Approximation Algorithms

Lecture 10:
MINIMUM-DEGREE SPANNING TREE
via Local Search

Part I:
MINIMUM-DEGREE SPANNING TREE

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Winter 2021/22

MINIMUM-DEGREE SPANNING TREE

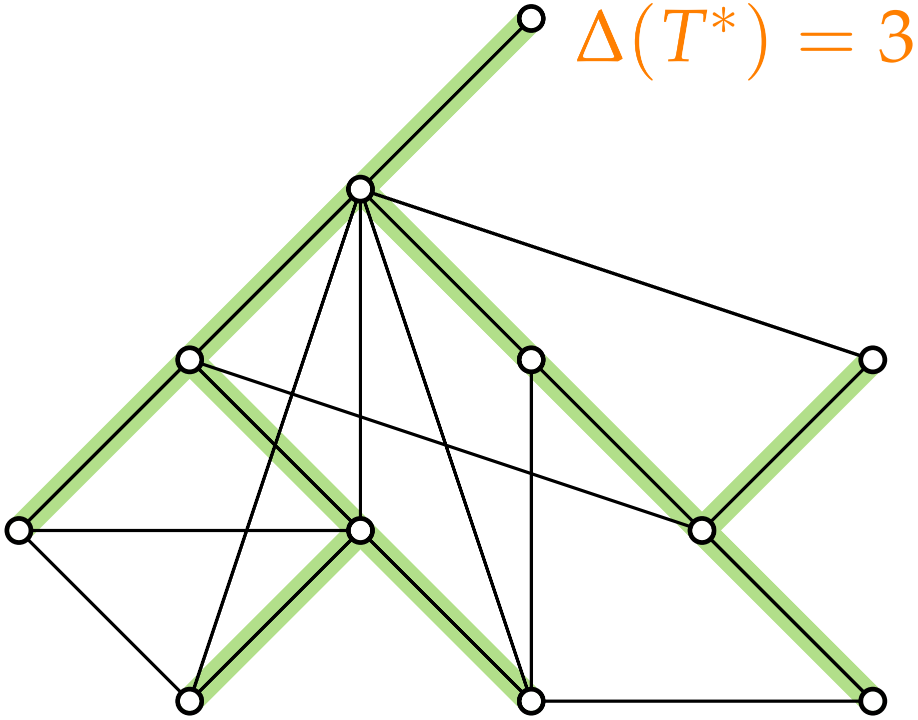
Given: A connected graph $G = (V, E)$

Task: Find a spanning tree T that has the minimum maximum degree $\Delta(T)$ among all spanning trees of G.

NP-hard ☹

Why?

Special case of Hamiltonian Path!



$\Delta(T^*) = 3$

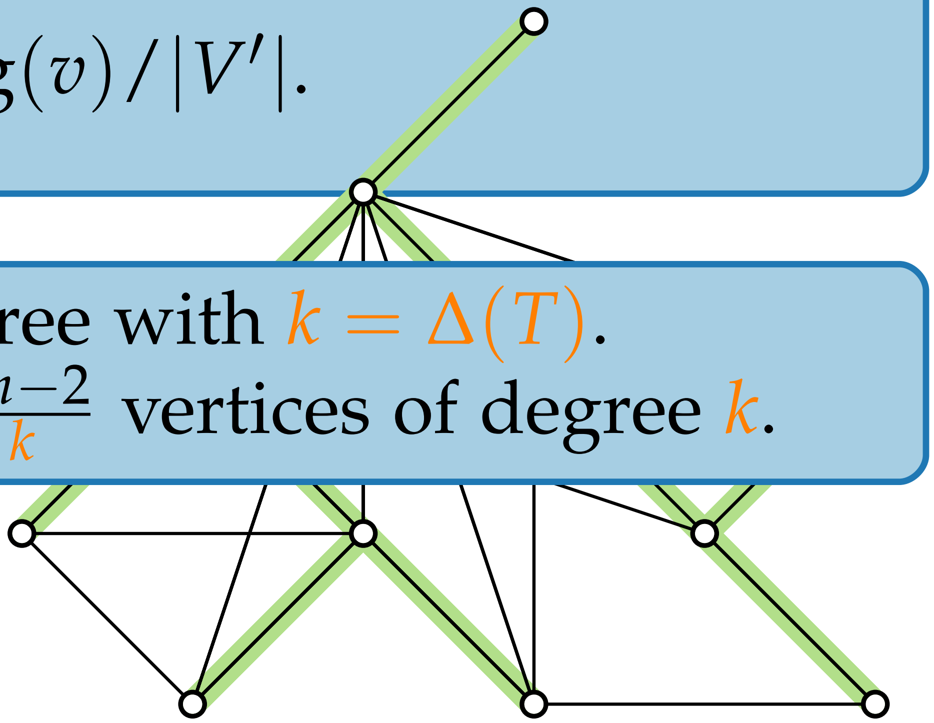
Warmup

Obs. A spanning tree T has...

- n vertices and $n-1$ edges,
- sum of degrees $\sum_{v \in V} \deg_T(v) = 2n-2$,
- average degree < 2.

Obs. Let $V' \subseteq V(G)$.
Then $\Delta(G) \geq \sum_{v \in V'} \deg(v) / |V'|$.

Obs. Let T be a spanning tree with $k = \Delta(T)$.
Then T has at most $\frac{2n-2}{k}$ vertices of degree k.



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Part II:
Edge Flips and Local Search

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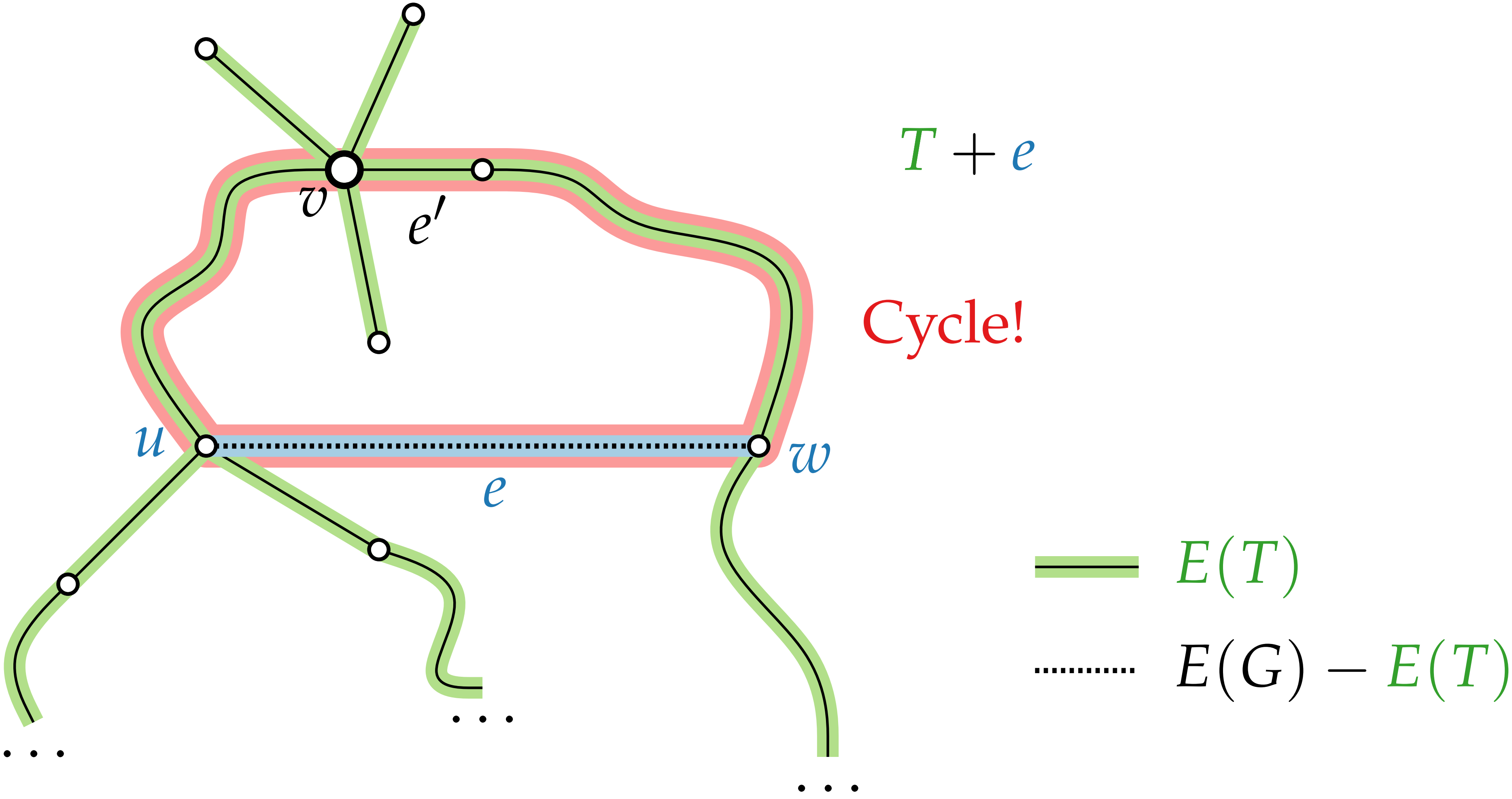
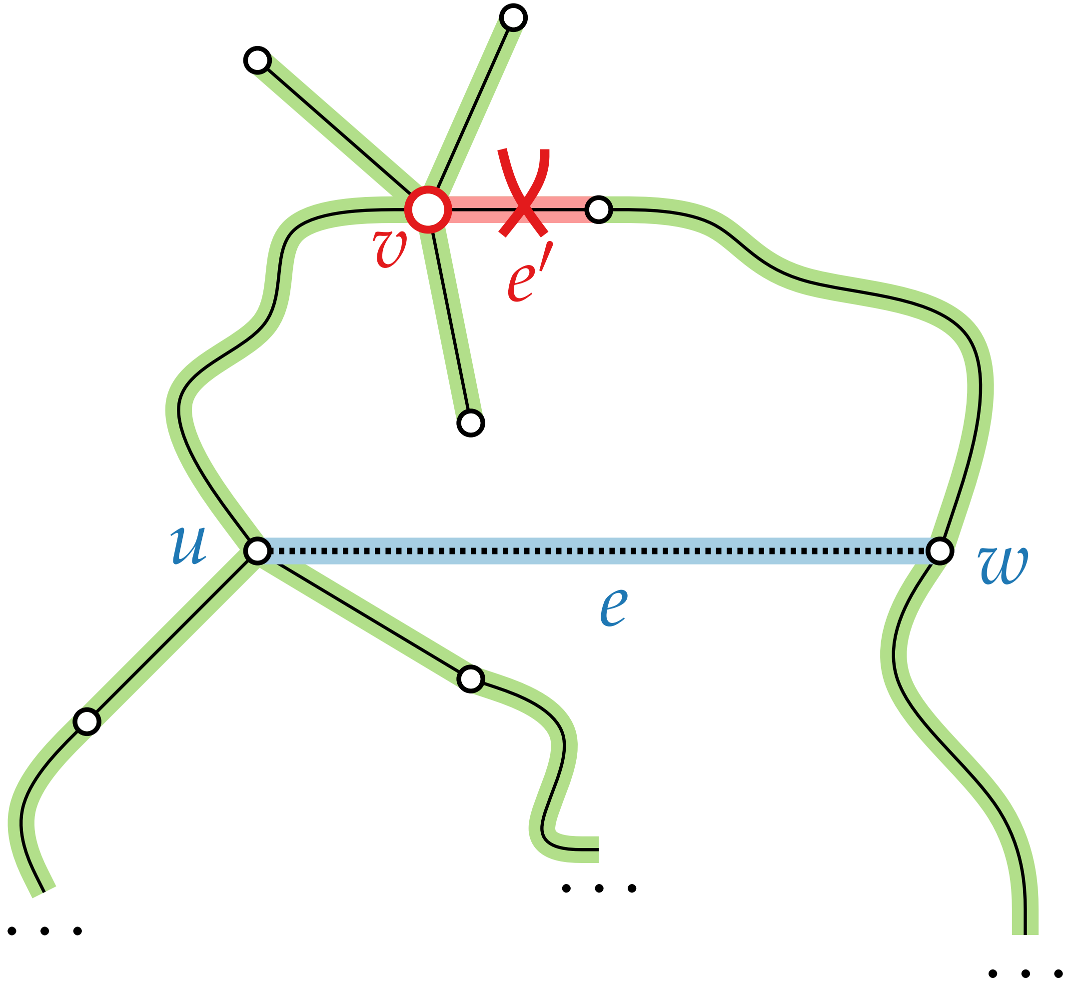
Edge Flips



Edge Flips

Def. An **improving flip** in T for a vertex v and an edge $uw \in E(G) \setminus E(T)$ is a flip with $\deg_T(v) > \max\{\deg_T(u), \deg_T(w)\} + 1$.



$T + e - e'$

is a new spanning tree

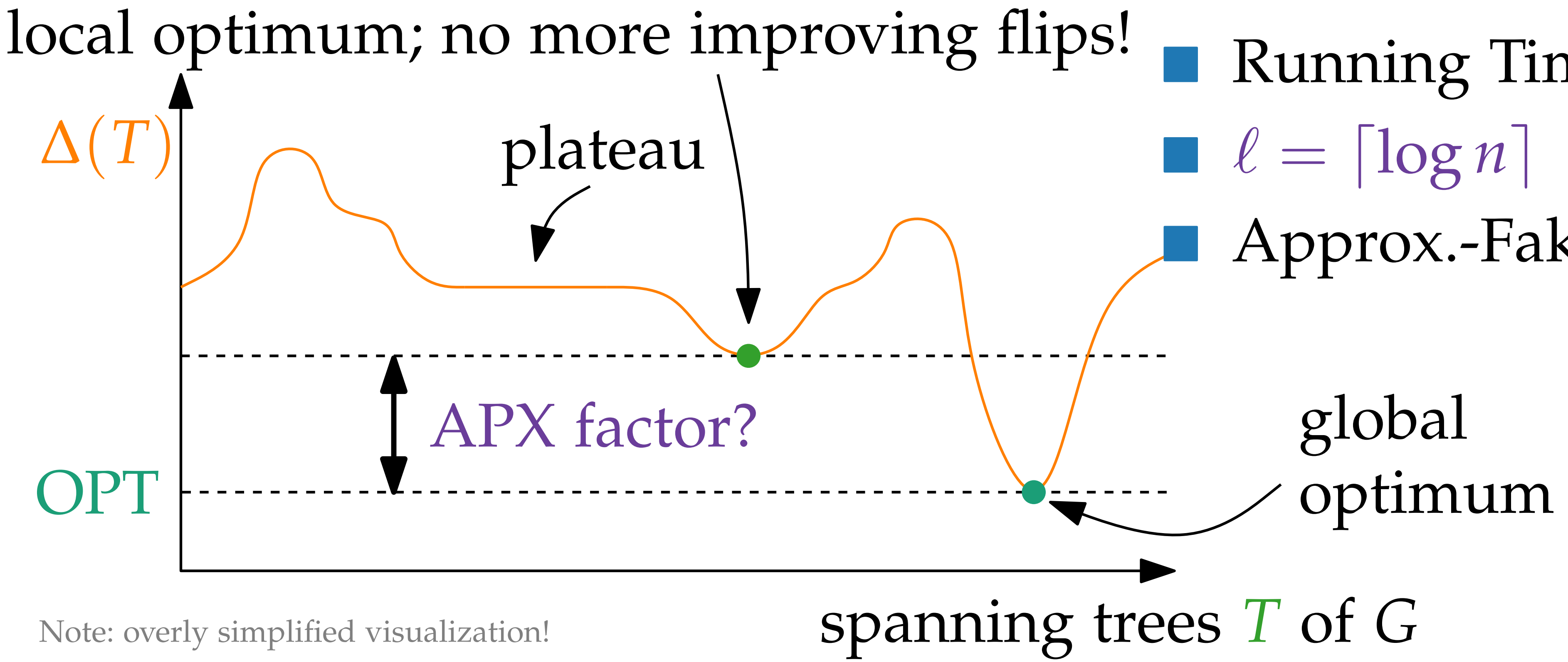
🟩 $E(T)$

┈ $E(G) - E(T)$

Local Search

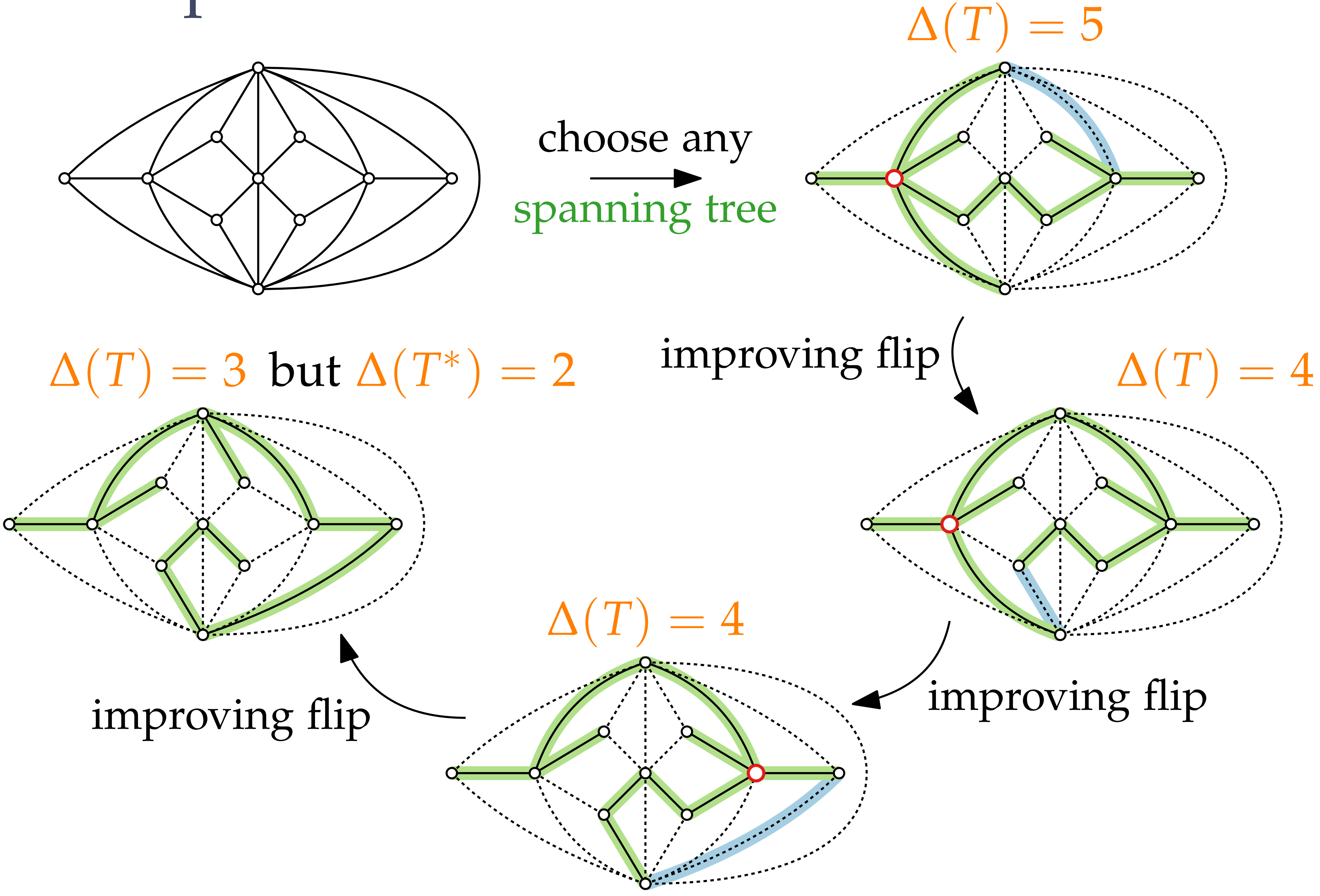
```
MinDegSpanningTreeLocalSearch($G$)
    $T \leftarrow$ any spanning tree of $G$
    while $\exists$ improving flip in $T$ for a vertex $v$
            with $\deg_T(v) \geq \Delta(T) - \ell$ do
        do the improving flip
```

- Termination?
- Running Time?
- $\ell = \lceil \log n \rceil$
- Approx.-Faktor?



Note: overly simplified visualization!

Example



choose any

spanning tree

$\Delta(T) = 5$

improving flip

$\Delta(T) = 4$

improving flip

$\Delta(T) = 4$

improving flip

$\Delta(T) = 3$ but $\Delta(T^*) = 2$

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Part III:
Lower Bound

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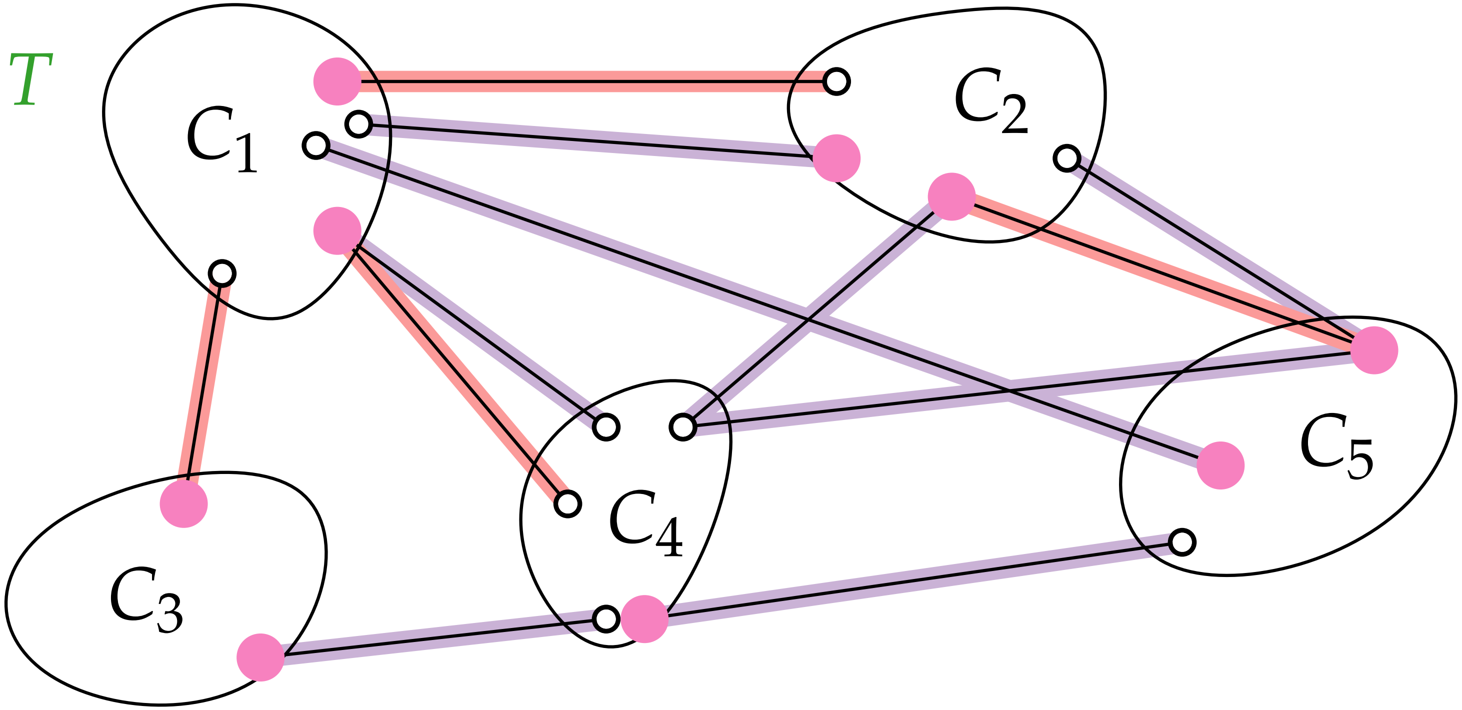
Decomposition $\Rightarrow$ Lower Bound for OPT

- Removing k edges decomposes T into $k+1$ components
- $E' :=$ {edges is G btw. different components $C_i \neq C_j$}.
- $S :=$ vertex cover of E'.



- $|E(T^*) \cap E'| \geq k$ for opt. spanning tree T^*
- $\sum_{v \in S} \deg_{T^*}(v) \geq k$

Lemma 1.
$\Rightarrow \mathrm{OPT} \geq k/|S|$

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Part IV:
More Lemmas

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More Lemmas

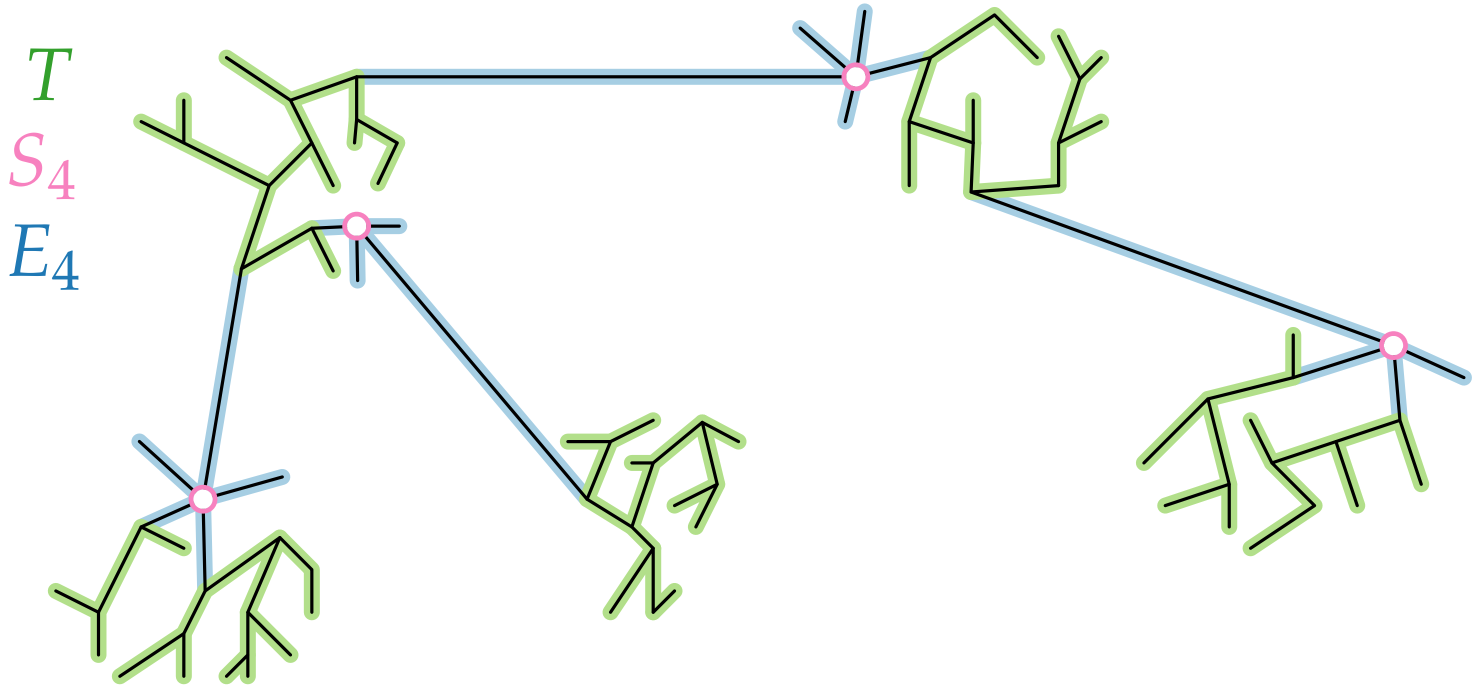
Let S_i be the vertices v in T with $\deg_T(v) \geq i$.
Let E_i be the edges in T incident to S_i.

$\Rightarrow S_1 \supseteq S_2 \supseteq \dots$
$\Rightarrow S_1 = V(G)$
$\Rightarrow E_1 = E(T)$

Lemma 2. There is some $i \geq \Delta(T) - \ell + 1$ with $|S_{i-1}| \leq 2|S_i|$.

Proof. $|S_{\Delta(T)-\ell}| > 2^{\ell}|S_{\Delta(T)}| \underset{\ell = \lceil \log_2 n \rceil}{=} 2^{\lceil \log_2 n \rceil}|S_{\Delta(T)}| \geq n|S_{\Delta(T)}|$ ↯

Otherwise

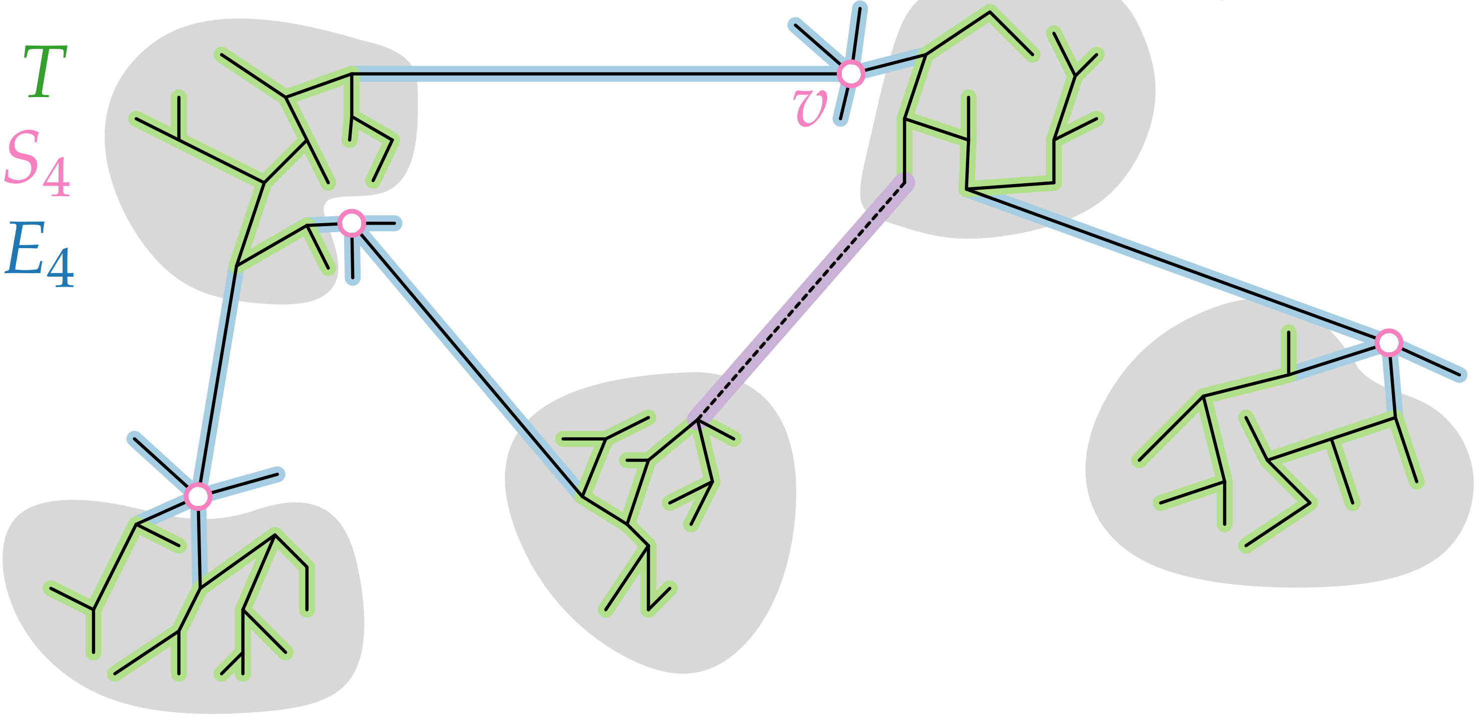


More Lemmas

Lemma 3. For $i \geq \Delta(T) - \ell + 1$,
(i) $|E_i| \geq (i-1)|S_i| + 1$,
(ii) Each $e \in E(G) \setminus E_i$ connecting distinct components of $T \setminus E_i$ is incident to a node of S_{i-1}.

Proof. (i) $|E_i| \geq \underset{\text{vertex-deg}}{i|S_i|} - \underset{\text{counted twice?}}{(|S_i| - 1)} = (i-1)|S_i| + 1$

(ii) Otherwise, there is an improving flip for $v \in S_i$.



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Part V:
Approximation Factor

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Approximation Factor

[Fürer & Raghavachari: SODA'92, JA'94]

Theorem. Let T be a locally optimal spanning tree.
Then $\Delta(T) \leq 2 \cdot \mathrm{OPT} + \ell$, where $\ell = \lceil \log_2 n \rceil$.

Proof. Let S_i be the vertices v in T with $\deg_T(v) \geq i$.
Let E_i be the edges in T incident to S_i.

Lemma 1. $\mathrm{OPT} \geq k/|S|$, $k = |\text{rem. edges}|$, S vert. cover

Lemma 2. There is an $i \geq \Delta(T) - \ell + 1$ with $|S_{i-1}| \leq 2|S_i|$.

Lemma 3. For $i \geq \Delta(T) - \ell + 1$,
(i) $|E_i| \geq (i-1)|S_i| + 1$,
(ii) Each $e \in E(G) \setminus E_i$ connecting distinct components of $T \setminus E_i$ is incident to a node of S_{i-1}.

Remove E_i for this i! $\Rightarrow$ S_{i-1} covers edges btw. comp.

$$\mathrm{OPT} \underset{\text{Lemma 1}}{\geq} \frac{k}{|S|} = \frac{|E_i|}{|S_{i-1}|} \underset{\text{Lemma 3}}{\geq} \frac{(i-1)|S_i|+1}{|S_{i-1}|} \underset{\text{Lemma 2}}{\geq} \frac{(i-1)|S_i|+1}{2|S_i|} > \frac{(i-1)}{2} \geq \frac{(\Delta(T)-\ell)}{2}$$

□

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Part VI:
Termination, Running Time & Extensions

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Termination and Running Time

Theorem. The algorithm finds a locally optimal spanning tree after $O(n^4)$ iterations.

Proof. Via potential function $\Phi(T)$ measuring the value of a solution where (hopefully): $\Phi(T) = \sum_{v \in V(G)} 3^{\deg_T(v)}$

- each iteration decreases the potential of a solution.

Lemma. After each flip $T \to T'$, $\Phi(T') \leq (1 - \frac{2}{27n^3})\Phi(T)$.

Homework

- the function is bounded both from above and below.

Lemma. For each spanning tree T, $\Phi(T) \in [3n, n3^n]$.

- executing $f(n)$ iterations would exceed this lower bound.

  Let $f(n) = \frac{27}{2}n^4 \cdot \ln 3$. How does $\Phi(T)$ change?

  decreases by: $(1 - \frac{2}{27n^3})^{f(n)} \leq (e^{-\frac{2}{27n^3}})^{f(n)} = e^{-n \ln 3} = 3^{-n}$

  Goal: After $f(n)$ iterations: $\Phi(T) = n < 3n$ □

Extensions

[Fürer & Raghavachari:
SODA'92, JA'94]

Corollary. For any constant $b > 1$ and $\ell = \lceil \log_b n \rceil$, the local search algorithm runs in polynomial time and produces a spanning tree T with $\Delta(T) \leq b \cdot \mathrm{OPT} + \lceil \log_b n \rceil$.

Proof. Similar to previous pages. Homework □

Theorem. There is a local search algorithm that runs in $O(EV\alpha(E,V)\log V)$ time and produces a spanning tree T with $\Delta(T) \leq \mathrm{OPT} + 1$.