Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part I:
Organizational

Alexander Wolff

Winter 2022/23

Organizational

Lectures: on site (in German :-)

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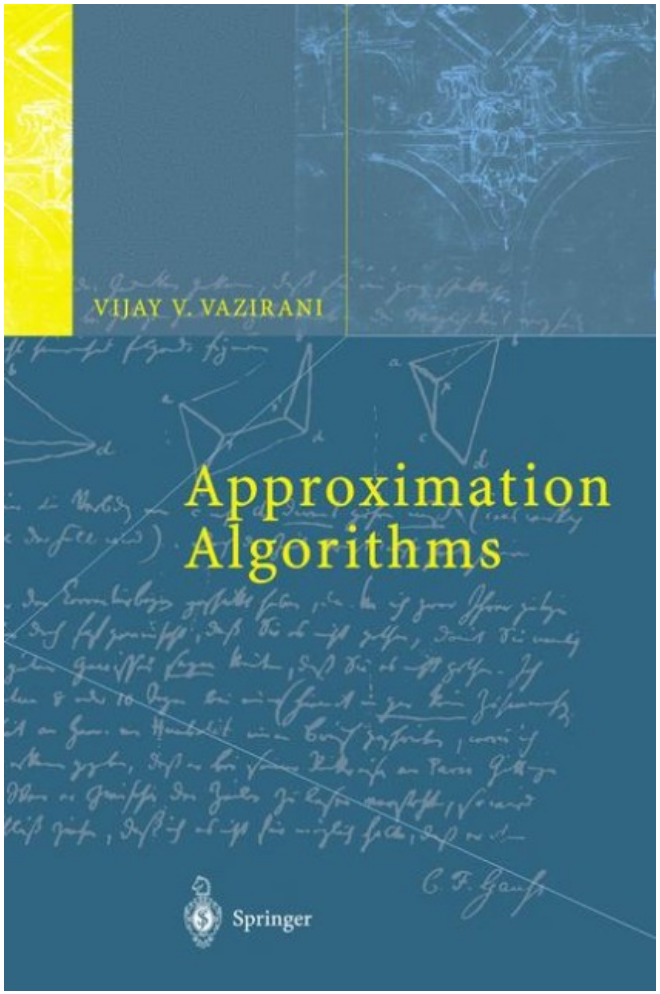
Most slides are due to Joachim Spoerhase. Thanks!

Textbooks



Vijay V. Vazirani:
Approximation Algorithms
Springer-Verlag, 2003.

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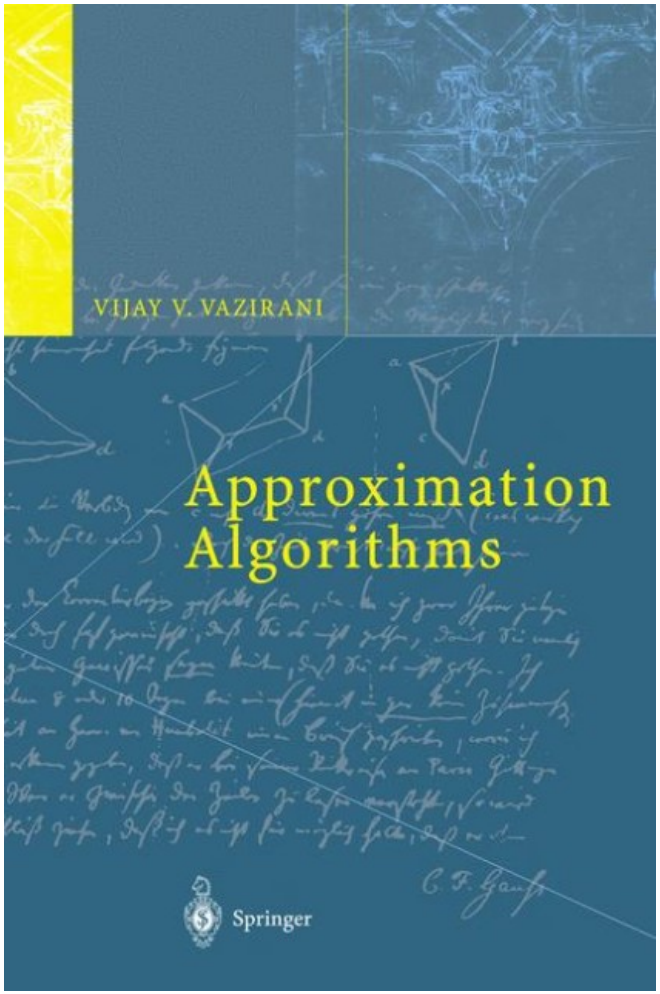

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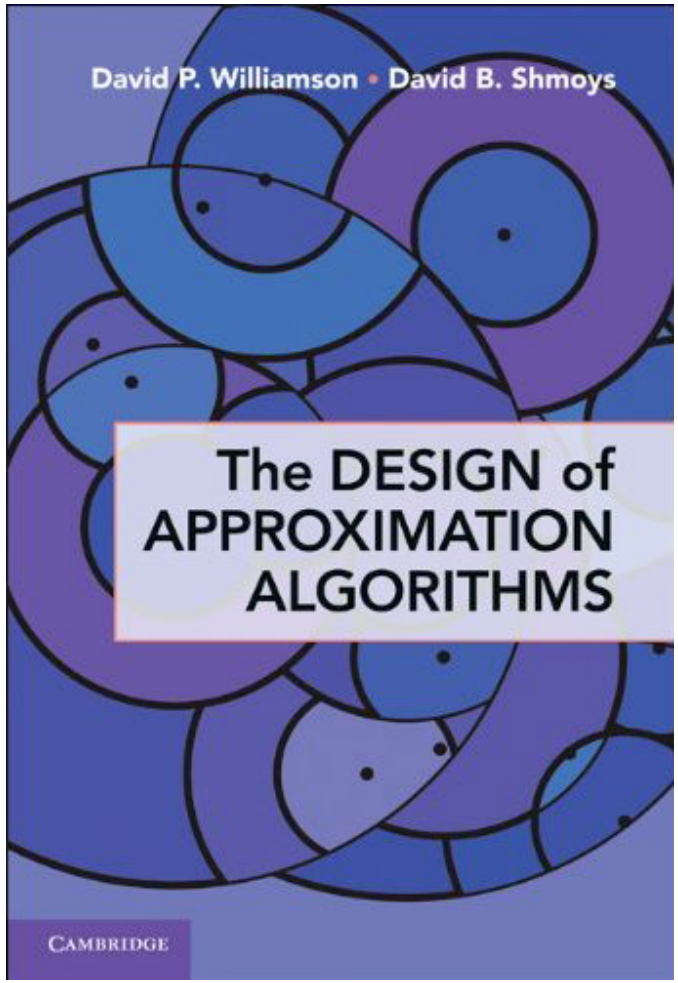

D. P. Williamson & D. B. Shmoys:
The Design of Approximation Algorithms
Cambridge-Verlag, 2011.

http://www.designofapproxalgs.com/

Approximation Algorithms

„All exact science is dominated by the idea of approximation.“

– Bertrand Russell
(1872 – 1970)

Approximation Algorithms

- Many optimization problems are NP-hard!
  (For example, the traveling salesperson problem.)

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- Many optimization problems are NP-hard!
  (For example, the traveling salesperson problem.)
- $\rightsquigarrow$ an optimal solution cannot be efficiently computed unless P=NP.
- However, good approximate solutions can often be found efficiently!
- **Techniques** for the design and analysis of approximation algorithms arise from studying specific optimization problems.

Overview

Combinatorial algorithms

- Introduction (Vertex Cover)
- Set Cover via Greedy
- Shortest Superstring
  via reduction to SC
- Steiner Tree via MST
- Multiway Cut via Greedy
- k-Center via Parametrized Pruning
- Min-Degree Spanning Tree
  and local search
- Knapsack via DP and Scaling
- Euclidean TSP via Quadtrees

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LP-based algorithms

- introduction to LP-Duality
- Set Cover via LP Rounding
- Set Cover via Primal–Dual Schema
- Maximum Satisfiability
- Scheduling und Extreme Point Solutions
- Steiner Forest via Primal–Dual

Approximation Algorithms

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Introduction and Vertex Cover

Part II:
(Cardinality) Vertex Cover

Alexander Wolff

Winter 2022/23

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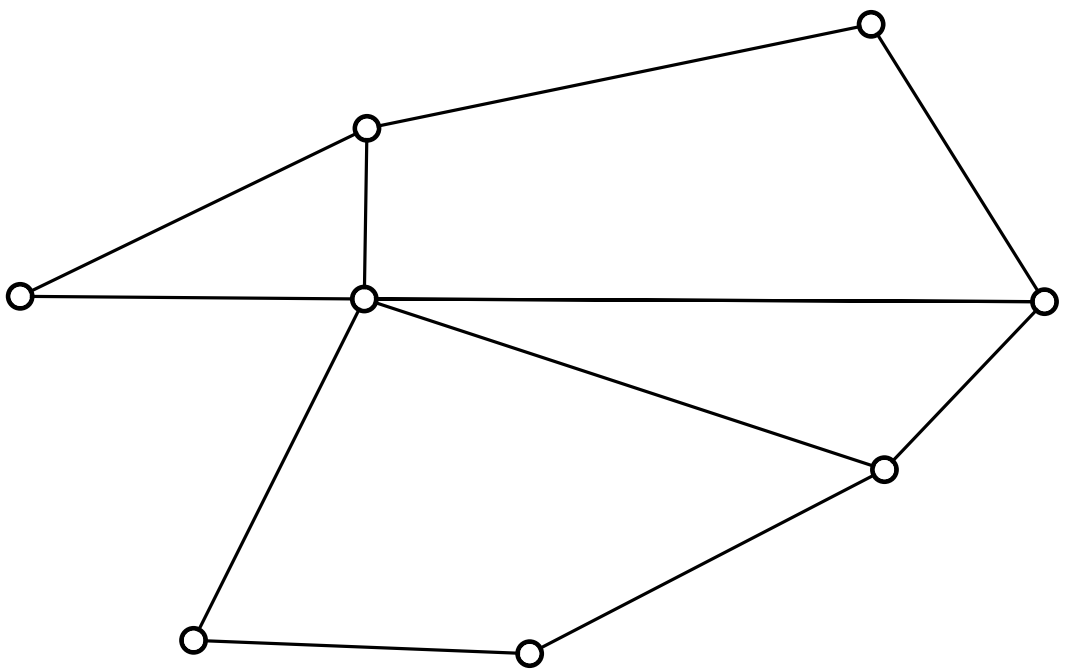

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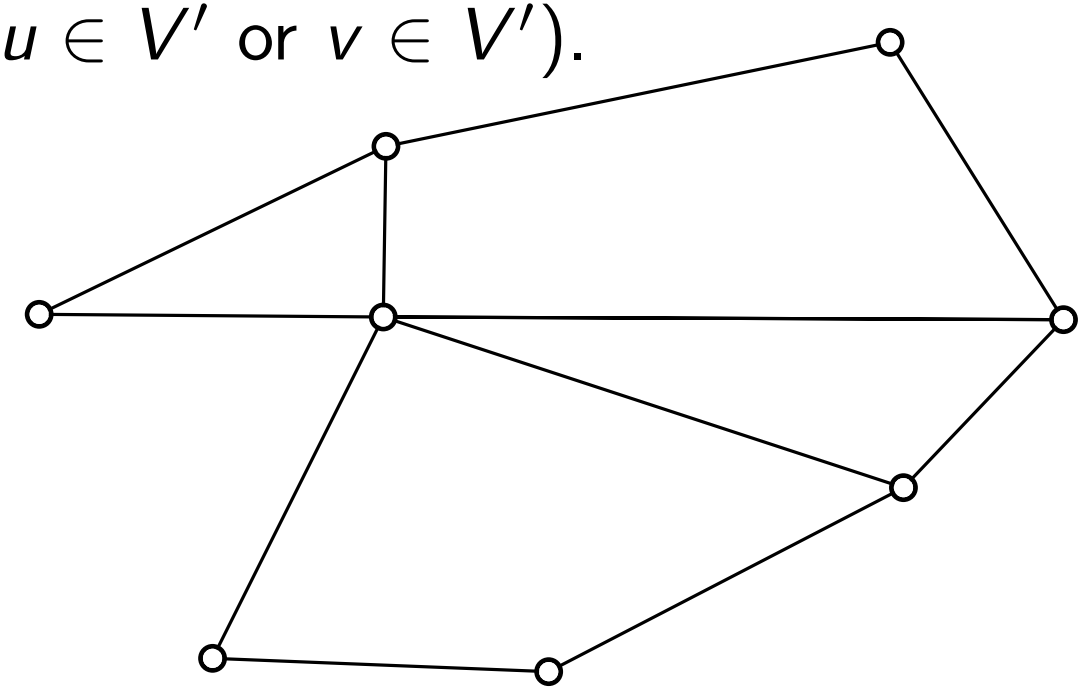

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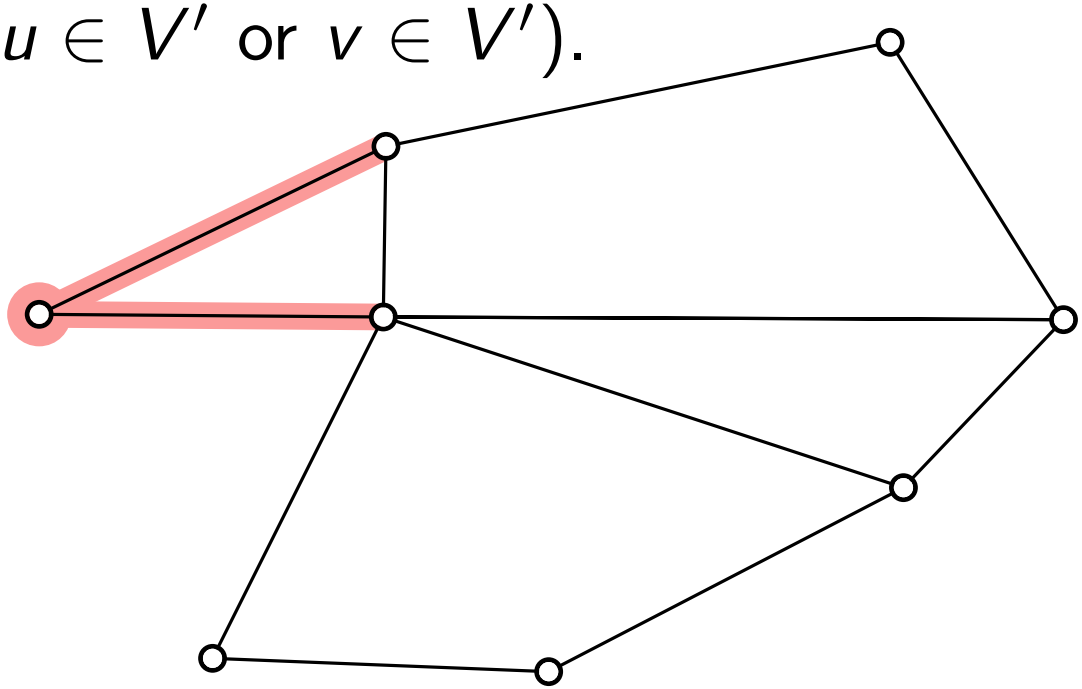

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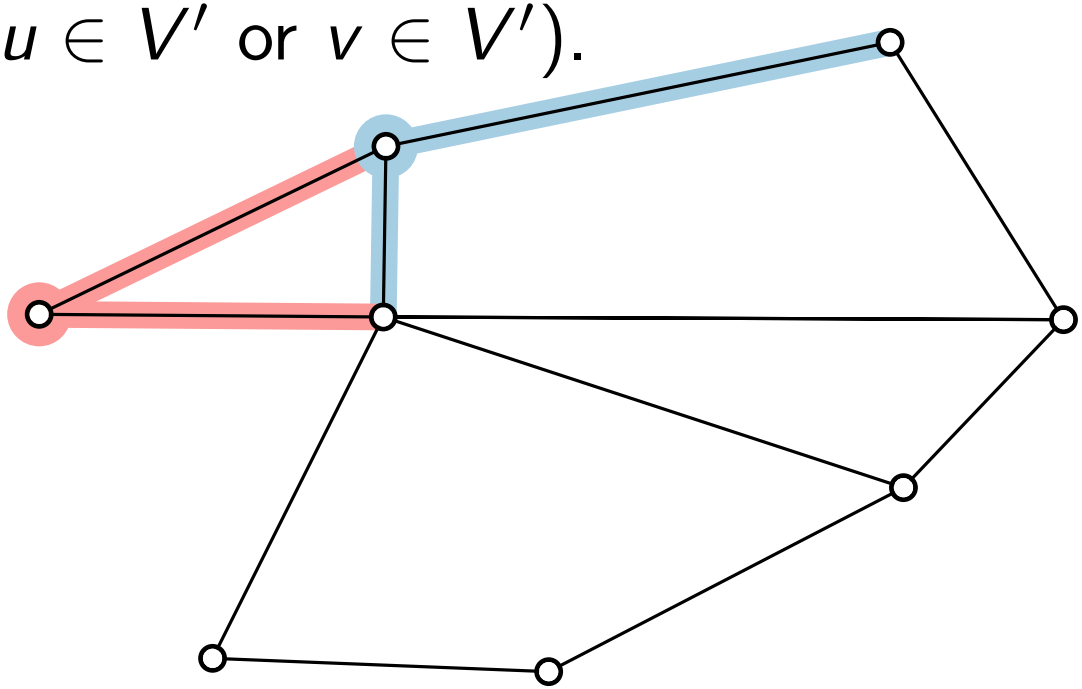

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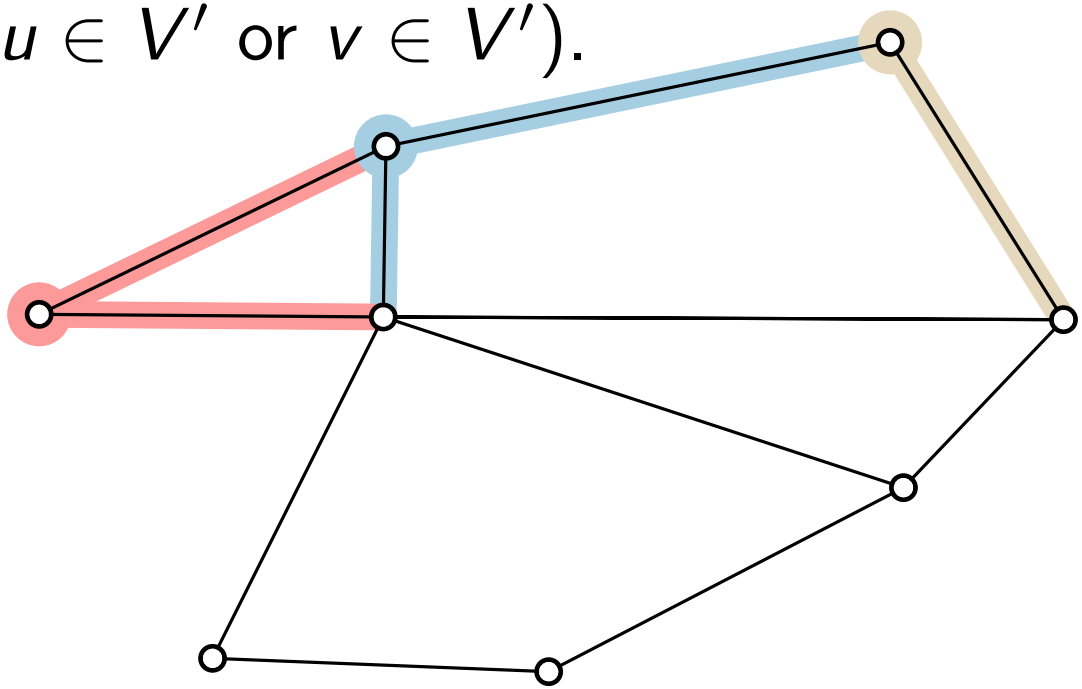

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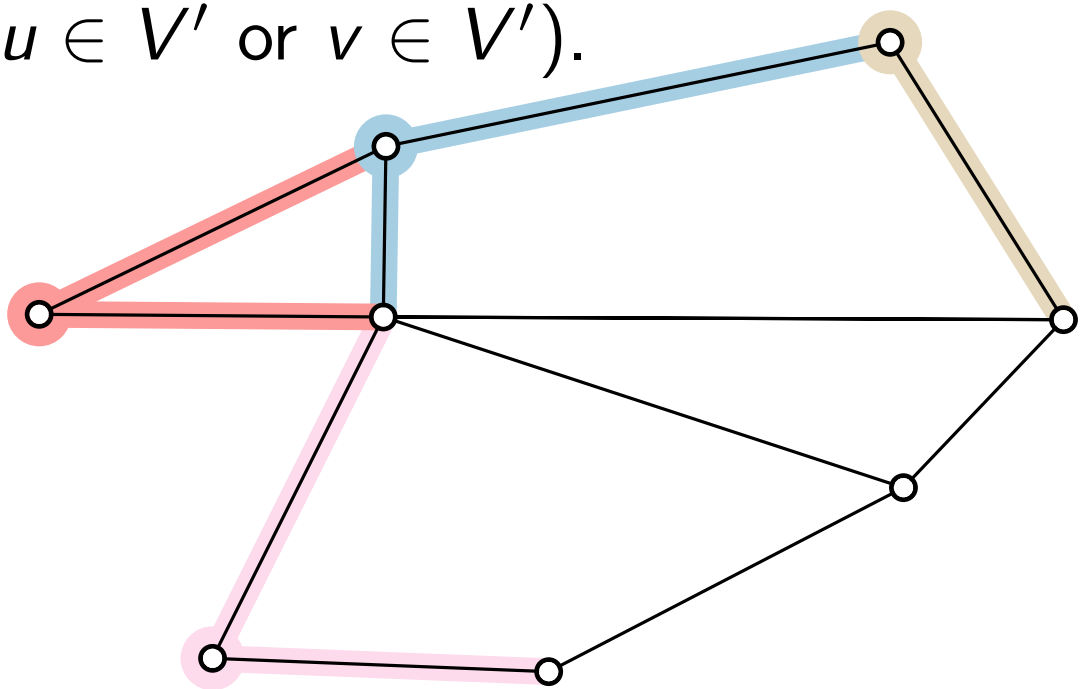

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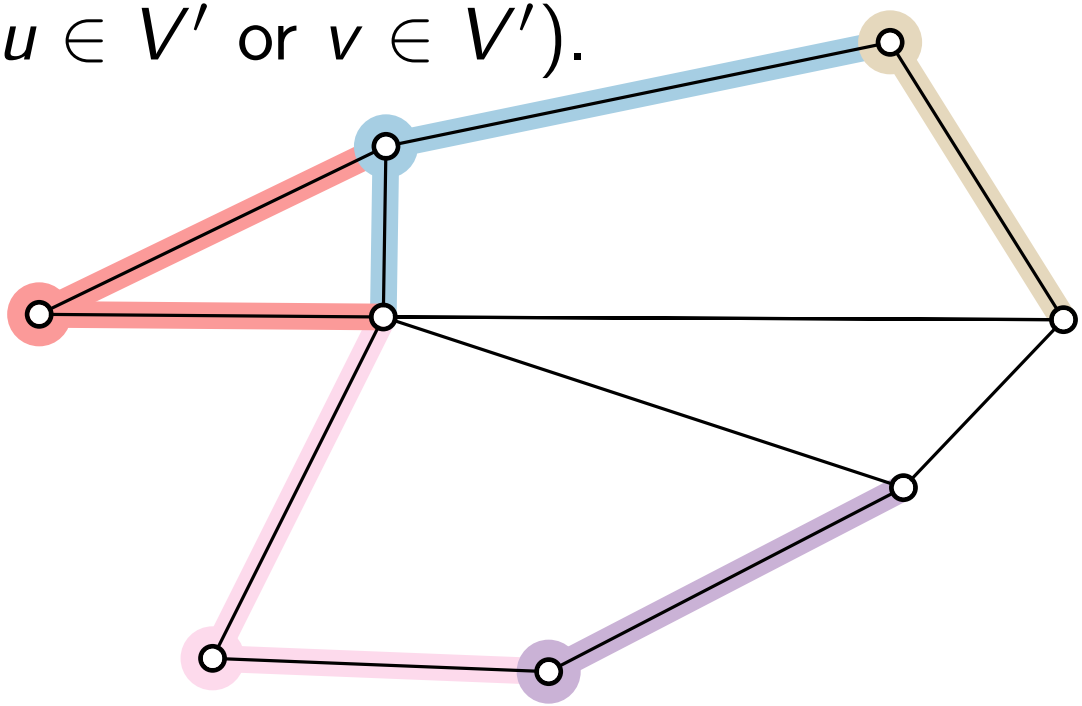

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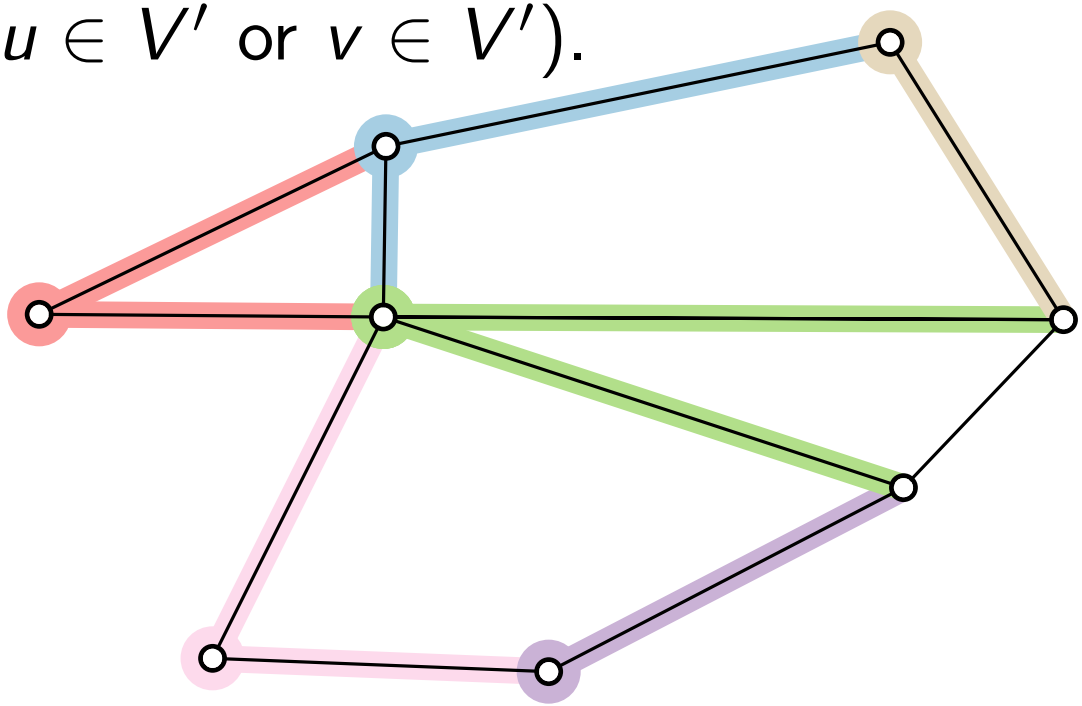

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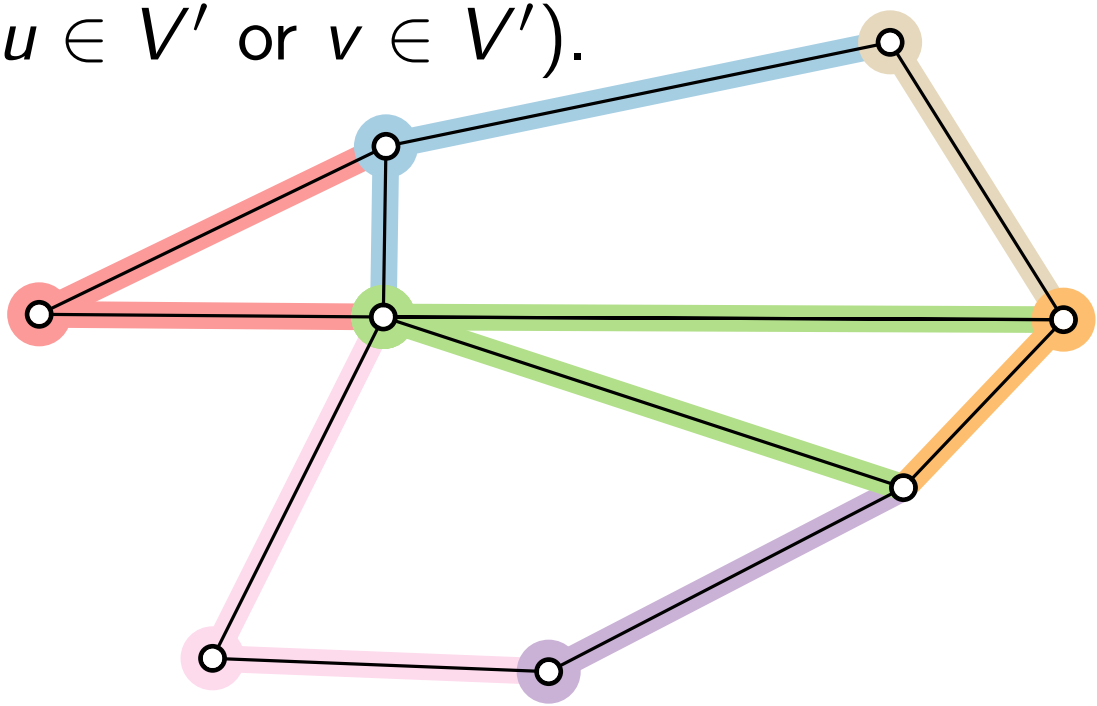

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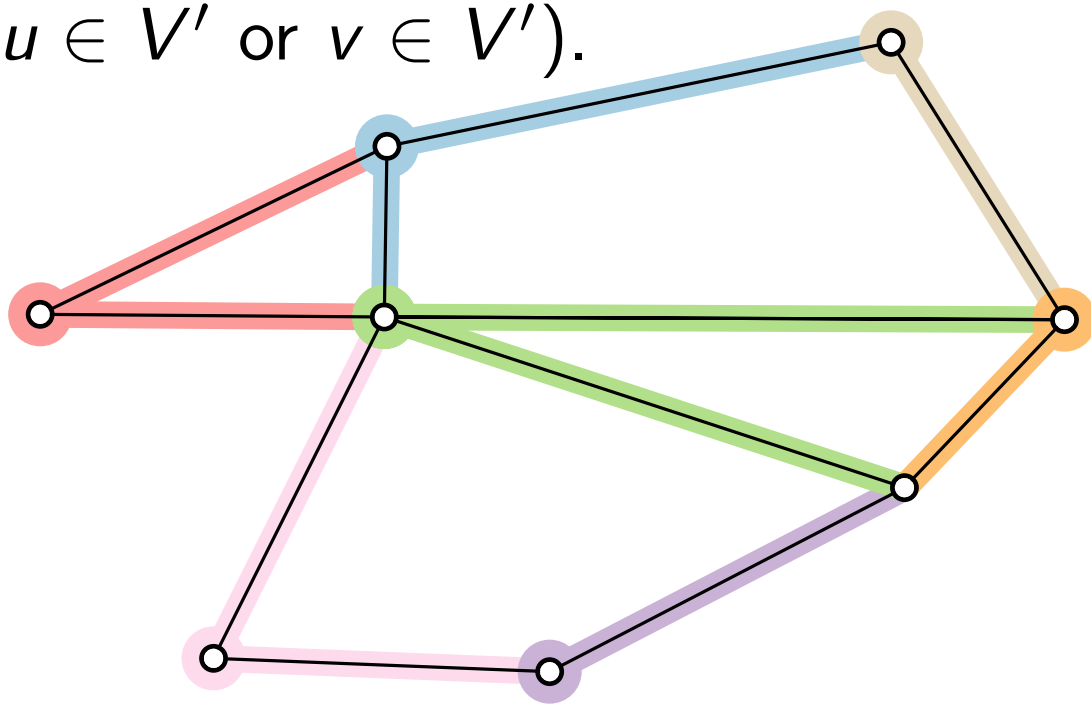

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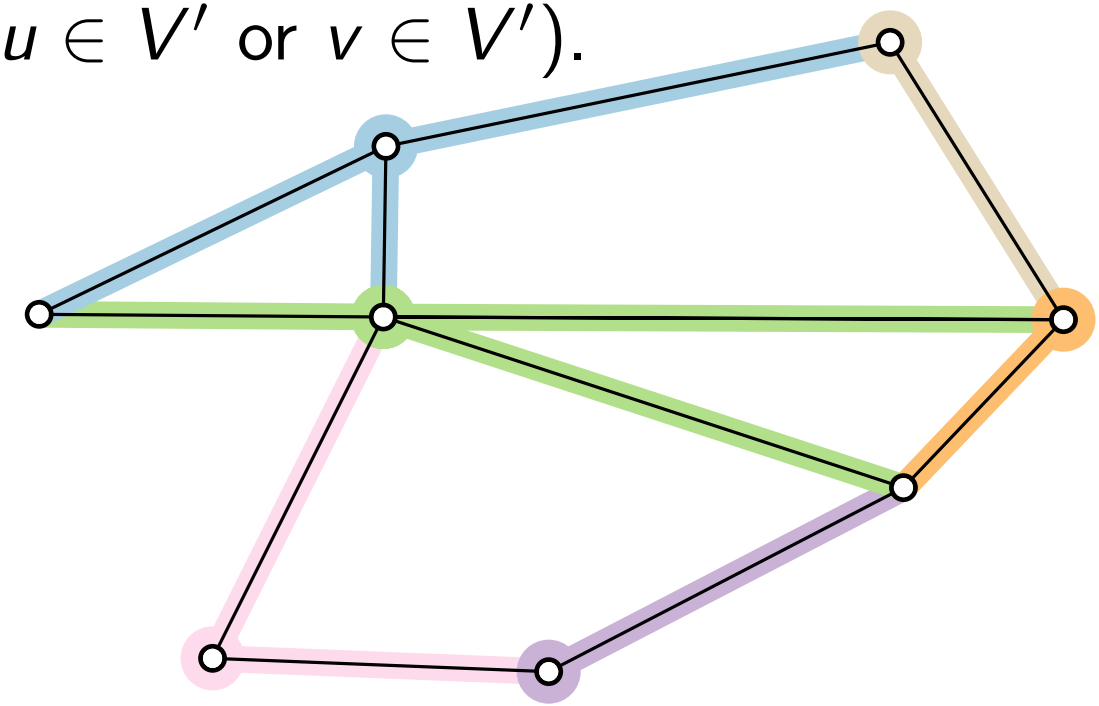

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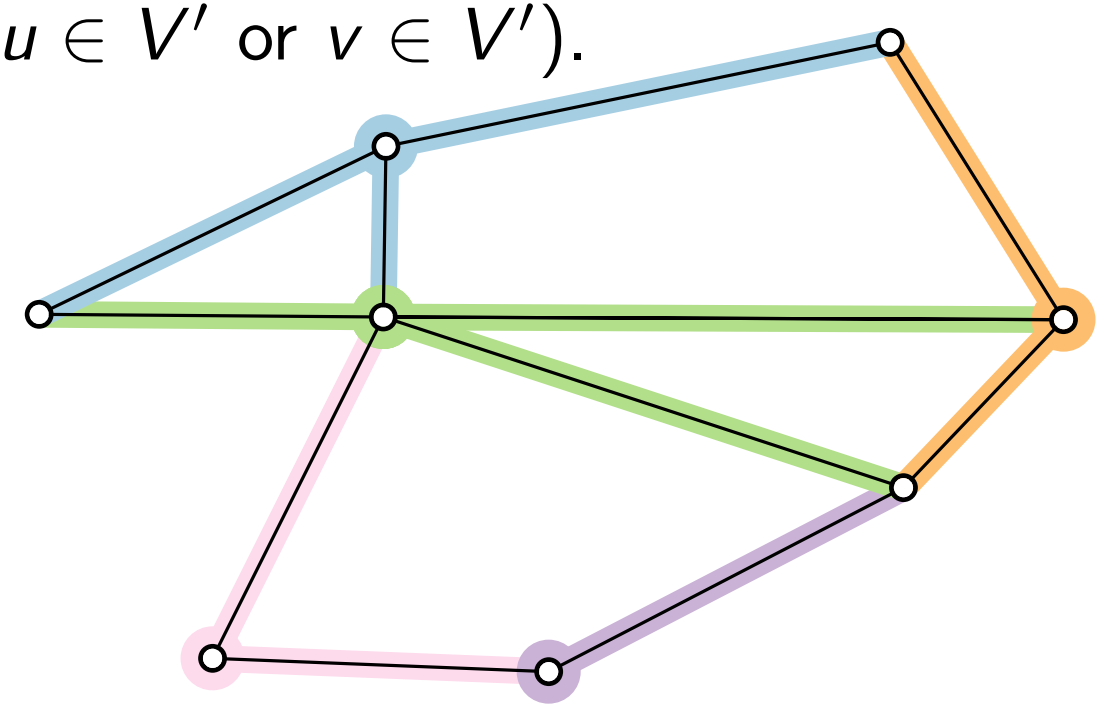

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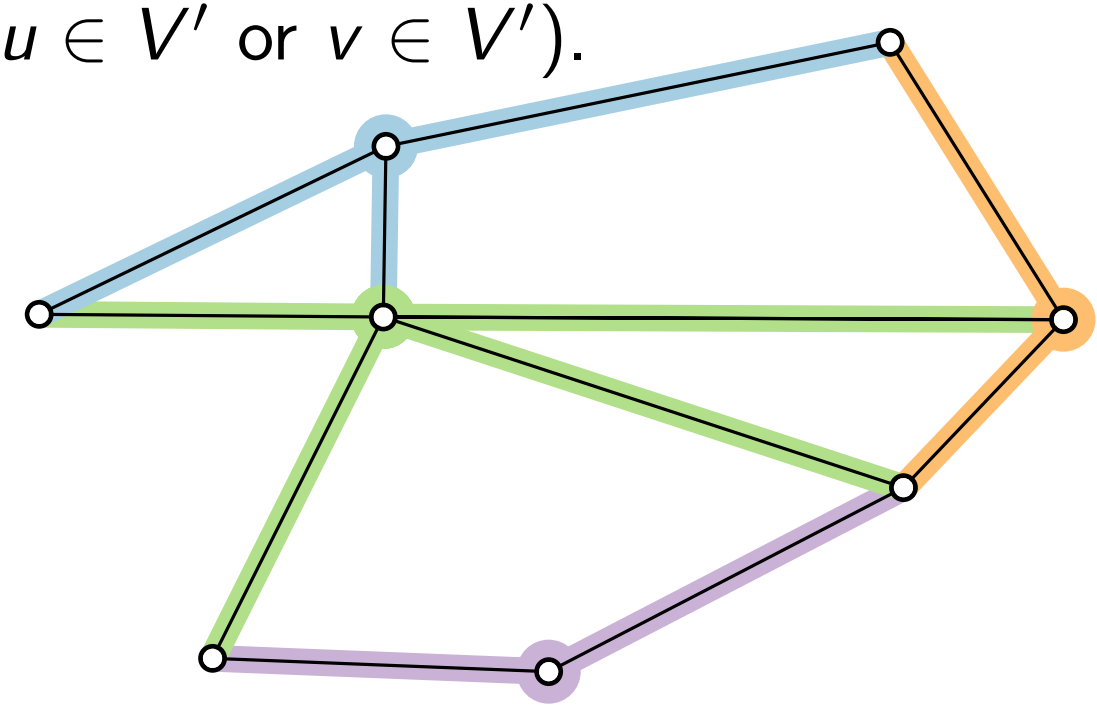

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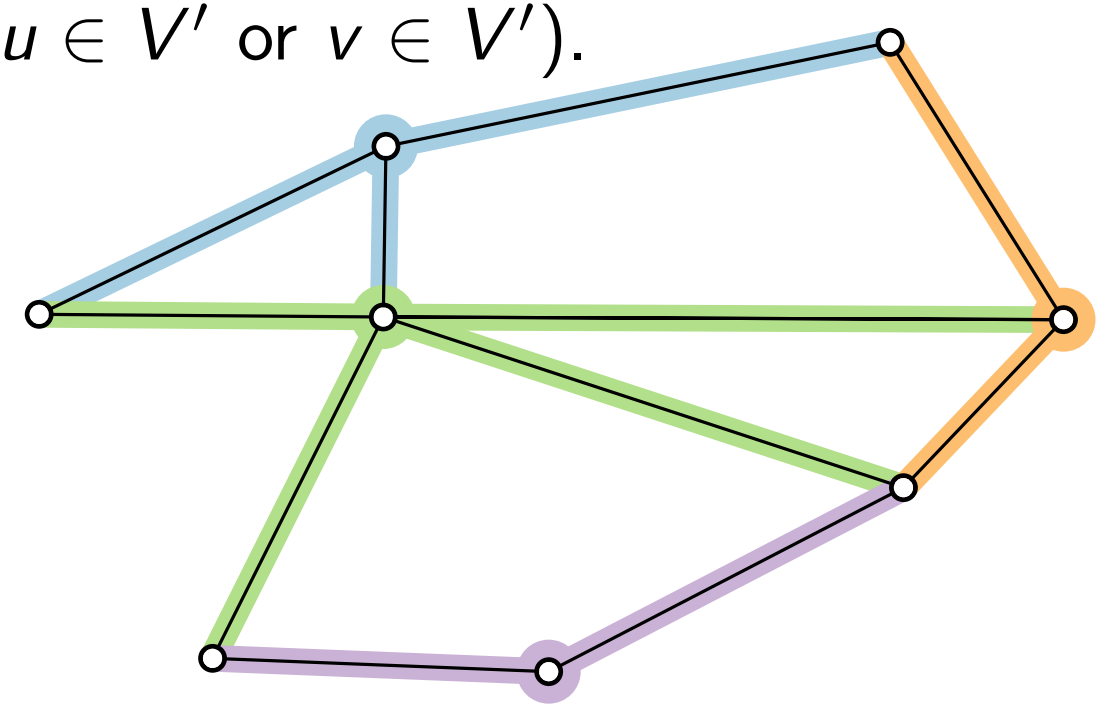

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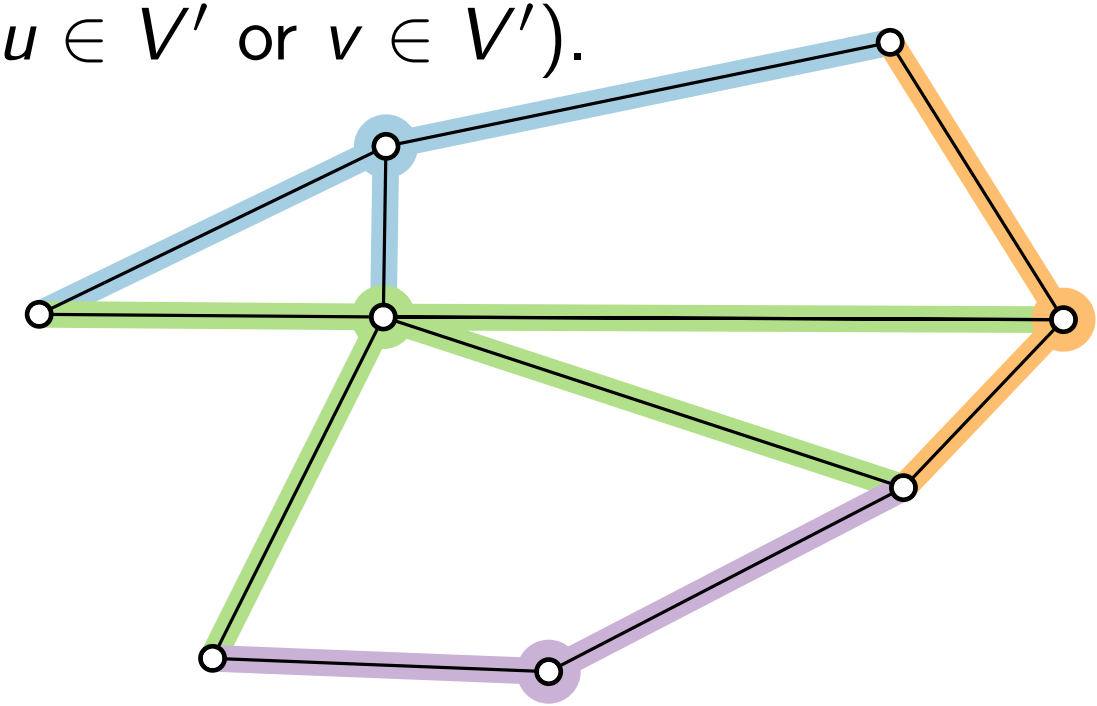

Optimum (OPT $= 4$) – but in general NP-hard to find :-(

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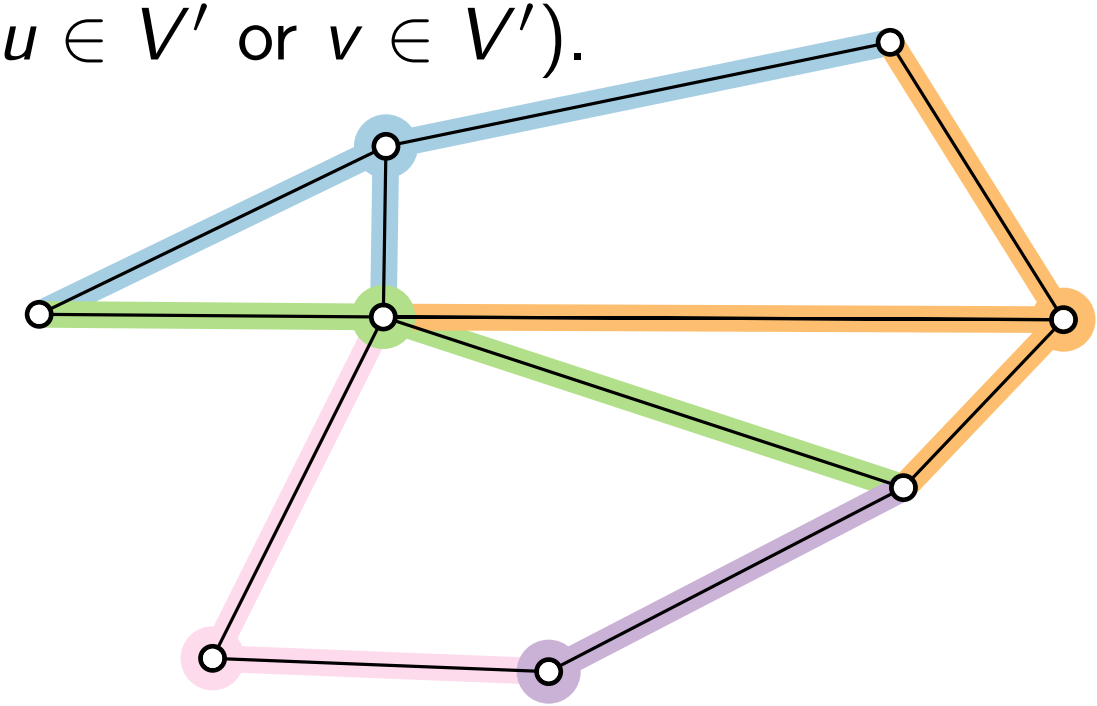

"good" (5/4-) approximate solution

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Part III:
NP-Optimization Problem

Alexander Wolff

Winter 2022/23

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- Π is either a minimization or maximization problem.

VertexCover: NP-Optimization Problem

Task: Fill in the gaps for $\Pi =$ Vertex Cover.

$D_\Pi =$

For $I \in D_\Pi$: $|I| =$

$S_\Pi(I) =$

- Why is $|s| \in \text{poly}(|I|)$ for every $s \in S_\Pi(I)$?

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Π is a minimization problem.

Optimum and Optimal Objective Value

Let Π be a minimization problem and $I \in D_\Pi$ an instance of Π.

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The optimal value $\text{obj}_\Pi(I, s^*)$ of the objective function is denoted by $\text{OPT}_\Pi(I)$ or simply by OPT in context.

Approximation Algorithms

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Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part IV:
Approximation Algorithm for VertexCover

Alexander Wolff

Winter 2022/23

Approximation Alg. for VertexCover

Ideas?

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Ideas?

- Edge-Greedy

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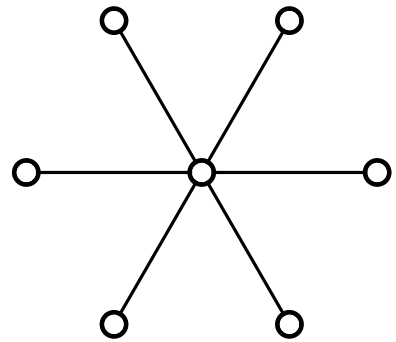

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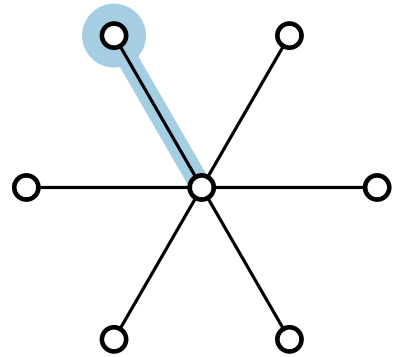

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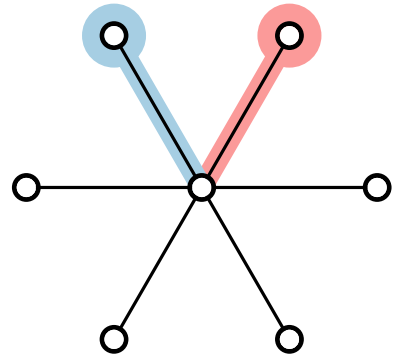

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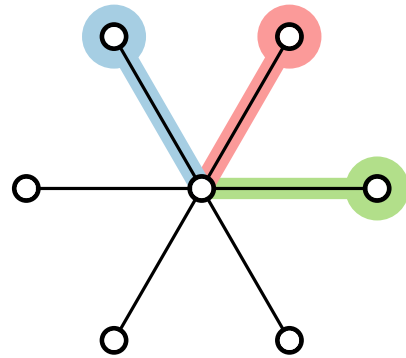

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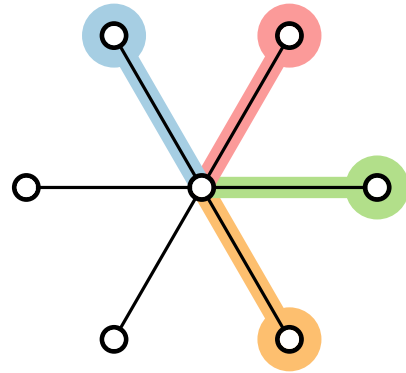

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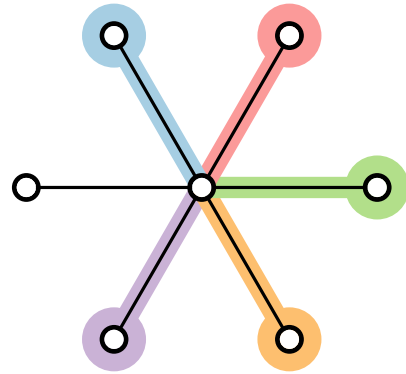

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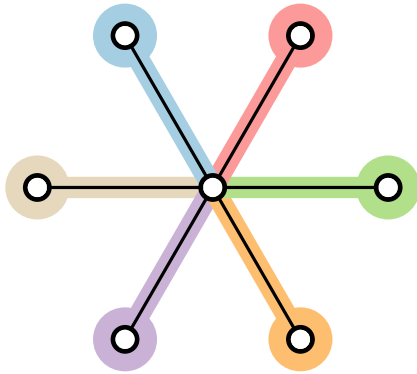

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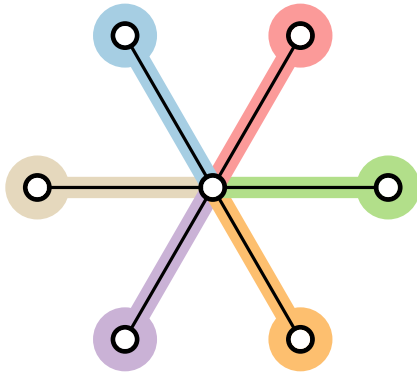

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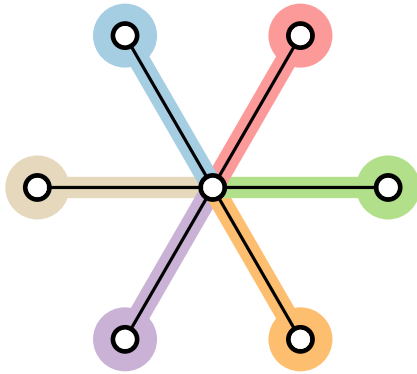

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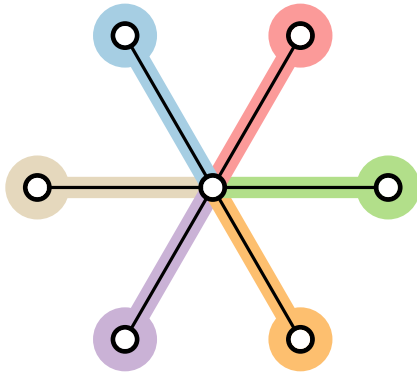

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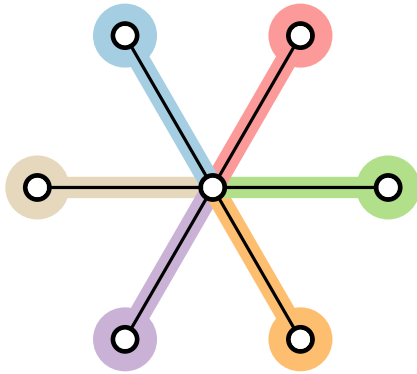

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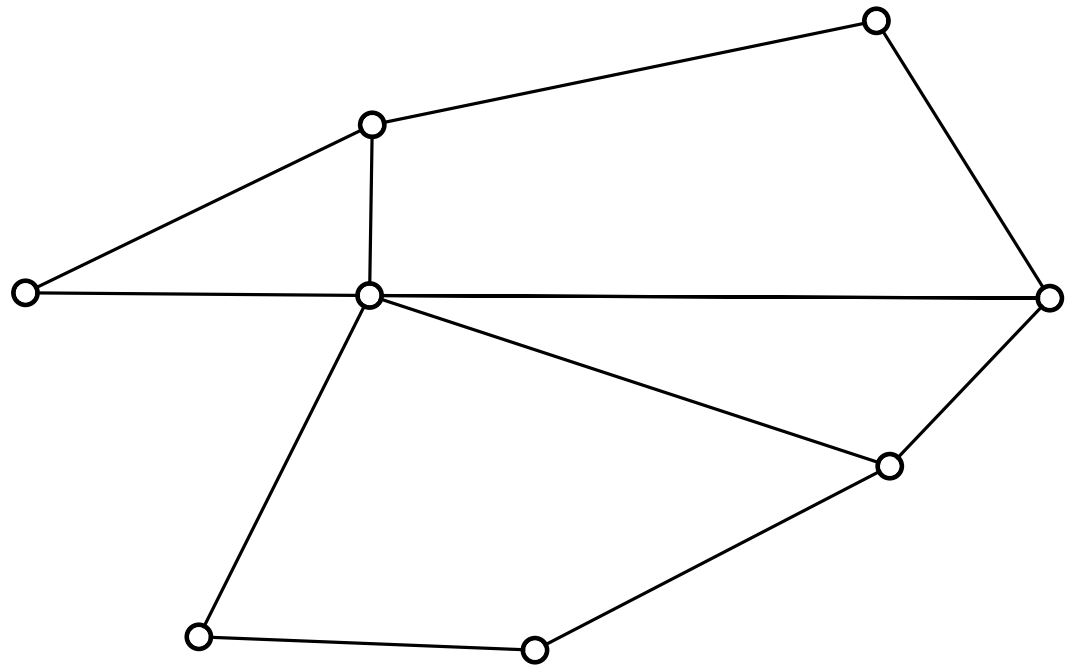
Lower Bound by Matchings



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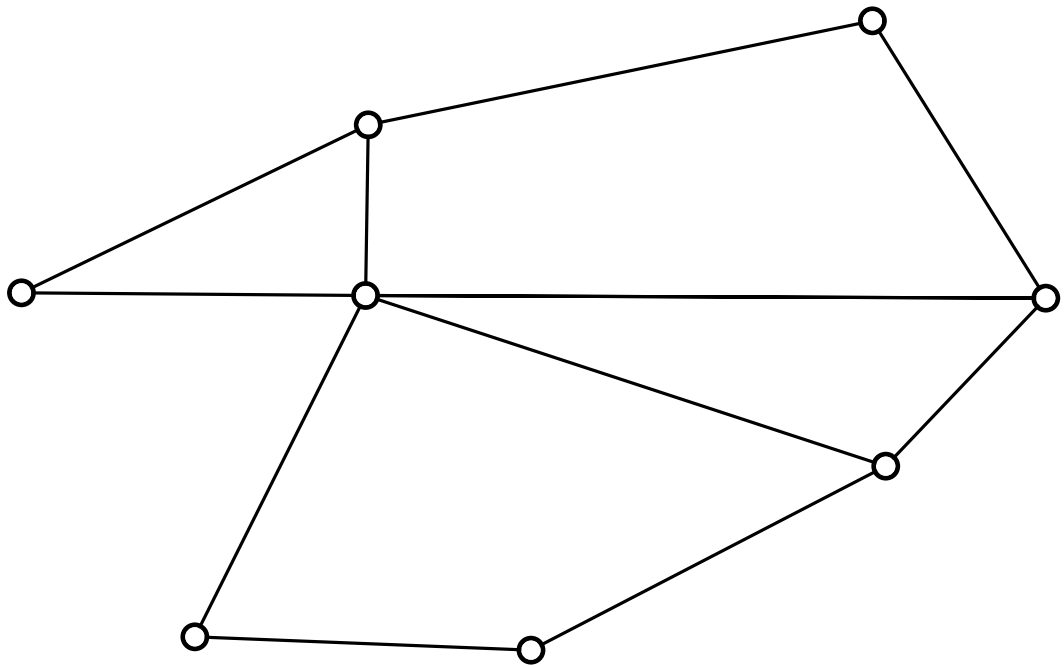

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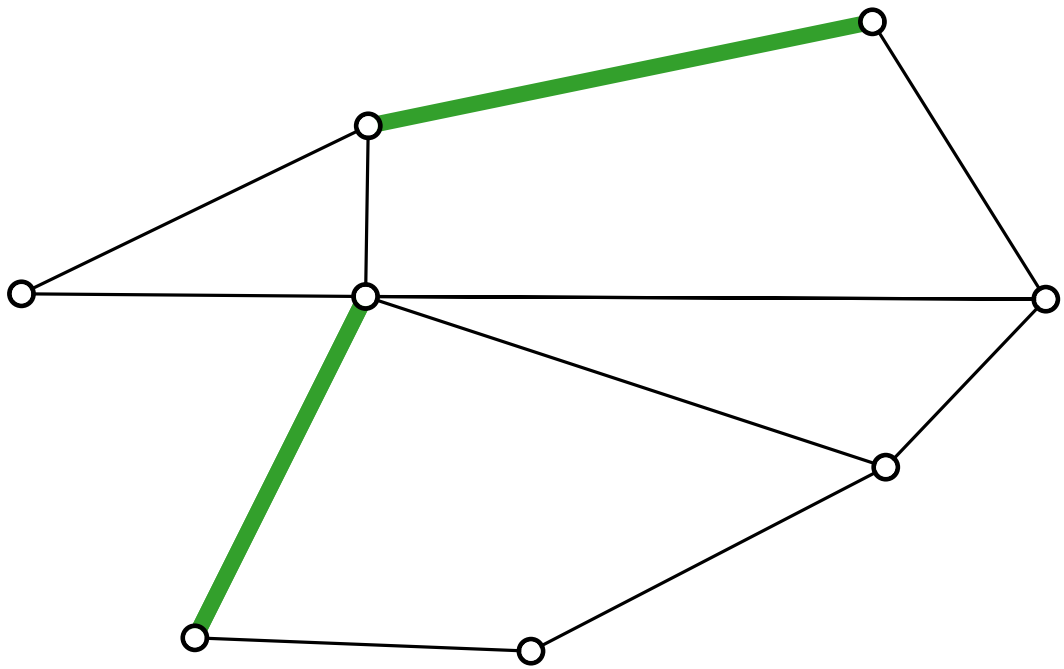

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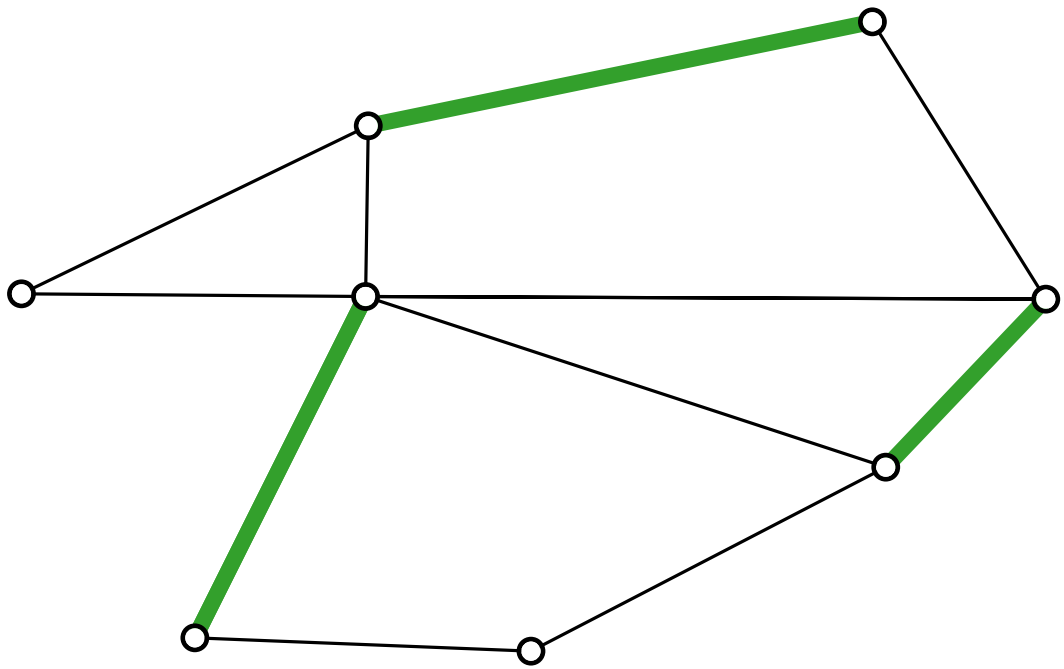

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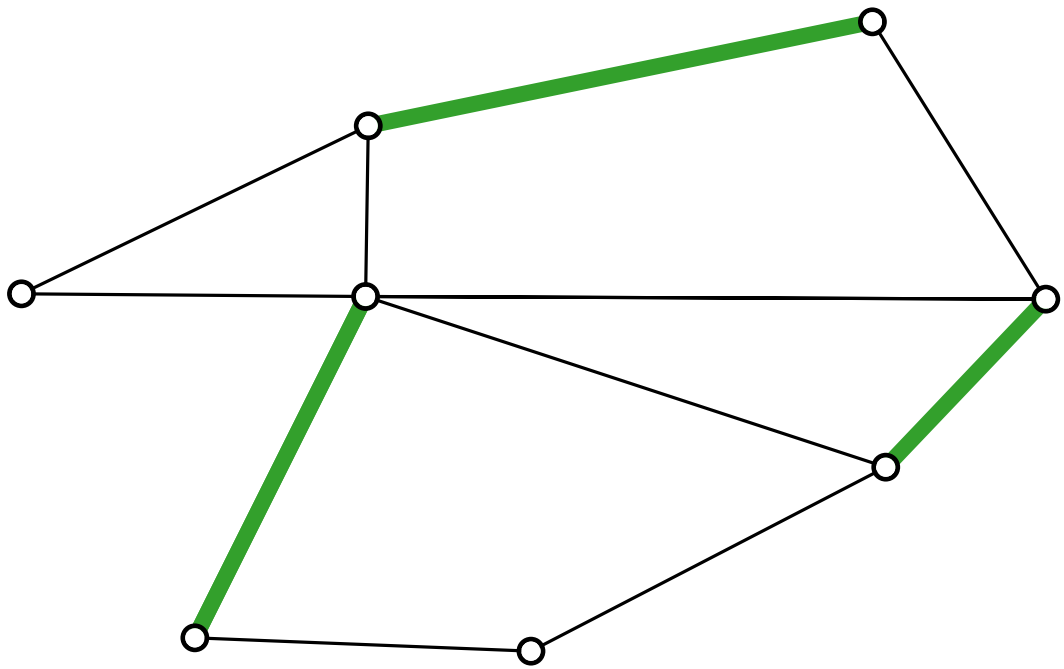

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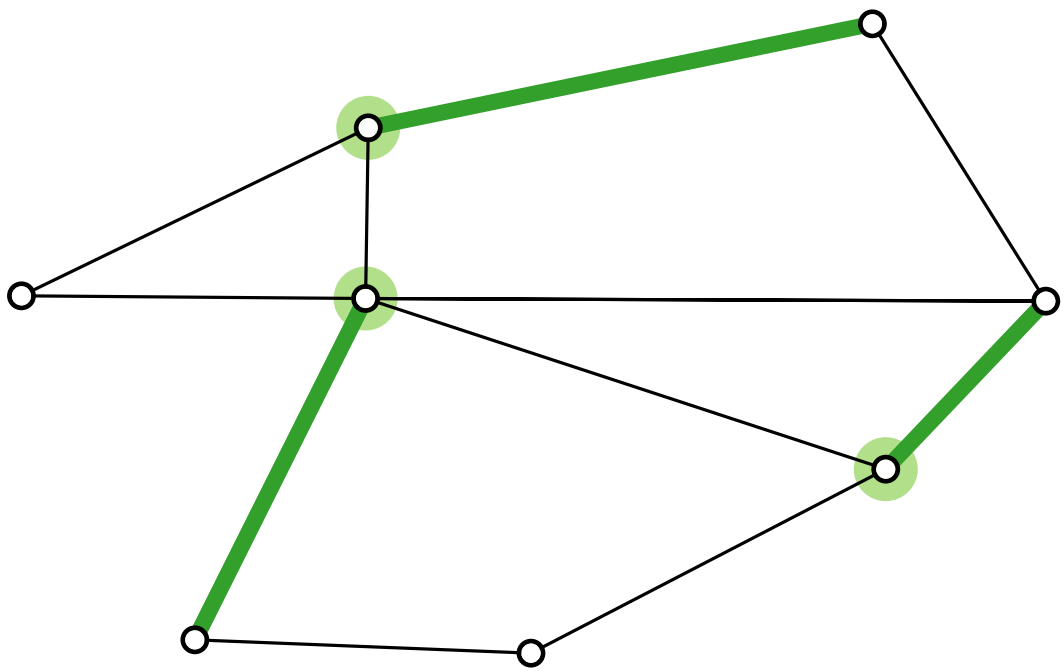
Vertex cover of M



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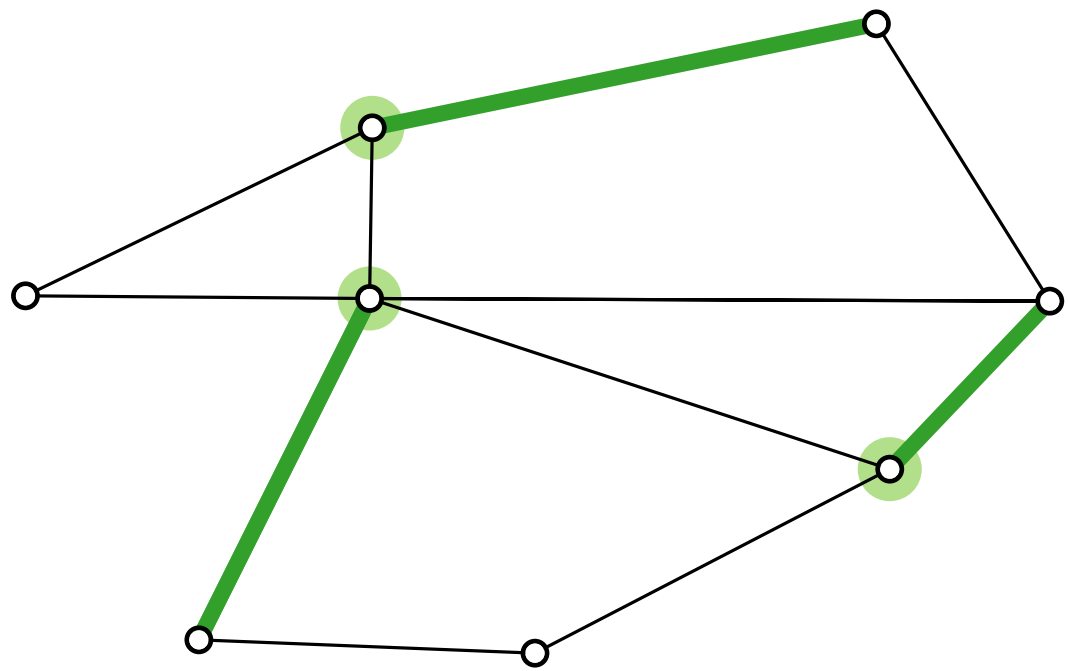

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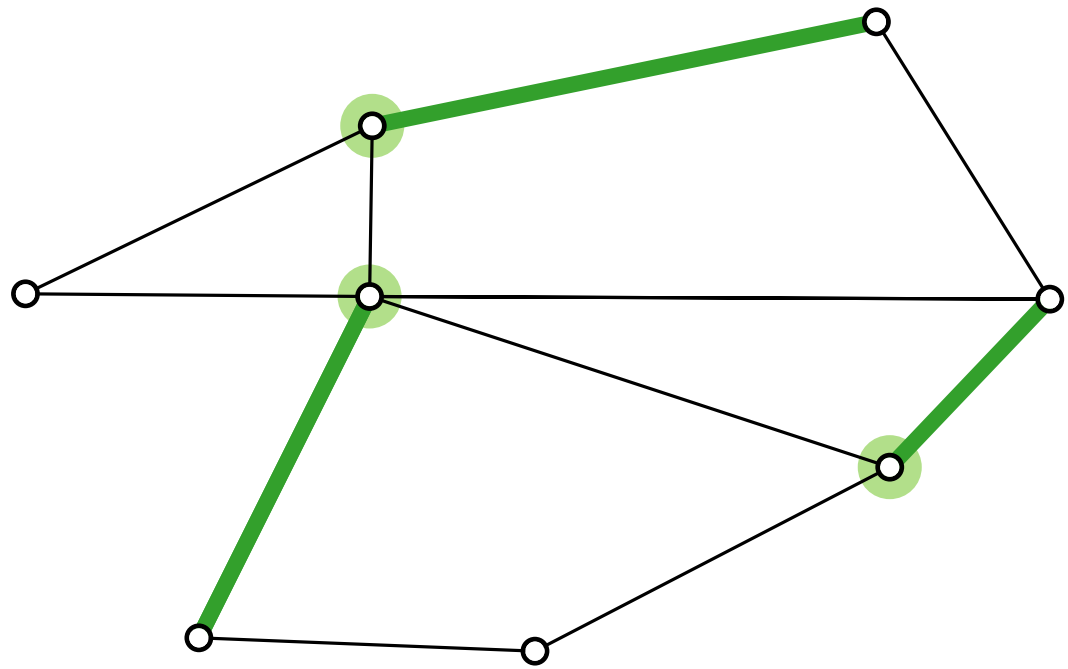

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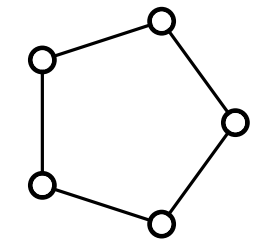

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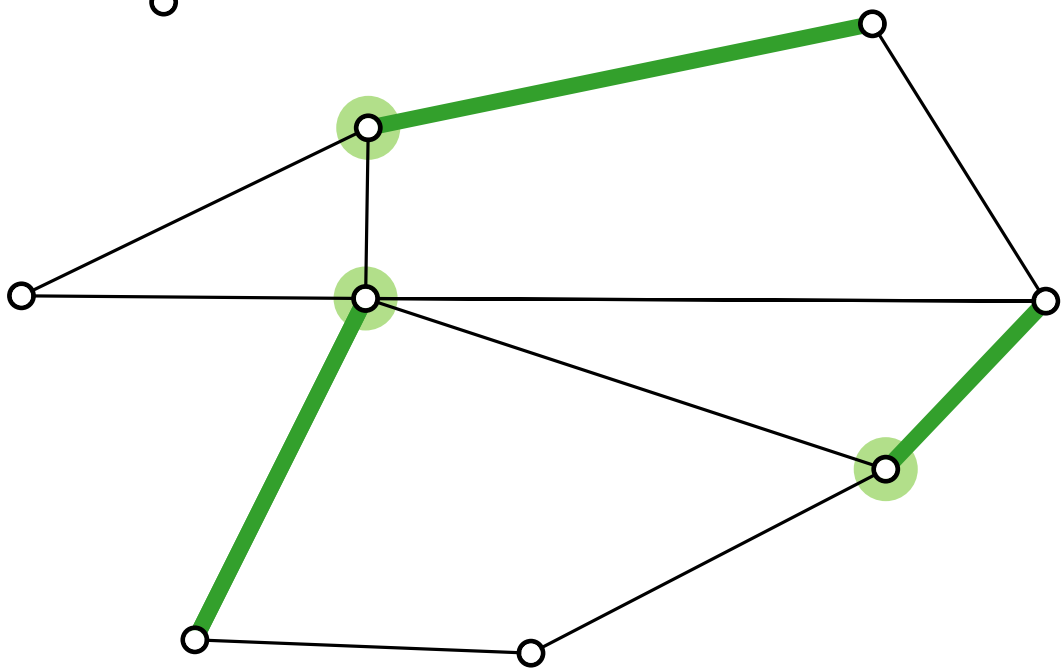

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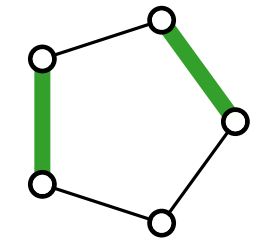

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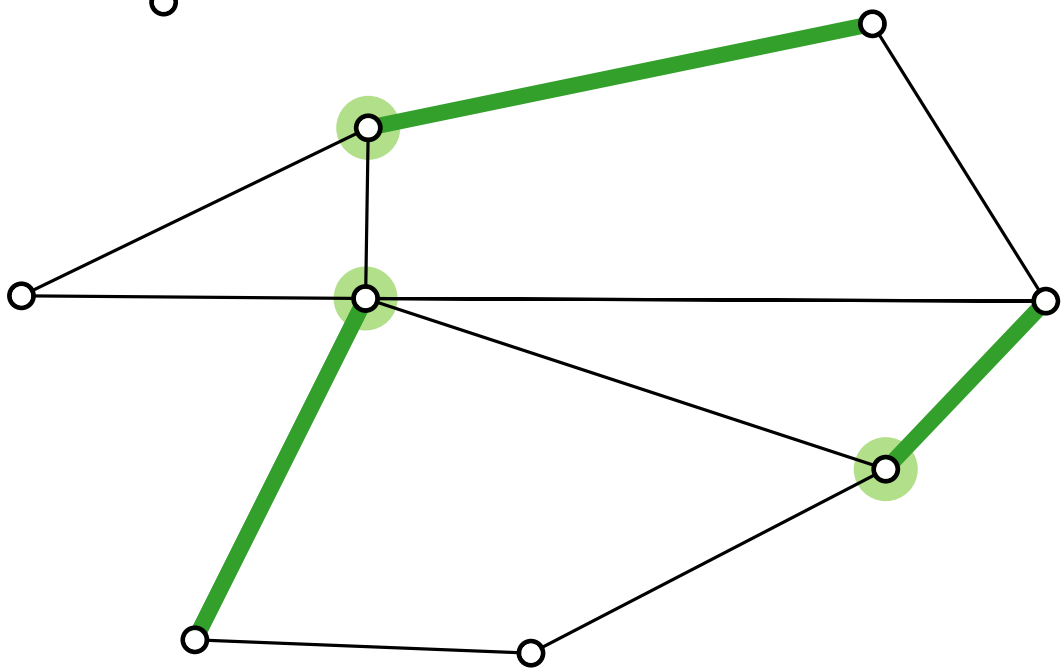

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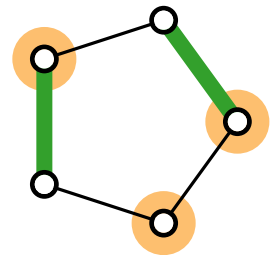

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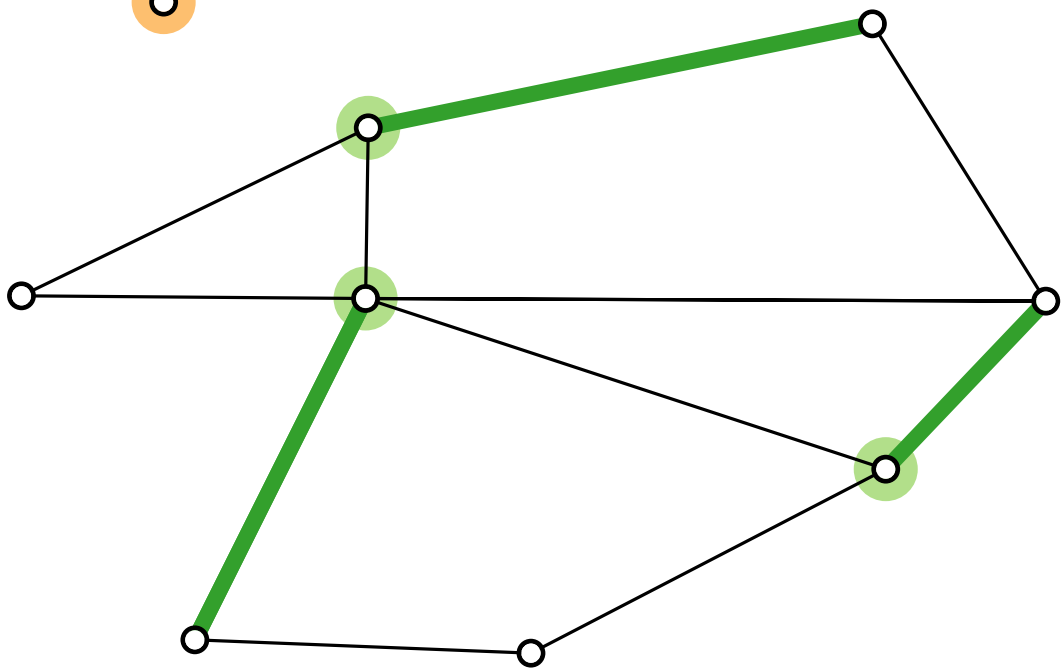

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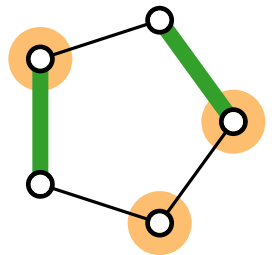

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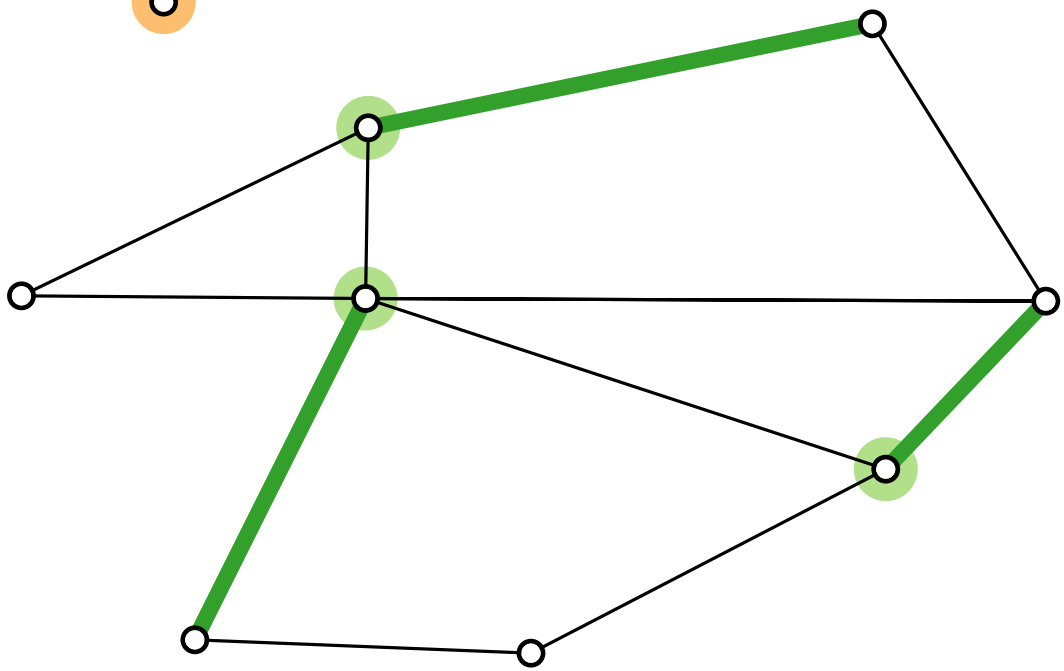

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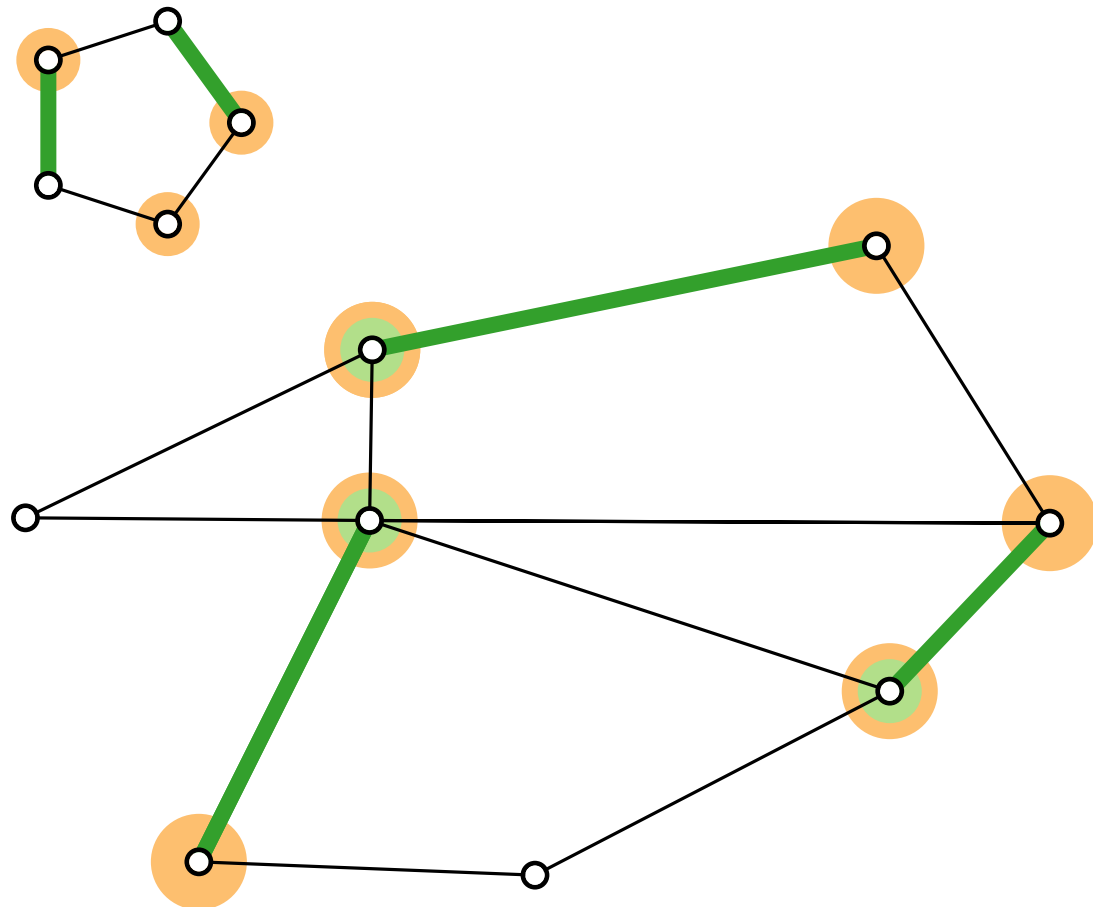
Vertex cover of M

Vertex cover of E



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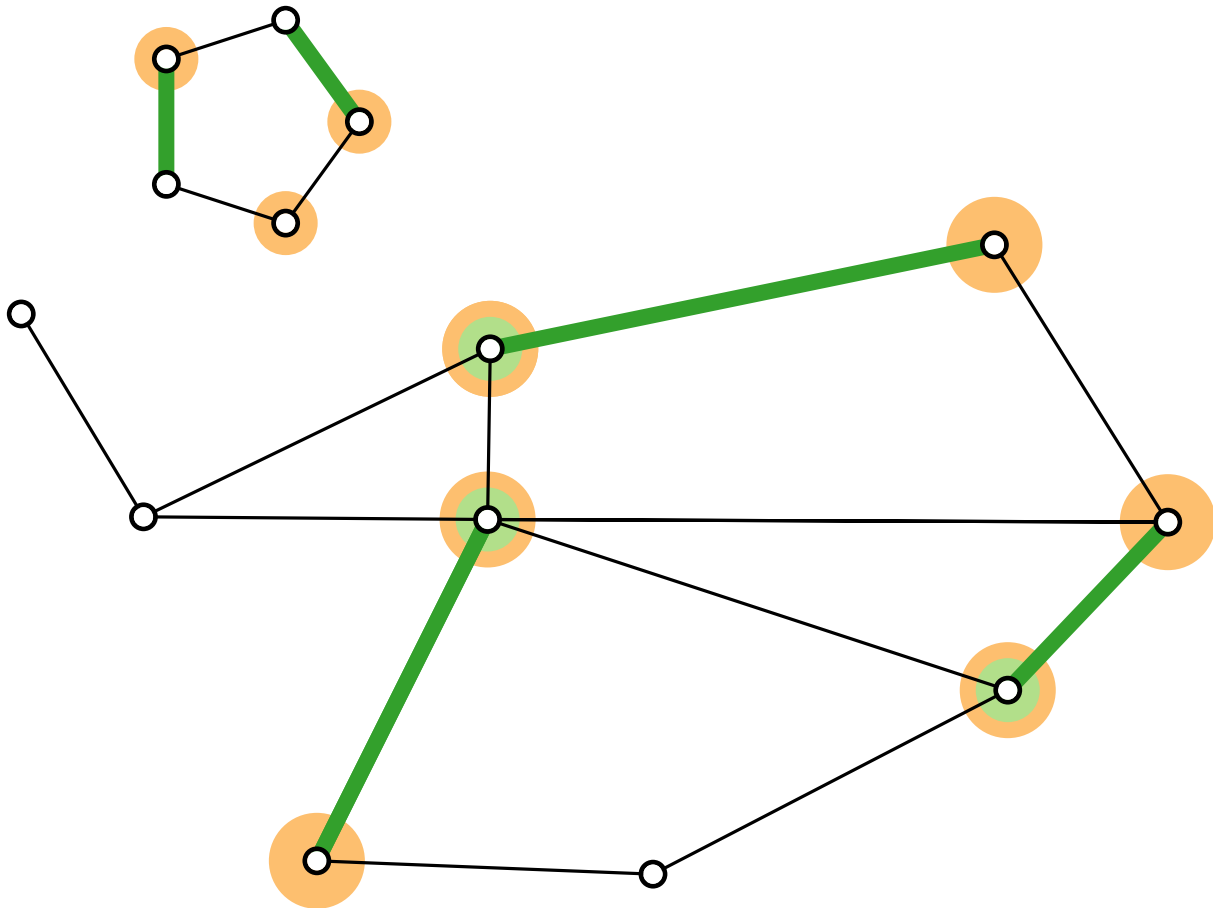

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Vertex cover of M

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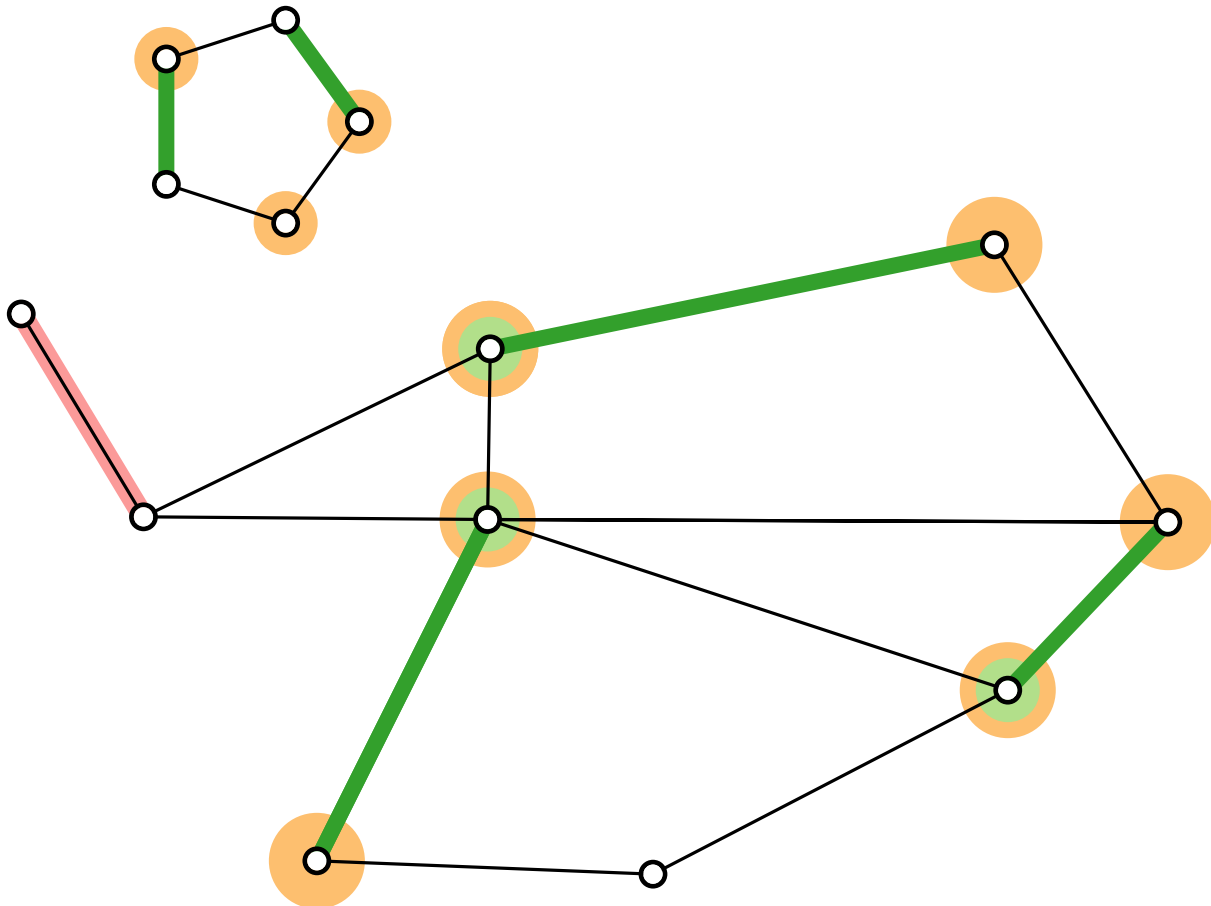

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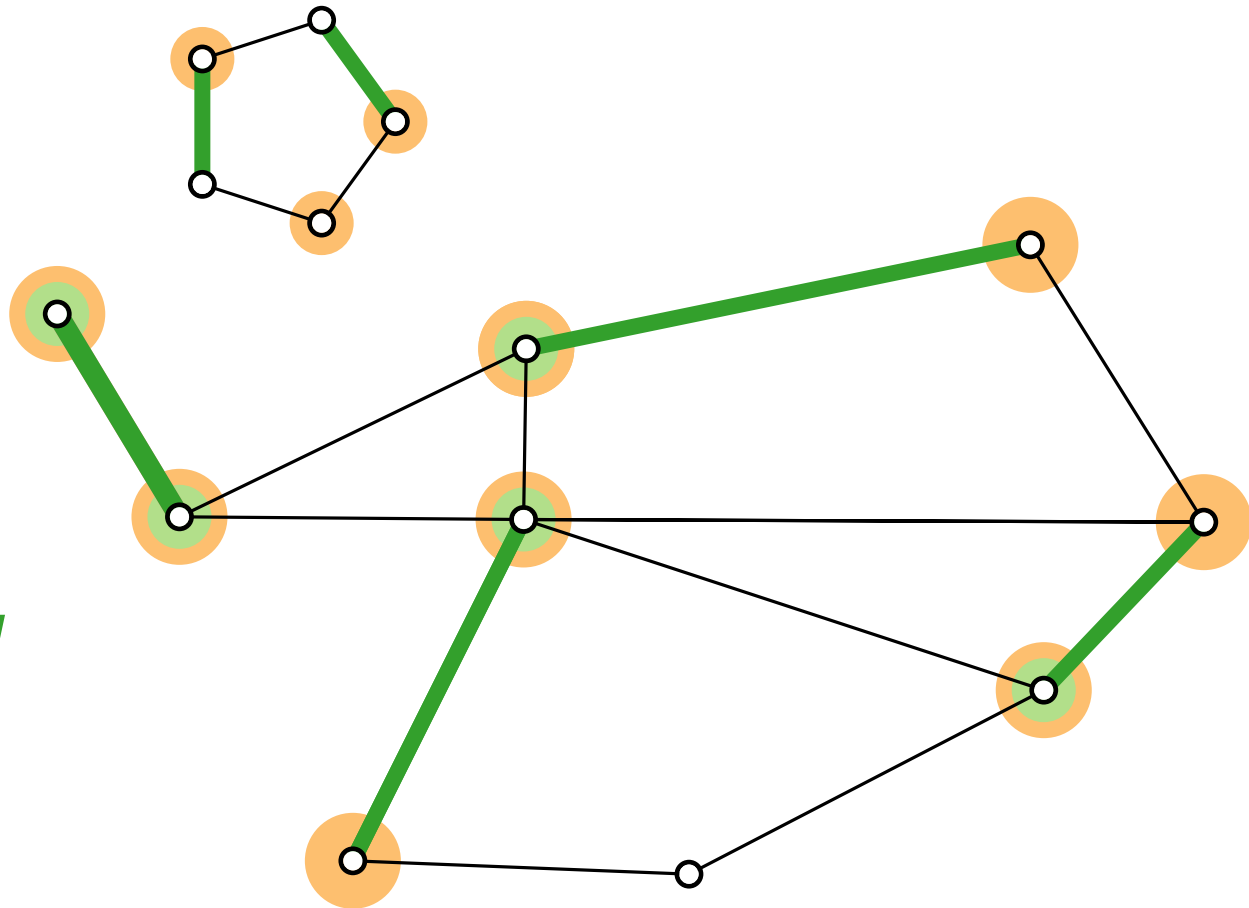

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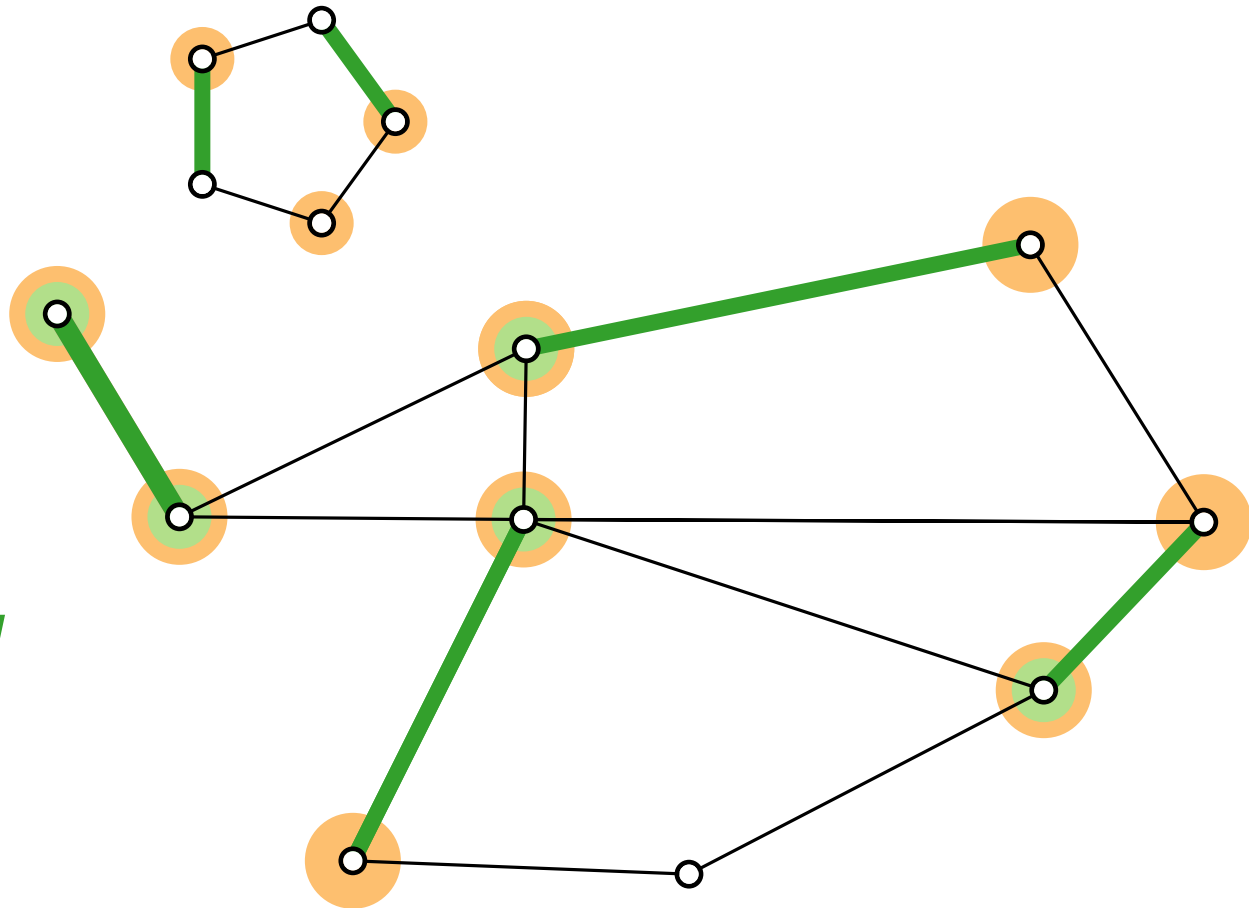
Vertex cover of M

Vertex cover of E

$\mathrm{ALG} = 2 \cdot |M| \leq$



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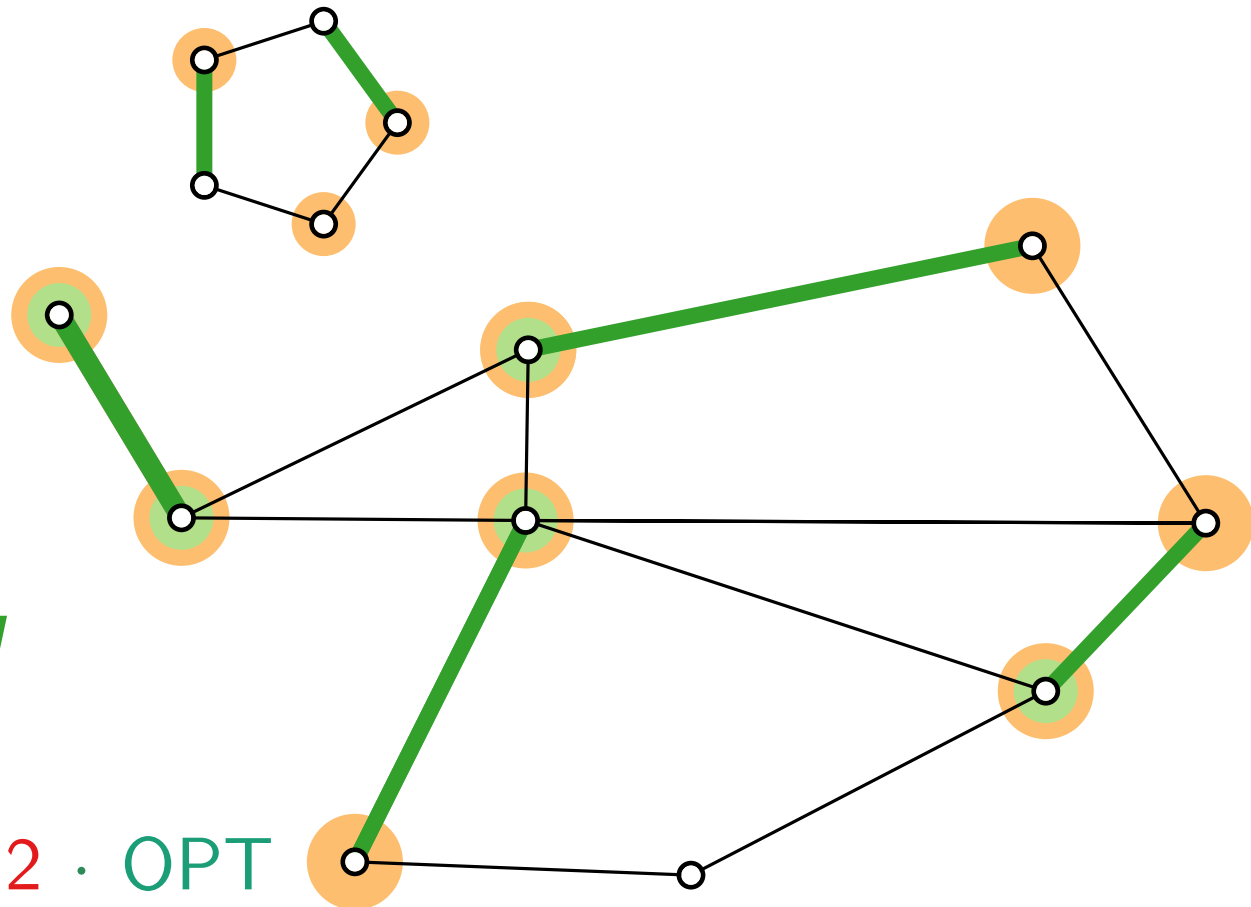

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Vertex cover of M

Vertex cover of E

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Approximation Alg. for VertexCover

Algorithm VertexCover(G)

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Theorem. The above algorithm is a factor-2 approximation algorithm for VertexCover.

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VertexCover cannot be approximated within a factor of $2 - \Theta(1)$ – if the *Unique Games Conjecture* holds.