# Approximation Algorithms

## Lecture 9: An Approximation Scheme for EuclideanTSP

### Part I: TravelingSalesmanProblem

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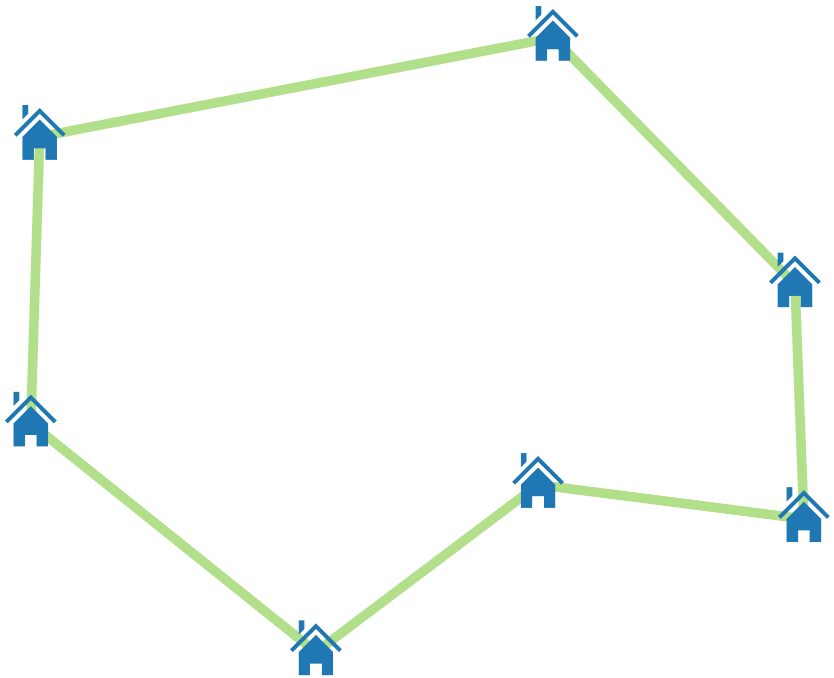
Winter 2021/22

# TRAVELINGSALESMANPROBLEM (TSP)

**Question:** What's the fastest way to deliver all parcels to their destination?

**Given:** A set of $n$ houses (points) in $\mathbb{R}^2$.

**Task:** Find a tour (Hamiltonian cycle) of min. length.



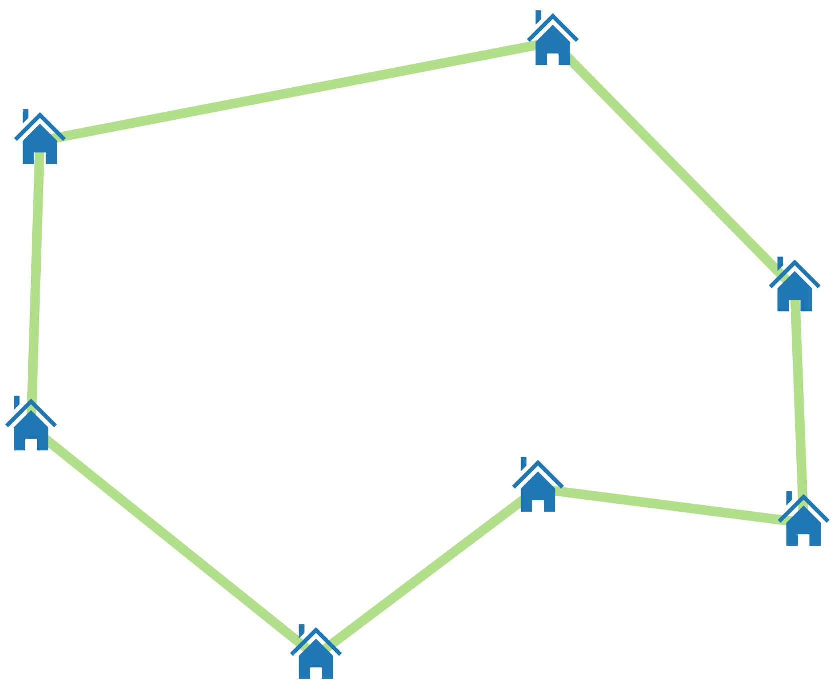
# TravelingSalesmanProblem (TSP)

**Question:** What's the fastest way to deliver all parcels to their destination?

**Given:** A set of $n$ houses (points) in $\mathbb{R}^2$.

**Task:** Find a tour (Hamiltonian cycle) of min. length.

Distance between two points?



For every polynomial $p(n)$, TSP cannot be approximated within factor $2^{p(n)}$ (unless P=NP).

There is a $3/2$-approximation algorithm for MetricTSP.

MetricTSP cannot be approximated within factor $123/122$ (unless P=NP).

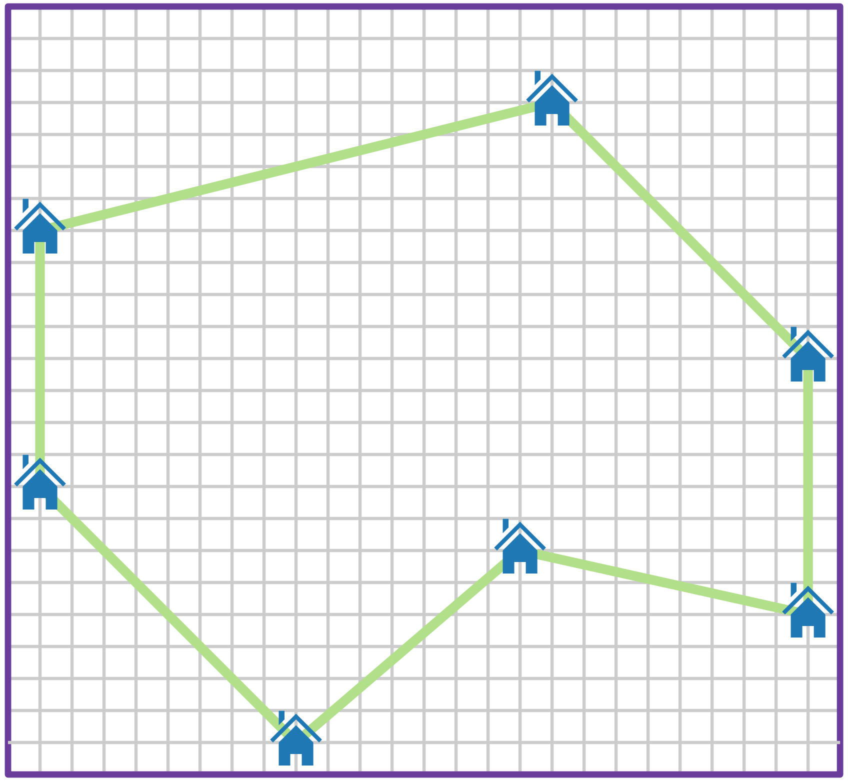
# TravelingSalesmanProblem (TSP)

**Question:** What's the fastest way to deliver all parcels to their destination?

**Given:** A set of $n$ houses (points) in $\mathbb{R}^2$.

**Task:** Find a tour (Hamiltonian cycle) of min. length.

The Salesman can fly $\Rightarrow$ Euclidean distance.



**Simplifying Assumptions**

- Houses inside $(L \times L)$-square
- $L := 4n^2 = 2^k$; $k = 2 + 2\log_2 n$
- integer coordinates

("justification": homework)

Goal: $(1+\varepsilon)$-approximation!

# Approximation Algorithms

## Lecture 9:
## PTAS for EuclideanTSP

### Part II:
### Dissection

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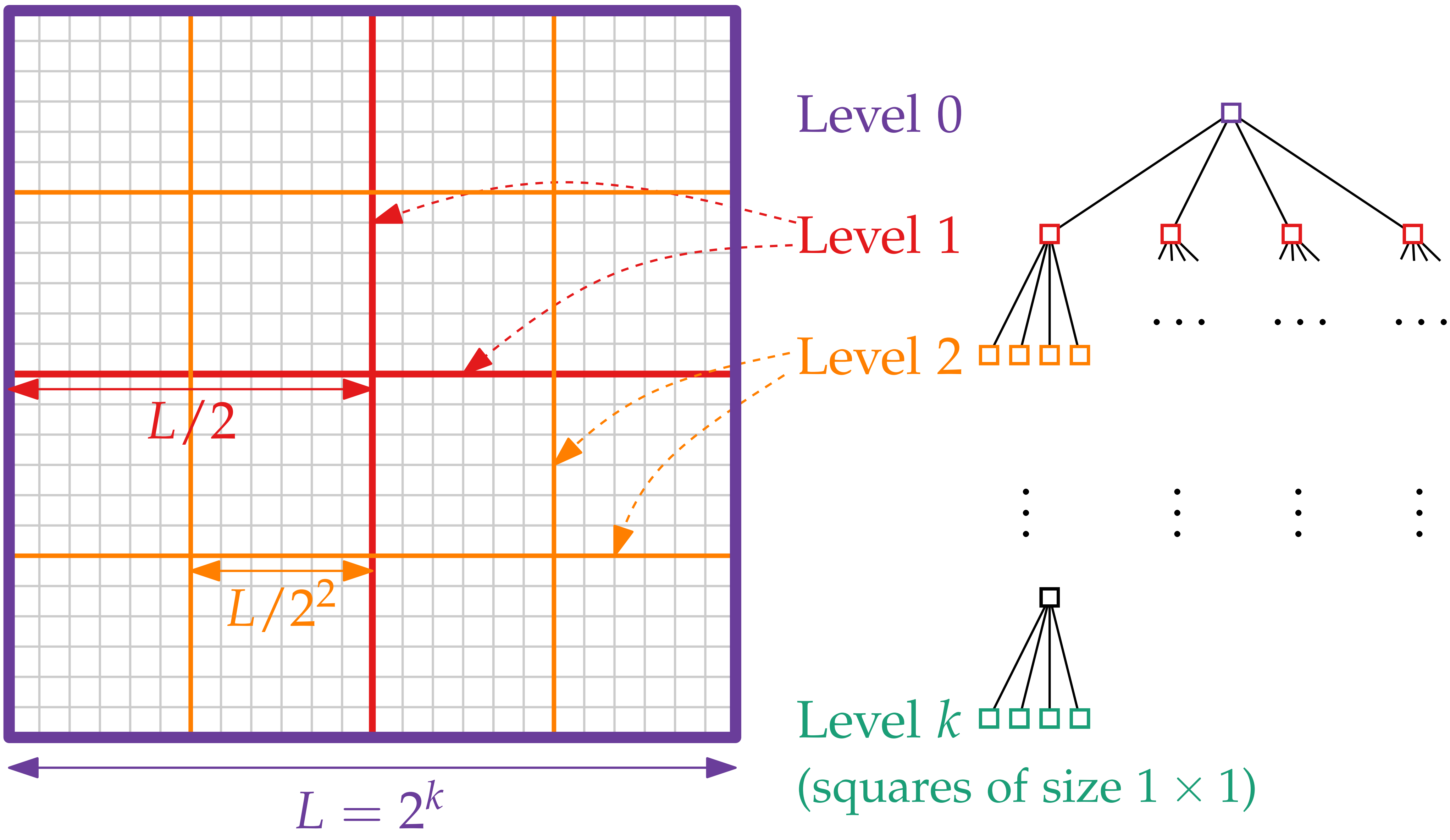
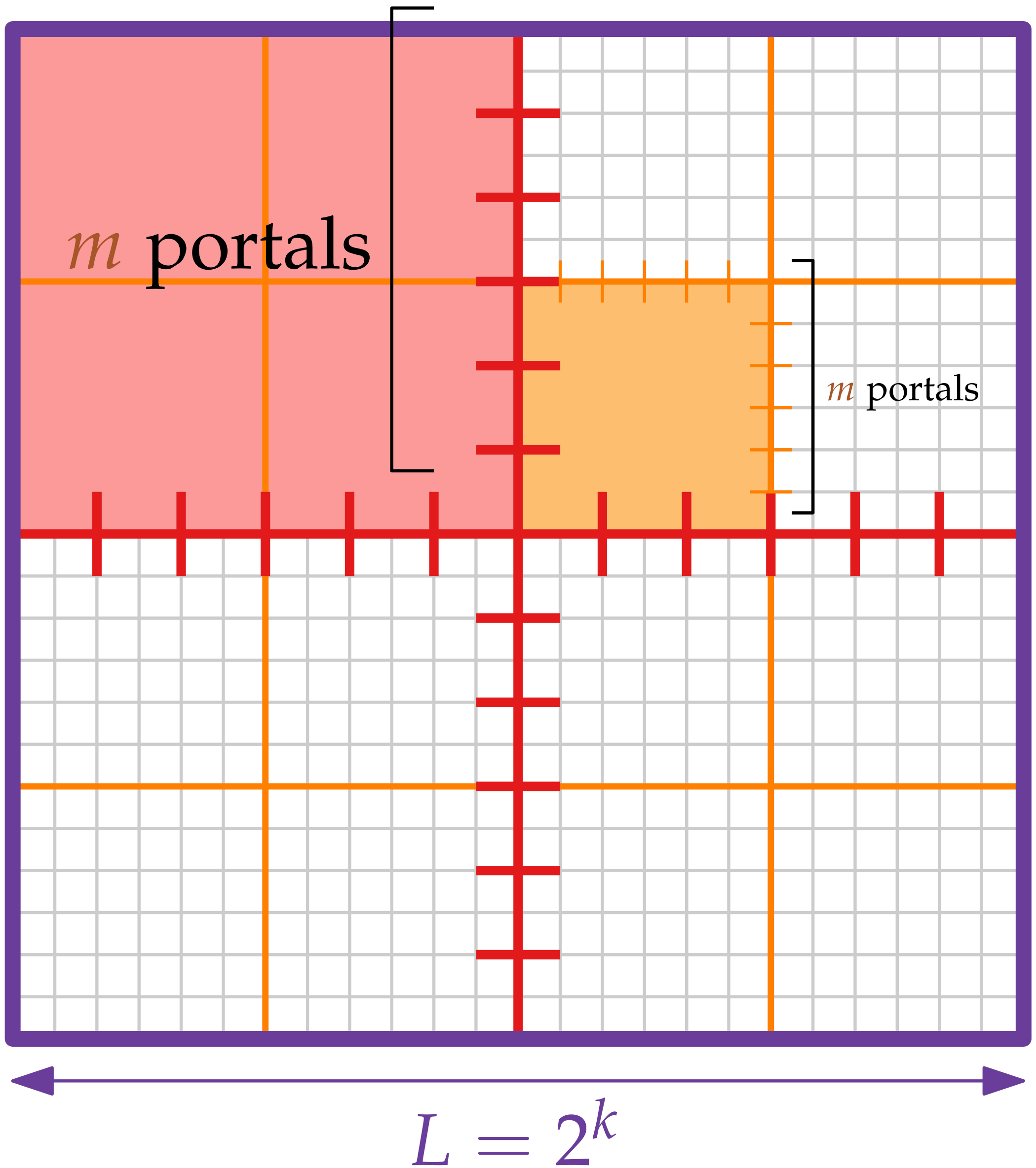
# Basic Dissection



# Portals




- $m$ power of two in interval $[k/\varepsilon, 2k/\varepsilon]$

  $k = 2 + 2\log_2 n$
  $\Rightarrow m = O((\log n)/\varepsilon)$

- **Portals** on level-$i$-line with distance $L/(2^i m)$

- Level-$i$-square: size $L/2^i \times L/2^i$

- Level-$i$-square has at most $4m$ portals on its boundary.

# Approximation Algorithms

## Lecture 9:
## PTAS for EuclideanTSP

### Part III:
### Well Behaved Tours

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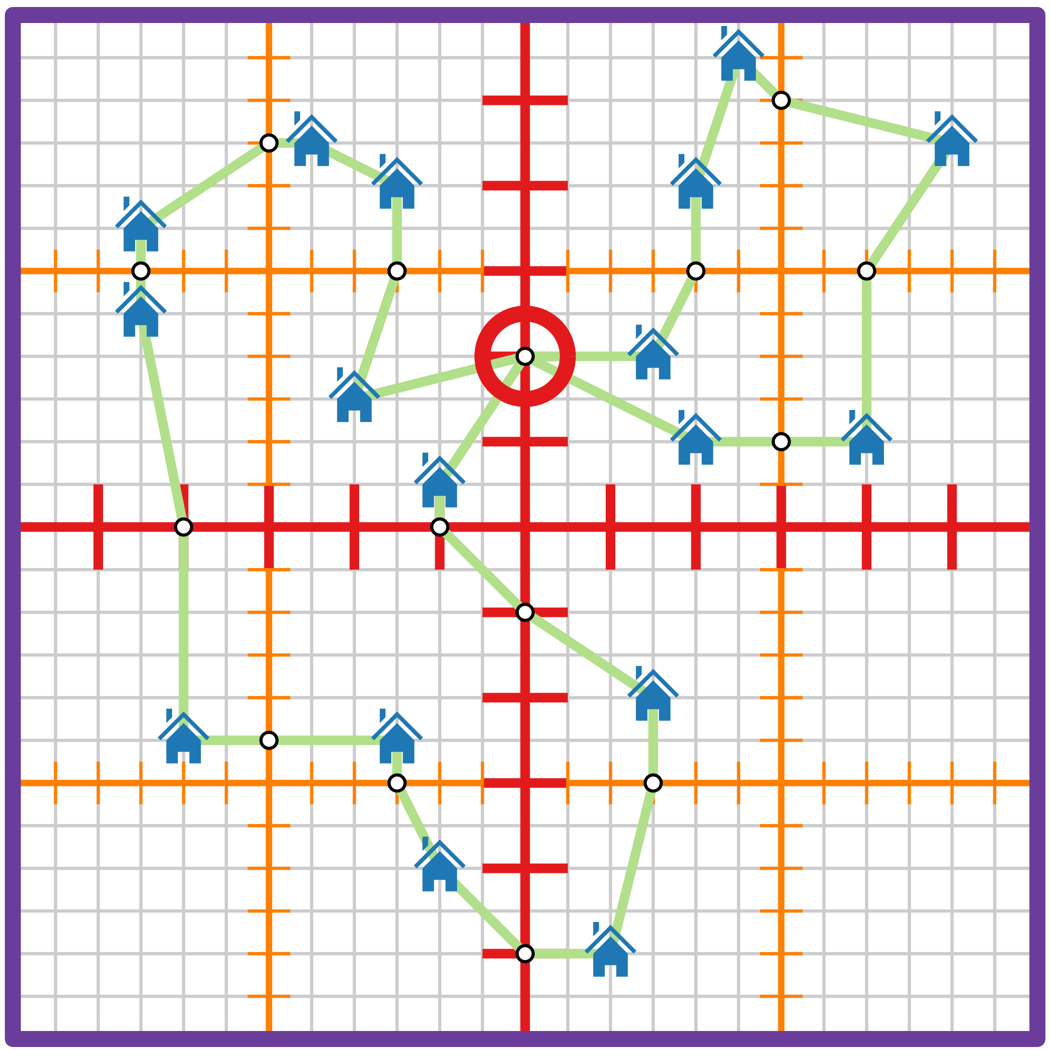
# Well Behaved Tours

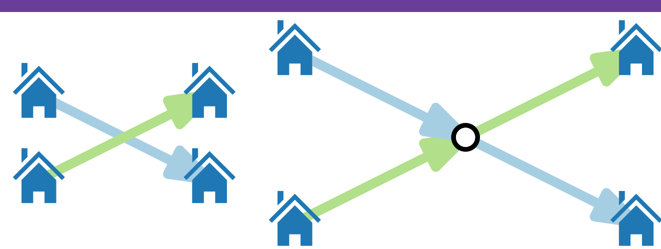


A tour is *well behaved* if

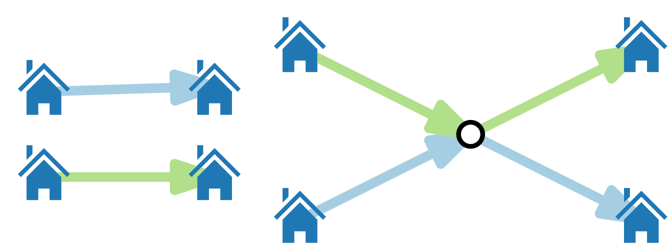
- it involves all houses and a subset of the portals,
- no edge of the tour crosses a line of the basic dissection,
- it is crossing-free.

W.l.o.g. (homework):
No portal visited more than twice

Crossing



No crossing



# Computing a Well Behaved Tour

**Lemma.** An optimal well behaved tour can be computed in $2^{O(m)} = n^{O(1/\varepsilon)}$ time.

**Sketch.**

- Dynamic Programming!
- Compute sub-structure of an optimal tour for each square in the dissection tree.
- These solutions can be efficiently propagated bottom-up through the dissection tree.

# Approximation Algorithms

## Lecture 9:
## PTAS for EuclideanTSP

### Part IV:
### Dynamic Program

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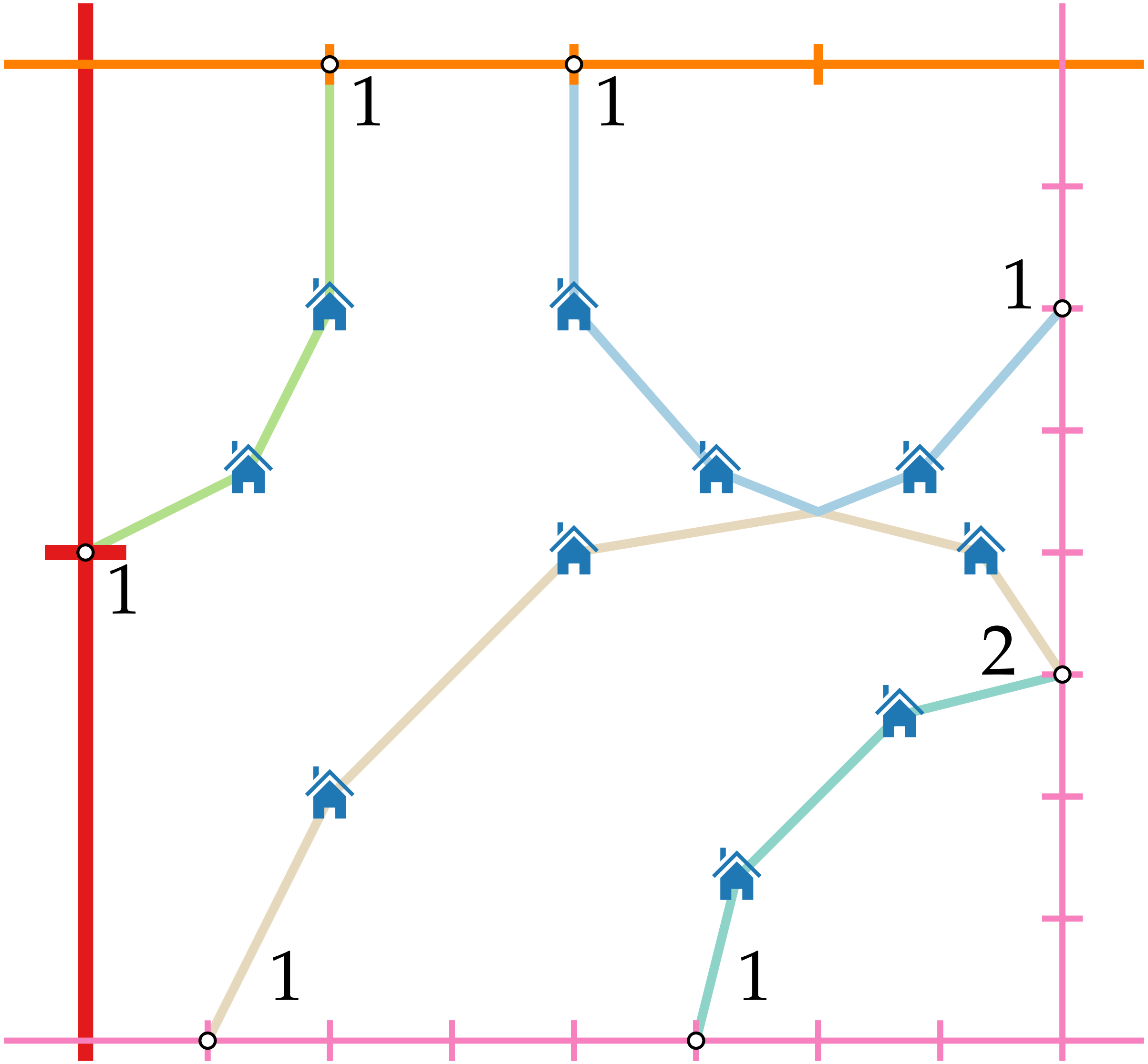
# Dynamic Program (I)



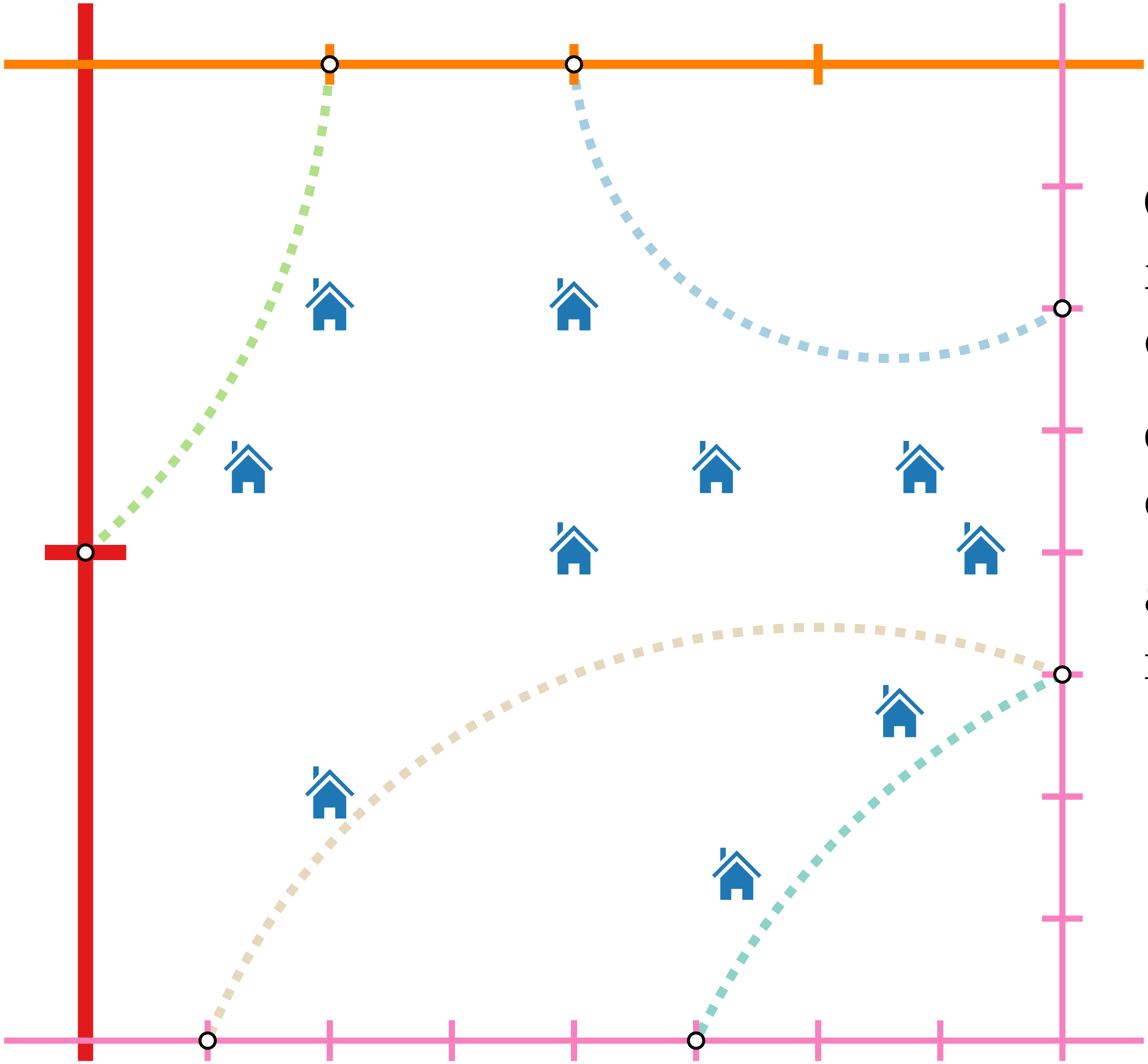

Each well behaved tour induces the following in each square $Q$ of the dissection:

- A path cover of the houses in $Q$
- Each portal of $Q$ is visited 0,1 or 2 times by this path cover

$$\Rightarrow \text{max. } 3^{4m} = 3^{O((\log n)/\varepsilon)} = n^{O(1/\varepsilon)} \text{ possibilities}$$

$m = O((\log n)/\varepsilon)$

# Dynamic Program (II)

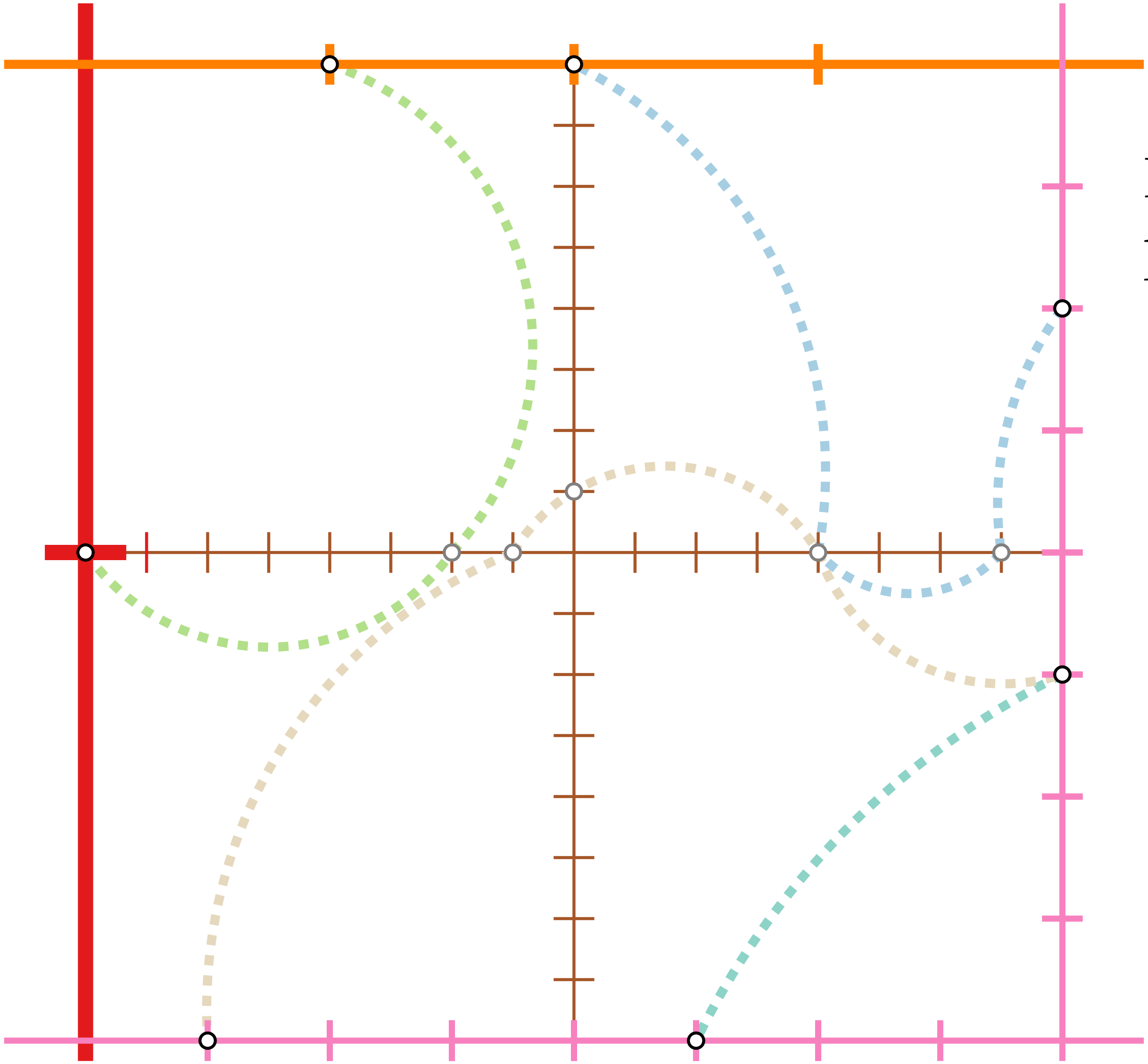


Compute:
for each square $Q$ in the dissection and

each crossing-free pairing $P$ of $Q$

an optimal path cover that respects $P$.

# Dynamic Program (III)



For a given square $Q$ and pairing $P$:

- Iterate over all $(n^{O(1/\varepsilon)})^4 = n^{O(1/\varepsilon)}$ crossing-free pairings of the child-squares
- Minimize the cost over all such pairings that respect $P$
- Correctness by induction

**Lemma.** An optimal well behaved tour can be computed in $2^{O(m)} = n^{O(1/\varepsilon)}$ time.

# Approximation Algorithms

## Lecture 9:
## PTAS for EuclideanTSP

### Part V:
### Shifted Dissections

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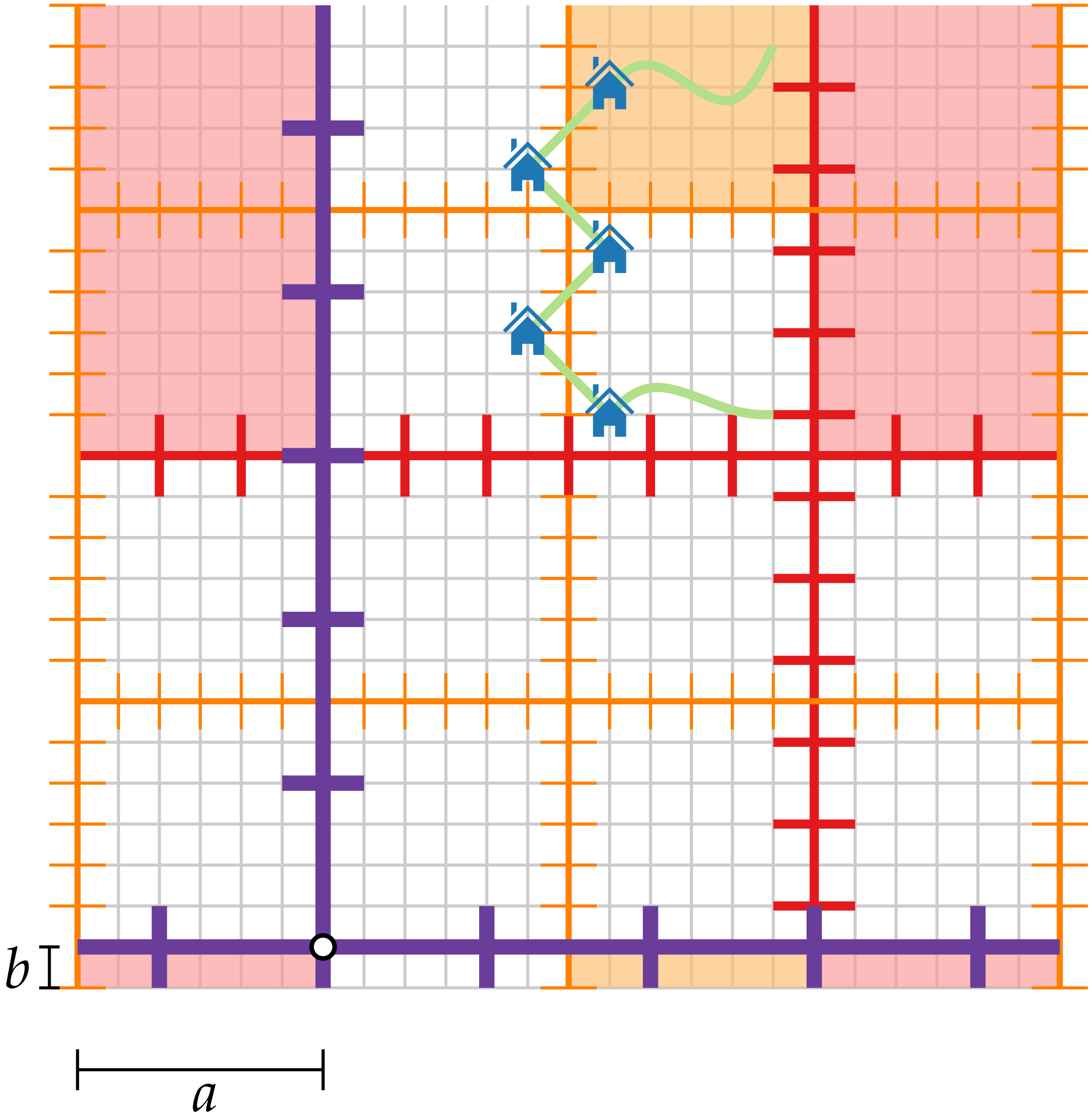
# Shifted Dissections




- The best well behaved tour can be a bad approximation.
- Consider an $(a, b)$-shifted dissection:

$$x \mapsto (x + a) \bmod L$$
$$y \mapsto (y + b) \bmod L$$

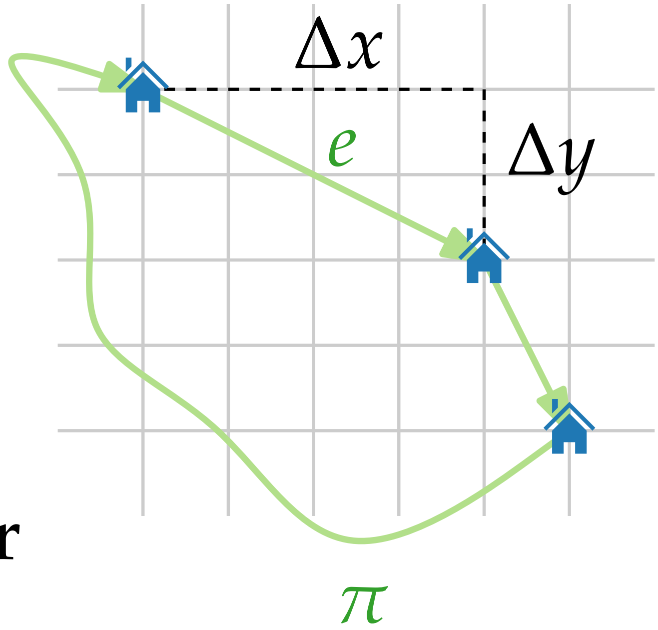
- Squares in the dissection tree are "wrapped around".
- Dynamic program must be modified accordingly.

# Shifted Dissections (II)

**Lemma.** Let $\pi$ be an optimal tour and $N(\pi)$ be the number of crossings of $\pi$ with the lines of the $(L \times L)$-grid. Then we have $N(\pi) \leq \sqrt{2} \cdot \mathrm{OPT}$.

**Proof.**

- Consider a tour as an ordered cyclic sequence.
- Each edge $e$ generates $N_e \leq \Delta x + \Delta y$ crossings.
- Crossings at the endpoint of an edge are counted for the next edge.




- $N_e{}^2 \leq (\Delta x + \Delta y)^2 \leq 2(\Delta x^2 + \Delta y^2) = 2|e|^2$.
- $N(\pi) = \sum_{e \in \pi} N_e \leq \sum_{e \in \pi} \sqrt{2|e|^2} = \sqrt{2} \cdot \mathrm{OPT}$. $\square$

# Approximation Algorithms

## Lecture 9:
## PTAS for EuclideanTSP

### Part VI:
### Approximation Factor

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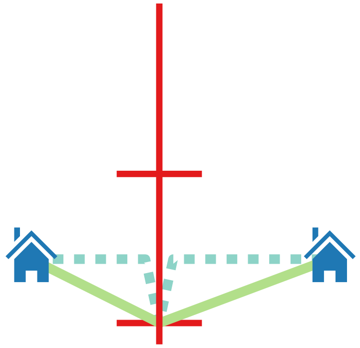
# Shifted Dissections (III)

**Theorem.** Let $a, b \in [0, L-1]$ be chosen independently and uniformaly at random. Then the expected cost of an optimal well behaved tour with respect to the $(a, b)$-shifted dissection is $\leq (1 + 2\sqrt{2}\varepsilon)\text{OPT}$.

**Proof.** Consider optimal tour $\pi$. Make $\pi$ well behaved by moving each intersection point with the $(L \times L)$-grid to the nearest portal.



Detour per intersection $\leq$ inter-portal distance.

# Shifted Dissections (III)

- Consider an intersection point between $\pi$ and a line $l$ of the $(L \times L)$-grid.
- With probability *at most* $2^i/L$, $l$ is a level-$i$-line $\rightsquigarrow$ an increase in tour length by a maximum of $L/(2^i m)$ (inter-portal distance).
- Thus, the expected increase in tour length due to this intersection is at most: $m \in [k/\varepsilon, 2k/\varepsilon]$

$$\sum_{i=0}^{k} \frac{2^i}{L} \cdot \frac{L}{2^i m} \leq \frac{k+1}{m} \leq 2\varepsilon.$$

- Summing over all $N(\pi) \leq \sqrt{2} \cdot \mathrm{OPT}$ intersection points, and applying linearity of expectation, provides the claim.

# Approximation Scheme

**Theorem.** Let $a, b \in [0, L-1]$ be chosen independently and uniformaly at random. Then the expected cost of an optimal well behaved tour with respect to the $(a, b)$-shifted dissection is $\leq (1 + 2\sqrt{2}\varepsilon)\text{OPT}$.

**Theorem.** There is a *deterministic* algorithm (PTAS) for EUCLIDEANTSP that provides for every $\varepsilon > 0$ a $(1+\varepsilon)$-approximation in $n^{O(1/\varepsilon)}$ time.

**Proof.** Try all $L^2$ many $(a, b)$-shifted dissections.
By the previous theorem, one of them is good enough. □