Approximation Algorithms

Lecture 8: Approximation Schemes and the Knapsack Problem

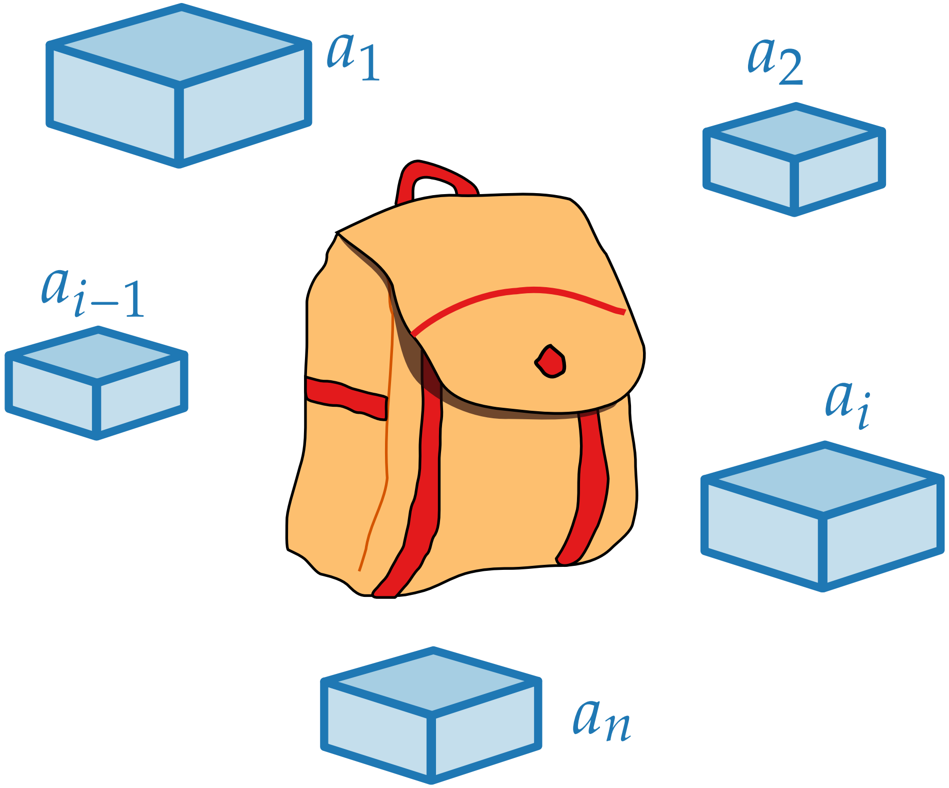
Part I: Knapsack

Joachim Spoerhase

Winter 2021/22

Knapsack

Given: ■ A set $S = \{a_1, \ldots, a_n\}$ of **objects**.



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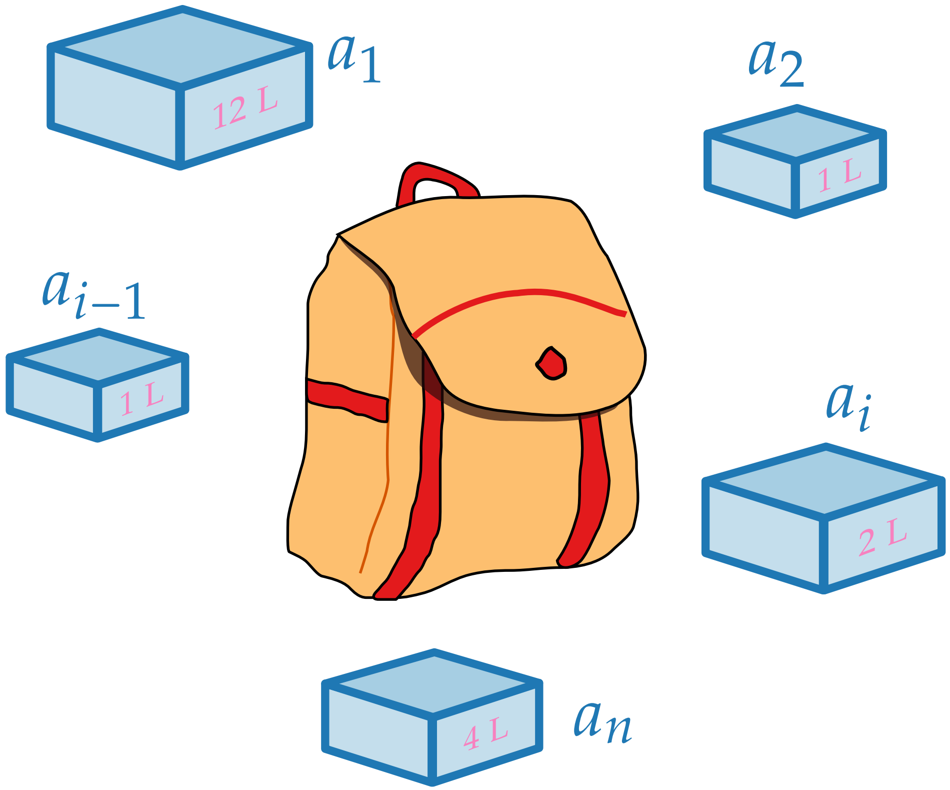
- For every object a_i a **size** $\mathrm{size}(a_i) \in \mathbb{N}^+$



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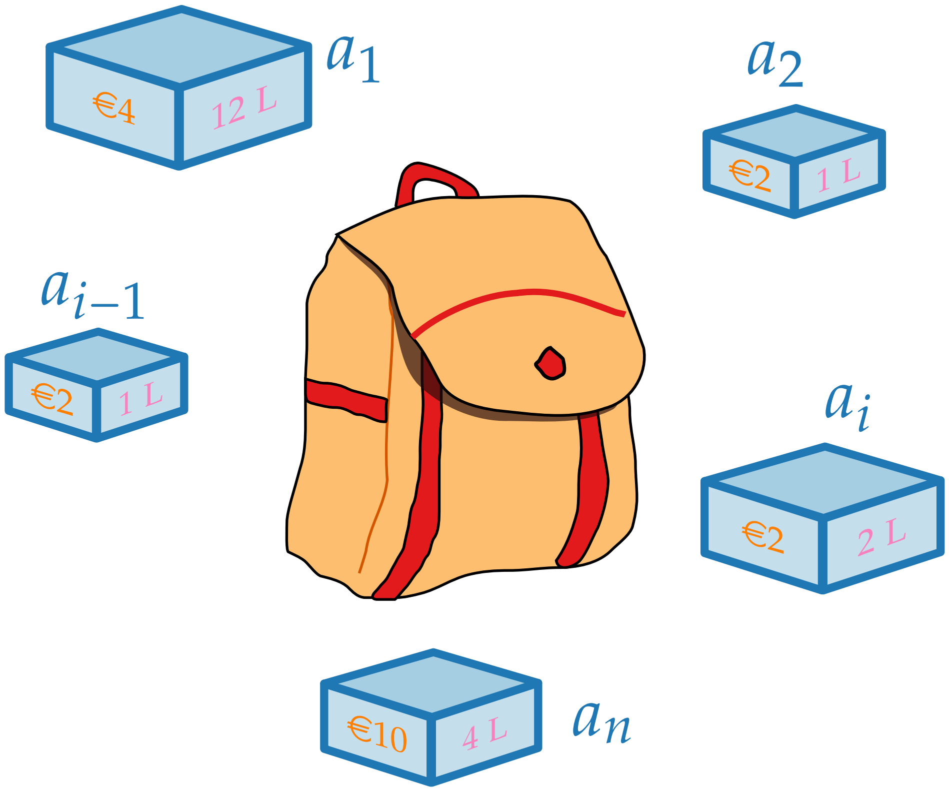

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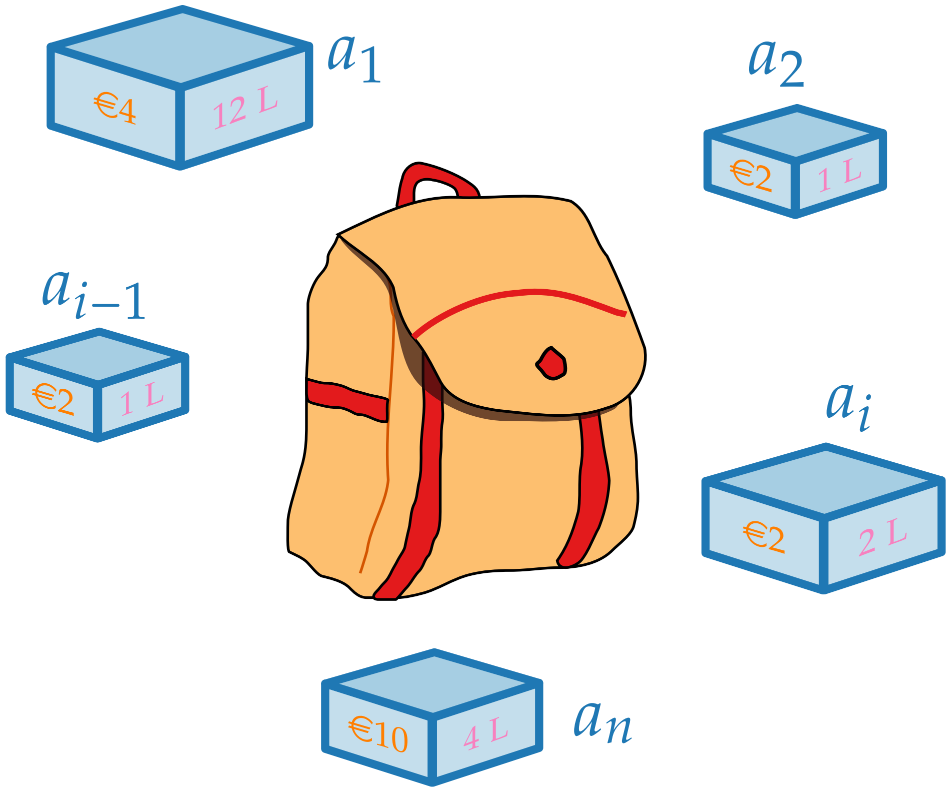

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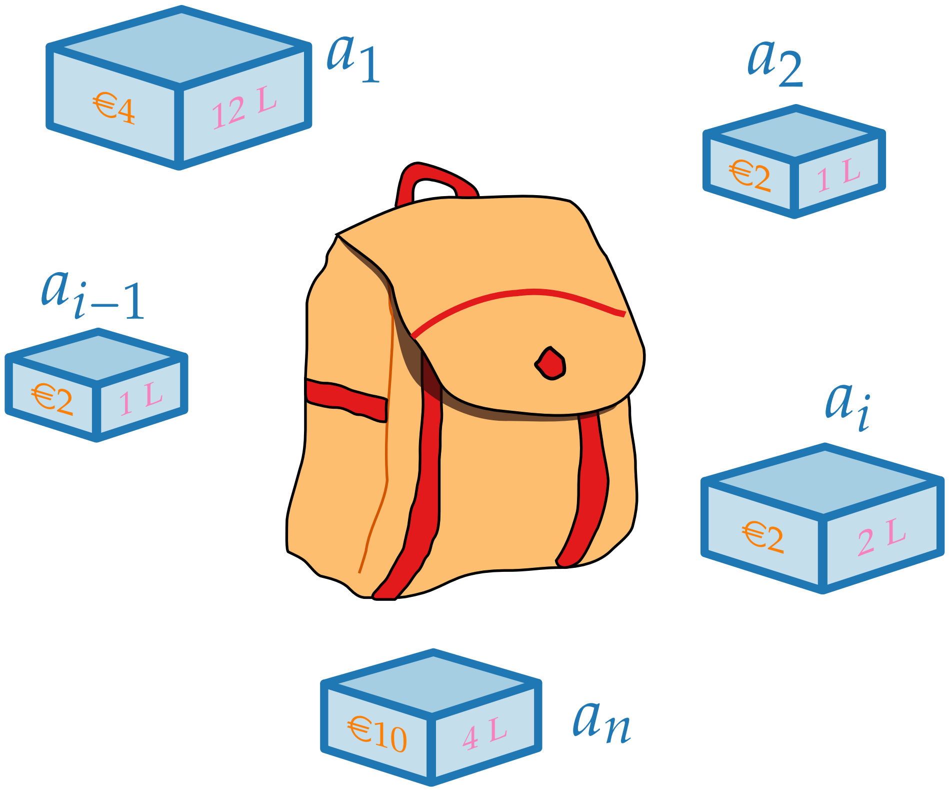

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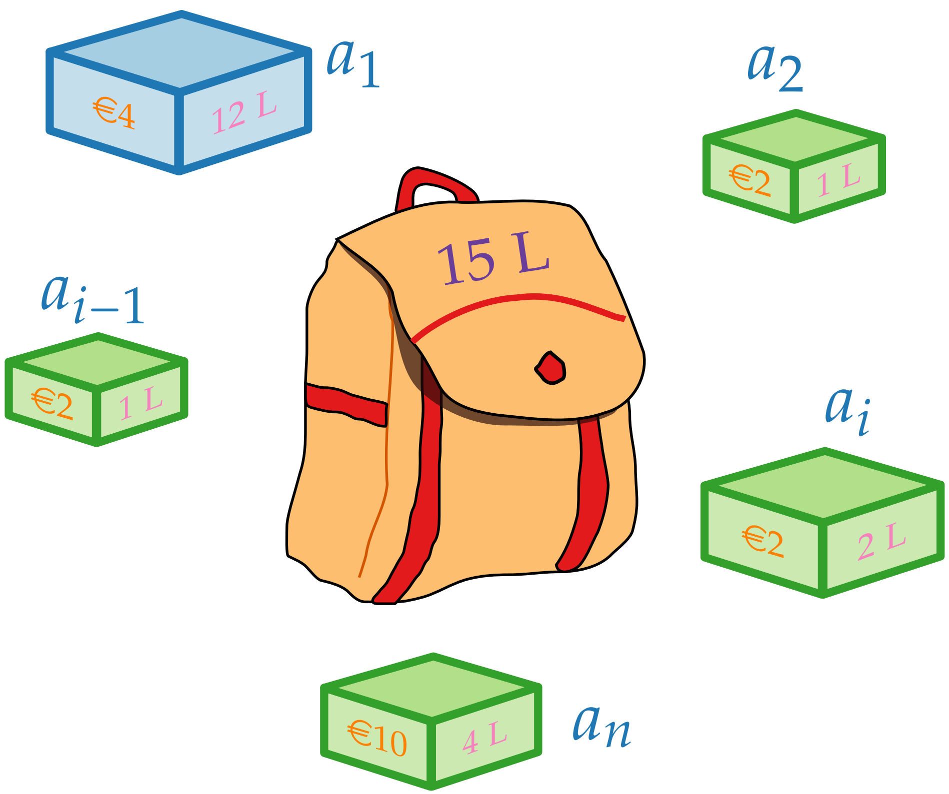

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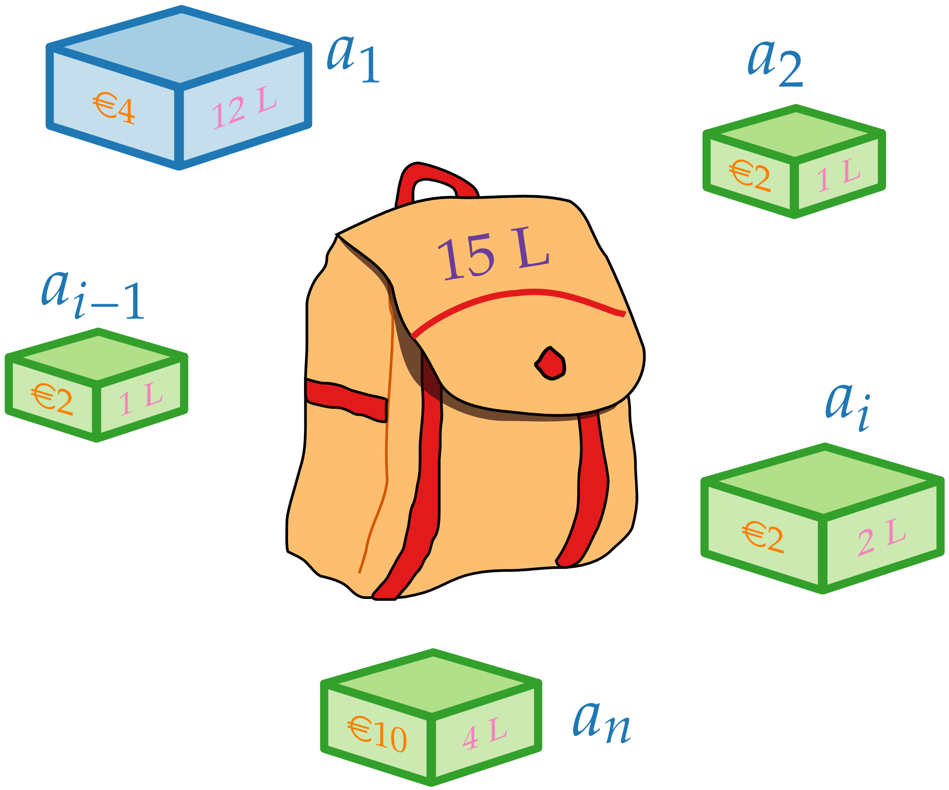


NP-hard

Approximation Algorithms

Lecture 8:
Approximation Schemes and the KNAPSACK Problem

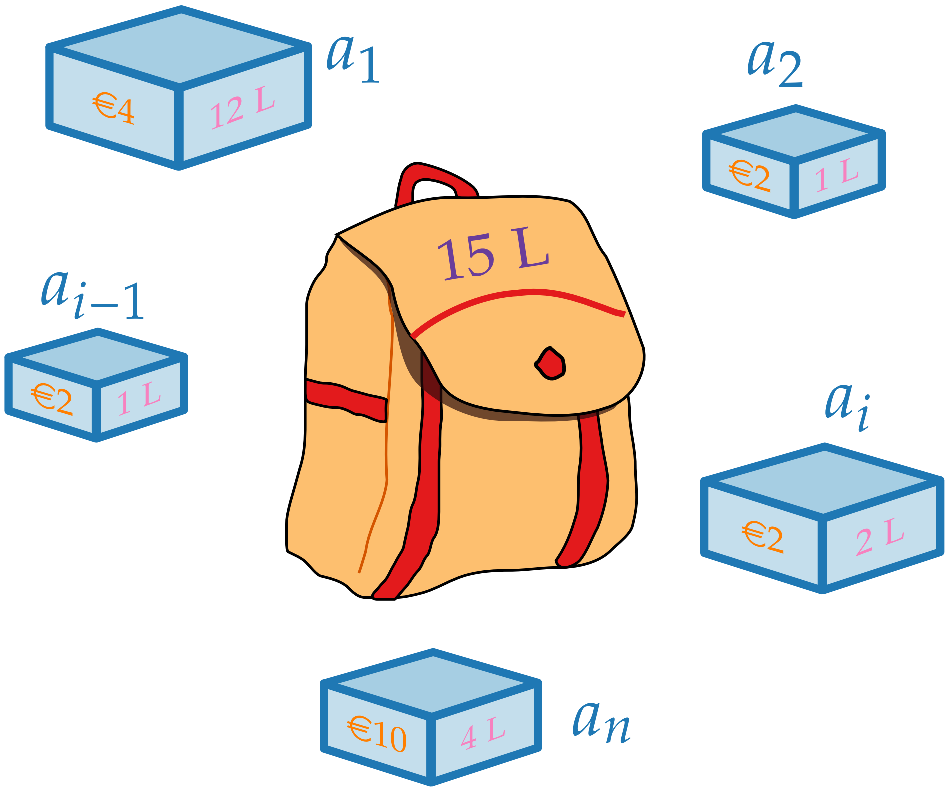
Part II:
Pseudo-Polynomial Algorithms and Strong NP-Hardness

Joachim Spoerhase

Winter 2021/22

Pseudo-Polynomial Algorithms

Let Π be an optimization problem whose instances can be represented by **objects** (such as sets, elements, edges, nodes) and **numbers** (such as costs, weights, profits).



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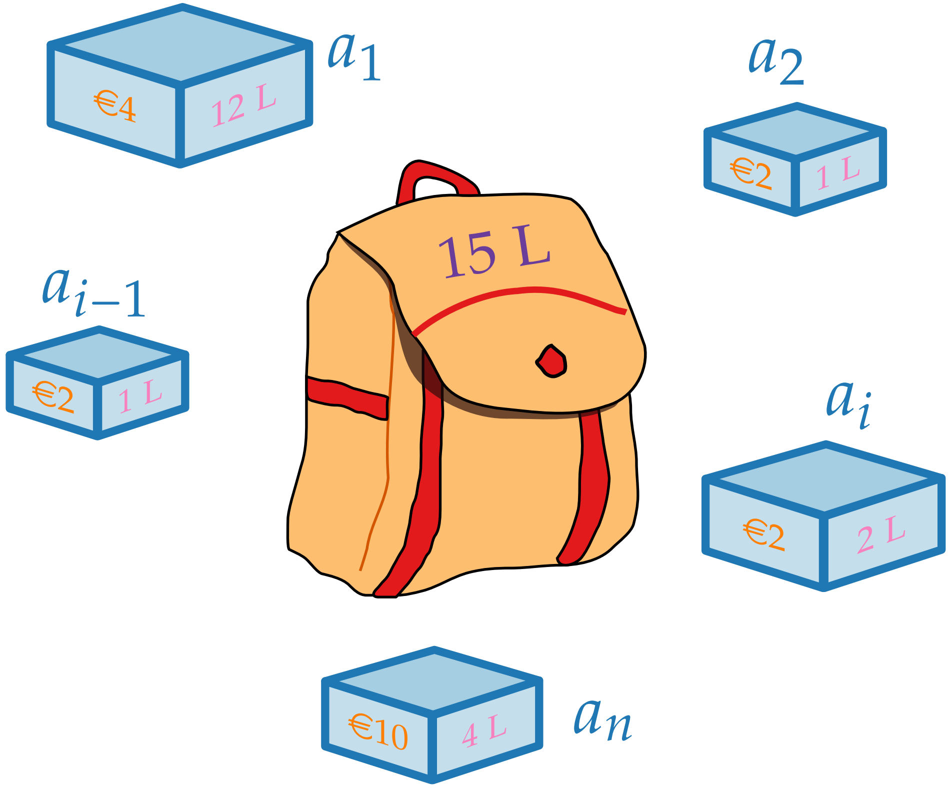
$|I|$: The size of an instance $I \in D_\Pi$, where all numbers in I are encoded in **binary**.



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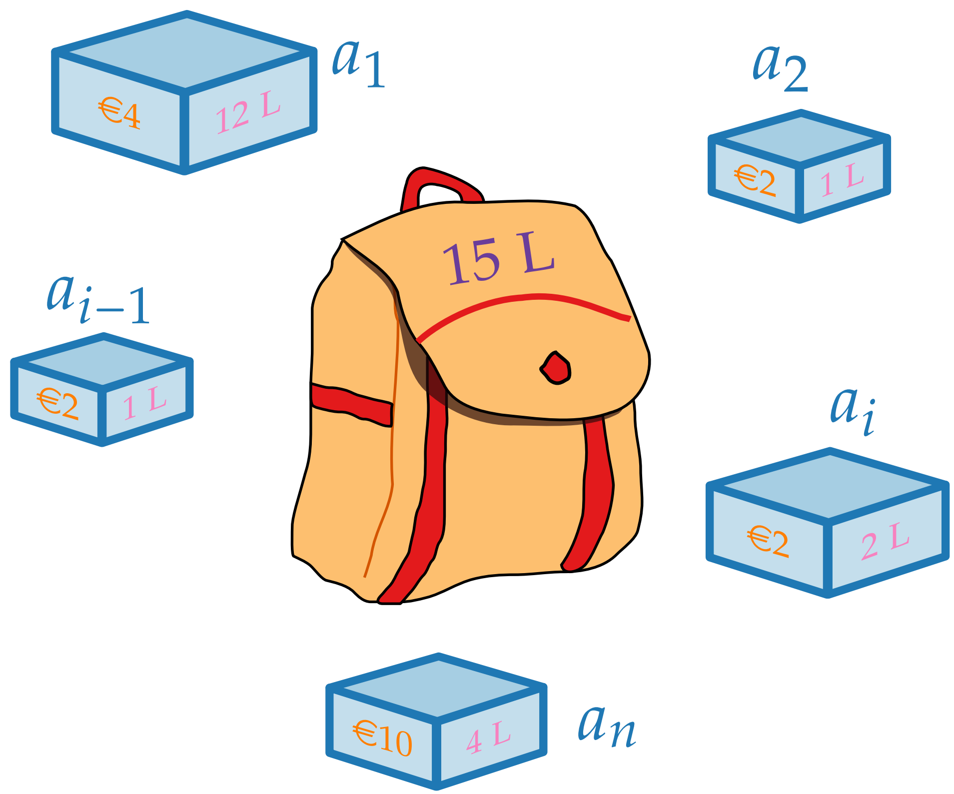

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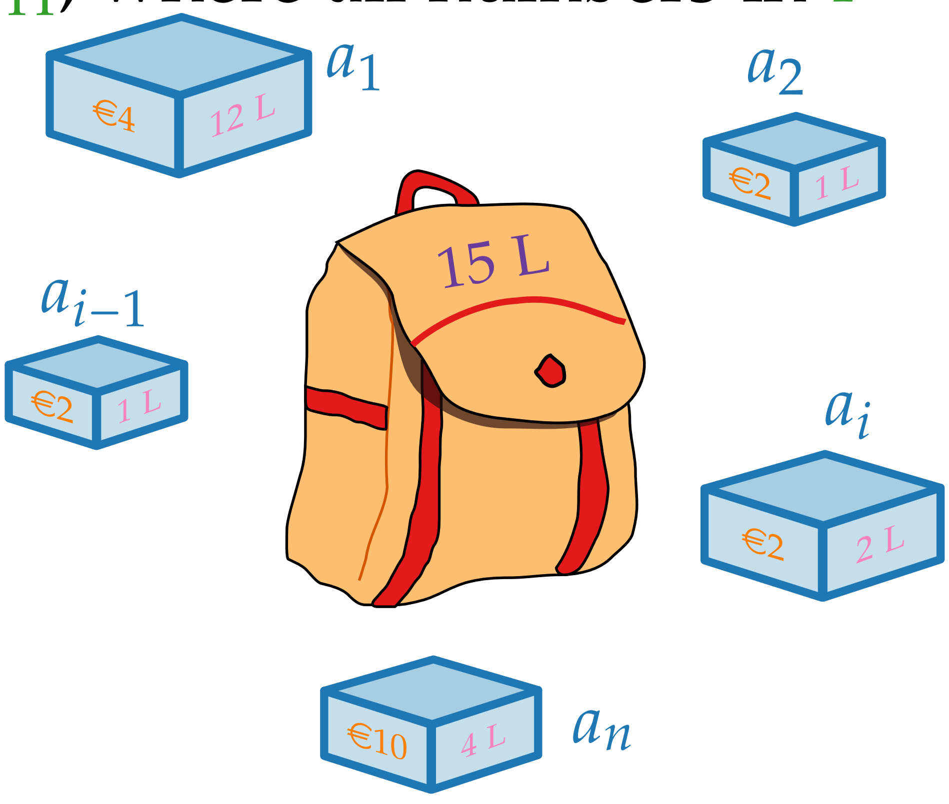

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The running time of a **pseudo-polynomial algorithm** is polynomial in $|I|_\mathrm{u}$.

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The running time of a polynomial algorithm for Π is polynomial in $|I|$.

The running time of a **pseudo-polynomial algorithm** is polynomial in $|I|_u$.

The running time of a pseudo-polynomial algorithm may not be polynomial in $|I|$.

Strong NP-Hardness

An optimization problem is called **strongly NP-hard** if it remains NP-hard under unary encoding.

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An optimization problem is called **strongly NP-hard** if it remains NP-hard under unary encoding.

An optimization problem is called **weakly NP-hard** if it is NP-hard under binary encoding but has a pseudo-polynomial algorithm.

Theorem. A strongly NP-hard problem has no pseudo-polynomial algorithm unless P = NP.

Approximation Algorithms

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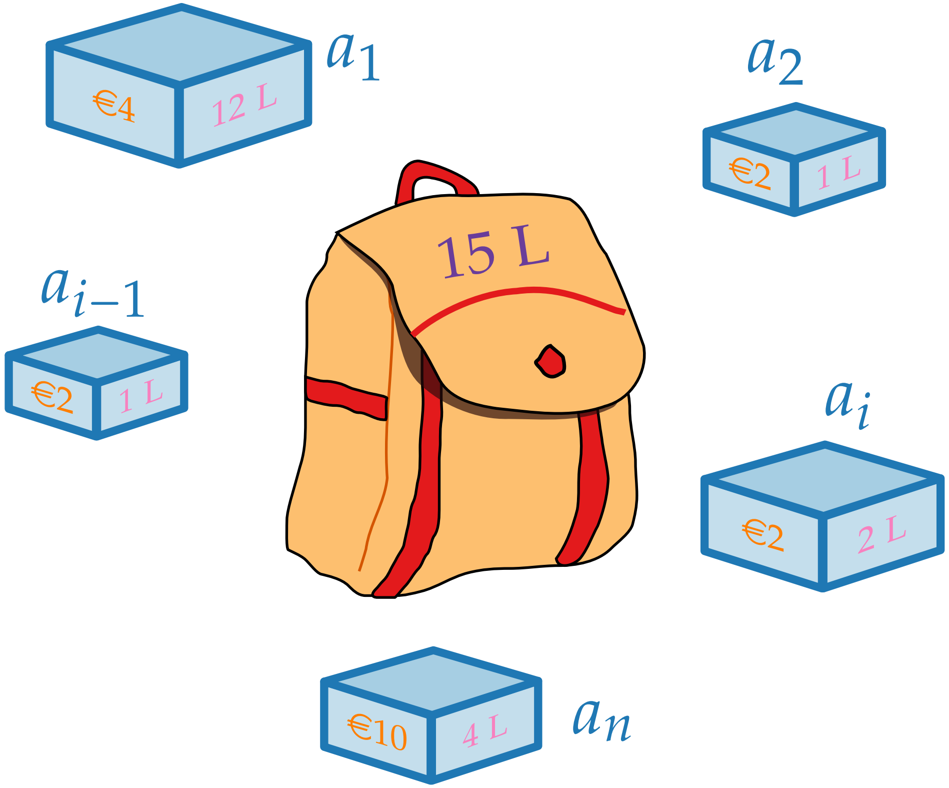
Part III: Pseudo-Polynomial Algorithm for Knapsack

Joachim Spoerhase

Winter 2021/22

Pseudo-Polynomial Alg. for KNAPSACK

Let $P := \max_i \text{profit}(a_i)$



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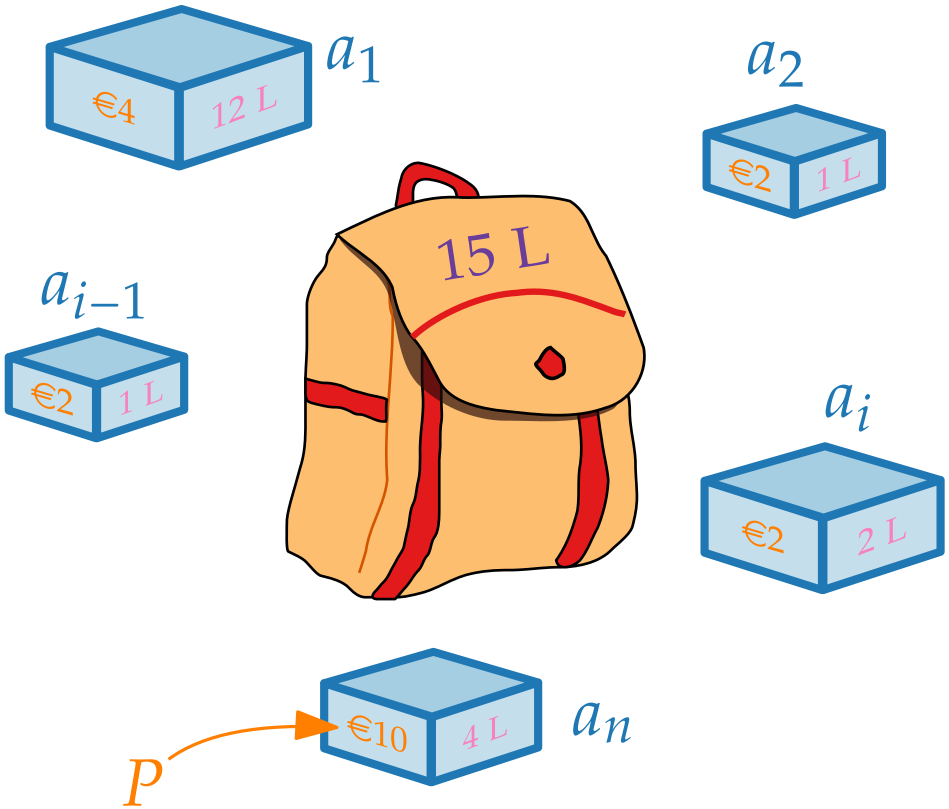

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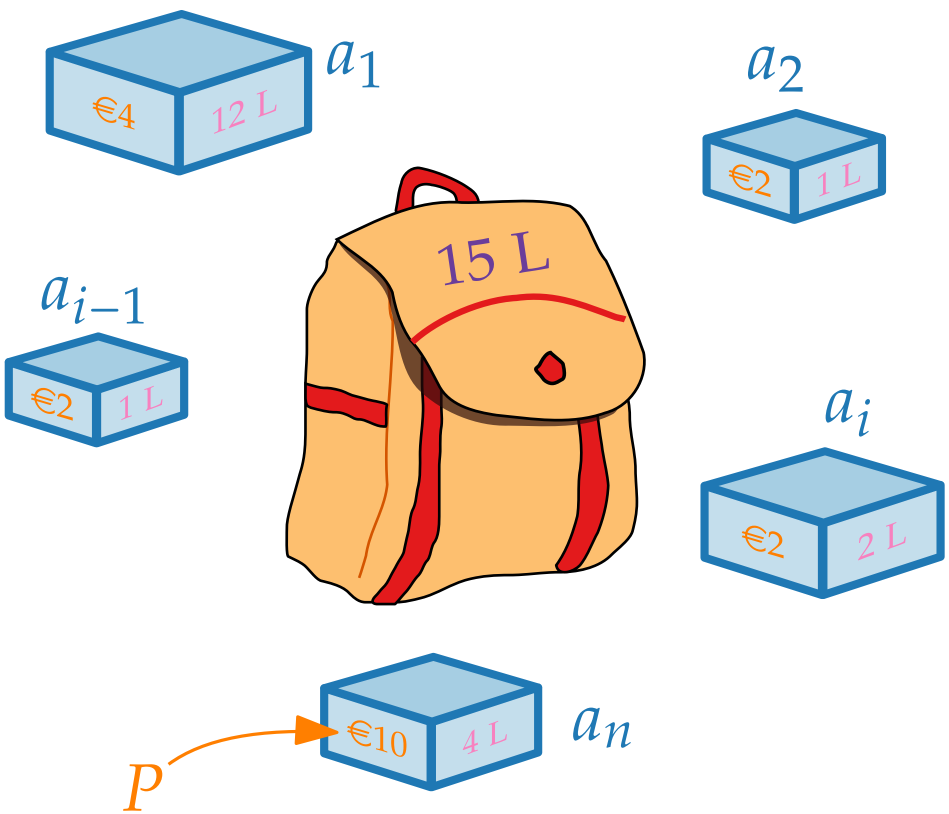

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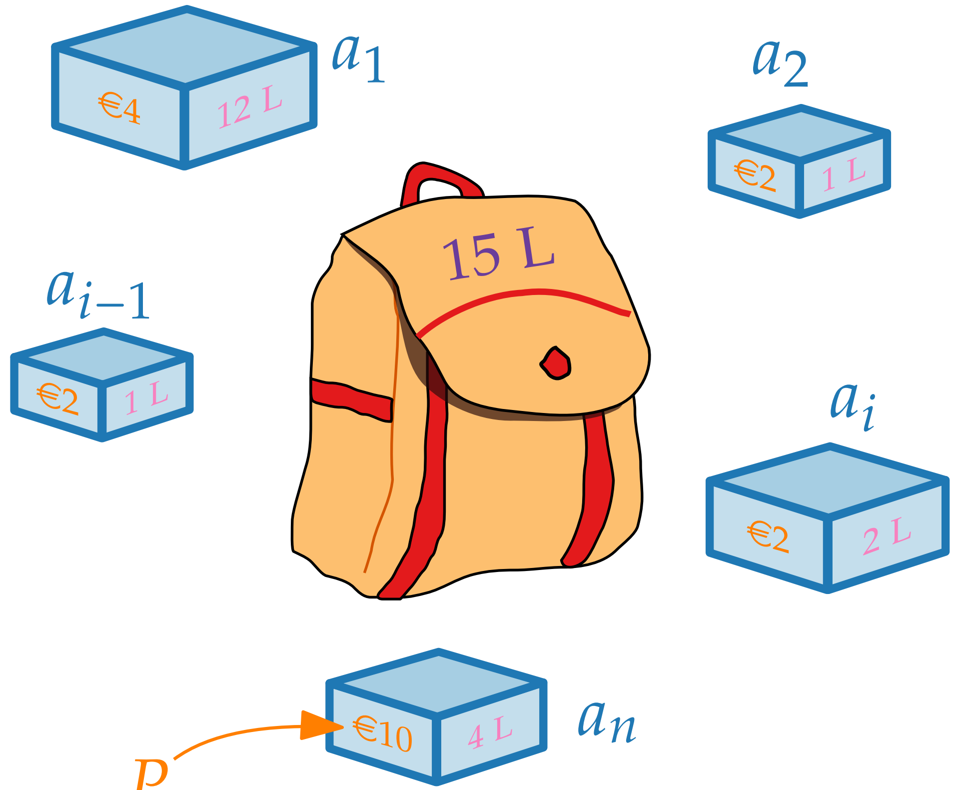

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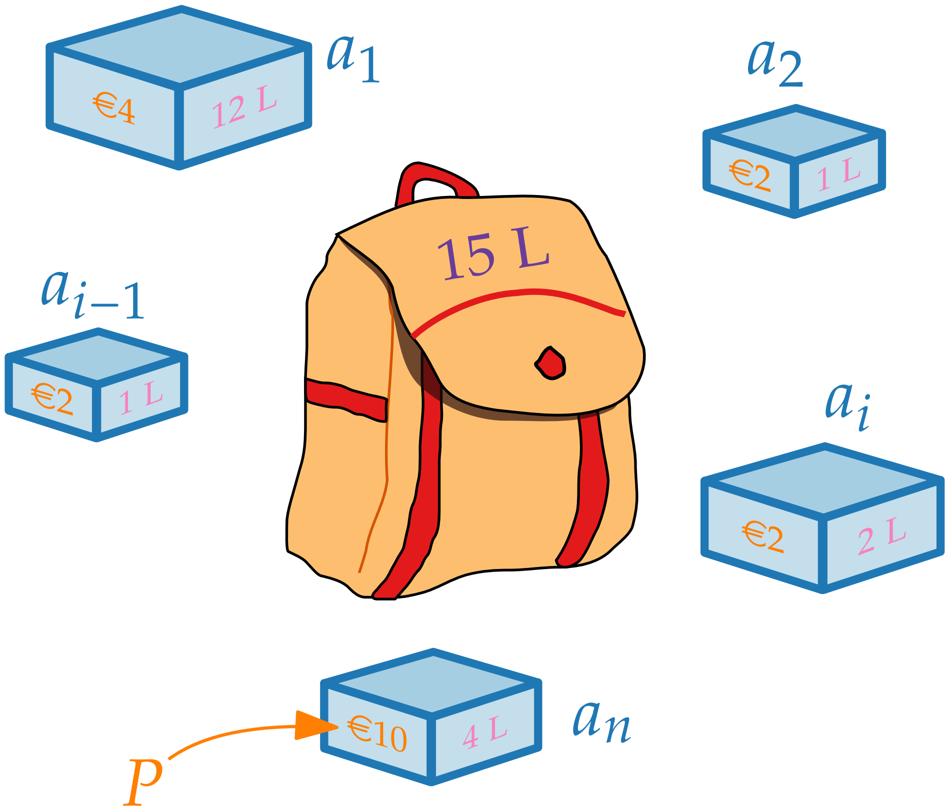

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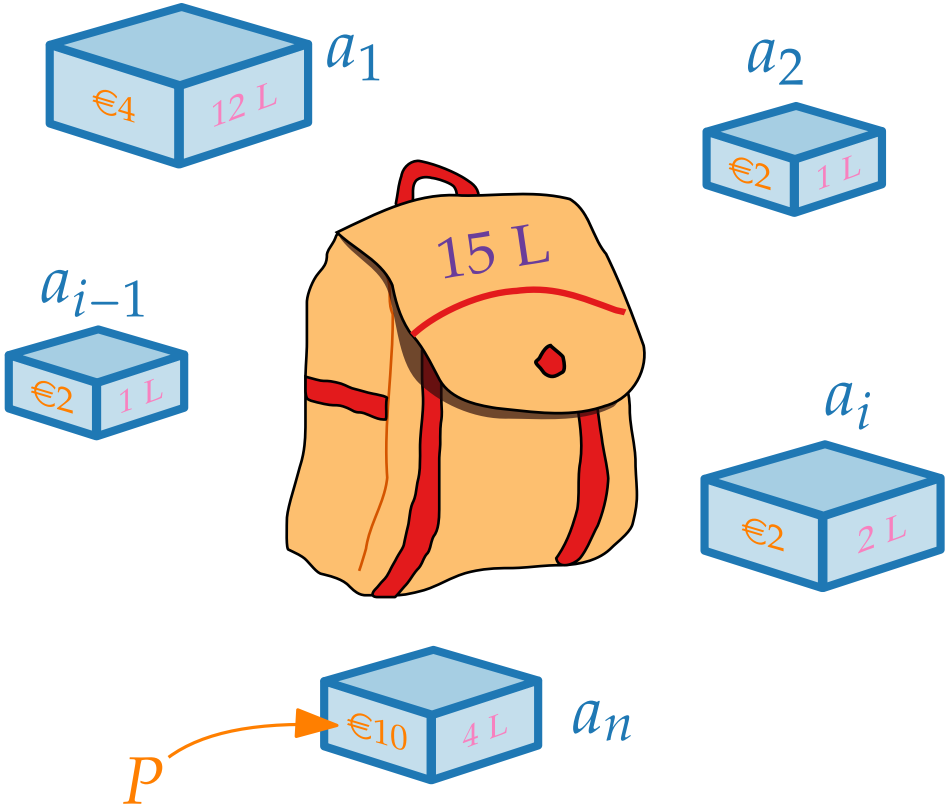

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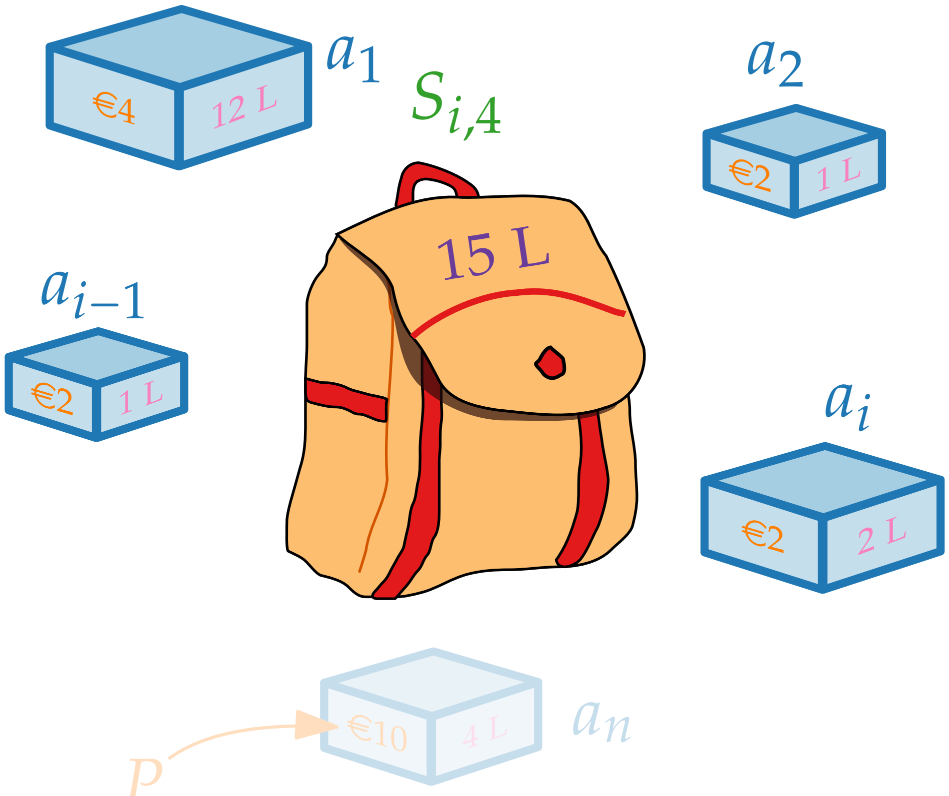

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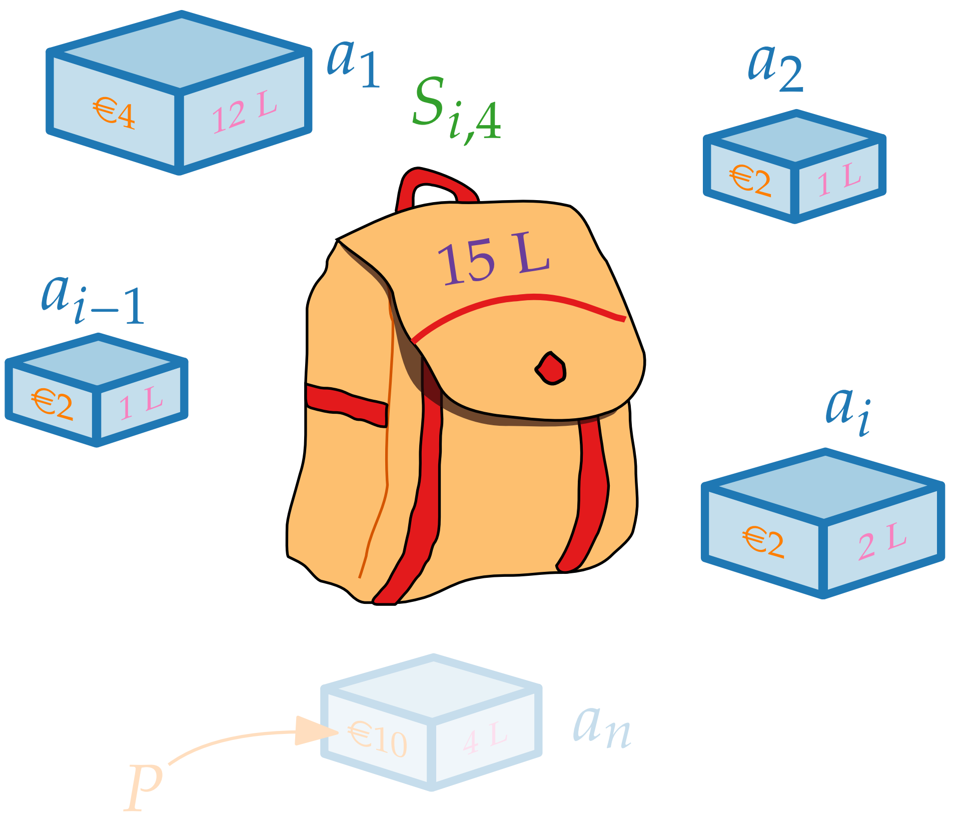

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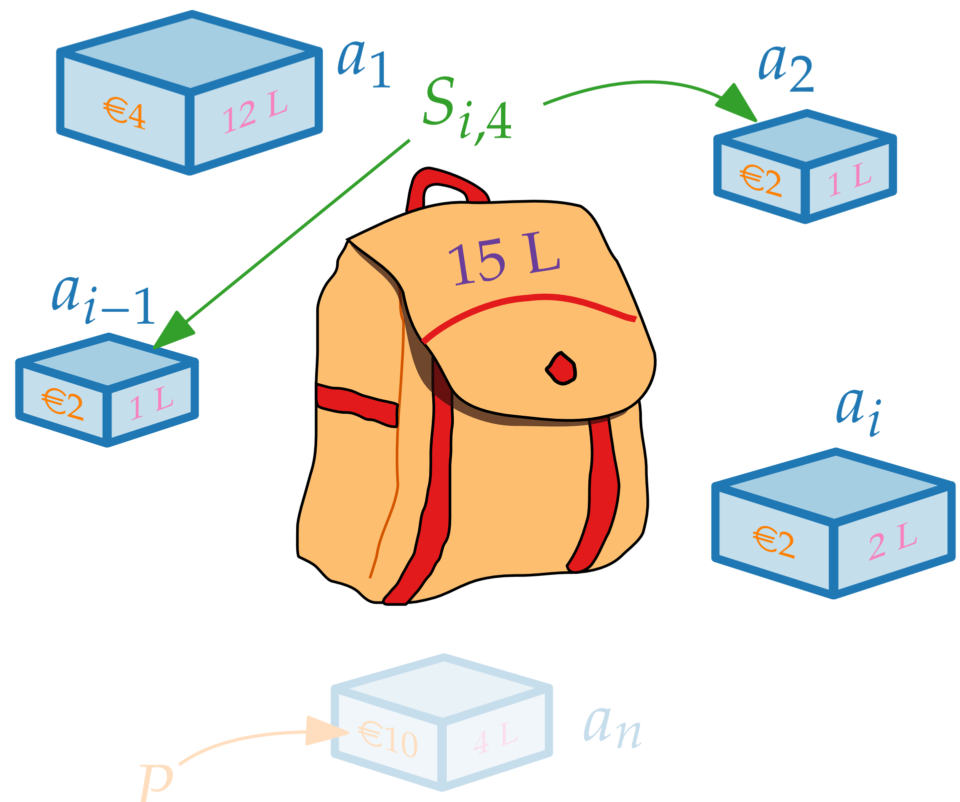

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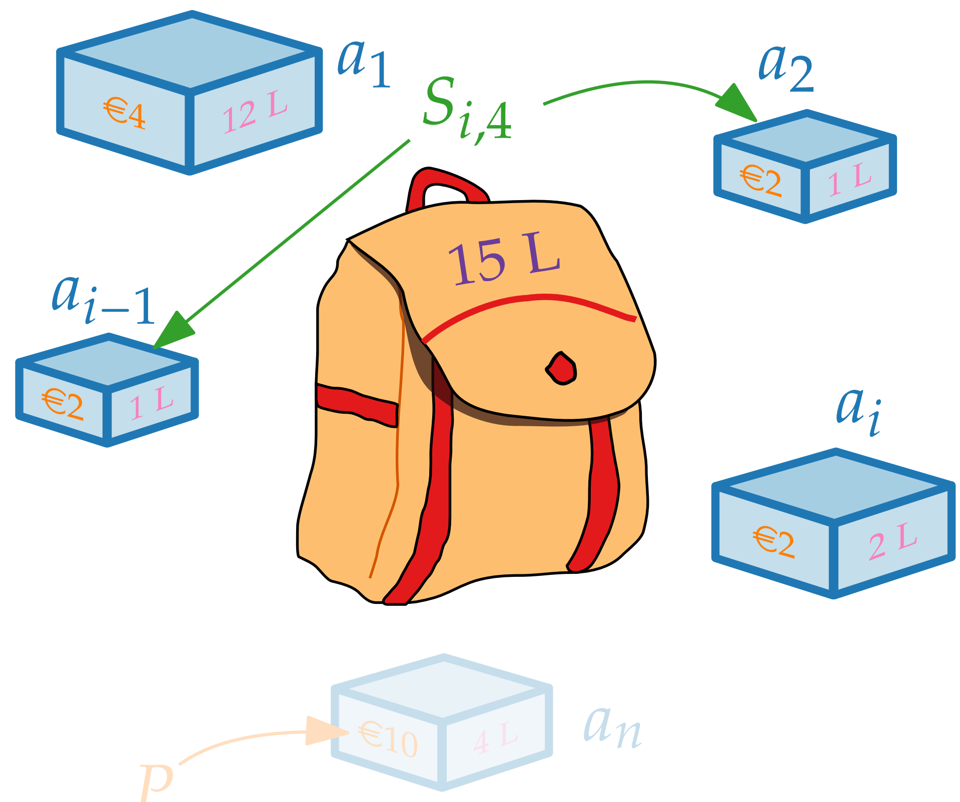

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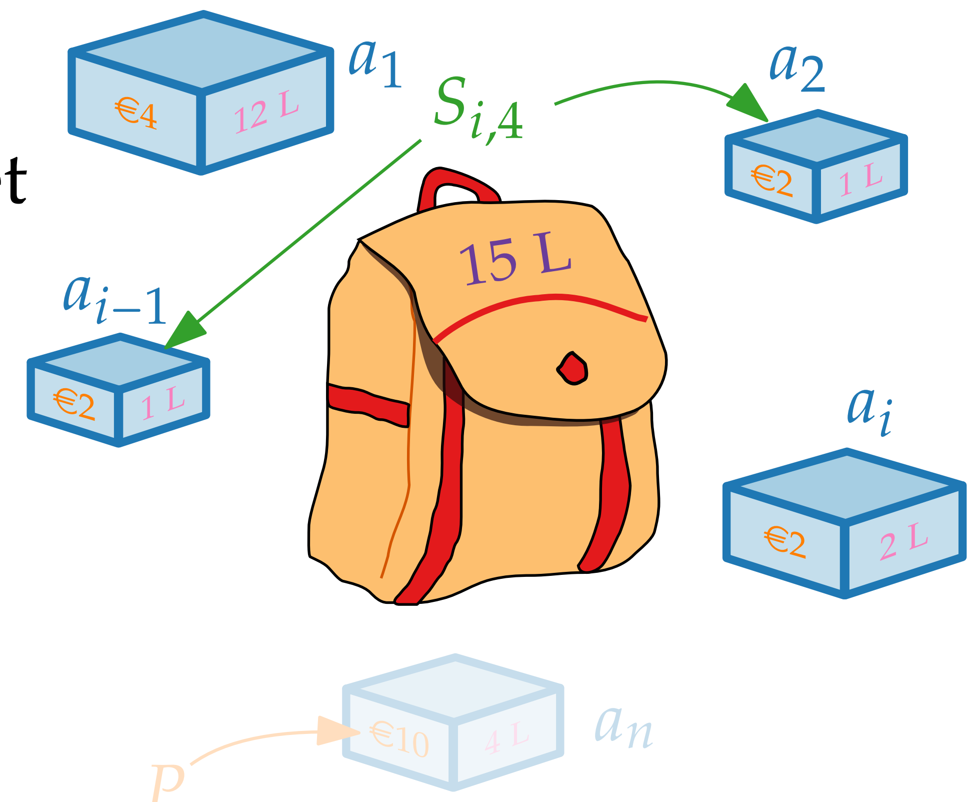

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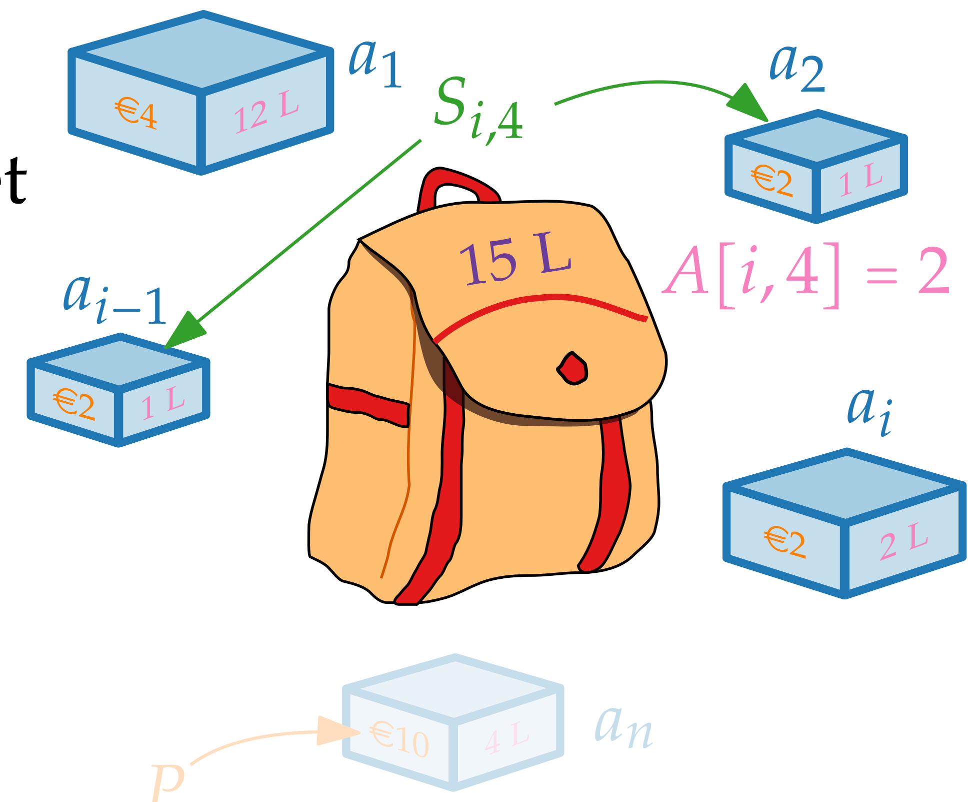

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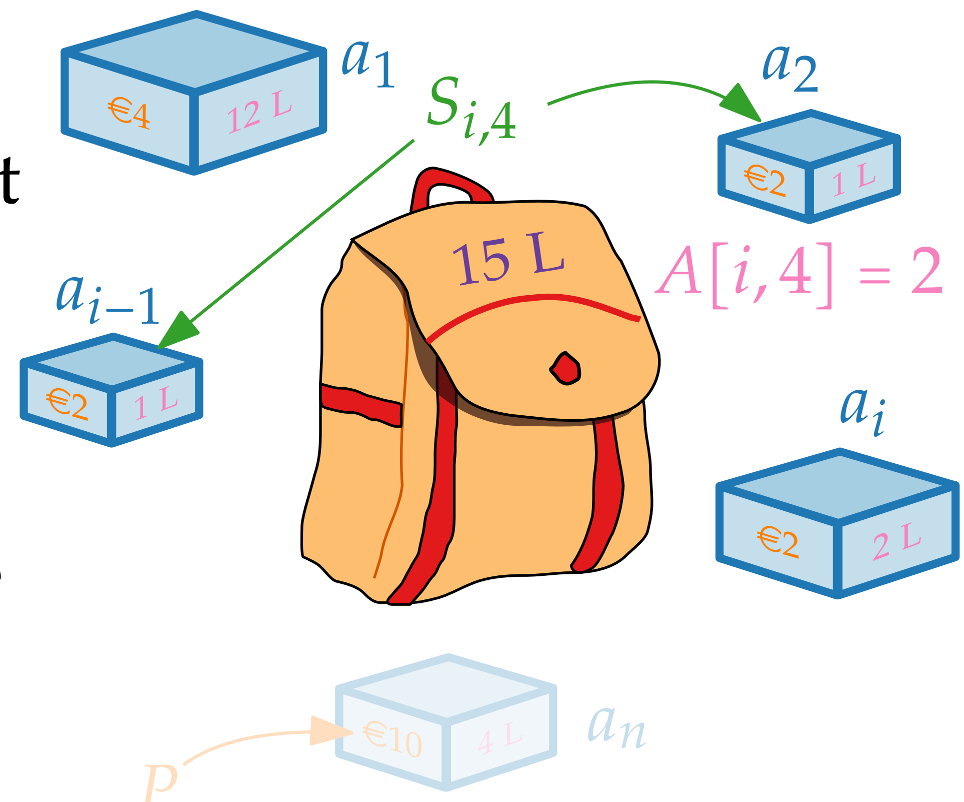

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If all $A[i,p]$ are known, then we can compute

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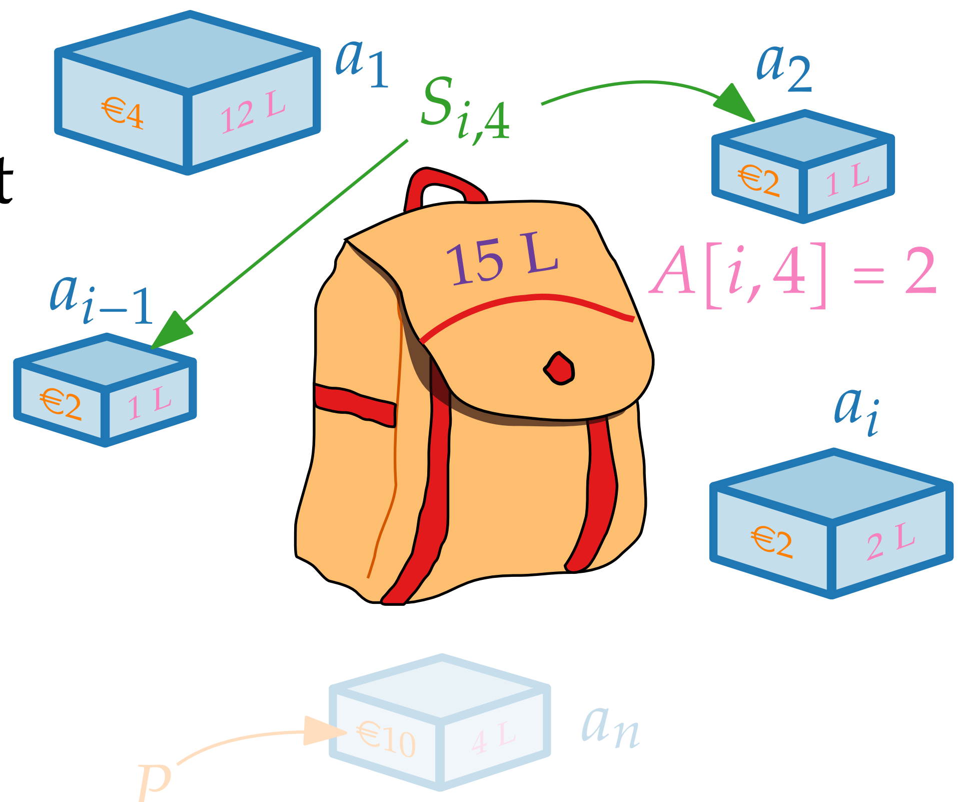

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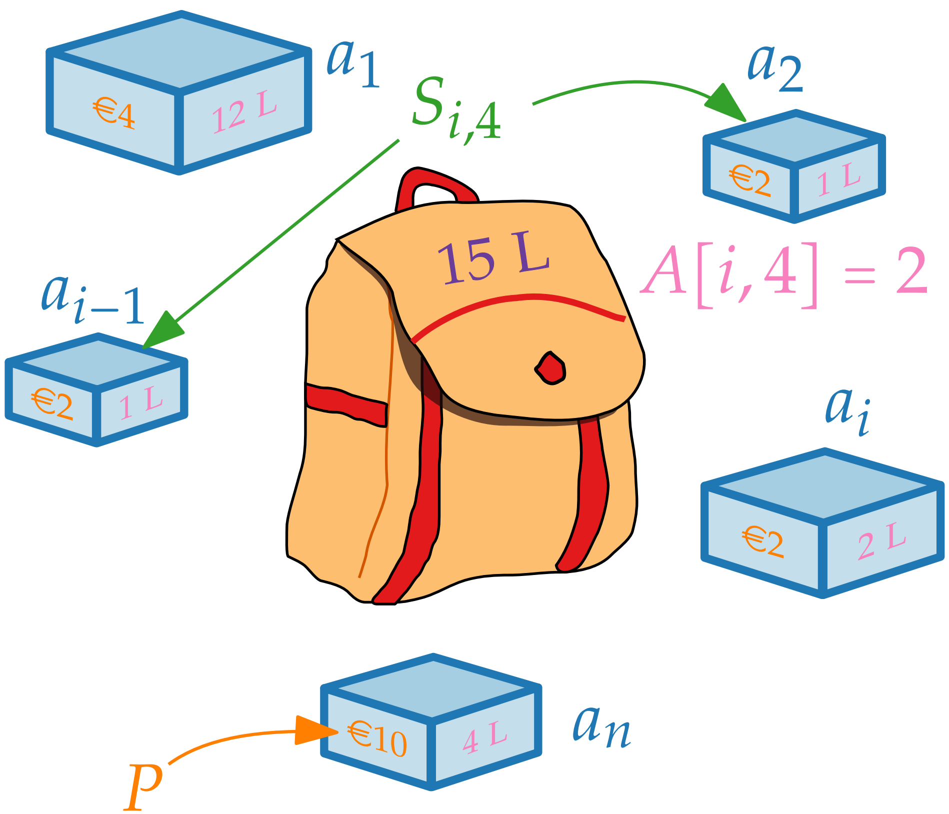
If all $A[i,p]$ are known, then we can compute $\text{OPT} = \max\{\, p \mid A[n,p] \leq B \,\}$.

Pseudo-Polynomial Alg. for KNAPSACK

$A[1, p]$ can be computed for all $p \in \{0, \dots, nP\}$.



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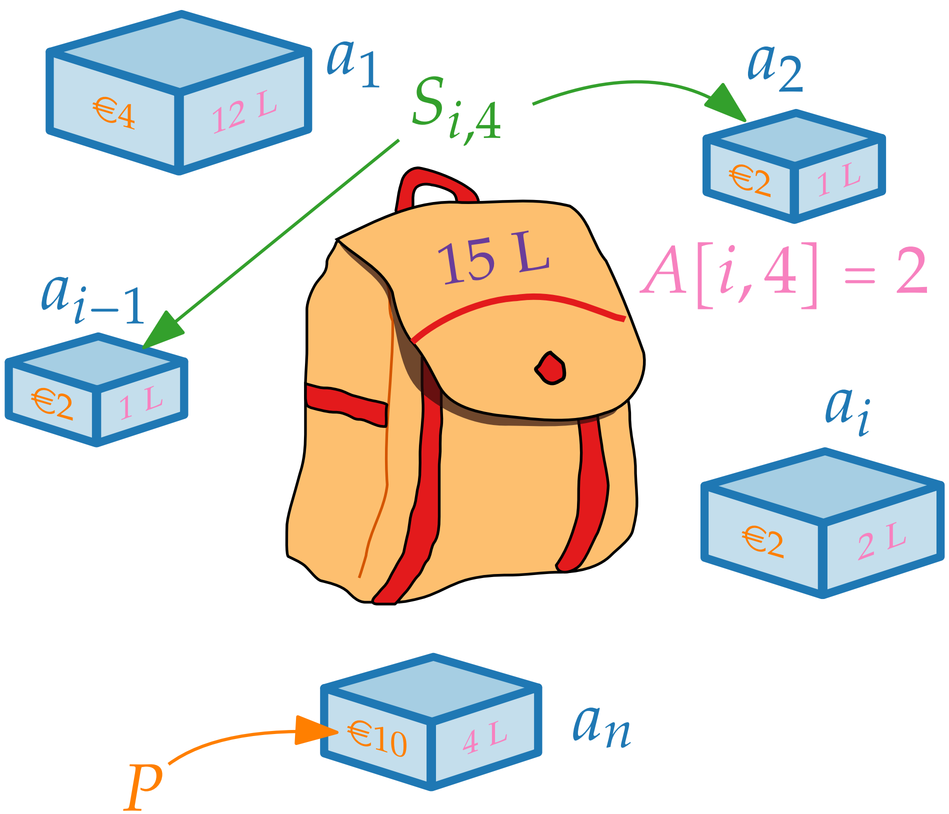

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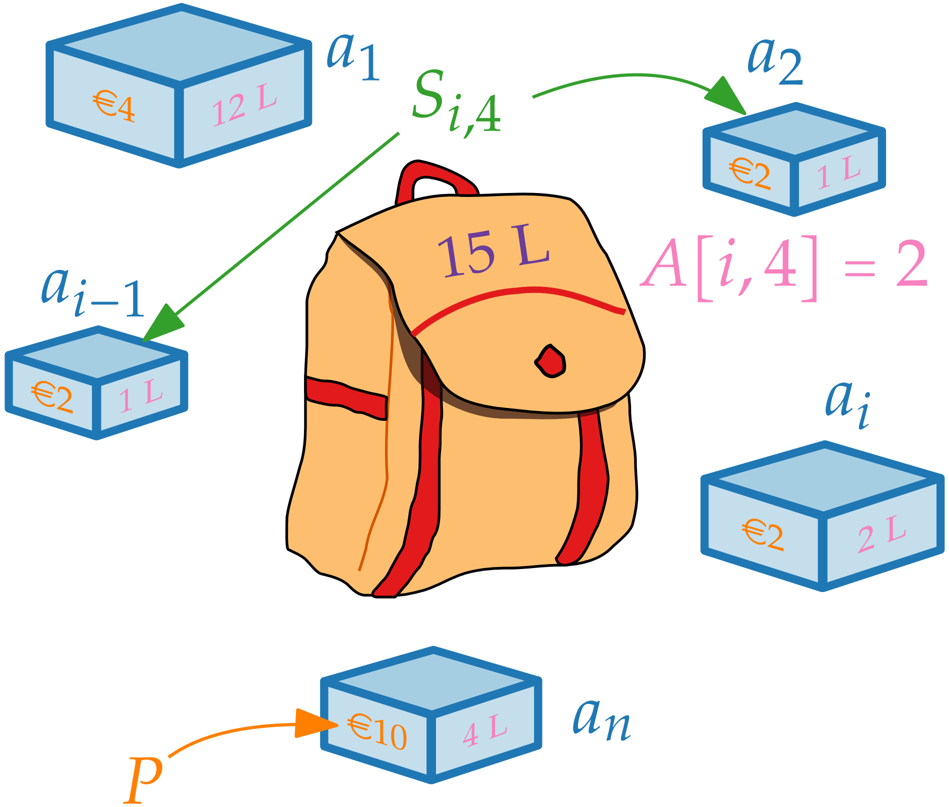
$A[i + 1, p] =$



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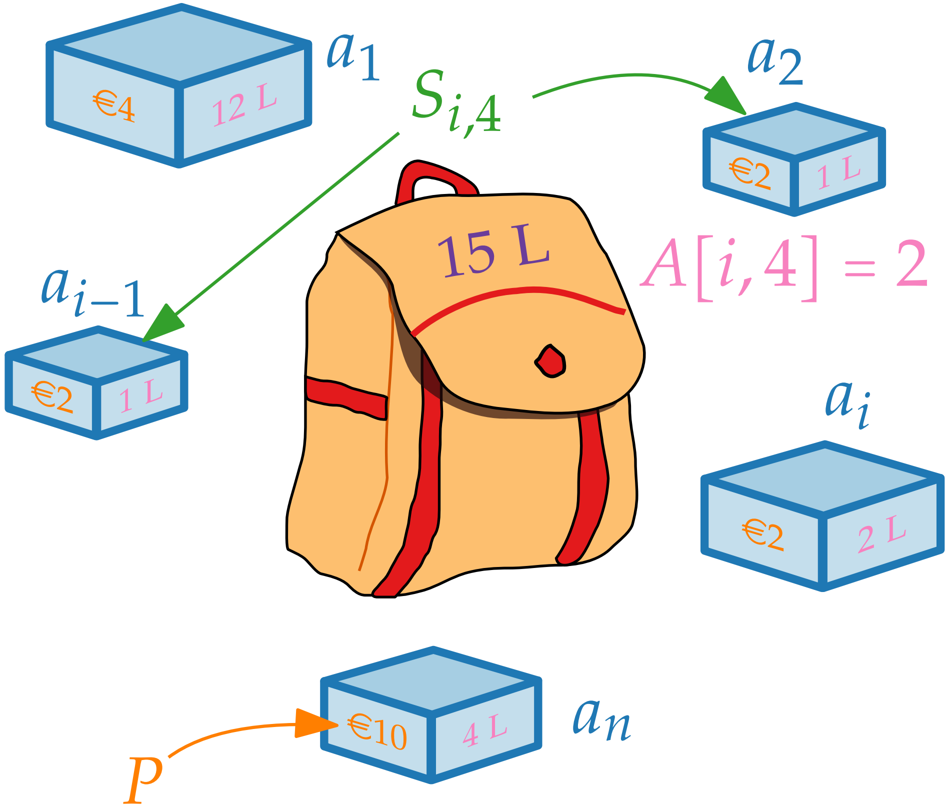

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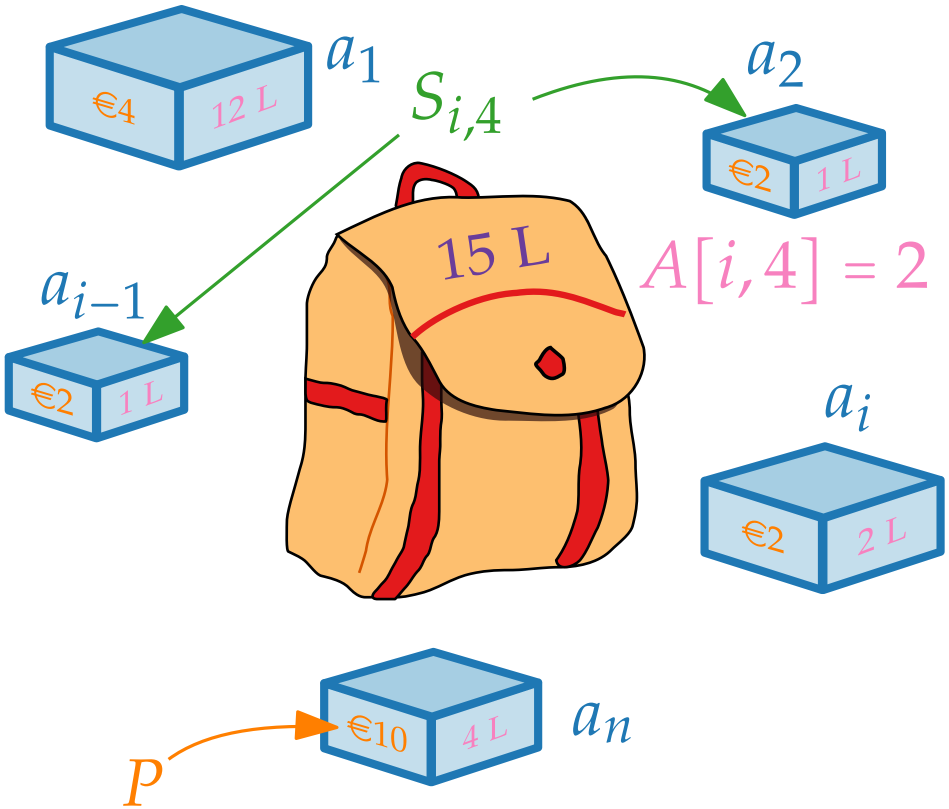

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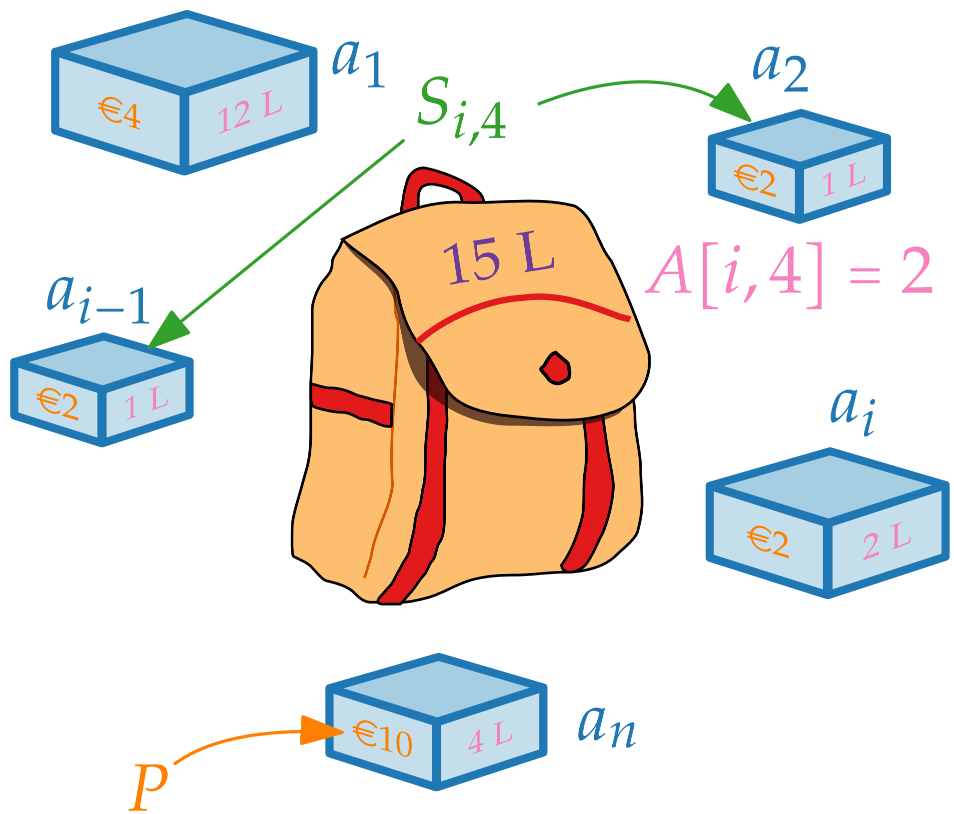
Set $A[i,p] := \infty$ for $p < 0$

$A[i+1,p] = \min\{A[i,p], \mathrm{size}(a_{i+1}) + \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \}$



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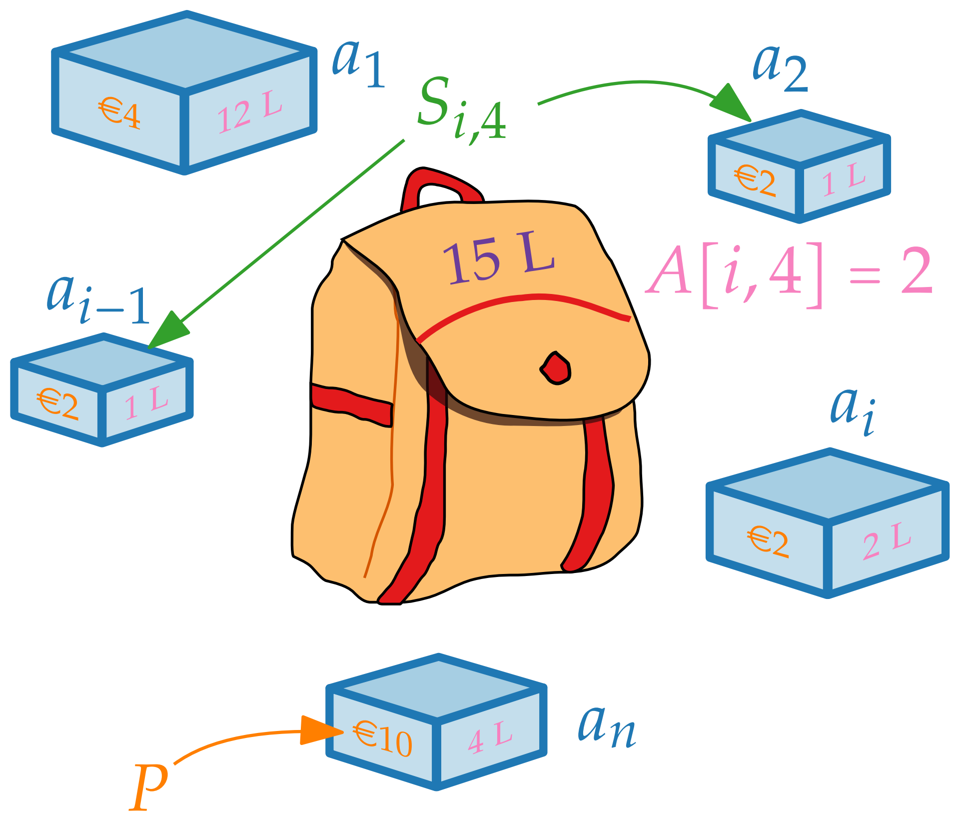

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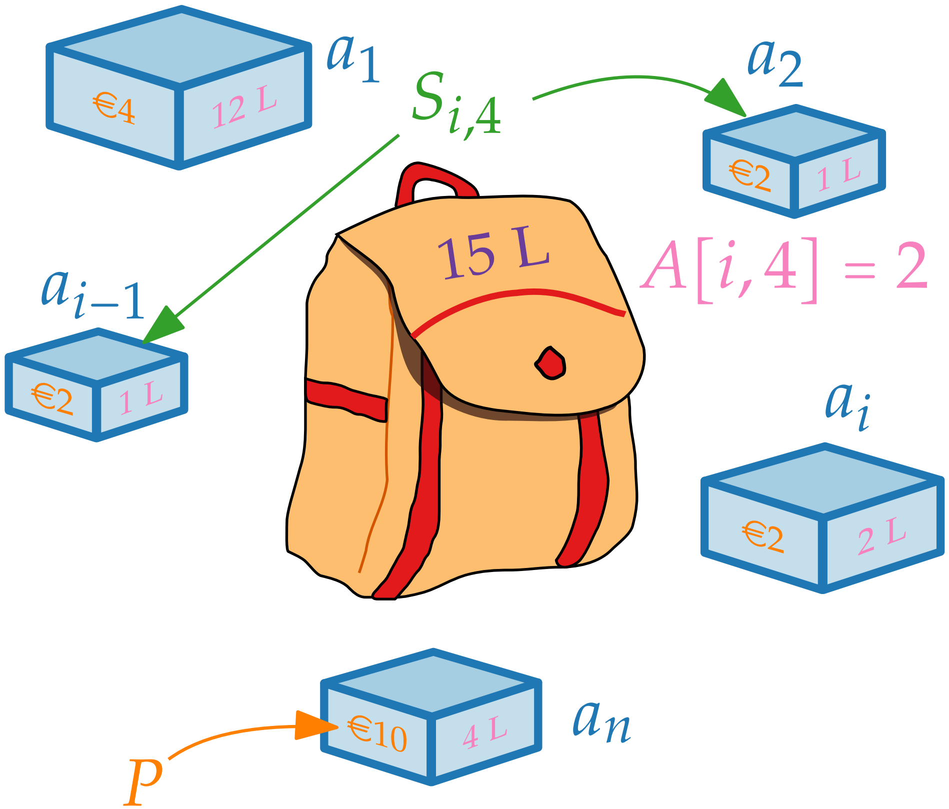
$\Rightarrow$ All values $A[i,p]$ can be computed in total time $O(n^2 P)$.



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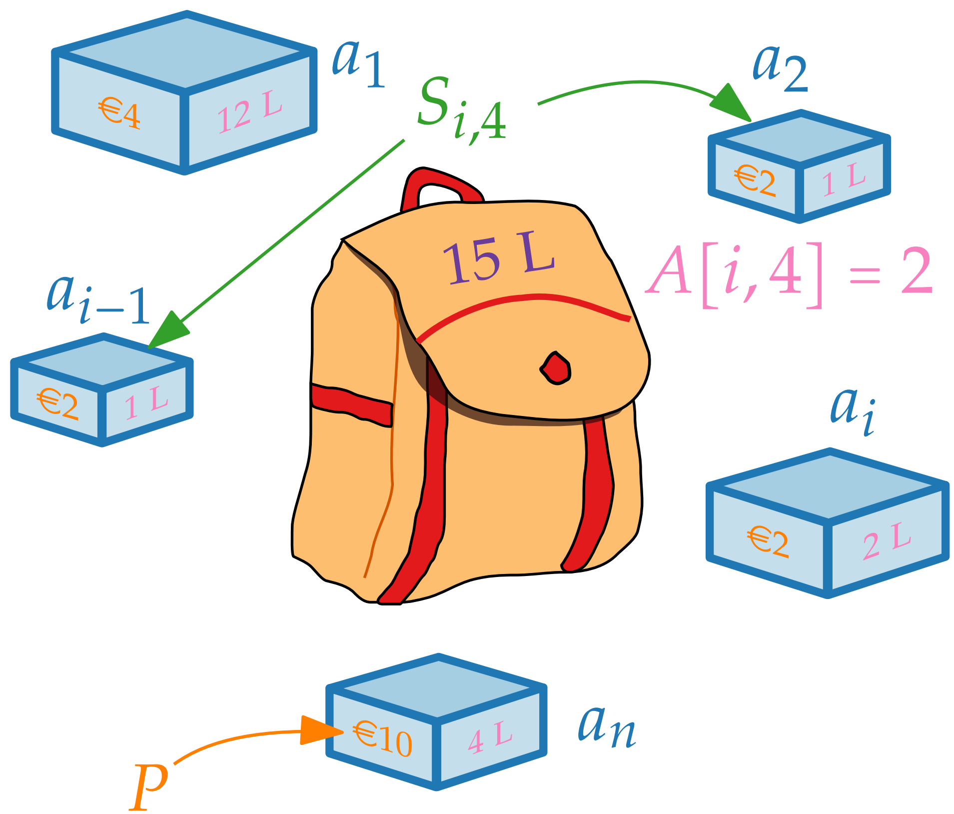
$\Rightarrow$ All values $A[i,p]$ can be computed in total time $O(n^2 P)$.

$\Rightarrow$ OPT can be computed in $O(n^2 P)$ time.



Pseudo-Polynomial Alg. for Knapsack

$A[1,p]$ can be computed for all $p \in \{0,\dots,nP\}$.

Set $A[i,p] := \infty$ for $p < 0$

$A[i+1,p] = \min\{A[i,p], \mathrm{size}(a_{i+1}) + A[i, p - \mathrm{profit}(a_{i+1})]\}$

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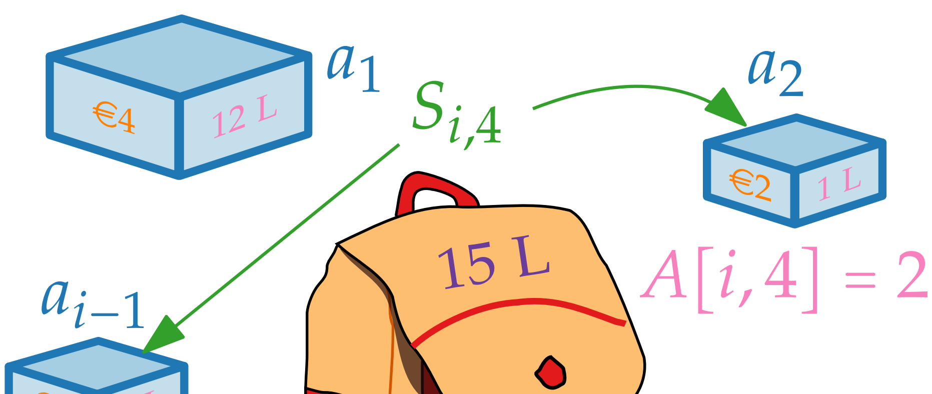
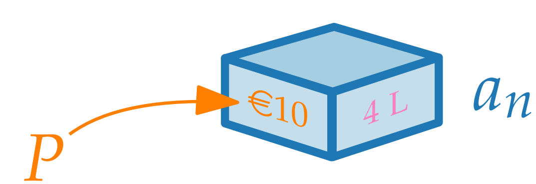


Theorem. Knapsack can be solved optimally in pseudo-polynomial time $O(n^2P)$.



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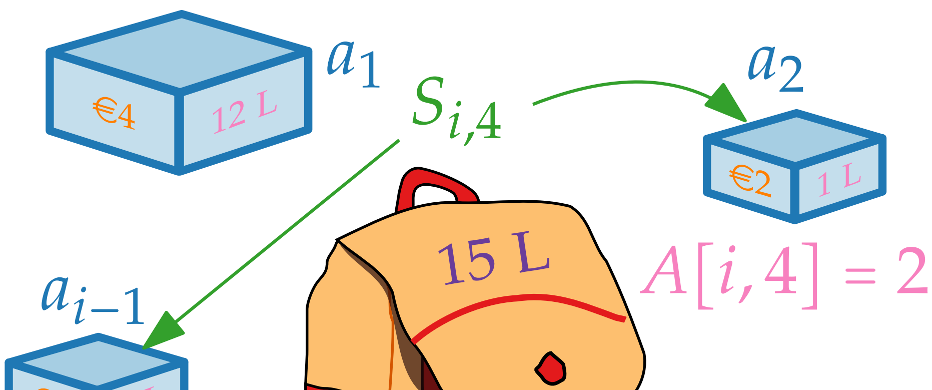

Theorem. Knapsack can be solved optimally in pseudo-polynomial time $O(n^2P)$.

Corollary. Knapsack is weakly NP-hard.

Pseudo-Polynomial Alg. for KNAPSACK

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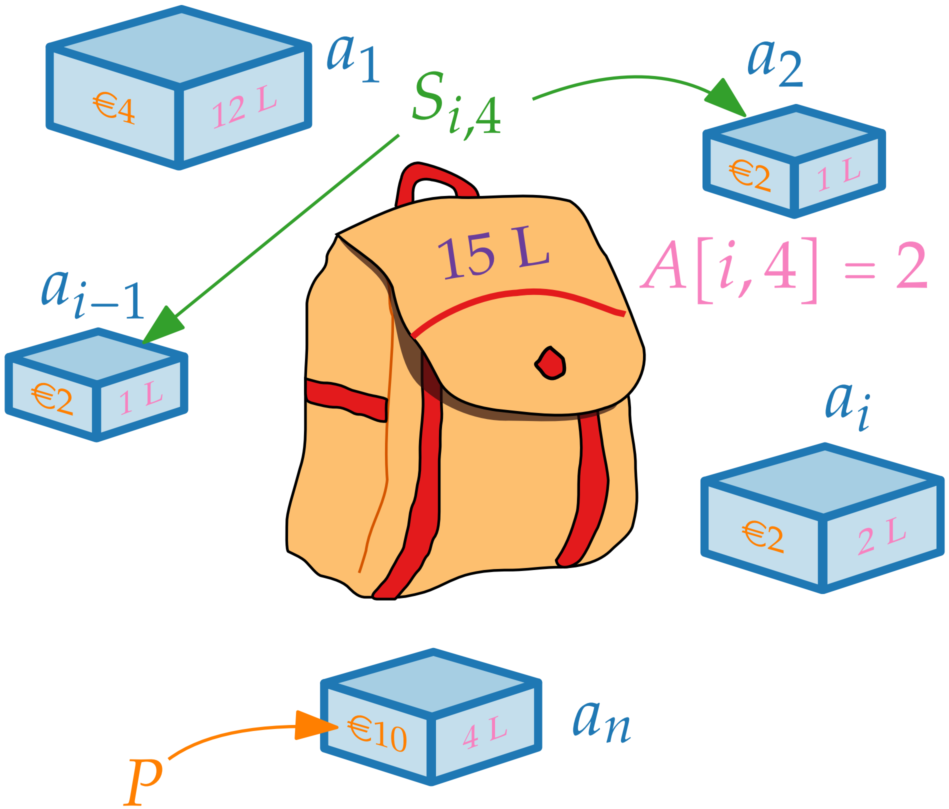

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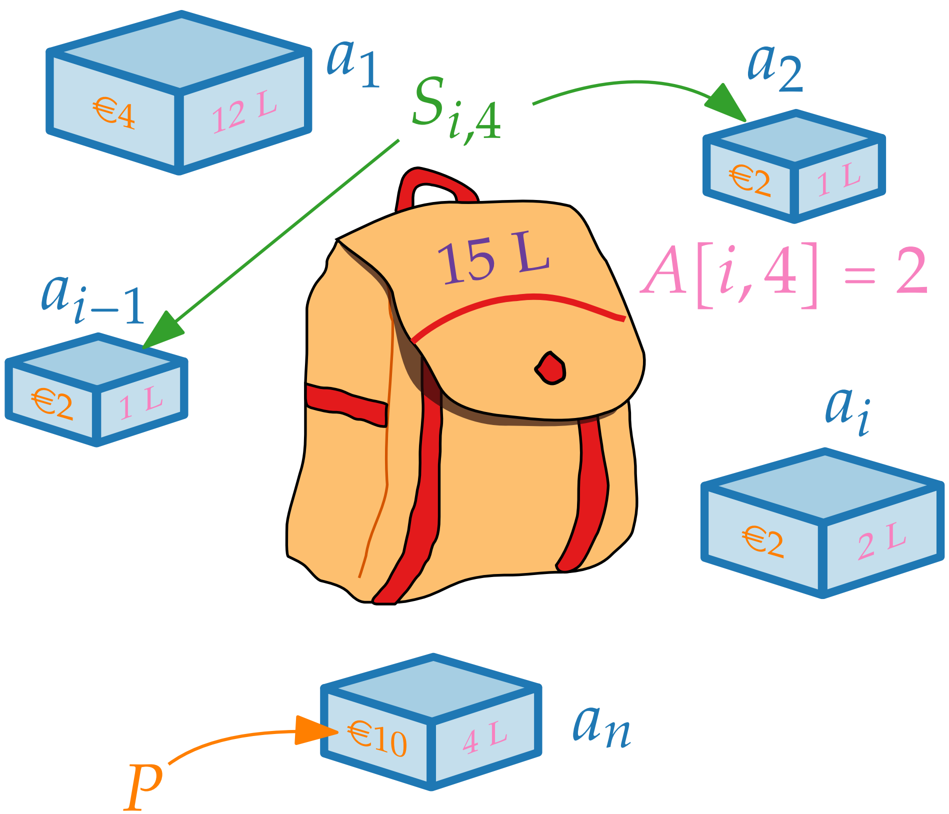
Examples. $P = n^5$



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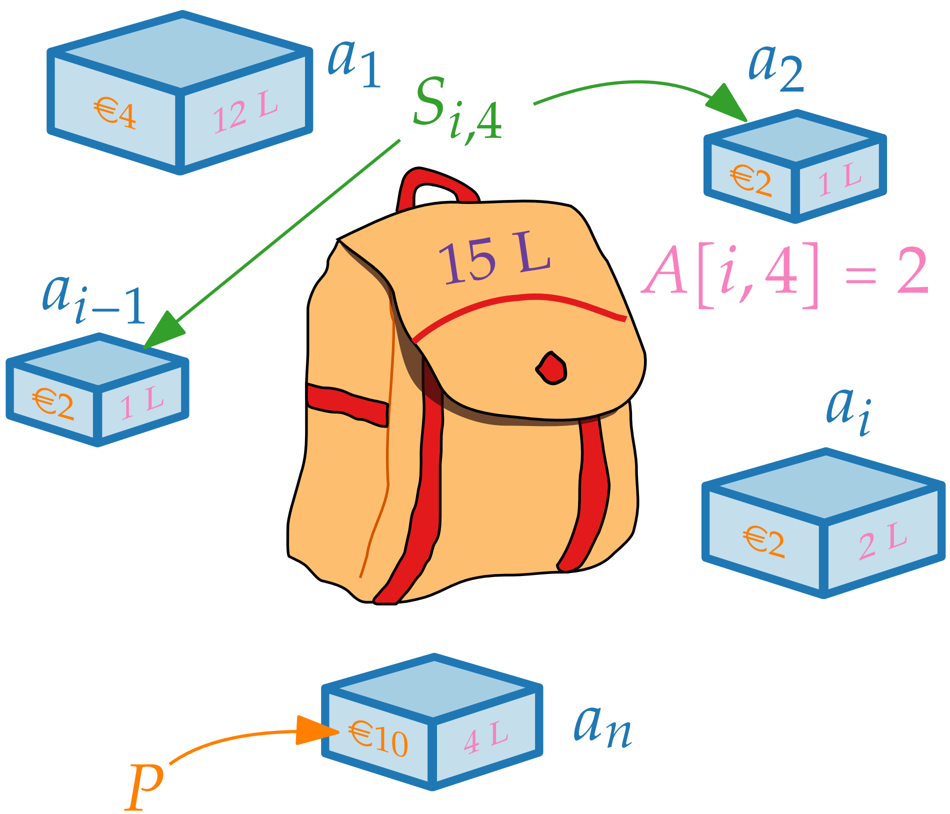
Examples. $P = n^5$ $\Rightarrow$ running time $O(n^7)$



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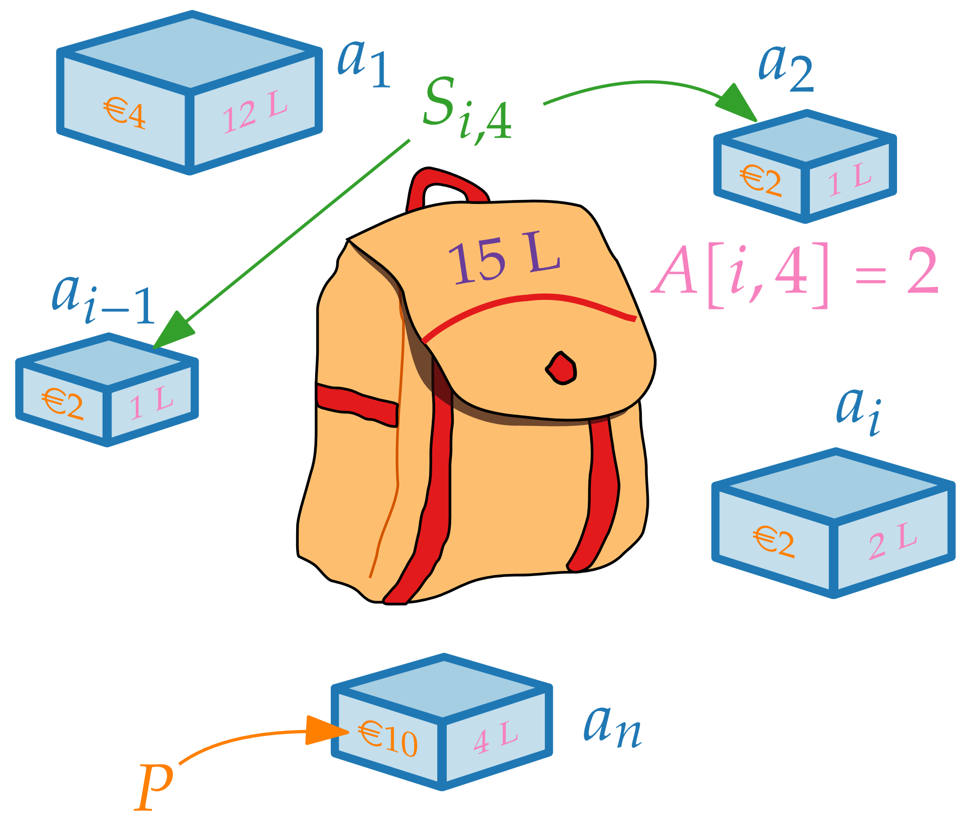

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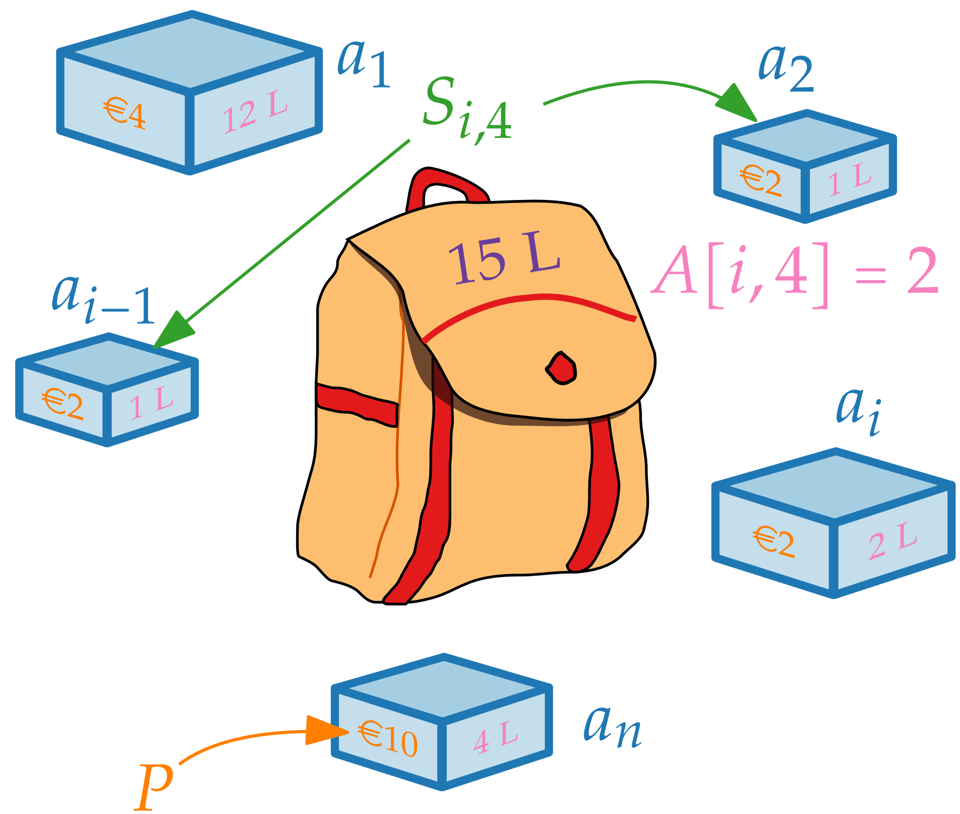

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Observe. Running time $O(n^2 P)$ poly in n if P poly in n.

Approximation Algorithms

Lecture 8:
Approximation Schemes and the Knapsack Problem

Part IV:
Approximation Schemes

Joachim Spoerhase

Winter 2021/22

Approximation Schemes

Let Π be an optimization problem.

Approximation Schemes

Let Π be an optimization problem. An algorithm $\mathcal{A}$ is called a **polynomial-time approximation scheme** (PTAS), if it outputs for every input (I, ε) with $I \in D_\Pi$ and $\varepsilon > 0$ a solution $s \in S_\Pi(I)$ such that the following holds:

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Approximation Algorithms

Lecture 8:
Approximation Schemes and the KNAPSACK Problem

Part V:
FPTAS for KNAPSACK

Joachim Spoerhase

Winter 2021/22

FPTAS for KNAPSACK via Scaling

FPTAS idea: **Scale** profits to polynomial size (as required by the error parameter ε)...

FPTAS for Knapsack via Scaling

KnapsackScaling (I, ε)

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Theorem. KnapsackScaling is an FPTAS for Knapsack with running time $O(n^3/\varepsilon) = O\left(n^2 \cdot \frac{P}{\varepsilon P/n}\right)$.

Approximation Algorithms

Lecture 8: Approximation Schemes and the Knapsack Problem

Part VI: Connections

Joachim Spoerhase

Winter 2021/22

FPTAS and Pseudo-Poly. Algorithms

Theorem. Let p be a polynomial and let Π be an NP-hard minimization problem

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Theorem. Let p be a polynomial and let Π be an NP-hard minimization problem with integral objective function

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Corollary. Let Π be an NP-hard optimization problem that fulfils the restrictions above. If Π is strongly NP-hard, then there is no FPTAS for Π (unless P = NP).