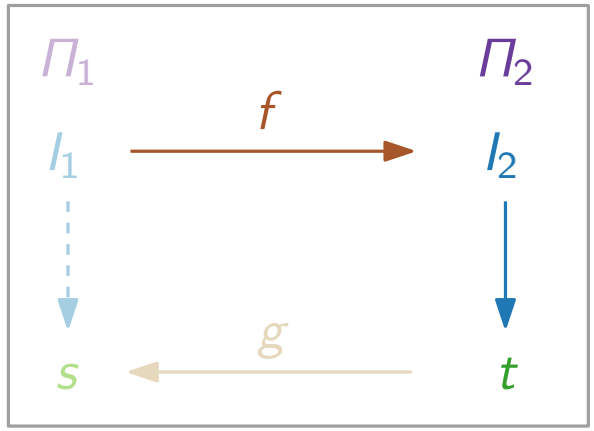

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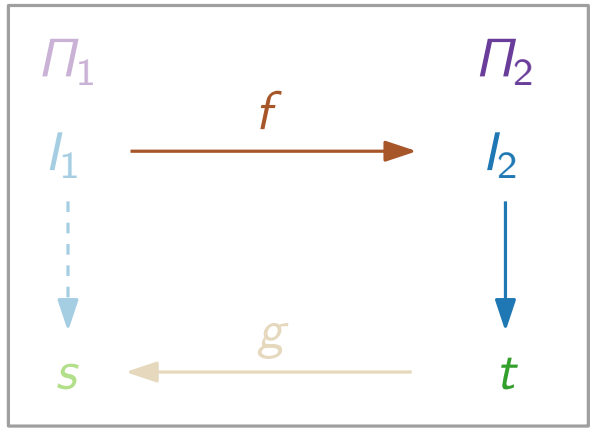

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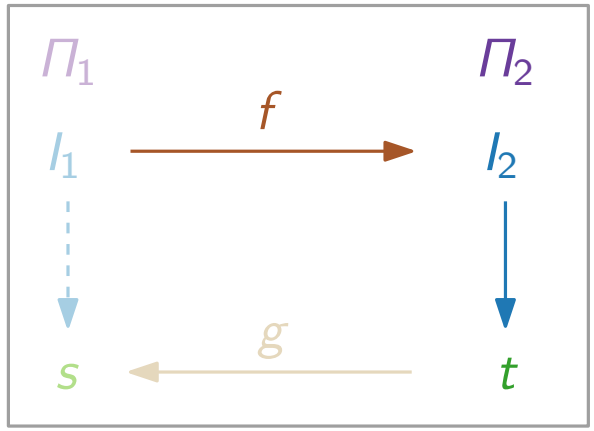

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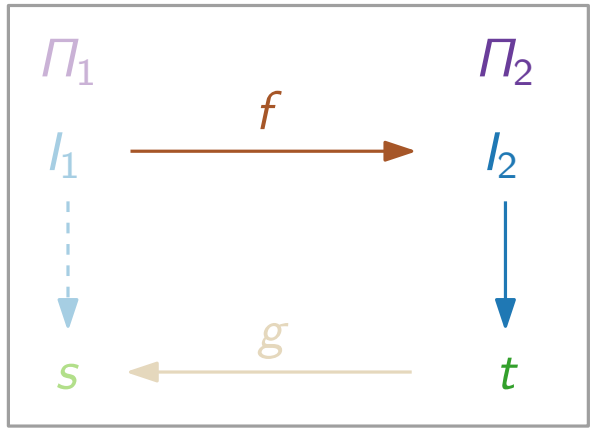

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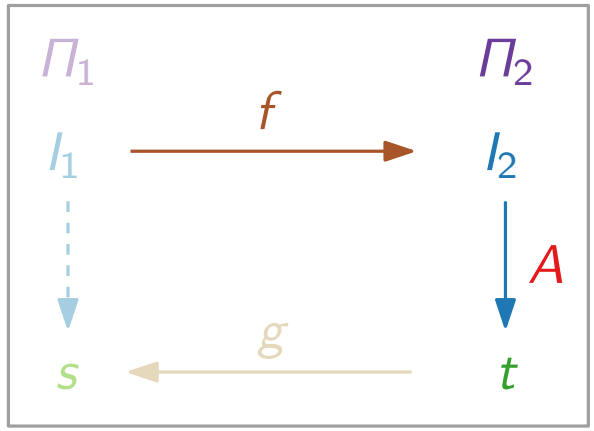

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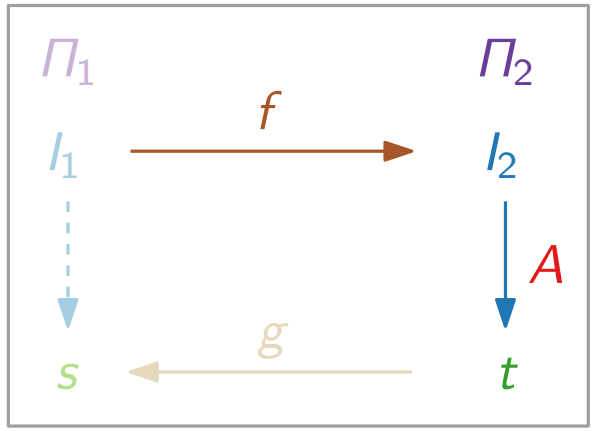

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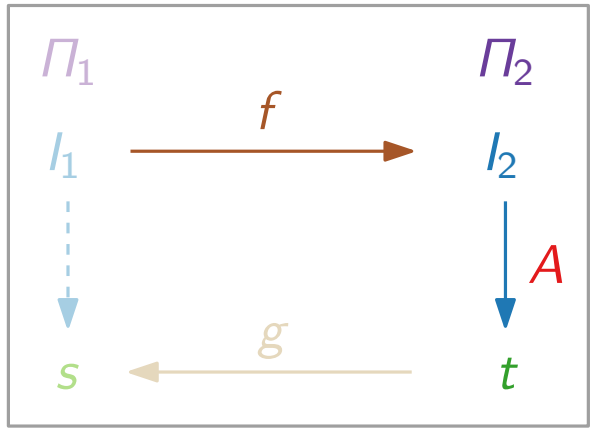

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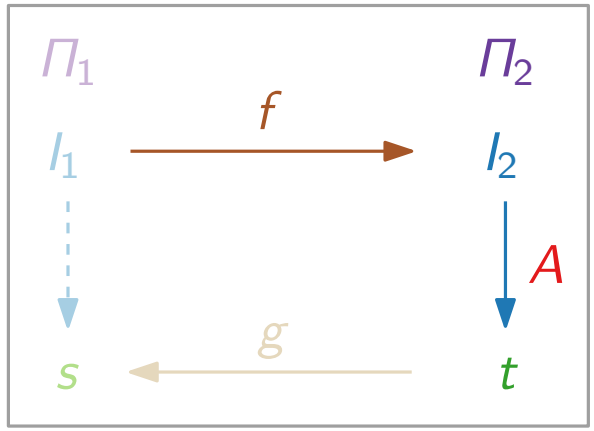

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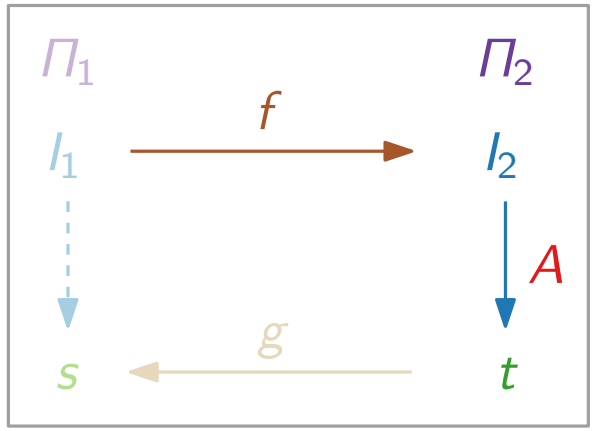

Proof.

Let A be a factor-α approx. alg. for Π_2.

Let I_1 be an instance of Π_1.

Set $I_2 := f(I_1)$, $t := A(I_2)$ and $s := g(I_1, t)$.

Then:

$\text{obj}_{\Pi_1}(I_1, s) \leq \text{obj}_{\Pi_2}(I_2, t) \leq \alpha \cdot \text{OPT}_{\Pi_2}(I_2) \leq$

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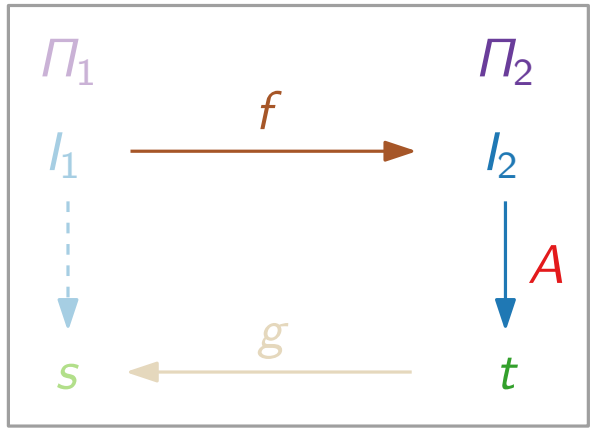

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Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part III:
Reduction to MetricSteinerTree

Alexander Wolff

Winter 2022/23

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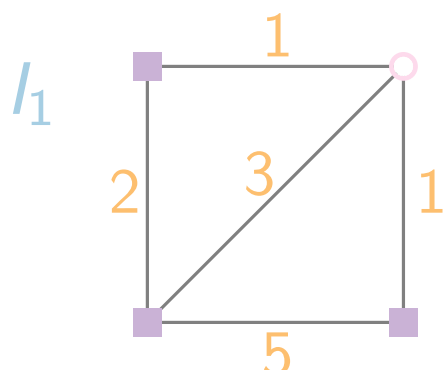

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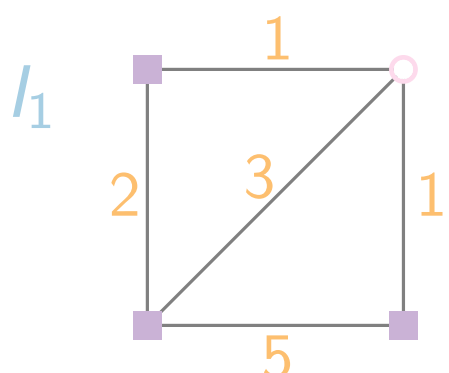

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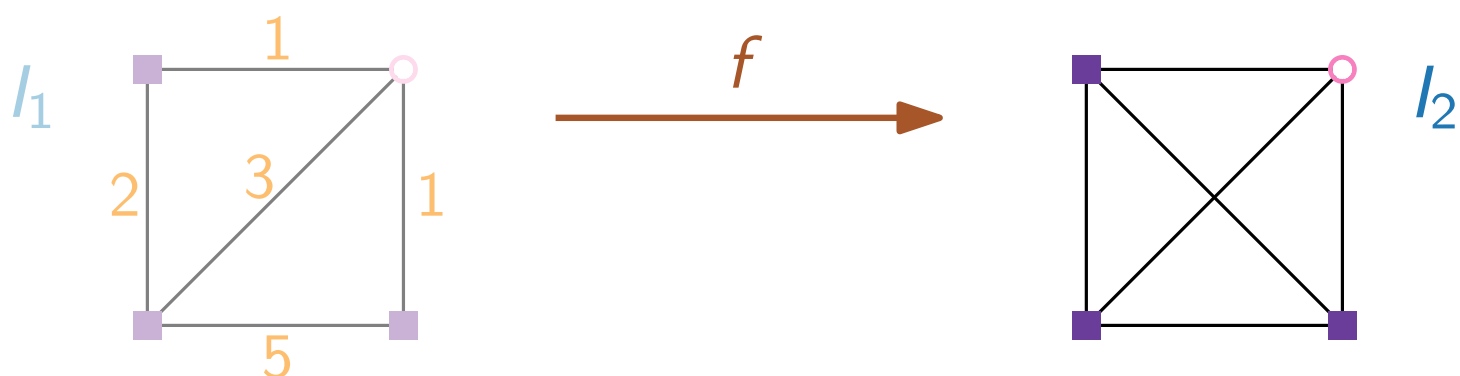

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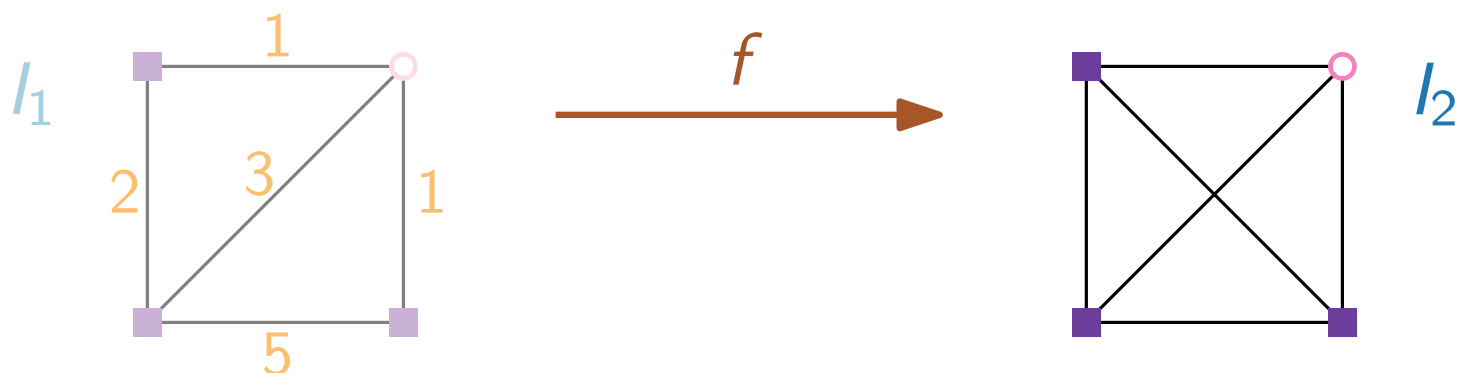

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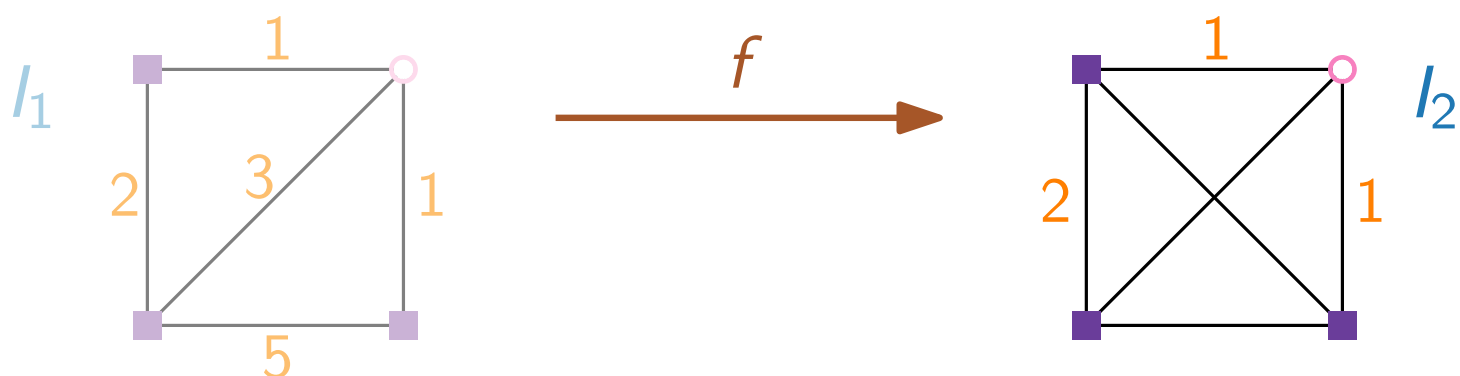

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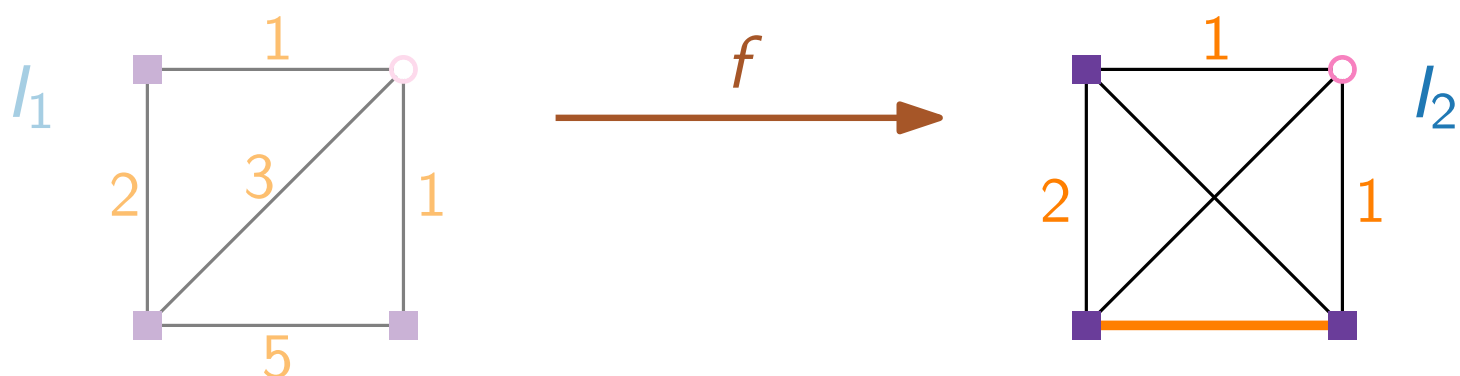

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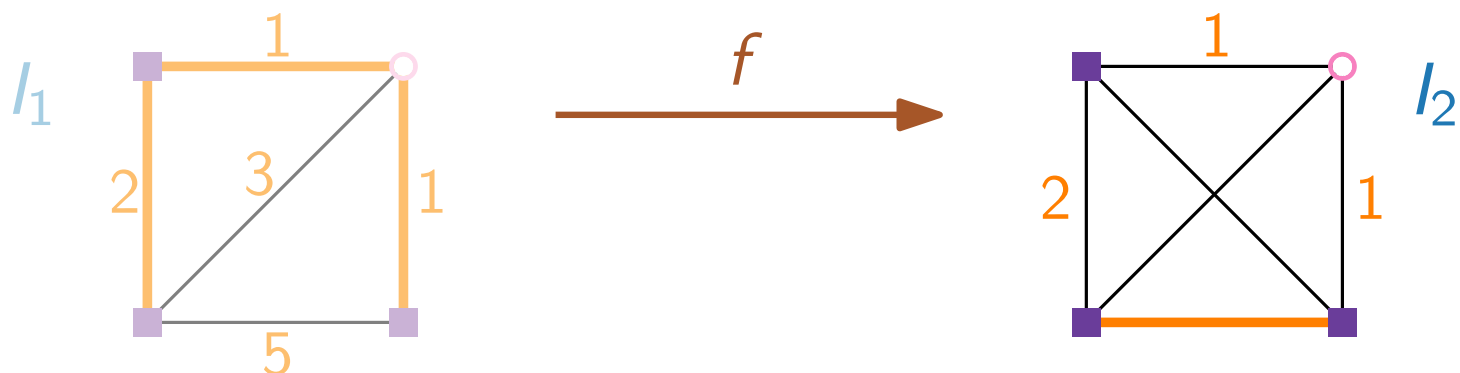

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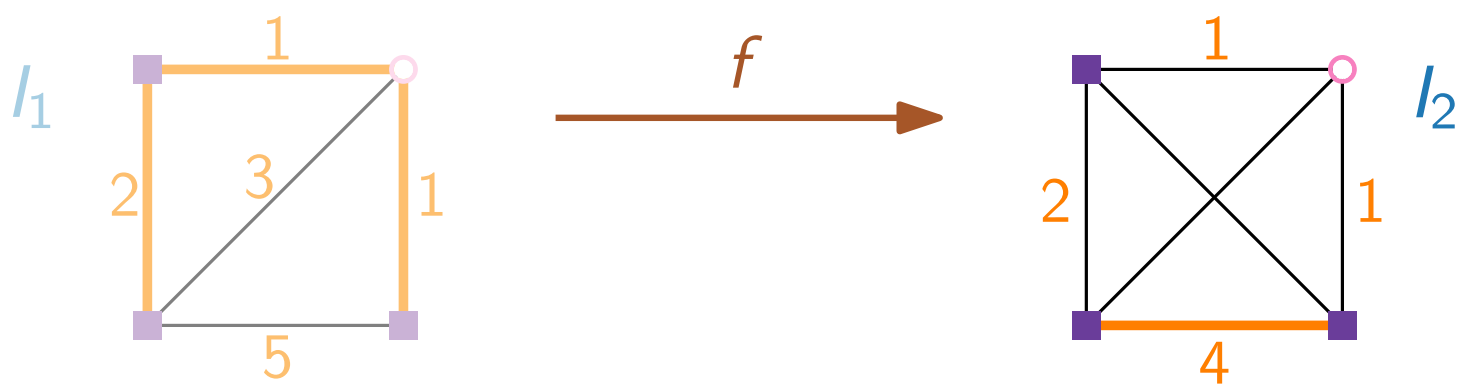

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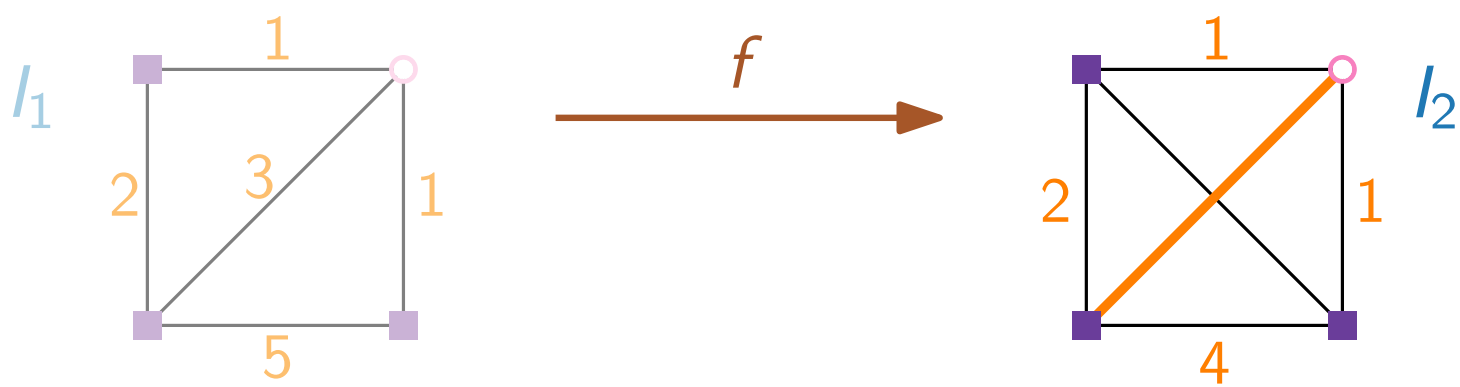

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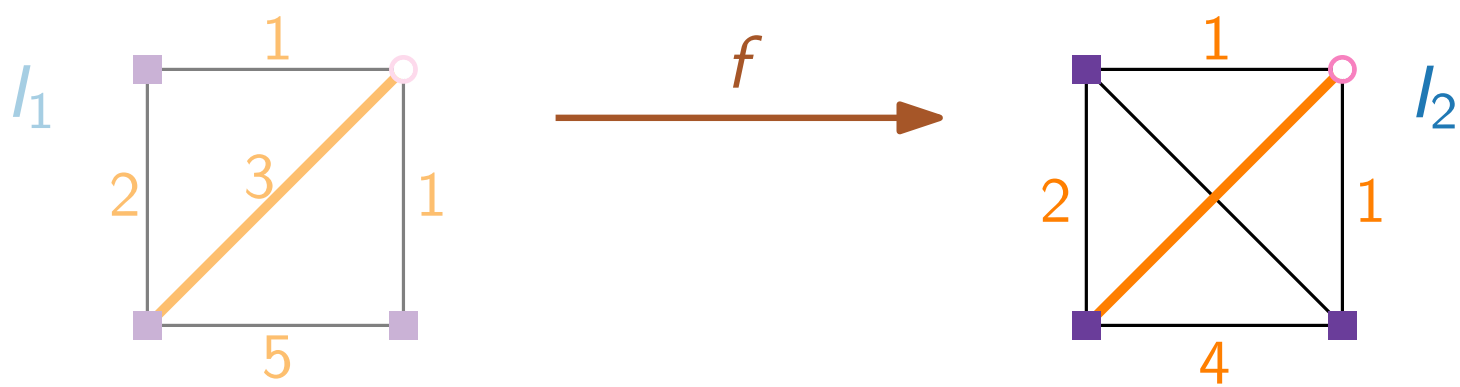

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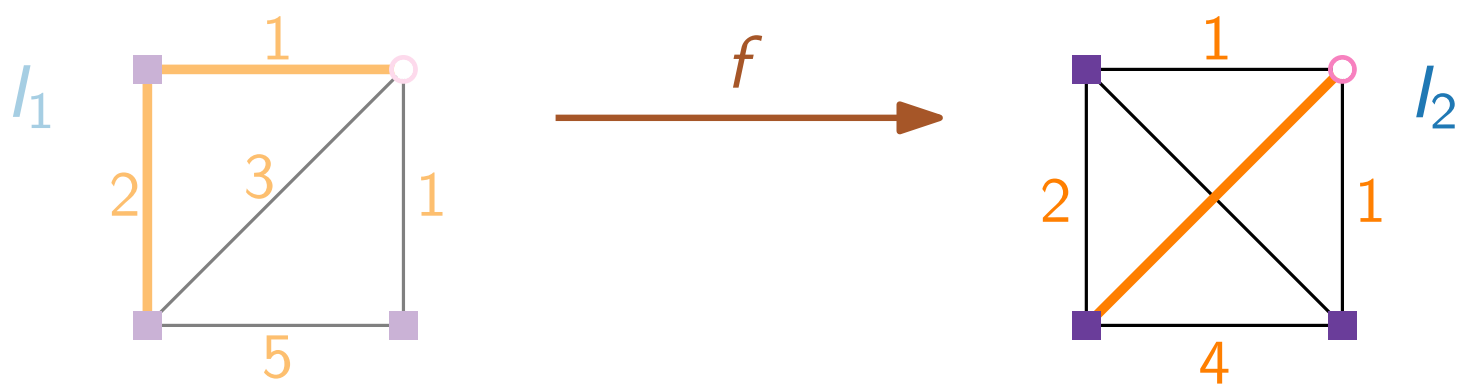

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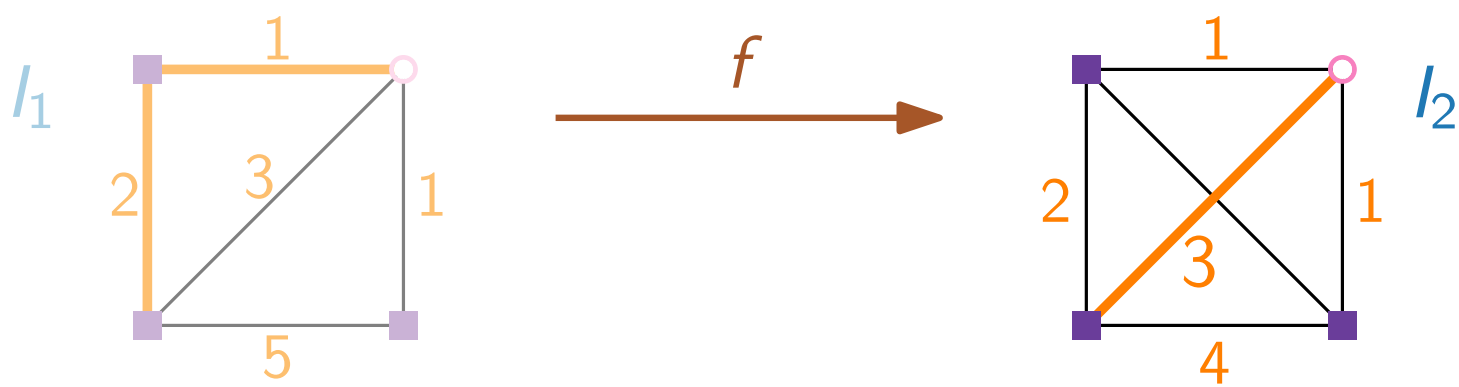

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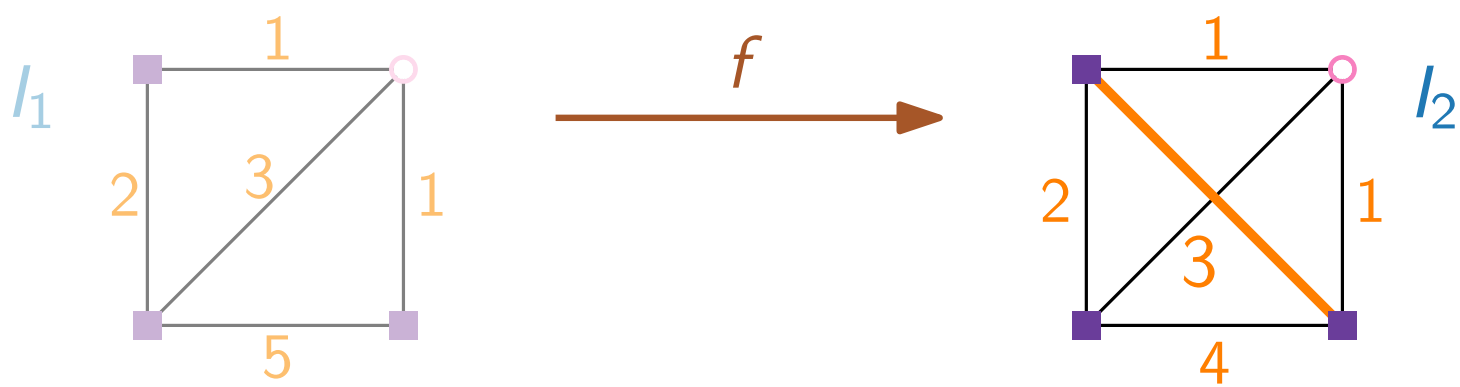

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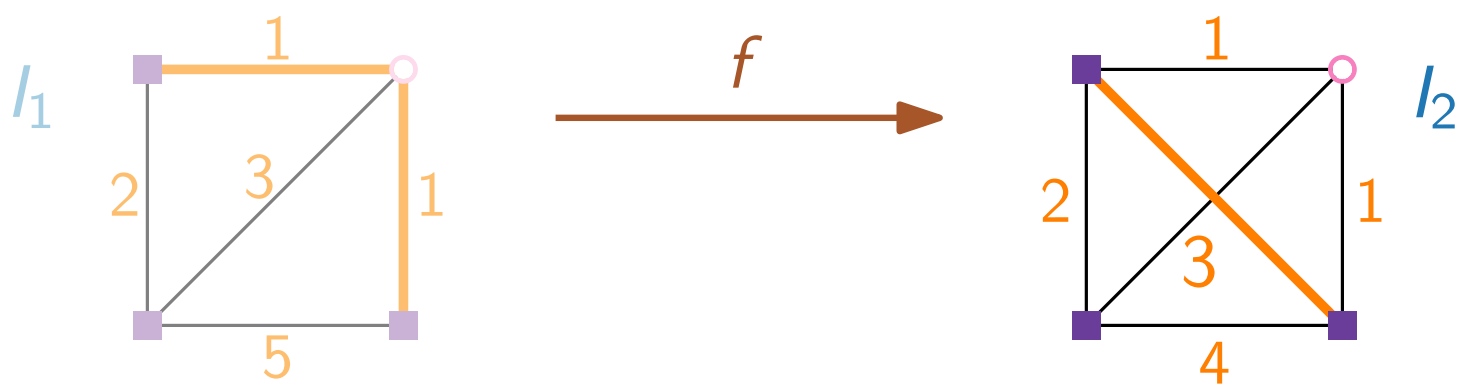

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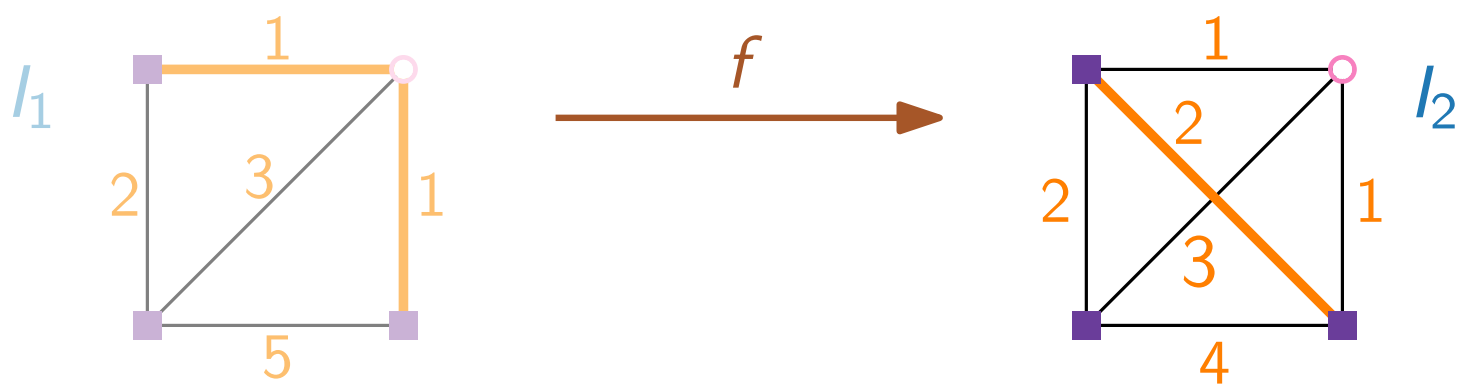

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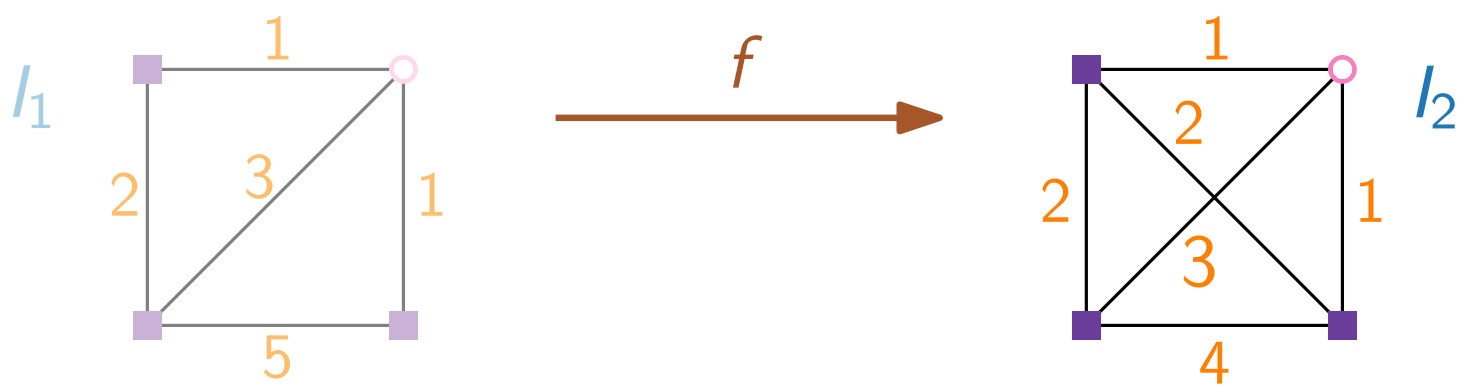

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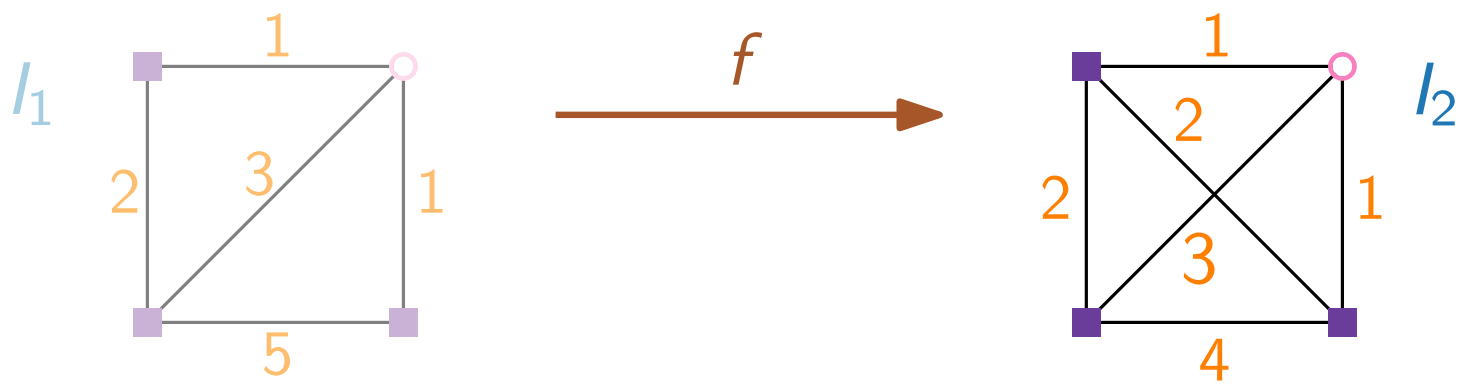

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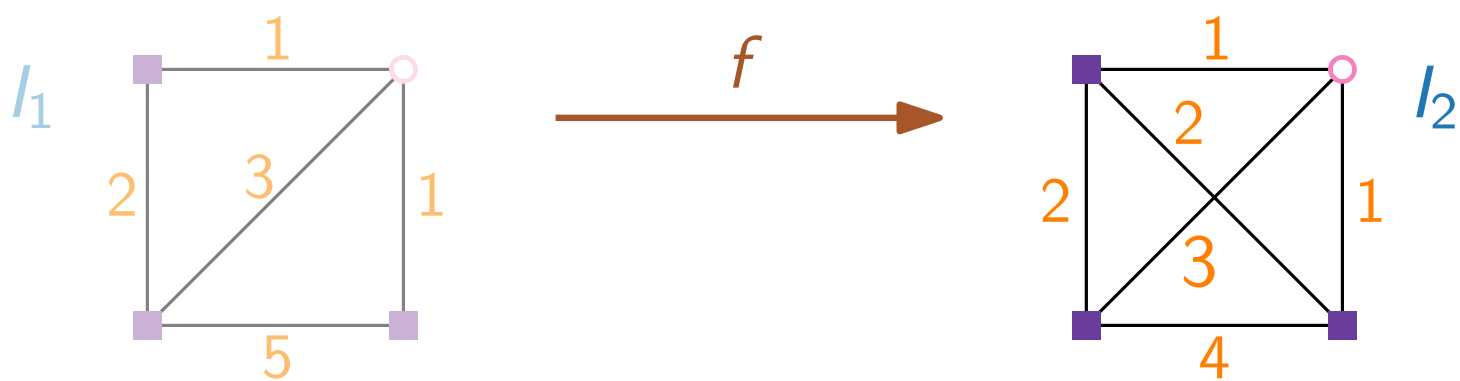

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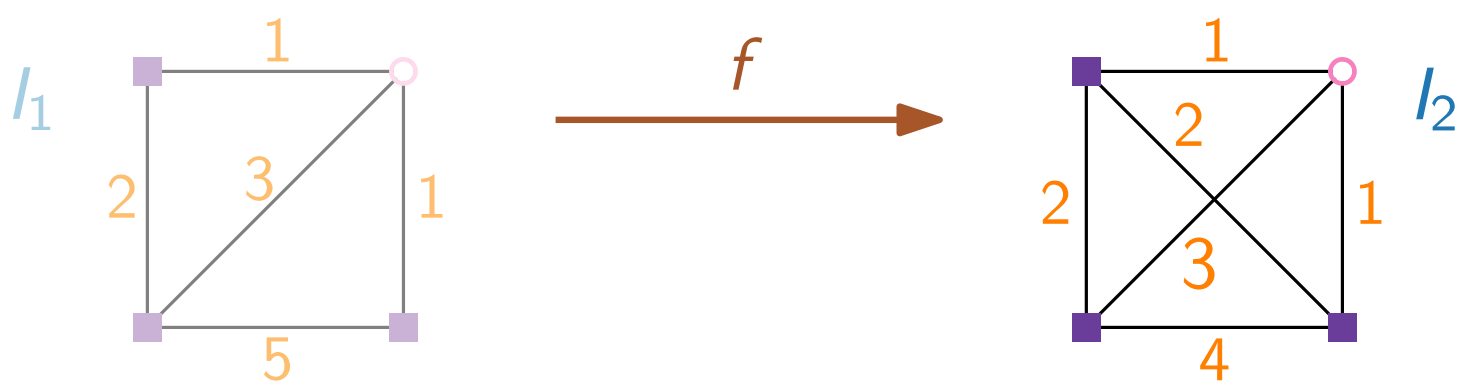

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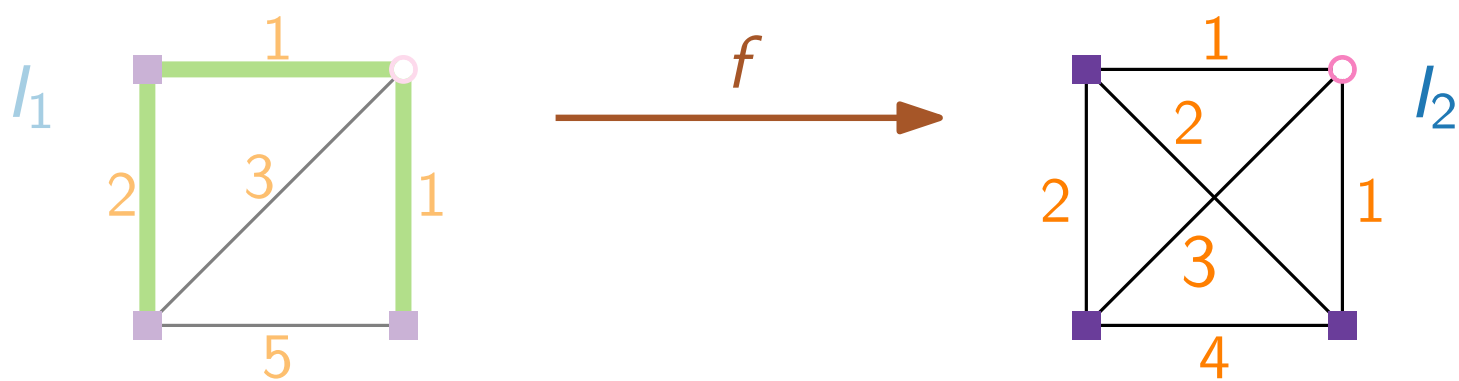

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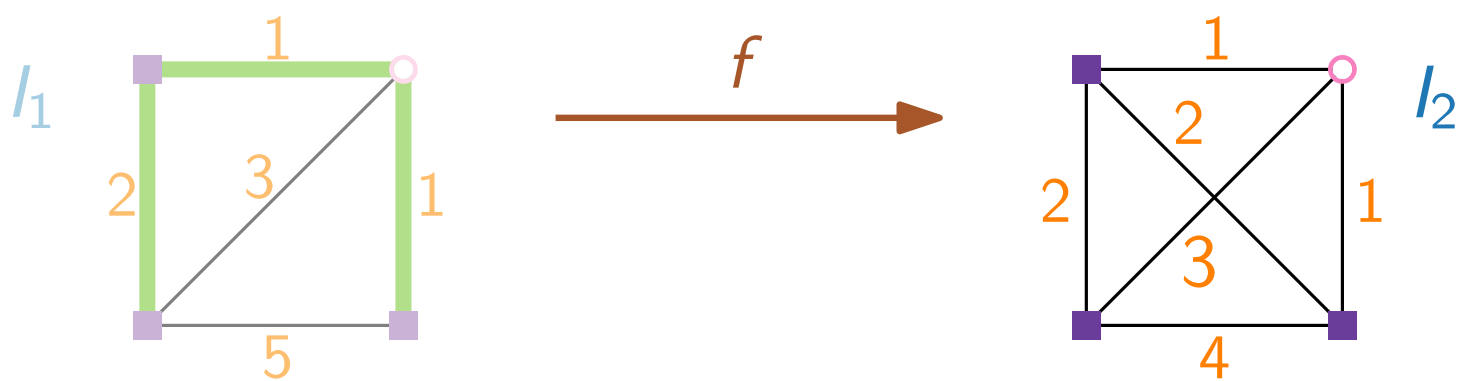

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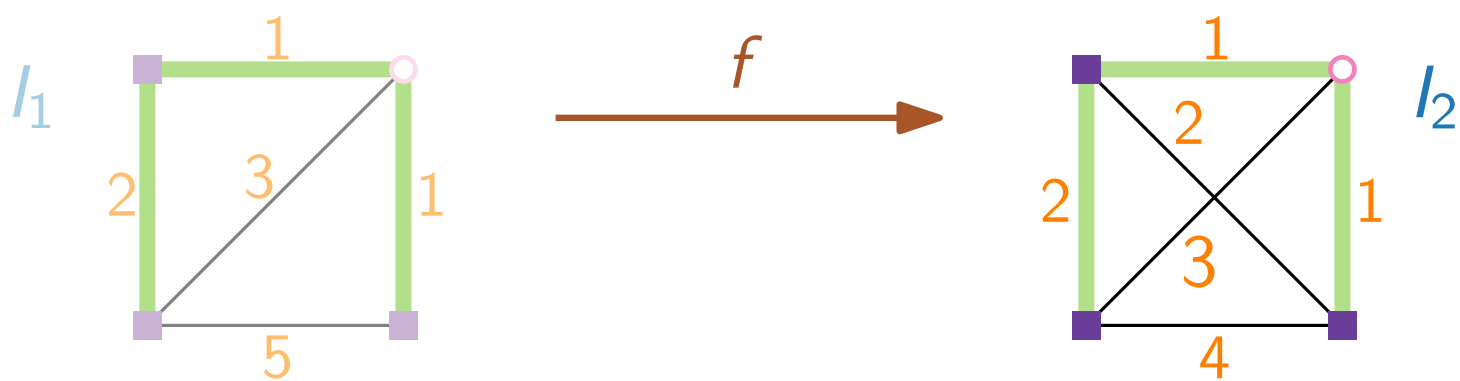

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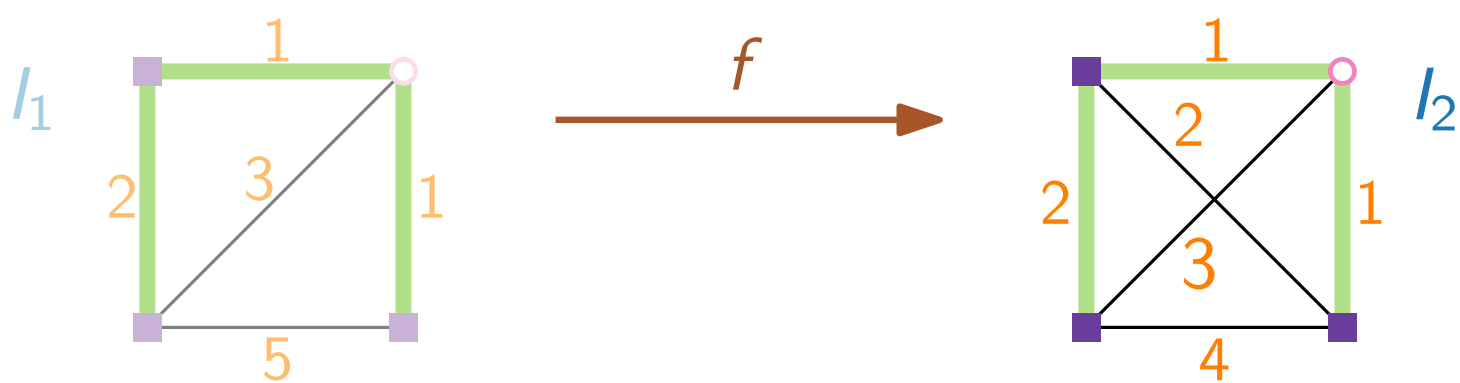

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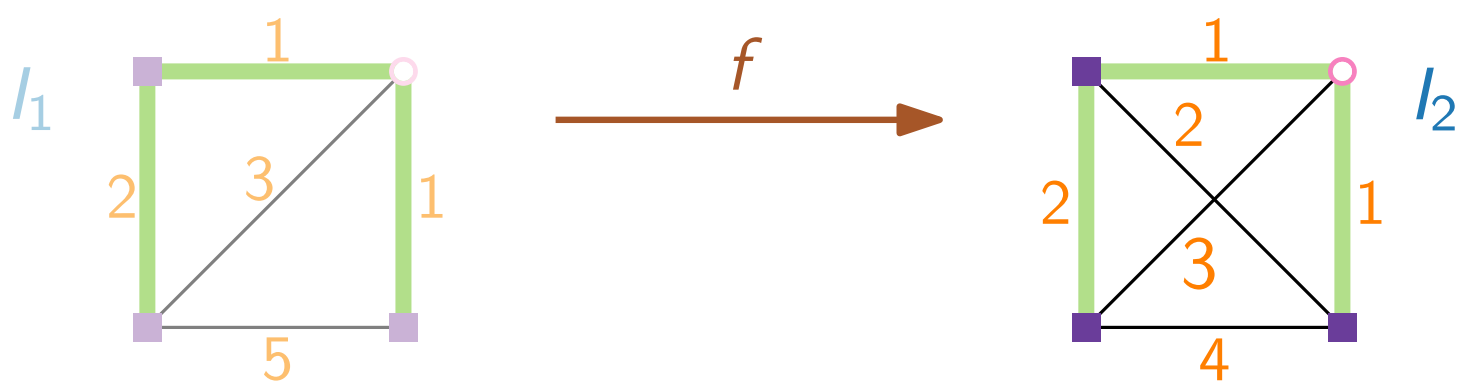

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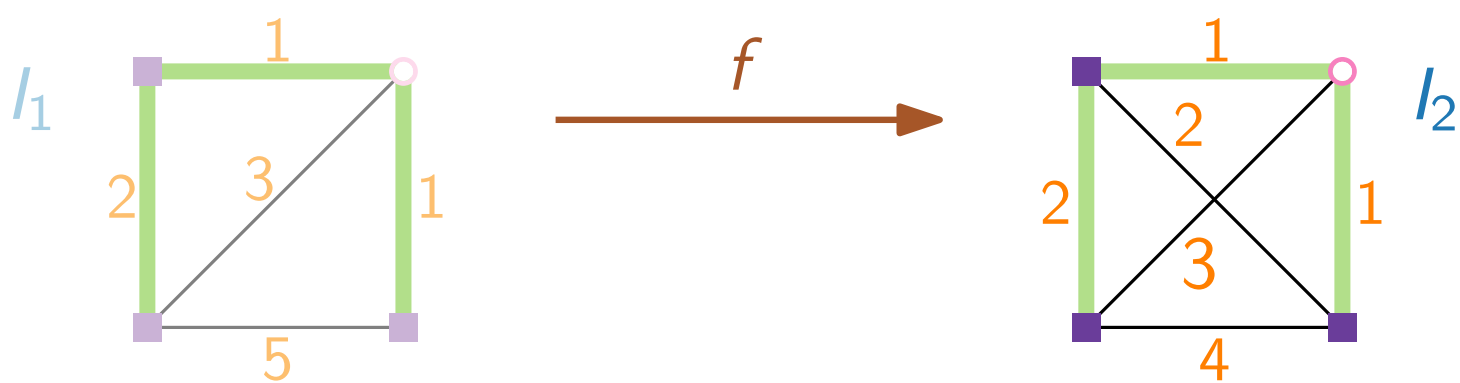

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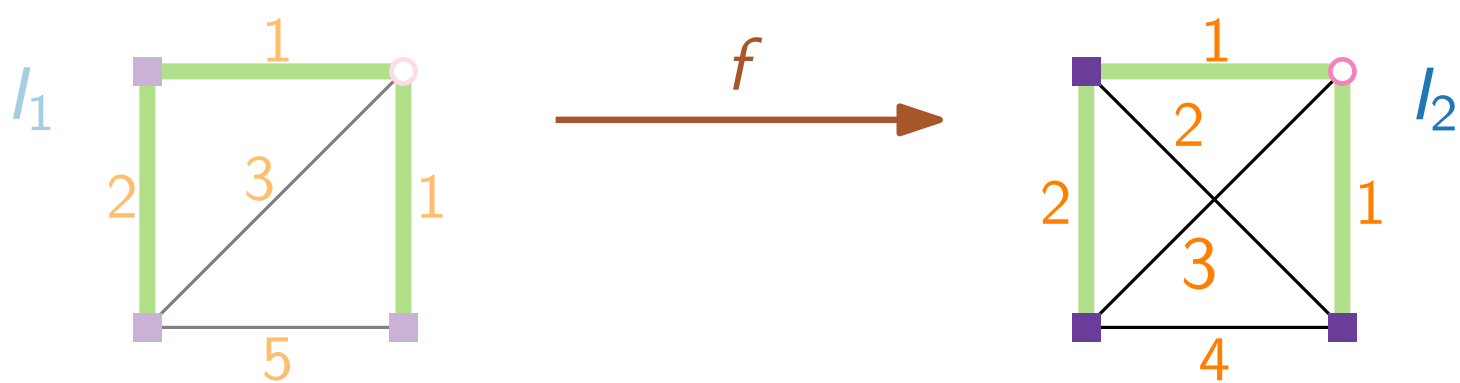

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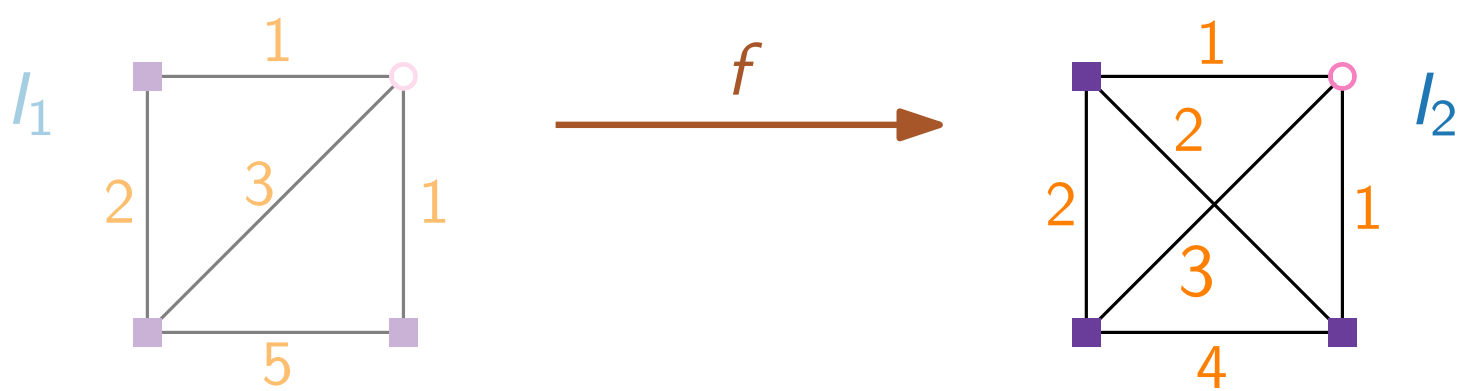

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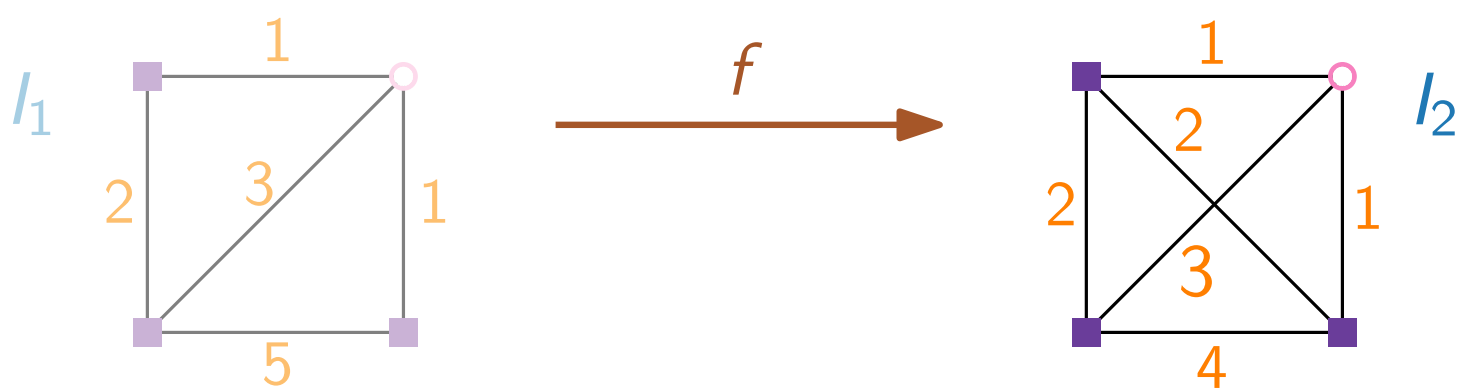

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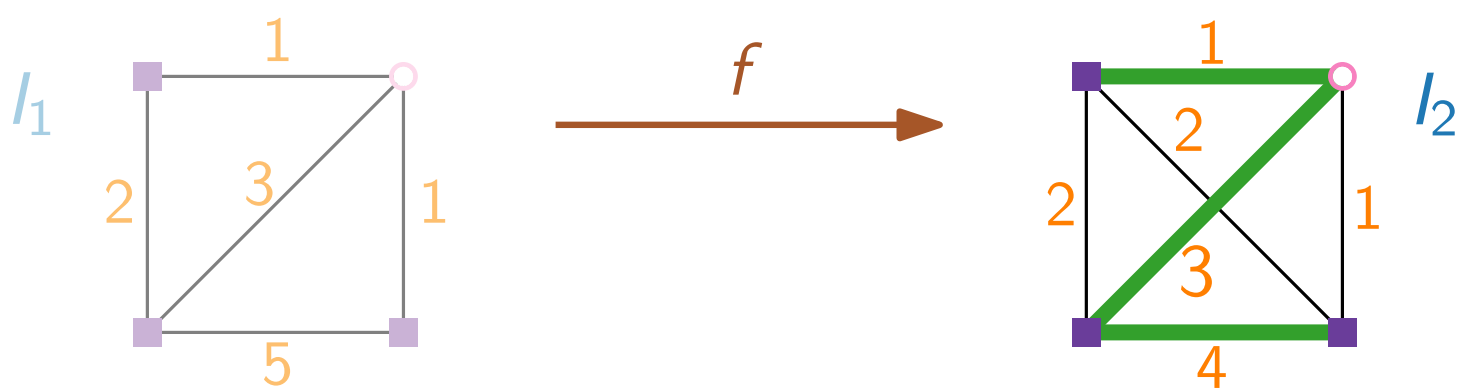

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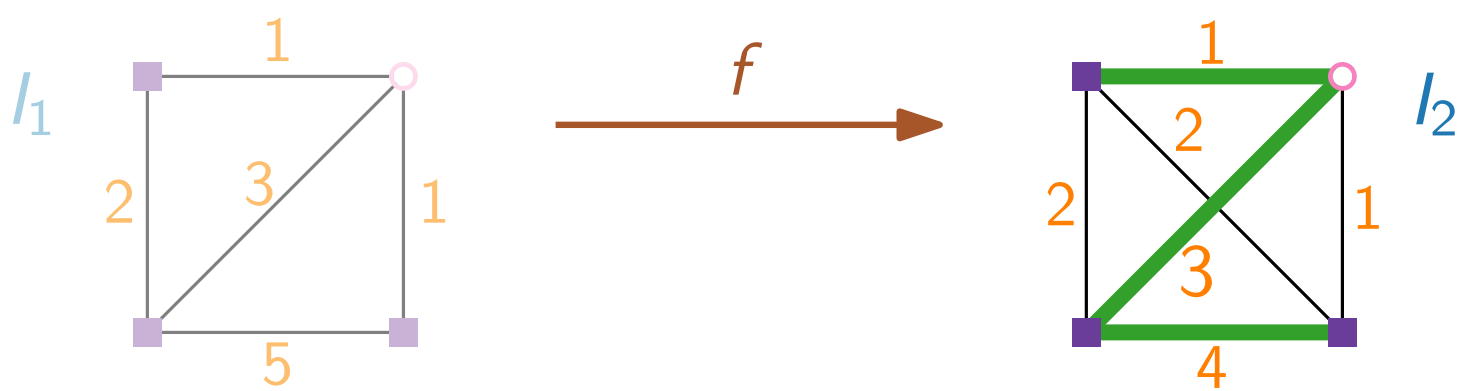

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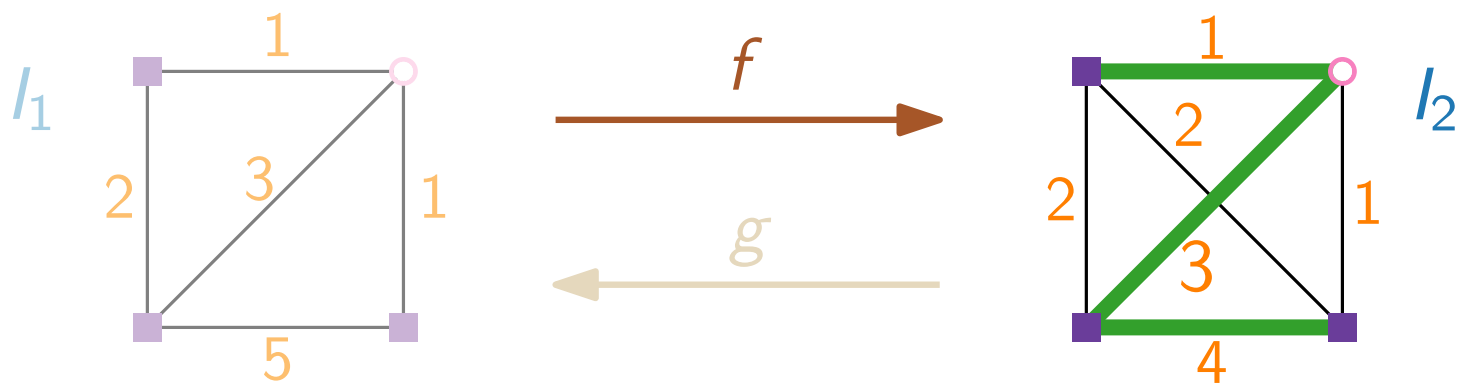

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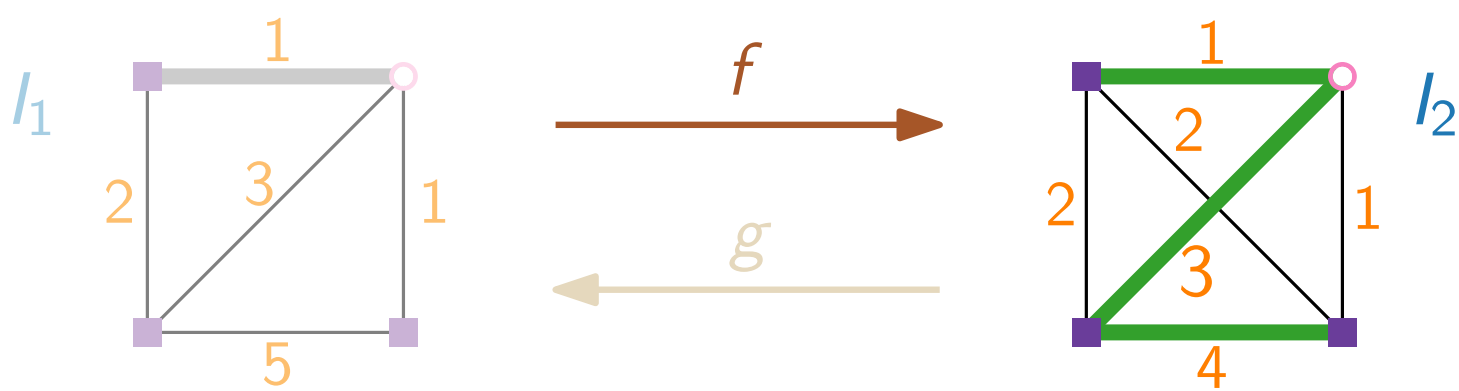

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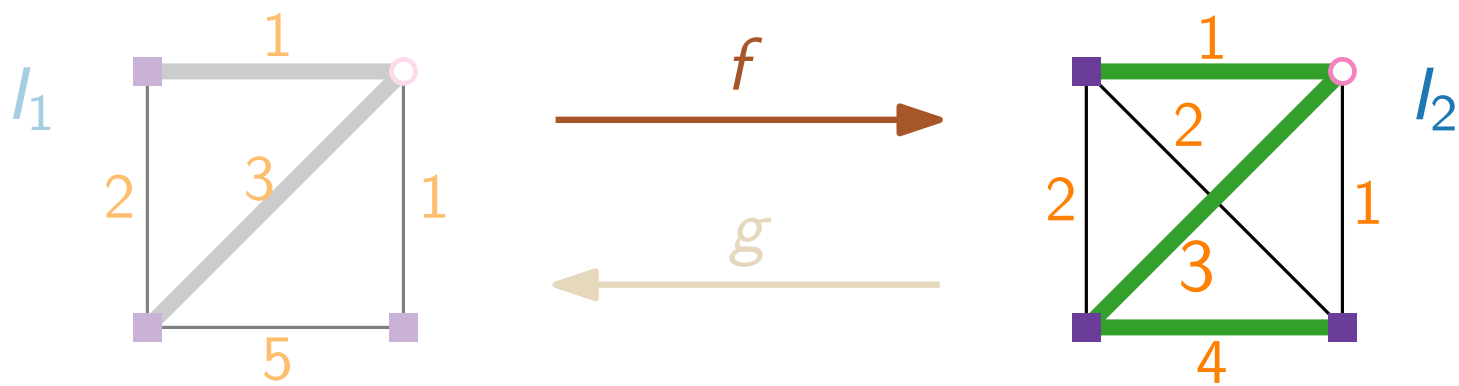

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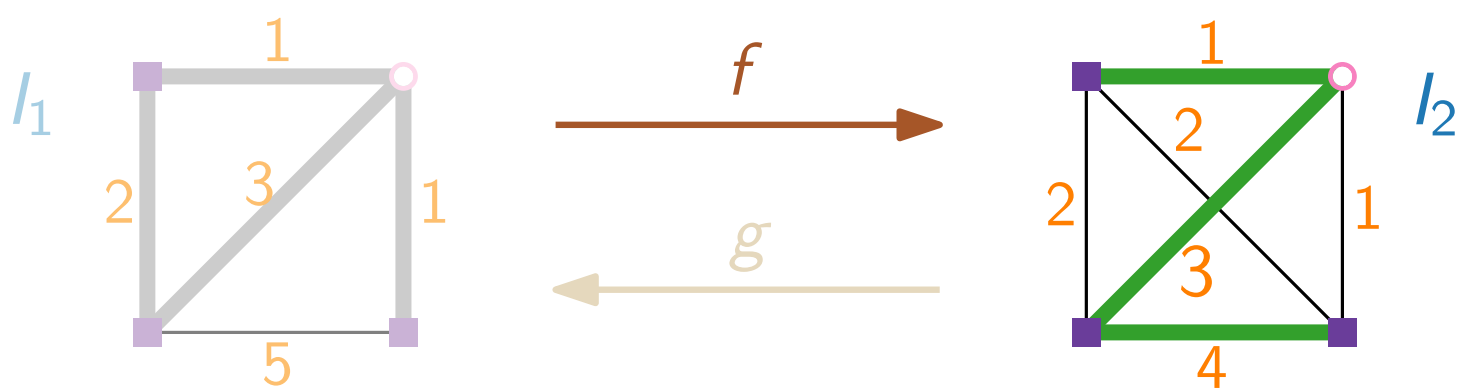

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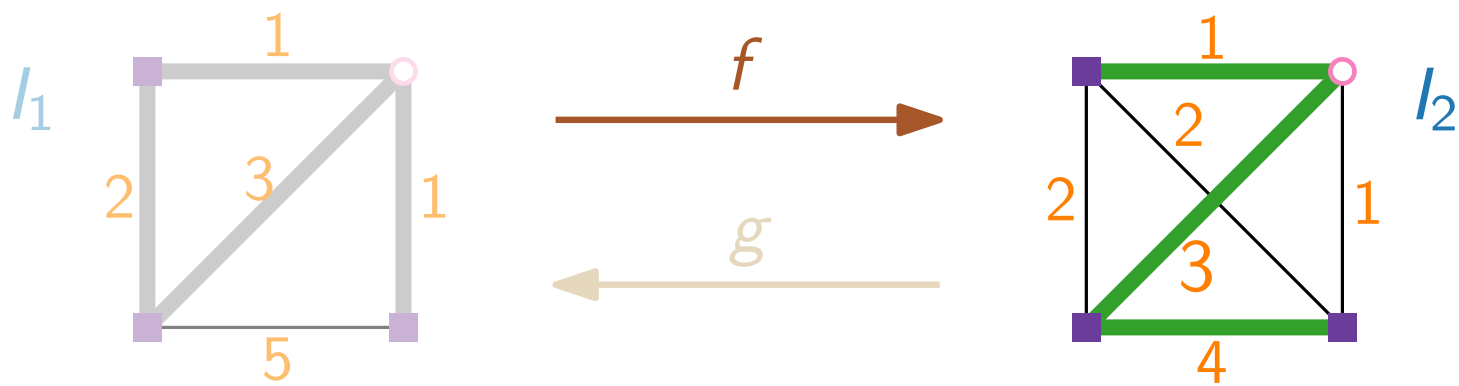

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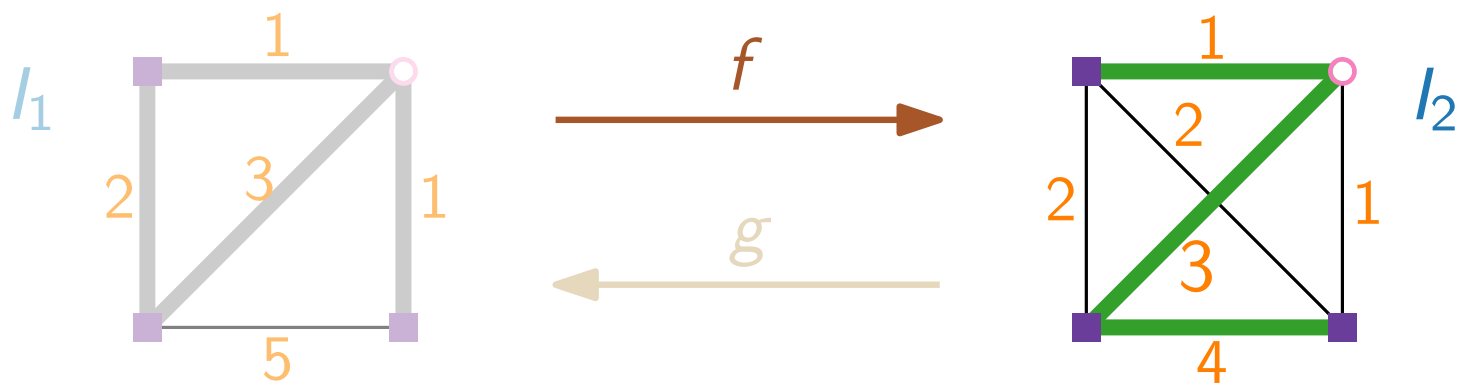

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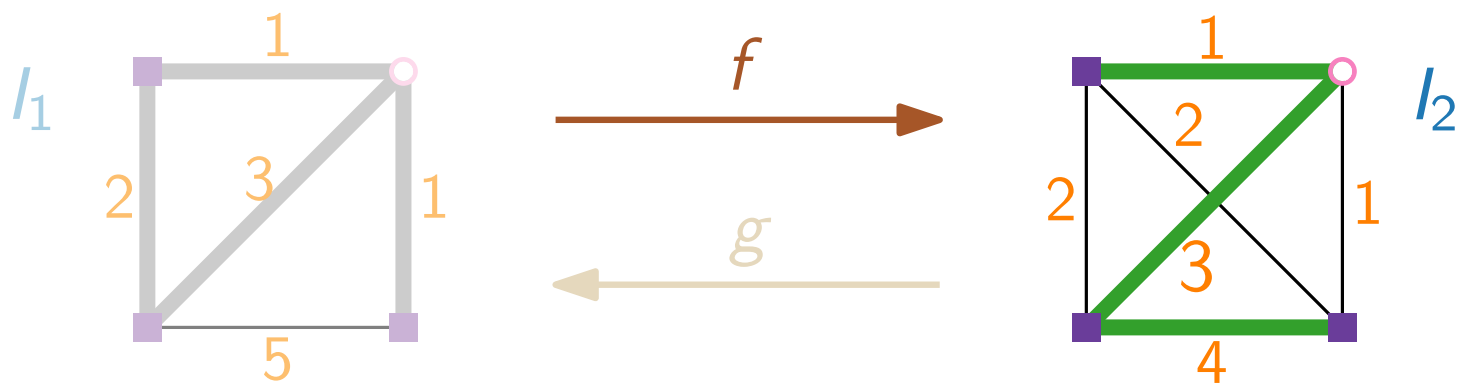

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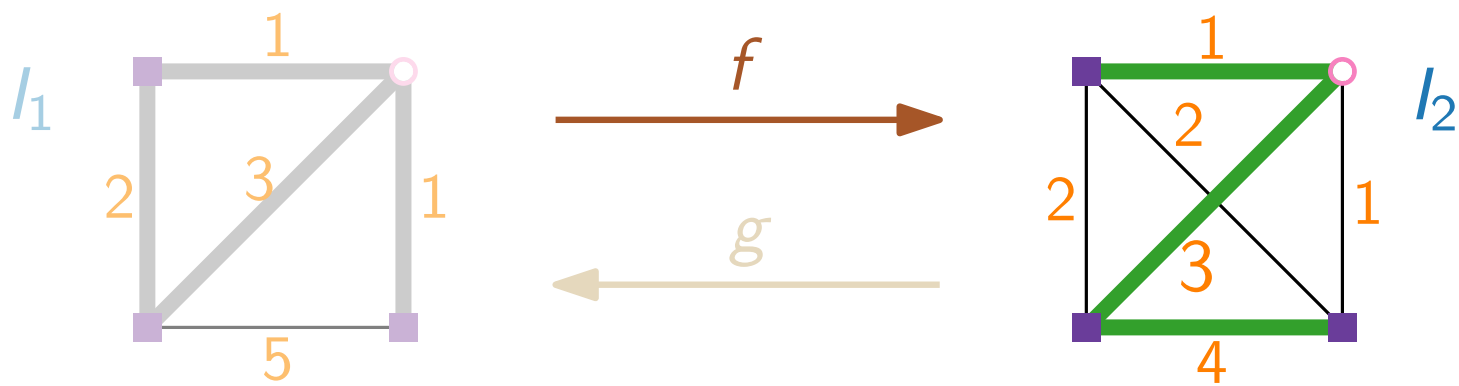

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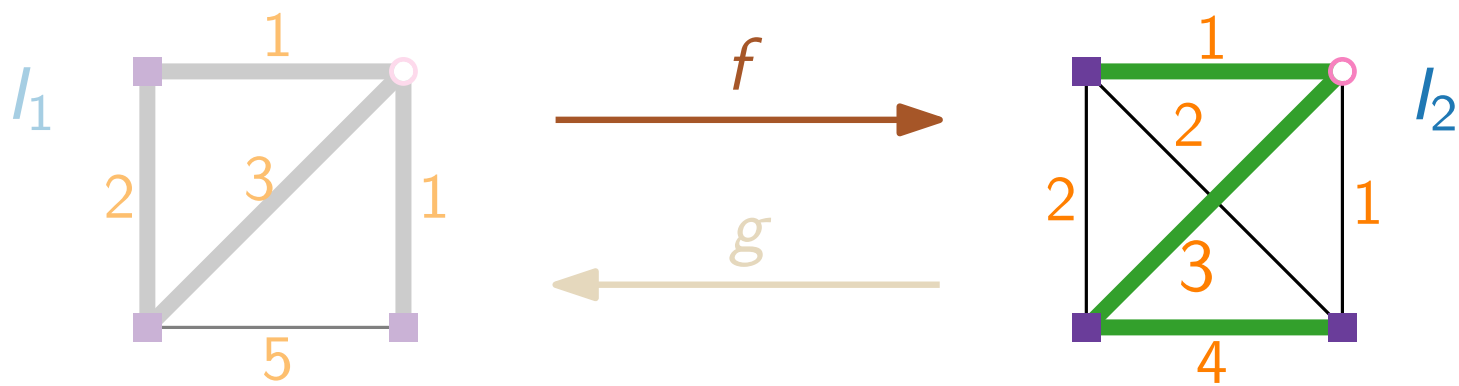

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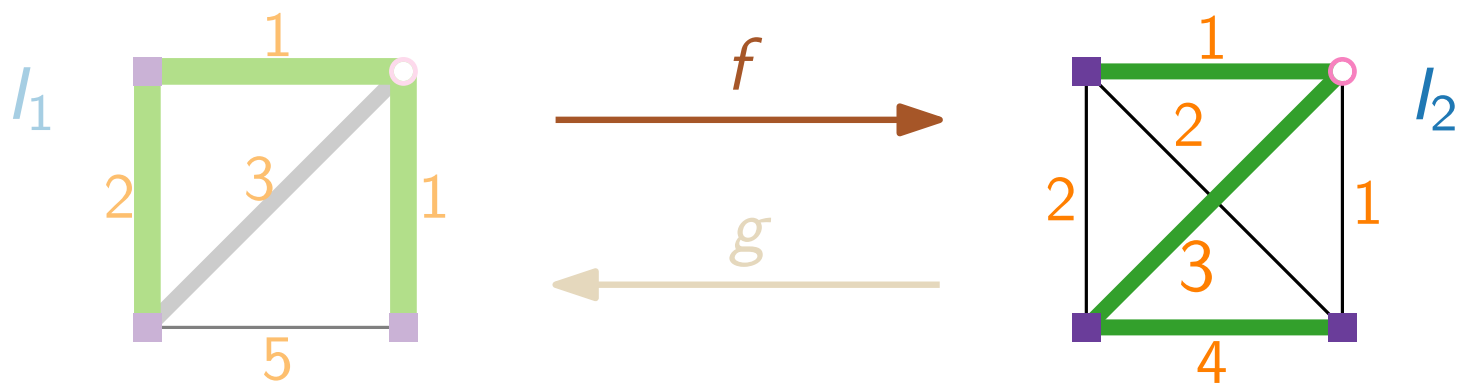

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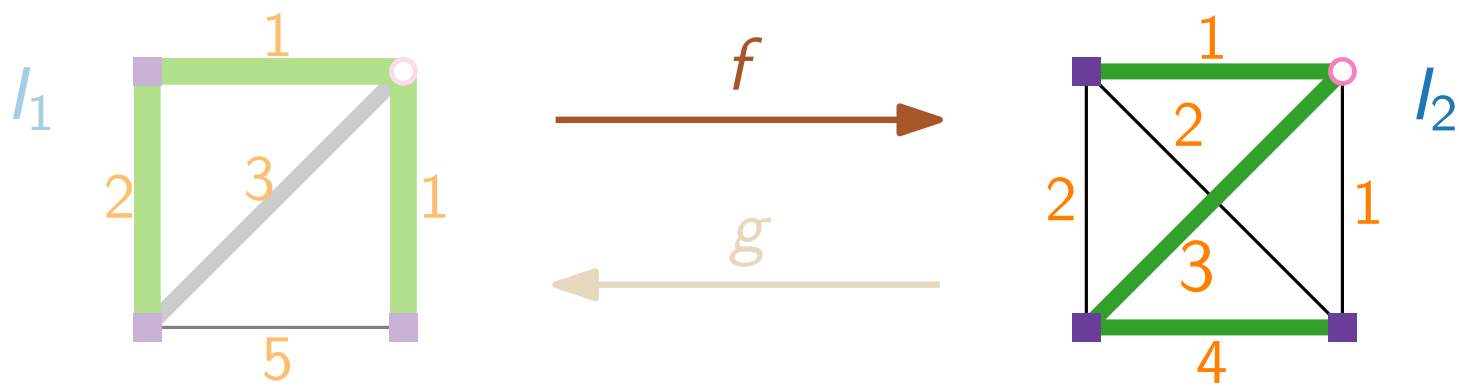

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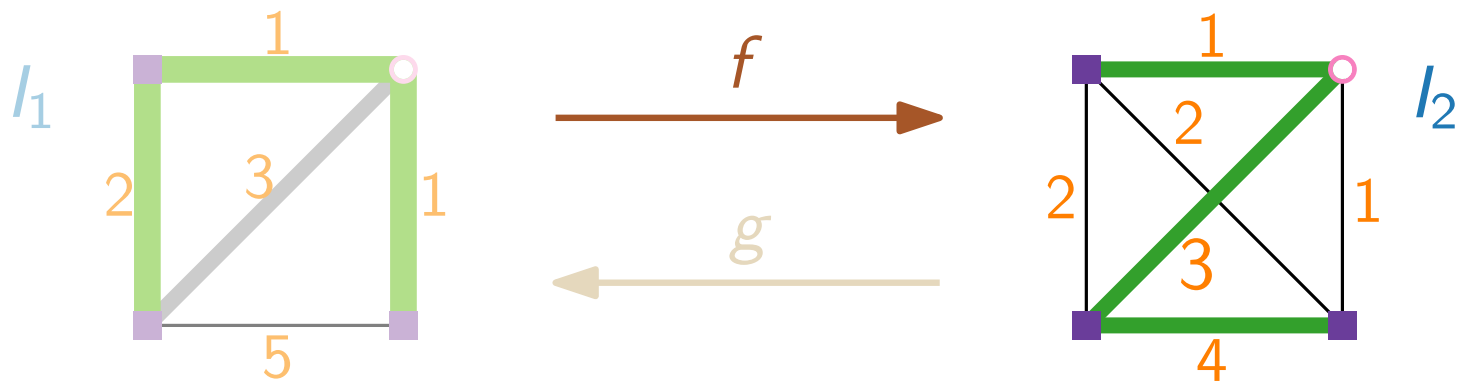
Note that $c_1(B_1) \leq c_1(G_1') \leq c_2(B_2)$.



Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part IV:
2-Approximation for SteinerTree

Alexander Wolff

Winter 2022/23

2-Approximation for SteinerTree

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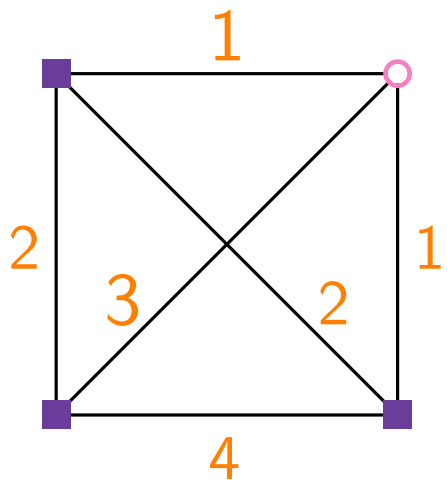

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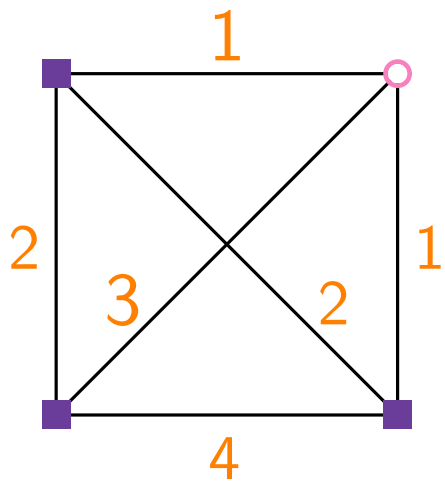

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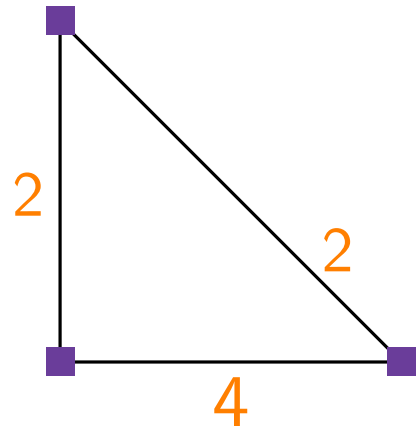

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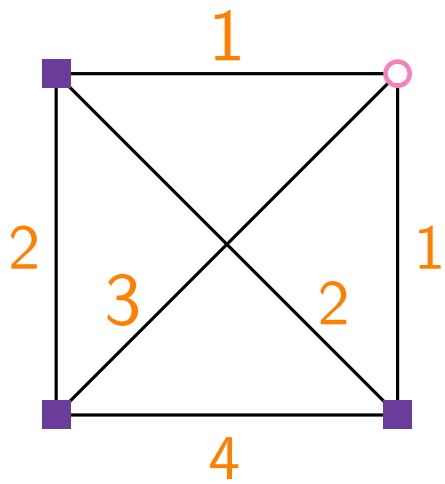

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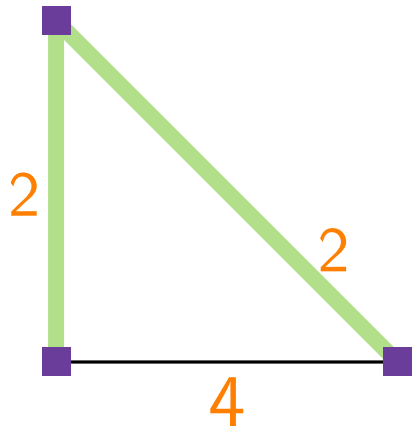

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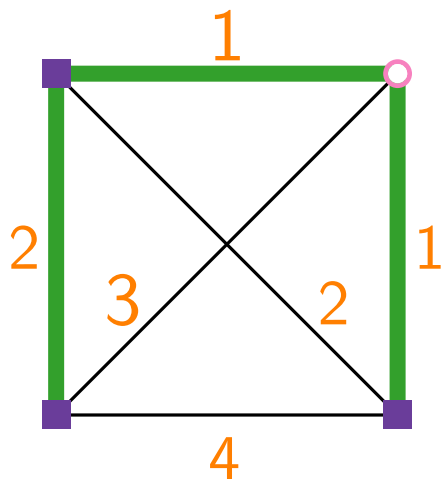

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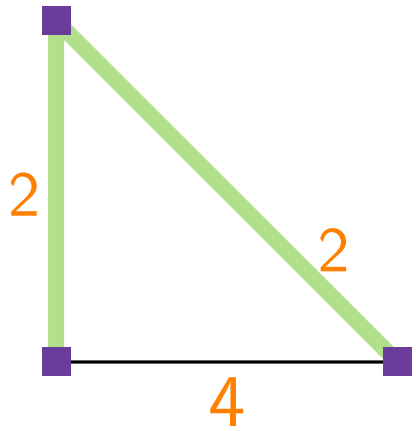

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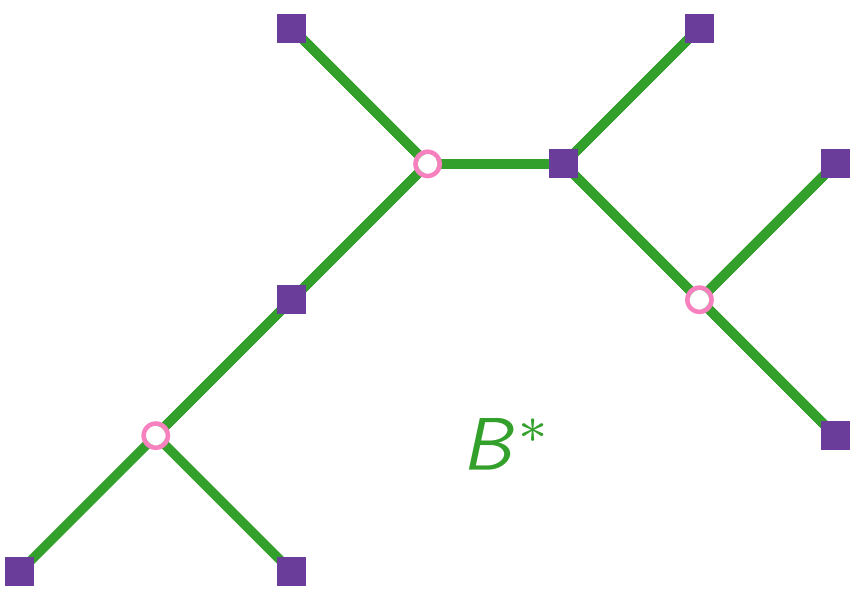

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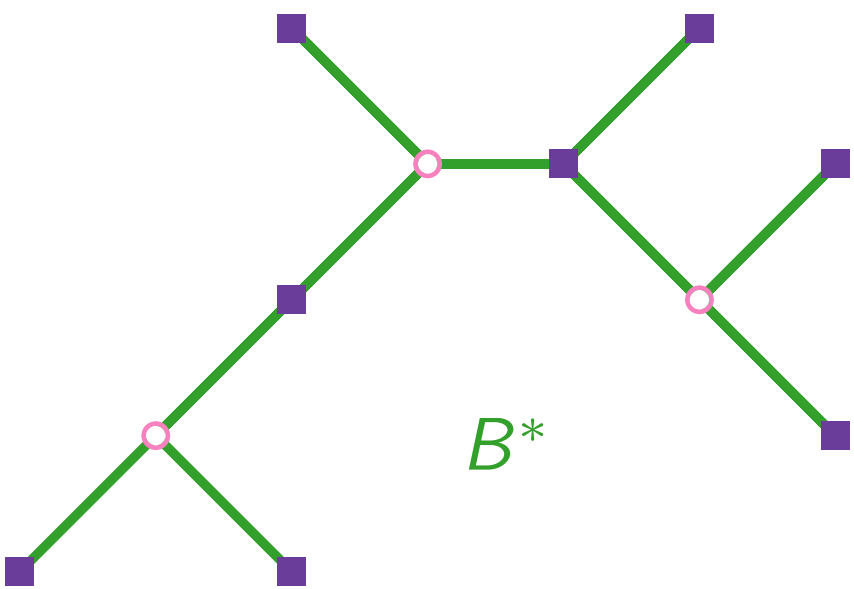

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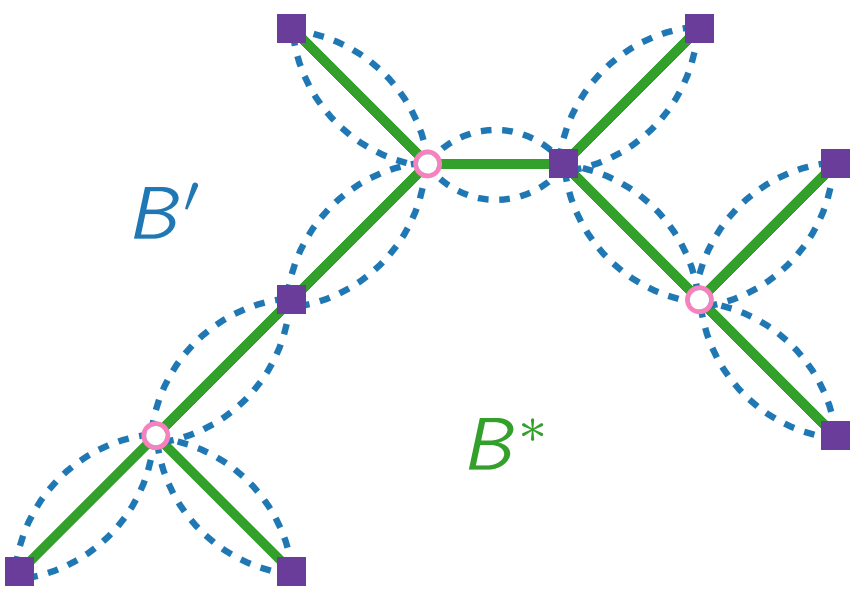

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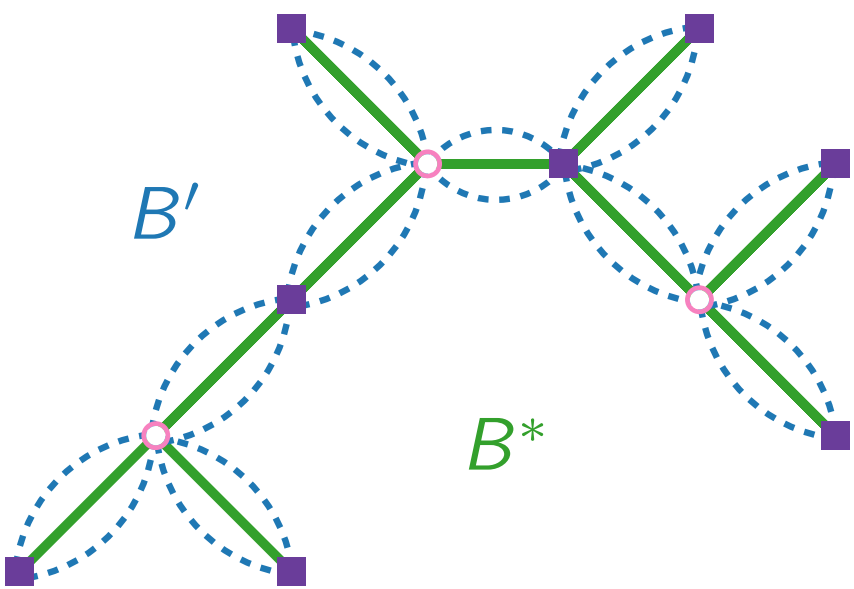

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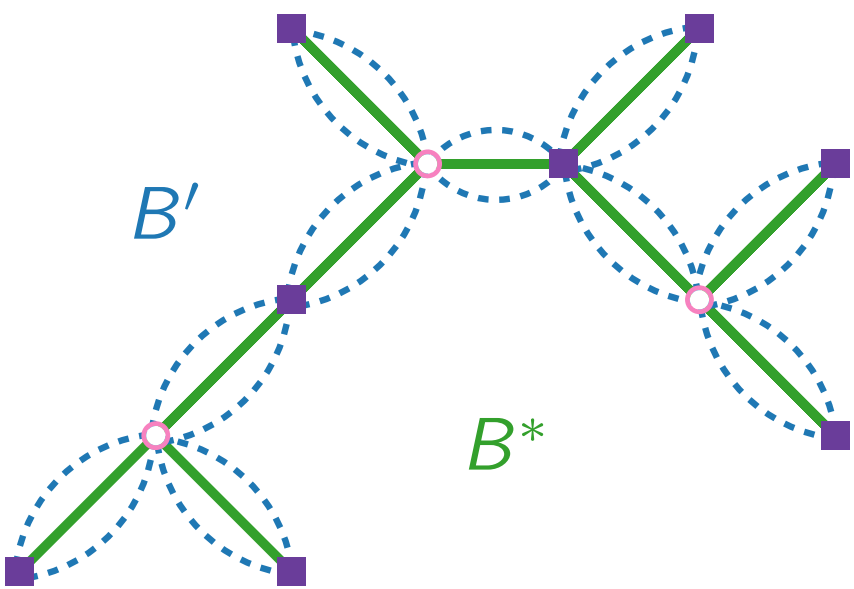

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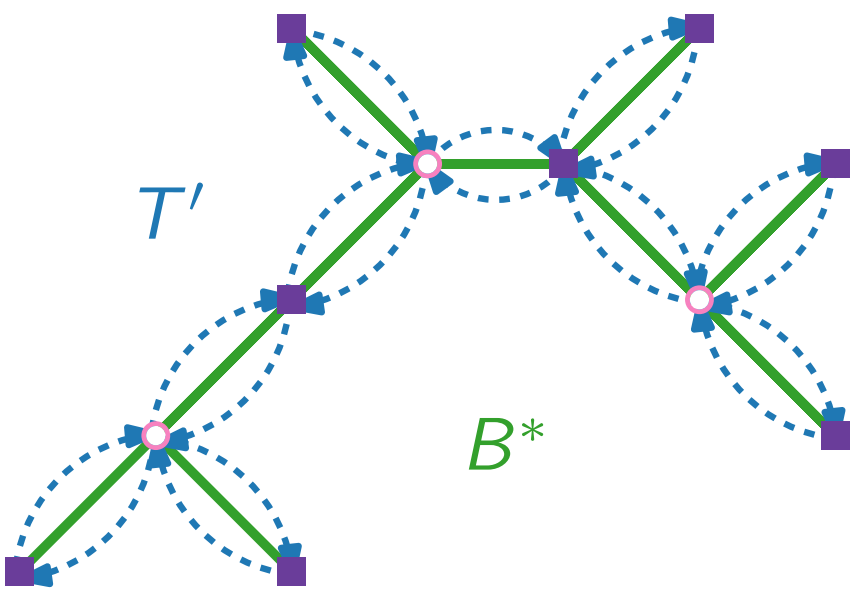

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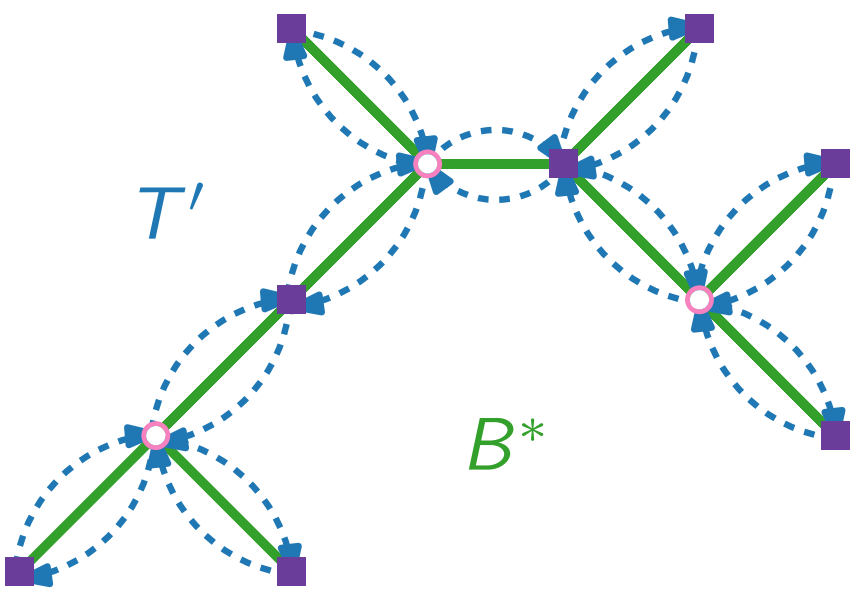

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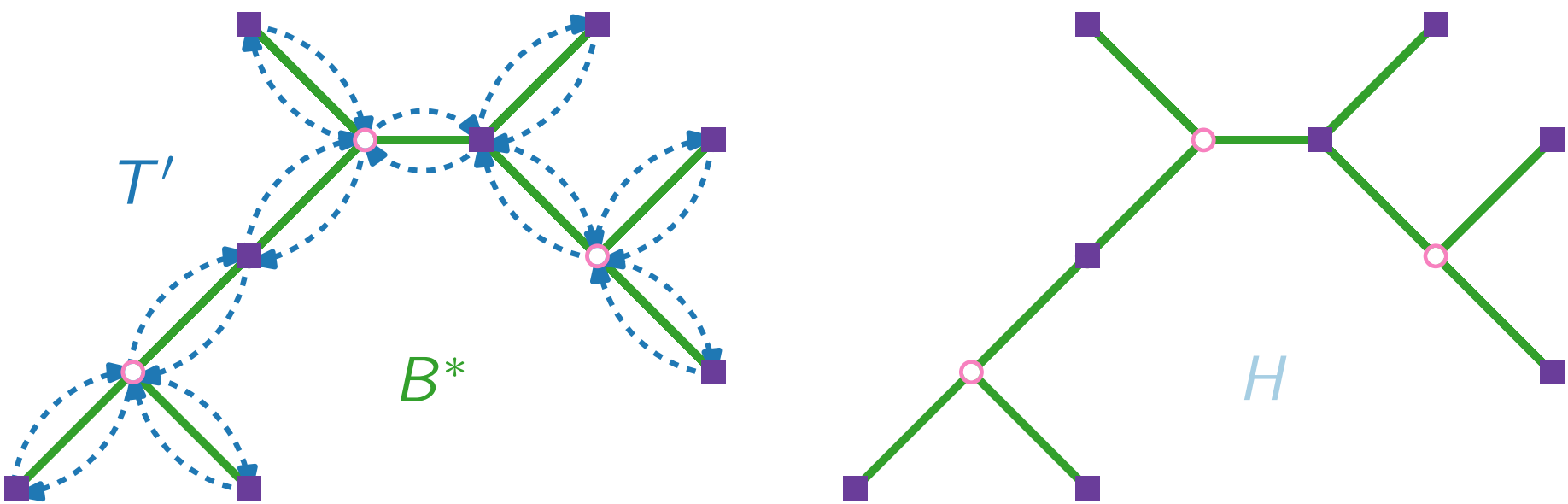

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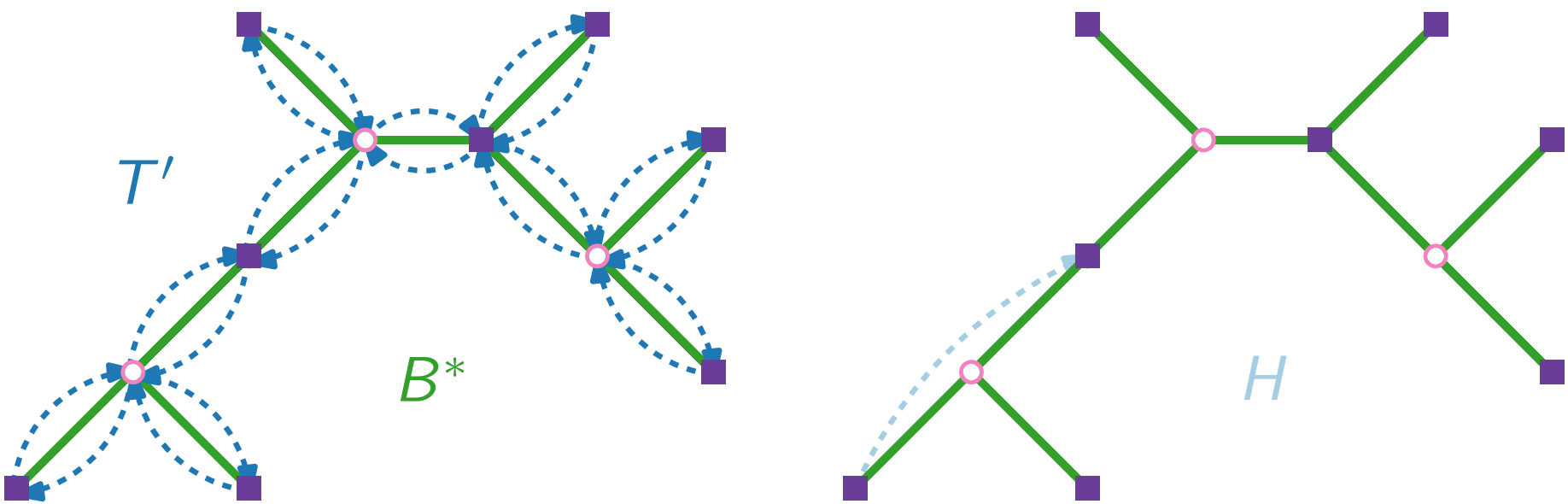

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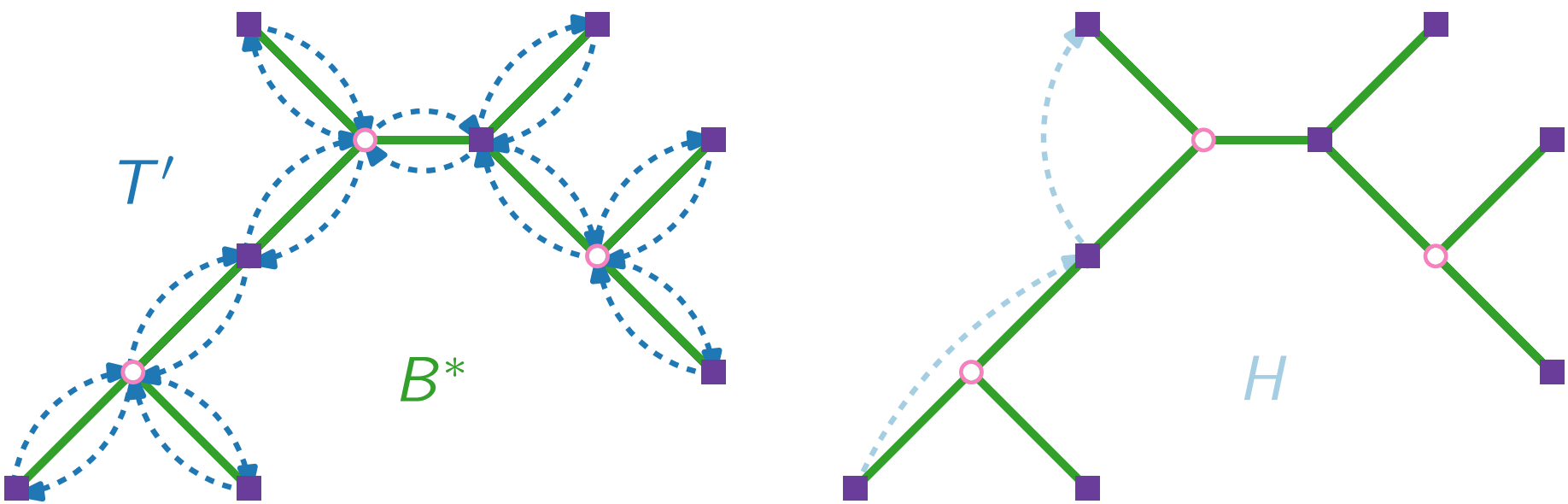

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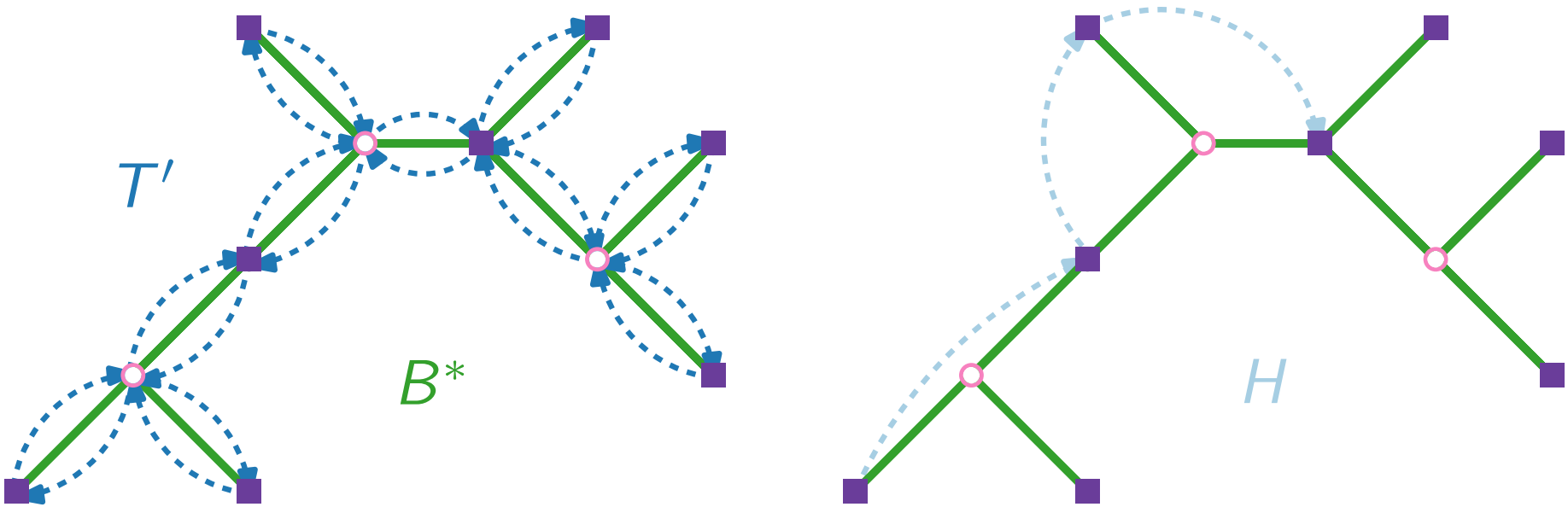

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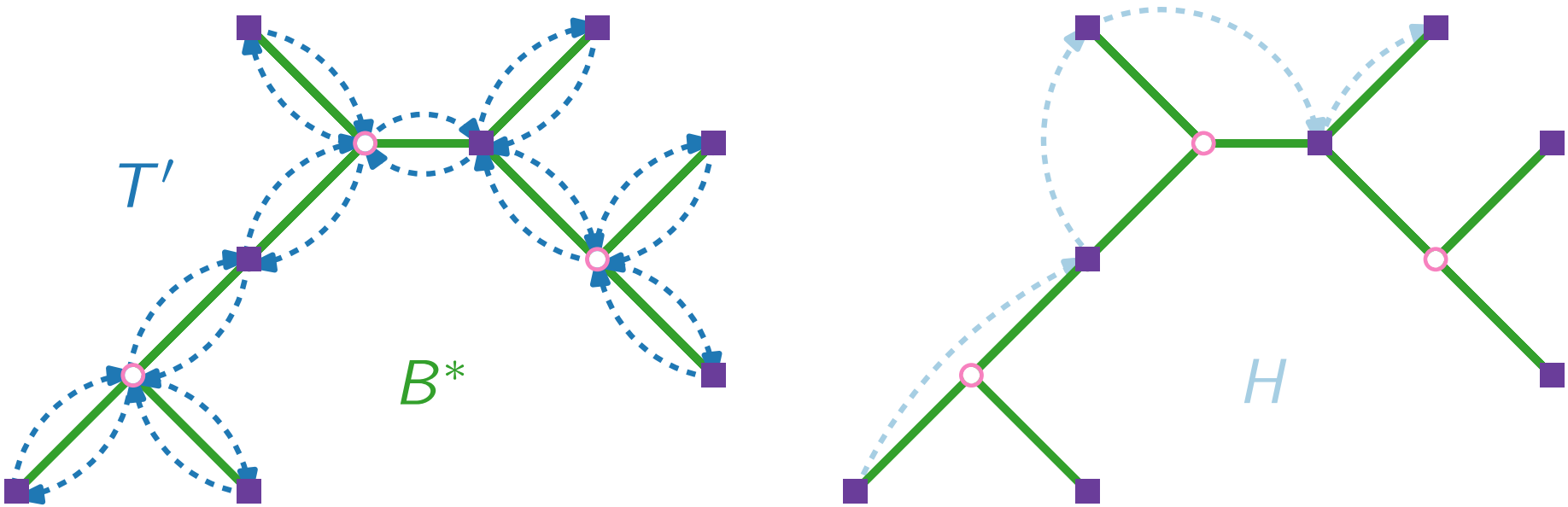

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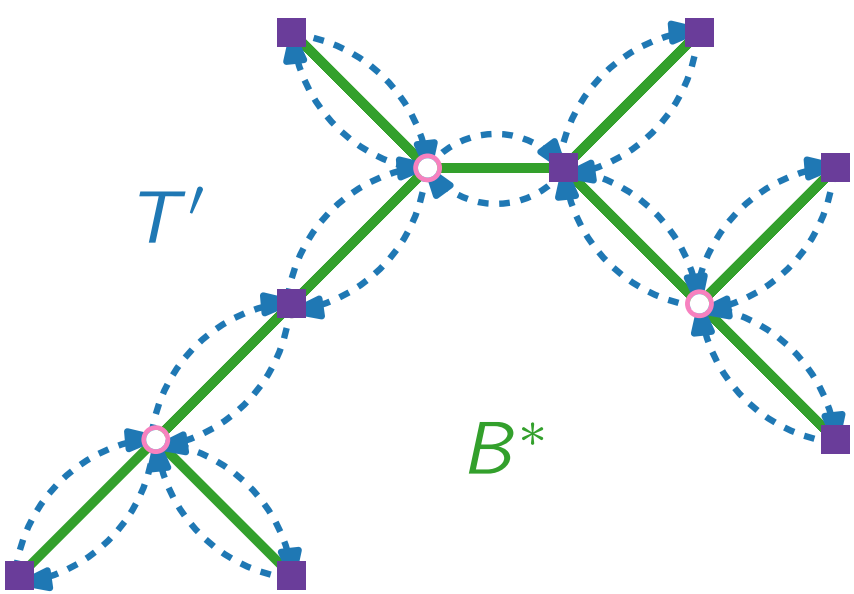

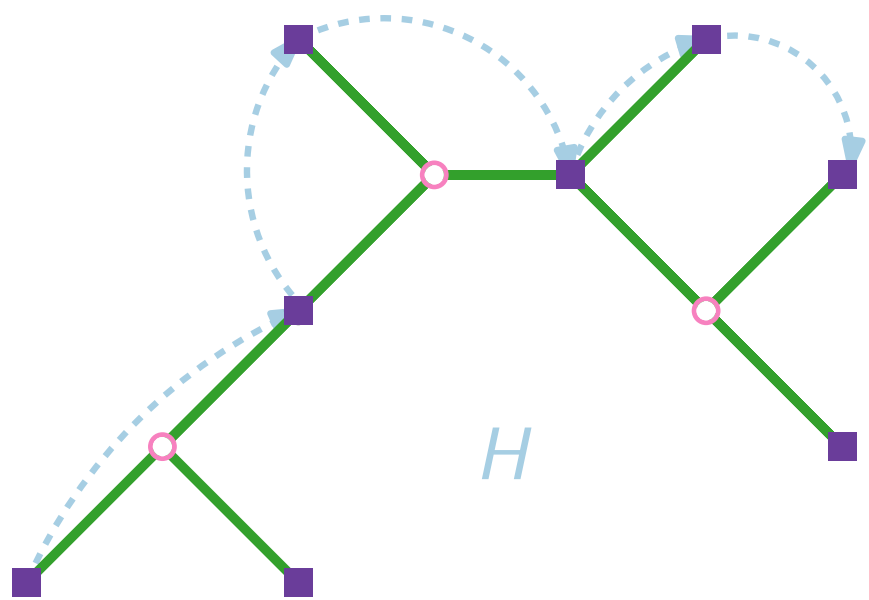

Proof of Approximation Factor

Consider an optimal Steiner tree B^*.

Duplicate all edges of B^*.

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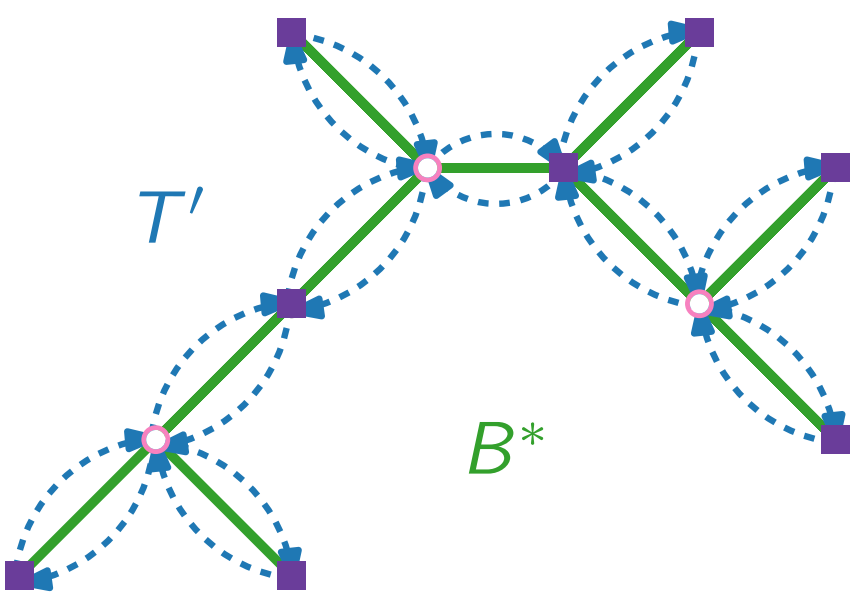

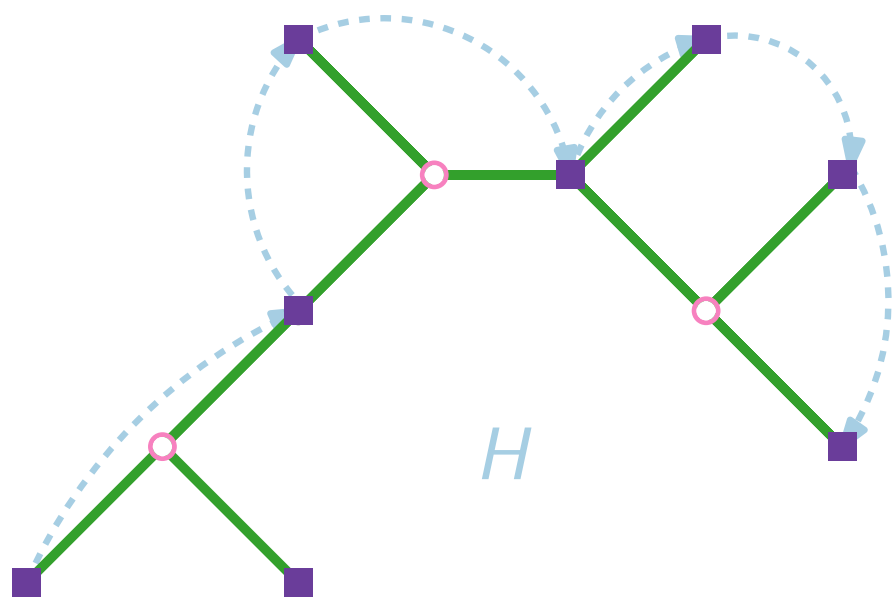

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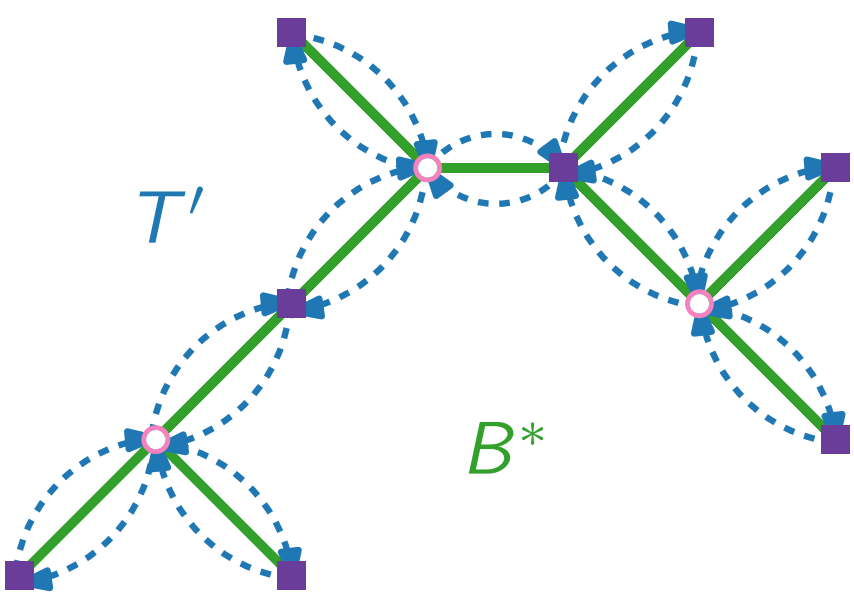

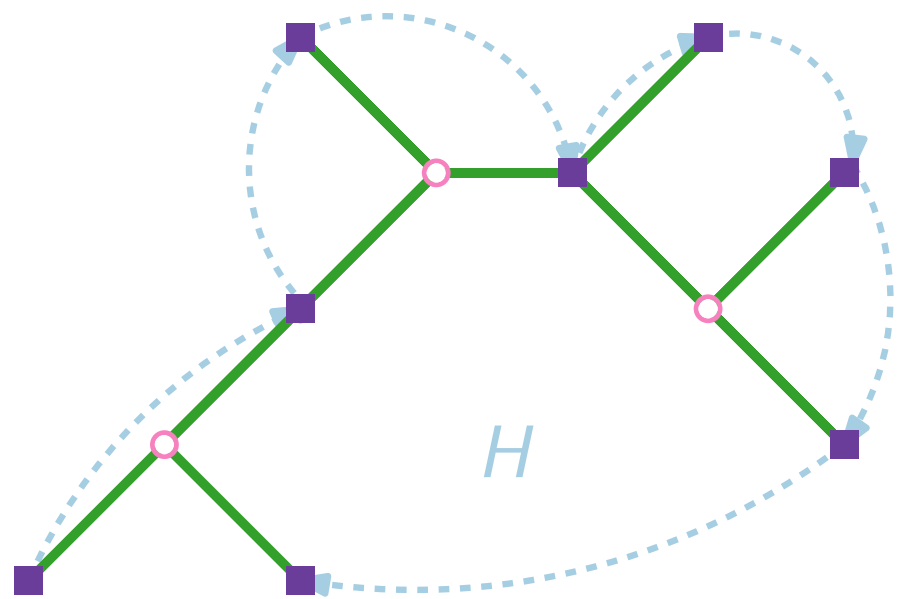

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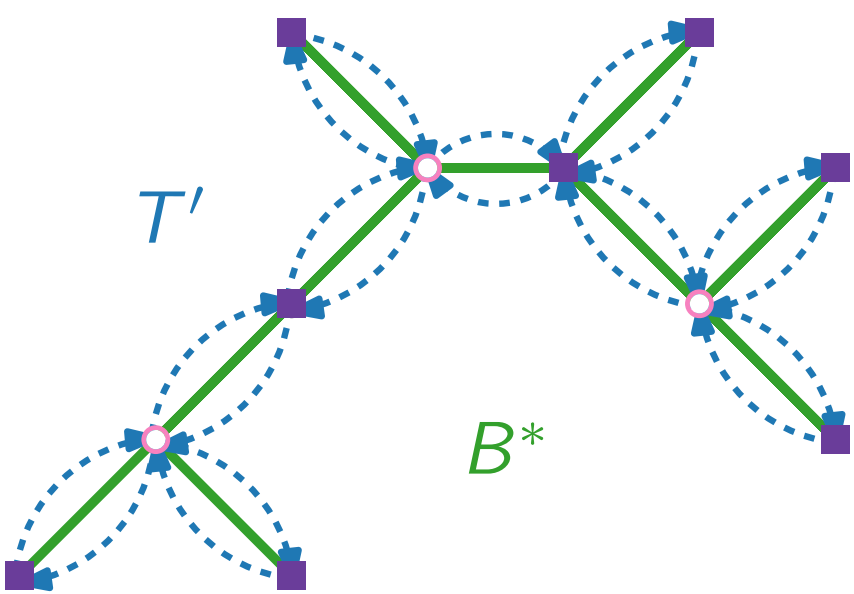

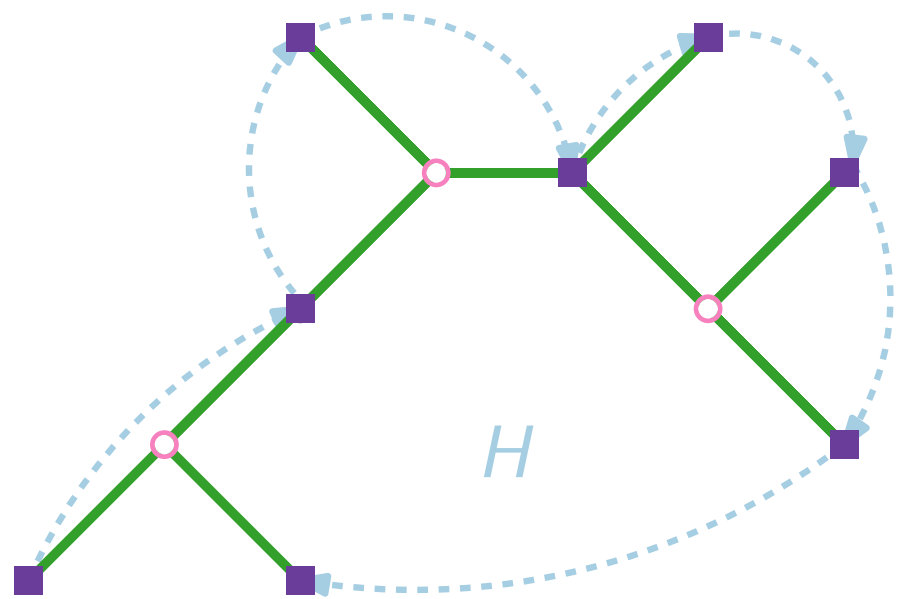

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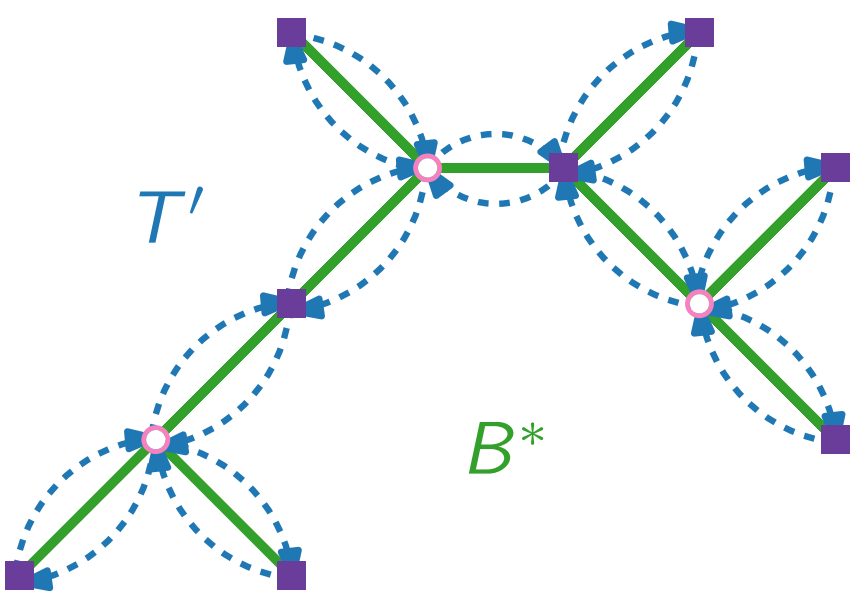

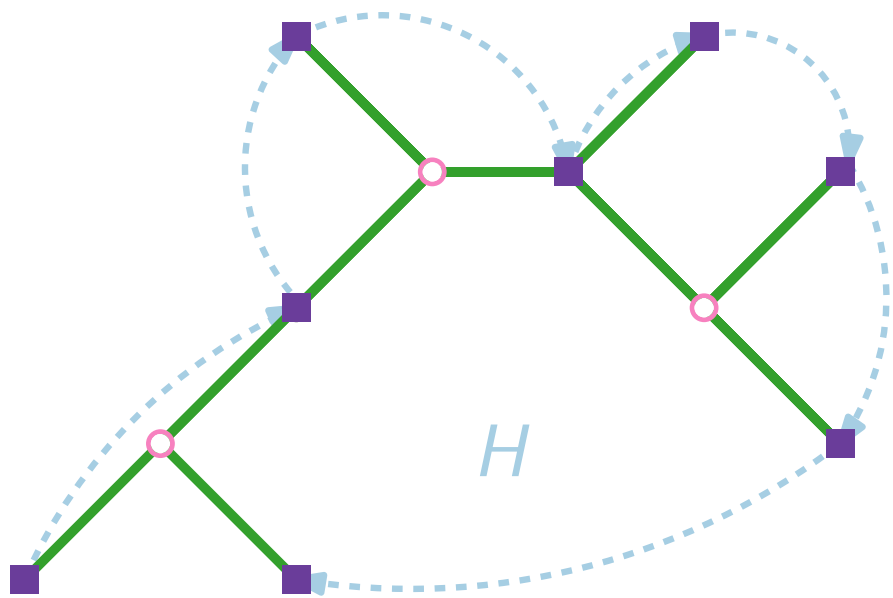

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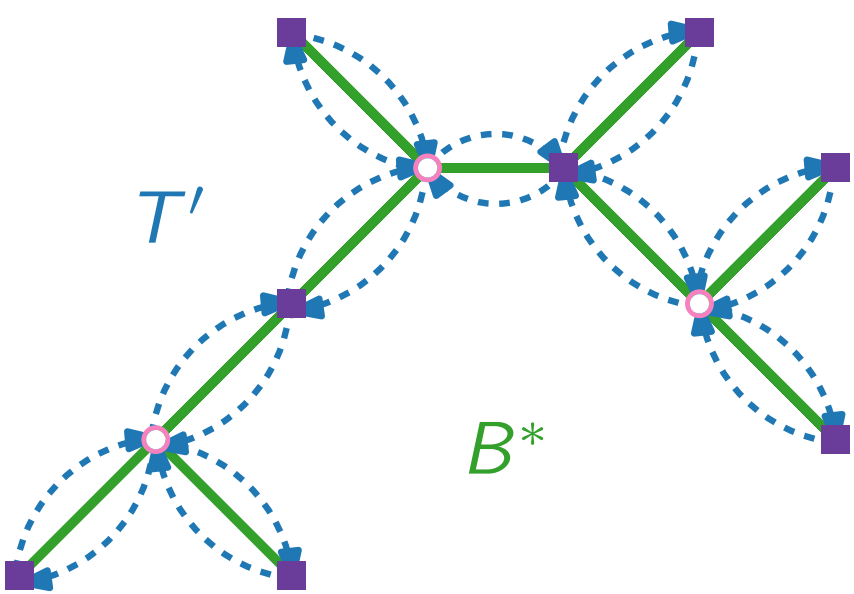
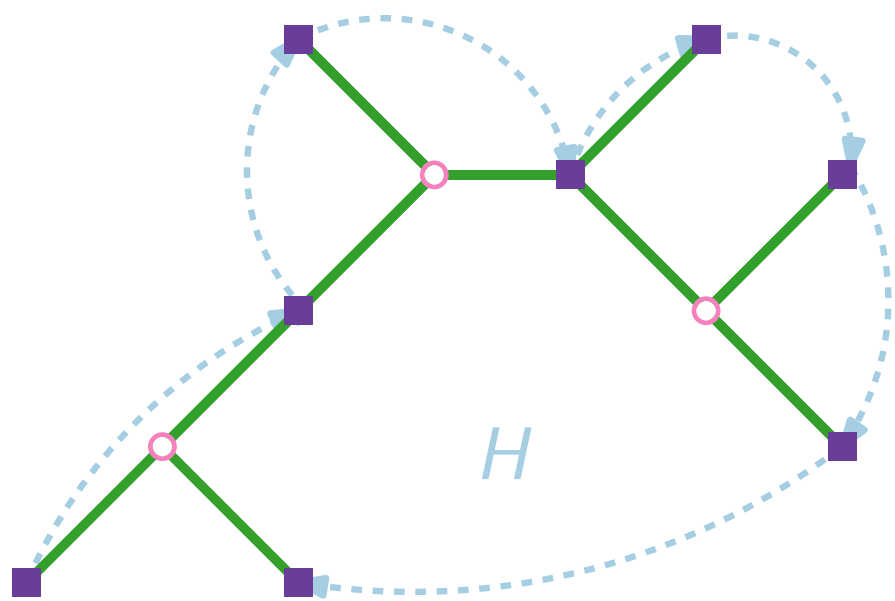


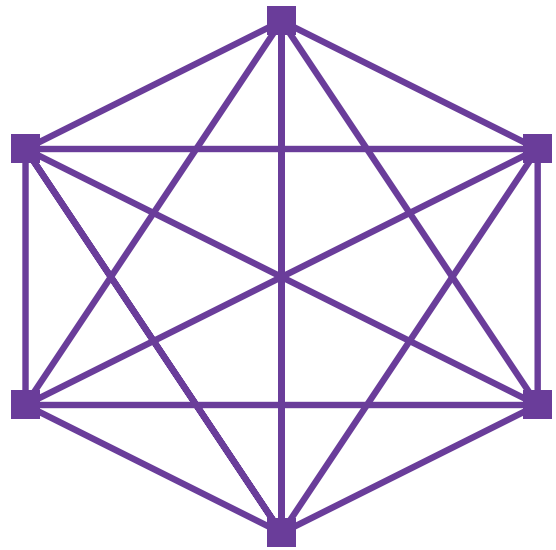


Analysis Tight?

Analysis Tight?

- terminal

K_n



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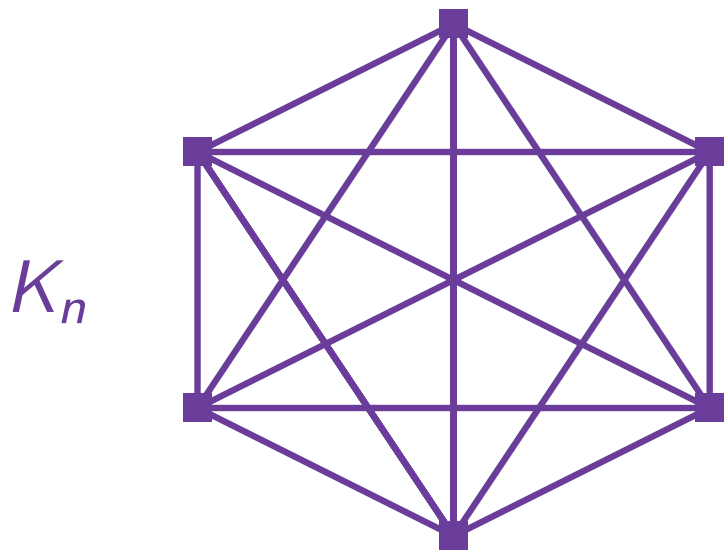
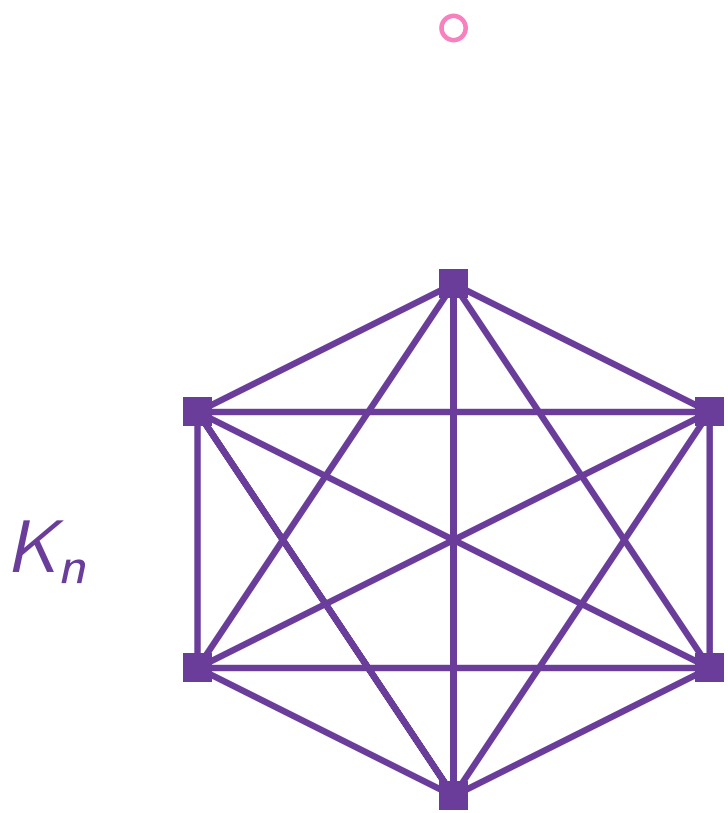


terminal

cost 2

Analysis Tight?

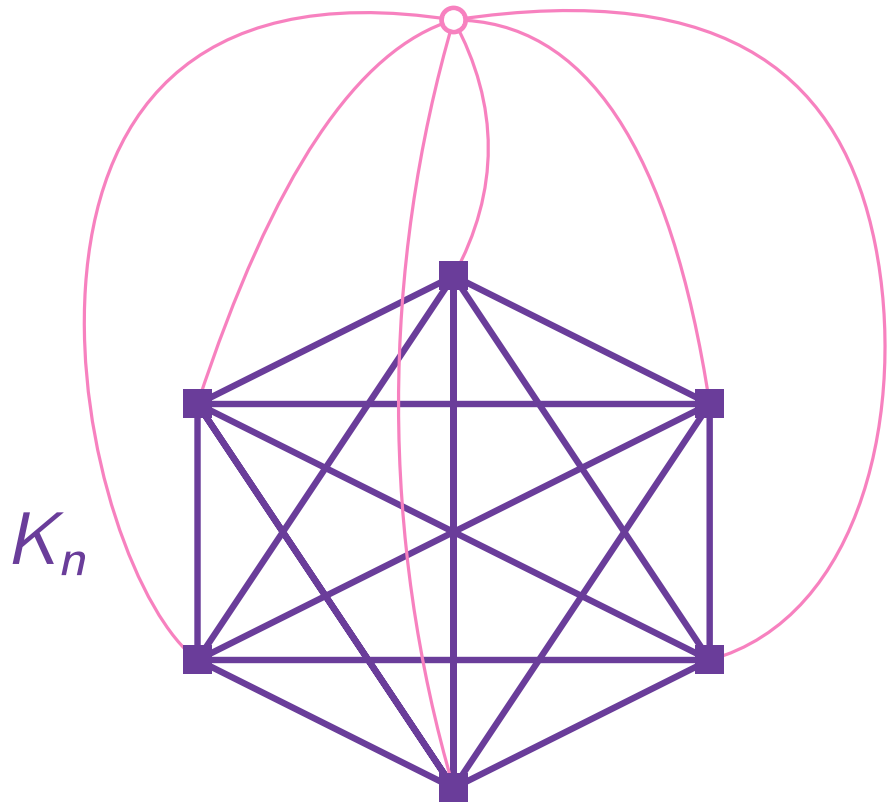


terminal

Steiner vertex

cost 2

Analysis Tight?



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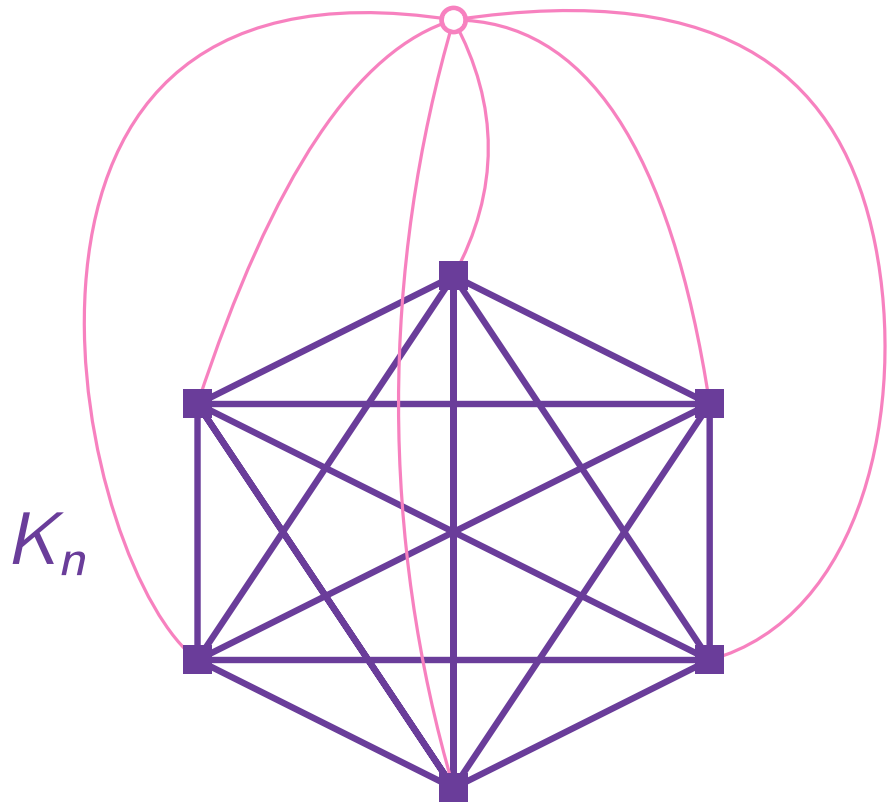


terminal

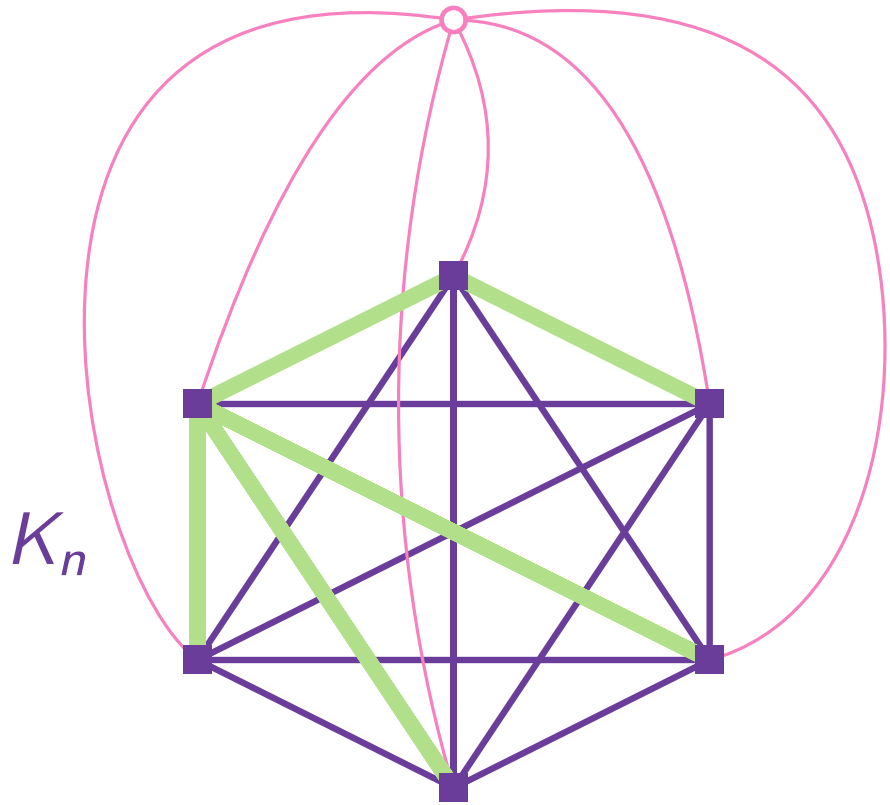
Steiner vertex

cost 1

cost 2

Analysis Tight?

MST of $G[T]$ with cost $2(n-1)$



terminal

Steiner vertex

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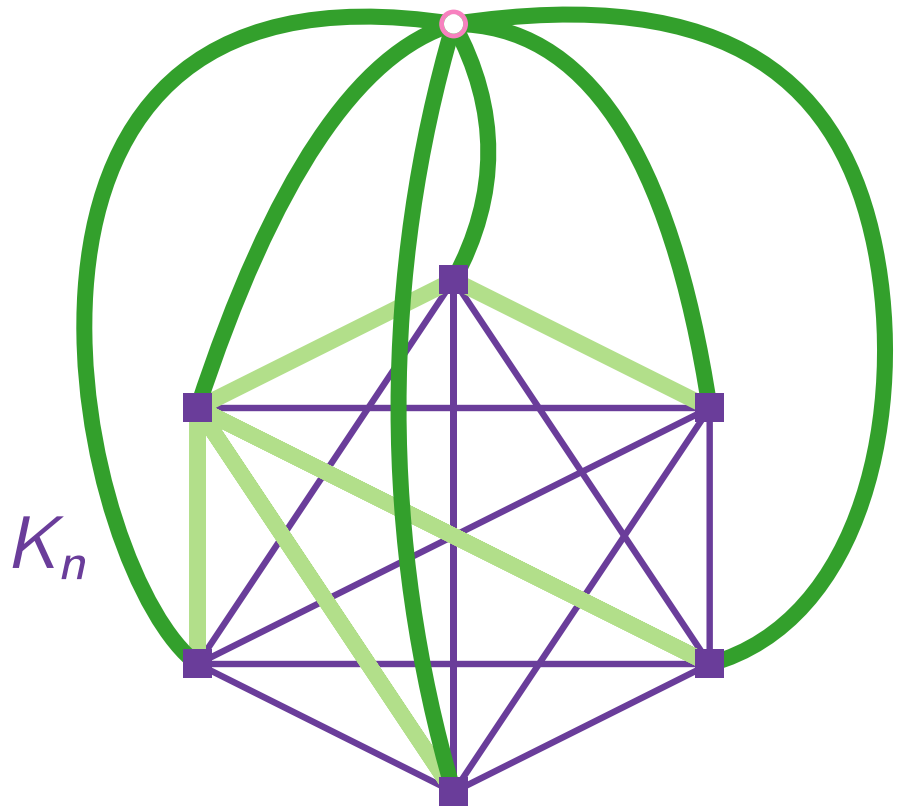
Analysis Tight?

MST of $G[T]$ with cost $2(n-1)$
Optimal solution with cost n



terminal

Steiner vertex

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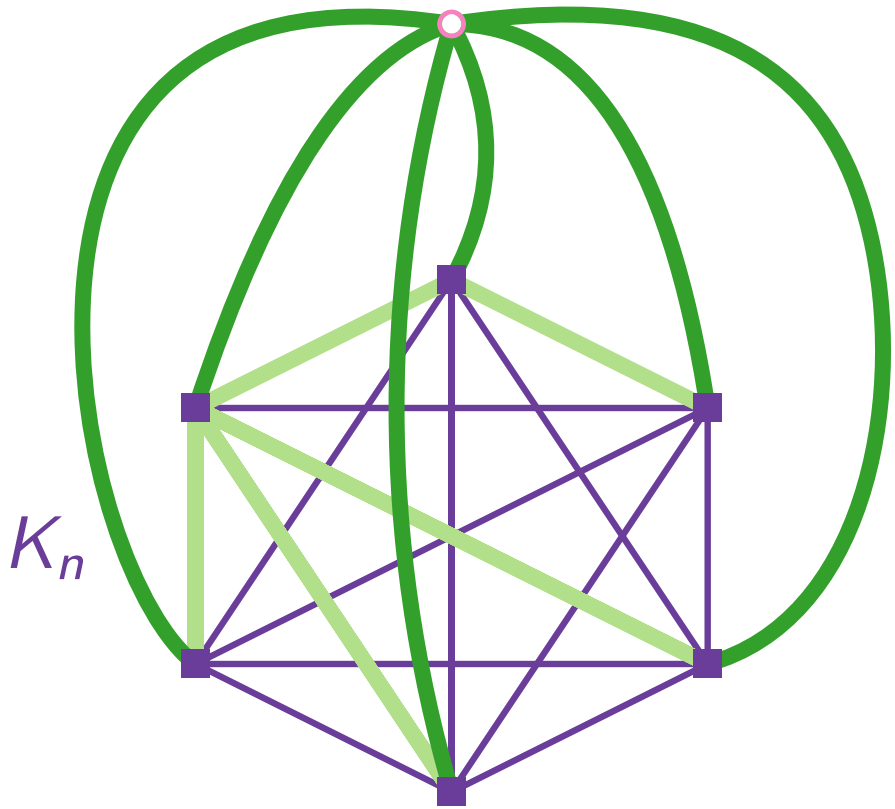
$$\frac{2(n-1)}{n} \to 2$$



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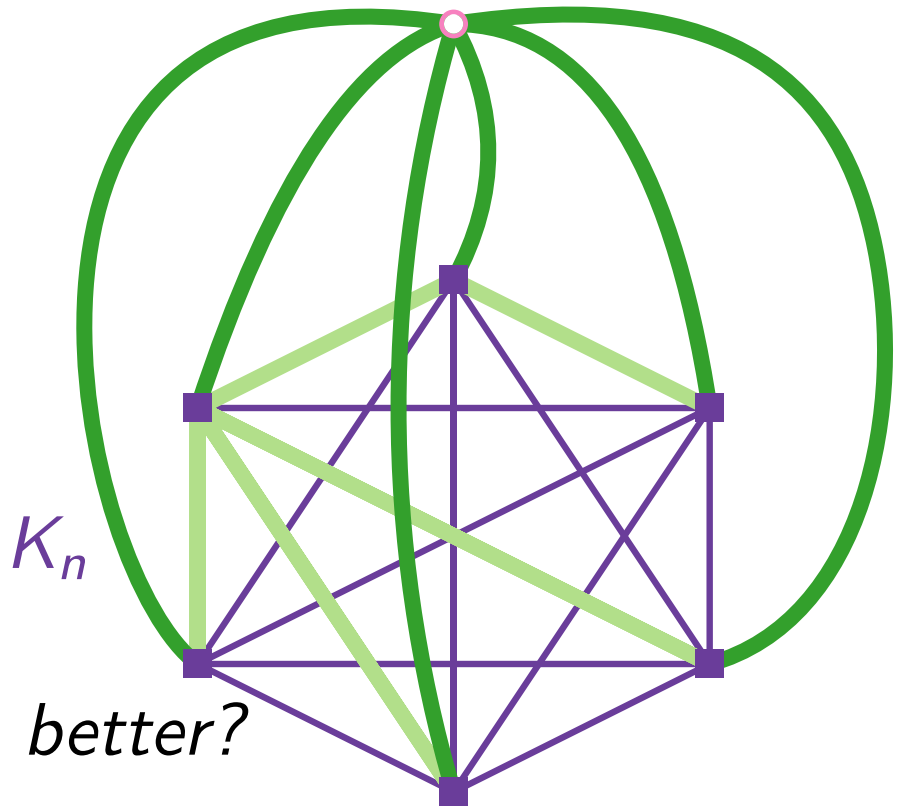

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Can we do better?

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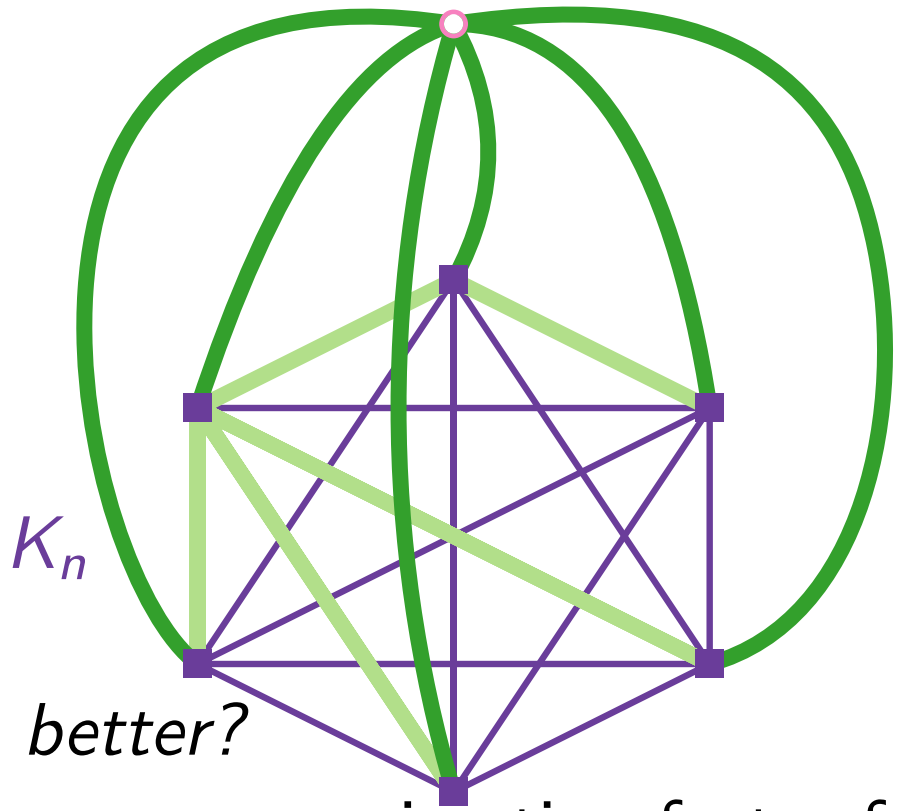

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The best known approximation factor for SteinerTree is $\ln(4) + \varepsilon \approx 1.39$

[Byrka, Grandoni, Rothvoß & Sanità, J. ACM'13]

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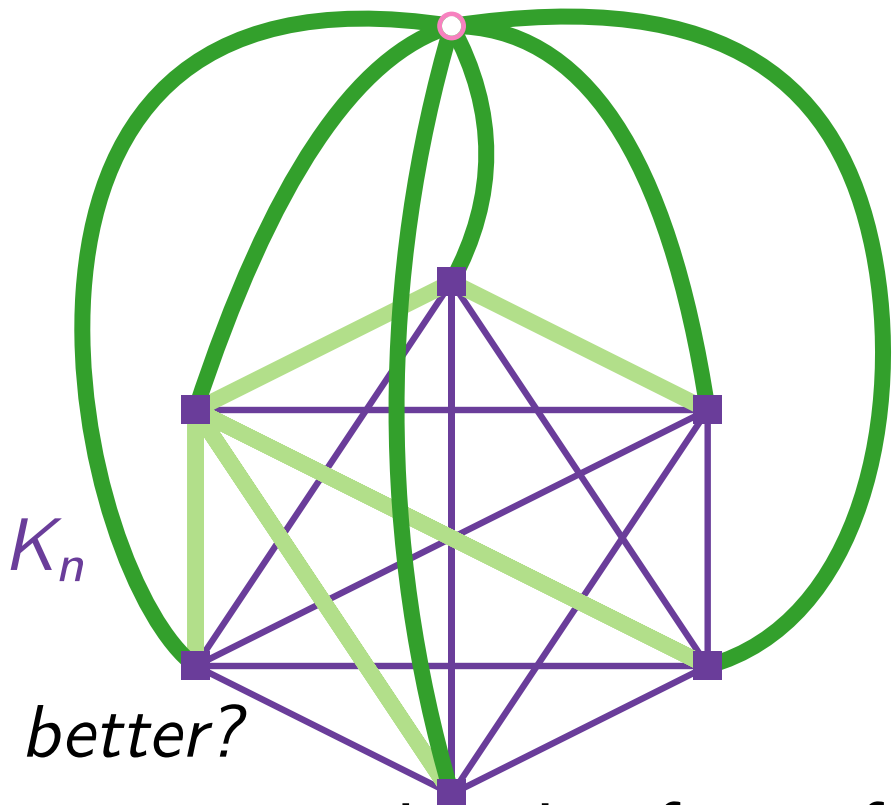

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The best known approximation factor for StEINERTREE is $\ln(4) + \varepsilon \approx 1.39$ [Byrka, Grandoni, Rothvoß & Sanità, J. ACM'13]

StEINERTREE cannot be approximated within factor $\frac{96}{95} \approx 1.0105$ (unless P=NP) [Chlebík & Chlebíková, TCS'08]

Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part V:
MultiwayCut

Alexander Wolff

Winter 2022/23

MultiwayCut

Given: A connected graph G

MultiwayCut

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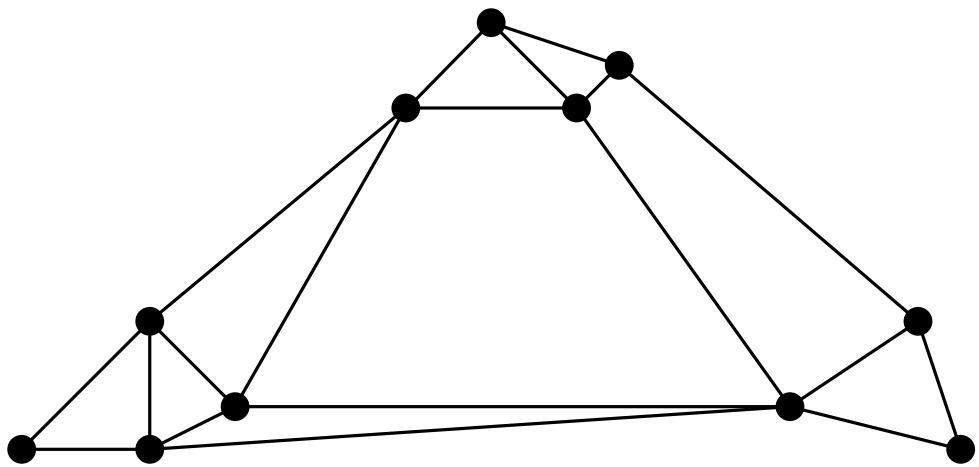
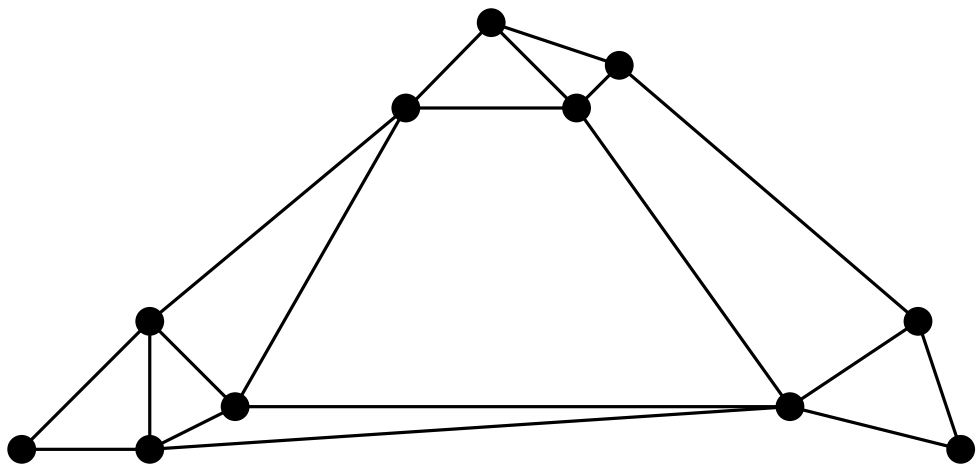


MultiwayCut

Given: A connected graph G with edge costs $c\colon E(G) \to \mathbb{Q}^+$



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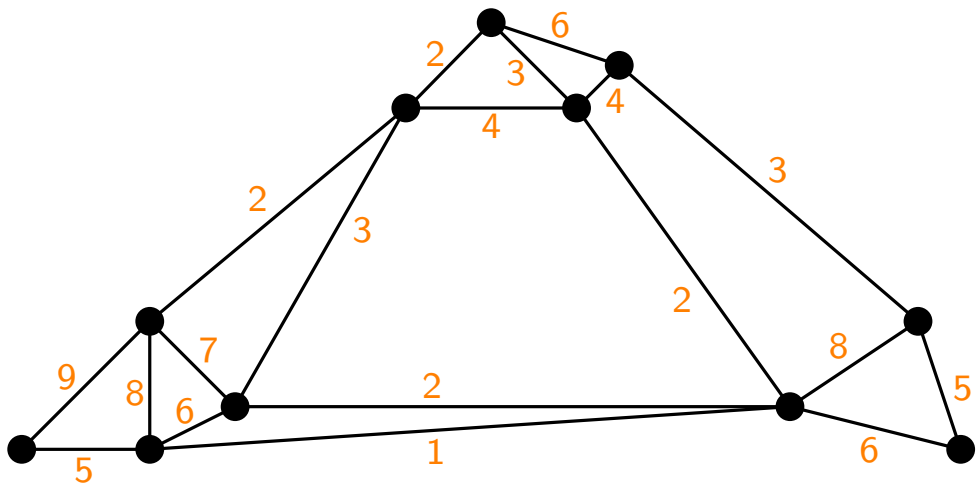

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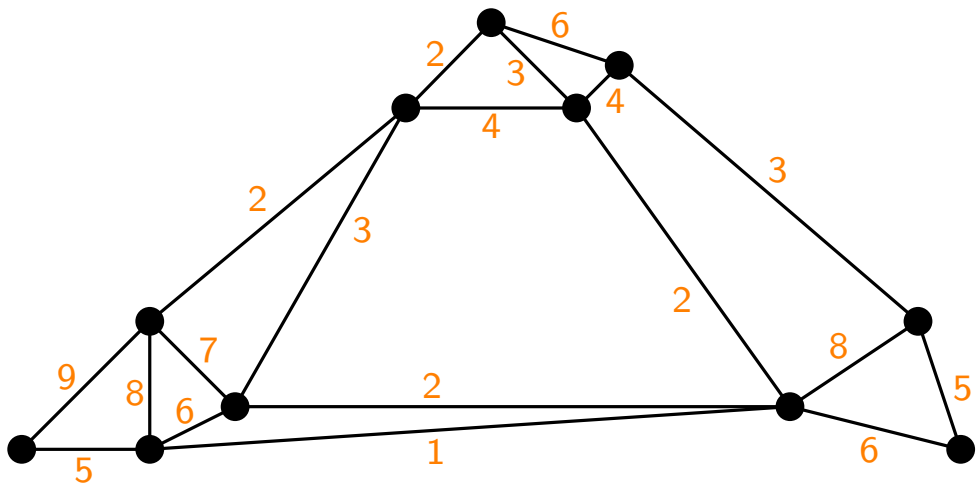

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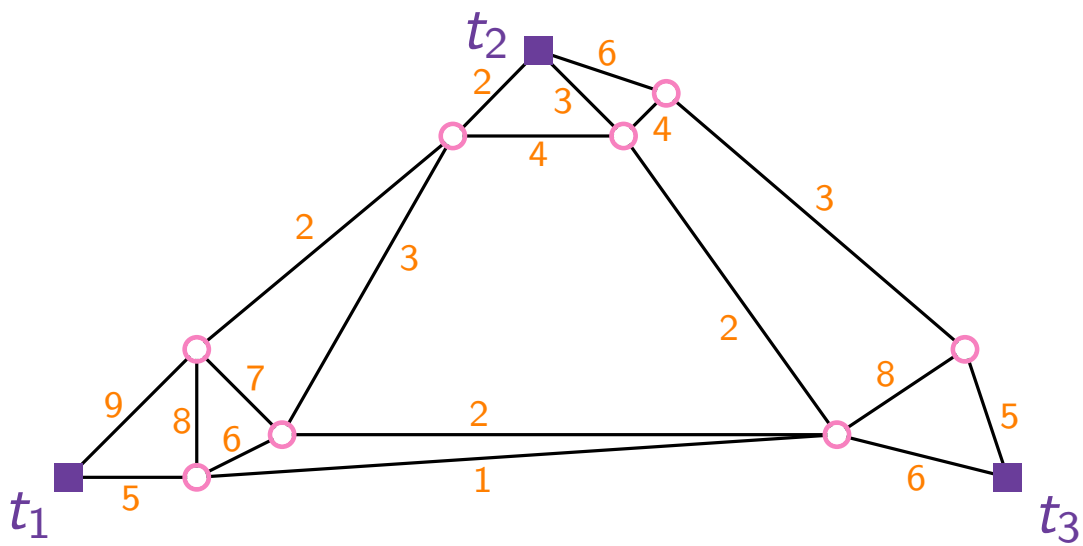

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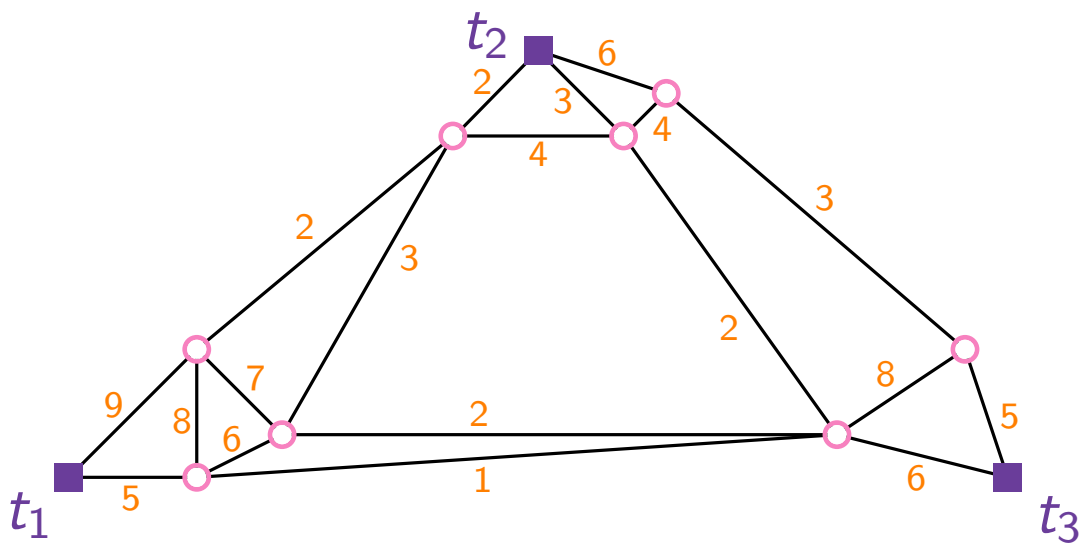

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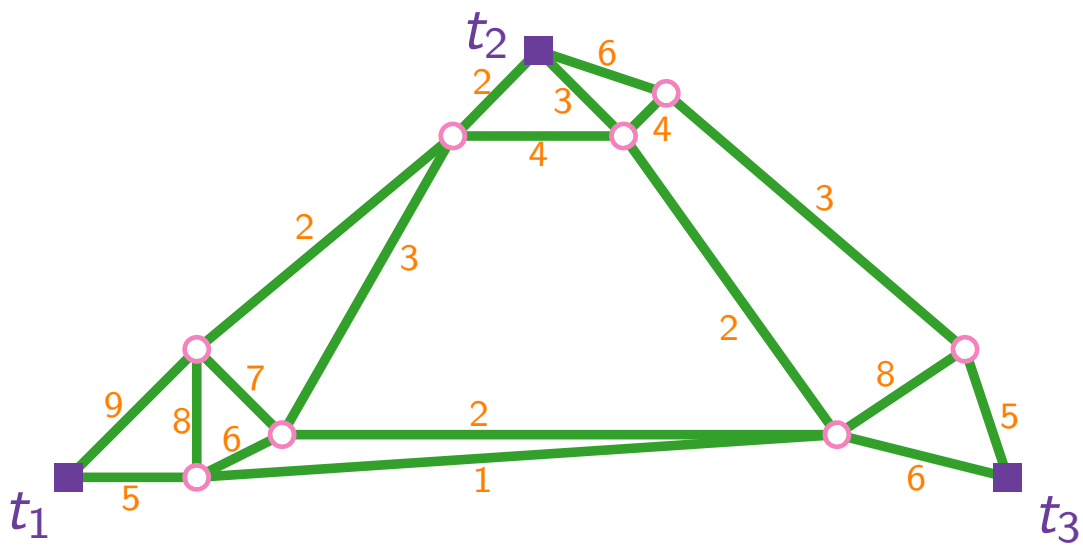

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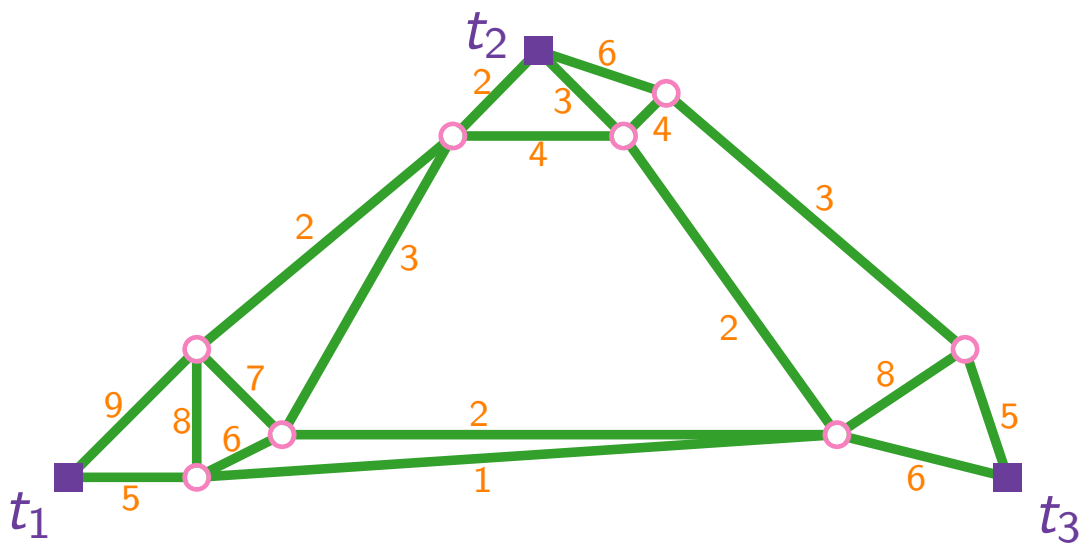

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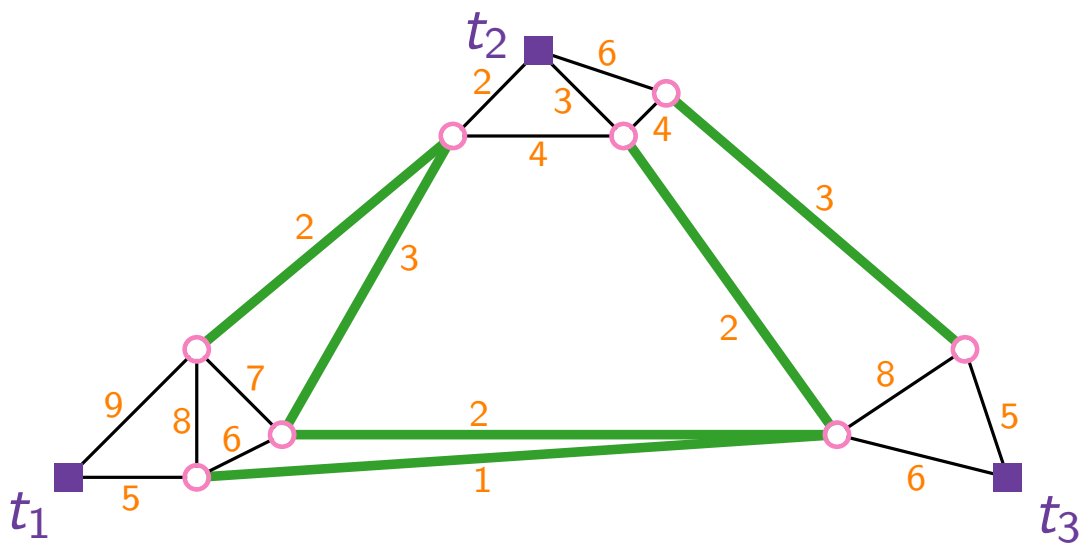

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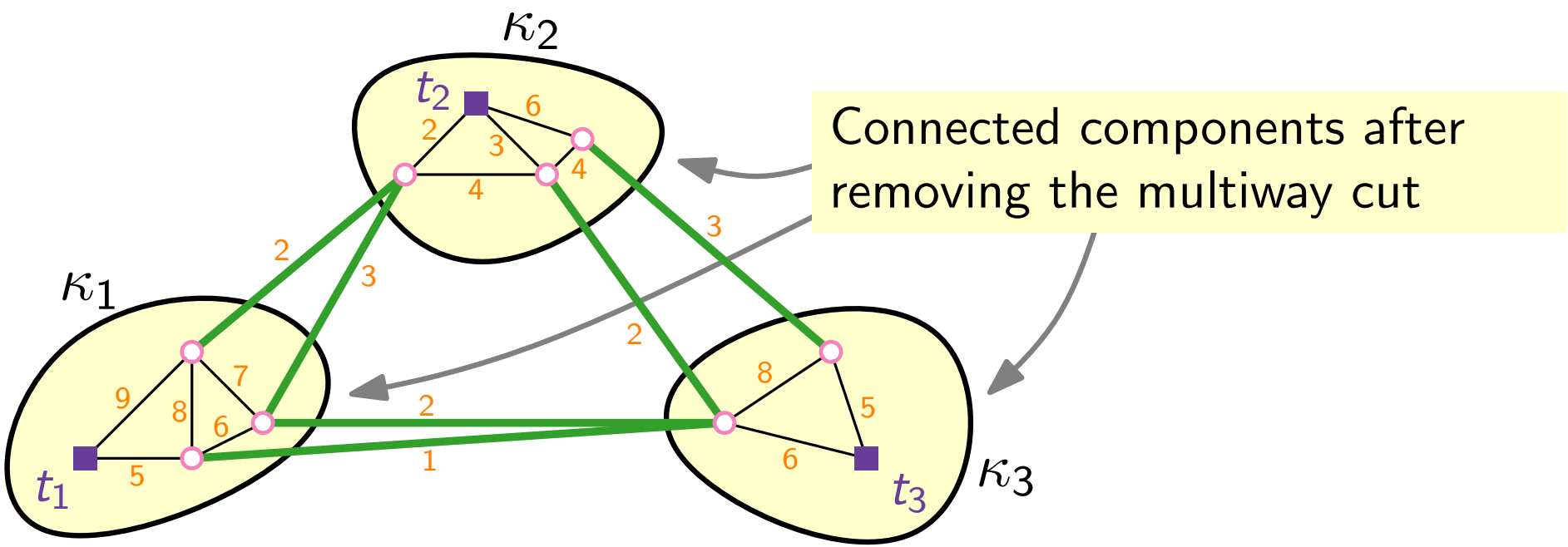

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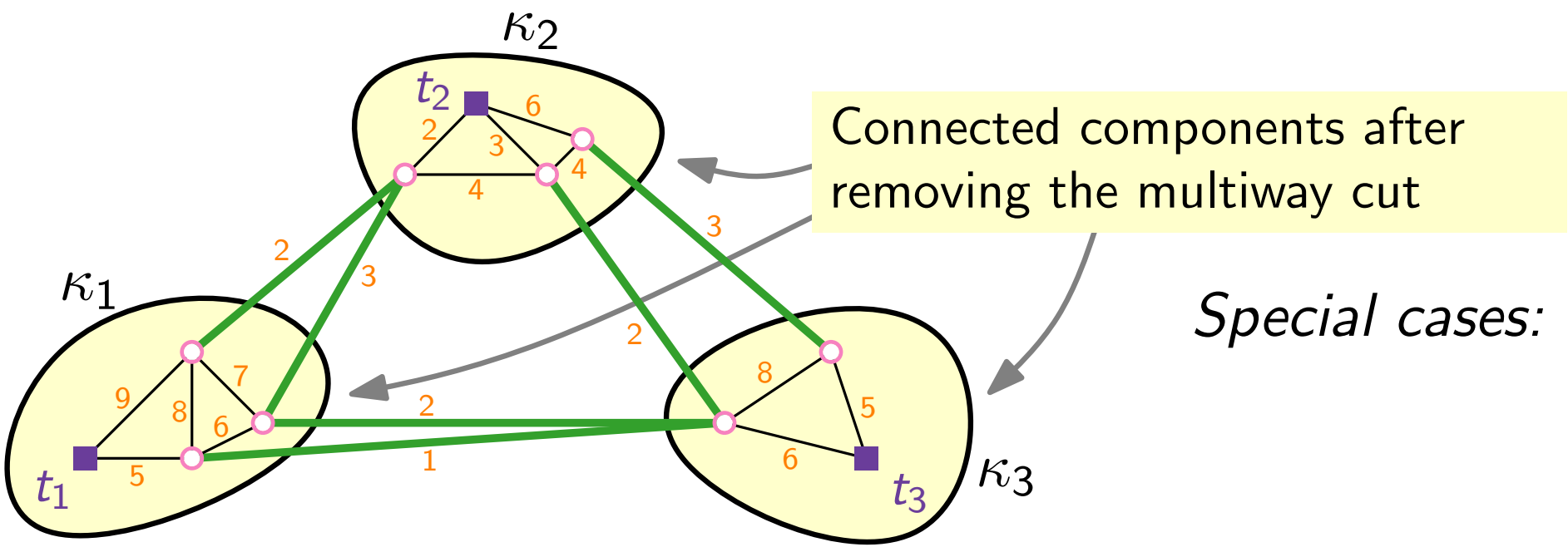

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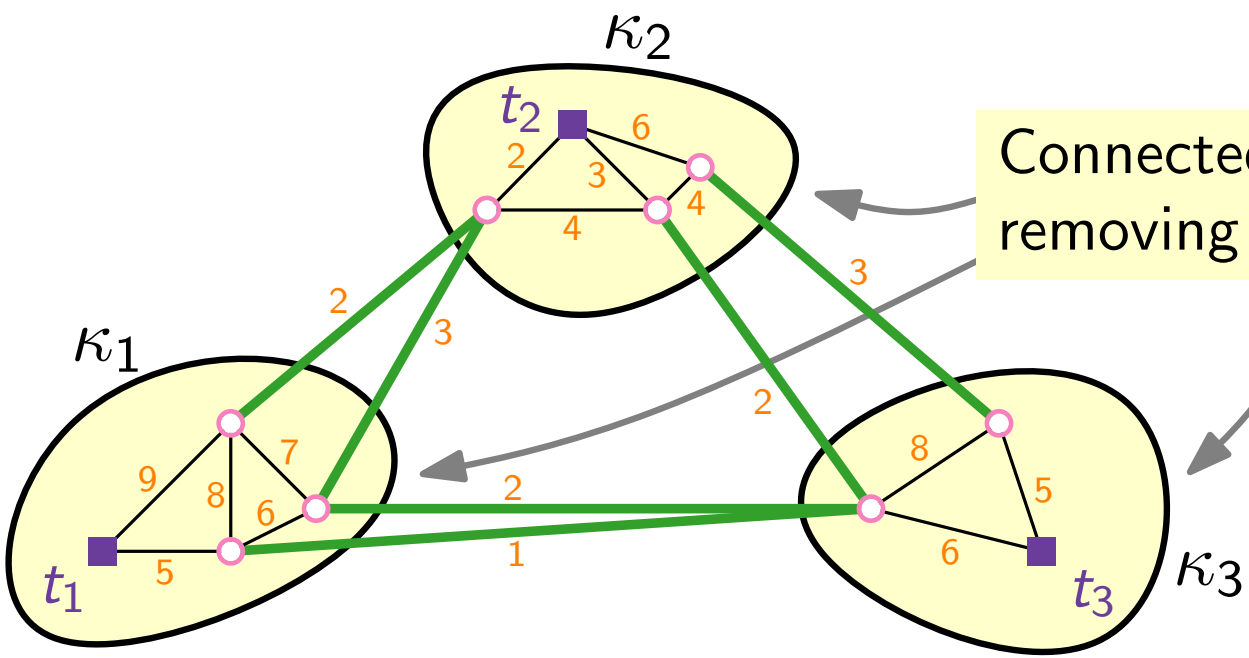

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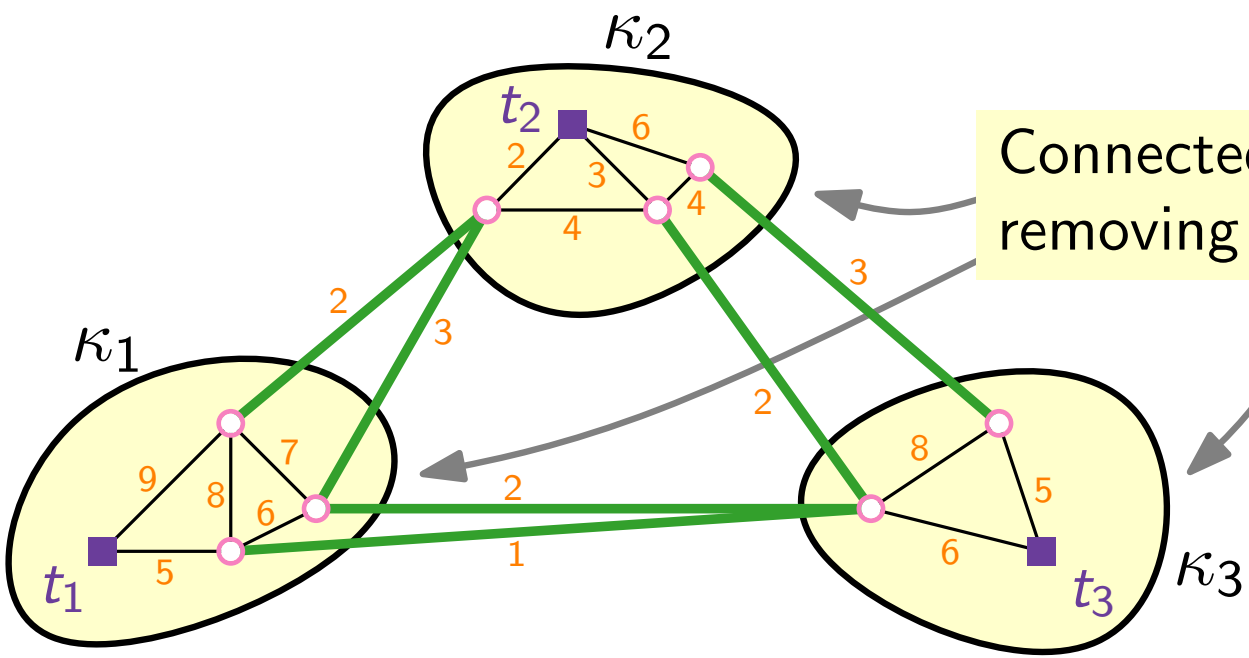

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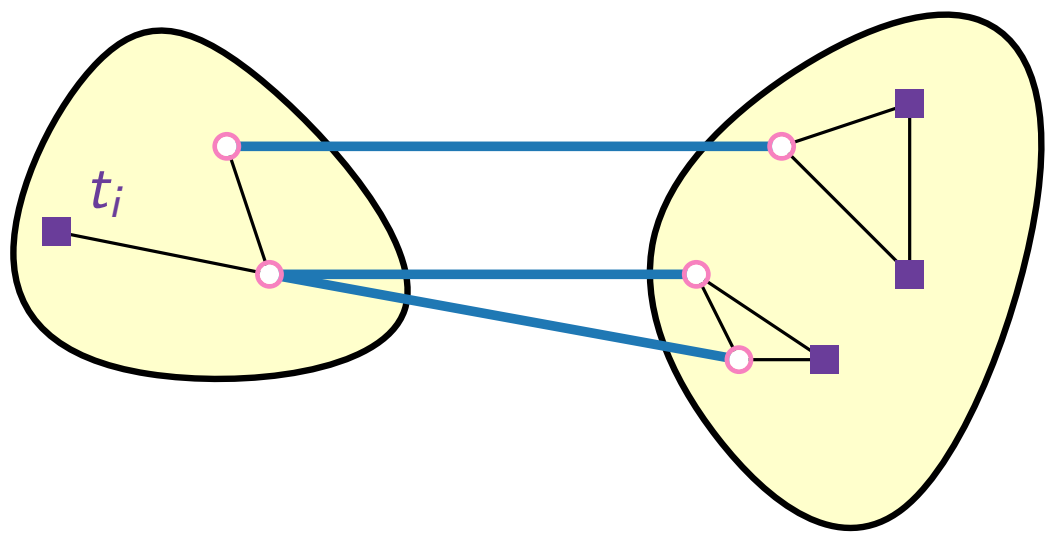

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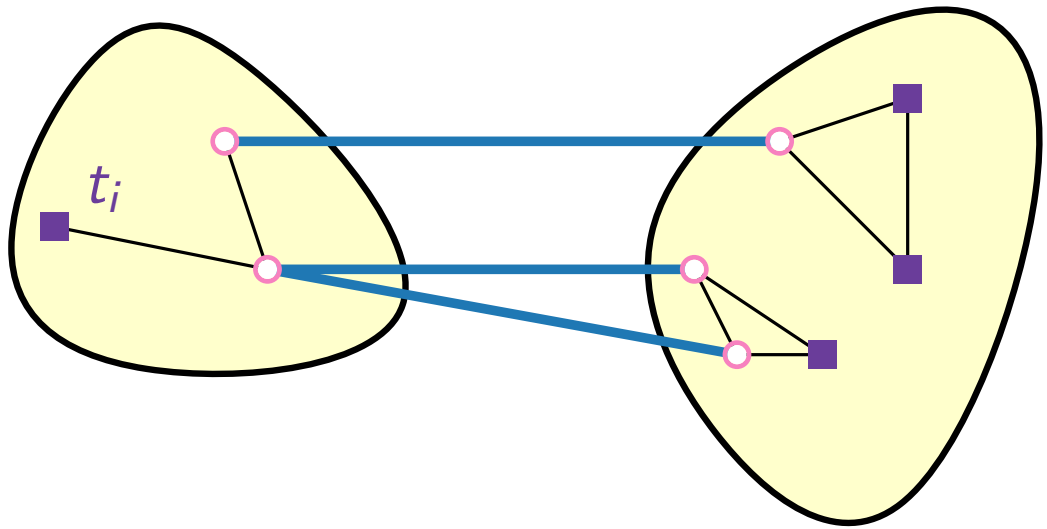

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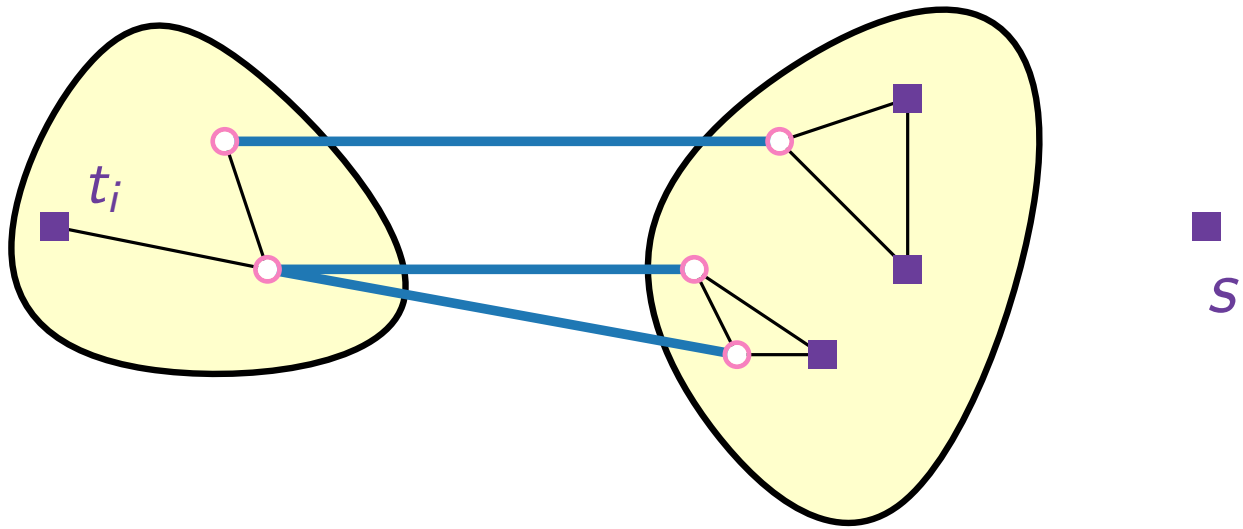

Add dummy terminal s

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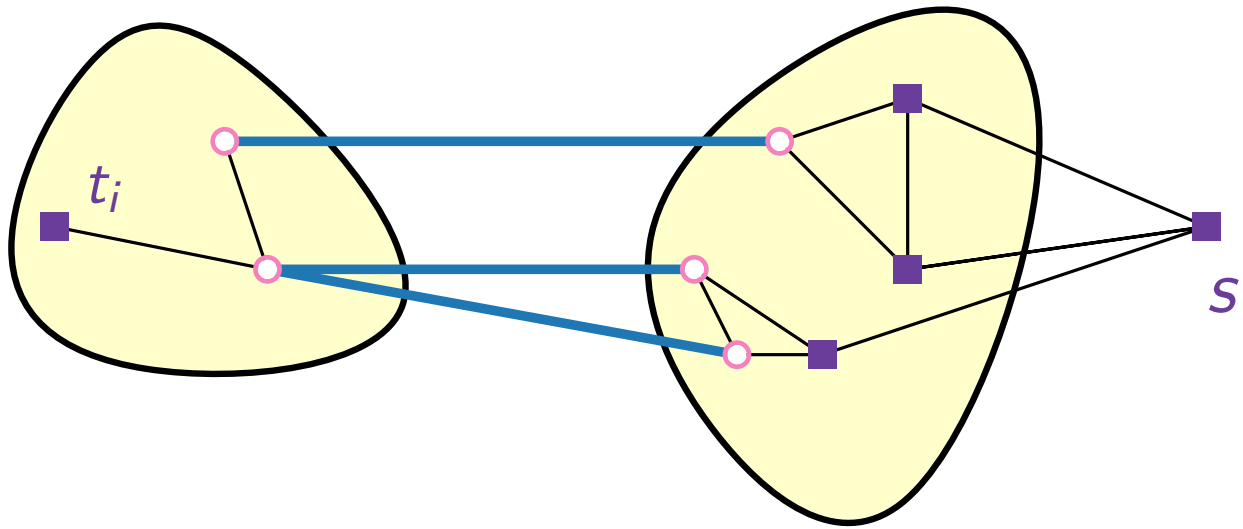

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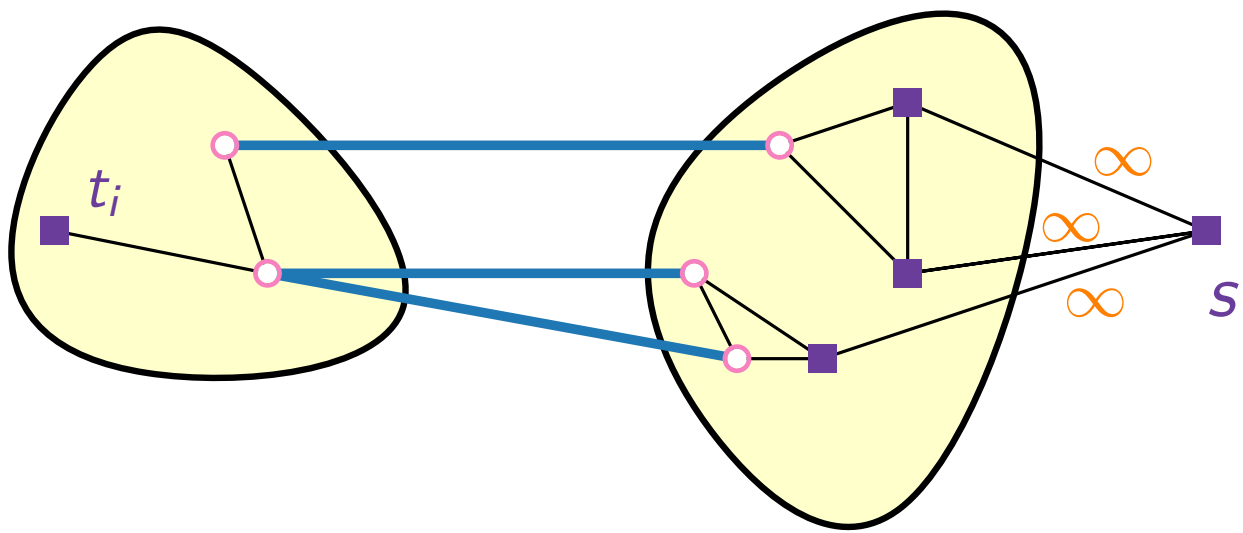

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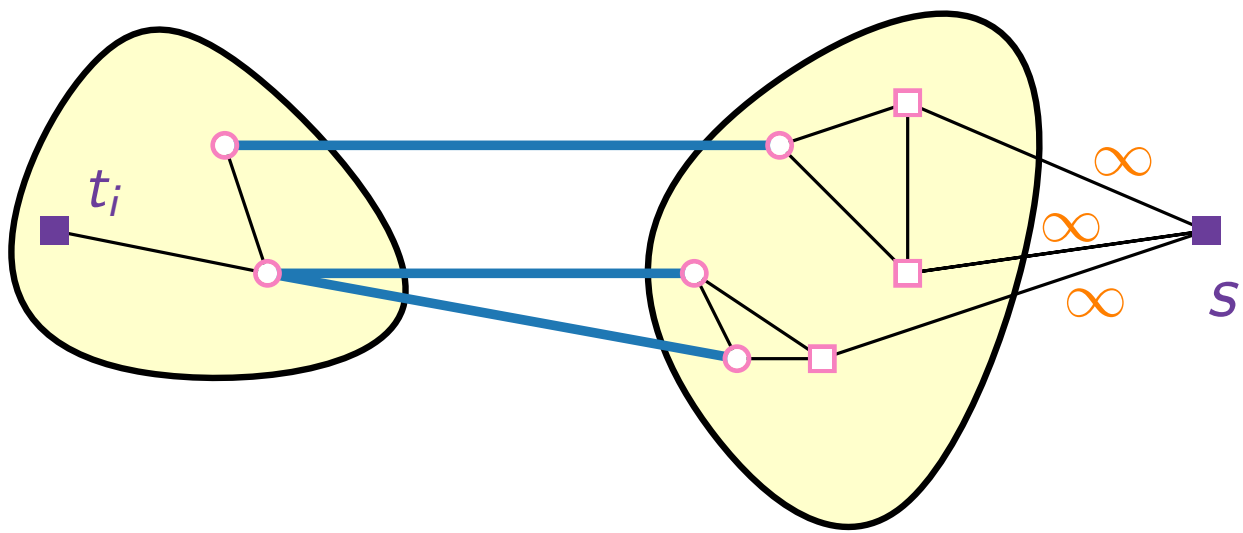

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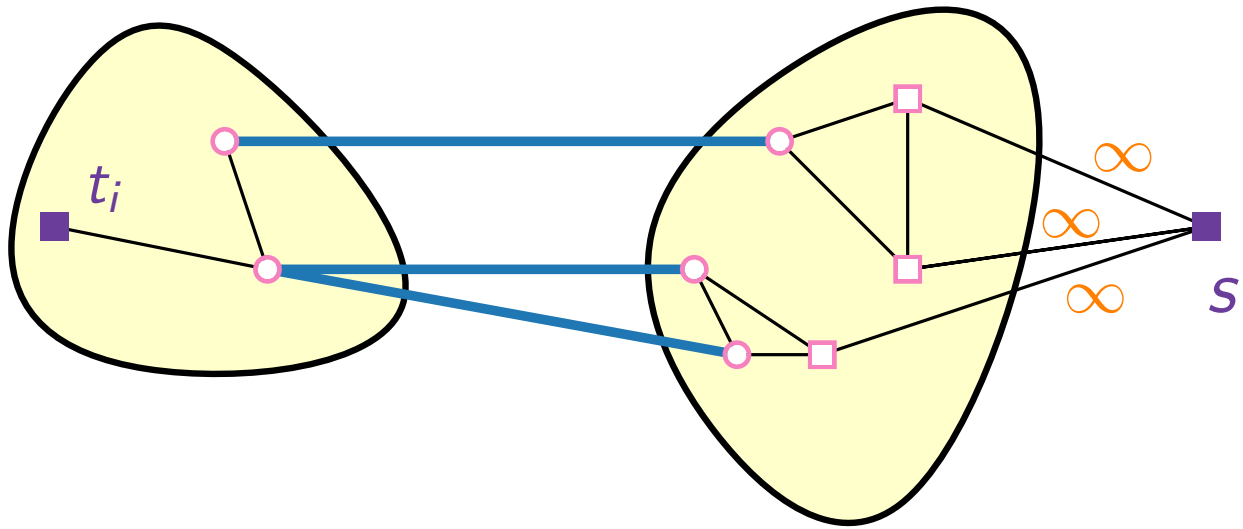

Add dummy terminal s and find a minimum-cost s–t_i cut.

Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part VI:
Algorithm for MultiwayCut

Alexander Wolff

Winter 2022/23

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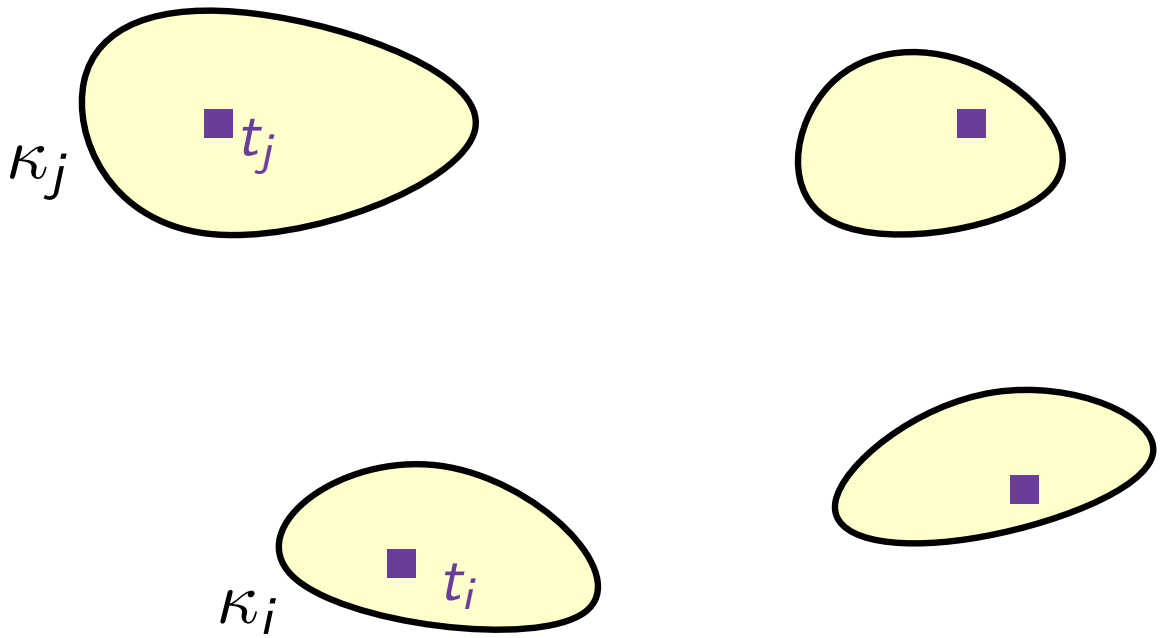

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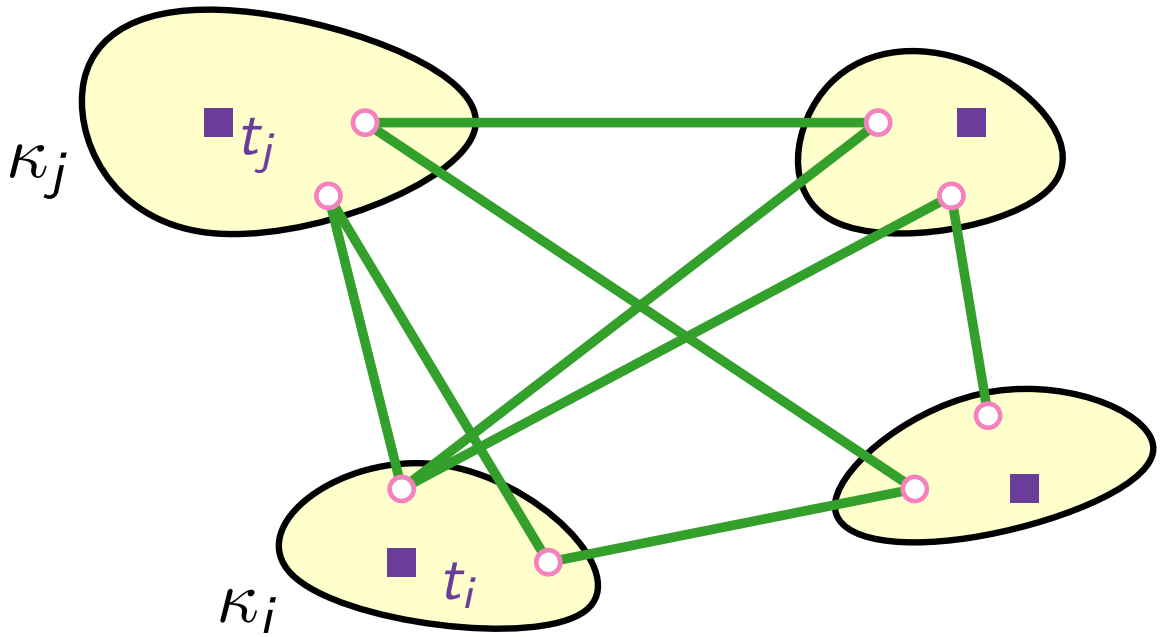

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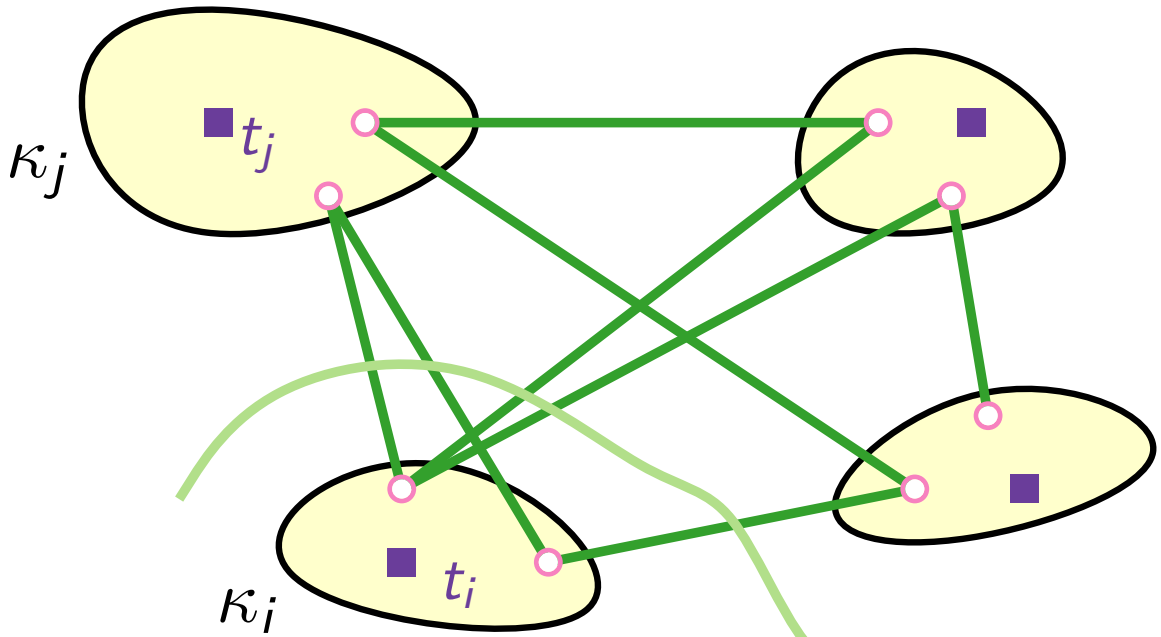

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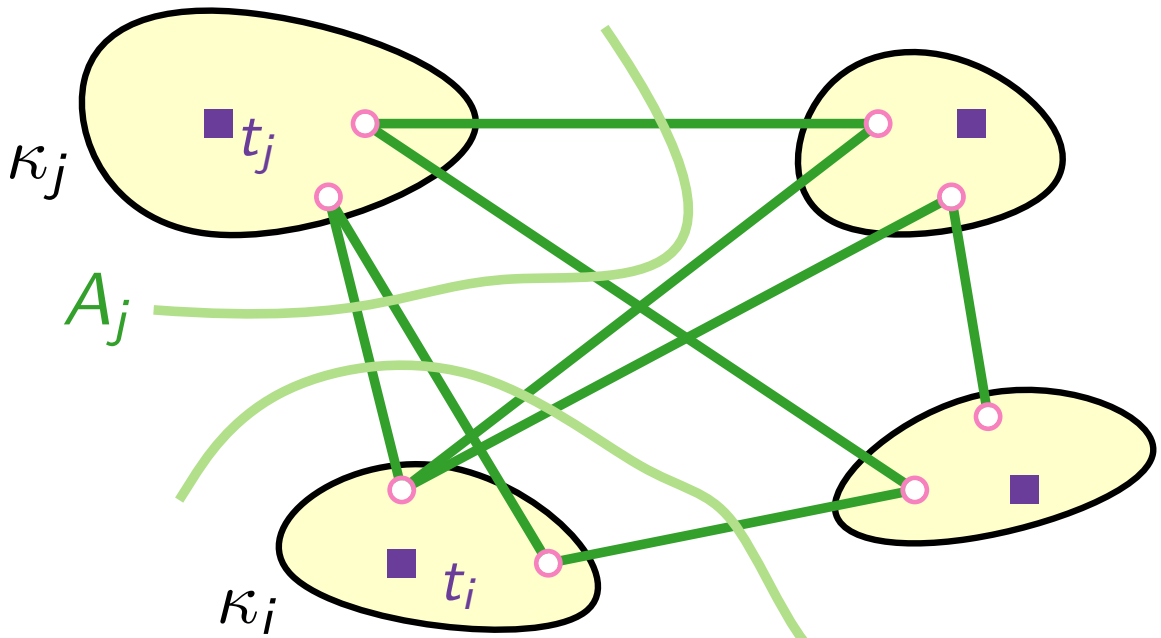

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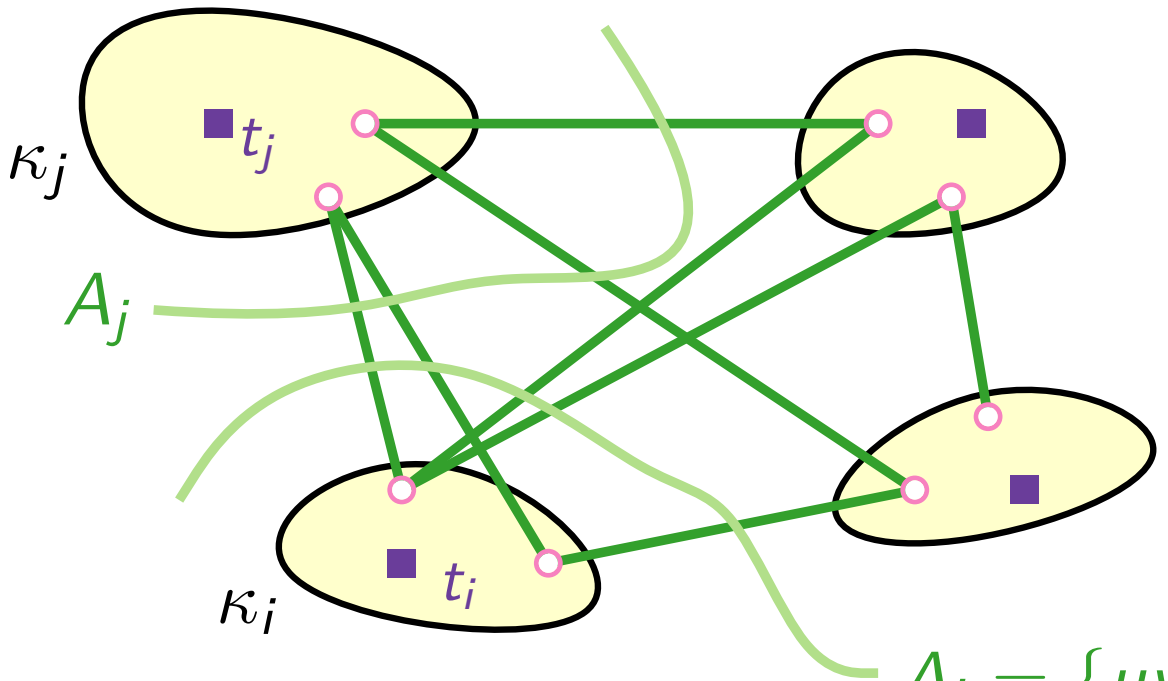

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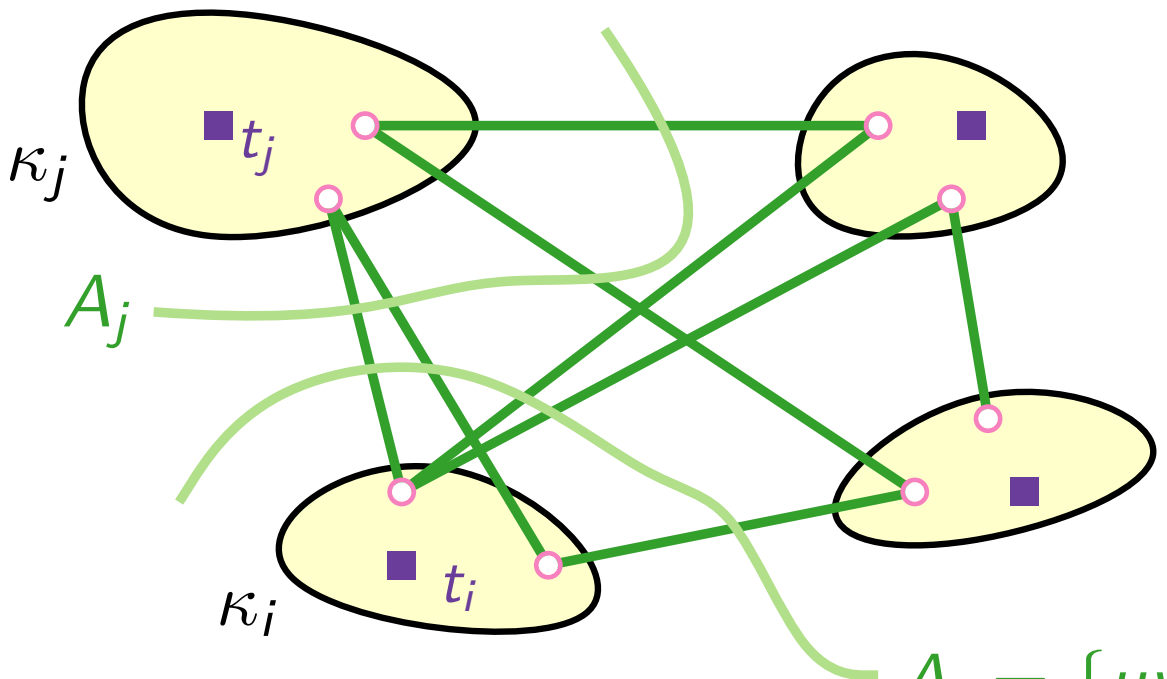

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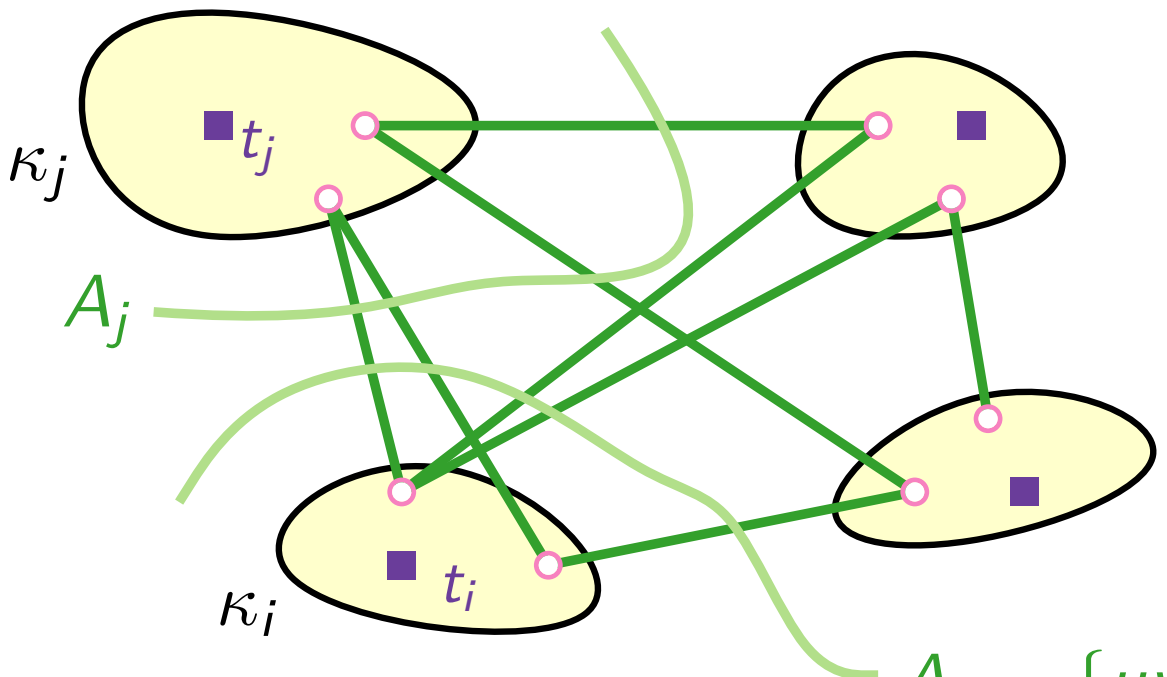

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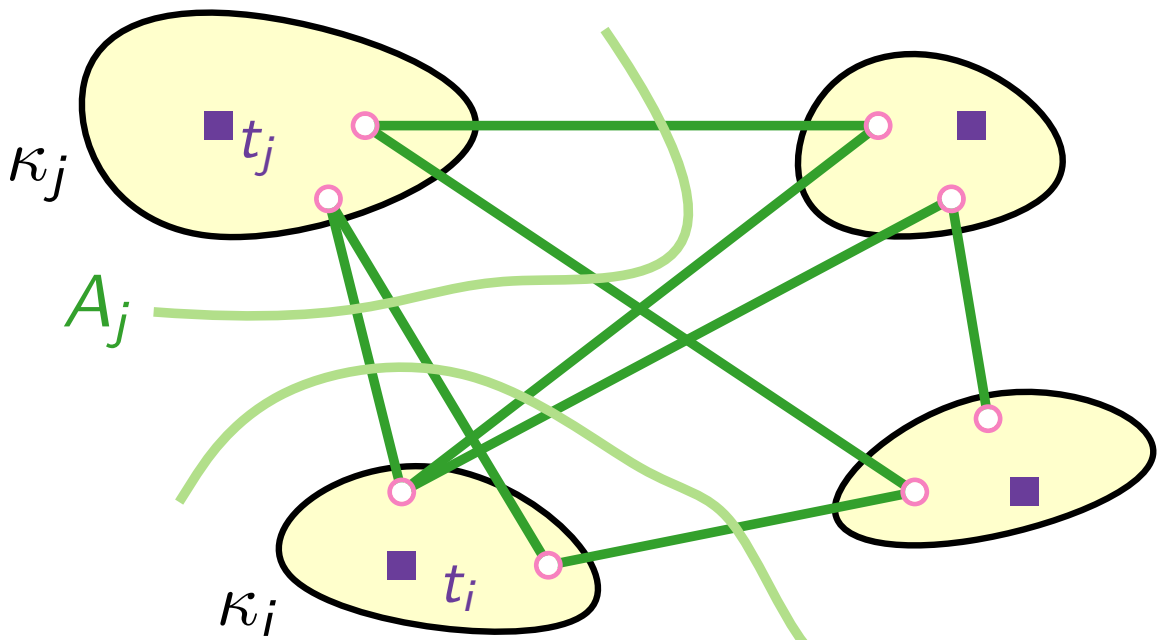

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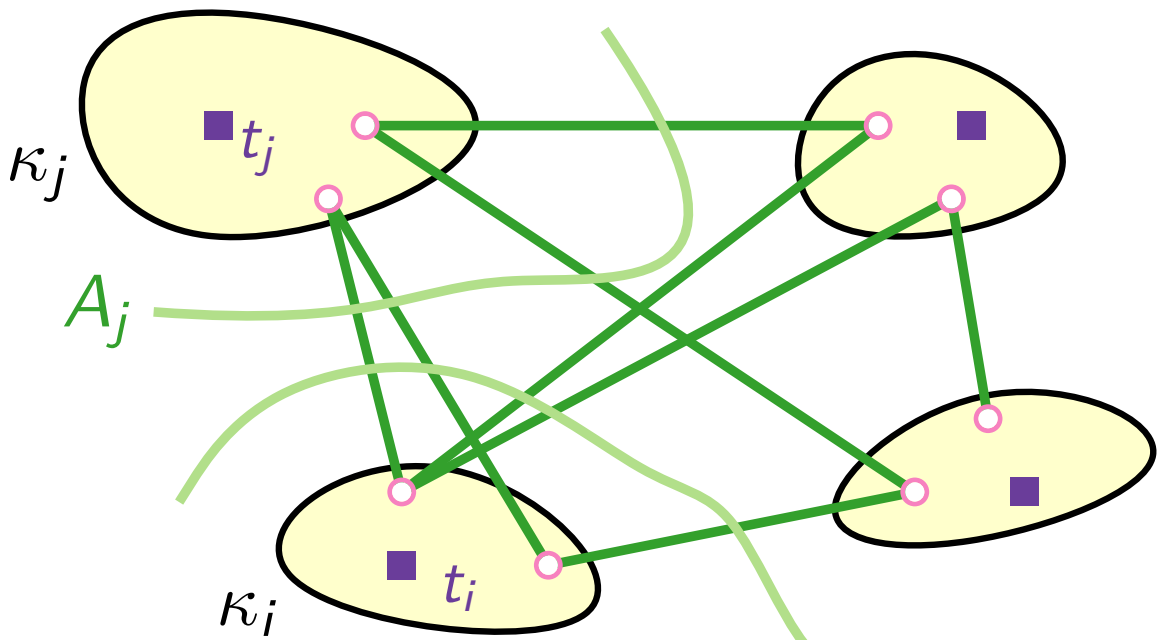

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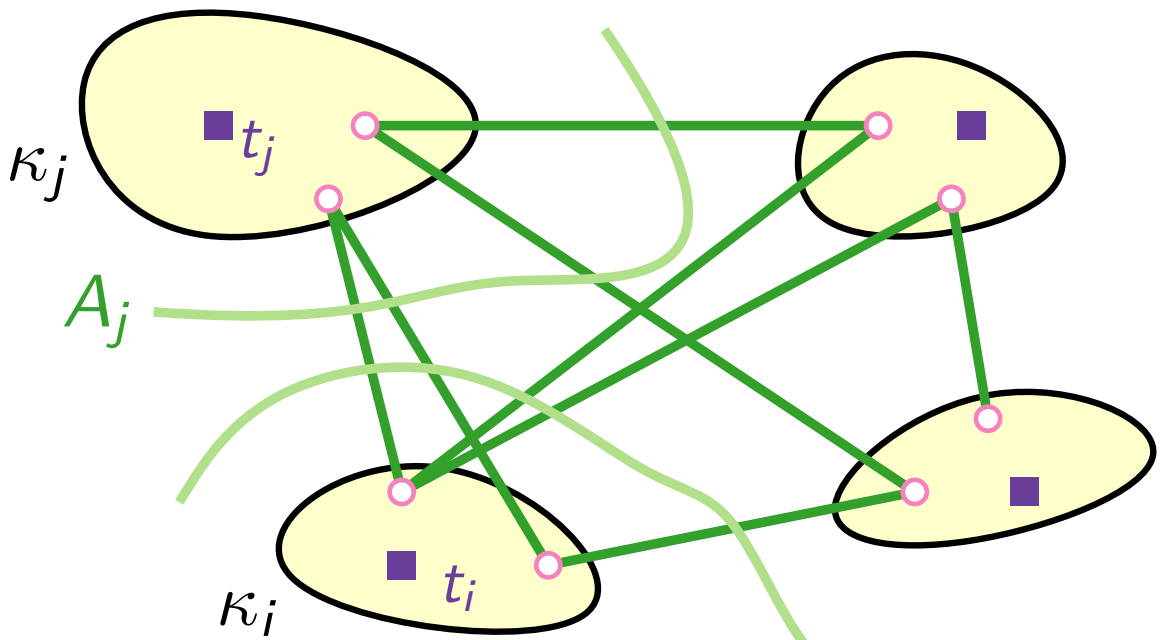

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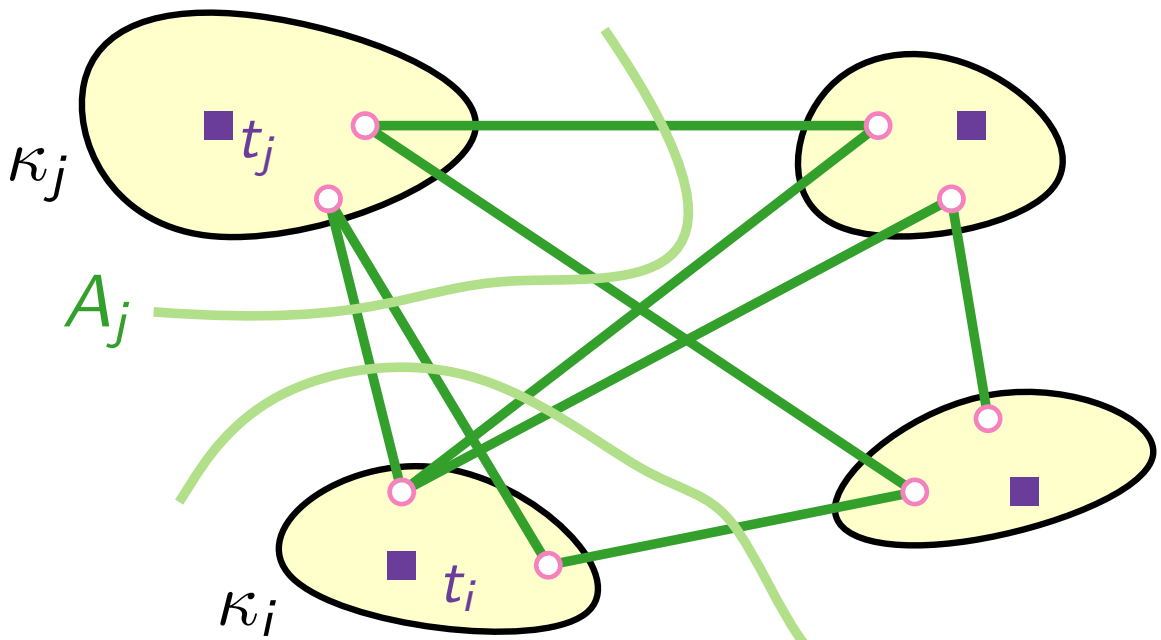

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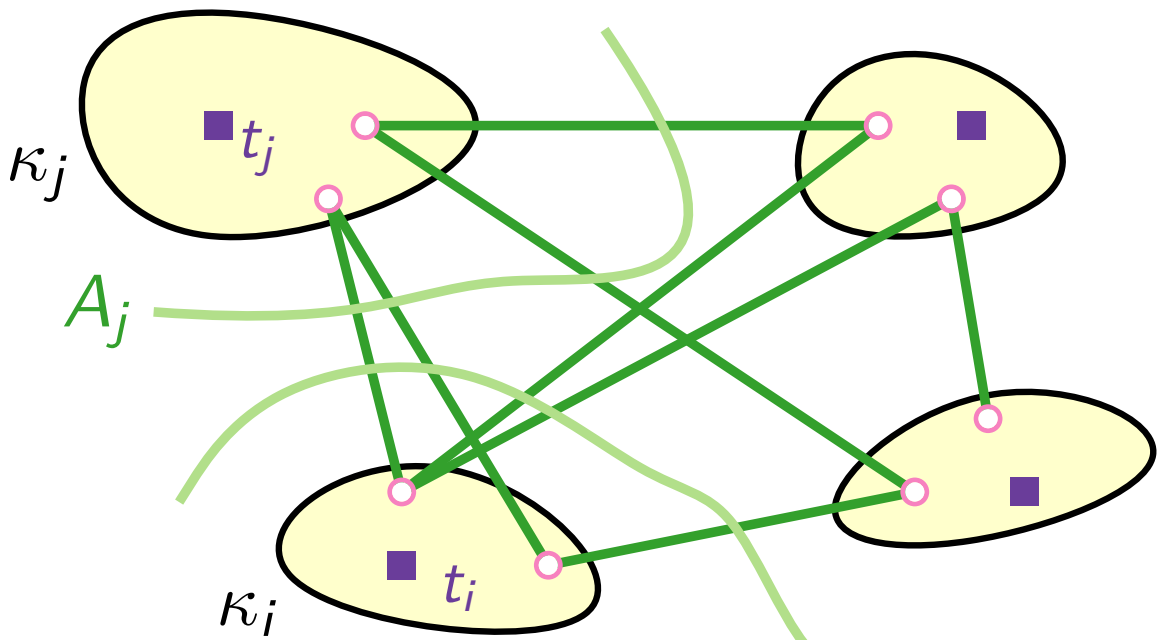

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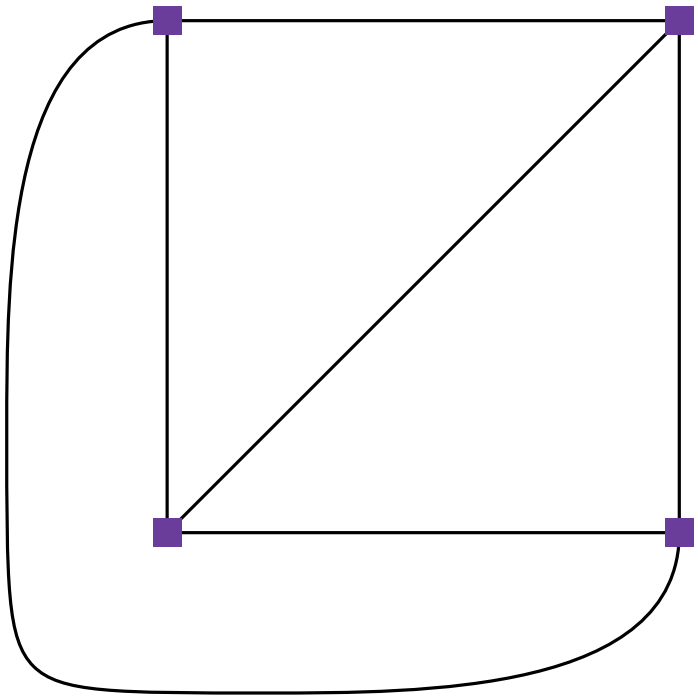

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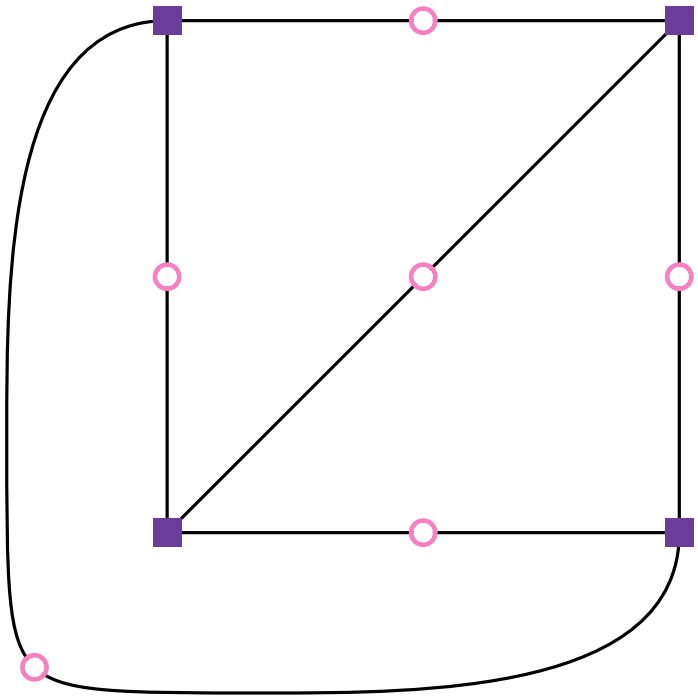

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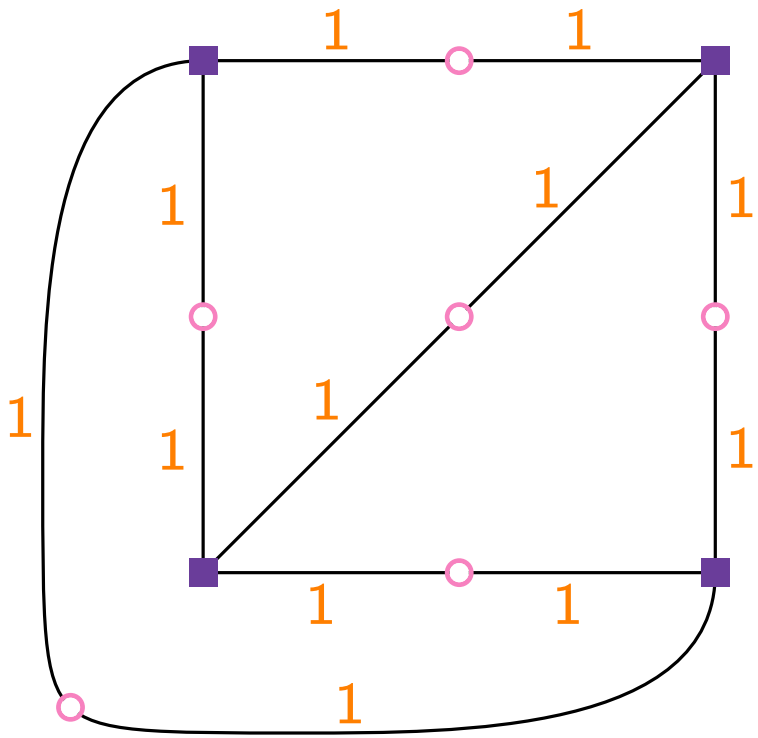

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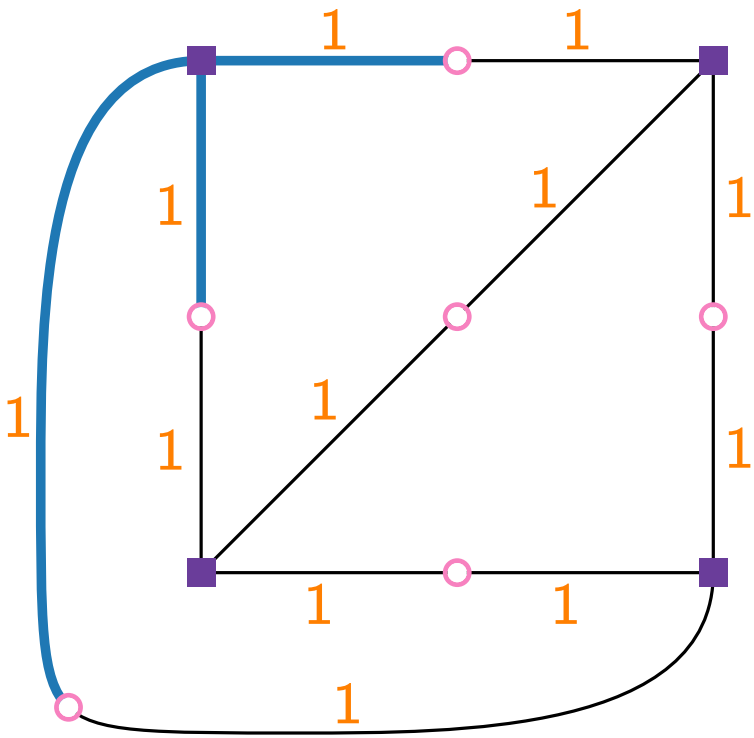

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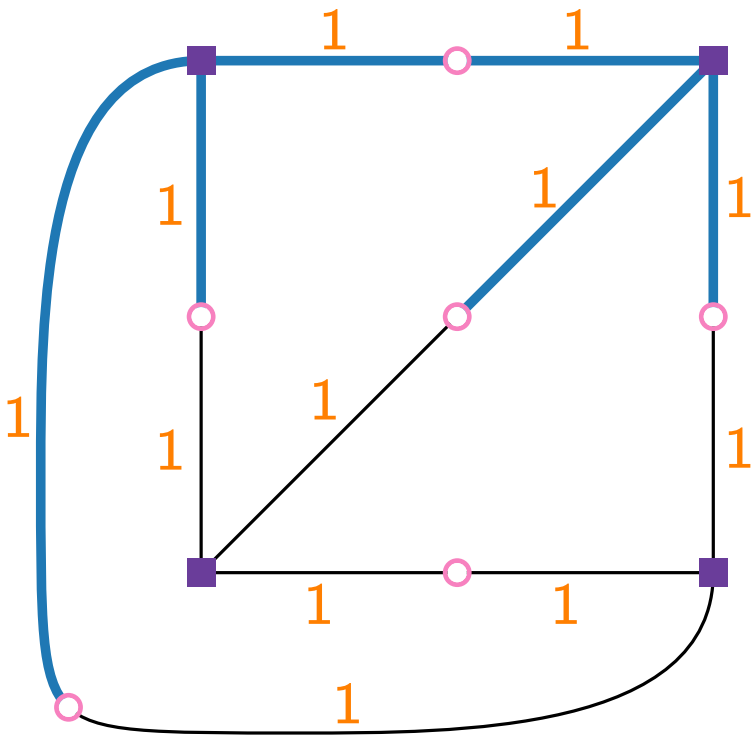

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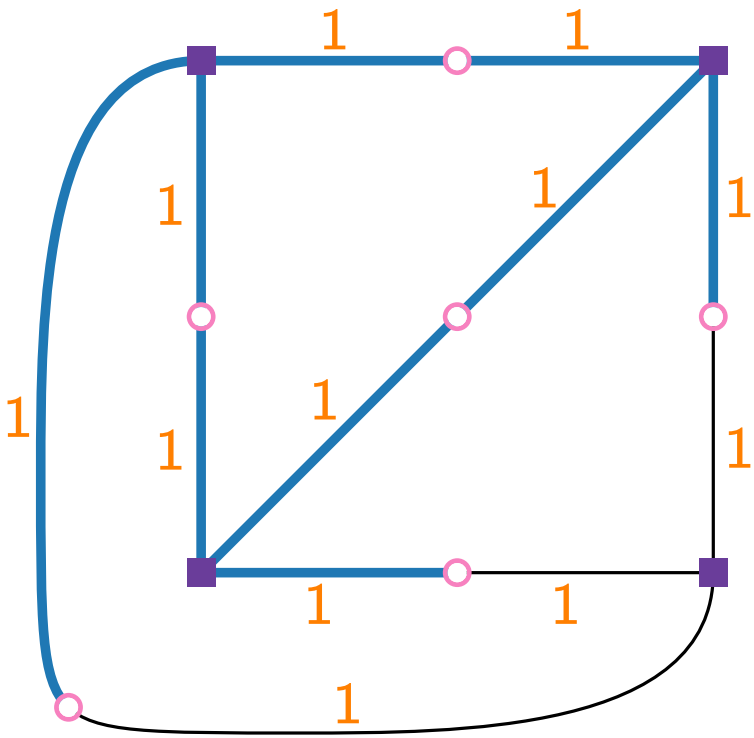

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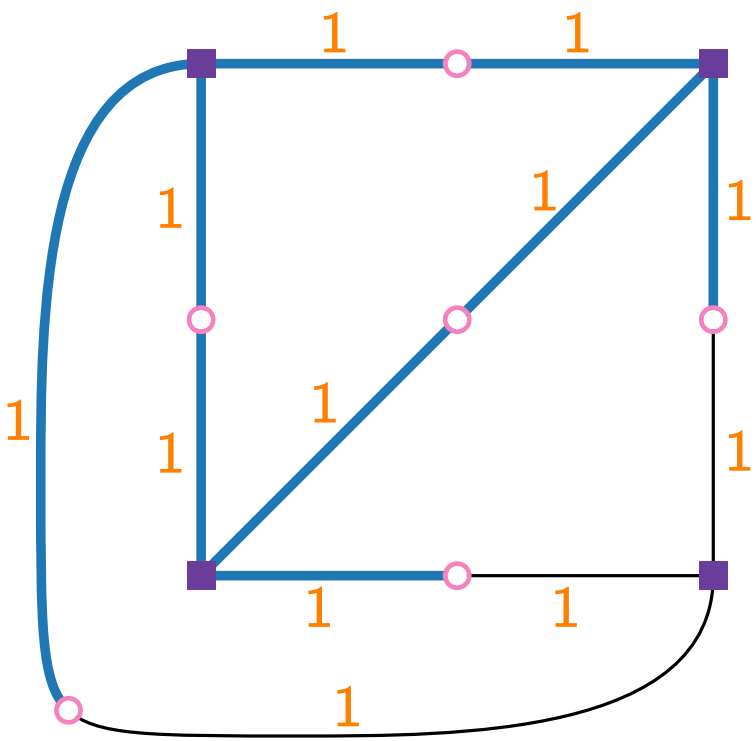

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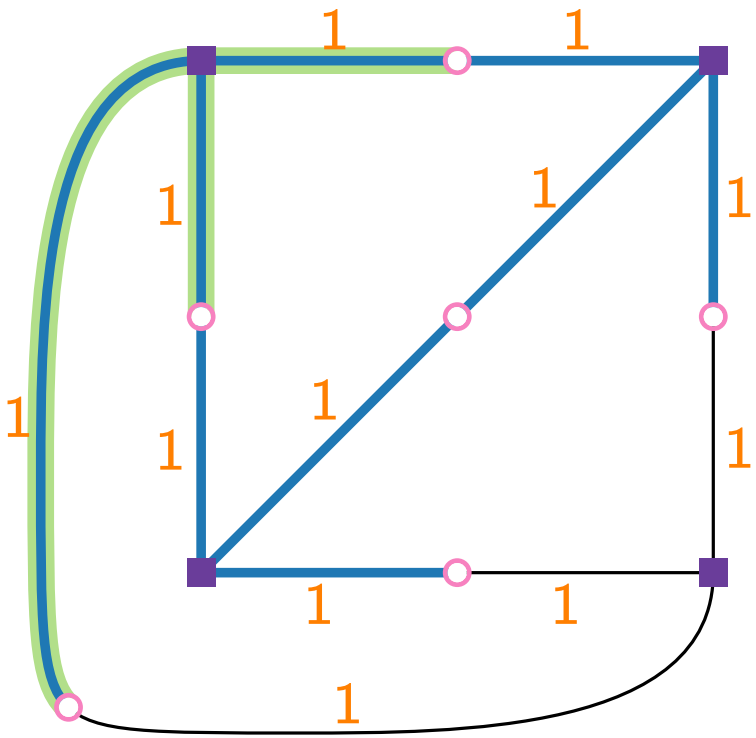

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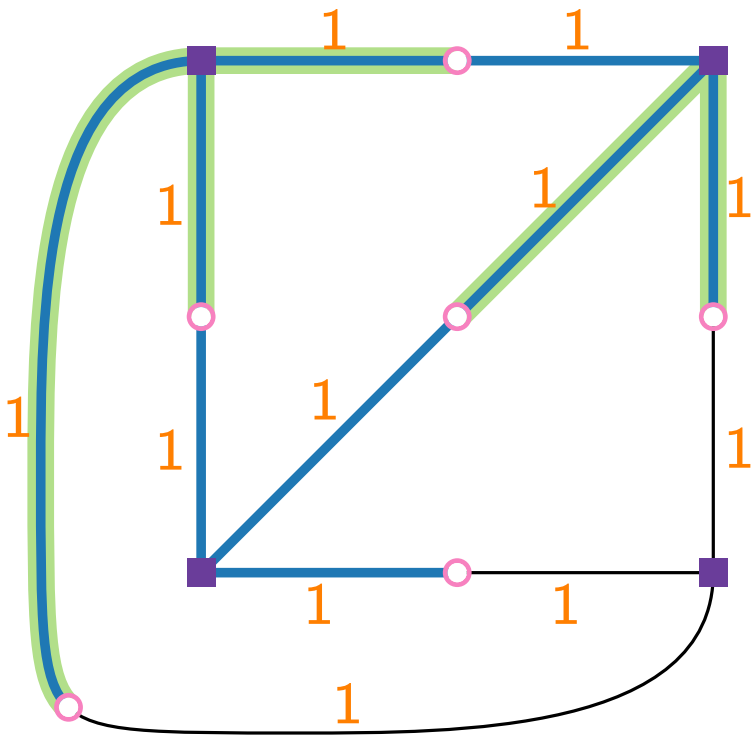

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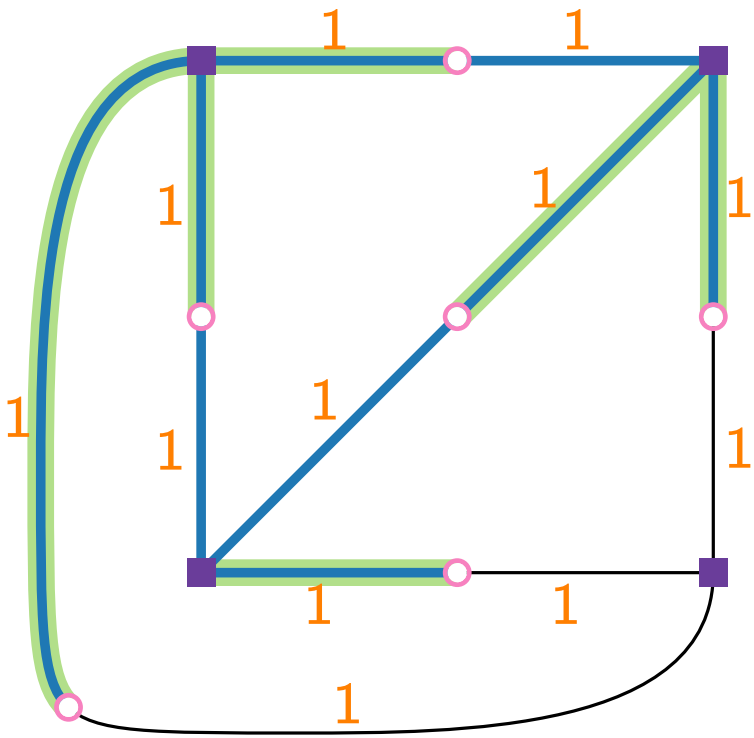

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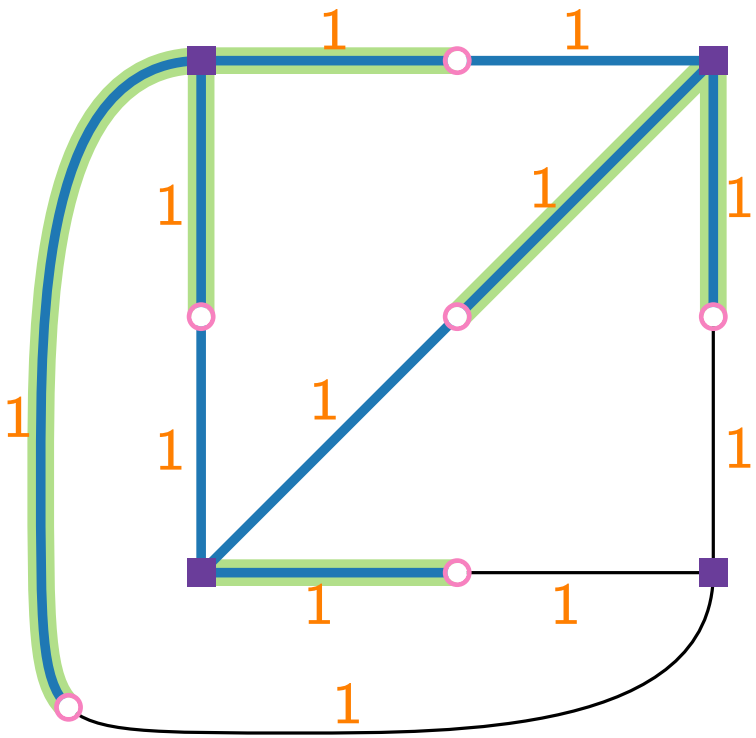

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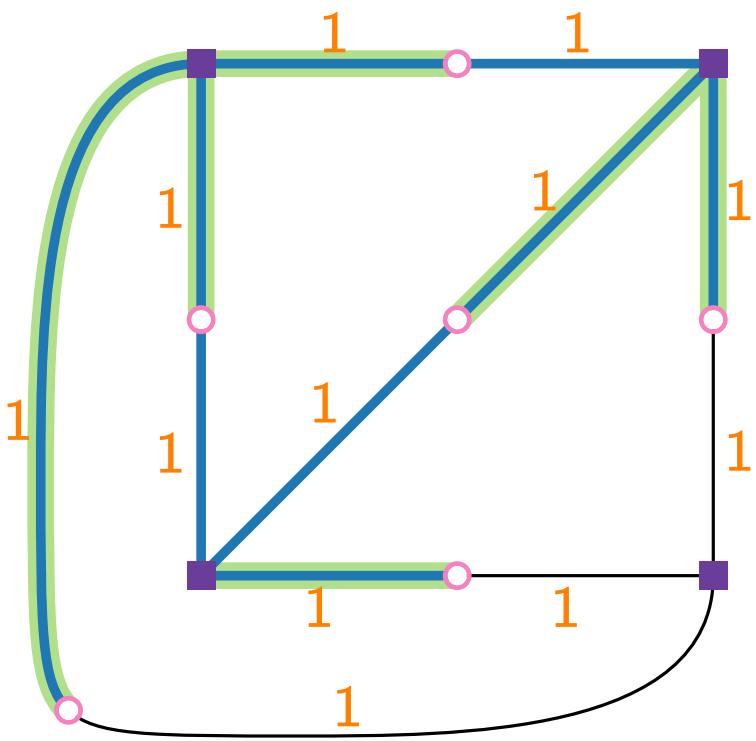

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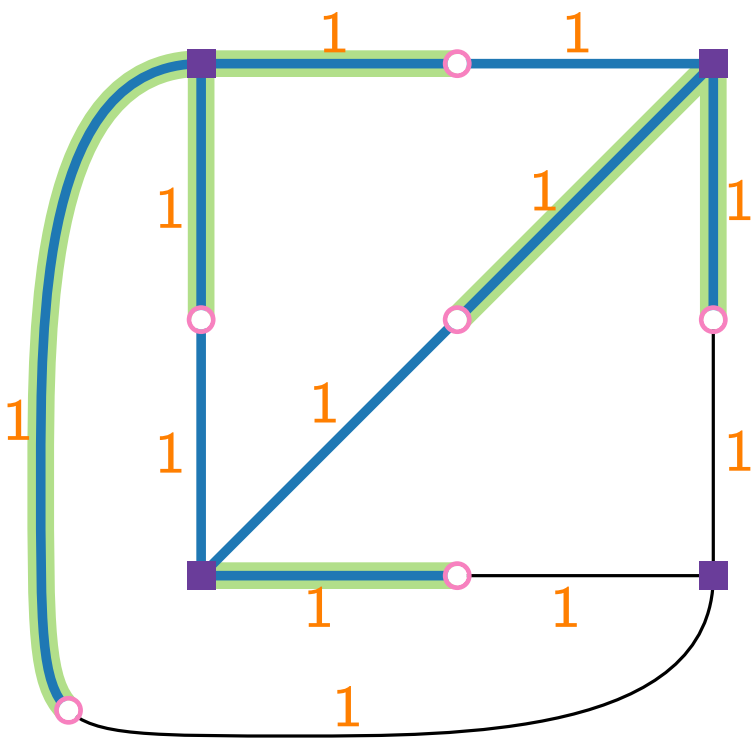

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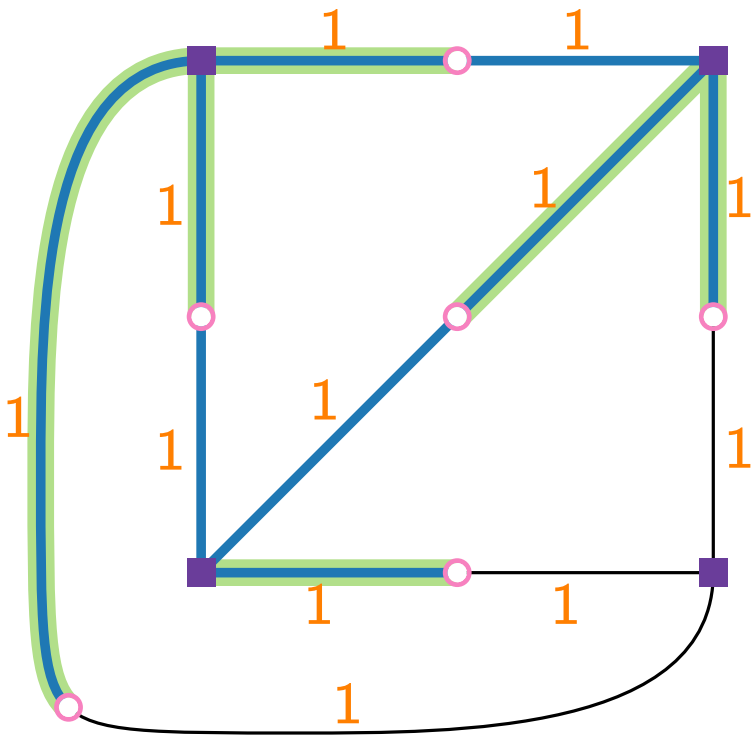

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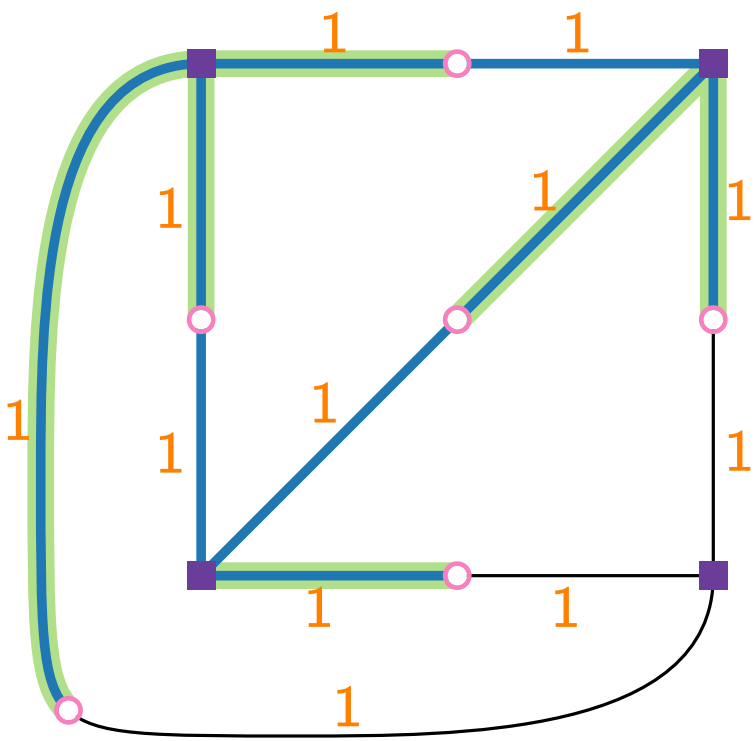

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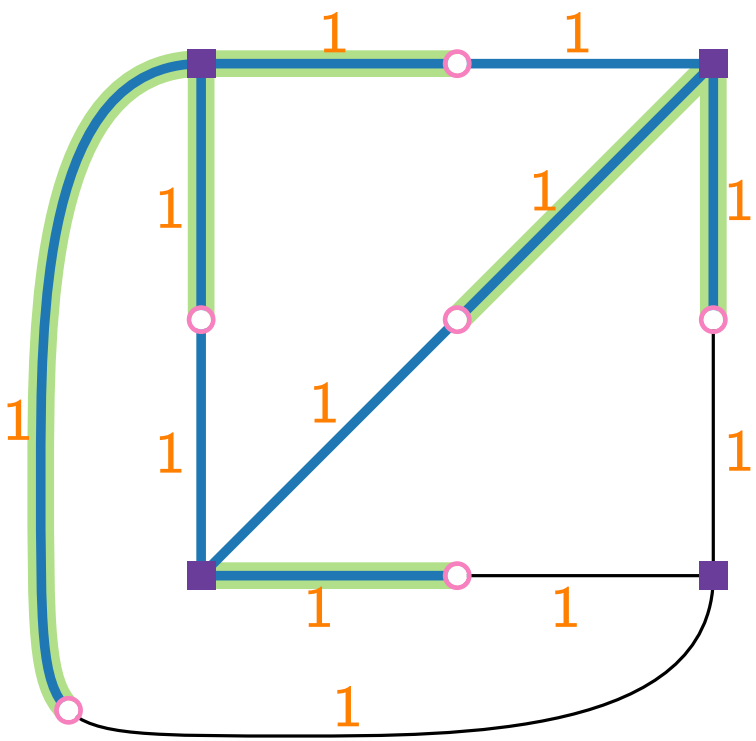

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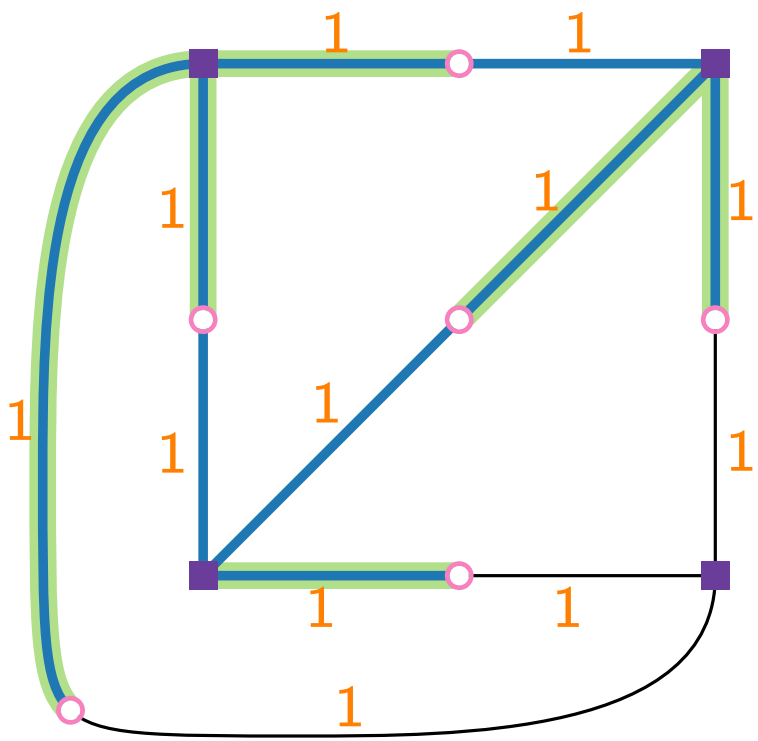

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Can we do better?

The best known approximation factor for MultiwayCut is $1.2965 - \frac{1}{k}$.

[Sharma & Vondrák, STOC'14]

Analysis Tight?

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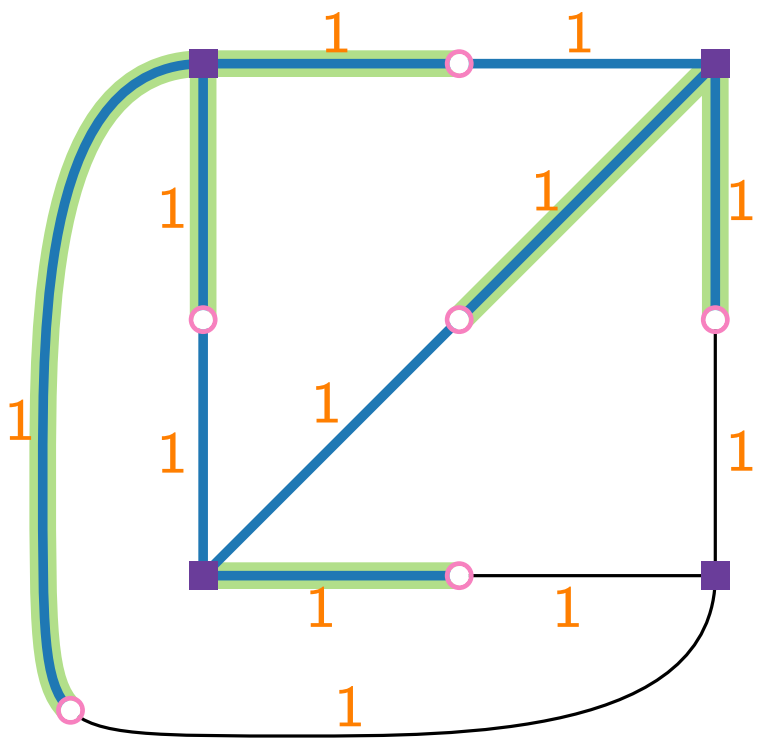

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$$\text{ALG}/\text{OPT} = \frac{2(k-1)}{k} = 2 - \frac{2}{k}$$

Can we do better?

The best known approximation factor for MultiwayCut is $1.2965 - \frac{1}{k}$.

[Sharma & Vondrák, STOC'14]

MultiwayCut cannot be approximated within factor $1.20016 - O(1/k)$ (unless P = NP).

[Bérczi, Chandrasekaran, Király & Madan, MP'18]