Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part I:
Maximum Satisfiability (MaxSat)

Joachim Spoerhase

Winter 2021/22

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Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part II:
A Simple Randomized Algorithm

Joachim Spoerhase

Winter 2021/22

A Simple Randomized Algorithm

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Thus, $E[W] \geq 1/2 \sum_{j=1}^{m} w_j \geq \mathrm{OPT}/2.$ □

Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part III:
Derandomization by Conditional Expectation

Joachim Spoerhase

Winter 2021/22

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Theorem. The previous algorithm can be derandomized, i.e., there is a deterministic $1/2$-approximation algorithm for MaxSat.

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Thus, the algorithm can be derandomized if the conditional expectation can be computed efficiently!

Consider a partial assignment $x_1 = b_1, \ldots, x_i = b_i$ and a clause C_j.

If C_j is already satisfied, then it contributes exactly to $E[W | x_1 = b_1, \ldots, x_i = b_i]$.

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Consider a partial assignment $x_1 = b_1, \ldots, x_i = b_i$ and a clause C_j.

If C_j is already satisfied, then it contributes exactly w_j to $E[W | x_1 = b_1, \ldots, x_i = b_i]$.

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If C_j is already satisfied, then it contributes exactly w_j to $E[W|x_1 = b_1, \ldots, x_i = b_i]$.

If C_j is not yet satisfied and contains k unassigned variables, then it contributes exactly to $E[W|x_1 = b_1, \ldots, x_i = b_i]$.

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The conditional expectation is simply the sum of the contributions from each clause. $\square$

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Standard procedure with which many randomized algorithms can be derandomized.

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Global optimization?

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Part IV:
Randomized Rounding

Joachim Spoerhase

Winter 2021/22

An ILP

maximize

subject to

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$ for $j = 1, \ldots, m$.

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$$y_i \in \{0,1\}, \qquad \text{for } i = 1,\ldots,n$$

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$$\textbf{maximize} \quad \sum_{j=1}^{m} w_j z_j$$

subject to

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$$
\begin{aligned}
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\textbf{subject to} \quad & \sum_{i \in P_j} \qquad + \sum_{i \in N_j} & \text{for } j = 1, \ldots, m \\
& y_i \in \{0, 1\}, & \text{for } i = 1, \ldots, n \\
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$$\begin{aligned}
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where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$ for $j = 1, \ldots, m$.

An ILP

$$
\begin{aligned}
\textbf{maximize} \quad & \sum_{j=1}^{m} w_j z_j \\
\textbf{subject to} \quad & \sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) && \text{for } j = 1, \ldots, m \\
& y_i \in \{0, 1\}, && \text{for } i = 1, \ldots, n \\
& z_j \in \{0, 1\}, && \text{for } j = 1, \ldots, m
\end{aligned}
$$

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$ for $j = 1, \ldots, m$.

An ILP

$$
\begin{array}{lll}
\textbf{maximize} & \displaystyle\sum_{j=1}^{m} w_j z_j & \\
\textbf{subject to} & \displaystyle\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq & \text{for } j = 1, \ldots, m \\
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An ILP

$$\begin{aligned}
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... and its Relaxation

$$\begin{array}{lll}
\textbf{maximize} & \displaystyle\sum_{j=1}^{m} w_j z_j & \\
\textbf{subject to} & \displaystyle\sum_{i \in P_j} y_i + \sum_{i \in N_j} (1 - y_i) \geq z_j & \text{for } j = 1, \ldots, m \\
& 0 \leq y_i \leq 1, & \text{for } i = 1, \ldots, n \\
& 0 \leq z_j \leq 1, & \text{for } j = 1, \ldots, m
\end{array}$$

where $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{i \in N_j} \bar{x}_i$ for $j = 1, \ldots, m$.

Randomized Rounding

Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation.

Randomized Rounding

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Randomized Rounding

Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation. Independently setting each variable x_i to 1 with probability y_i^* provides a $(1 - 1/e)$-approximation for MaxSat.

Randomized Rounding

Theorem. Let (y^*, z^*) be an optimal solution to the LP-relaxation. Independently setting each variable x_i to 1 with probability y_i^* provides a $\underbrace{(1 - 1/e)}_{\approx 0.63}$-approximation for MaxSat.

Approximation Algorithms

Lecture 11:
MaxSat via Randomized Rounding

Part V:
Randomized Rounding – Proof

Joachim Spoerhase

Winter 2021/22

Mathematical Toolkit

Let f be function that is concave on $[0, 1]$

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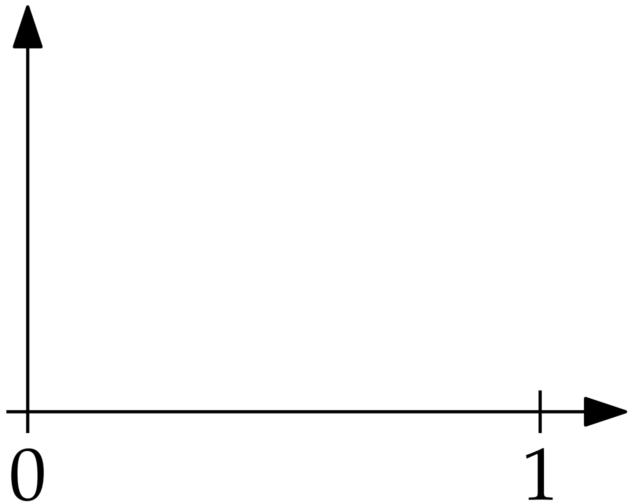

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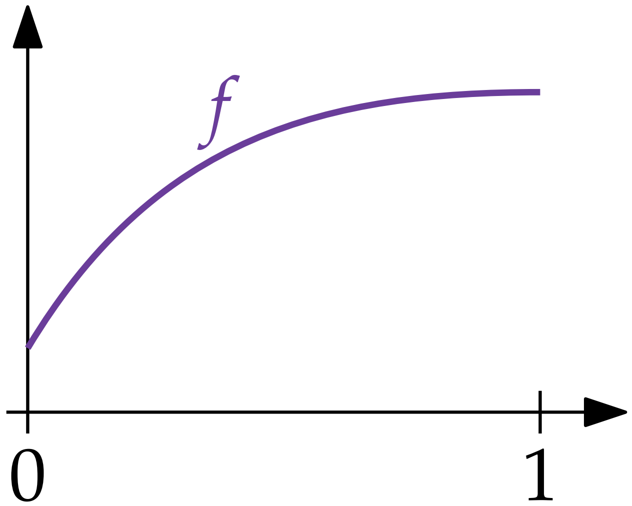

Mathematical Toolkit

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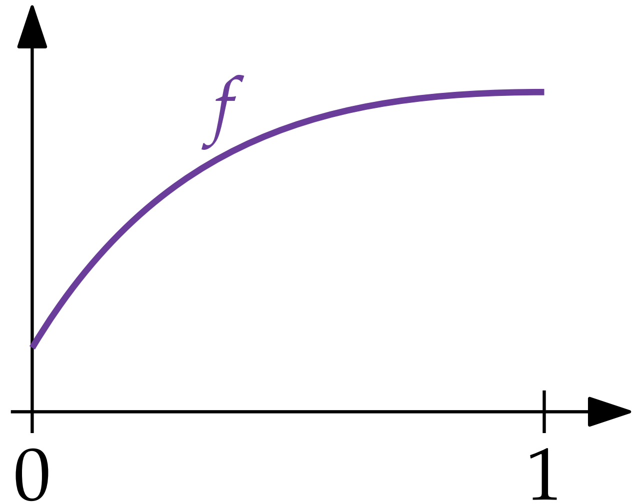

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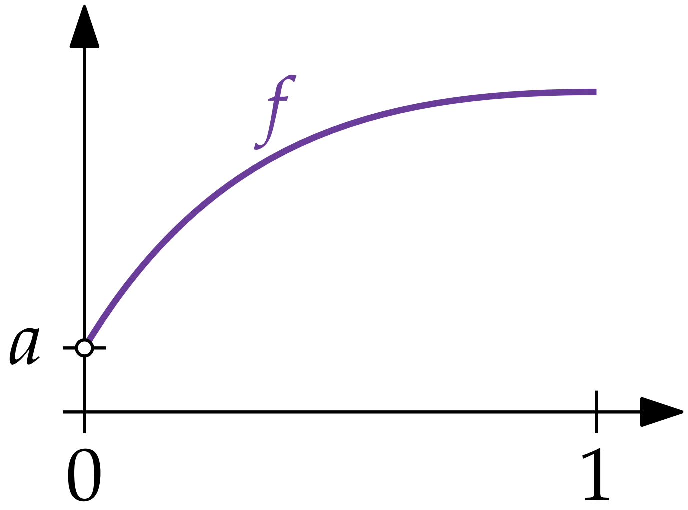

Mathematical Toolkit

Let f be function that is concave on $[0, 1]$
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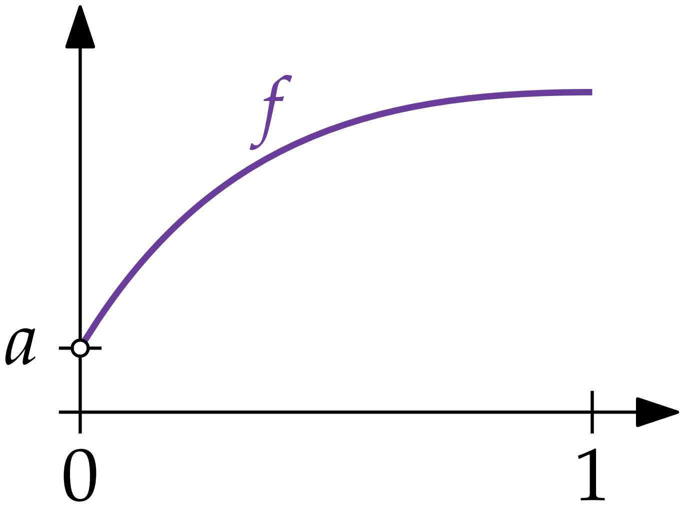

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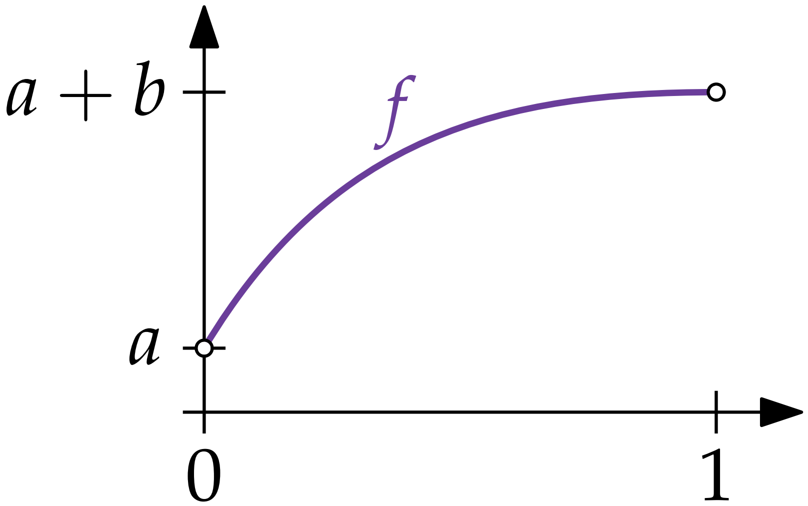

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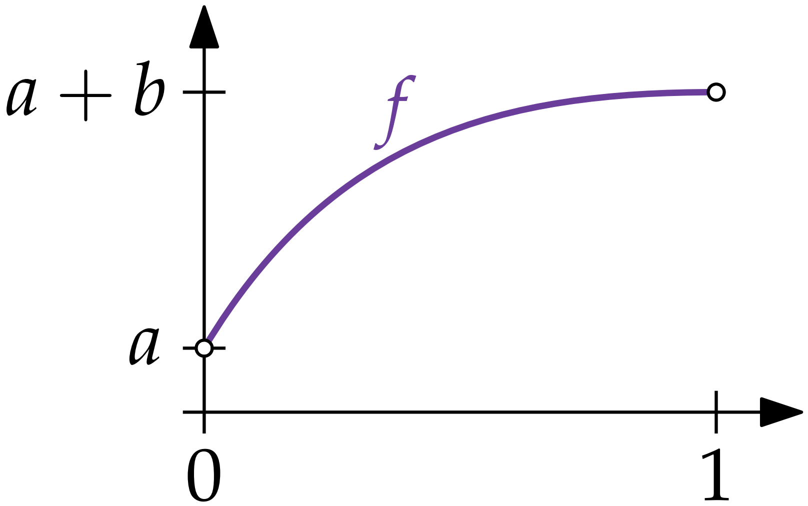
$$\Rightarrow f(x) \geq bx + a \text{ for } x \in [0,1].$$



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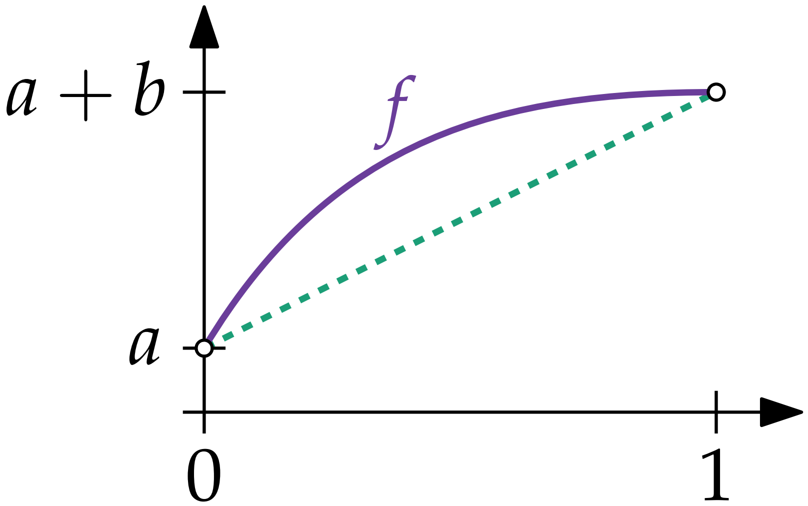

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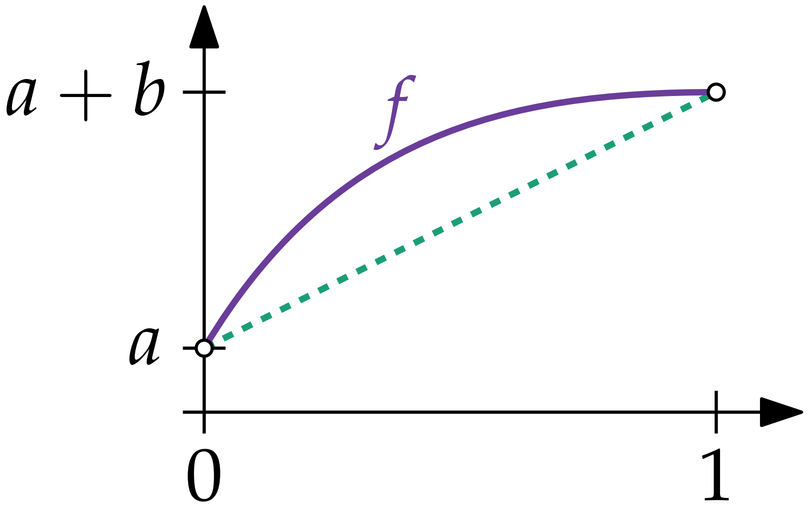

Arithmetic-Geometric Mean Inequality (AGMI):

Mathematical Toolkit

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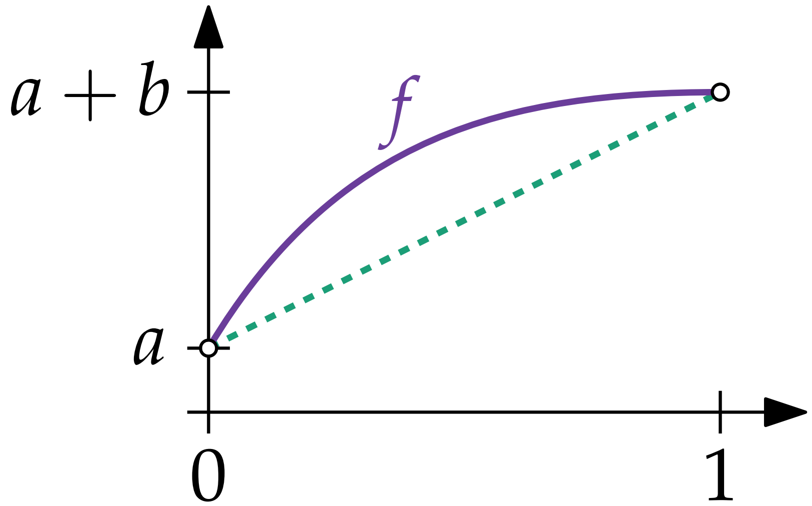

Arithmetic-Geometric Mean Inequality (AGMI):

For all non-negative numbers $a_1, \ldots, a_k$:

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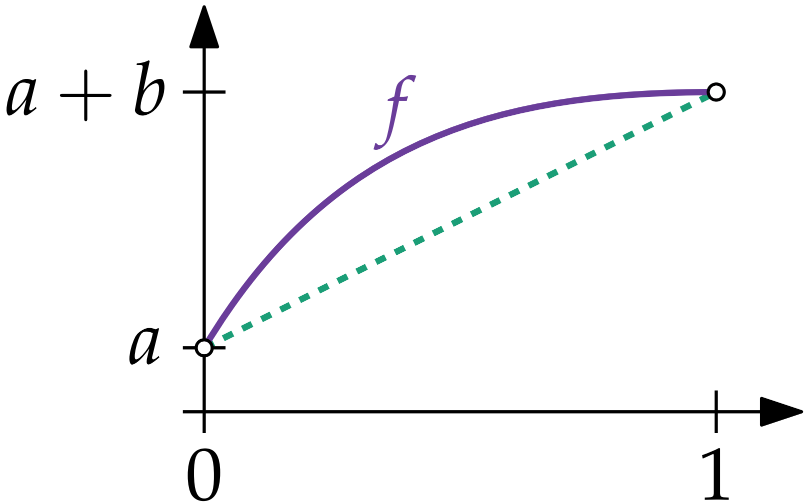

Arithmetic-Geometric Mean Inequality (AGMI):

For all non-negative numbers $a_1, \ldots, a_k$:

$$\left(\prod_{i=1}^{k} a_i\right)^{1/k} \leq$$

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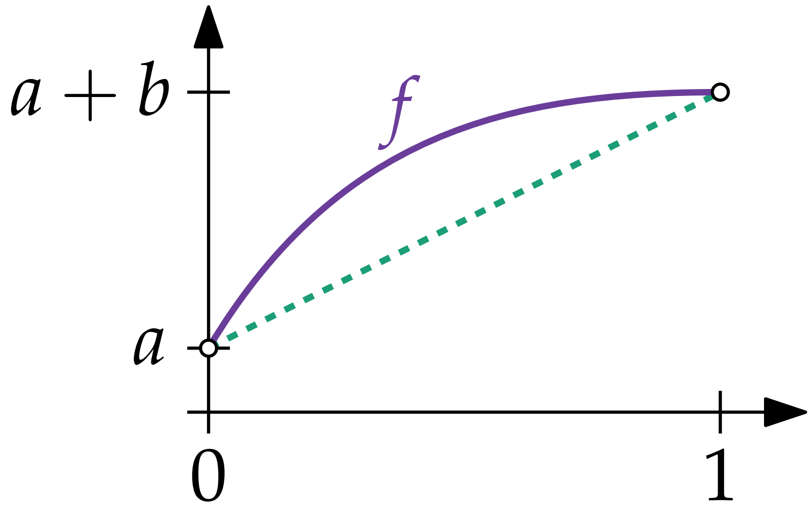

Arithmetic-Geometric Mean Inequality (AGMI):

For all non-negative numbers $a_1, \ldots, a_k$:

$$\left(\prod_{i=1}^{k} a_i\right)^{1/k} \leq \frac{1}{k}\left(\sum_{i=1}^{k} a_i\right)$$

Randomized Rounding (Proof)

Consider a fixed clause C_j of length l_j. Then we have:

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AGMI

$$\leq \left(\sum_{i \in P_j} (1 - y_i^*) + \sum_{i \in N_j} y_i^*\right)$$

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$$\leq \left[\frac{1}{l_j} \left(\sum_{i \in P_j} (1 - y_i^*) + \sum_{i \in N_j} y_i^* \right) \right]^{l_j}$$

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$$= \left[1 - \frac{1}{l_j} \left(\sum_{i \in P_j} y_i^* + \sum_{i \in N_j} (1 - y_i^*) \right) \right]^{l_j}$$

Randomized Rounding (Proof)

Consider a fixed clause C_j of length l_j. Then we have:

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by LP constraints

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$$\leq \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$$

Randomized Rounding (Proof)

The function $f(z_j^*) = 1 - \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$ is concave on $[0, 1]$.

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$$\Pr[C_j \text{ satisfied}] \geq f(z_j^*) \geq$$

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The function $f(z_j^*) = 1 - \left(1 - \frac{z_j^*}{l_j}\right)^{l_j}$ is concave on $[0, 1]$. Thus

$$\Pr[C_j \text{ satisfied}] \geq f(z_j^*) \geq f(1) \cdot z_j^* + f(0)$$

$$\geq$$

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$1 + x \leq e^x$

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$x = -\frac{1}{l_j}$

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$x = -\frac{1}{l_j} \Rightarrow 1 - \frac{1}{l_j} \leq e^{-1/l_j}$

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$$1 + x \leq e^x$$

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Randomized Rounding (Proof)

Therefore

$$E[W] = \sum_{j=1}^{m} \Pr[C_j \text{ satisfied}] \cdot w_j$$

$$\geq$$

Randomized Rounding (Proof)

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$$E[W] = \sum_{j=1}^{m} \Pr[C_j \text{ satisfied}] \cdot w_j$$

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$$=$$

Randomized Rounding (Proof)

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\end{aligned}
$$

□

Randomized Rounding (Proof)

Therefore

$$
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&\geq \left(1 - \frac{1}{e}\right) \mathrm{OPT}
\end{aligned}
$$

□

Theorem. The previous algorithm can be derandomized by the method of conditional expectation.

Approximation Algorithms

Lecture 11: MaxSat via Randomized Rounding

Part VI: Combining the Algorithms

Joachim Spoerhase

Winter 2021/22

Take the better of the two solutions!

Theorem. The better solution among the randomized algorithm and the randomized LP-rounding algorithm provides a   -approximation for MaxSat.

Take the better of the two solutions!

Theorem. The better solution among the randomized algorithm and the randomized LP-rounding algorithm provides a $3/4$-approximation for MaxSat.

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Theorem. The better solution among the randomized algorithm and the randomized LP-rounding algorithm provides a $3/4$-approximation for MaxSat.

Proof.

We use another probabilistic argument. With probability $1/2$ choose the solution of the first algorithm, otherwise the solution of the second algorithm.

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Theorem. The better solution among the randomized algorithm and the randomized LP-rounding algorithm provides a $3/4$-approximation for MaxSat.

Proof.

We use another probabilistic argument. With probability $1/2$ choose the solution of the first algorithm, otherwise the solution of the second algorithm.

The better solution is at least as good as the expectation of the above algorithm.

Take the better of the two solutions!

The probability that clause C_j is satisfied is at least:

Take the better of the two solutions!

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$$\frac{1}{2}\left[\qquad \qquad \qquad \qquad \qquad \qquad \right].$$

Take the better of the two solutions!

The probability that clause C_j is satisfied is at least:

$$\frac{1}{2}\left[\underbrace{\left(1-\left(1-\frac{1}{l_j}\right)^{l_j}\right) z_j^*}_{\text{LP-Rounding}} \qquad\qquad\qquad\right].$$

Take the better of the two solutions!

The probability that clause C_j is satisfied is at least:

$$\frac{1}{2}\left[\underbrace{\left(1-\left(1-\frac{1}{l_j}\right)^{l_j}\right) z_j^*}_{\text{LP-Rounding}}+\underbrace{\left(1-2^{-l_j}\right)}_{\text{rand. Alg.}}\right].$$

Take the better of the two solutions!

The probability that clause C_j is satisfied is at least:

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For $l_j \geq 3$, $1-(1-1/l_j)^{l_j} \geq (1-1/e)$ and $1-2^{-l_j} \geq \frac{7}{8}$.

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Thus, we have at least:

$$\frac{1}{2}\left[\left(1-\frac{1}{e}\right)+\frac{7}{8}\right]z_j^* \approx$$

□

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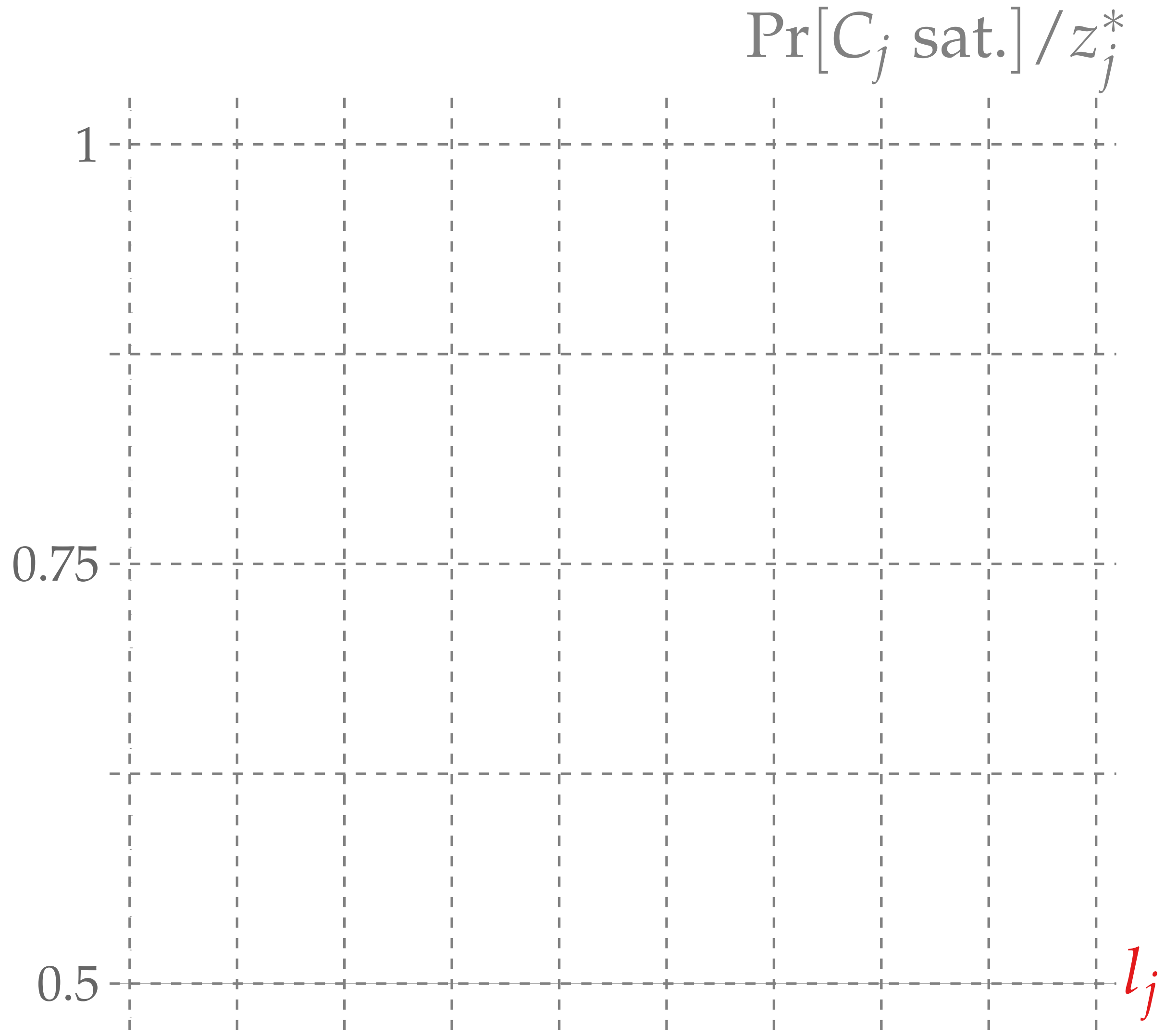
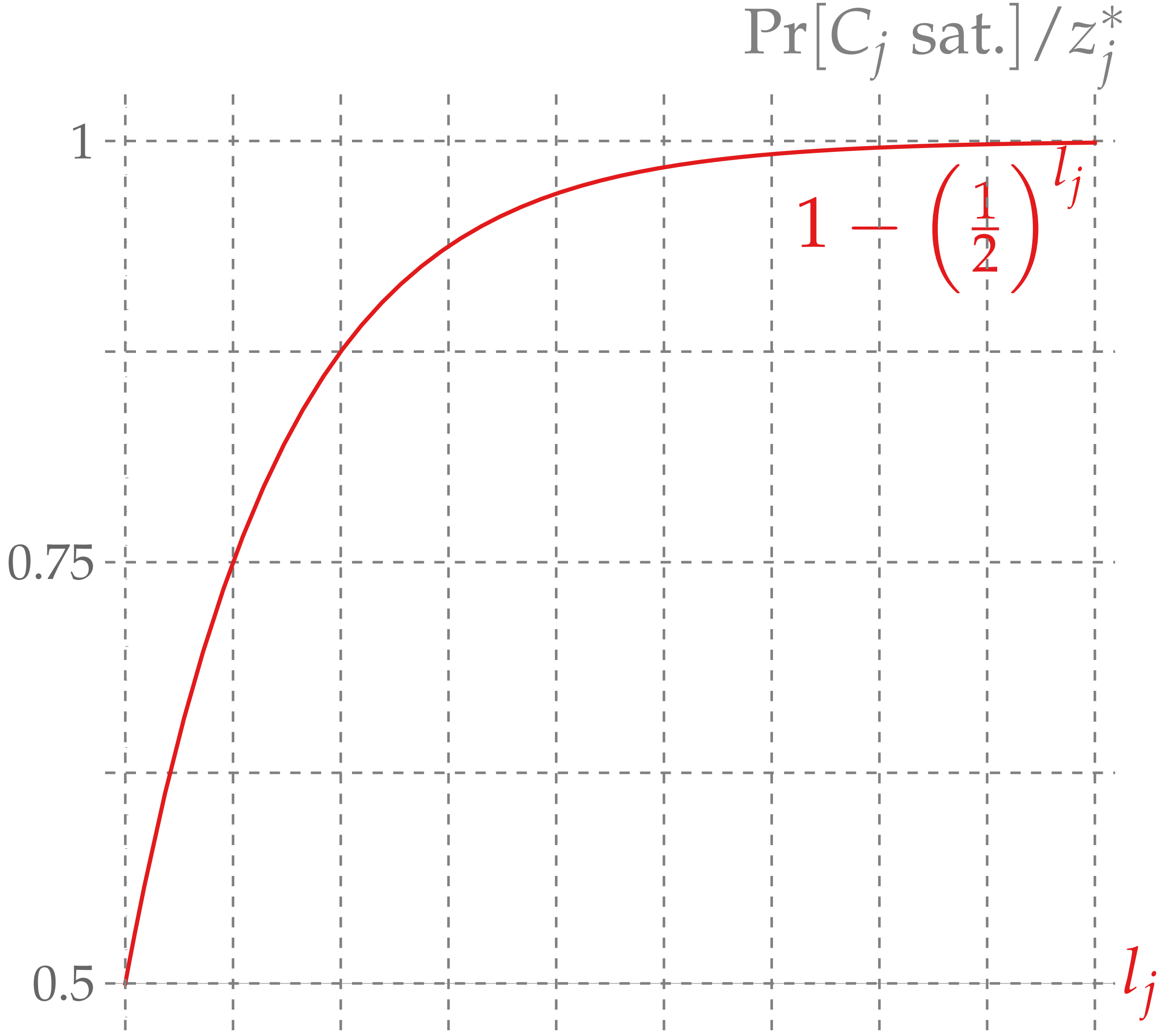
□

Visualization and Derandomization

Visualization and Derandomization



Visualization and Derandomization



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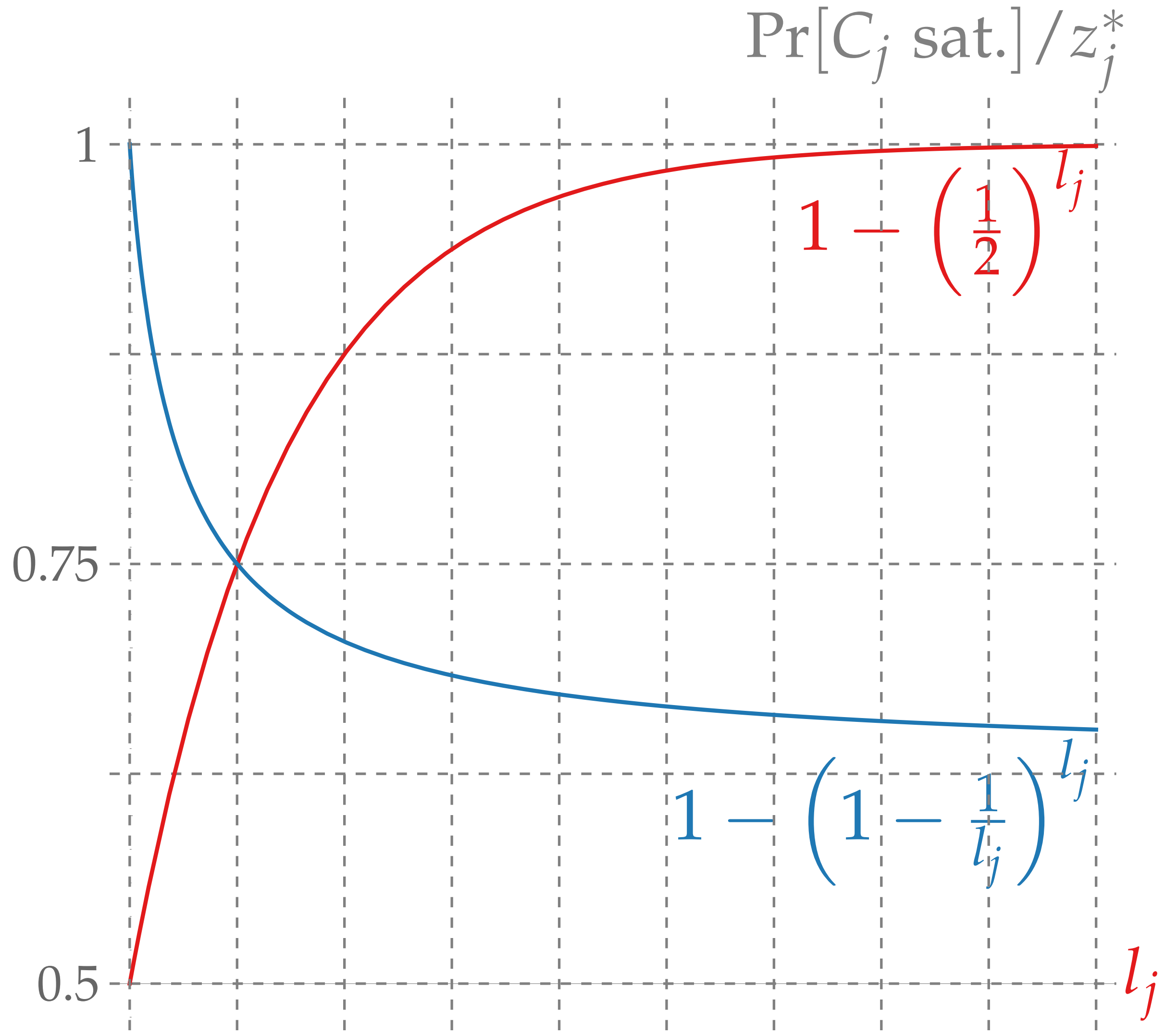

Visualization and Derandomization

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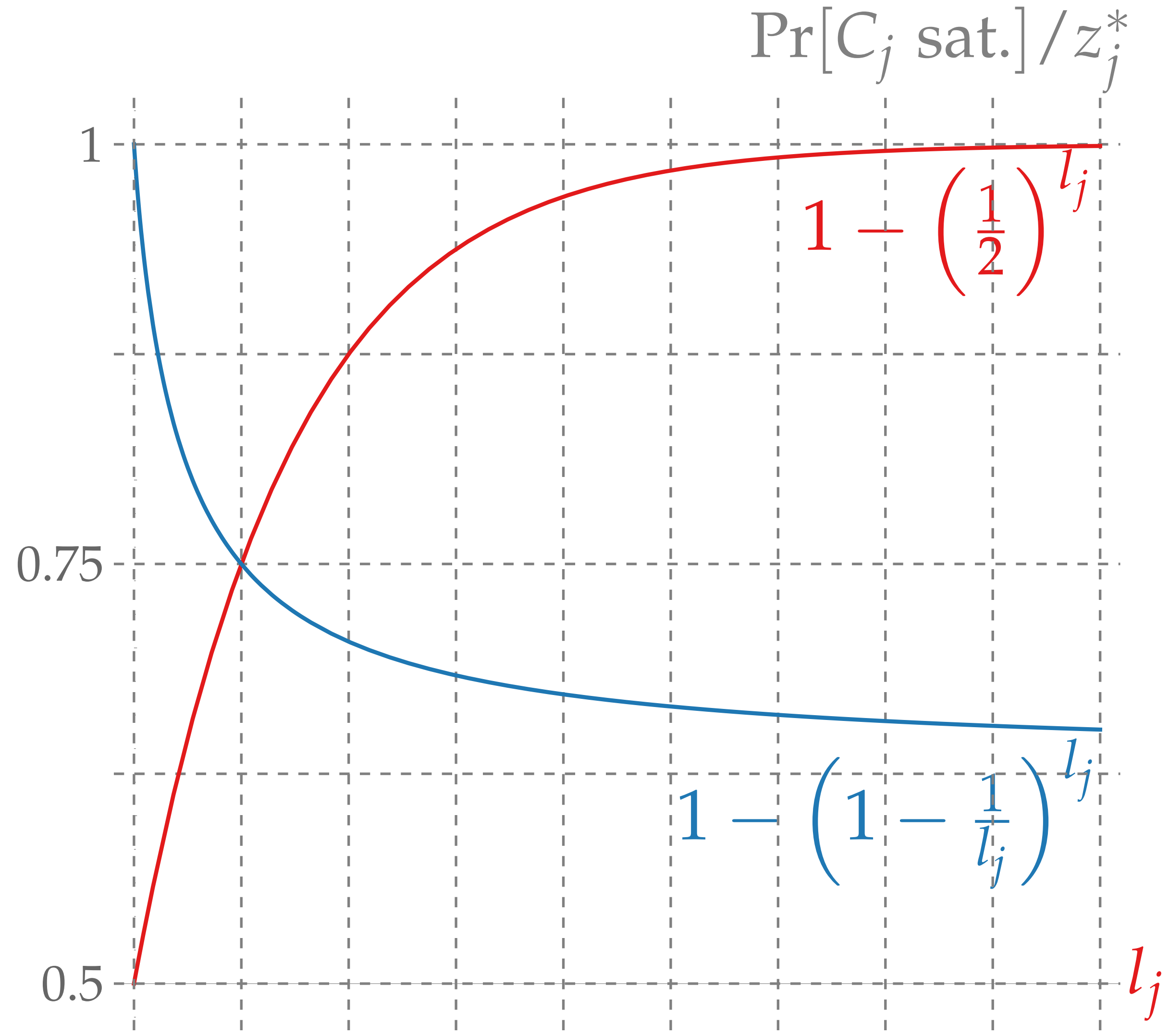

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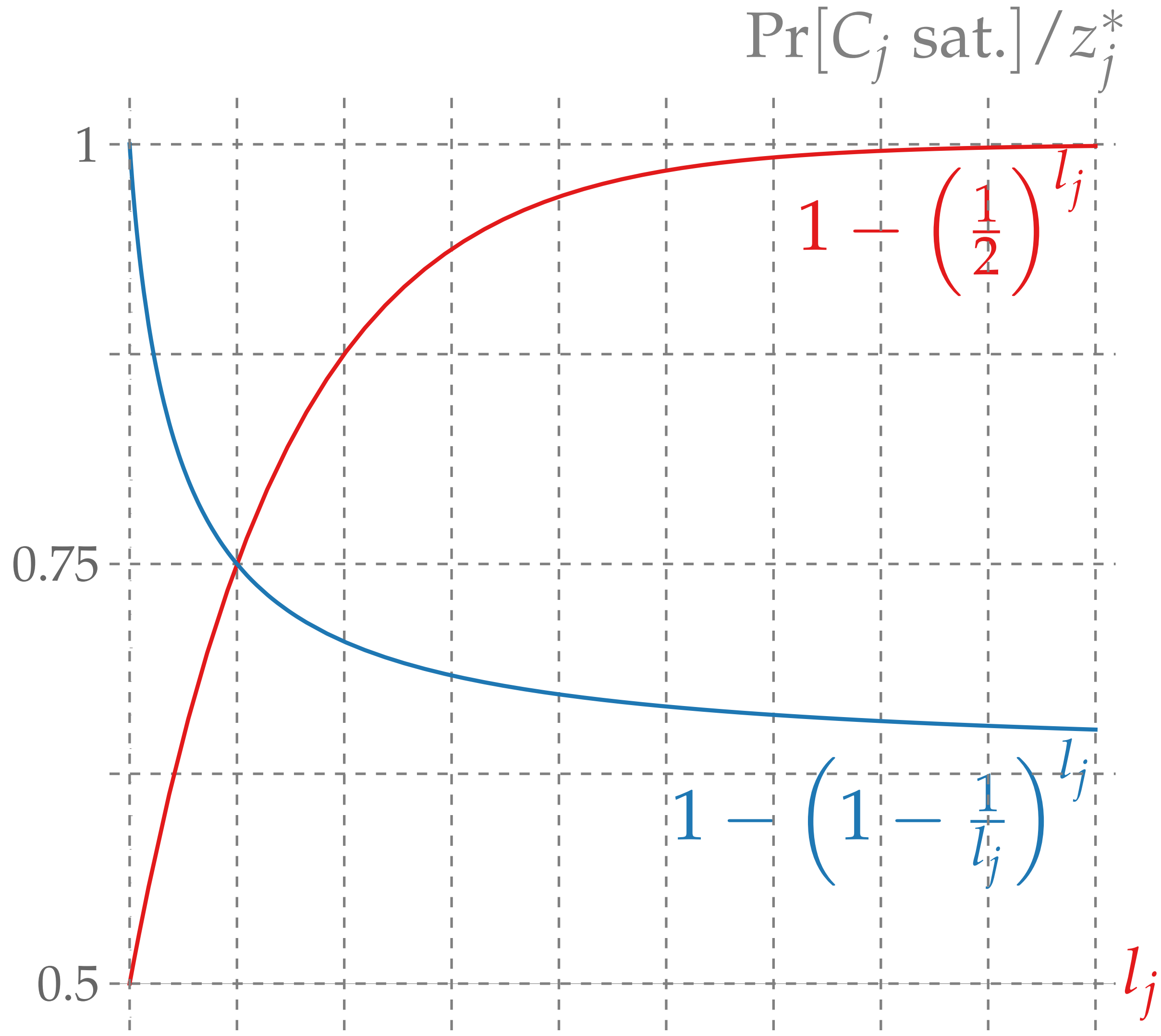

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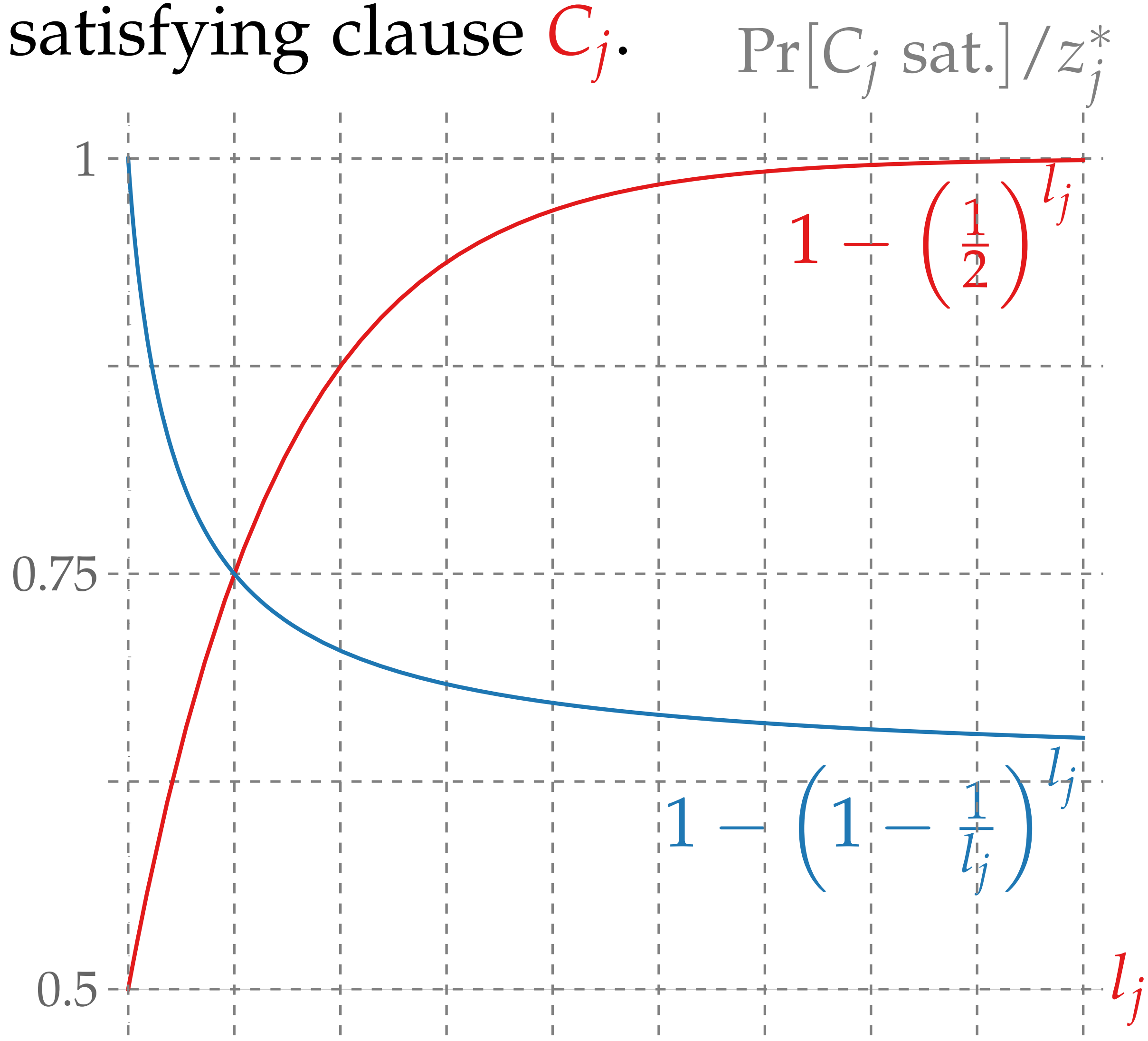

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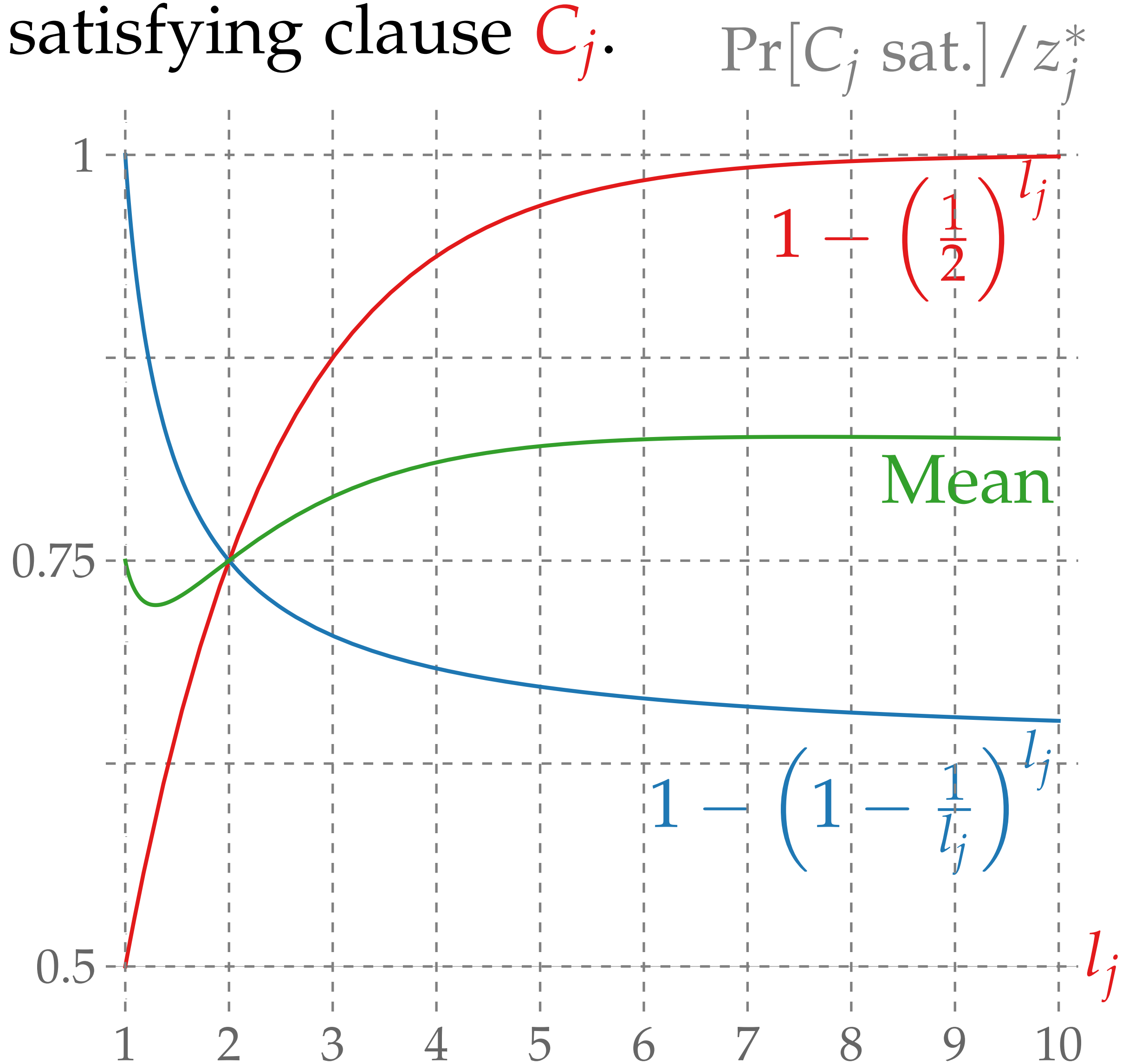

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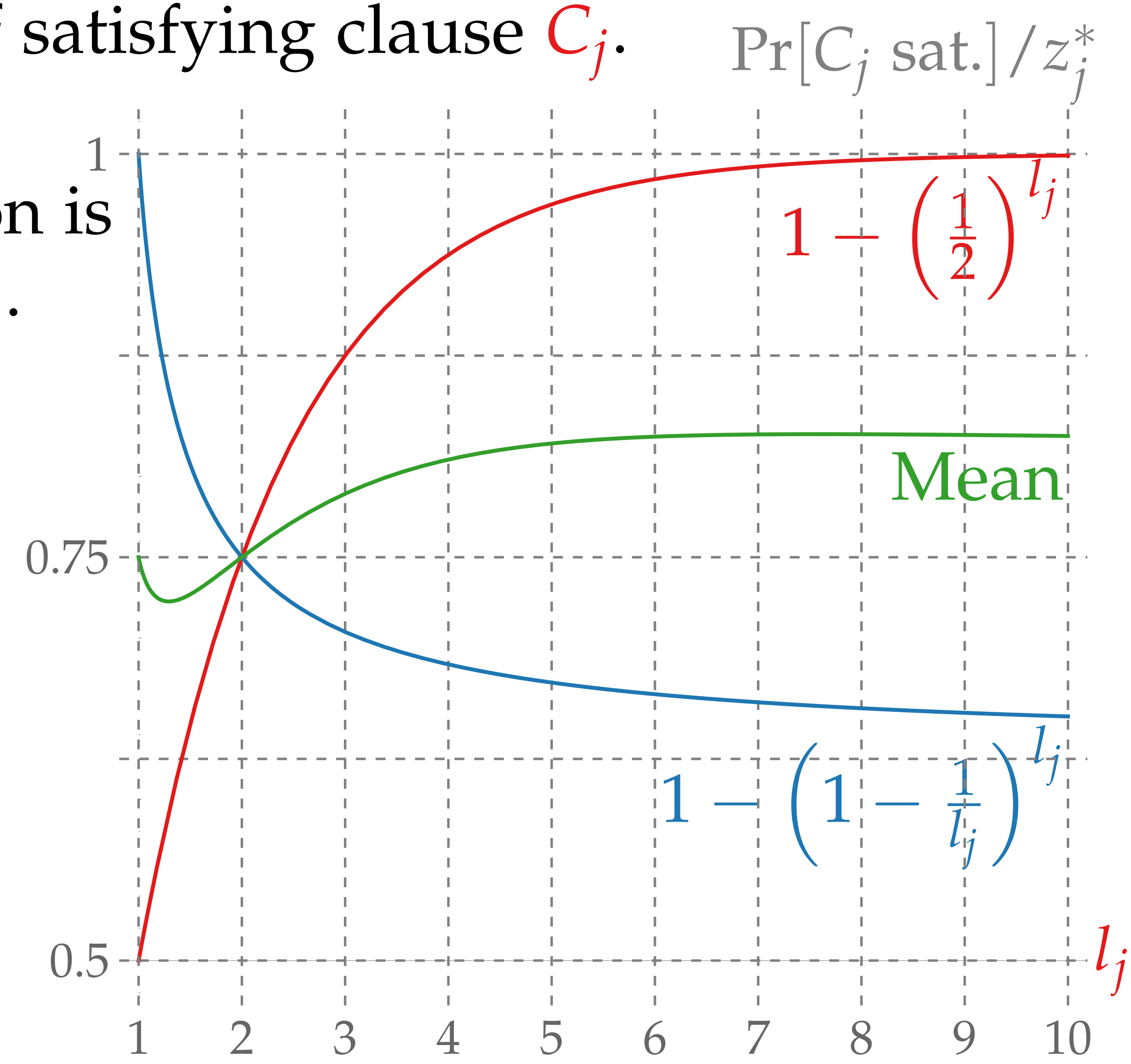
Mean of the two solution is at least 3/4 for *integer* l_j.



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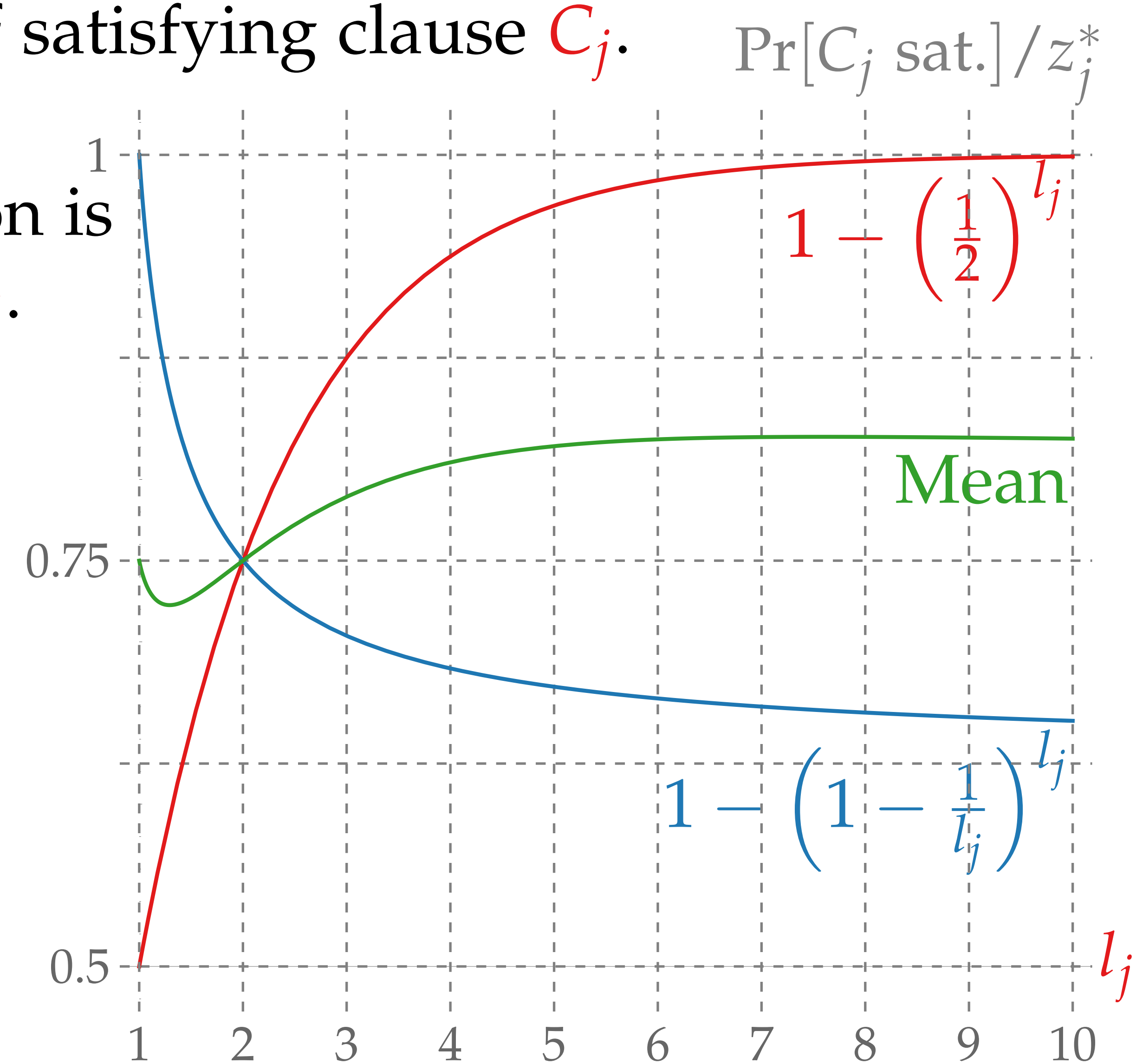
Maximum is at least as large as the mean.



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This algorithm can also be derandomized by conditional expectation.