Approximation Algorithms

Lecture 10:
Minimum-Degree Spanning Tree via Local Search

Part I:
Minimum-Degree Spanning Tree

Joachim Spoerhase

Winter 2021/22

Minimum-Degree Spanning Tree

Given: A connected graph $G = (V, E)$

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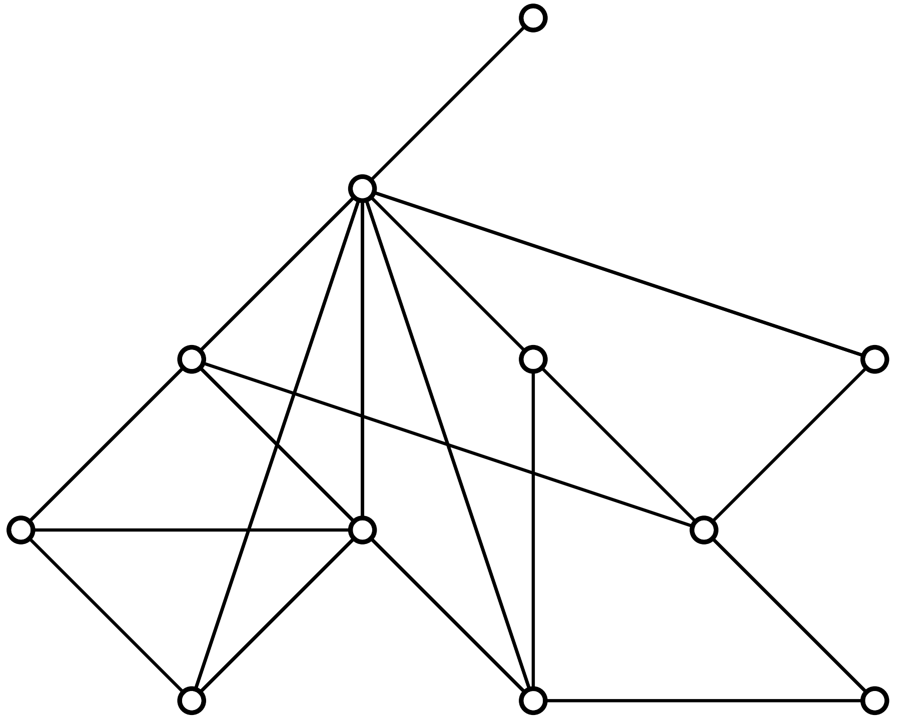

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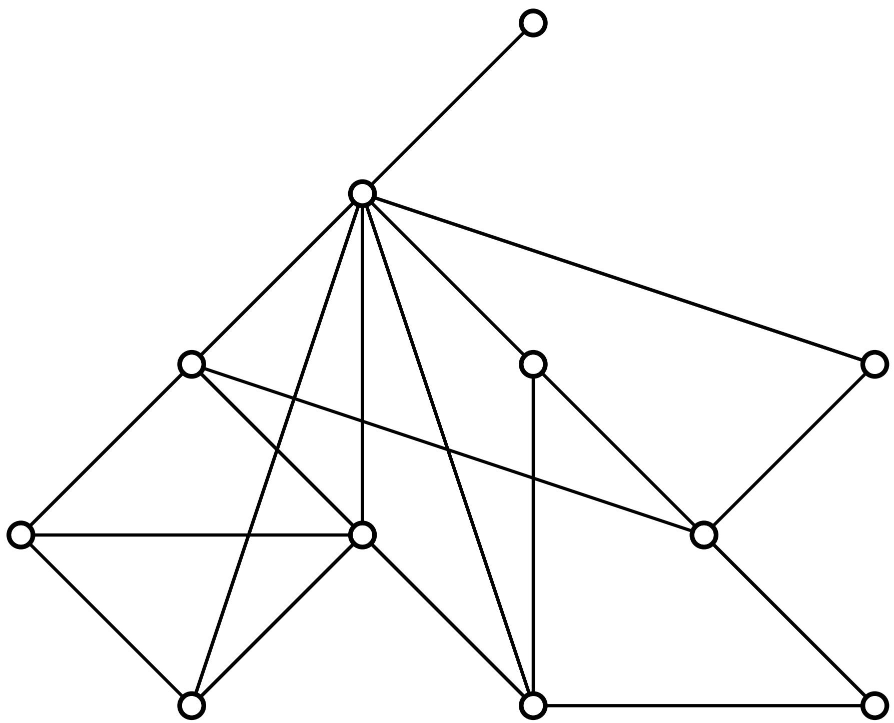
Given: A connected graph $G = (V, E)$

Task: Find a spanning tree T that has the minimum maximum degree $\Delta(T)$ among all spanning trees of G.



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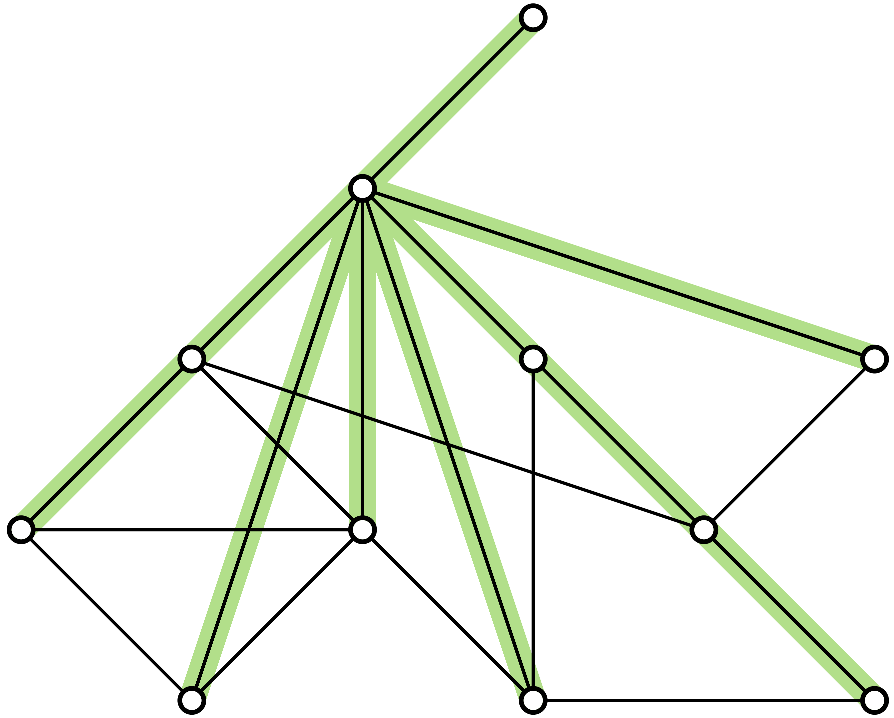

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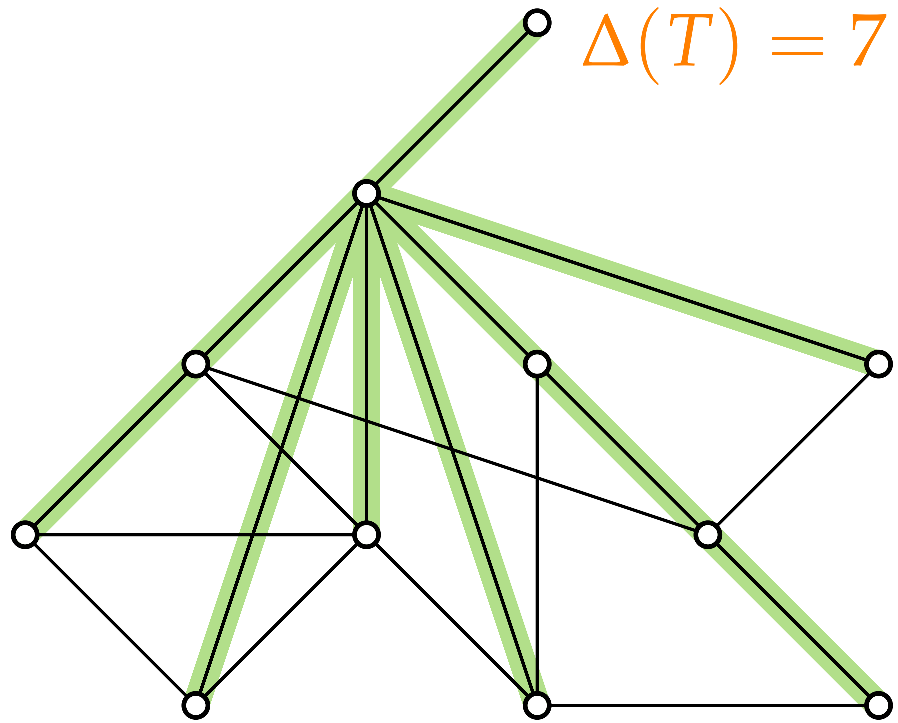

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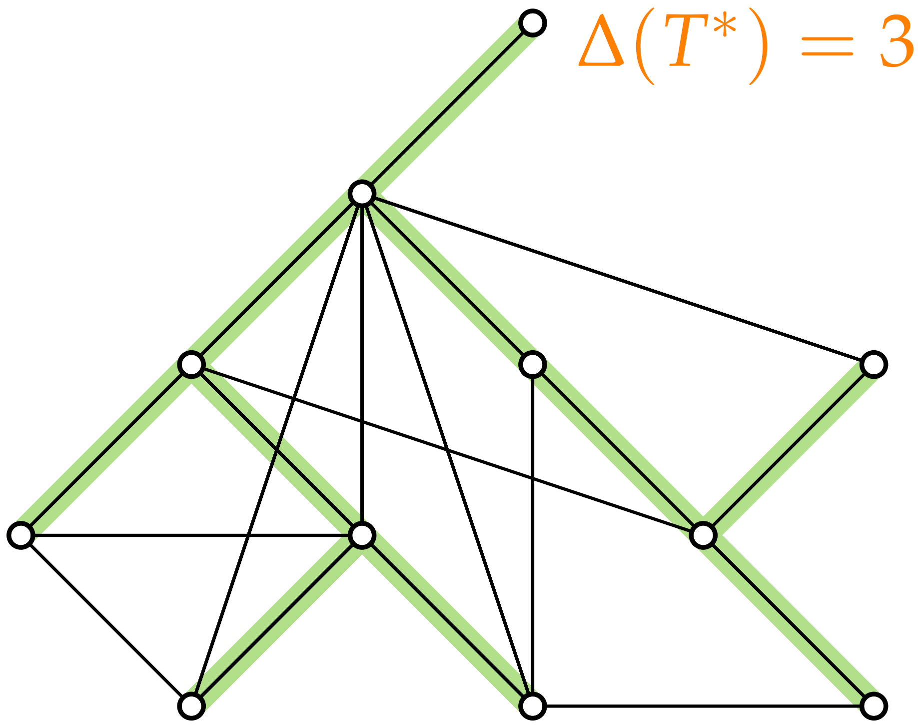

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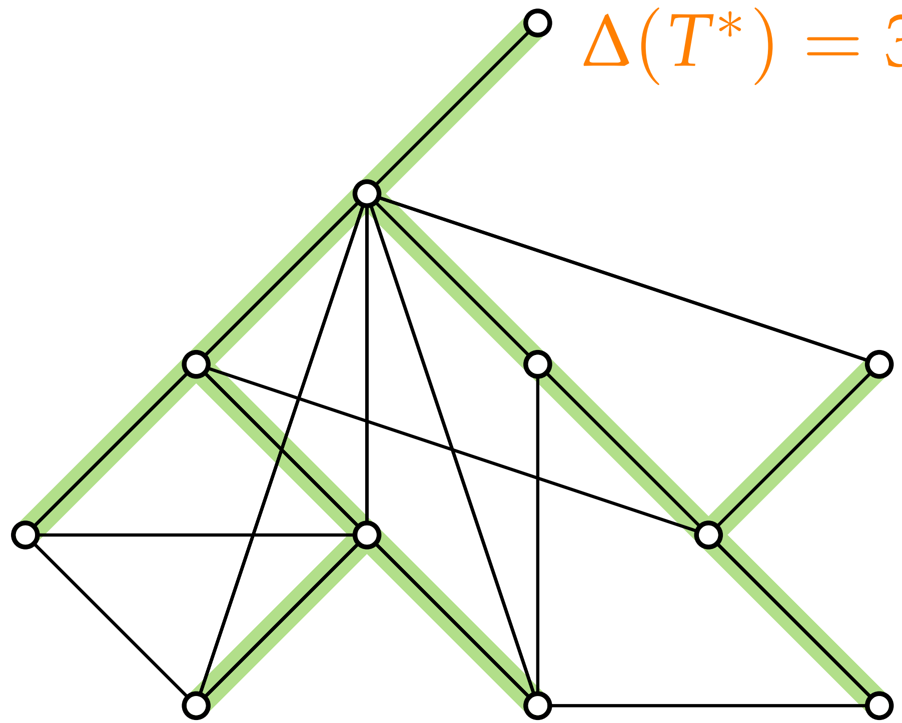
NP-hard 



$\Delta(T^*) = 3$

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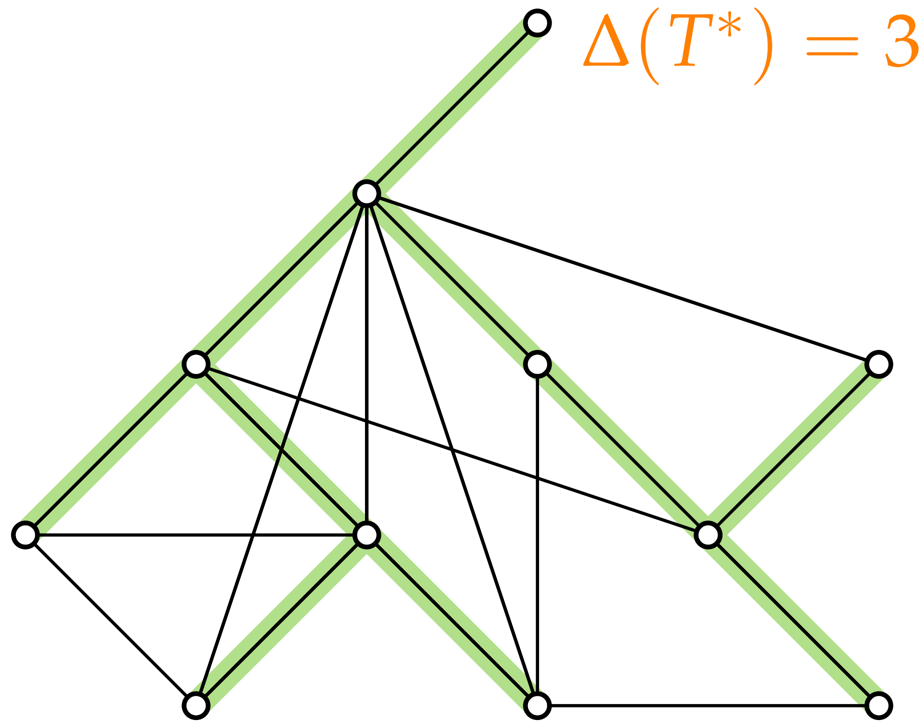
Why?



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Given: A connected graph $G = (V, E)$

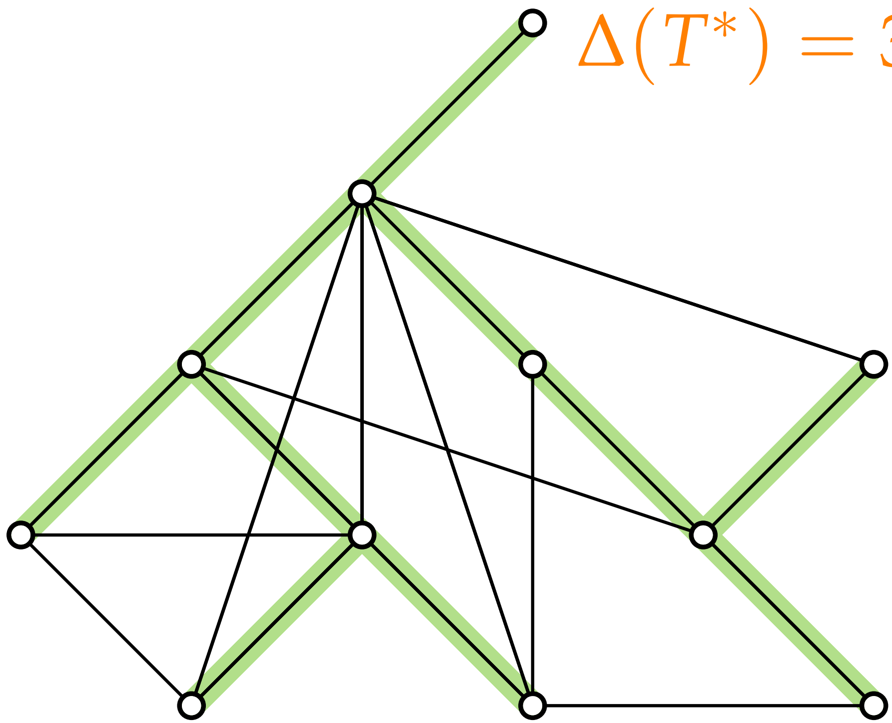
Task: Find a spanning tree T that has the minimum maximum degree $\Delta(T)$ among all spanning trees of G.

NP-hard 

Why?

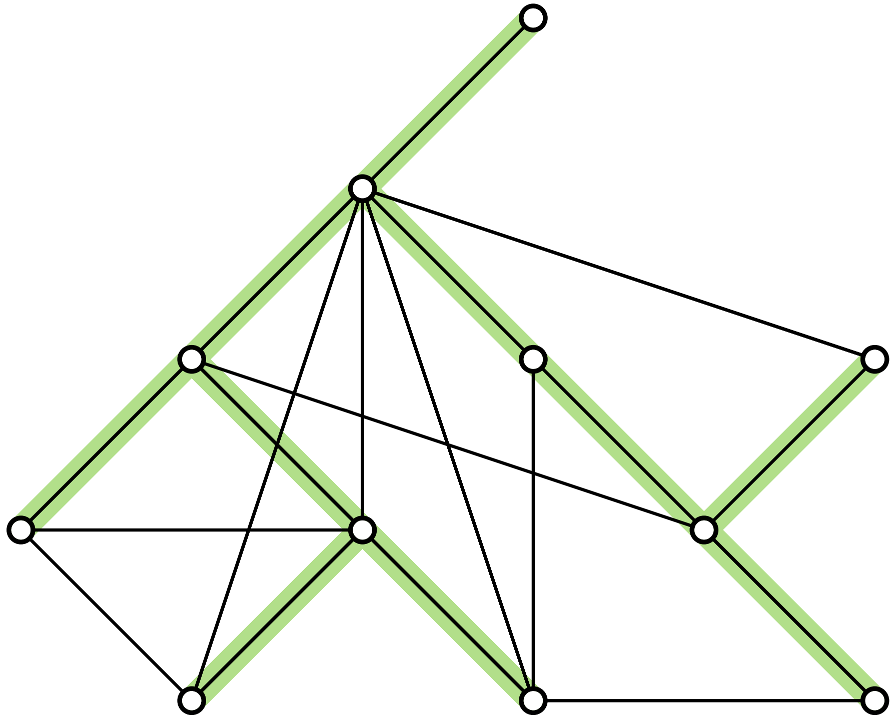
Special case of Hamiltonian Path!

$\Delta(T^*) = 3$



Warmup

Obs. A spanning tree T has...



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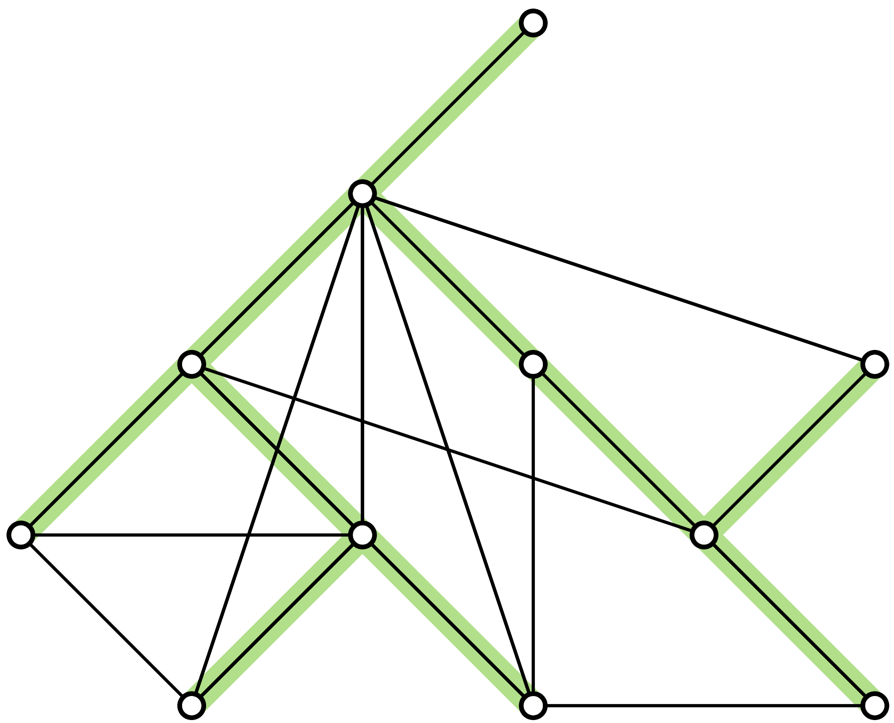
Obs. A spanning tree T has...

- n vertices and ? edges,



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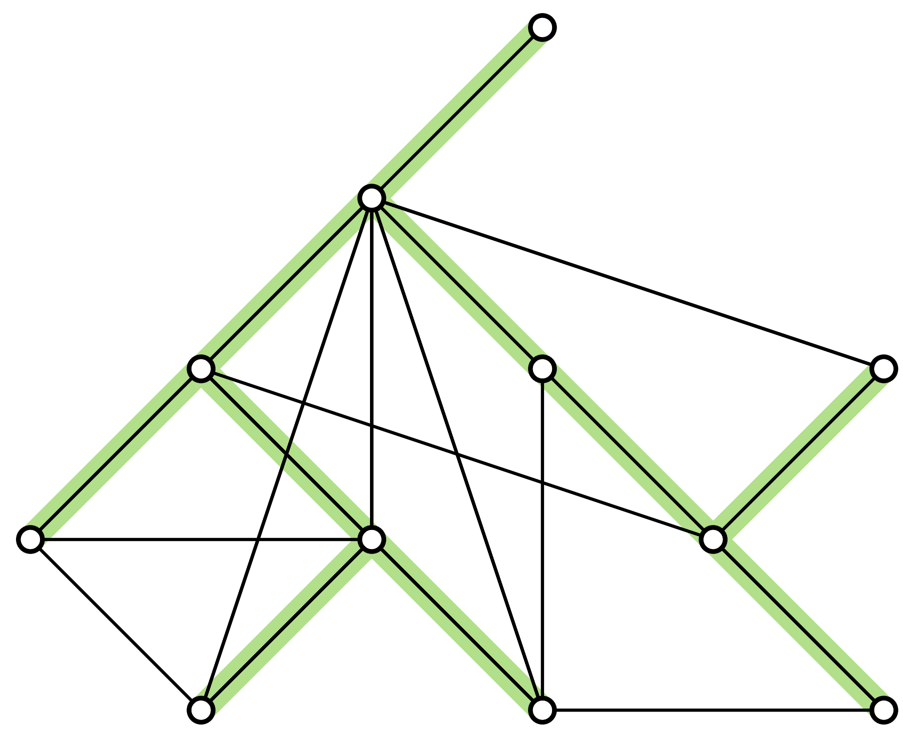
Obs. A spanning tree T has...

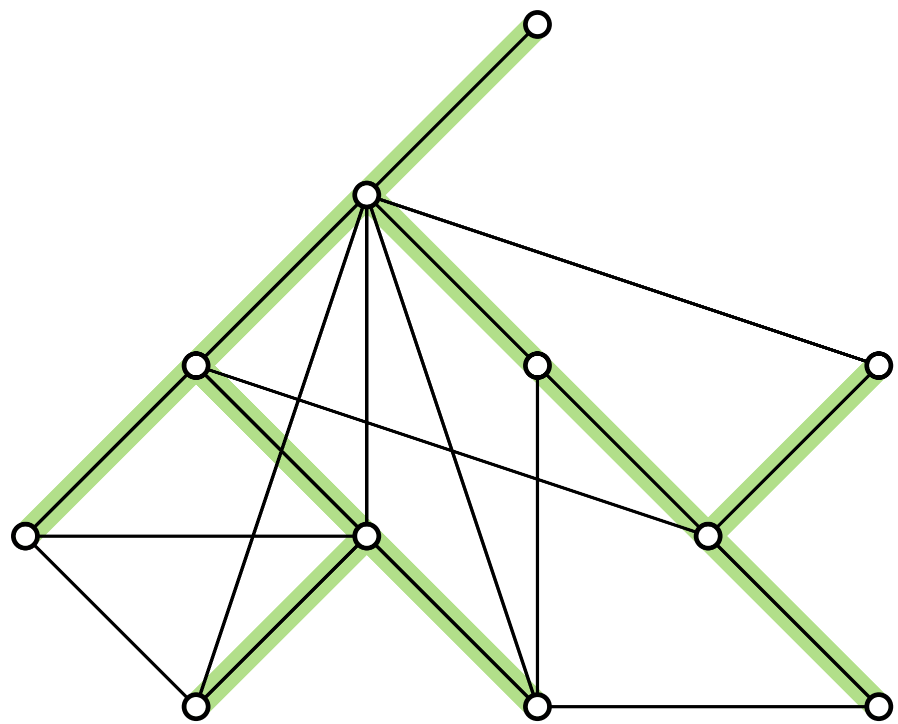
- n vertices and ? edges,
- sum of degrees $\sum_{v \in V} \deg_T(v) =$?



Warmup

Obs. A spanning tree T has...

- n vertices and ? edges,
- sum of degrees $\sum_{v \in V} \deg_T(v) =$?
- average degree ?



Warmup

Obs. A spanning tree T has...

- n vertices and $n-1$ edges,
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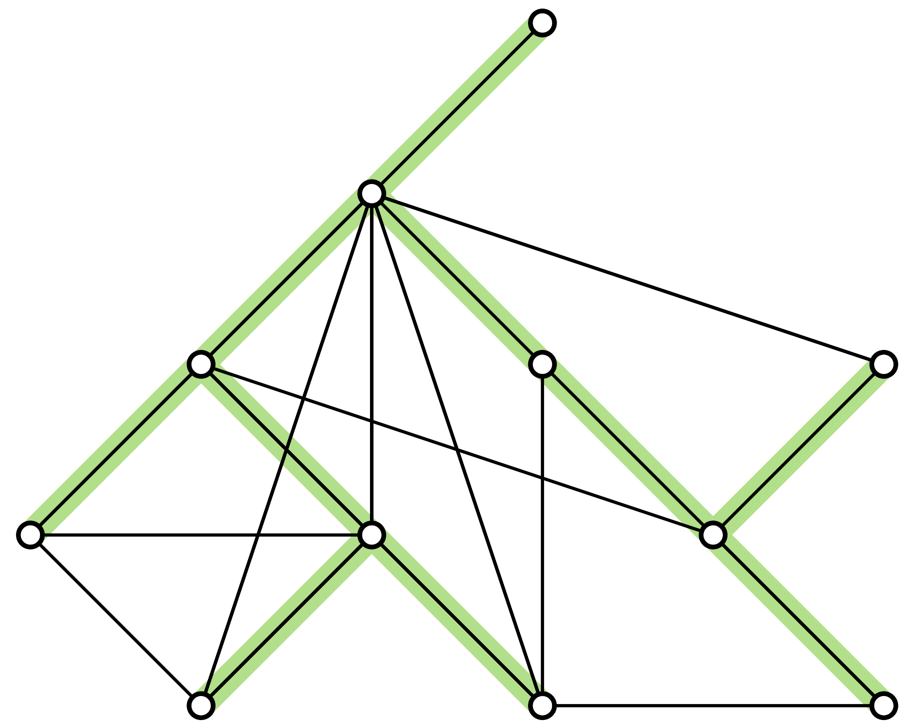

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Obs. A spanning tree T has...

- n vertices and $n-1$ edges,
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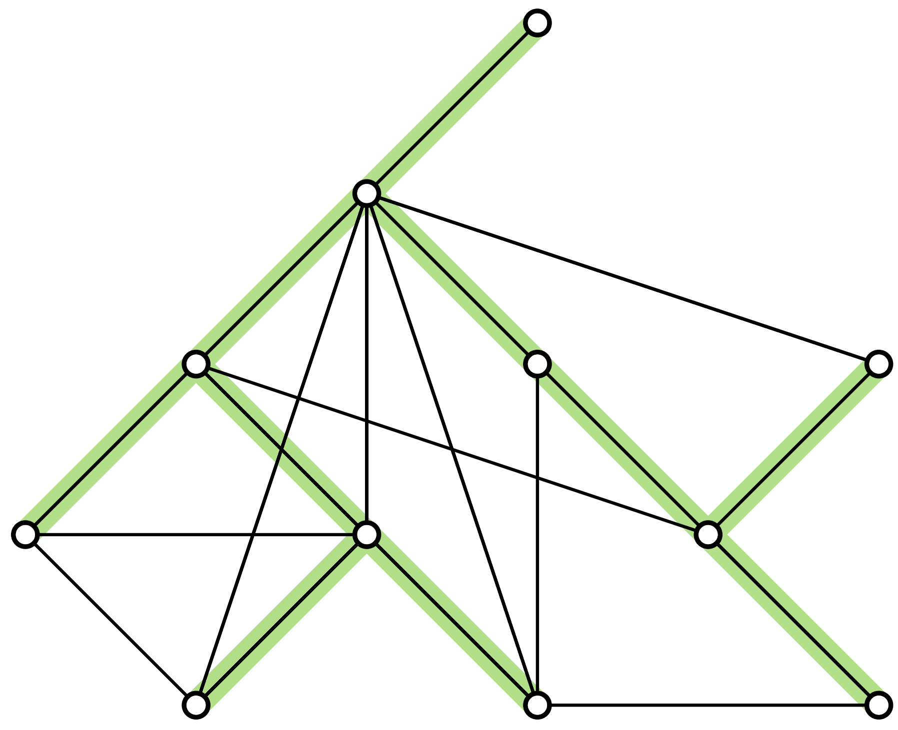
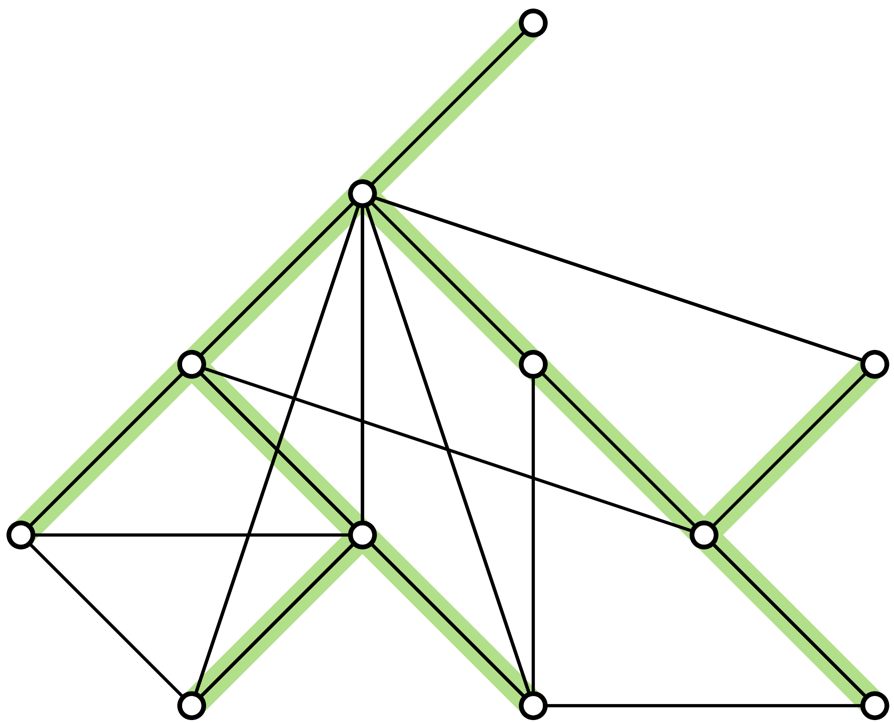
- average degree ?



Warmup

Obs. A spanning tree T has...

- n vertices and $n-1$ edges,
- sum of degrees $\sum_{v \in V} \deg_T(v) = 2n - 2$,
- average degree < 2.



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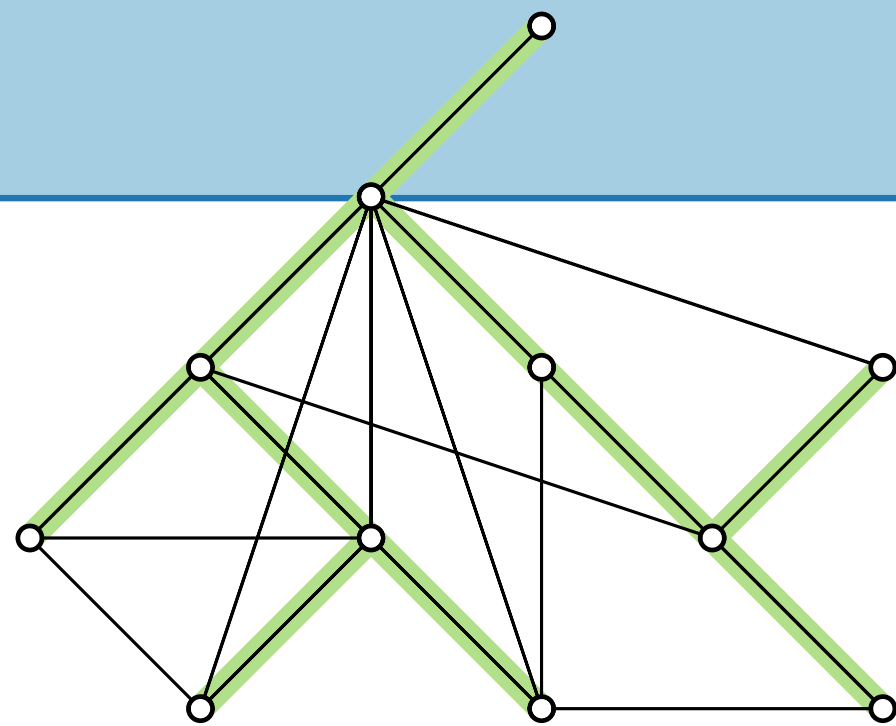
Obs. Let $V' \subseteq V(G)$.
Then $\Delta(G) \geq$?



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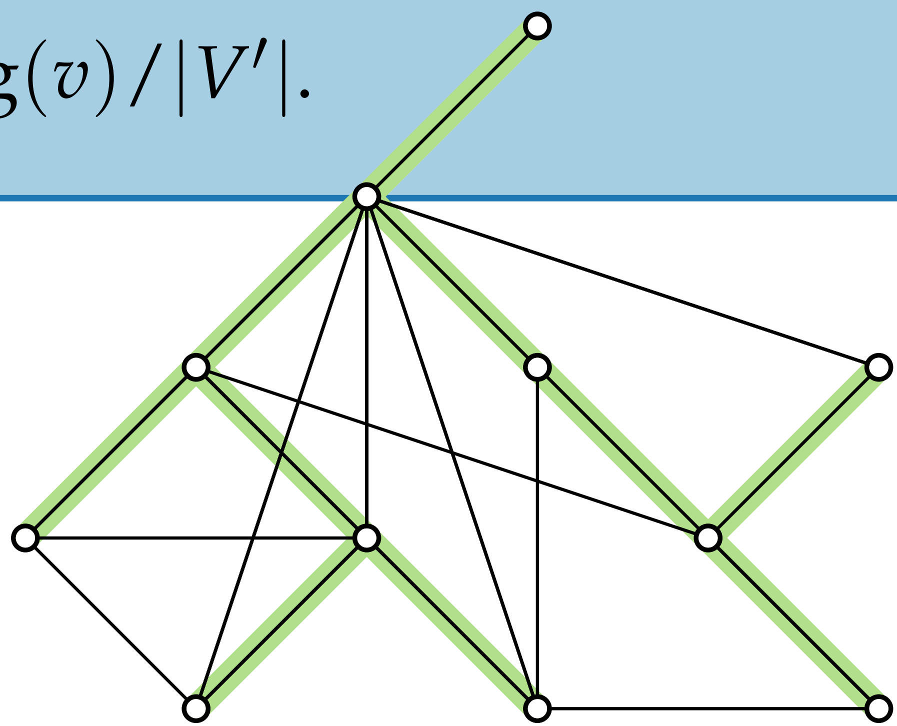
Obs. Let $V' \subseteq V(G)$.
Then $\Delta(G) \geq \sum_{v \in V'} \deg(v) / |V'|$.



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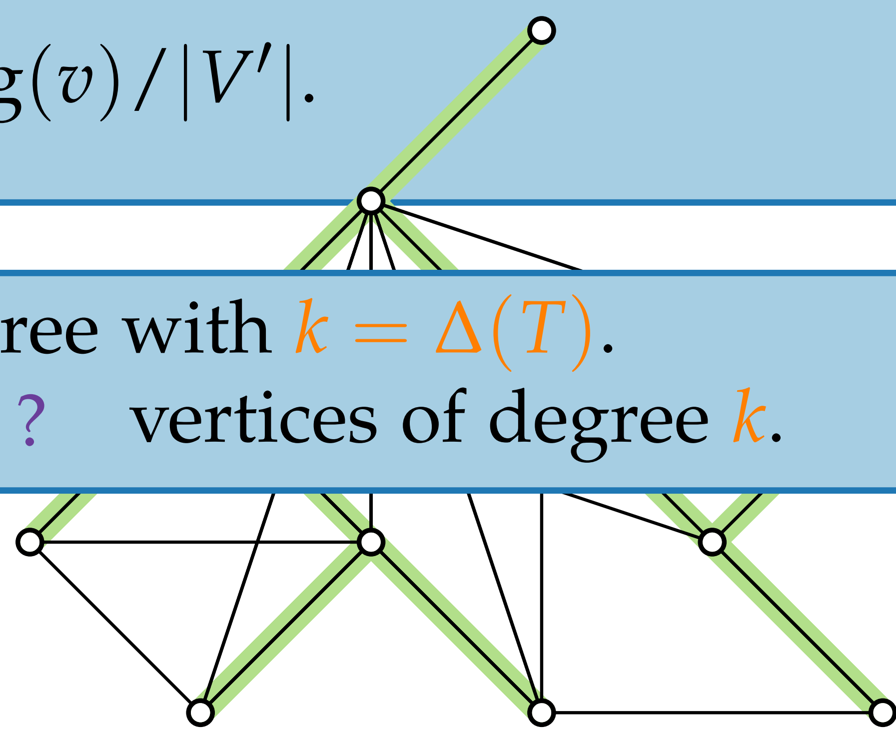
Obs. Let T be a spanning tree with $k = \Delta(T)$.
Then T has at most ? vertices of degree k.



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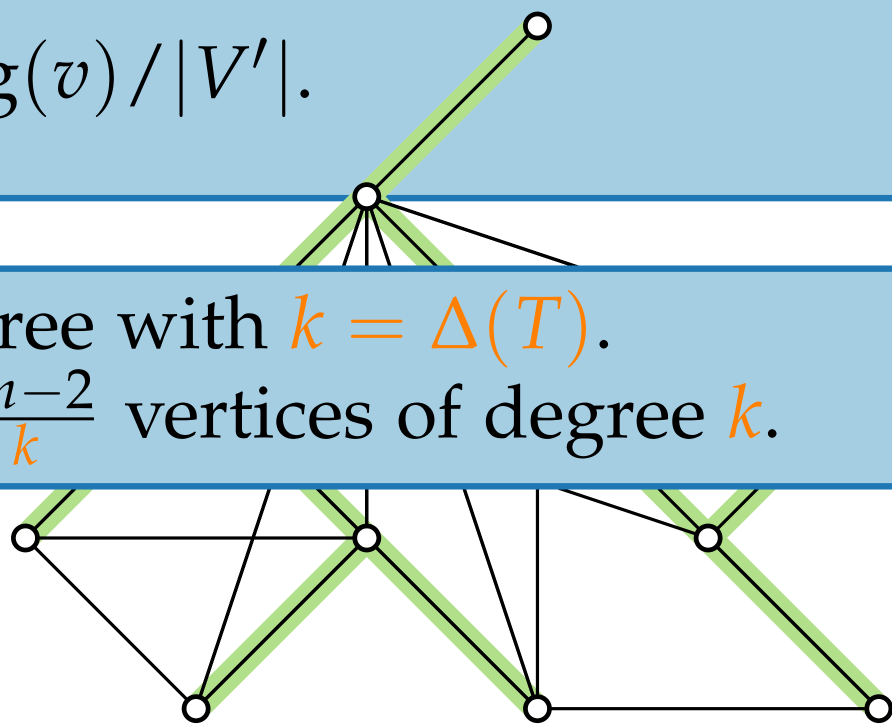
Then $\Delta(G) \geq \sum_{v \in V'} \deg(v) / |V'|$.

Obs. Let T be a spanning tree with $k = \Delta(T)$.
Then T has at most $\frac{2n-2}{k}$ vertices of degree k.



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Part II:
Edge Flips and Local Search

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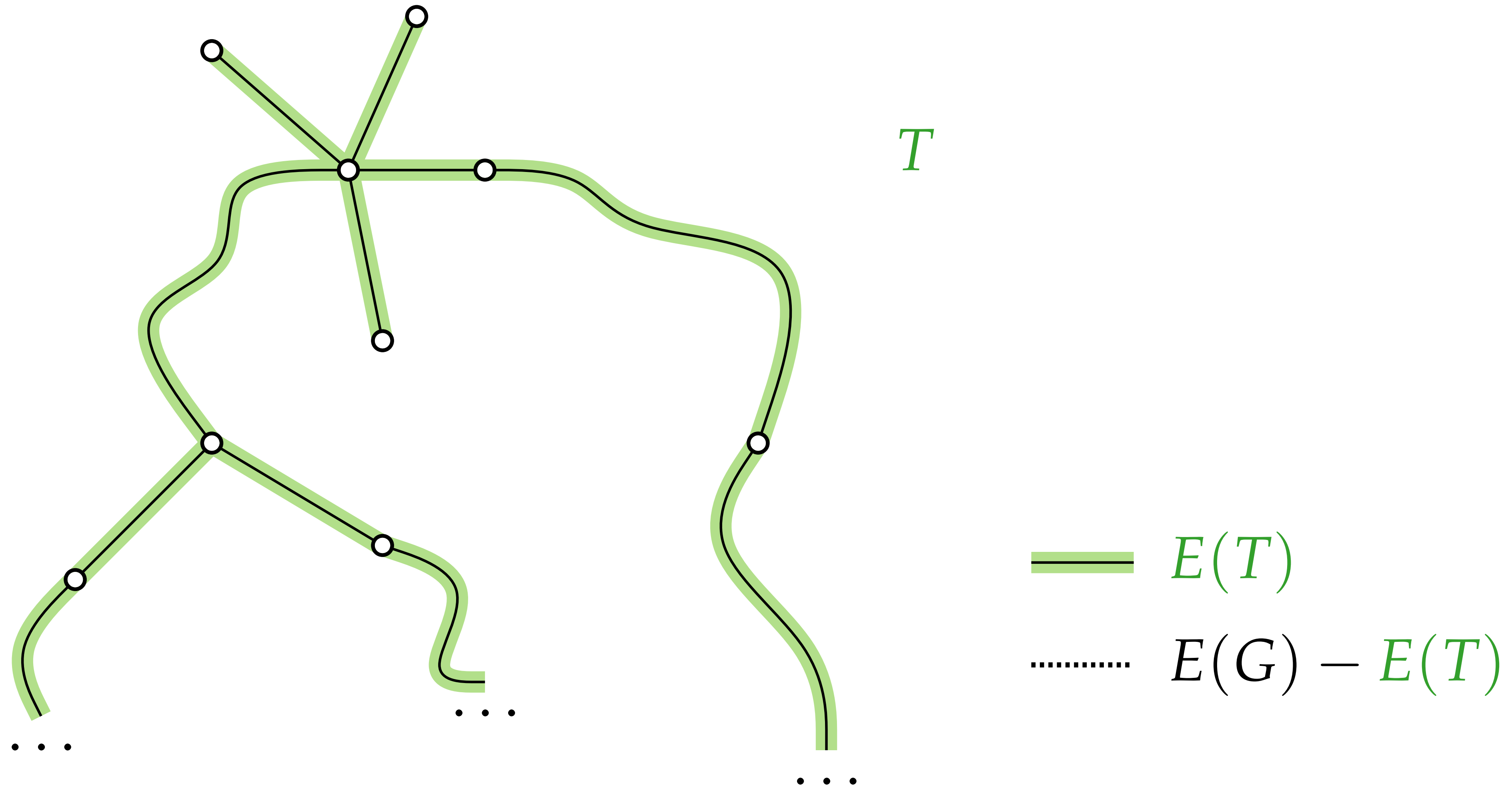
Winter 2021/22

Edge Flips



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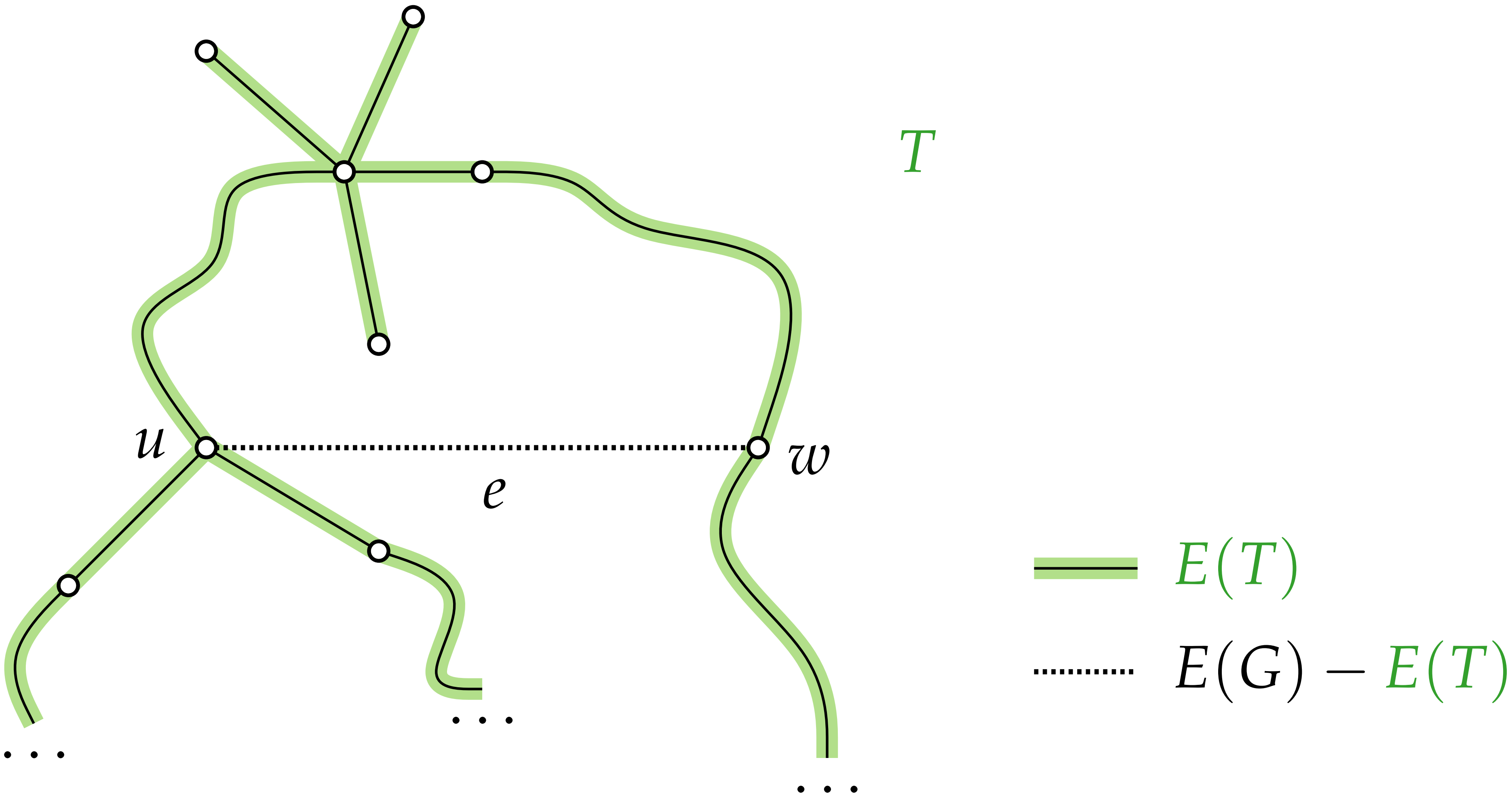

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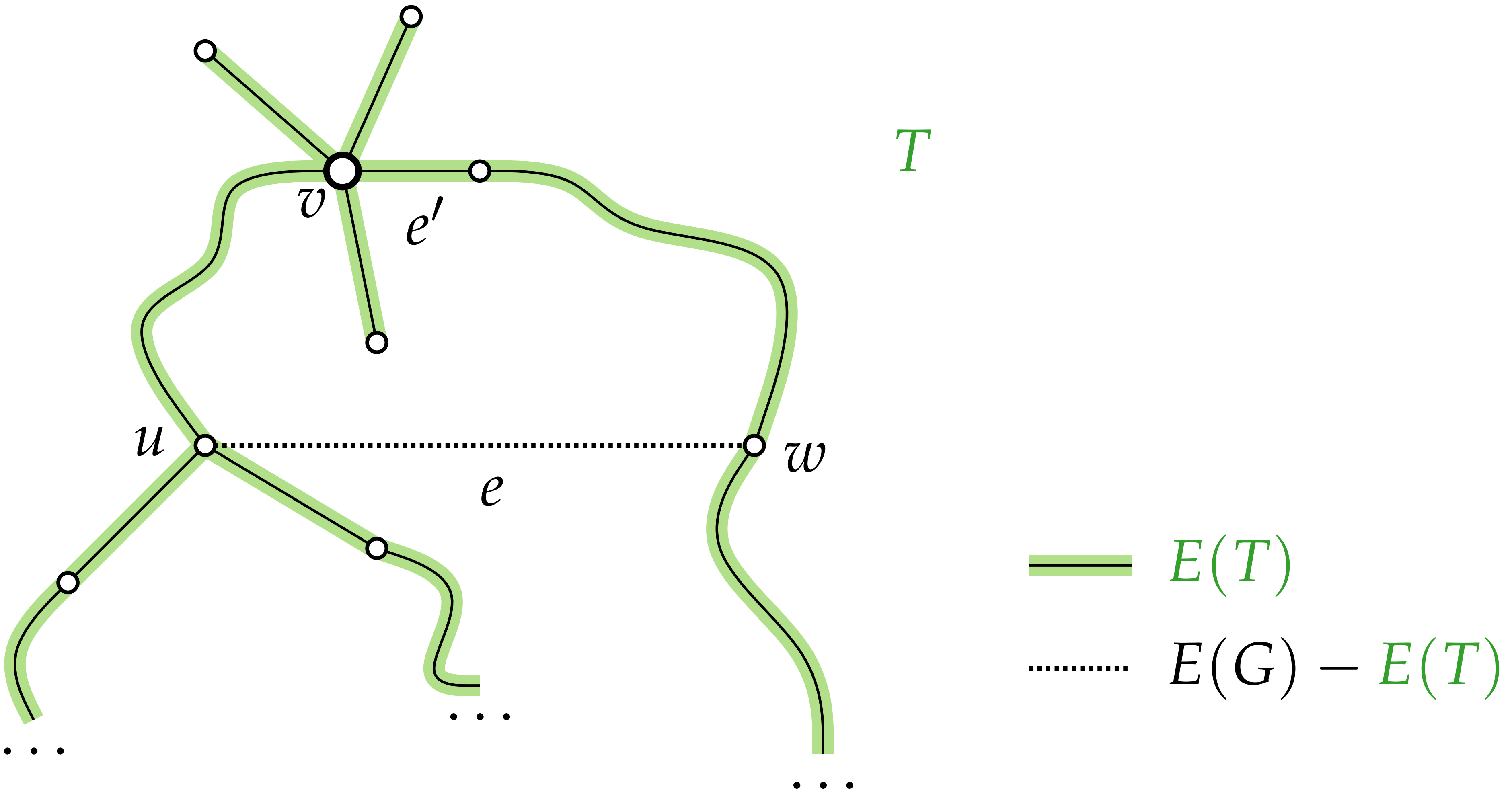

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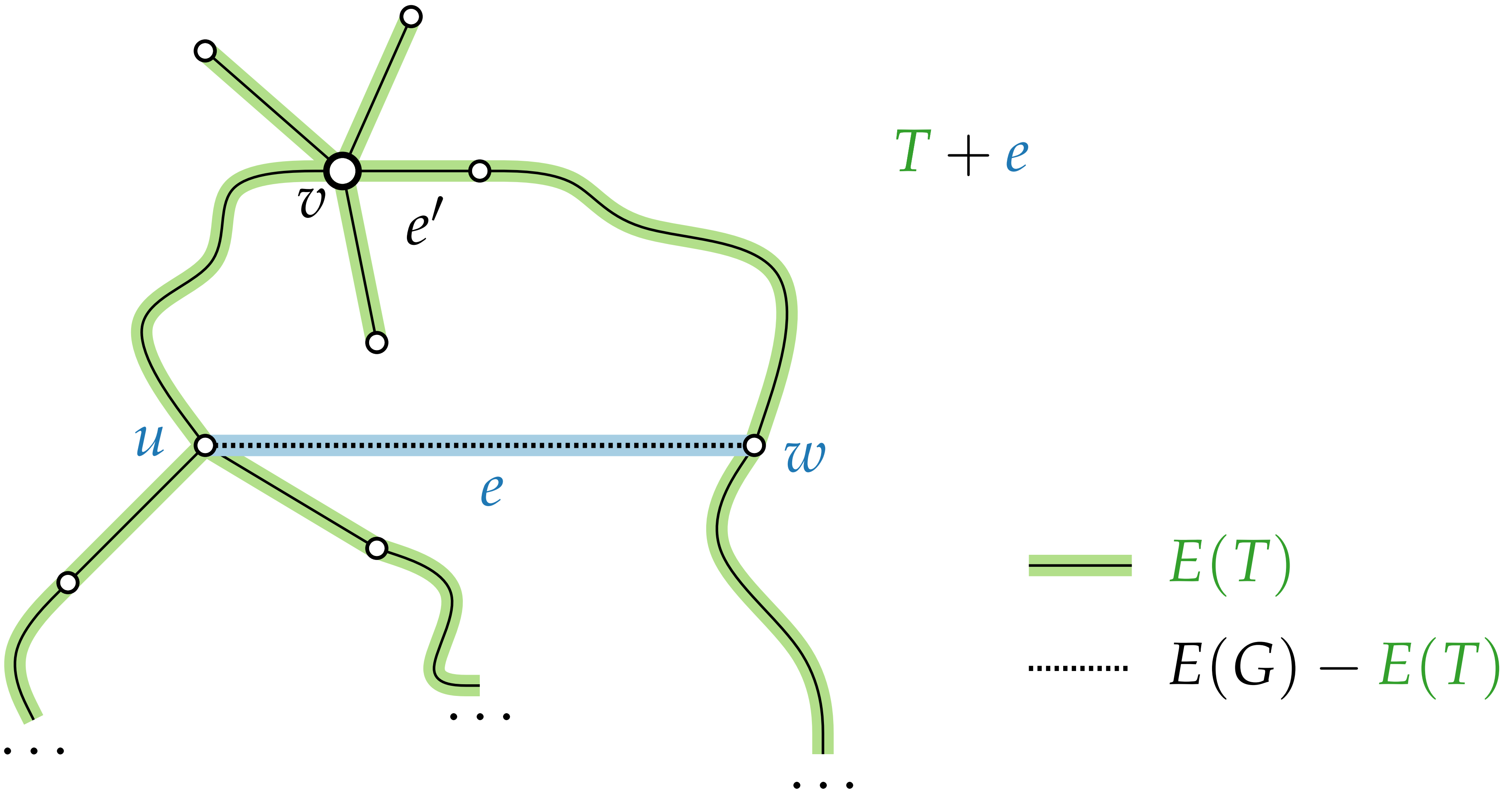

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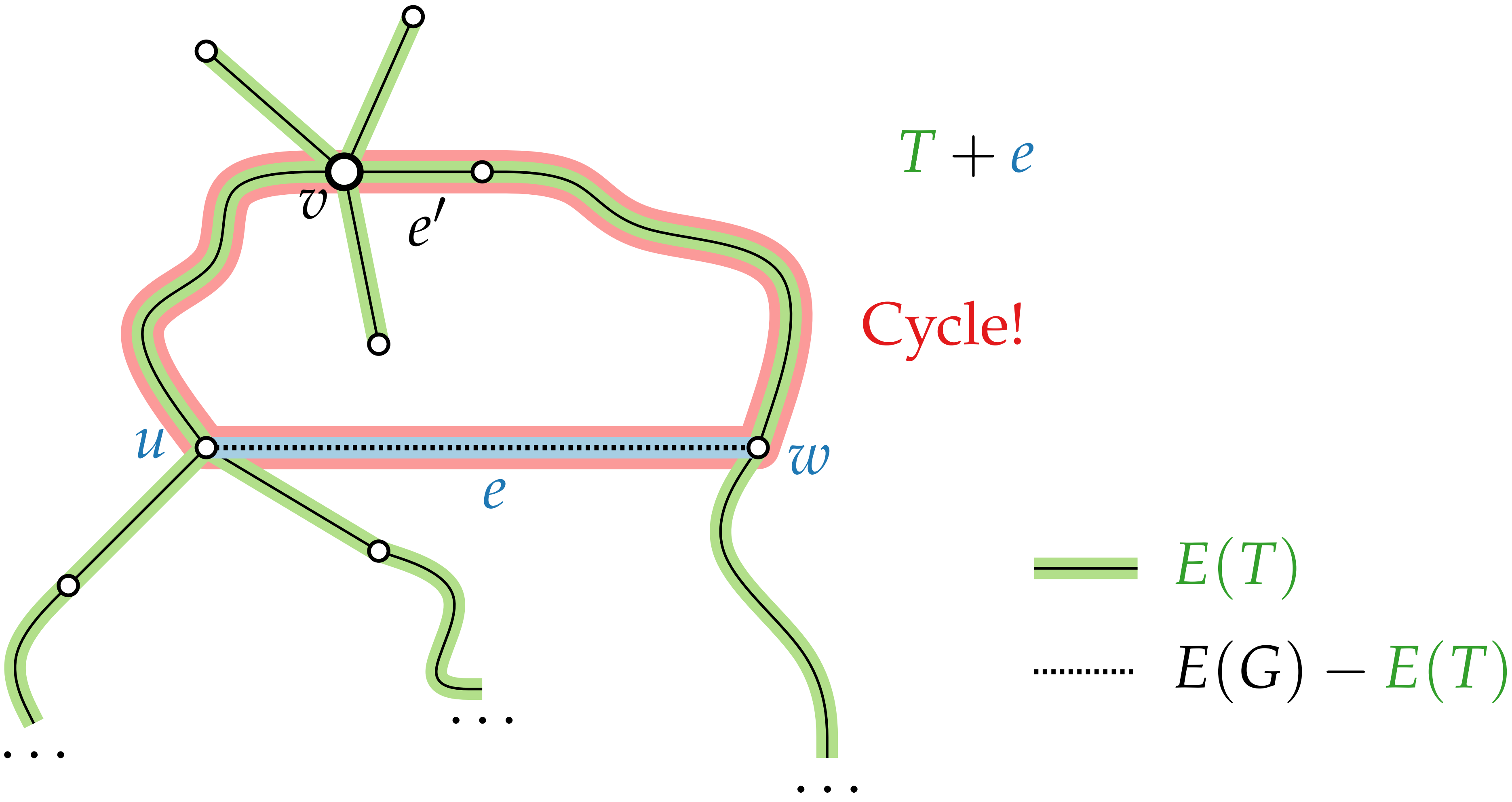

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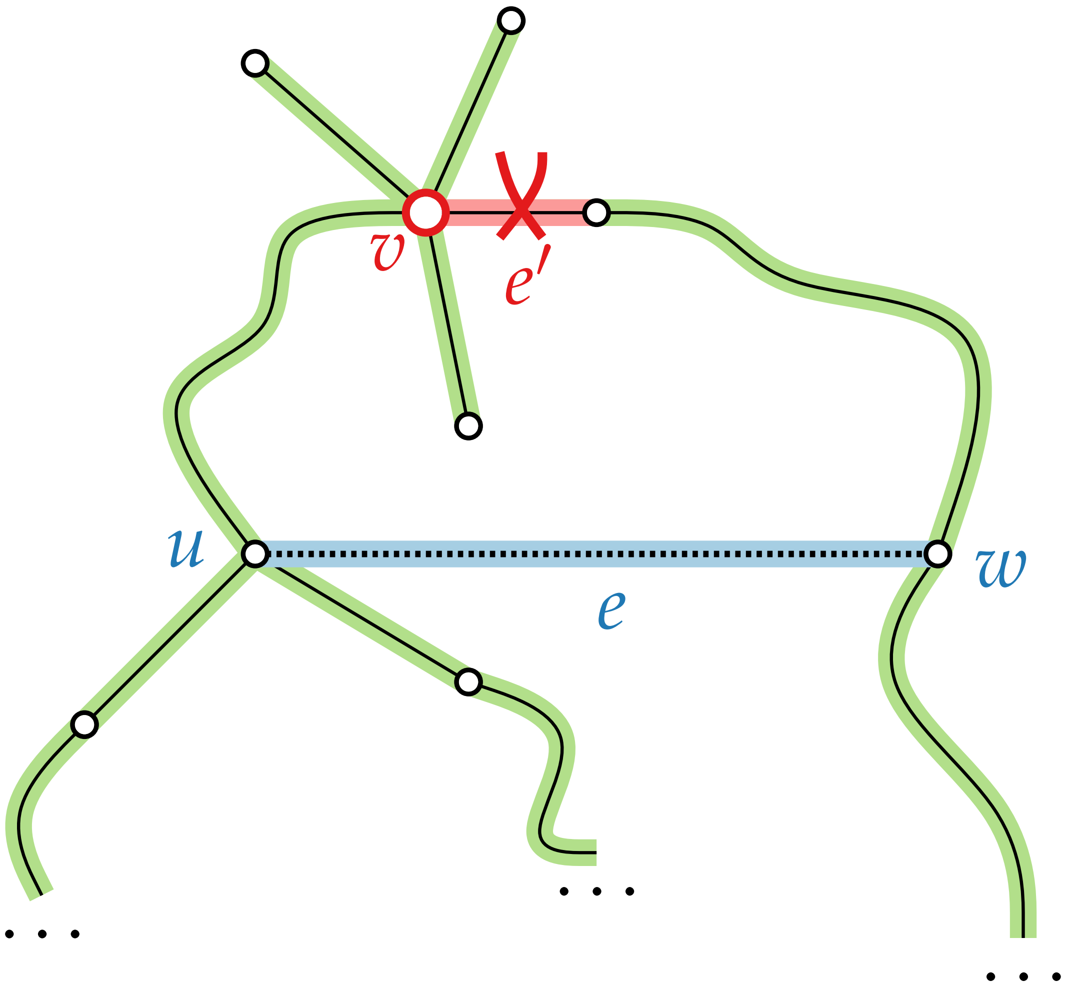
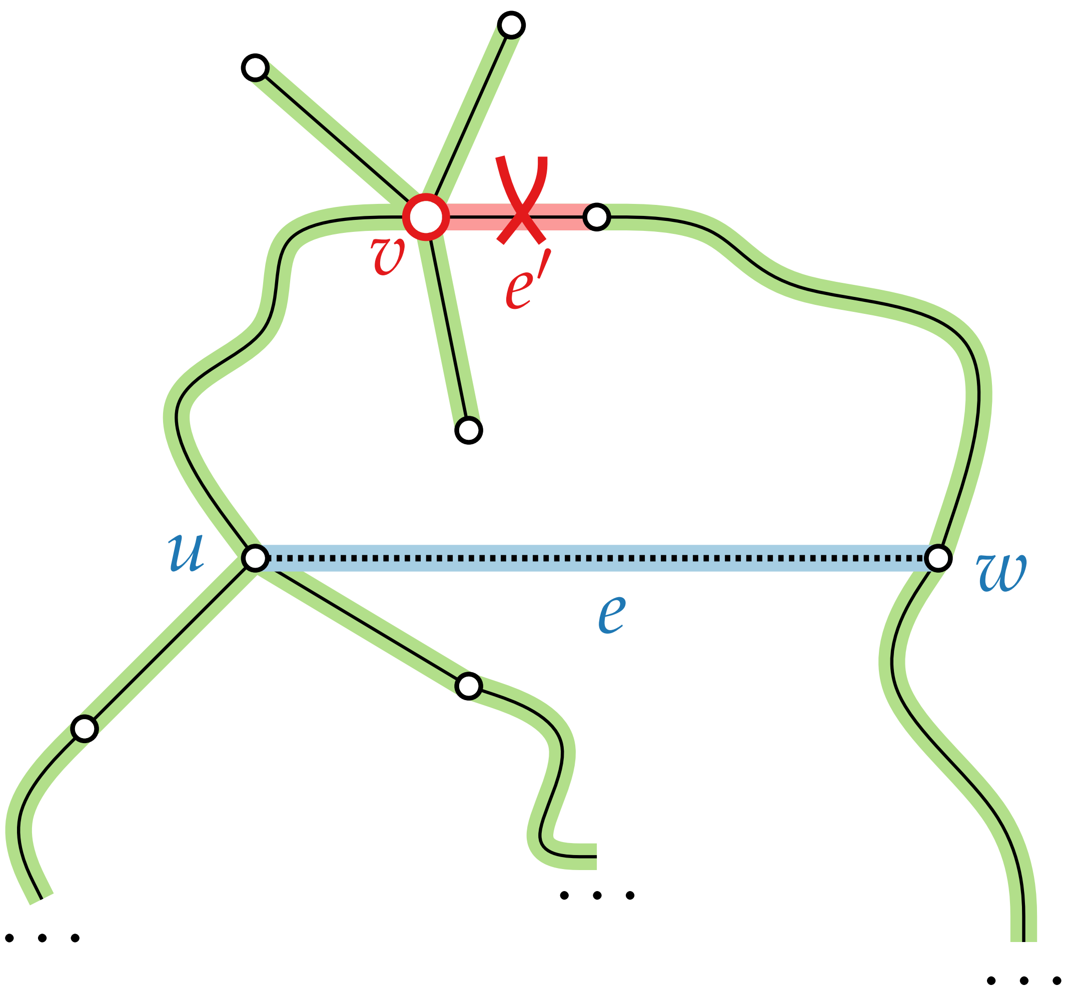


$T + e - e'$

is a new spanning tree

Edge Flips

Def. An **improving flip** in T for a vertex v and an edge $uw \in E(G) \setminus E(T)$ is a flip with
$\deg_T(v) >$



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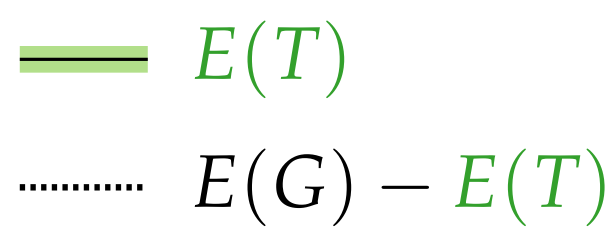
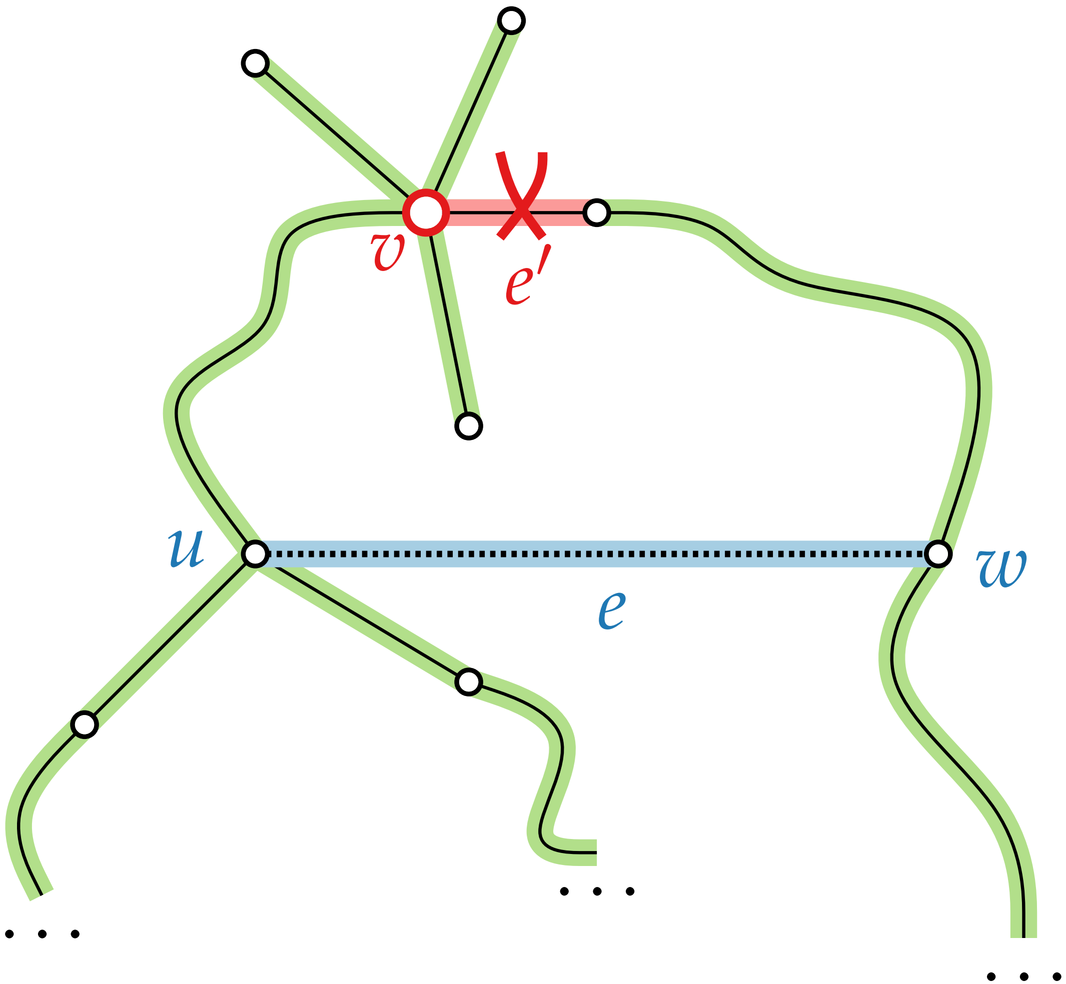


Edge Flips

Def. An **improving flip** in T for a vertex v and an edge $uw \in E(G) \setminus E(T)$ is a flip with $\deg_T(v) > \max\{\deg_T(u), \deg_T(w)\} + 1$.



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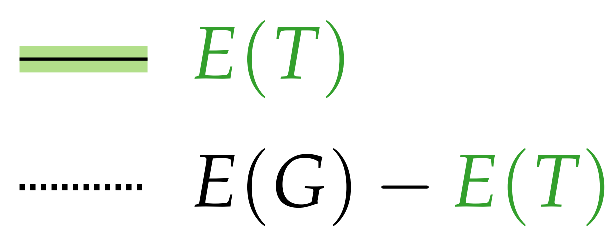

Local Search

MinDegSpanningTreeLocalSearch(G)

$T \leftarrow$ any spanning tree of G

while $\exists$ improving flip in T for a vertex v
with $\deg_T(v) \geq \Delta(T) - \ell$ **do**

  do the improving flip

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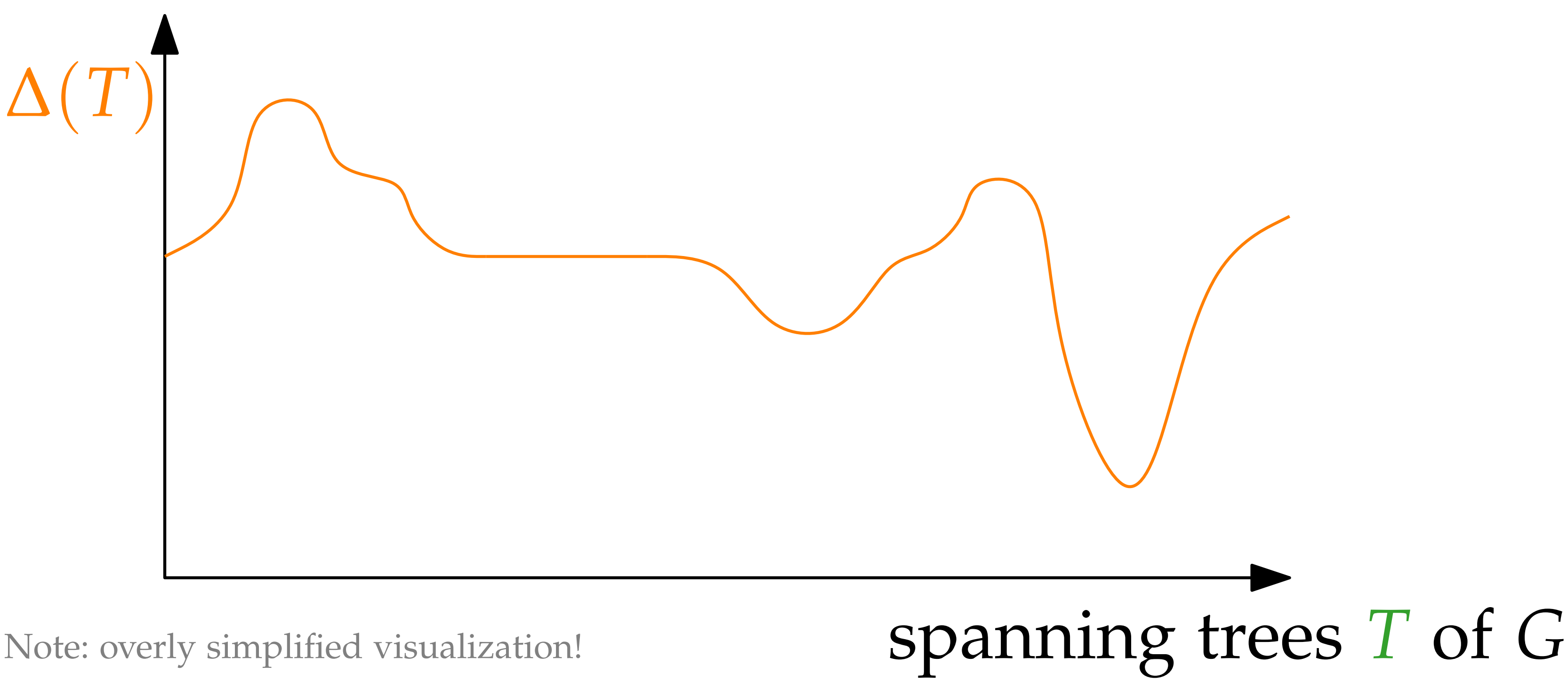

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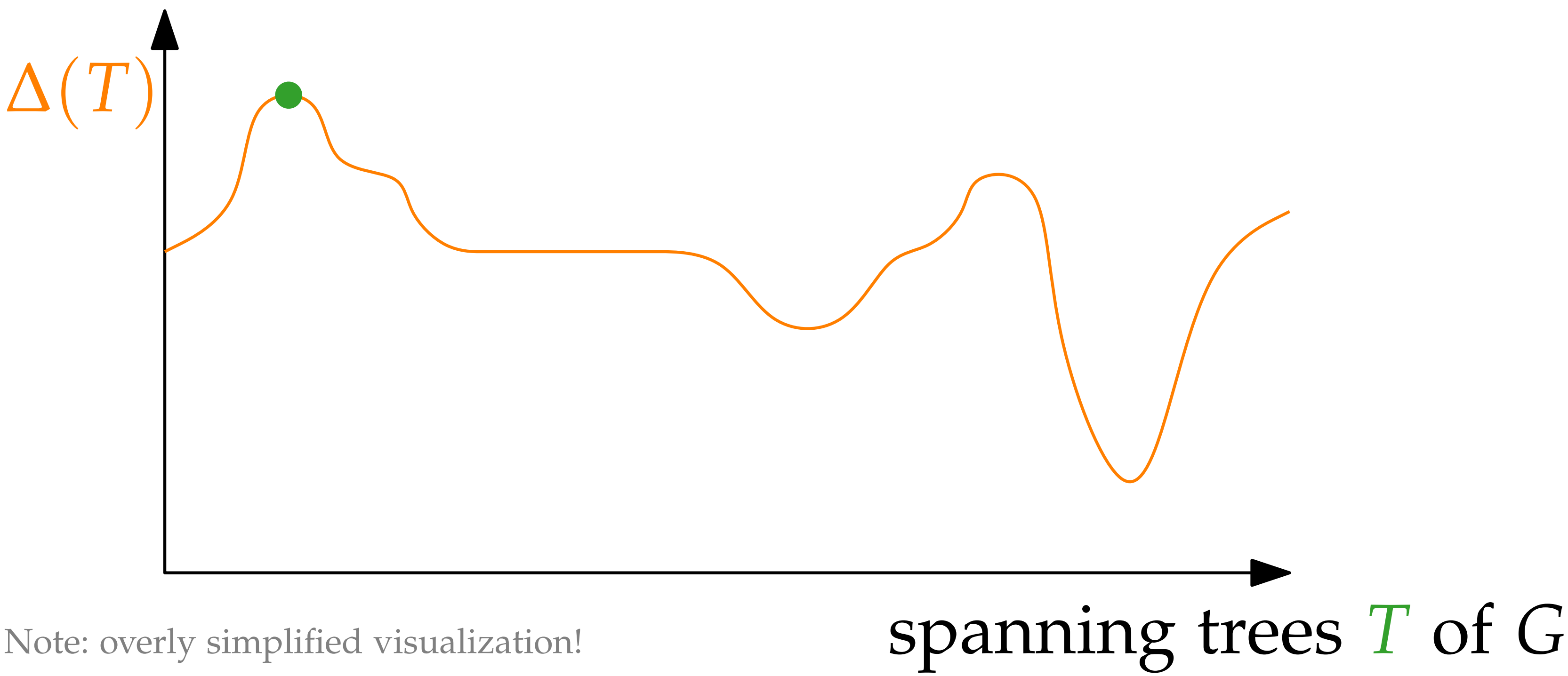

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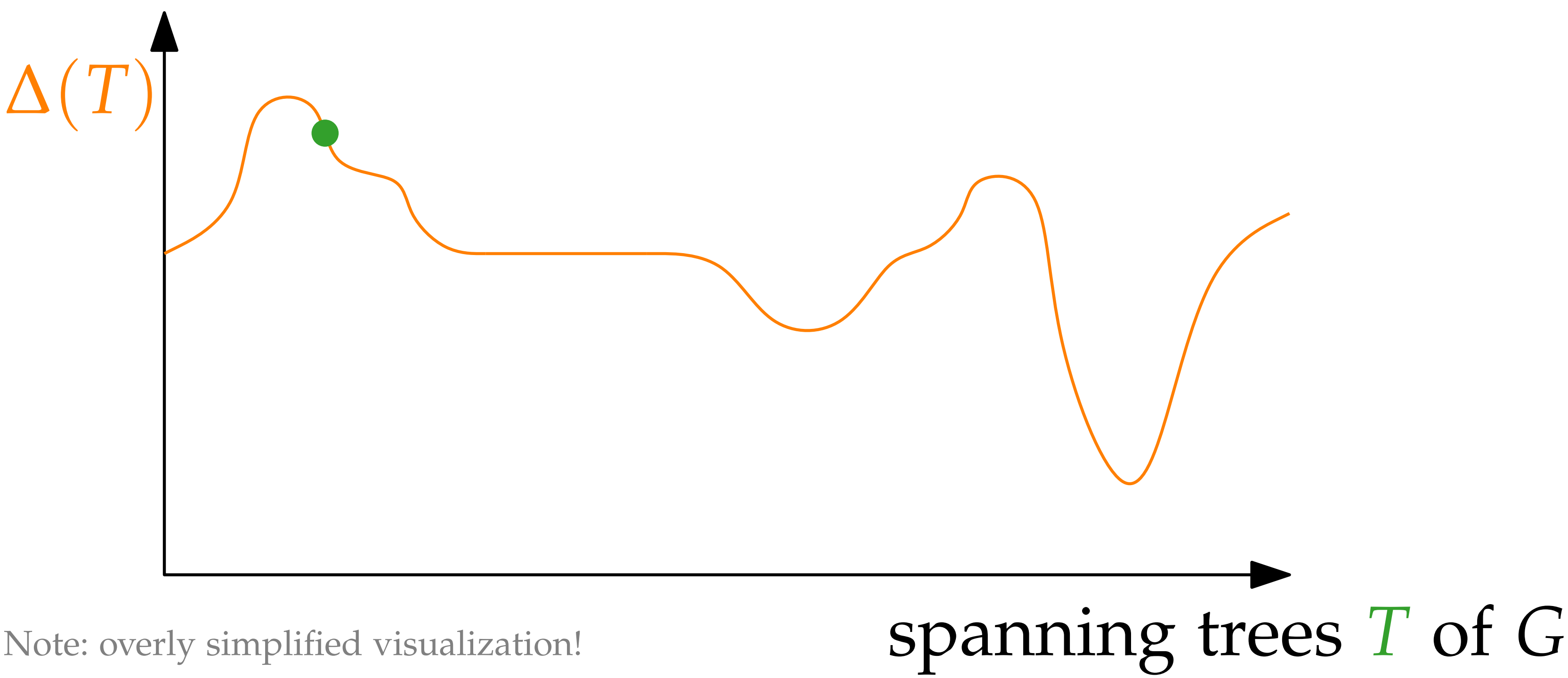

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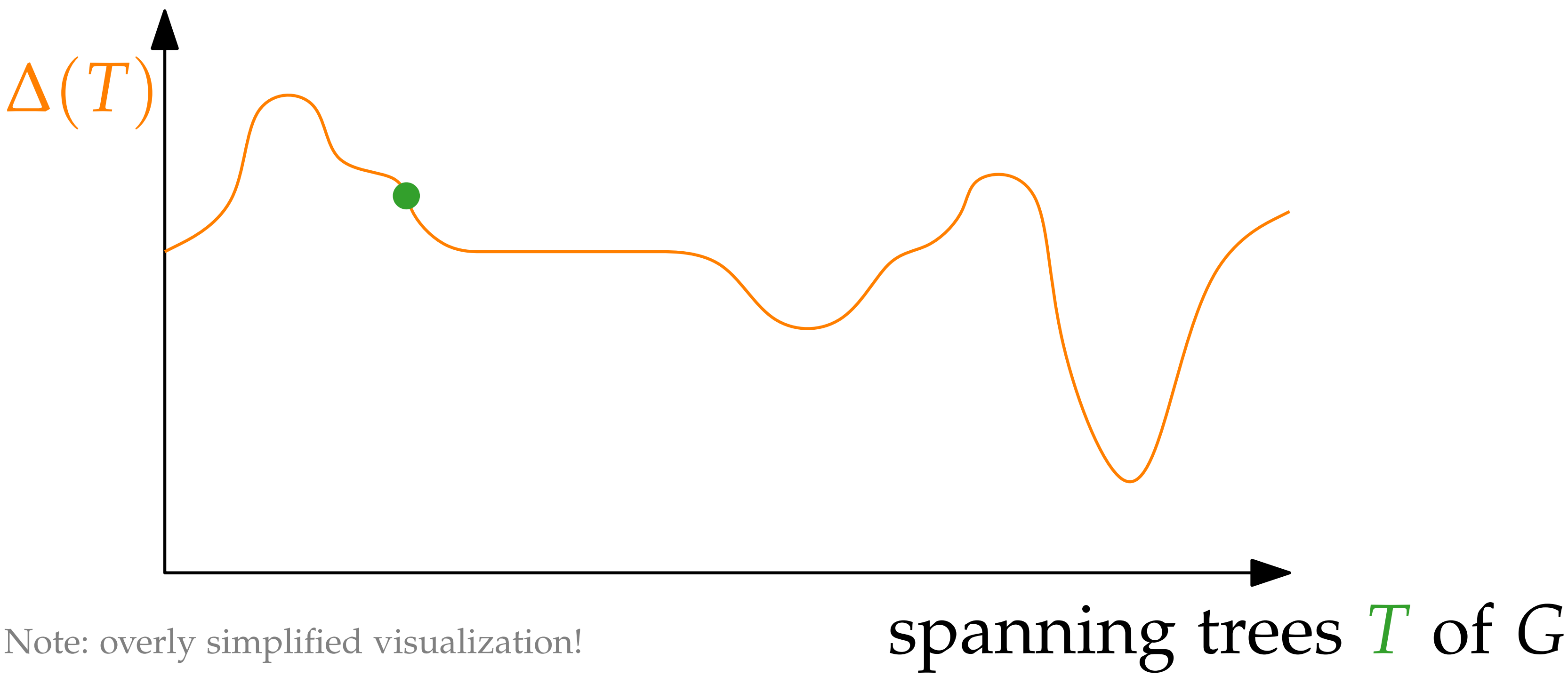

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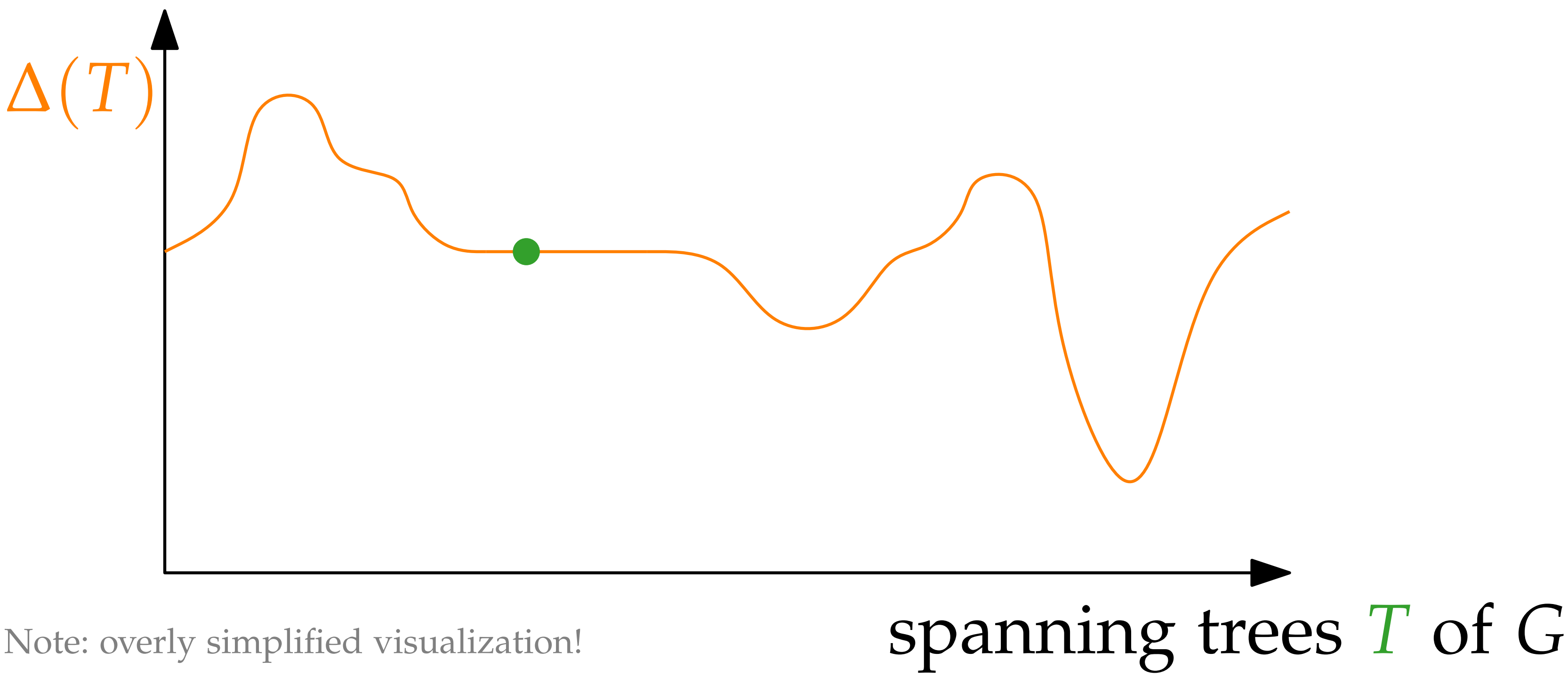

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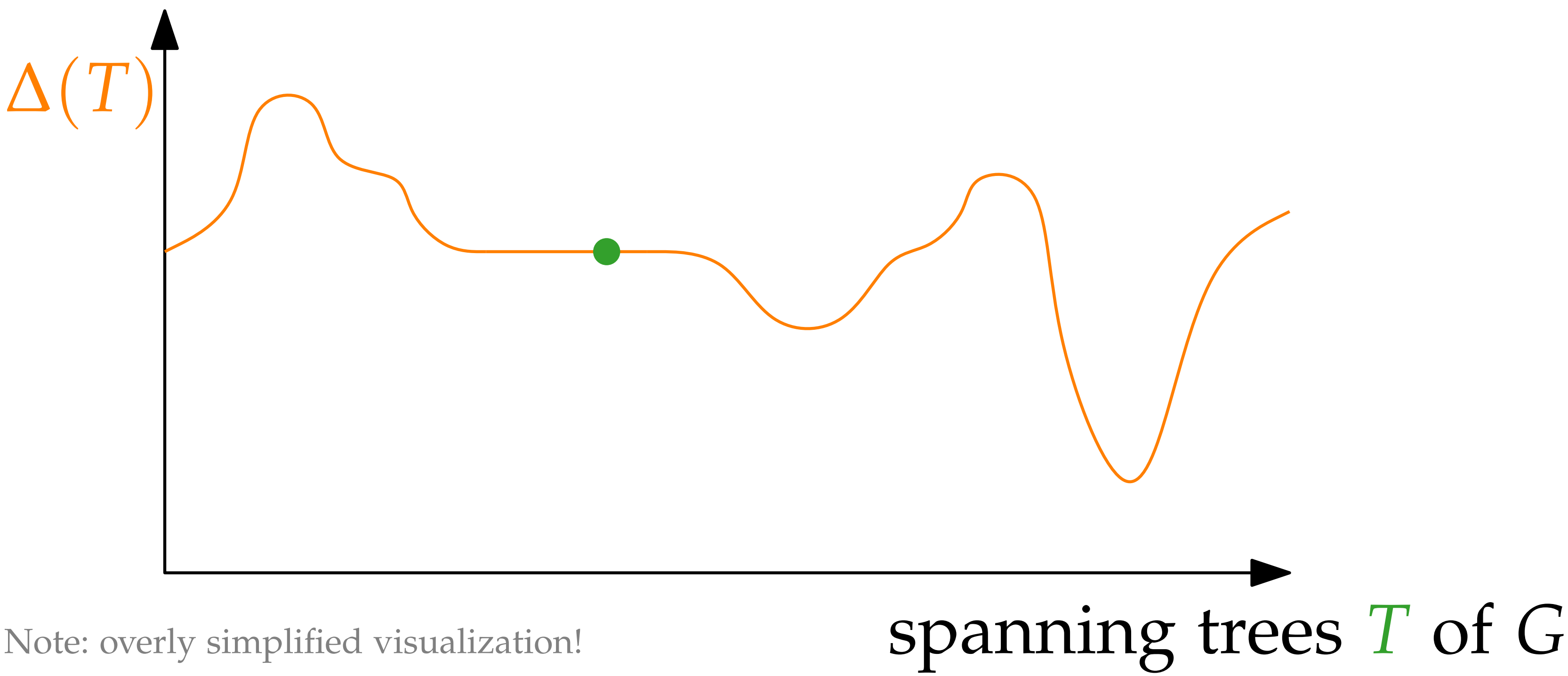

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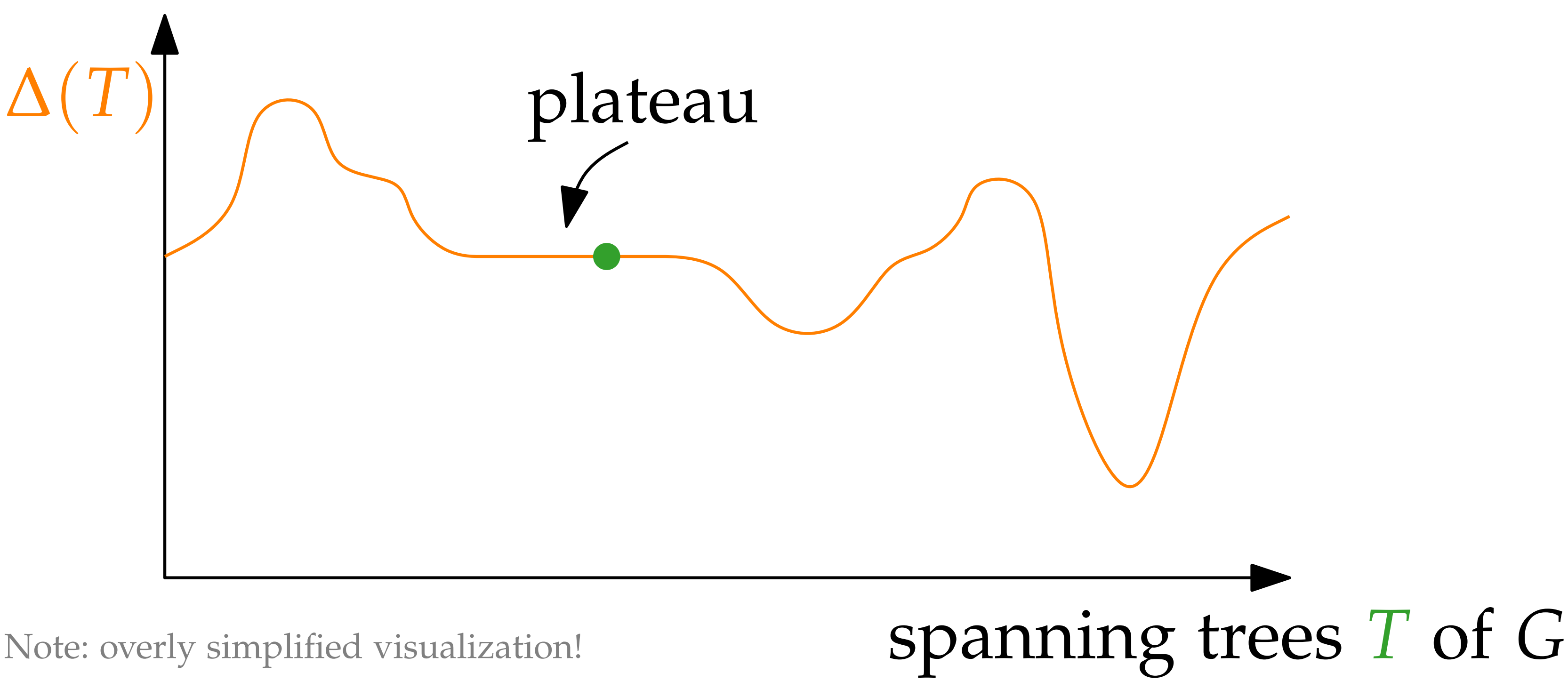

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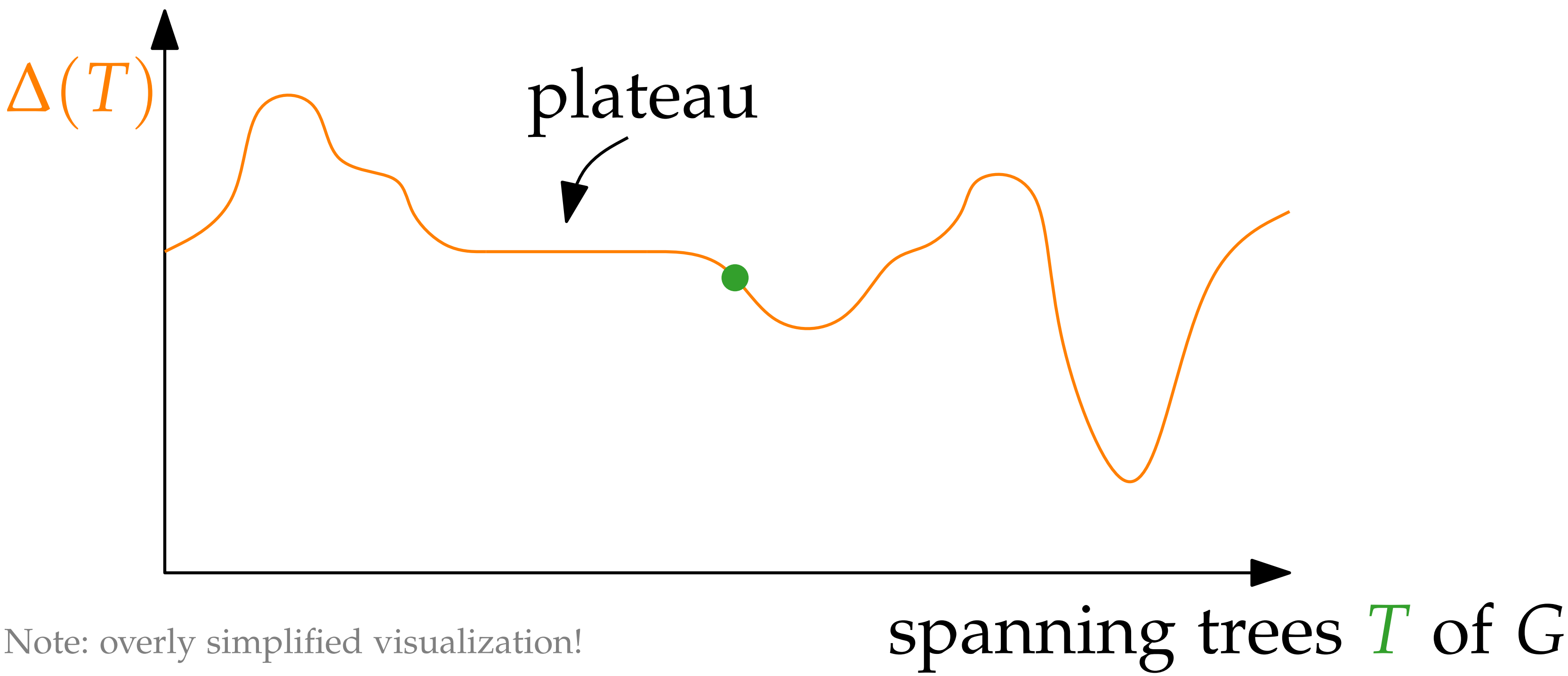

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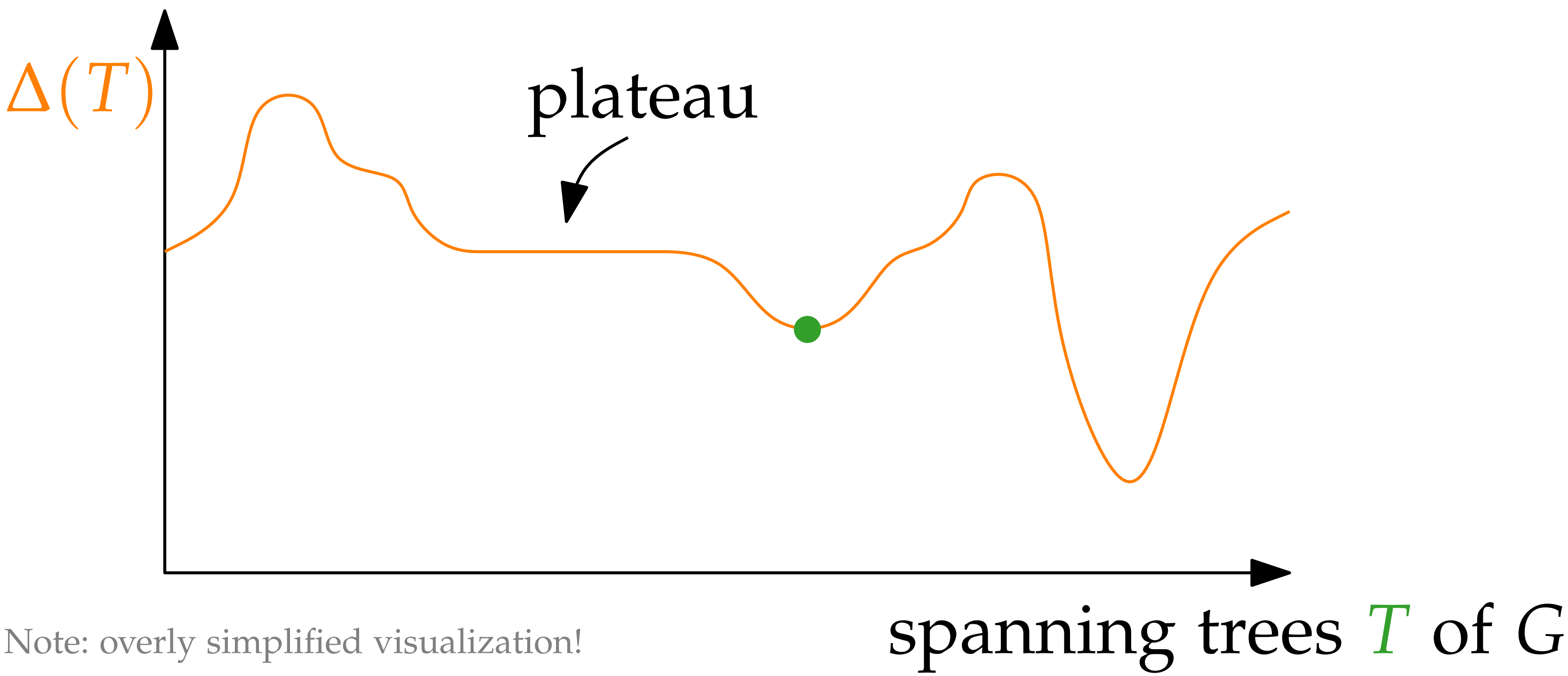

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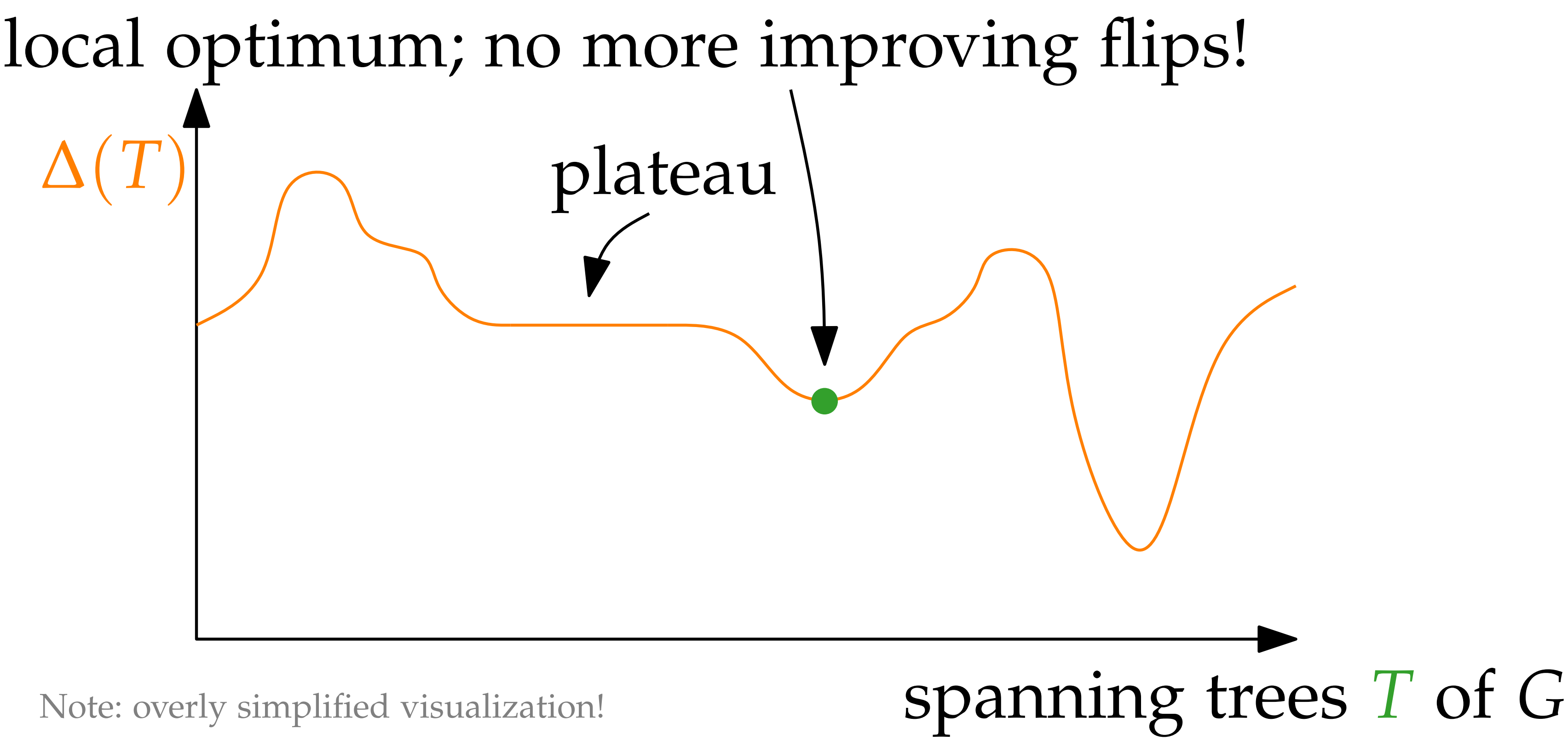

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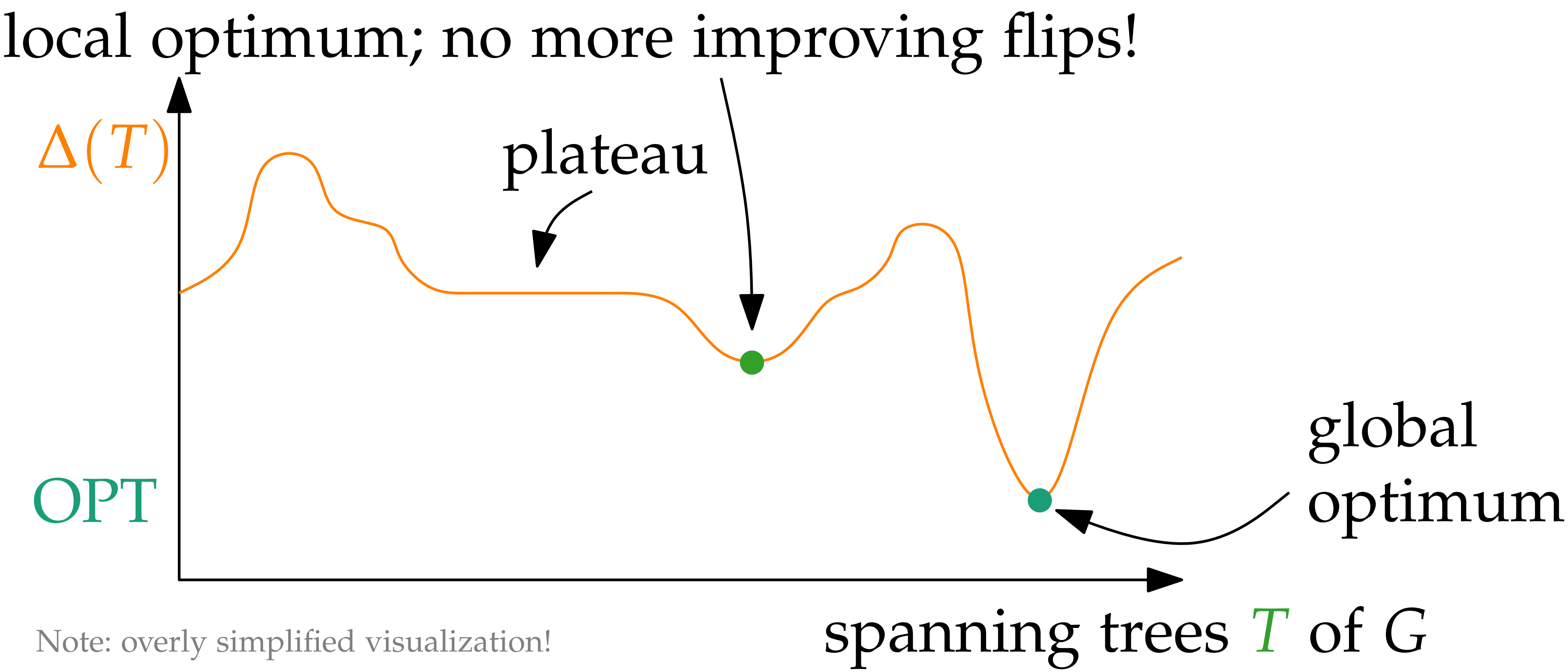

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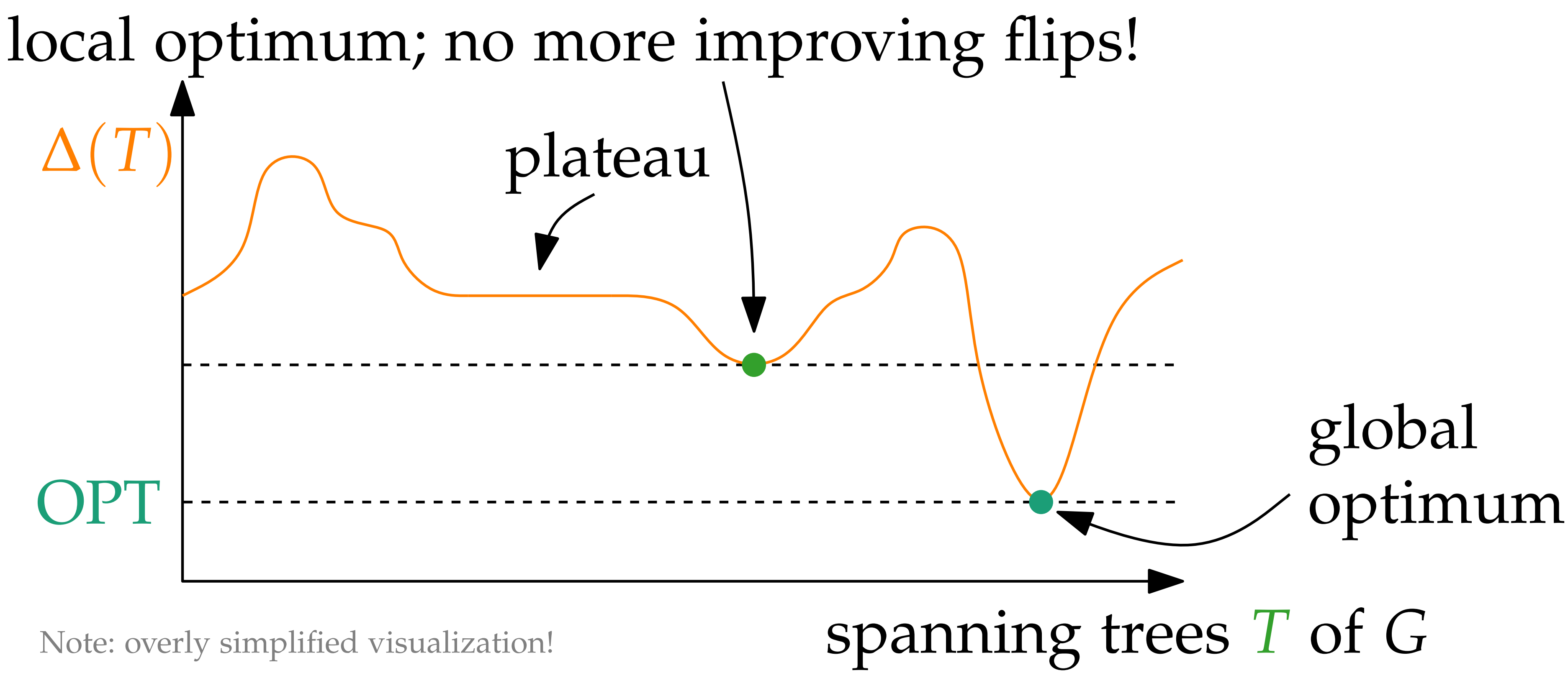

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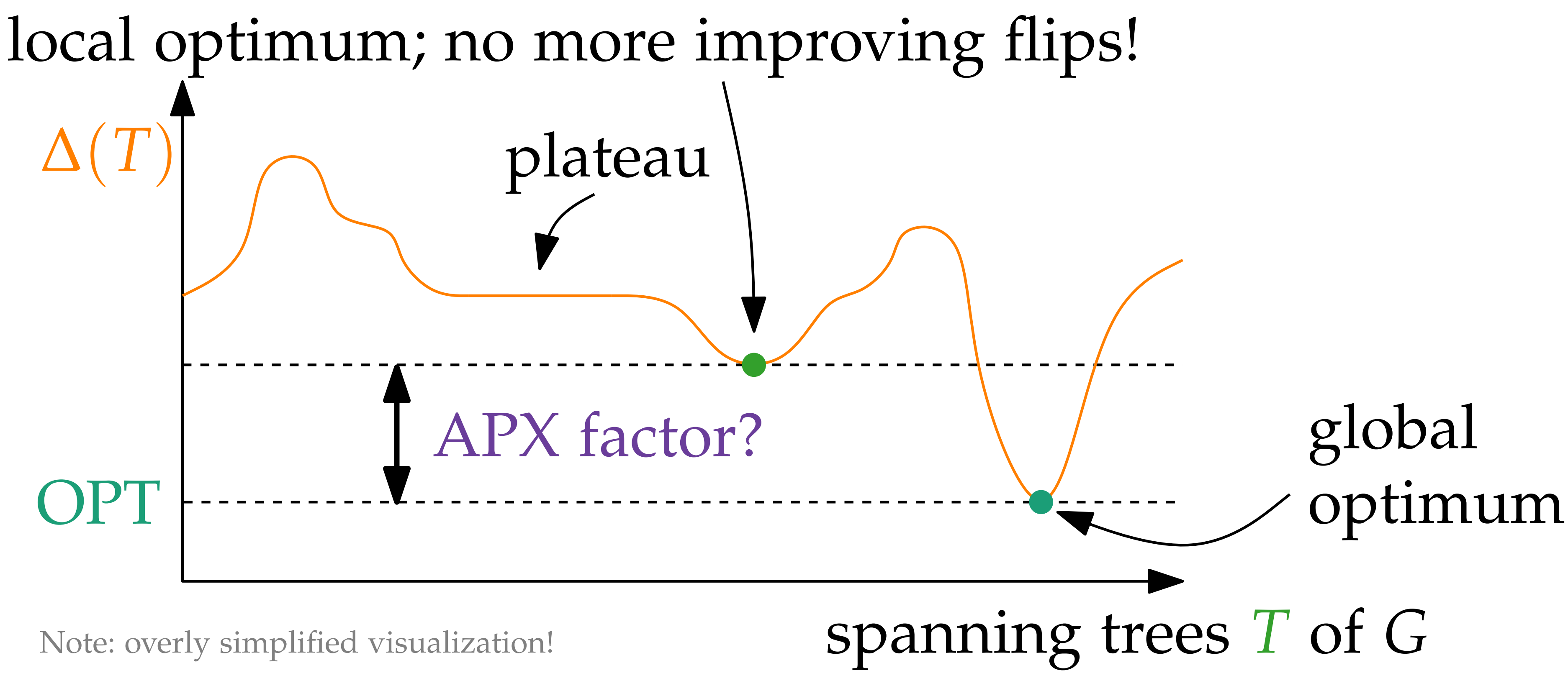

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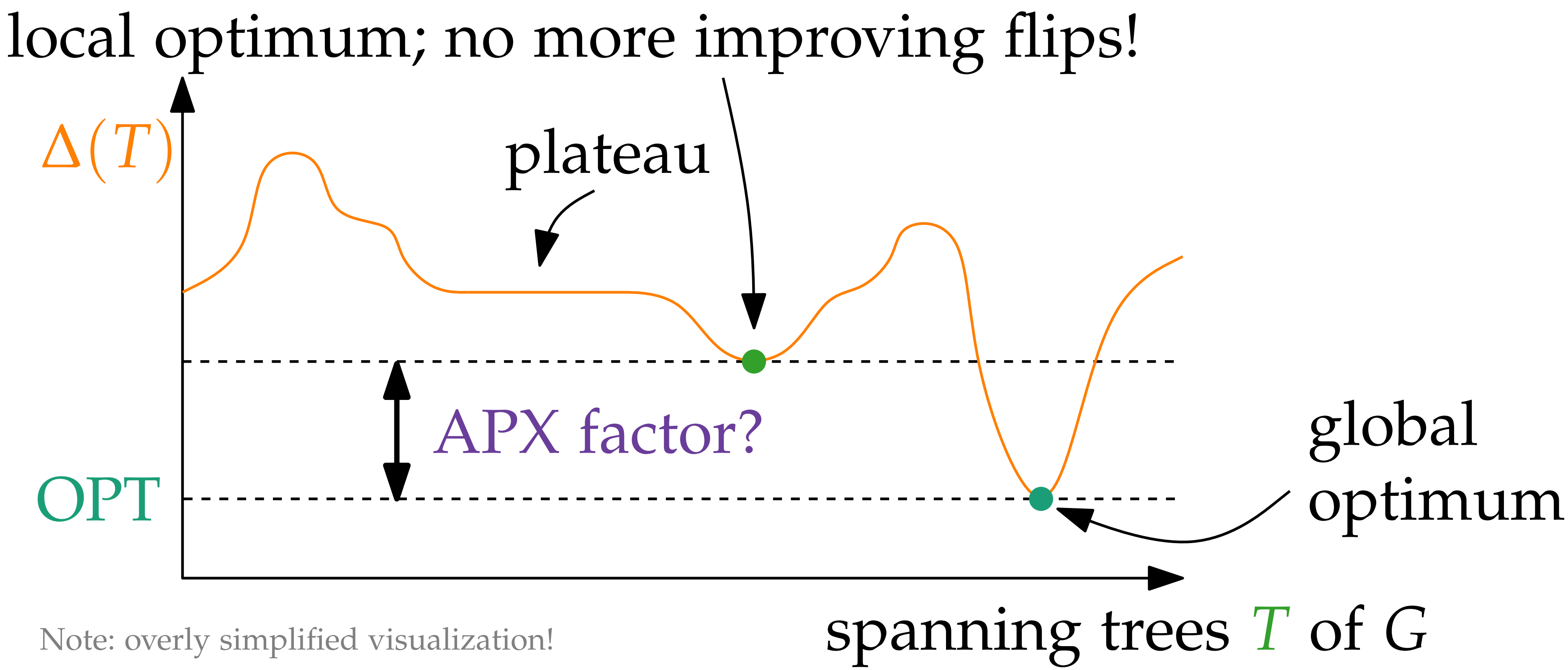
- Termination?



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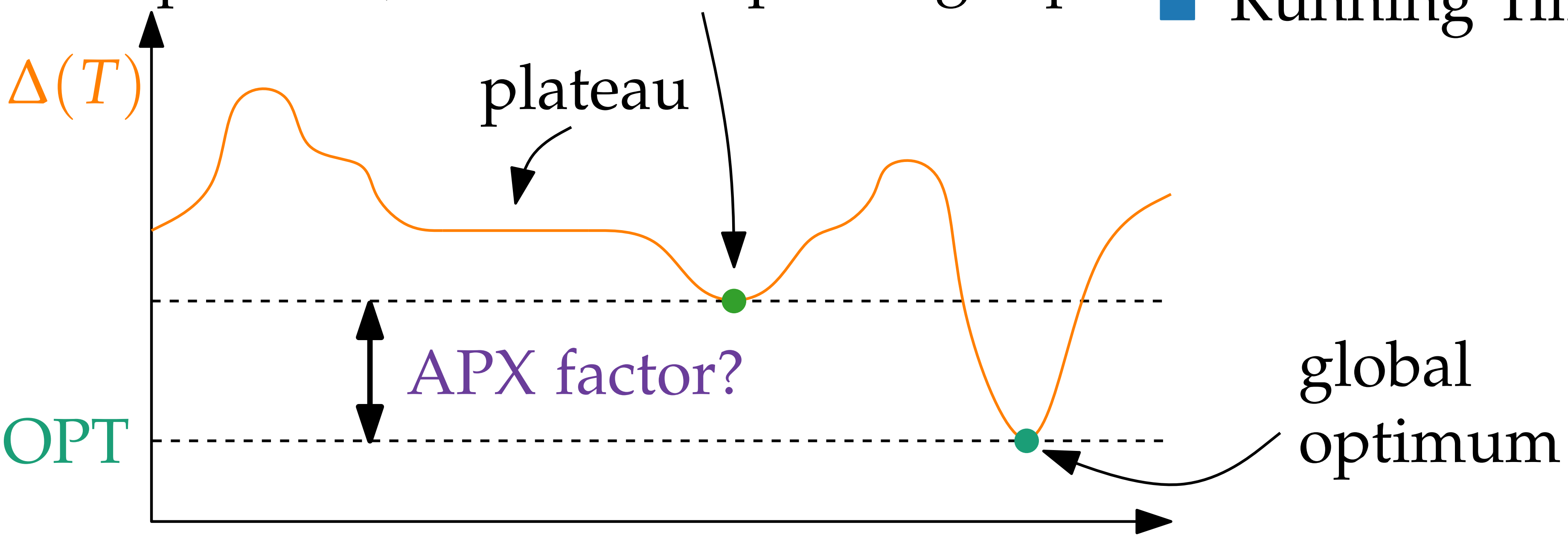
- Termination?
- Running Time?



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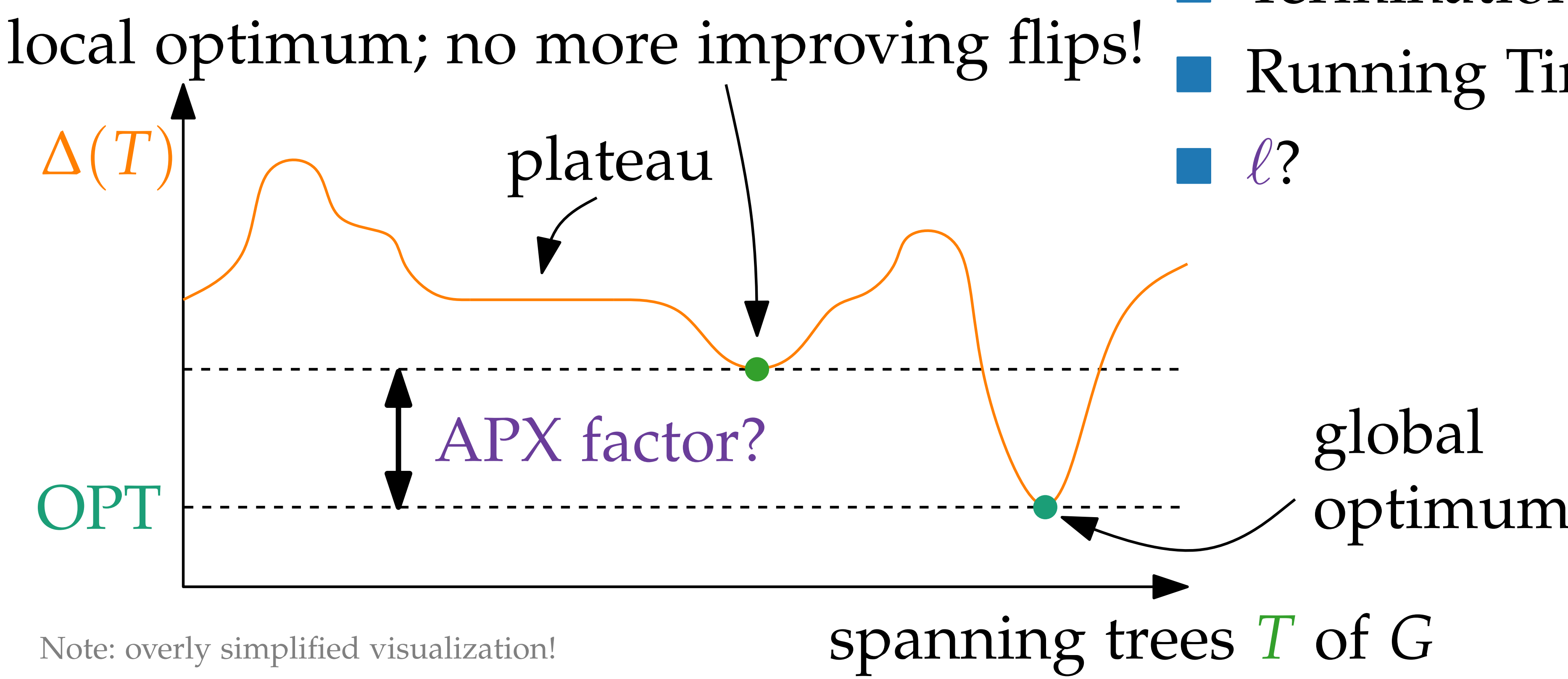
- ℓ?



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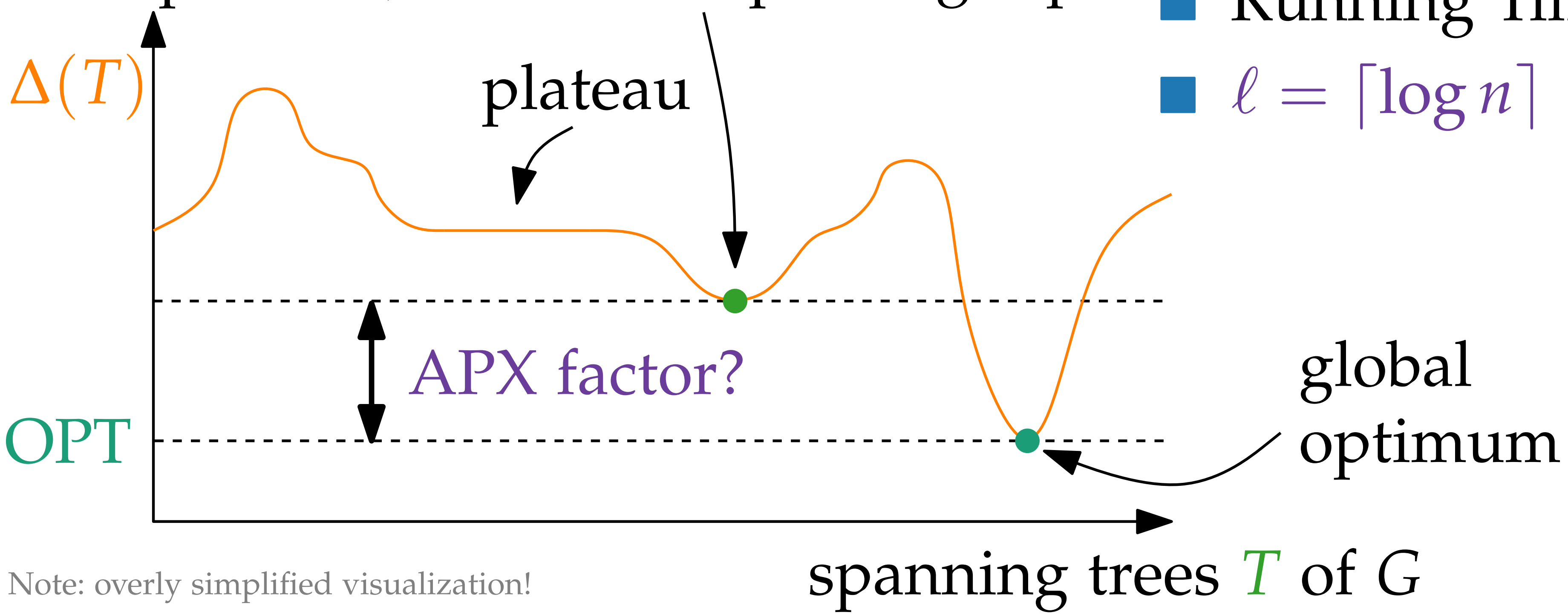
- Termination?
- Running Time?
- $\ell = \lceil \log n \rceil$

local optimum; no more improving flips!



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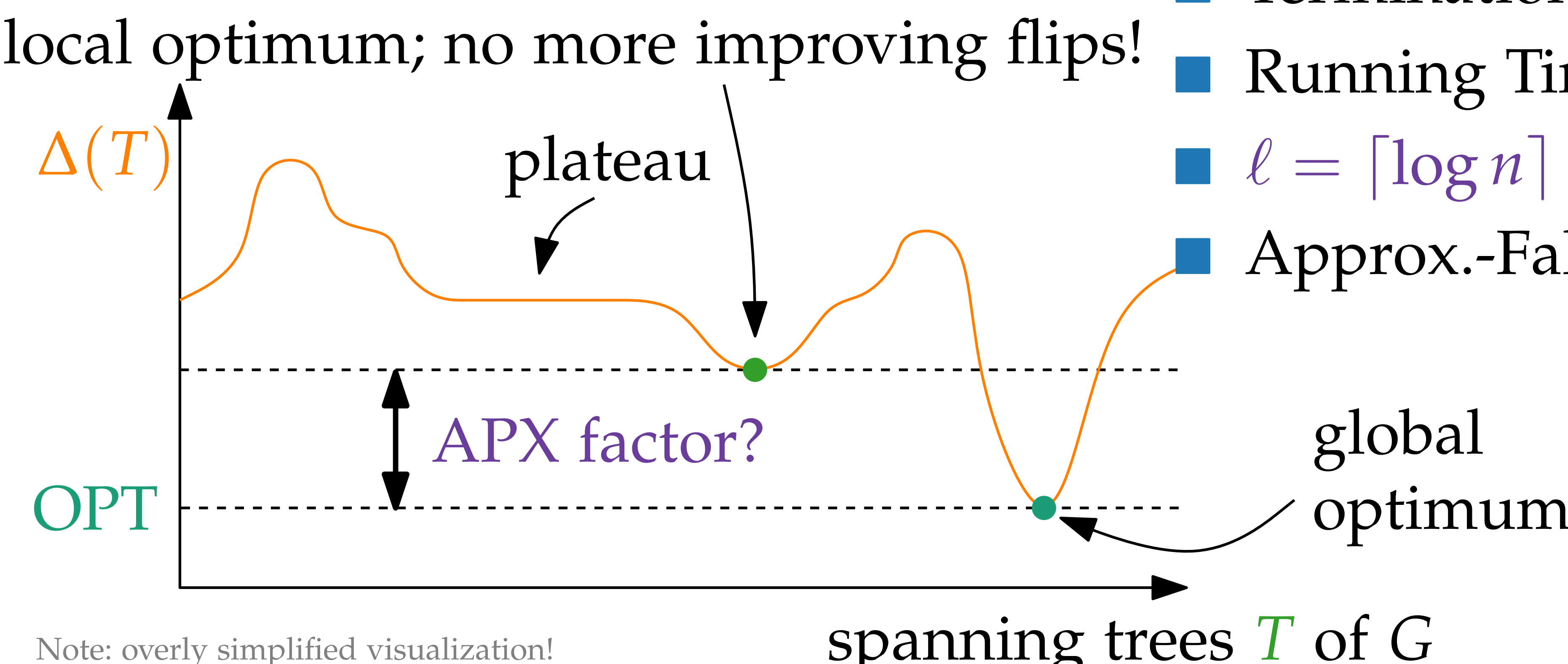
- Termination?
- Running Time?
- $\ell = \lceil \log n \rceil$
- Approx.-Faktor?



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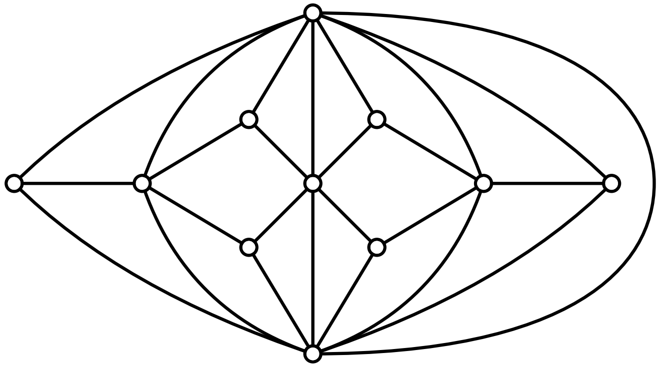
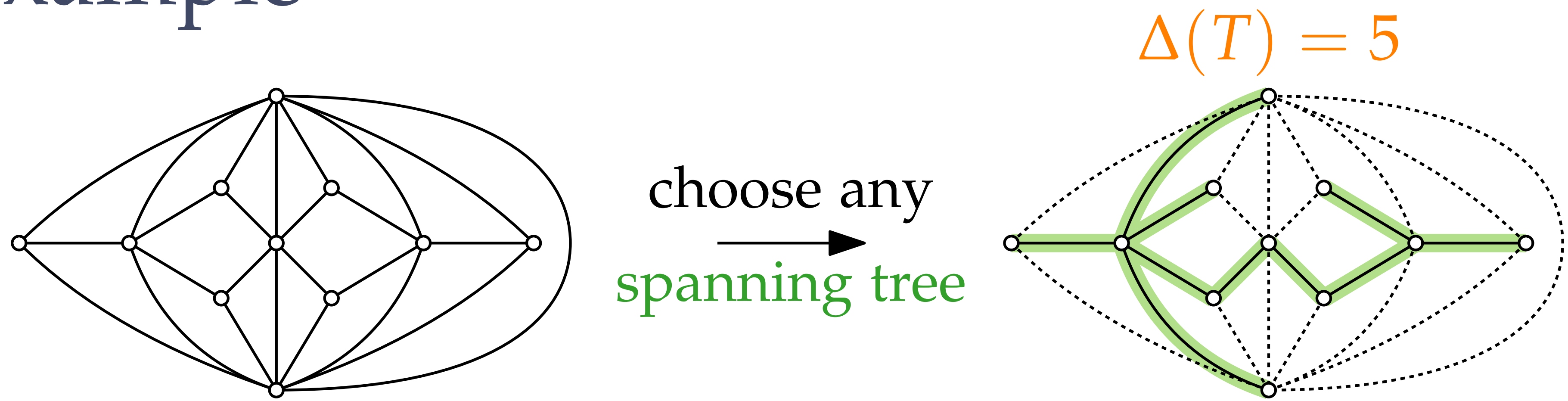
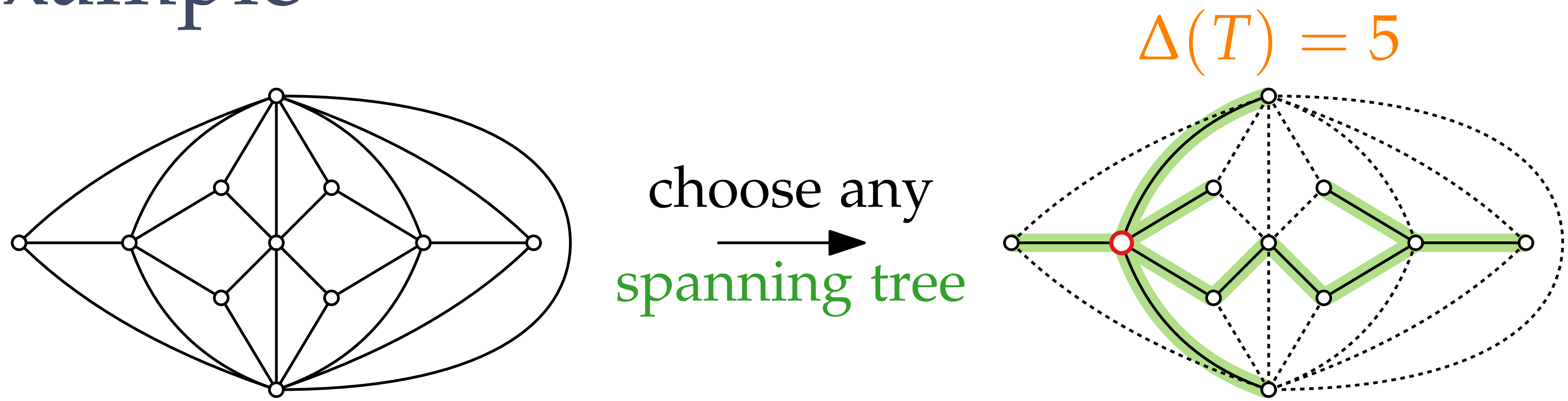
Example



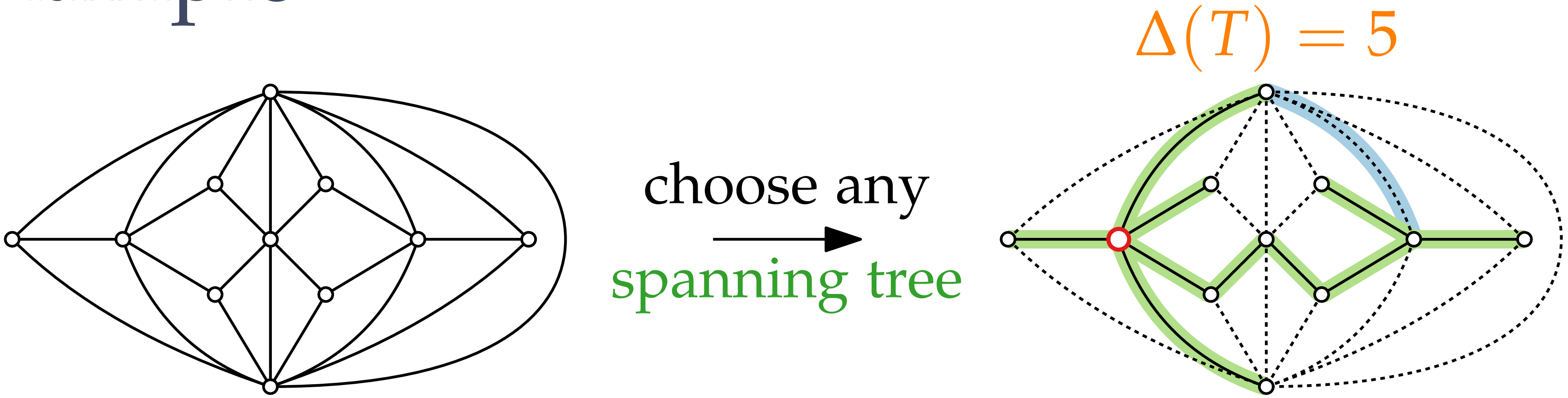
Example



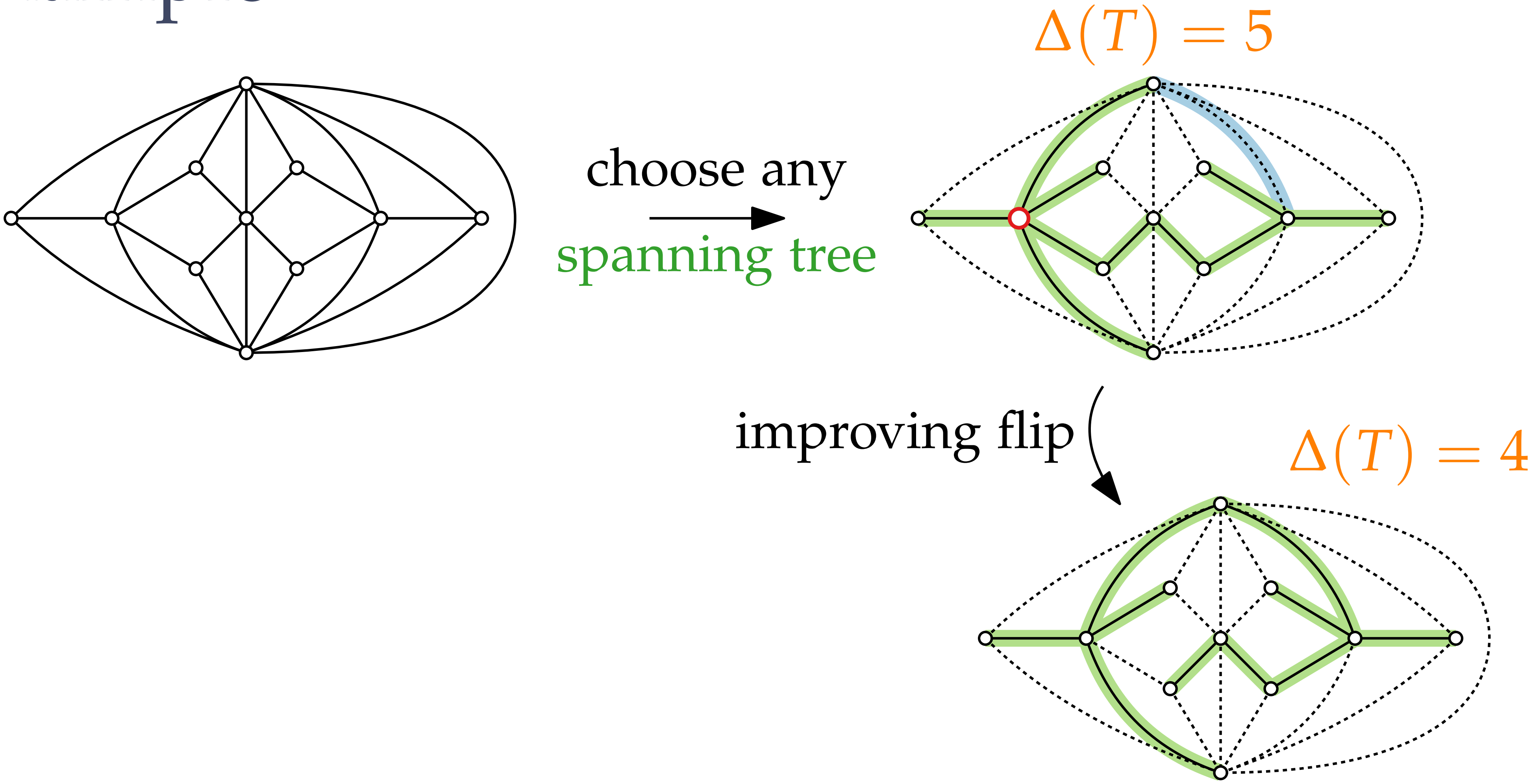
Example



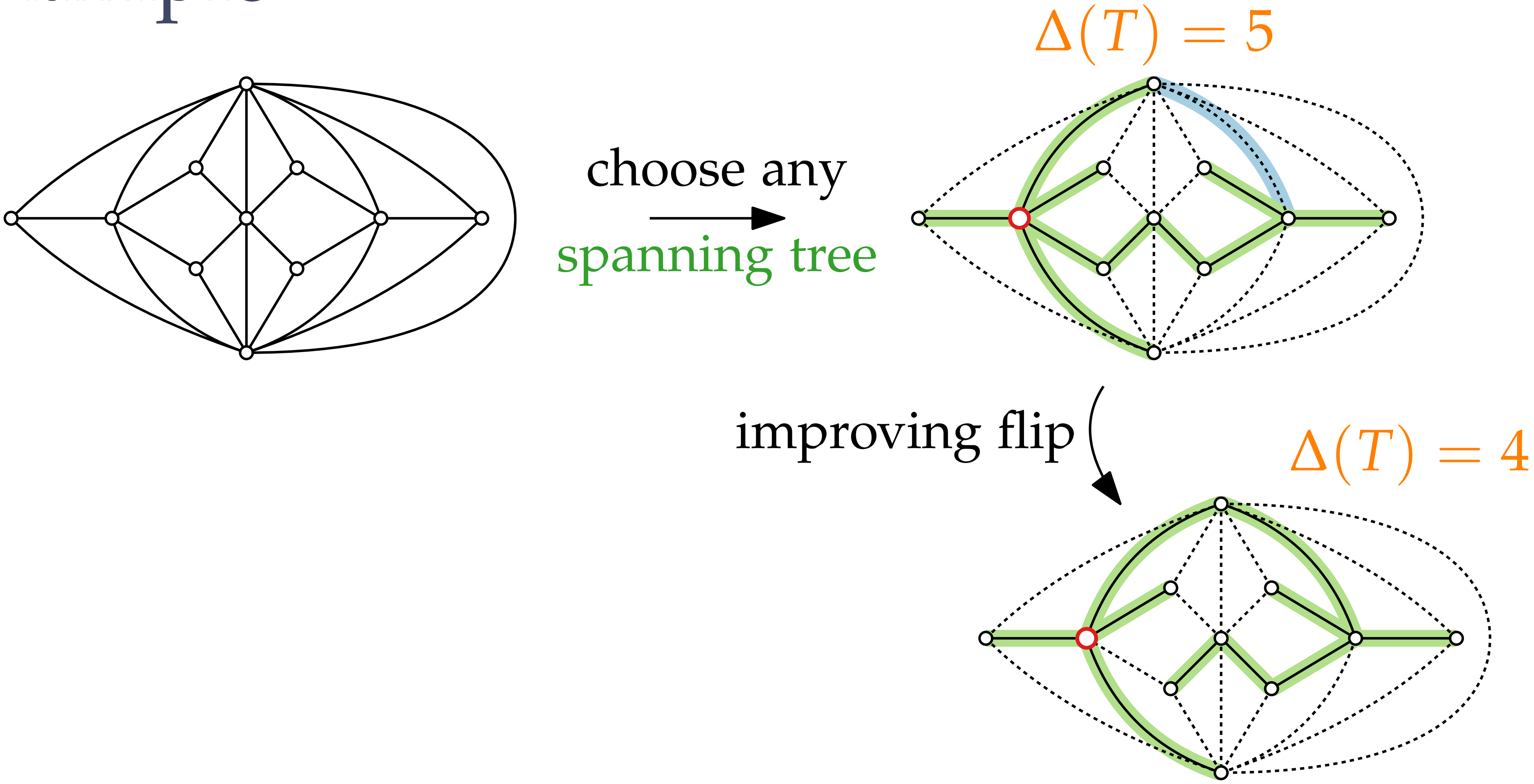
Example



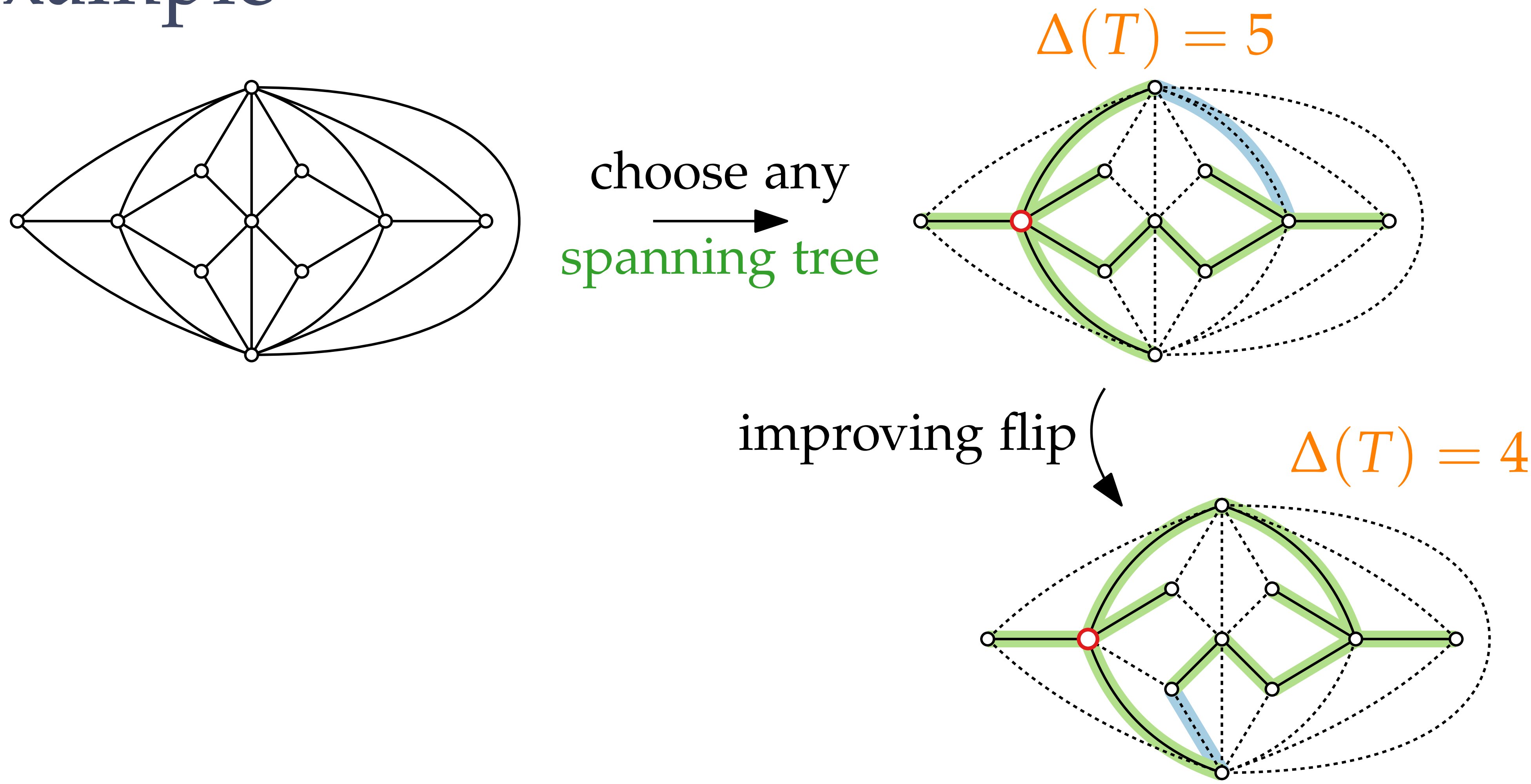
Example



Example



Example



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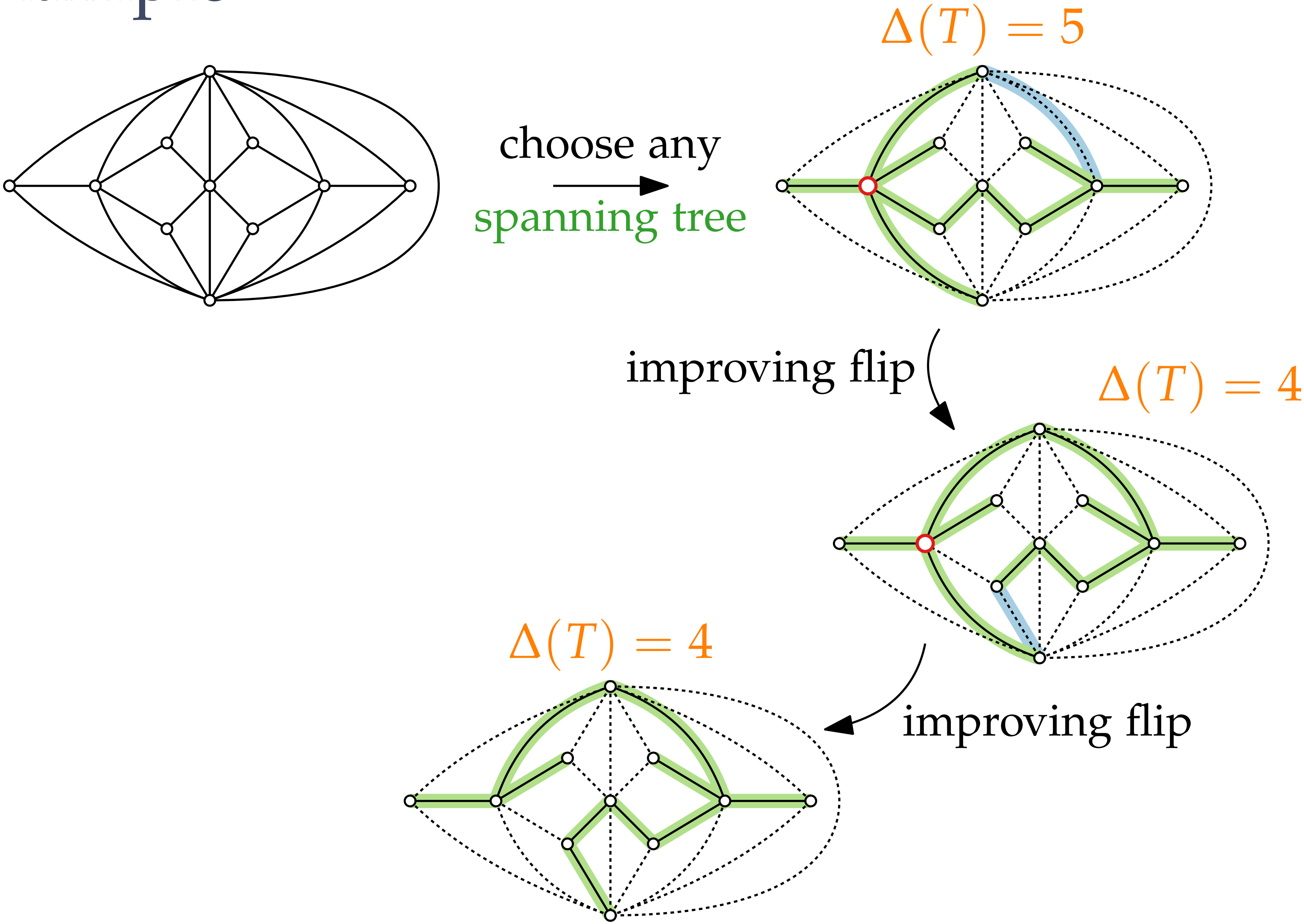
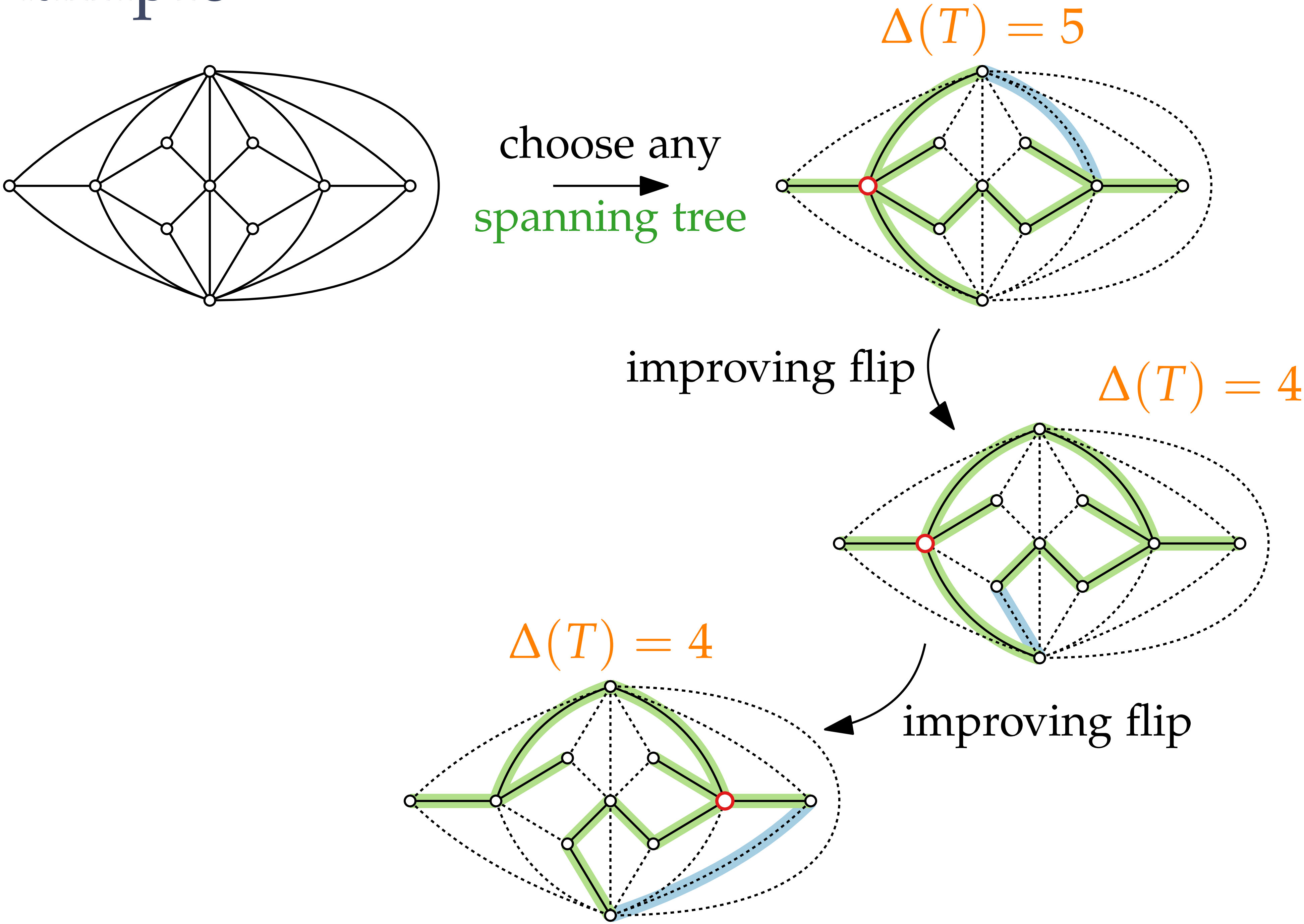


choose any
spanning tree
$\Delta(T) = 5$
improving flip
$\Delta(T) = 4$
improving flip
$\Delta(T) = 4$

Example



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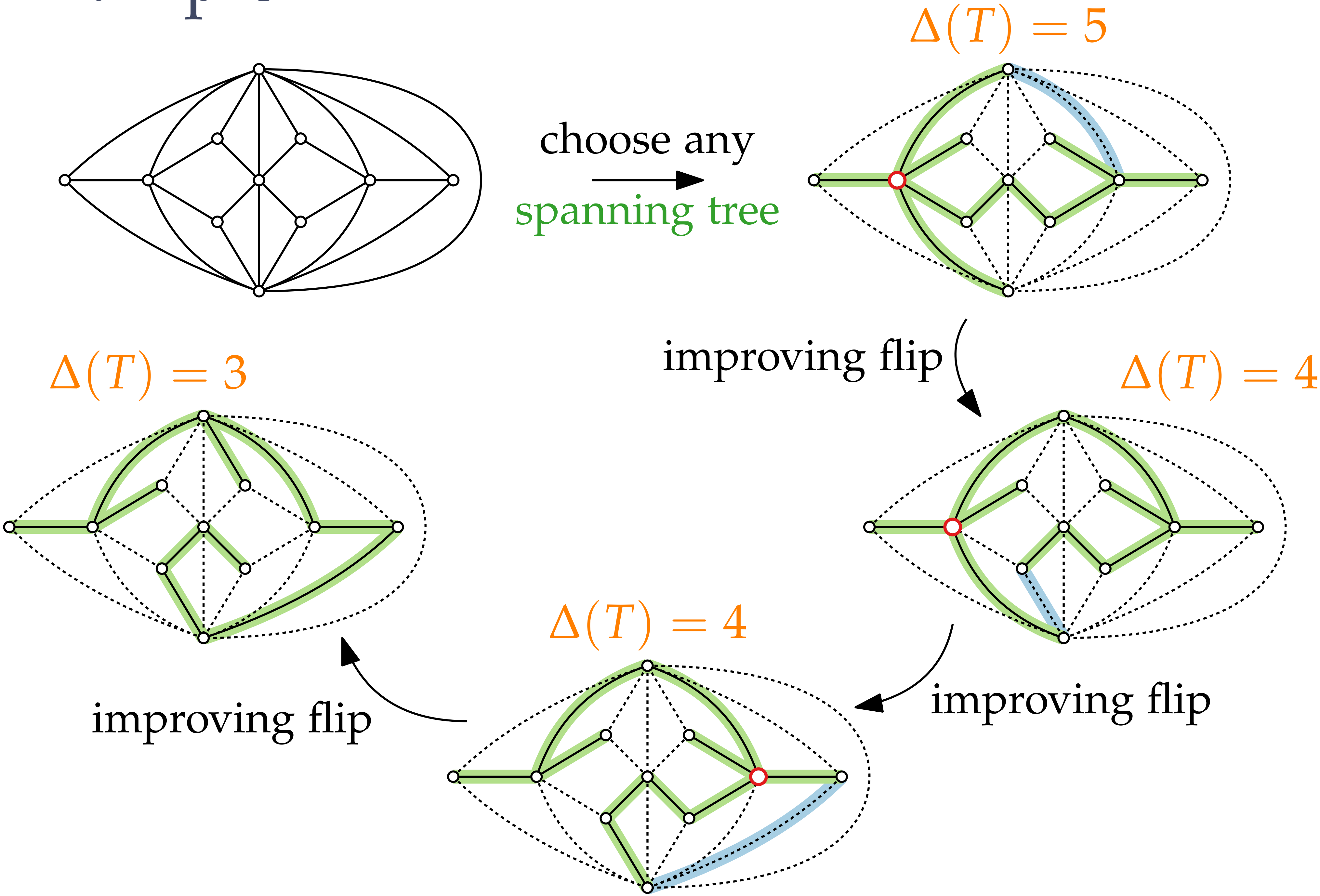
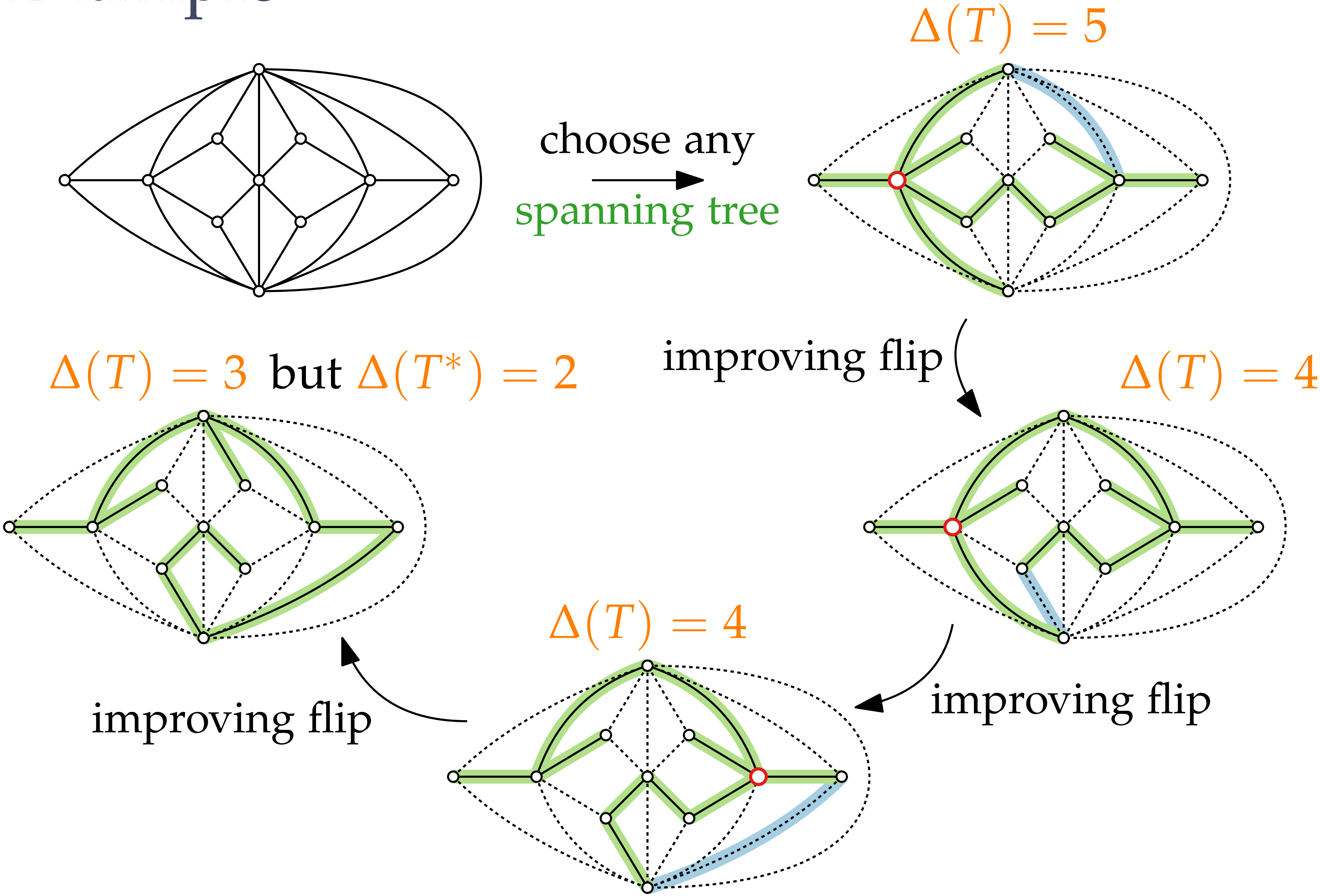
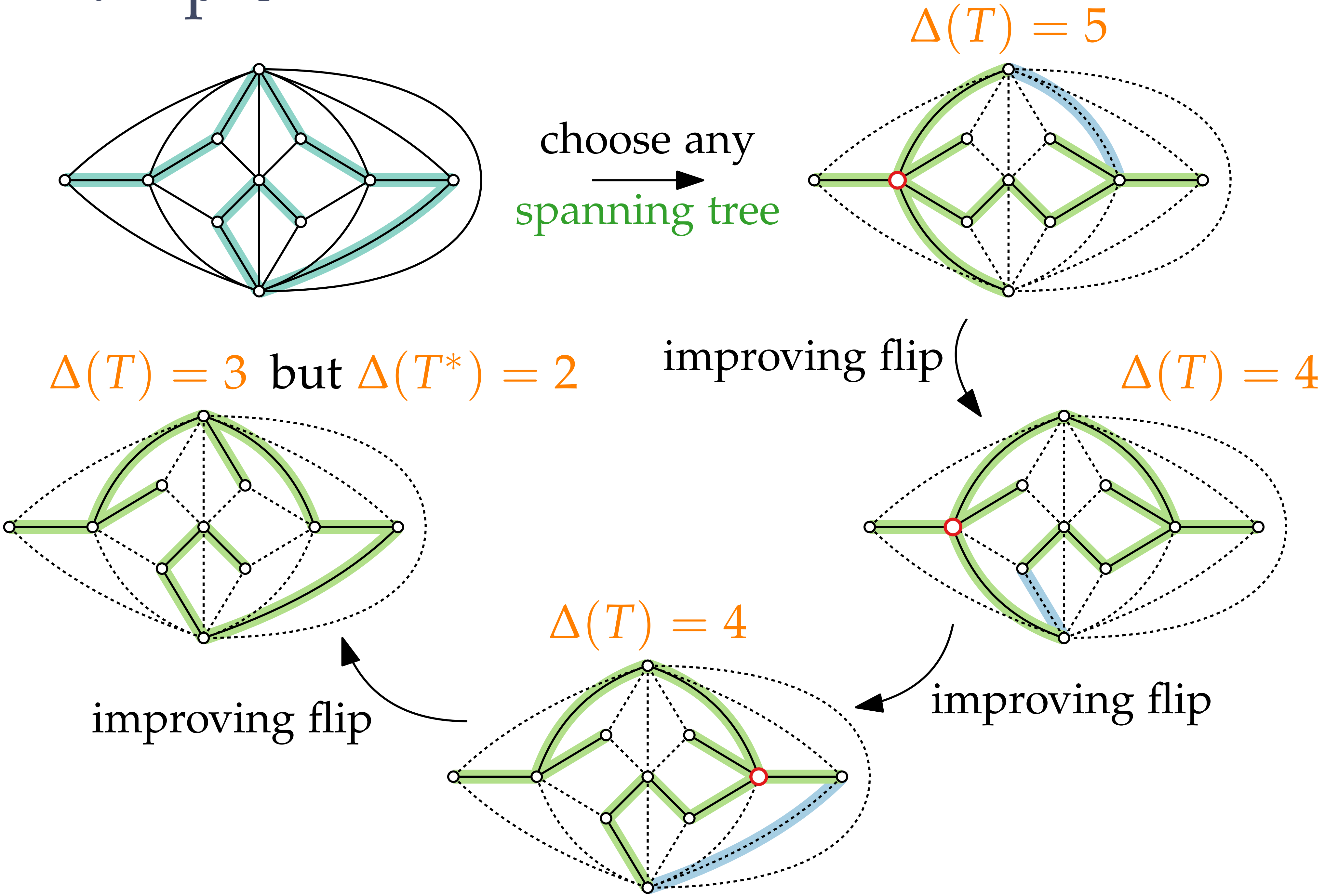


choose any
spanning tree
Δ(T) = 5
improving flip
Δ(T) = 4
improving flip
Δ(T) = 4
improving flip
Δ(T) = 3

Example


choose any
spanning tree
$\Delta(T) = 5$
improving flip
$\Delta(T) = 4$
improving flip
$\Delta(T) = 4$
improving flip
$\Delta(T) = 3$ but $\Delta(T^*) = 2$

Example



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Approximation Algorithms

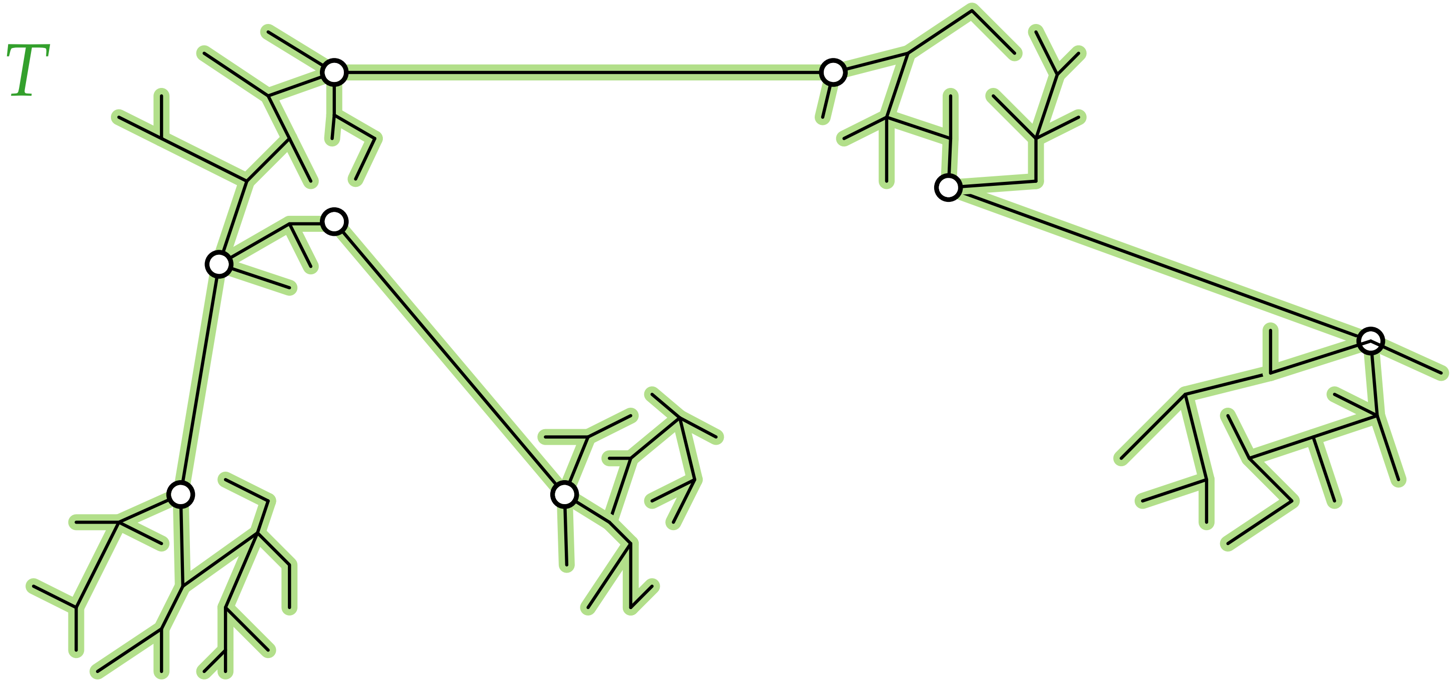
Lecture 10:
Minimum-Degree Spanning Tree via Local Search

Part III:
Lower Bound

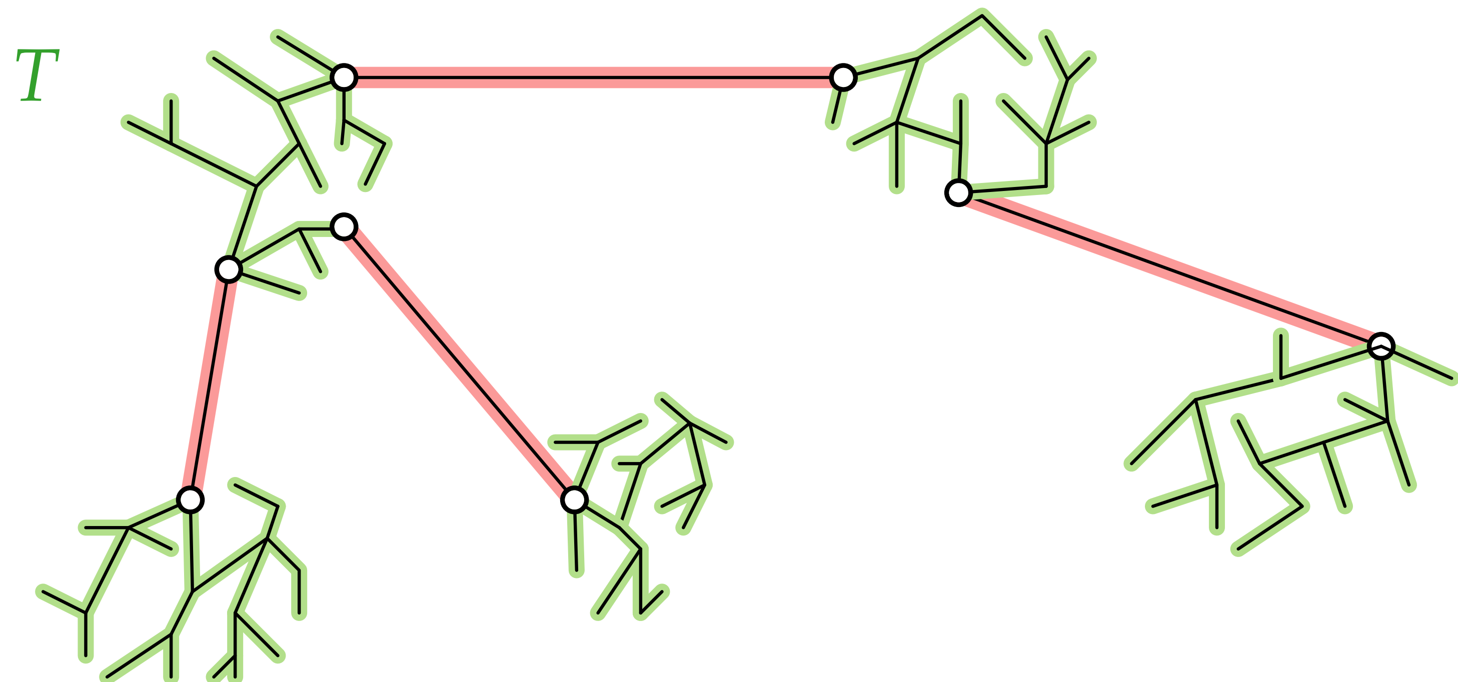
Joachim Spoerhase

Winter 2021/22

Decomposition

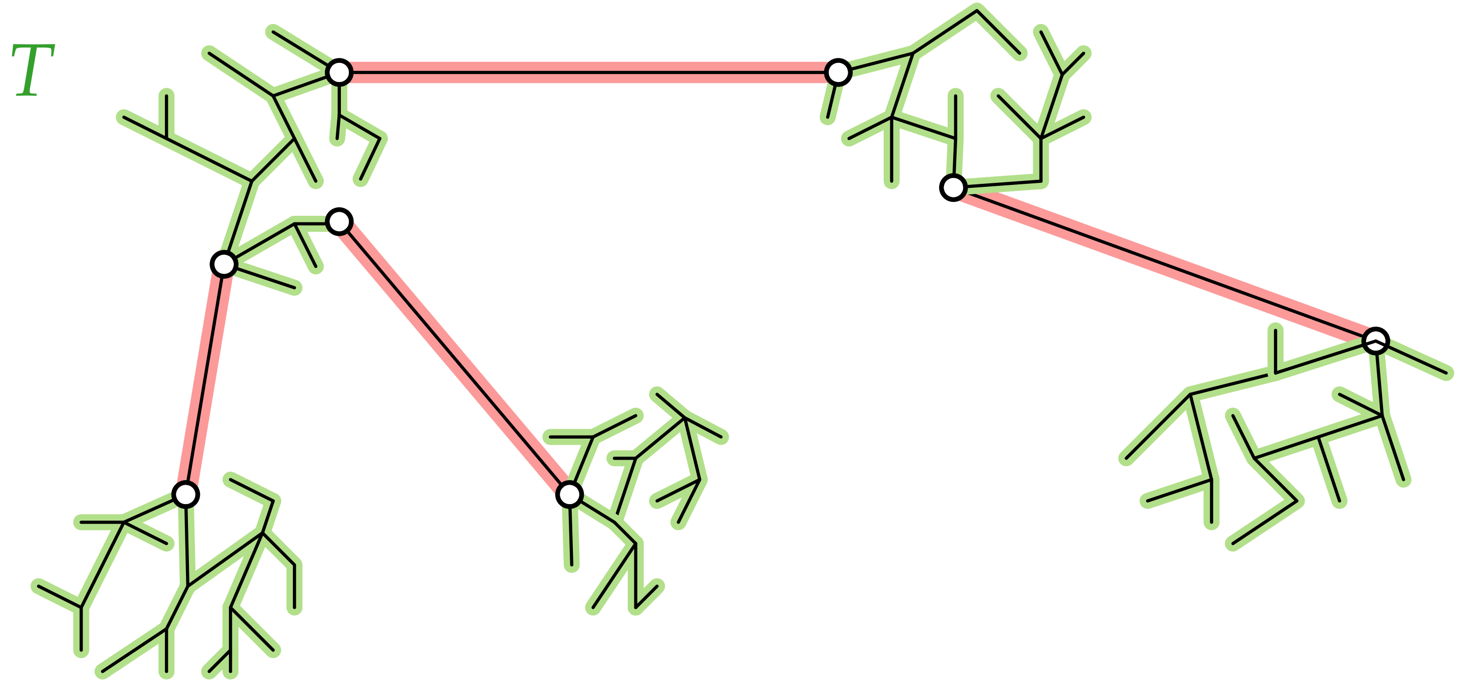


Decomposition



Decomposition

- Removing k edges decomposes T into $k+1$ components



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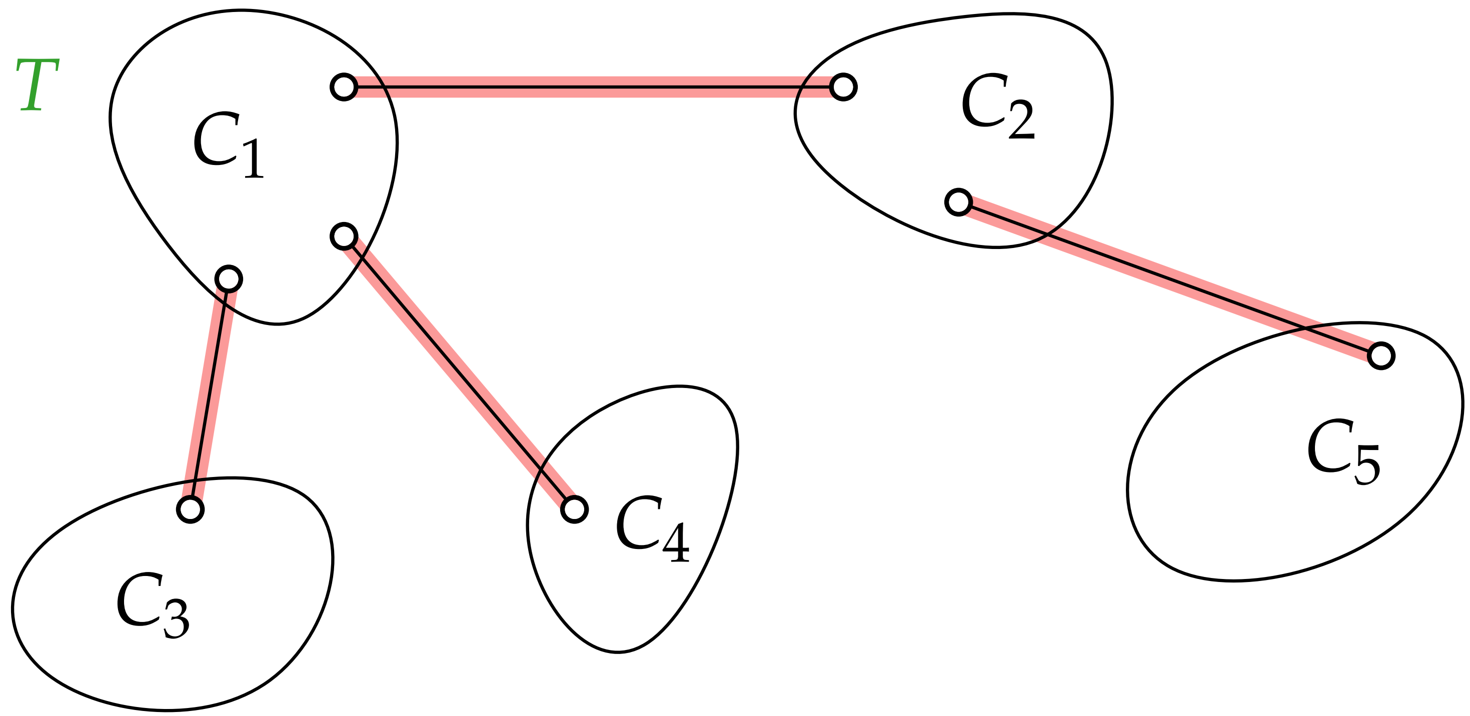
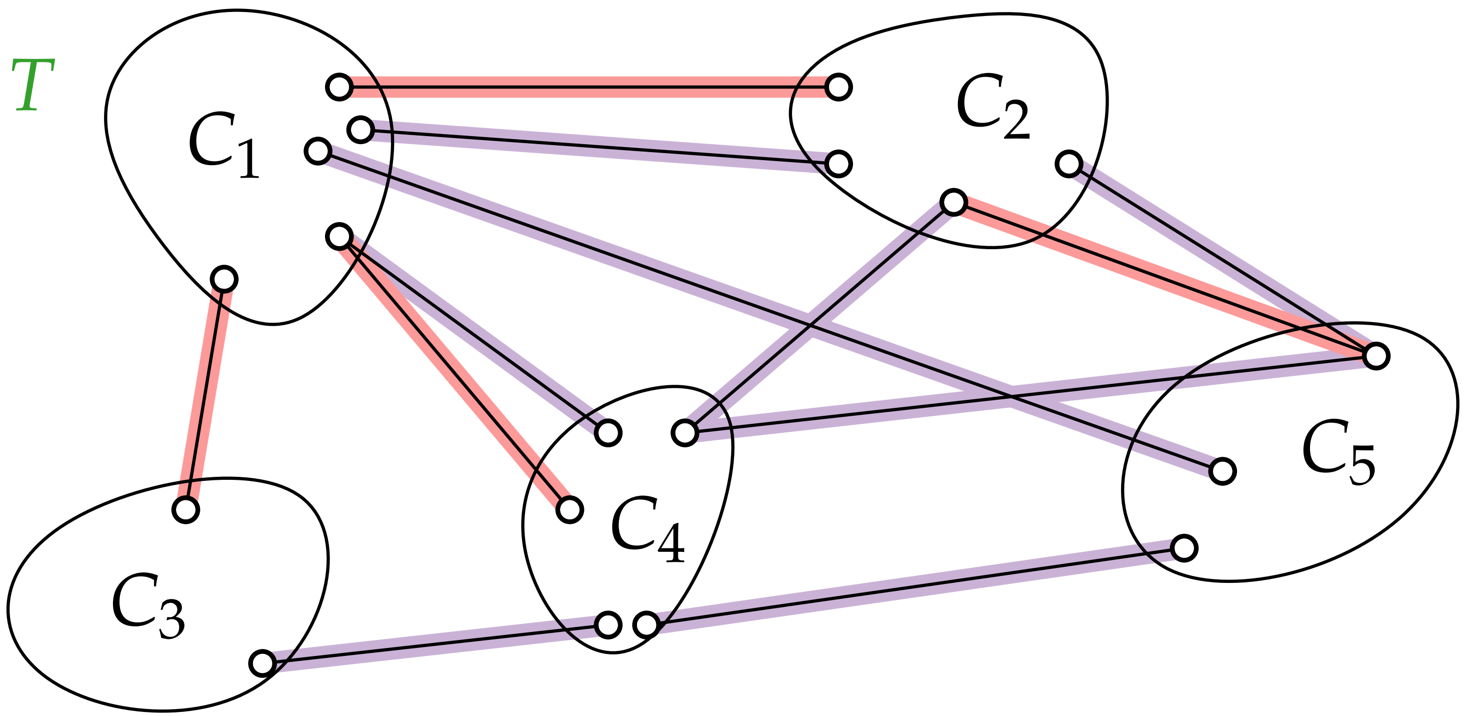


Decomposition

- Removing k edges decomposes T into $k+1$ components
- $E' :=$ {edges is G btw. different components $C_i \neq C_j$}.



Decomposition

- Removing k edges decomposes T into $k+1$ components
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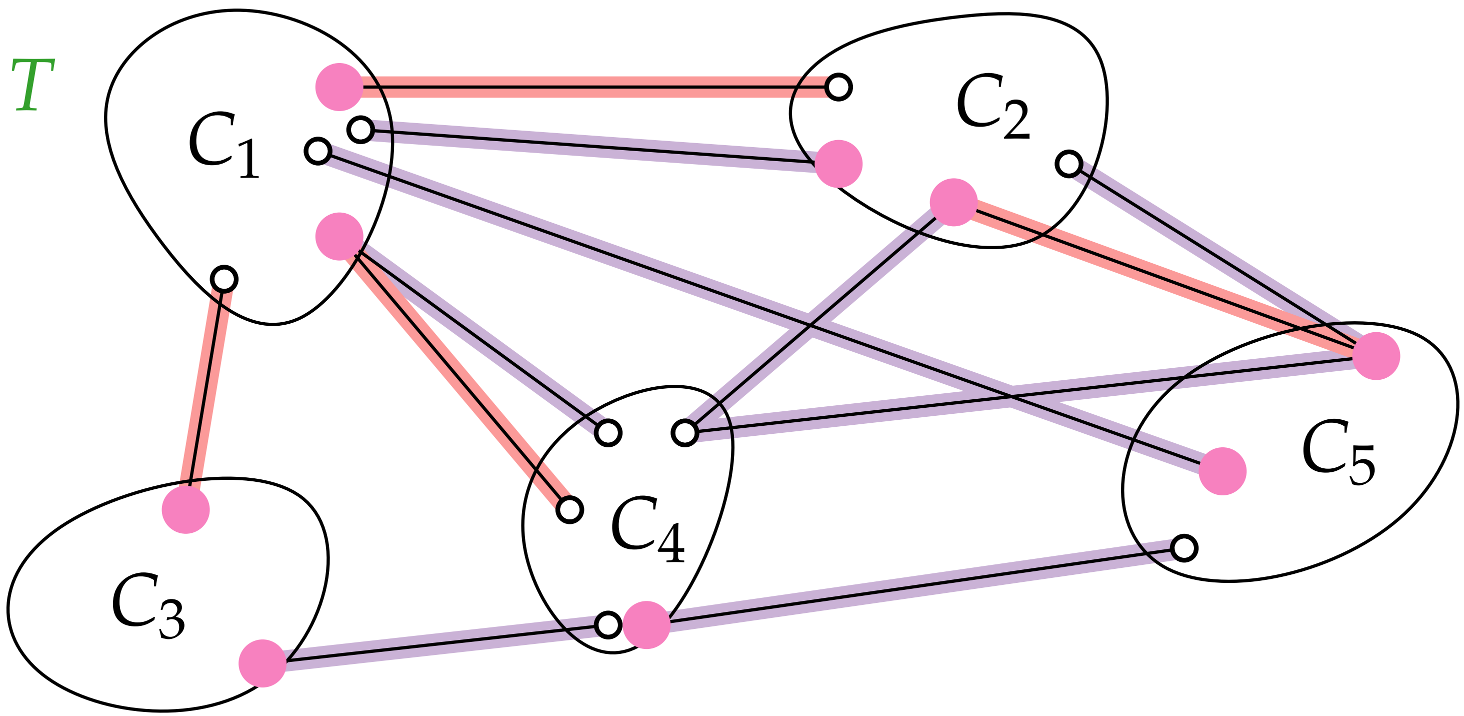
- $S :=$ vertex cover of E'.



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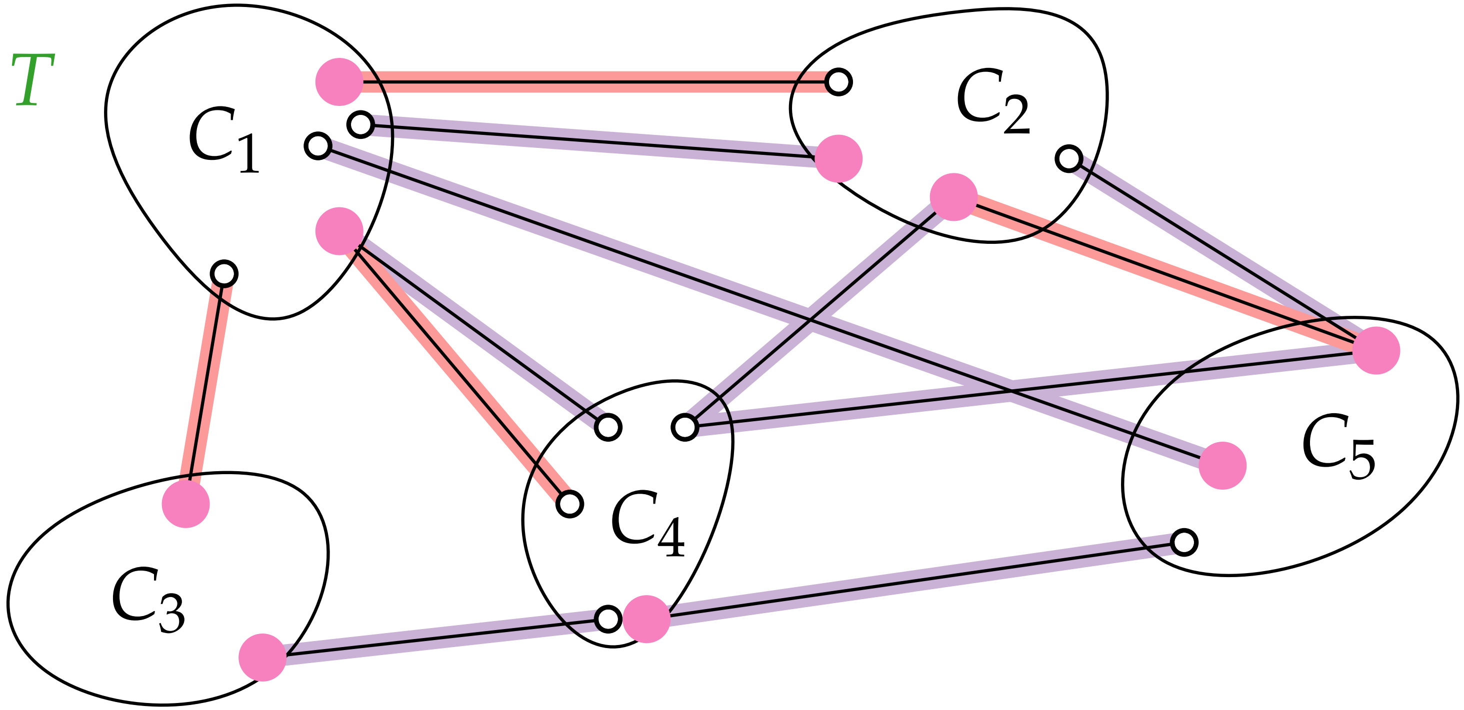

- $|E(T^*) \cap E'| \geq k$ for opt. spanning tree T^*

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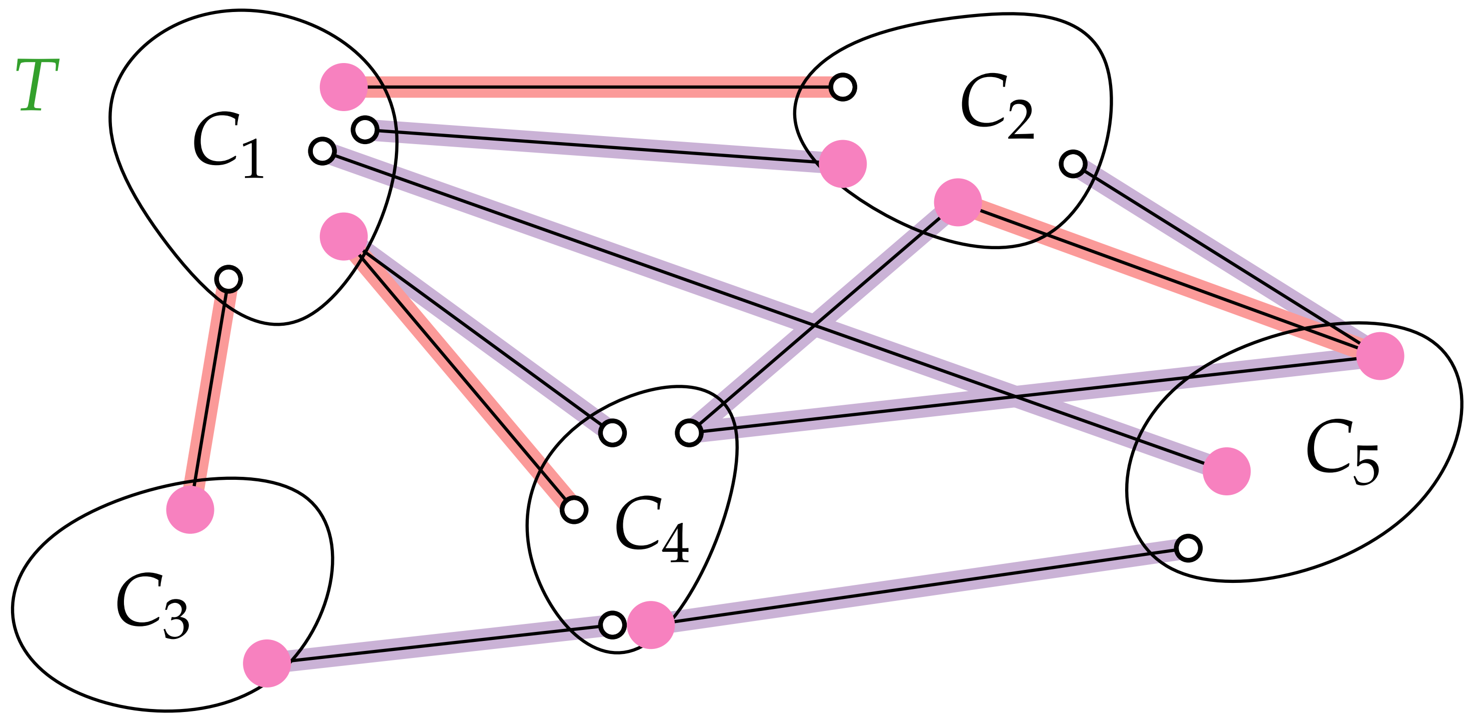

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- $\sum_{v \in S} \deg_{T^*}(v) \geq k$

Decomposition $\Rightarrow$ Lower Bound for OPT

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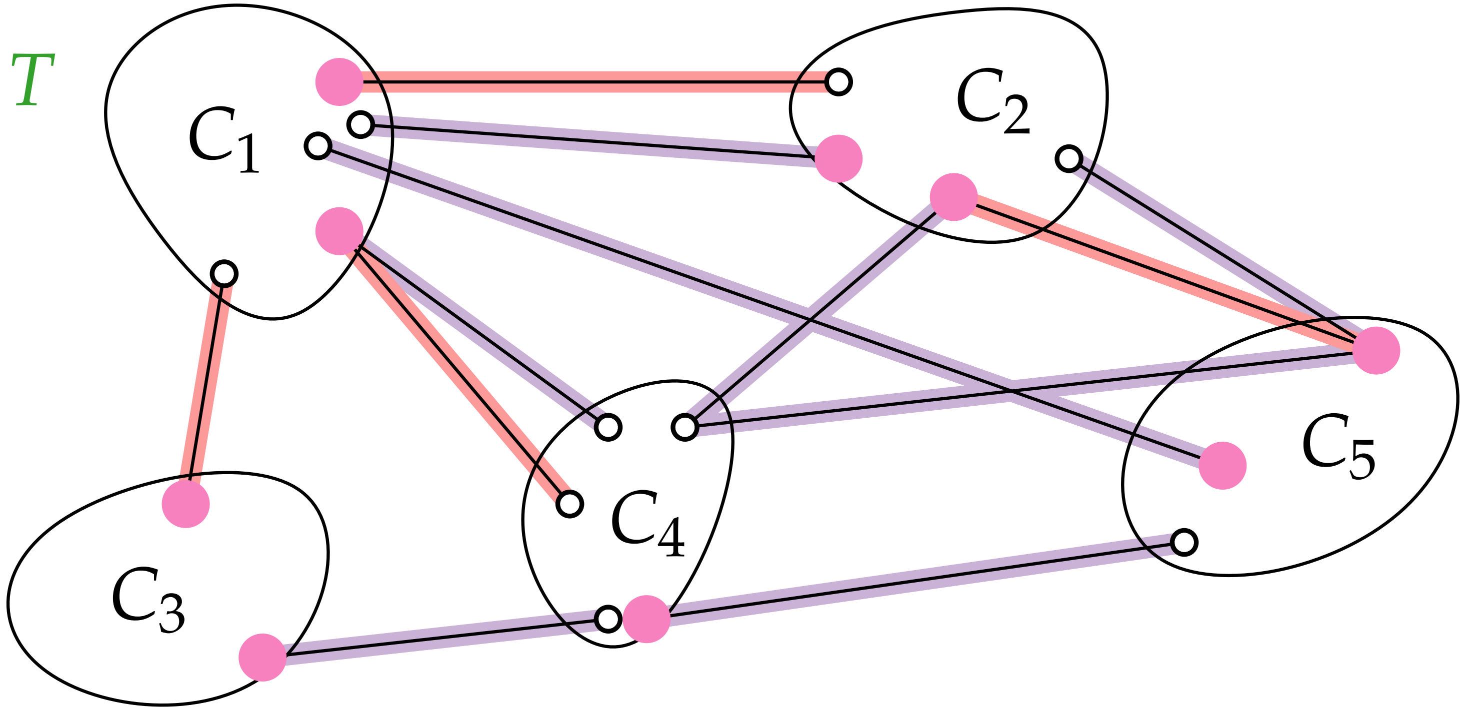

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Lemma 1.
$\Rightarrow$ OPT $\geq$

Decomposition $\Rightarrow$ Lower Bound for OPT

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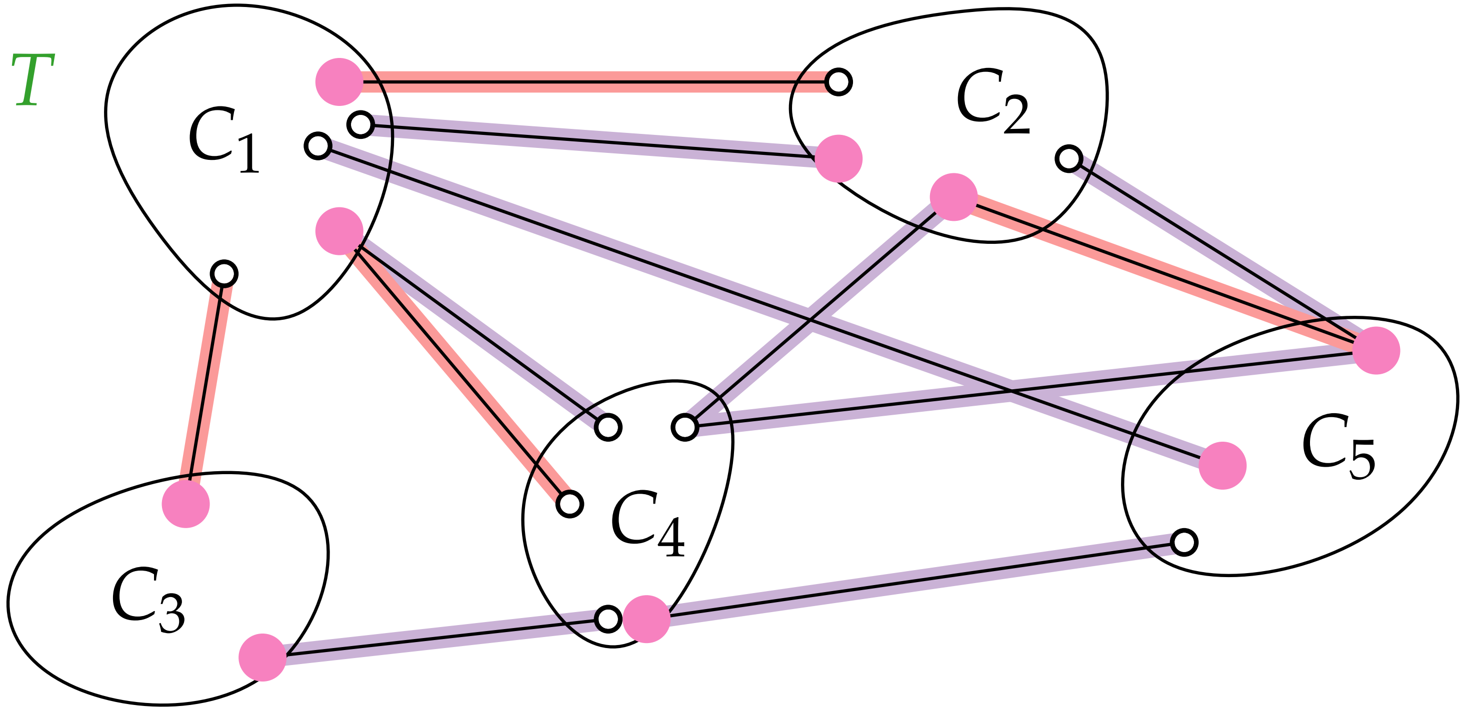

- $|E(T^*) \cap E'| \geq k$ for opt. spanning tree T^*
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Lemma 1.
$\Rightarrow \mathrm{OPT} \geq k / |S|$

Approximation Algorithms

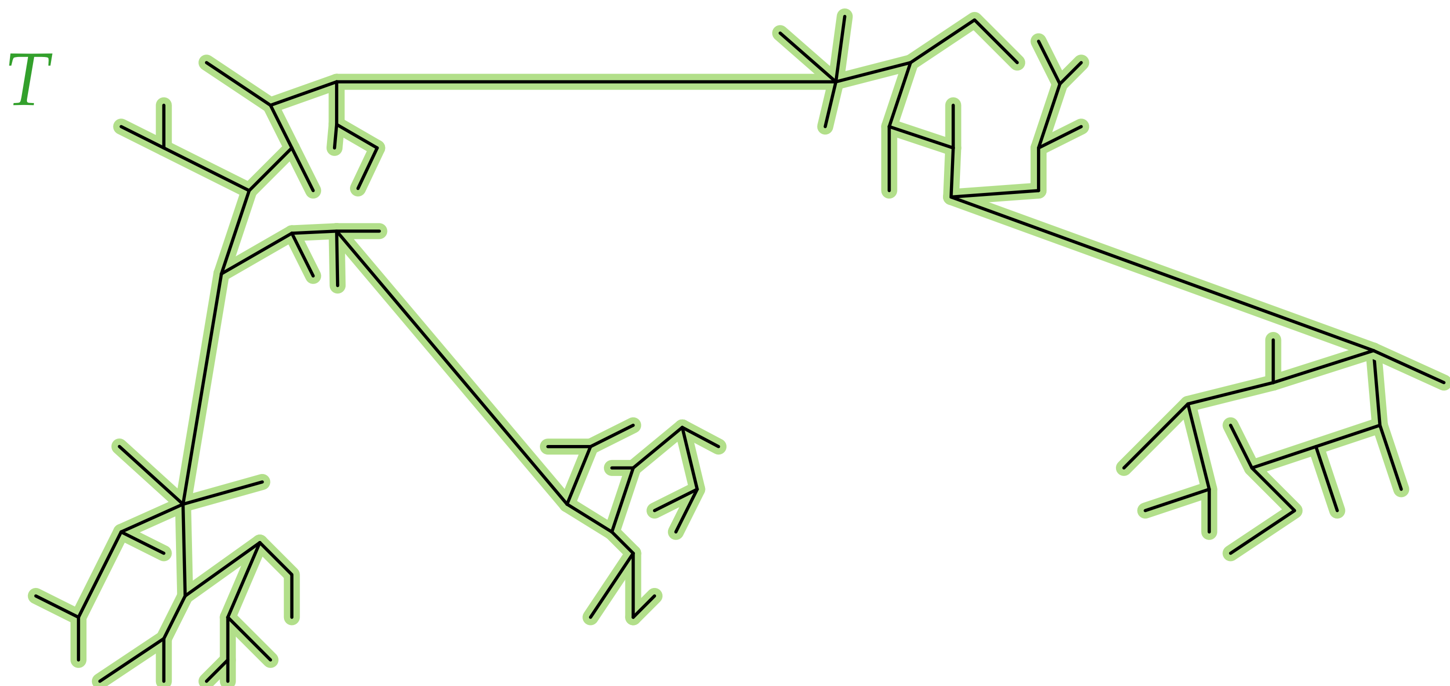
Lecture 10:
Minimum-Degree Spanning Tree
via Local Search

Part IV:
More Lemmas

Joachim Spoerhase

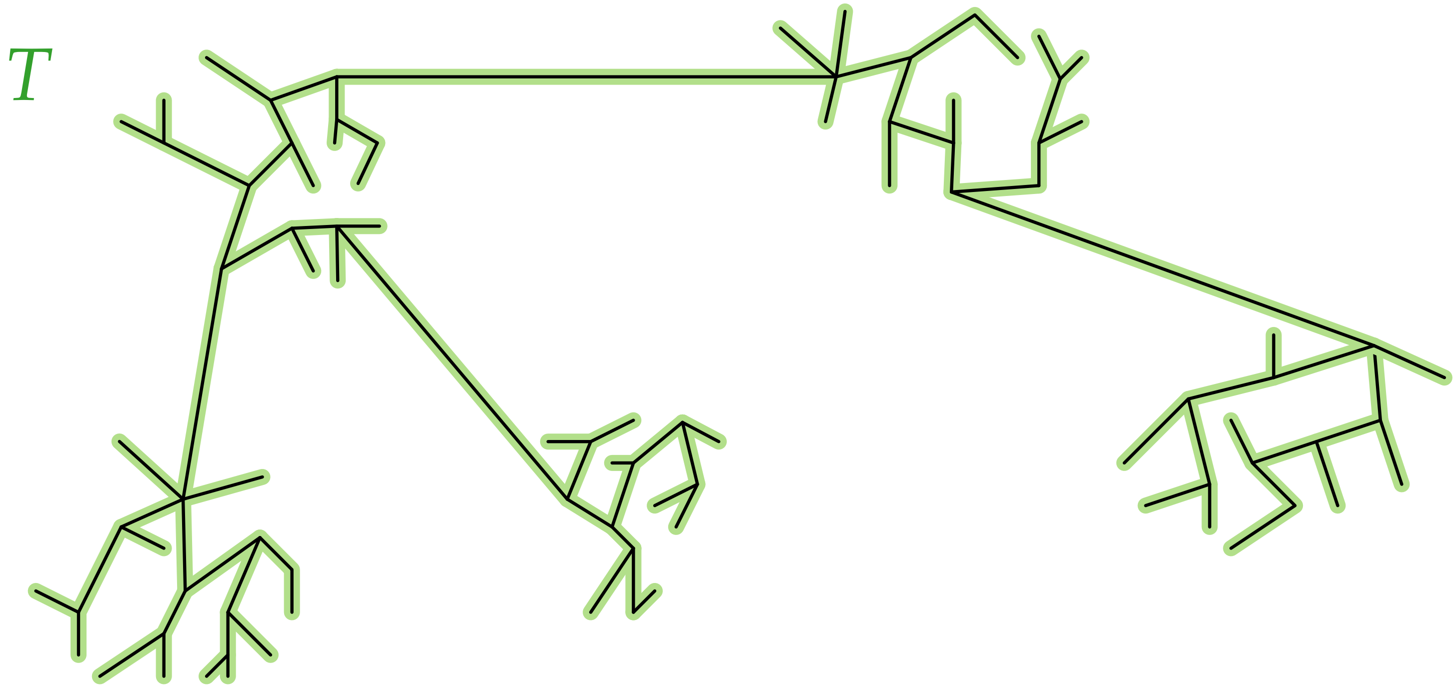
Winter 2021/22

More Lemmas



More Lemmas

Let S_i be the vertices v in T with $\deg_T(v) \geq i$.



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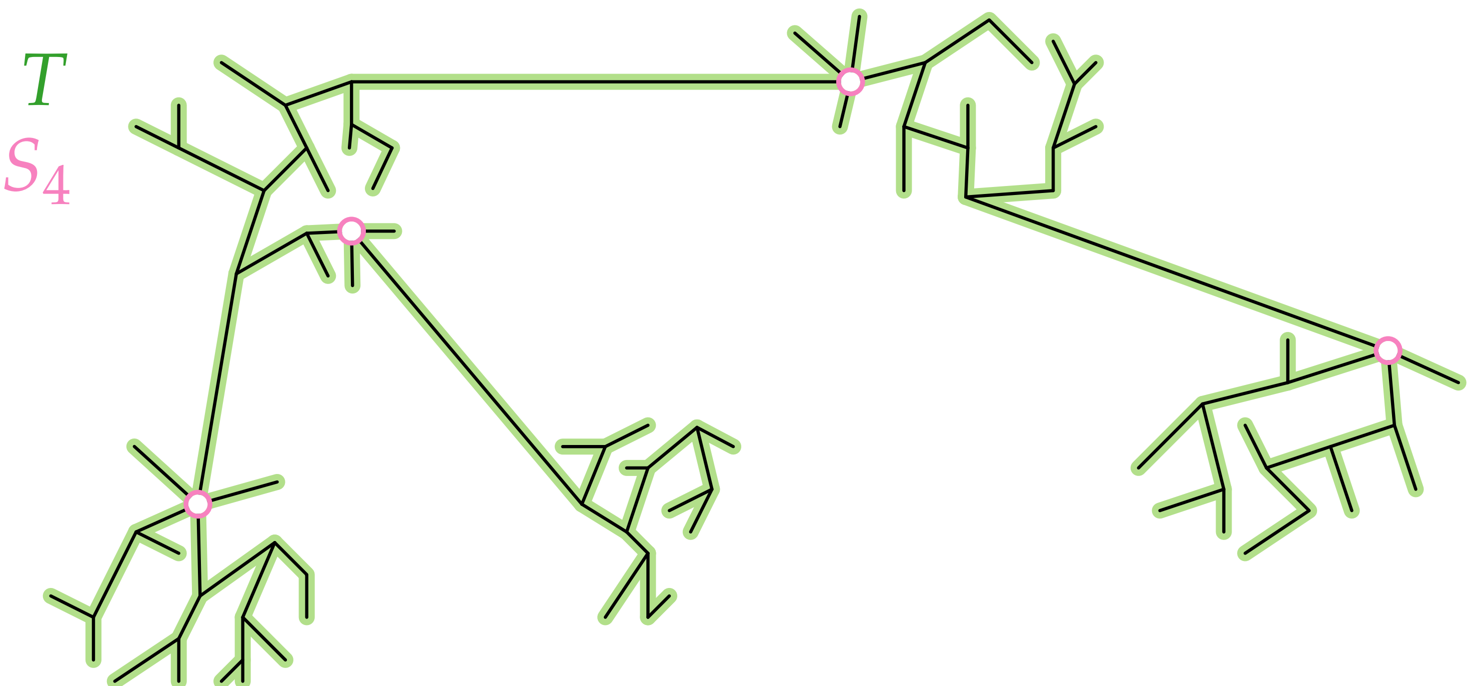

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Let S_i be the vertices v in T with $\deg_T(v) \geq i$.
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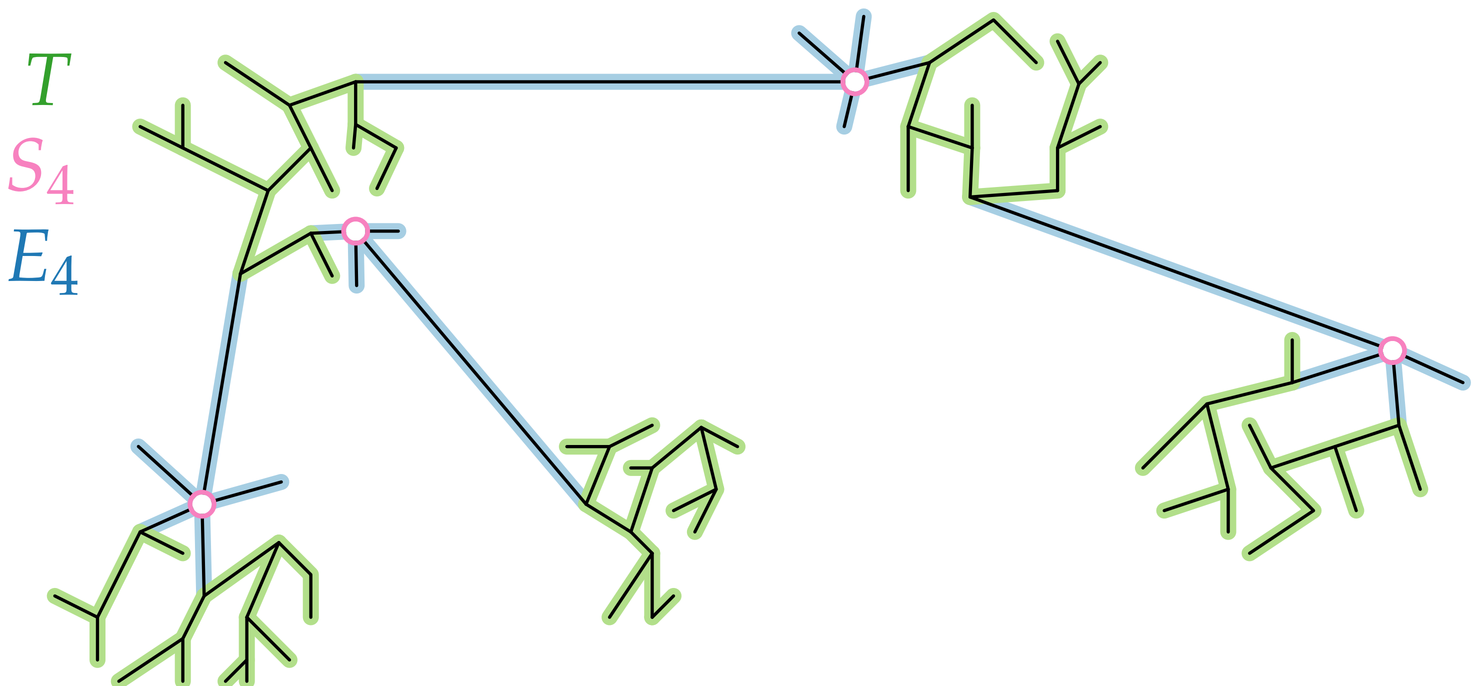

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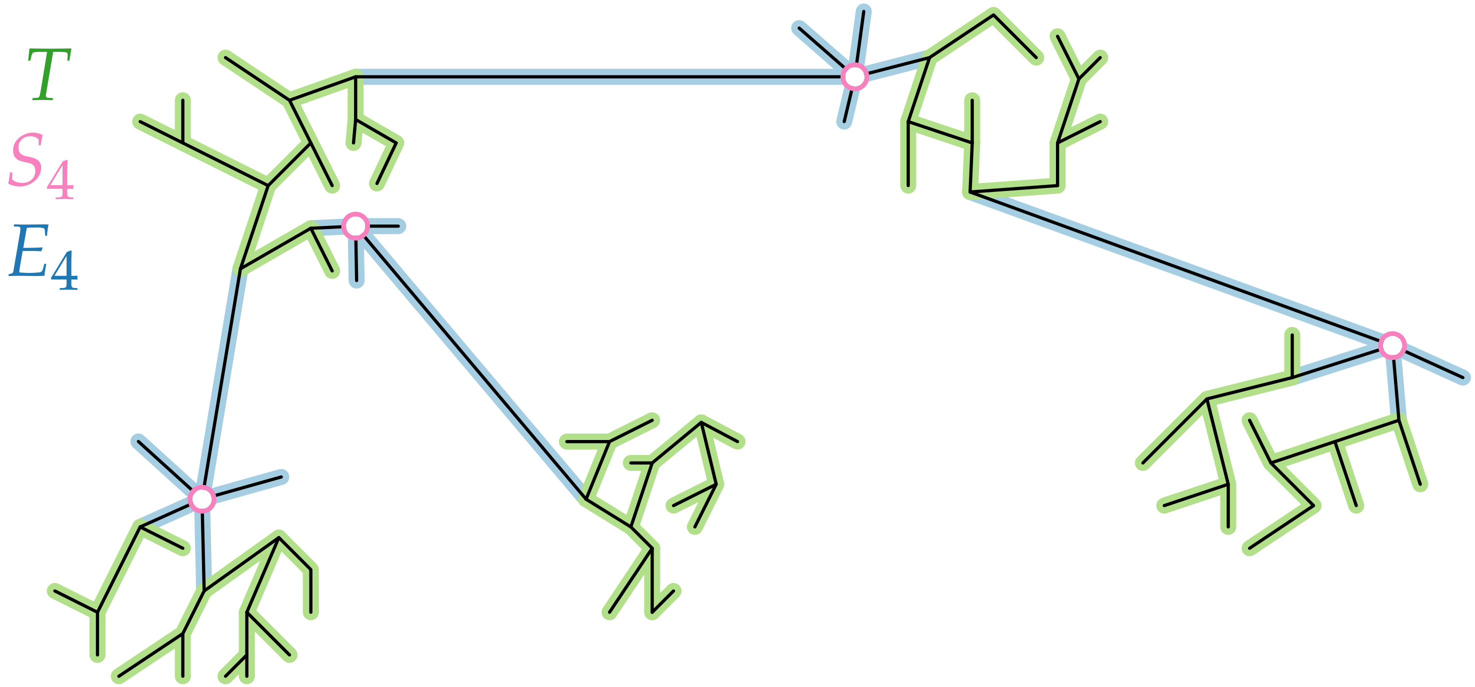
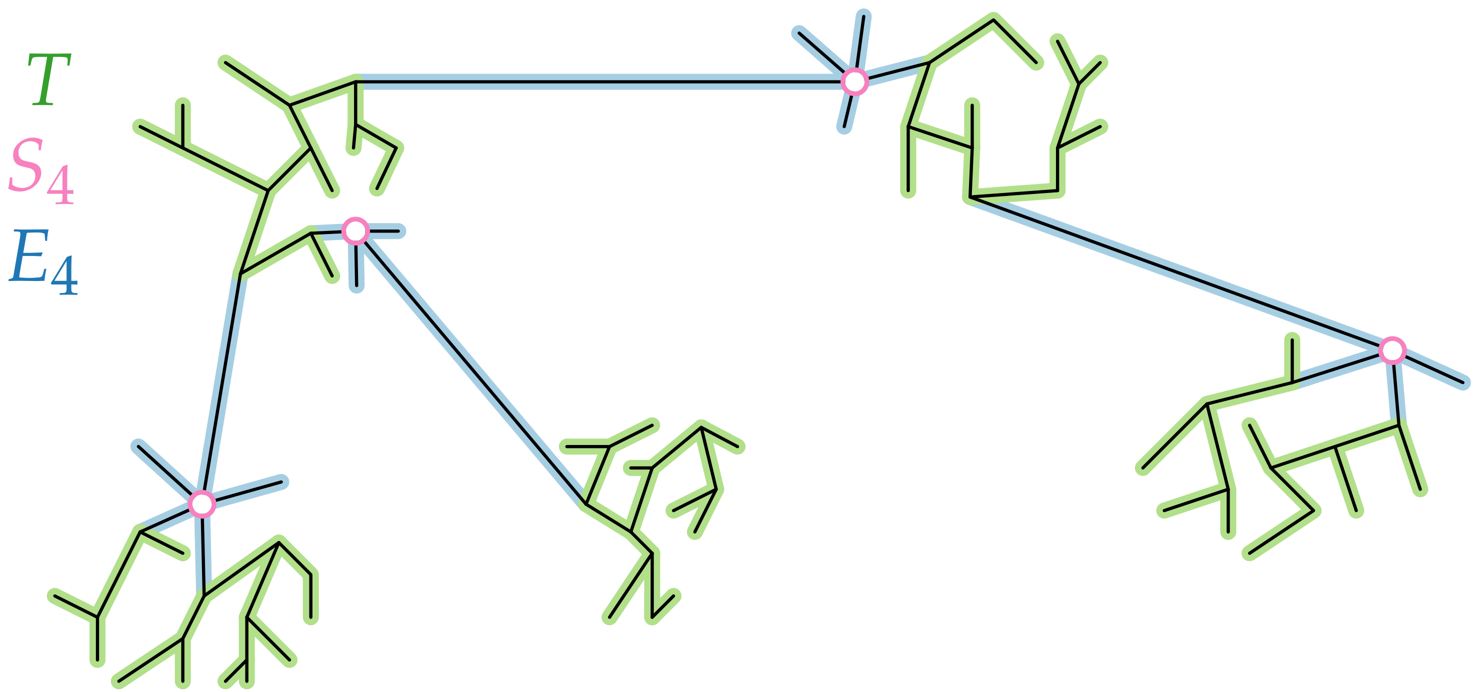


More Lemmas

$\Rightarrow S_1 \supseteq S_2 \supseteq \dots$

Let S_i be the vertices v in T with $\deg_T(v) \geq i$.
Let E_i be the edges in T incident to S_i.

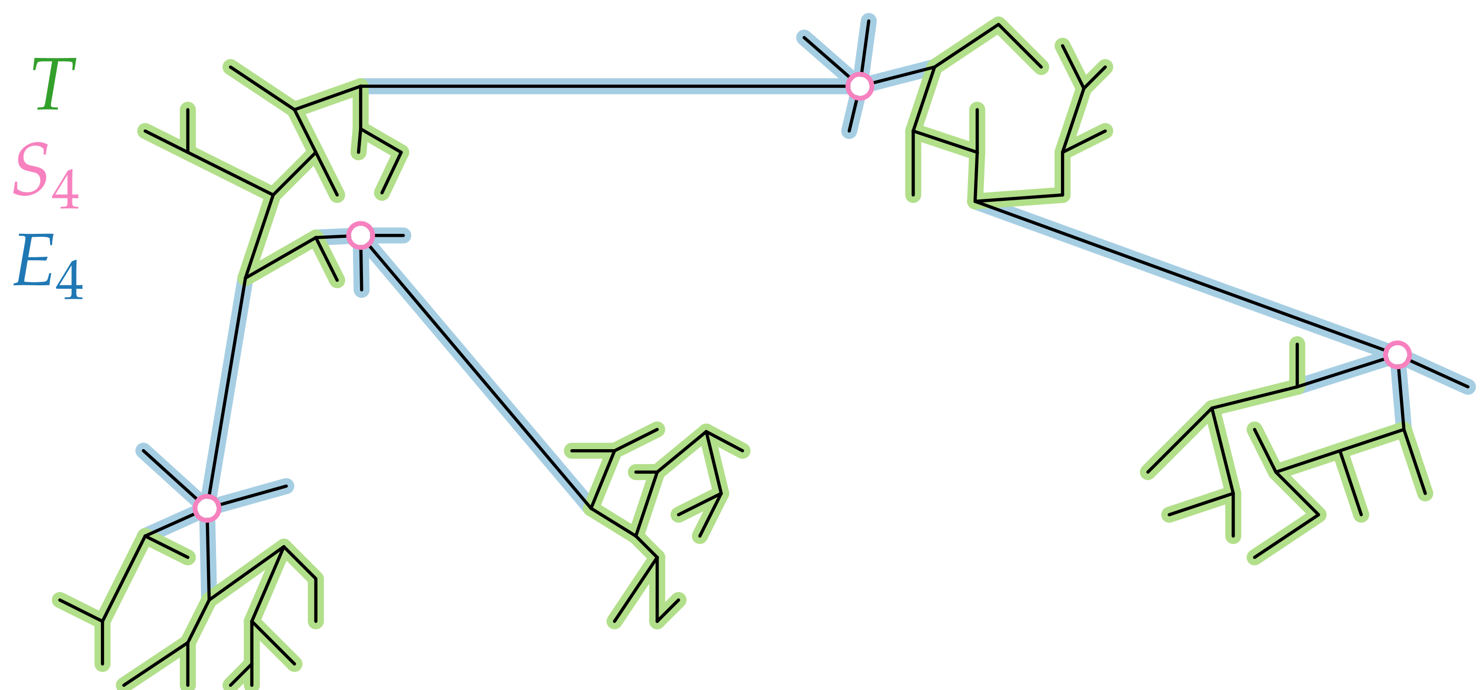


More Lemmas

Let S_i be the vertices v in T with $\deg_T(v) \geq i$. $\Rightarrow S_1 \supseteq S_2 \supseteq \dots$

$\Rightarrow S_1 = V(G)$

Let E_i be the edges in T incident to S_i.



More Lemmas

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$\Rightarrow S_1 \supseteq S_2 \supseteq \dots$
$\Rightarrow S_1 = V(G)$
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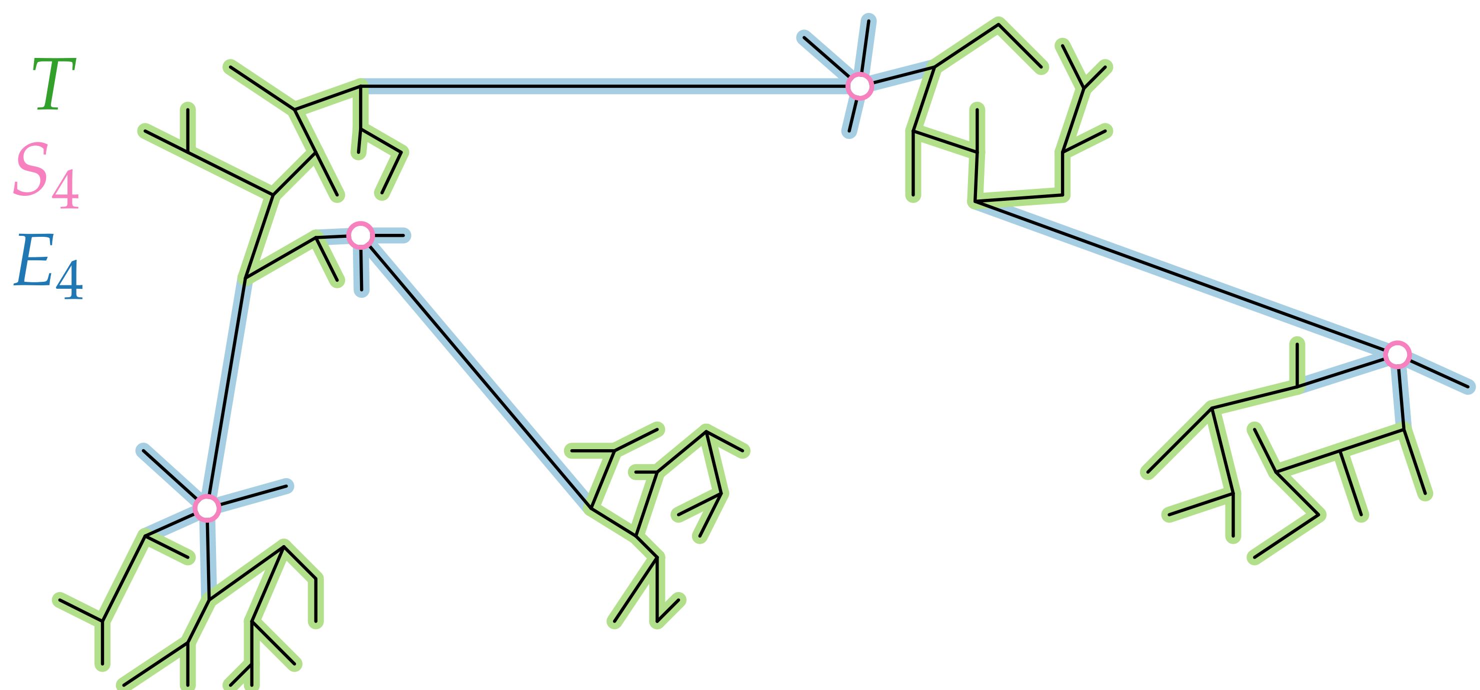

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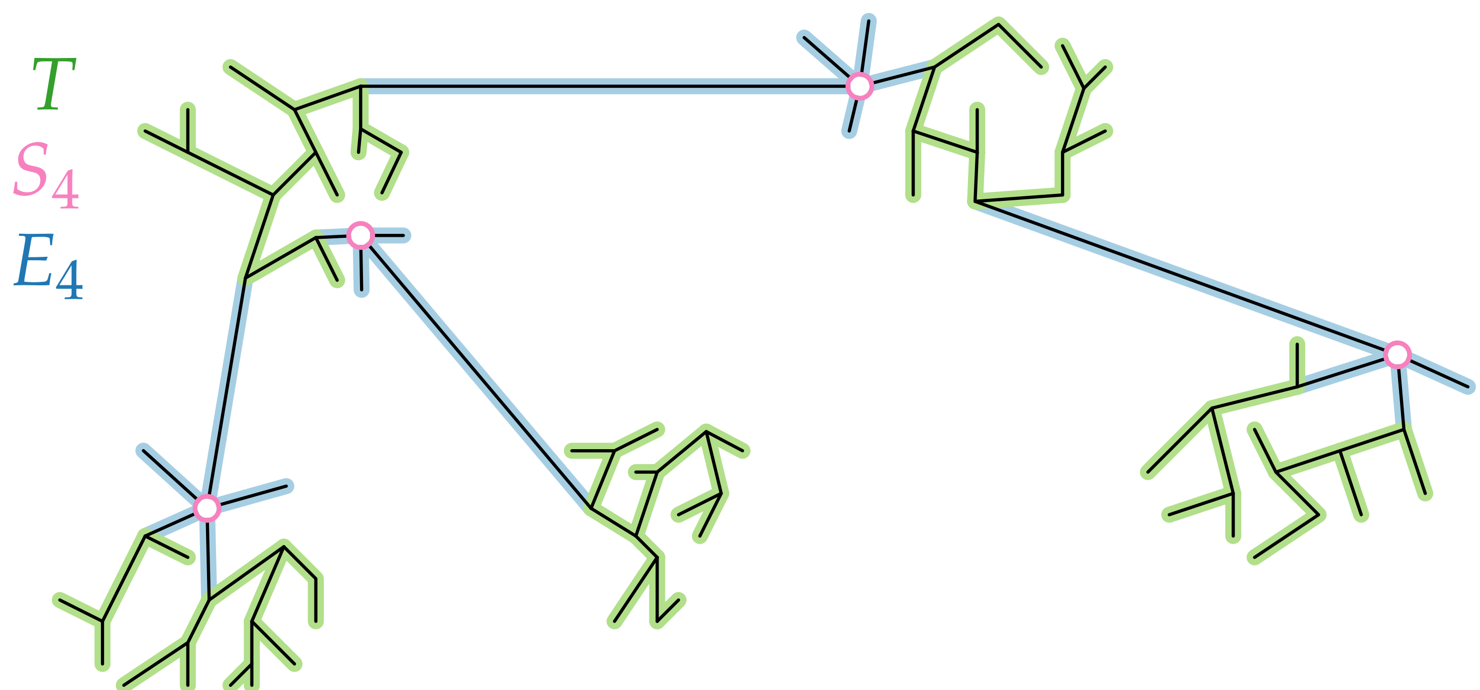
Lemma 2. There is some $i \geq \Delta(T) - \ell + 1$ with $|S_{i-1}| \leq 2|S_i|$.



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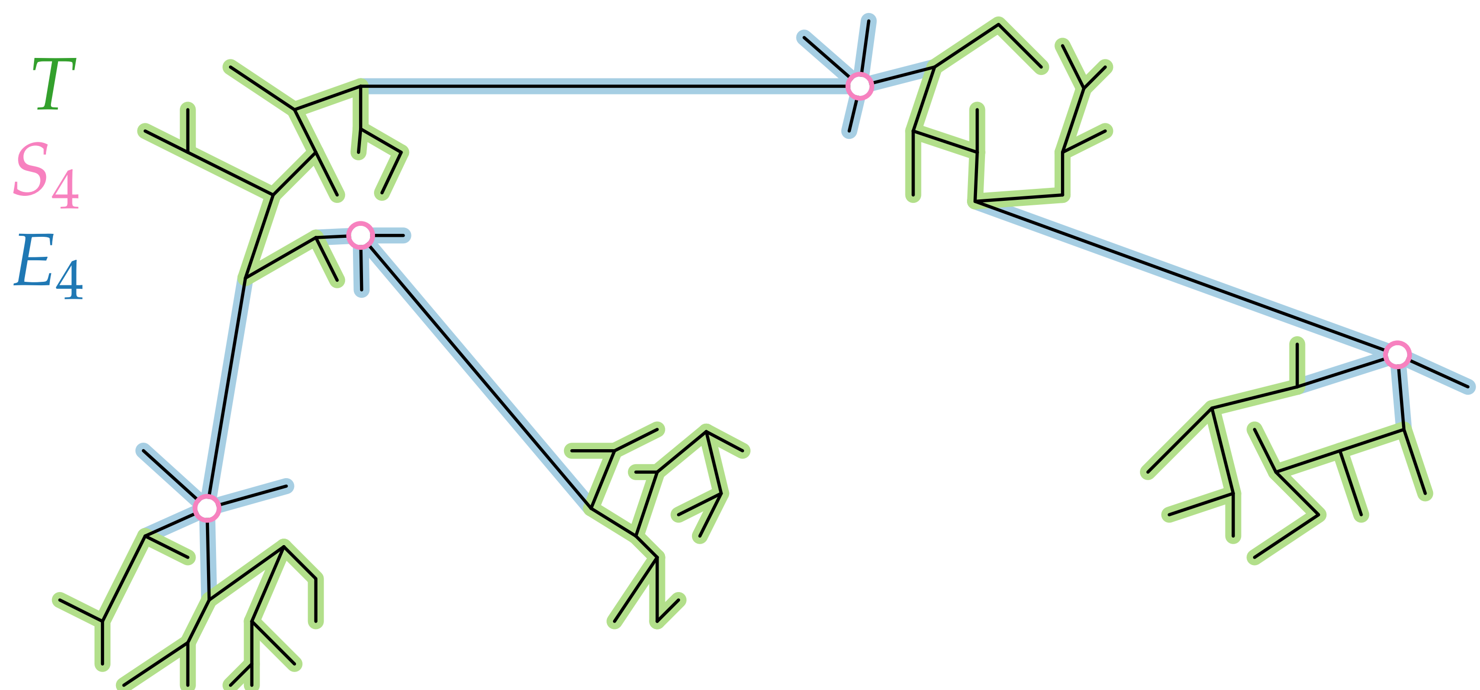
Proof. $|S_{\Delta(T)-\ell}| > 2^\ell |S_{\Delta(T)}|$

Otherwise



More Lemmas

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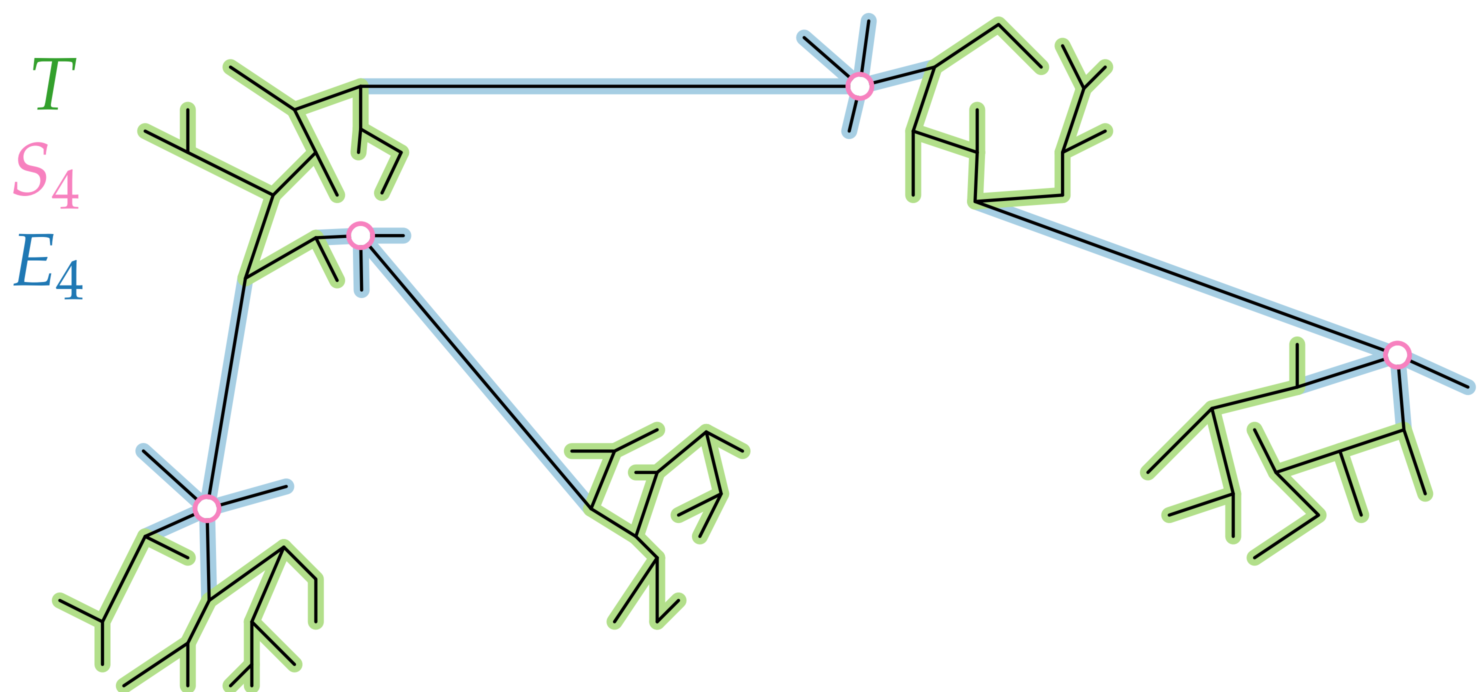
Proof. $|S_{\Delta(T)-\ell}| > 2^{\ell}|S_{\Delta(T)}| = 2^{\lceil \log_2 n \rceil}|S_{\Delta(T)}|$

$\ell = \lceil \log_2 n \rceil$

Otherwise



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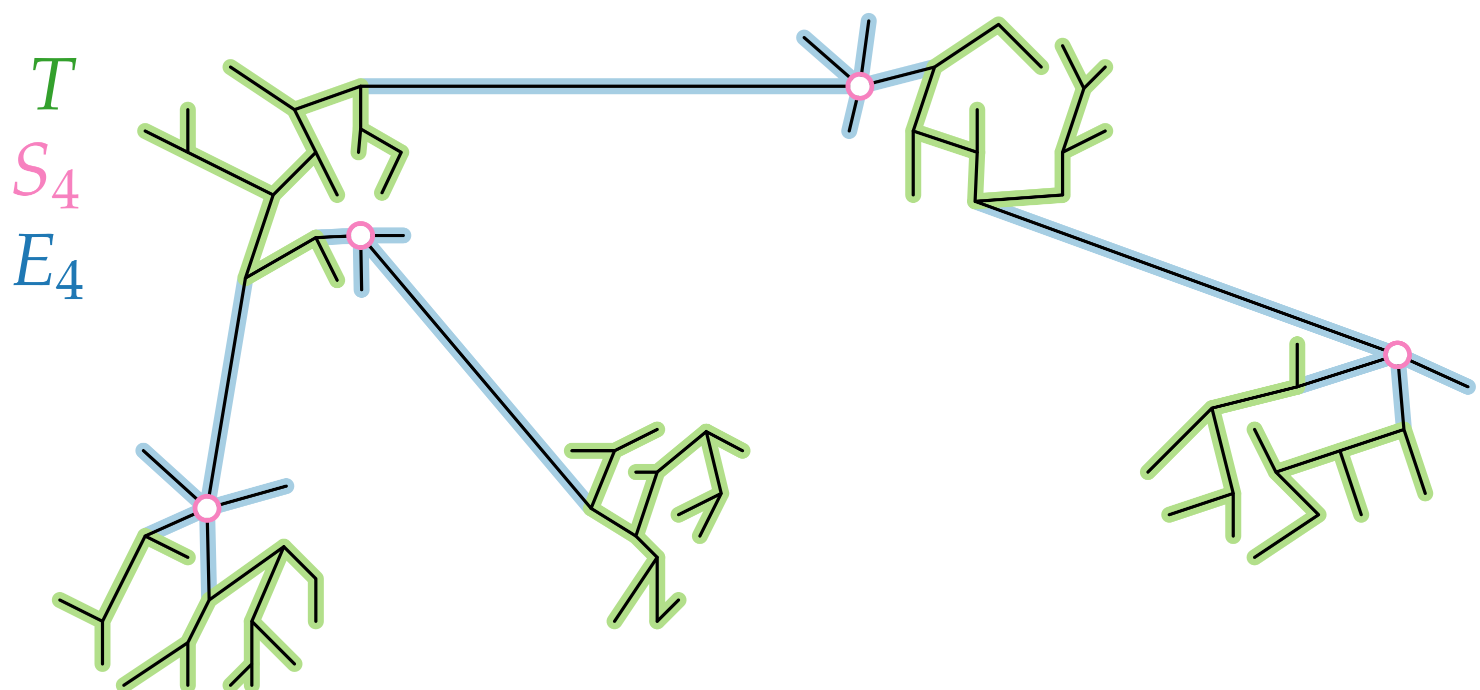

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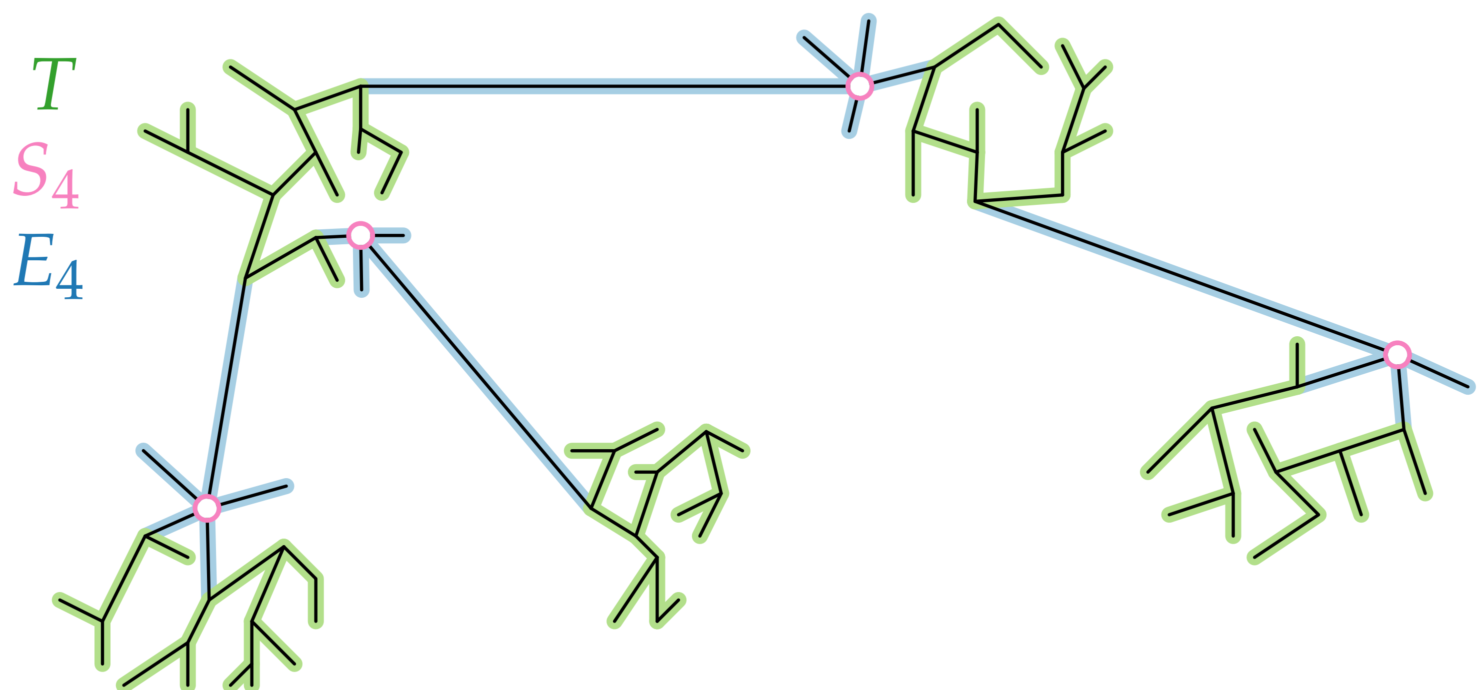

More Lemmas

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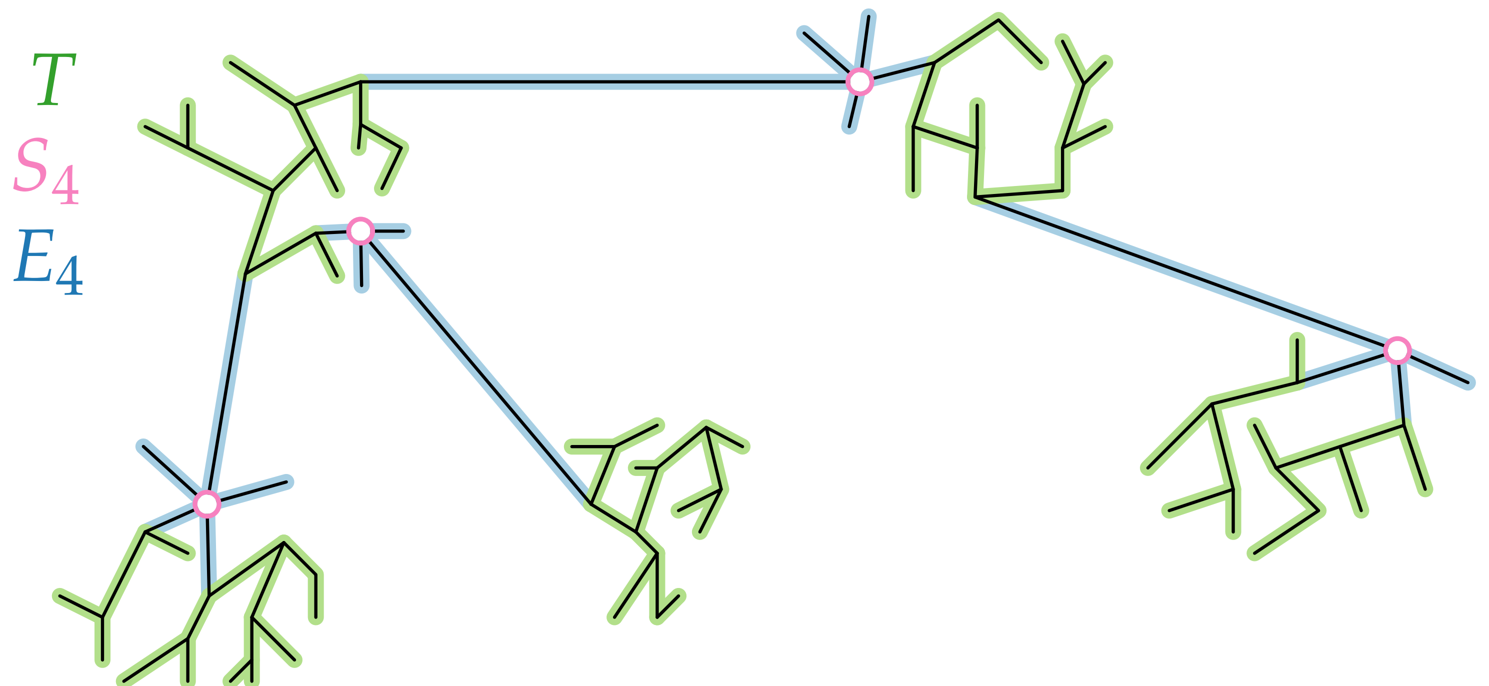

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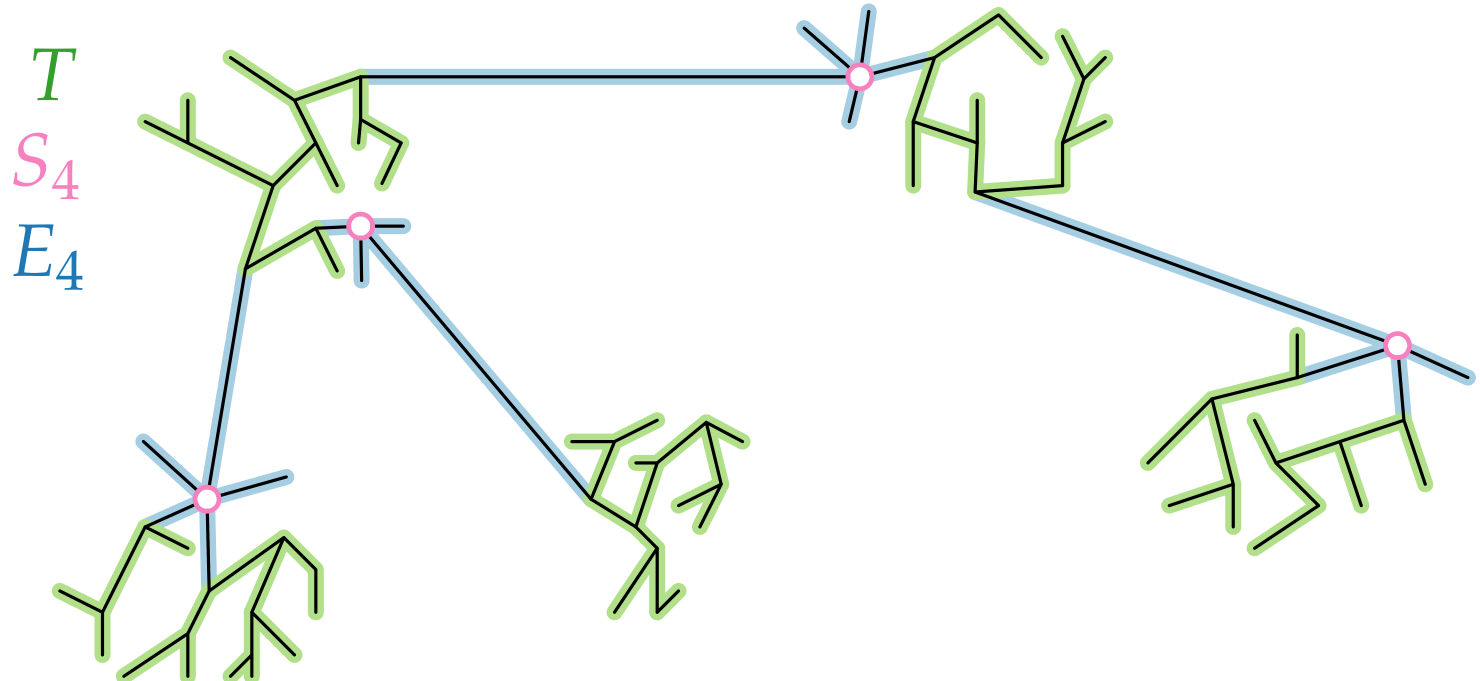

More Lemmas

Lemma 3. For $i \geq \Delta(T) - \ell + 1$,
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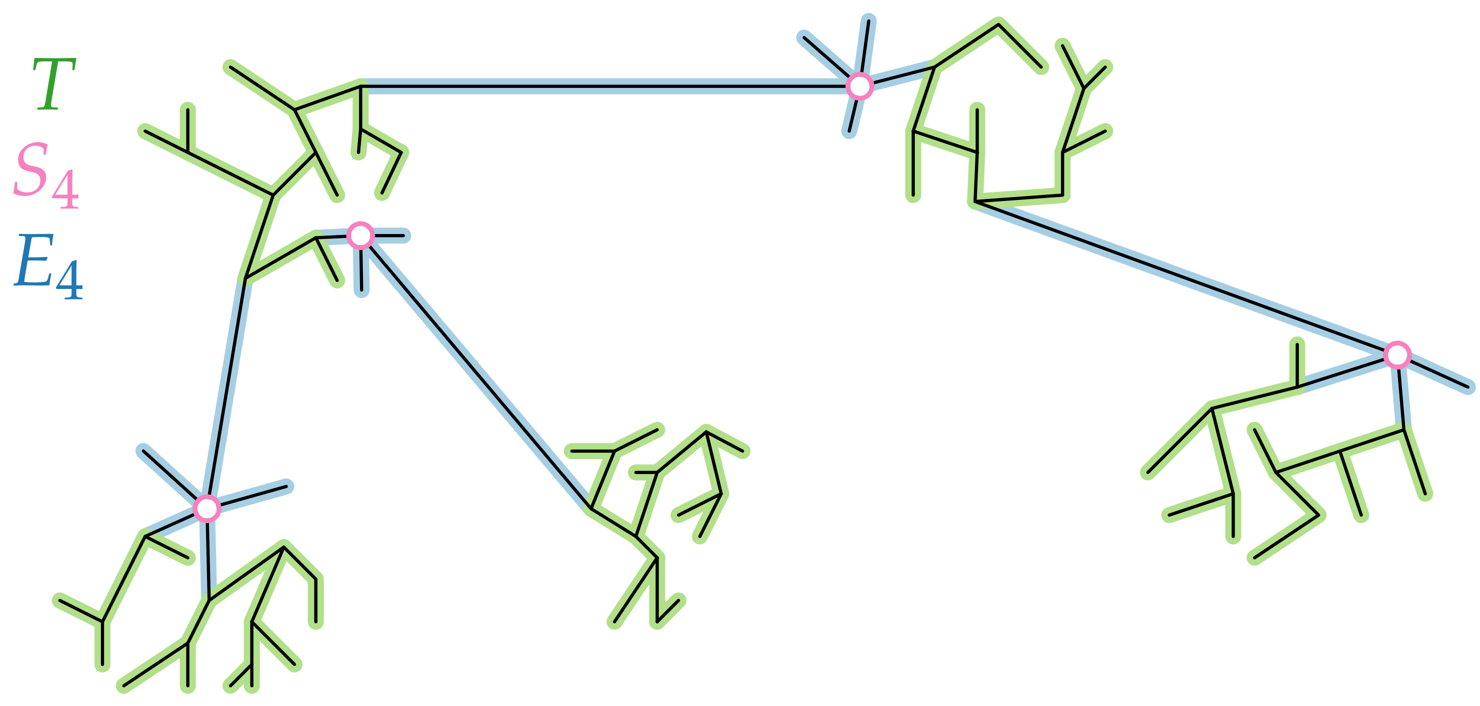

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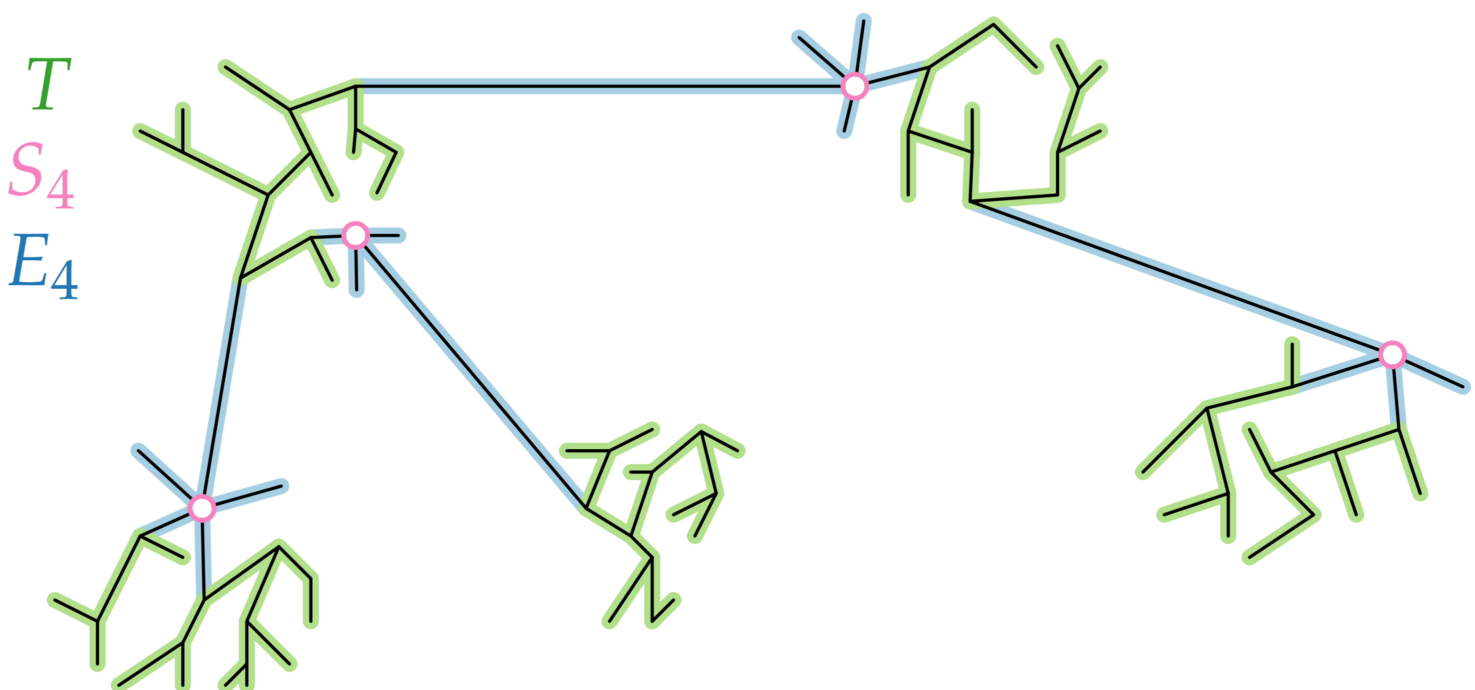
Proof. (i) $|E_i| \geq$



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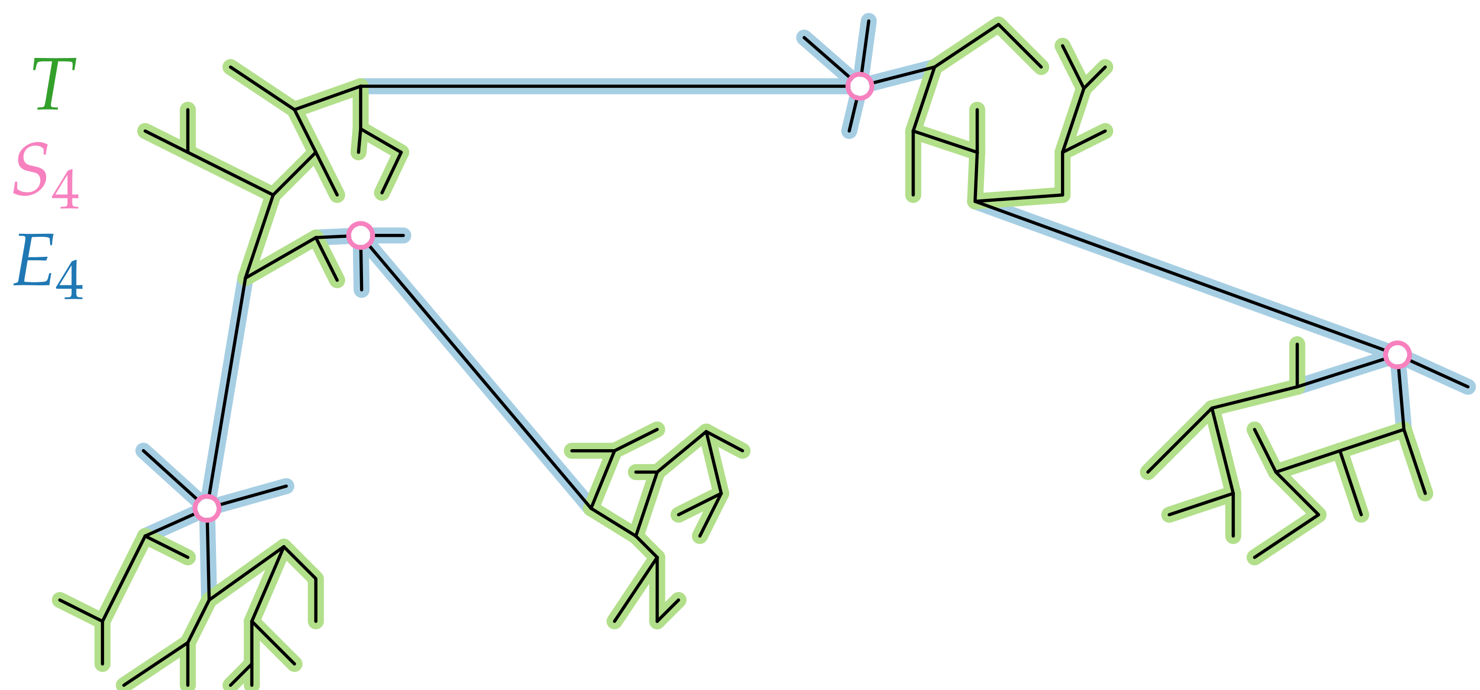
Proof. (i) $|E_i| \geq i|S_i|$

vertex-deg



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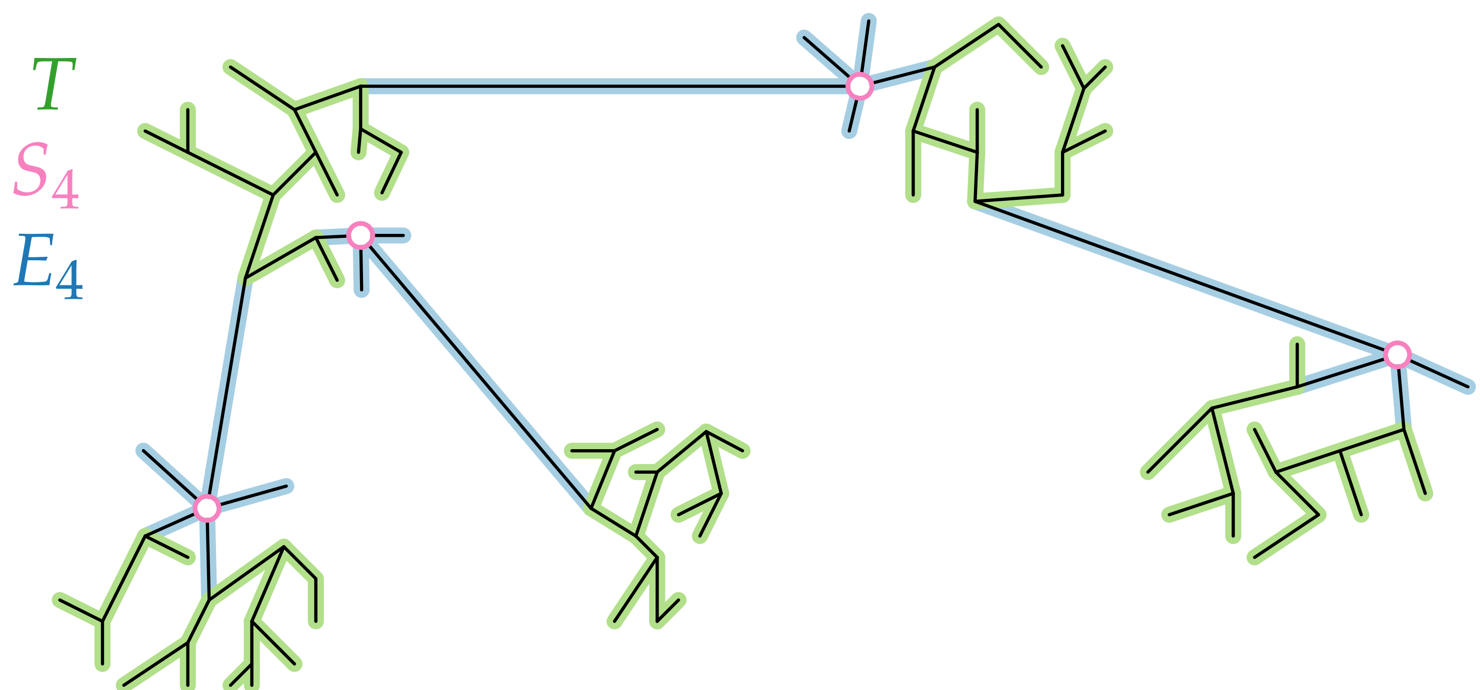
Proof. (i) $|E_i| \geq i|S_i| - (|S_i| - 1)$

vertex-deg counted twice?



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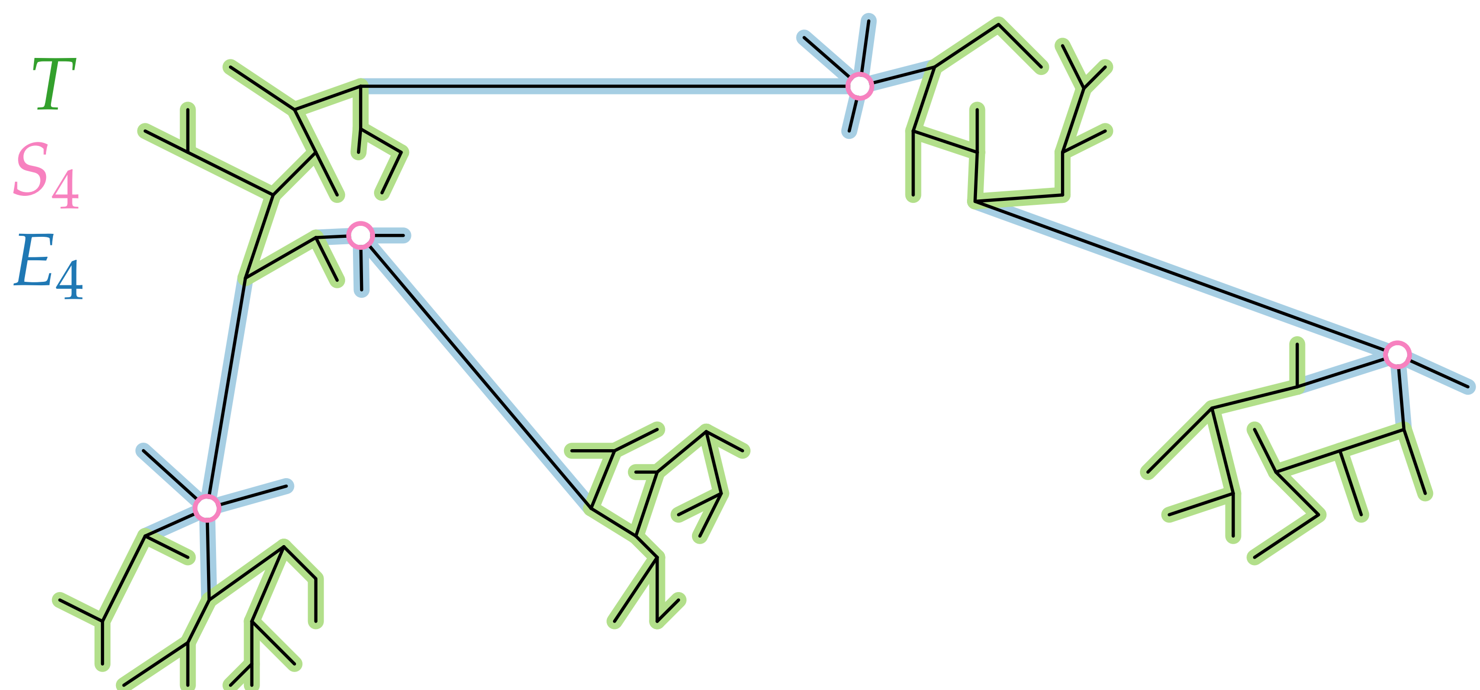
vertex-deg   counted twice?



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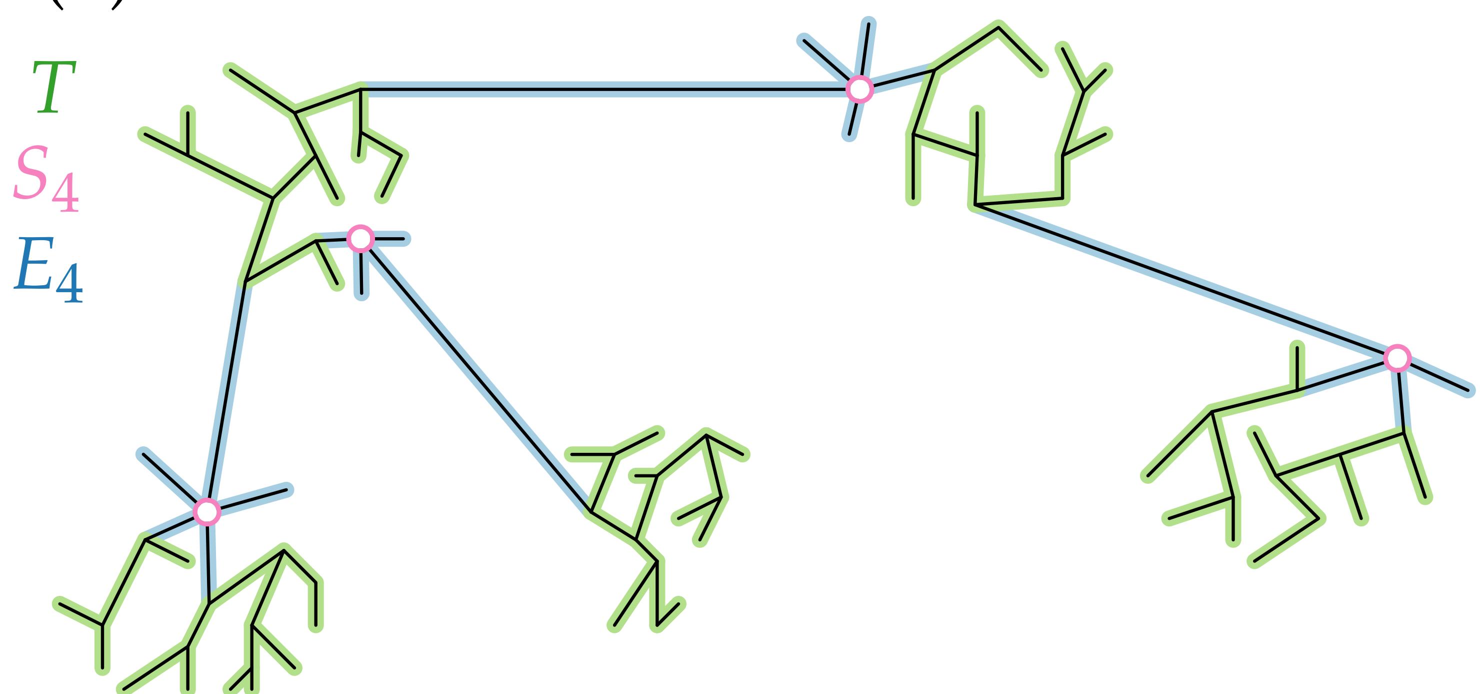
vertex-deg   counted twice?

(ii)



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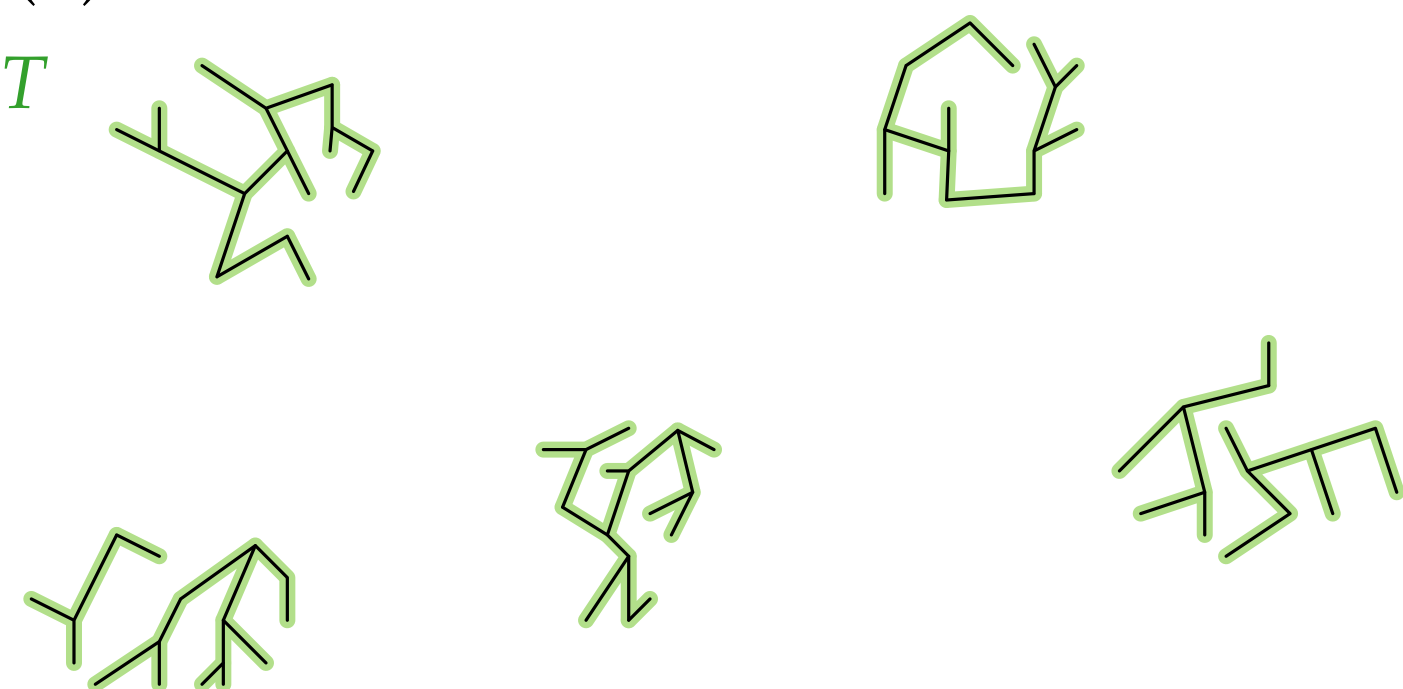
vertex-deg counted twice?

(ii)

T



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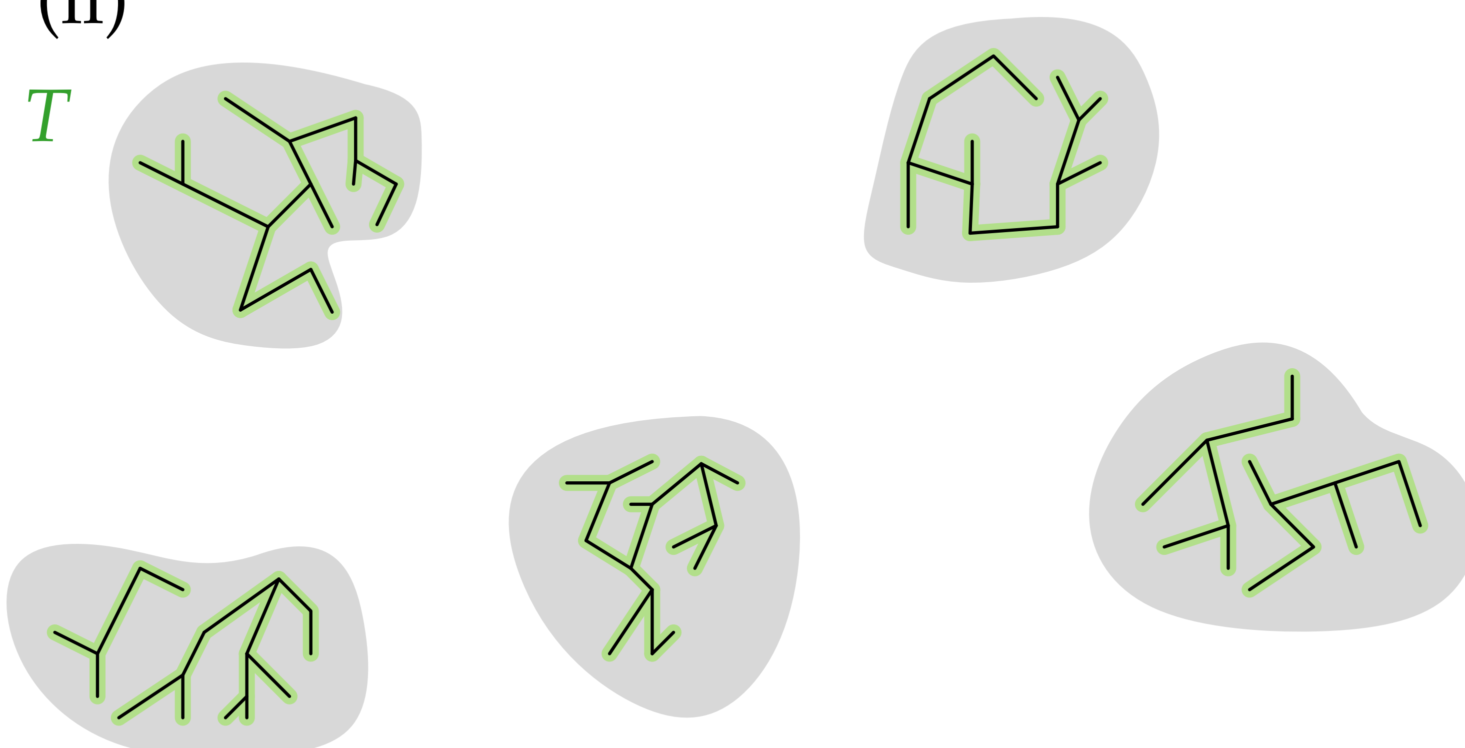

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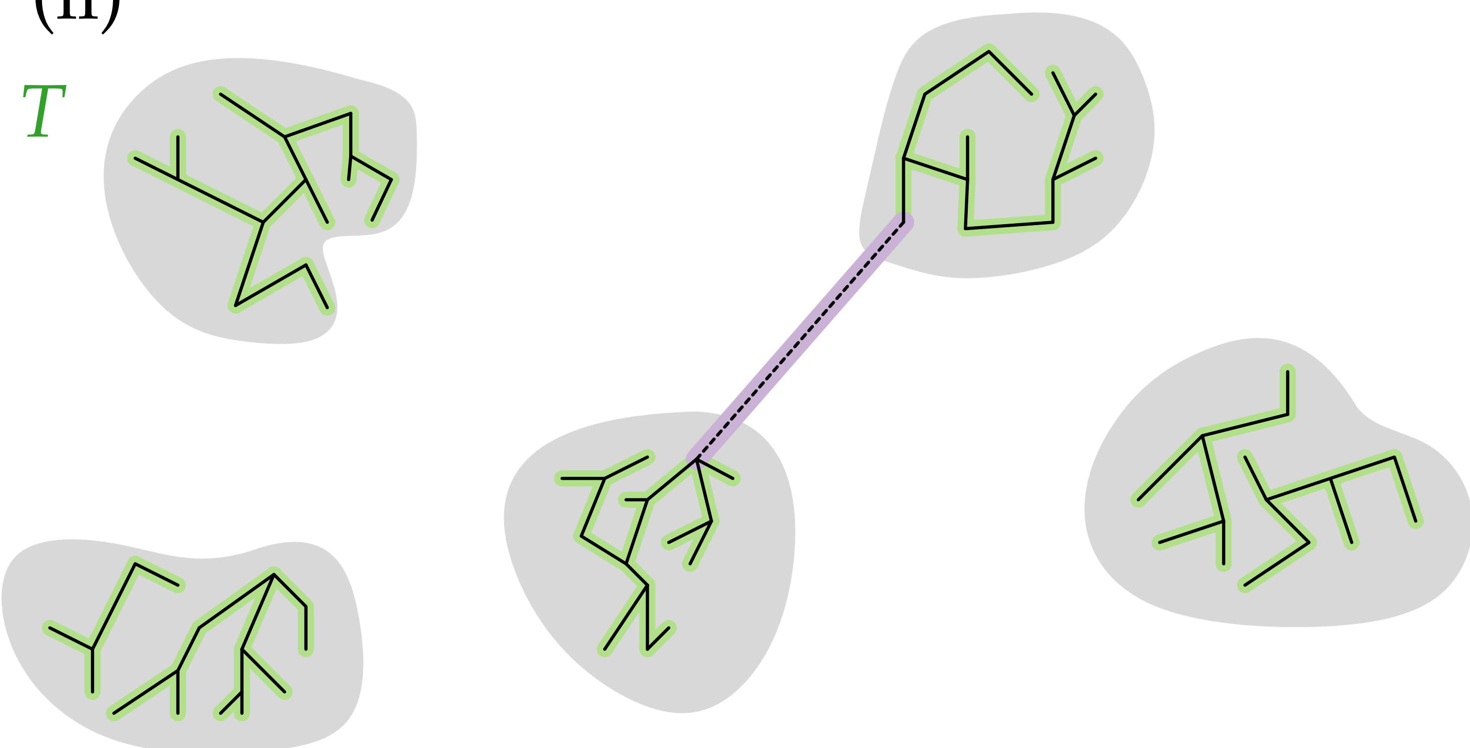
Proof. (i) $|E_i| \geq i|S_i| - (|S_i| - 1) = (i-1)|S_i| + 1$

vertex-deg counted twice?

(ii)



More Lemmas

Lemma 3. For $i \geq \Delta(T) - \ell + 1$,
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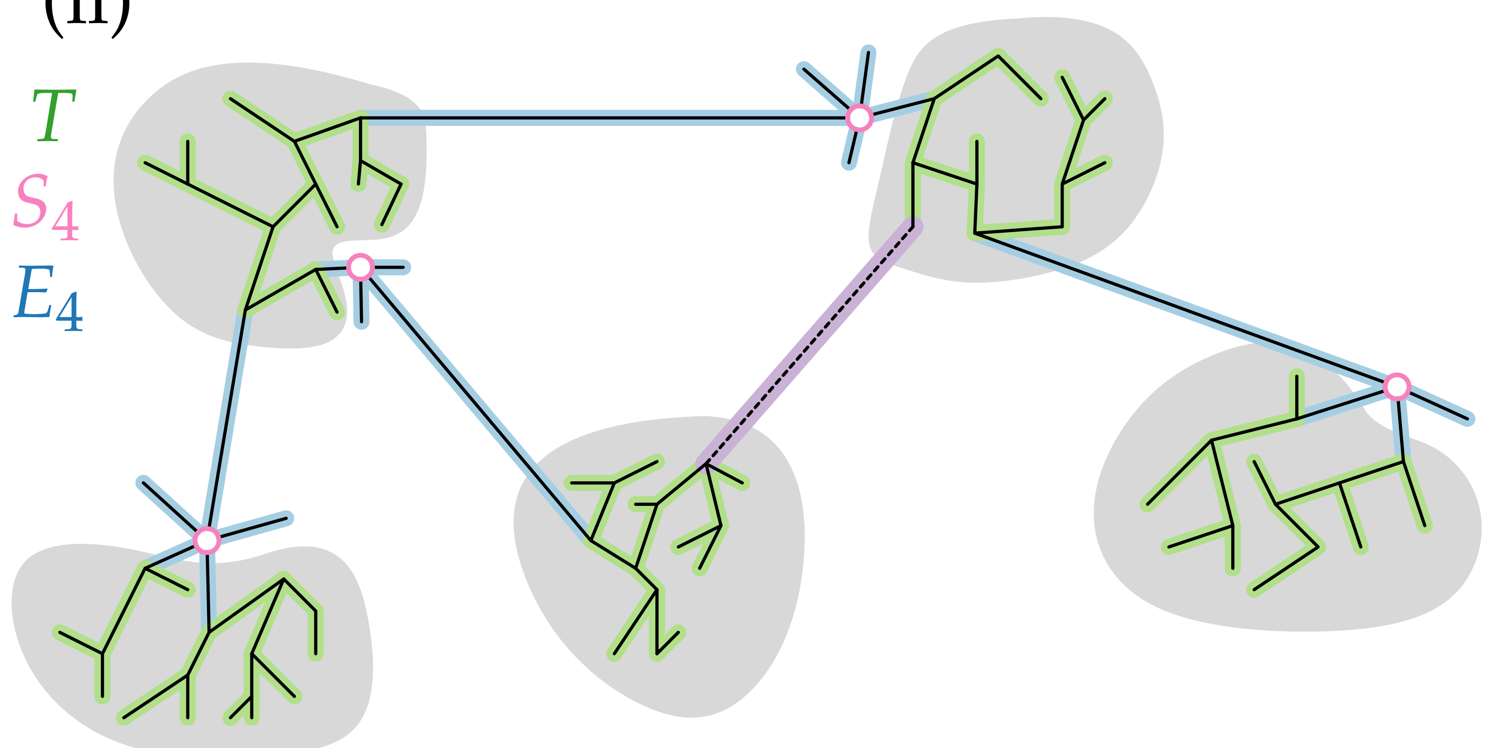

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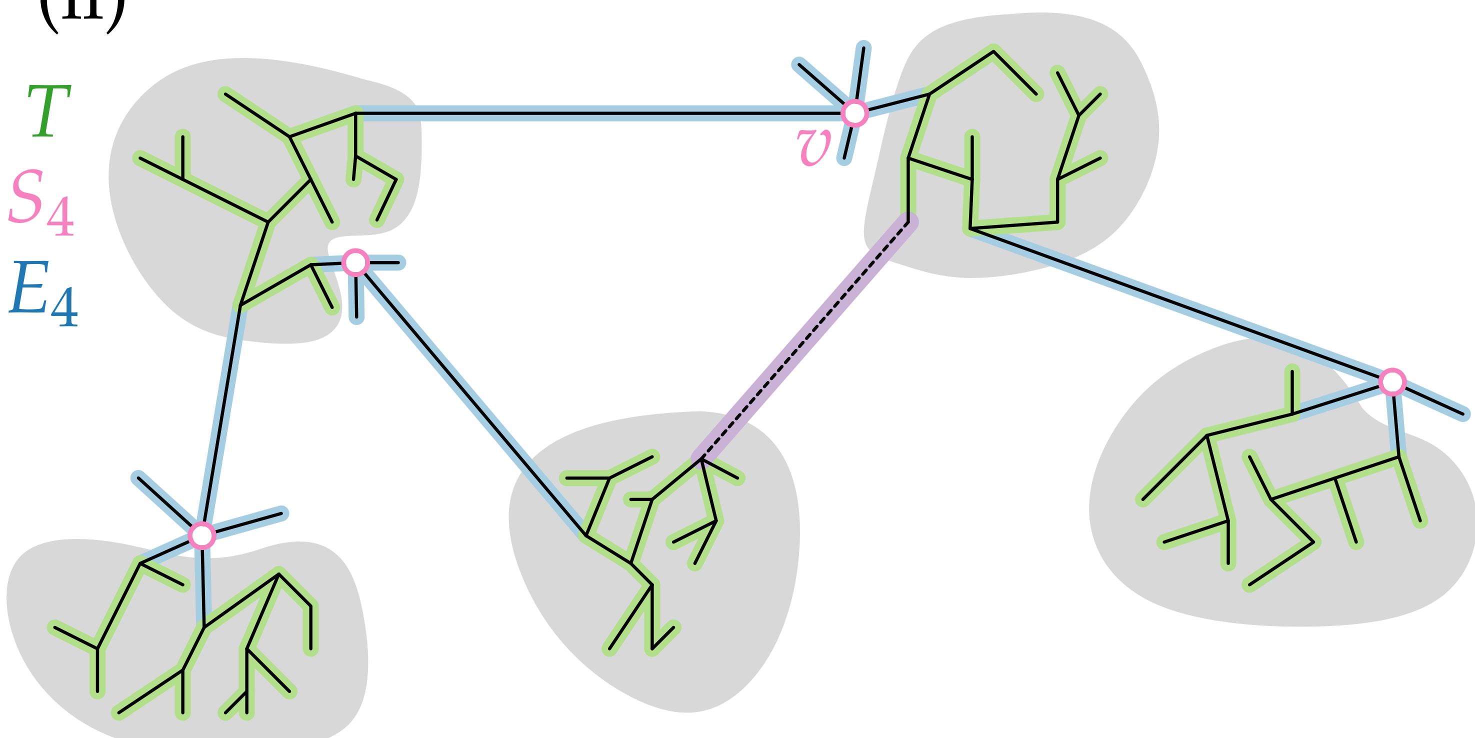

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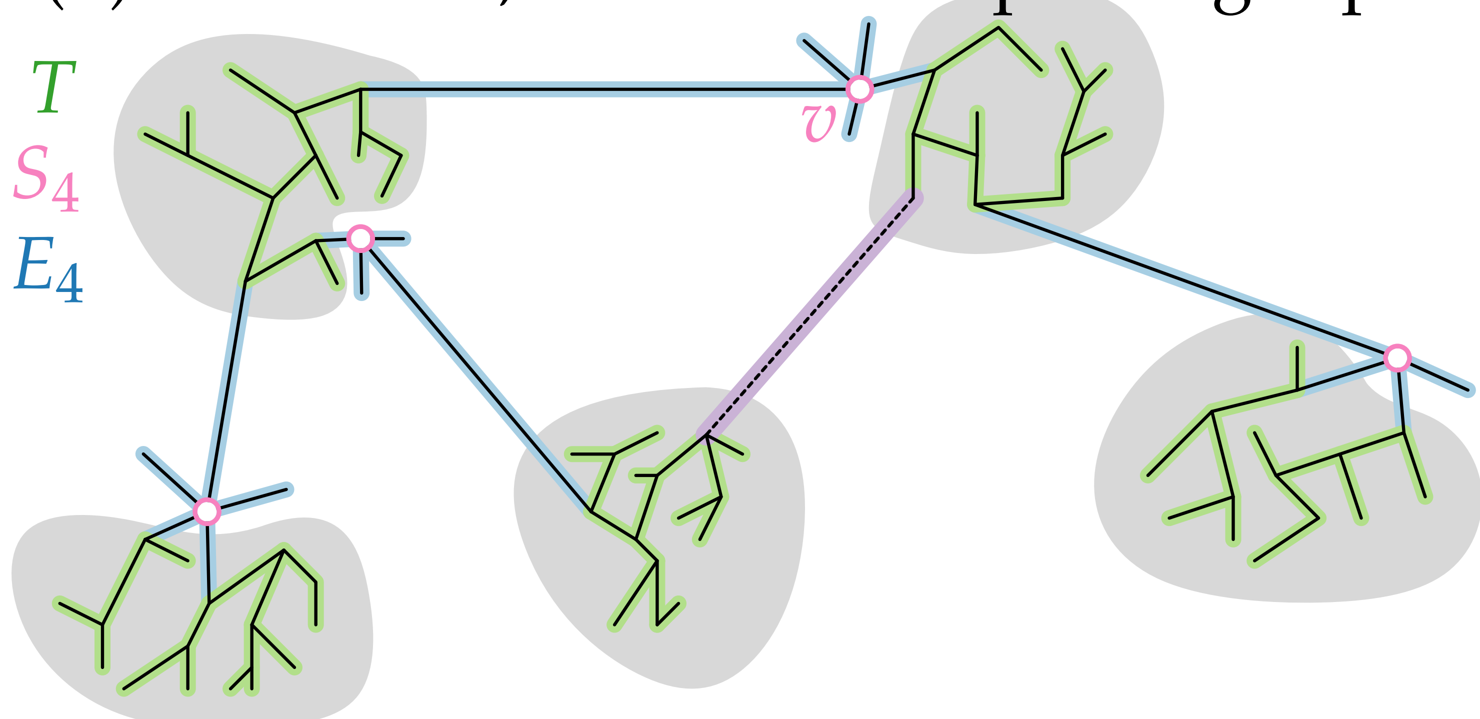
(ii) Otherwise, there is an improving flip for $v \in S_i$.



Approximation Algorithms

Lecture 10:
Minimum-Degree Spanning Tree
via Local Search

Part V:
Approximation Factor

Joachim Spoerhase

Winter 2021/22

Approximation Factor

Approximation Factor

[Fürer & Raghavachari: SODA'92, JA'94]

Theorem. Let T be a locally optimal spanning tree.
Then $\Delta(T) \leq 2 \cdot \mathrm{OPT} + \ell$, where $\ell = \lceil \log_2 n \rceil$.

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$\mathrm{OPT} \underset{\text{Lemma 1}}{\geq} \frac{k}{|S|}$

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Lemma 1 Lemma 3 Lemma 2

Approximation Factor

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□

Approximation Algorithms

Lecture 10: Minimum-Degree Spanning Tree via Local Search

Part VI: Termination, Running Time & Extensions

Joachim Spoerhase

Winter 2021/22

Termination and Running Time

Theorem. The algorithm finds a locally optimal spanning tree efficiently.

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Proof.

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Proof. Via potential function $\Phi(T)$ measuring the value of a solution where (hopefully):

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Termination and Running Time

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Proof. Via potential function $\Phi(T)$ measuring the value of a solution where (hopefully):

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- the function is bounded both from above and below.

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- executing $f(n)$ iterations would exceed this lower bound.

Termination and Running Time

Theorem. The algorithm finds a locally optimal spanning tree after at most $f(n)$ iterations.

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$\Phi(T) = \sum_{v \in V(G)} 3^{\deg_T(v)}$

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- each iteration decreases the potential of a solution.

Lemma. After each flip $T \to T'$, $\Phi(T') \leq (1 - \frac{2}{27n^3})\Phi(T)$.

Homework

- the function is bounded both from above and below.
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Goal: After $f(n)$ iterations: $\Phi(T) = n < 3n$

Termination and Running Time

Theorem. The algorithm finds a locally optimal spanning tree after at most $f(n)$ iterations.

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- the function is bounded both from above and below.

Lemma. For each spanning tree T, $\Phi(T) \in [3n, n3^n]$.

- executing $f(n)$ iterations would exceed this lower bound.

Let $f(n) = \frac{27}{2}n^4 \cdot \ln 3$. How does $\Phi(T)$ change?

decreases by: $(1 - \frac{2}{27n^3})^{f(n)} \leq (e^{-\frac{2}{27n^3}})^{f(n)}$

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  Goal: After $f(n)$ iterations: $\Phi(T) = n < 3n$ $\square$

Termination and Running Time

Theorem. The algorithm finds a locally optimal spanning tree after $O(n^4)$ iterations.

Proof. Via potential function $\Phi(T)$ measuring the value of a solution where (hopefully): $\Phi(T) = \sum_{v \in V(G)} 3^{\deg_T(v)}$

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Extensions

[Fürer & Raghavachari: SODA'92, JA'94]

Corollary. For any constant $b > 1$ and $\ell = \lceil \log_b n \rceil$, the local search algorithm runs in polynomial time and produces a spanning tree T with $\Delta(T) \leq b \cdot \mathrm{OPT} + \lceil \log_b n \rceil$.

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Proof. Similar to previous pages. Homework $\square$

Extensions

[Fürer & Raghavachari: SODA'92, JA'94]

Corollary. For any constant $b > 1$ and $\ell = \lceil \log_b n \rceil$, the local search algorithm runs in polynomial time and produces a spanning tree T with $\Delta(T) \leq b \cdot \mathrm{OPT} + \lceil \log_b n \rceil$.

Proof. Similar to previous pages. Homework $\square$

Theorem. There is a local search algorithm that runs in $O(EV\alpha(E,V)\log V)$ time and produces a spanning tree T with $\Delta(T) \leq \mathrm{OPT} + 1$.