Approximation Algorithms

Lecture 7: Scheduling Jobs on Parallel Machines

Part I: ILP & Parametric Pruning

Joachim Spoerhase

Winter 2020/21

Scheduling on Parallel Machines

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$$\mathcal{J} = \{J_1, J_2, \ldots, J_8\}$$

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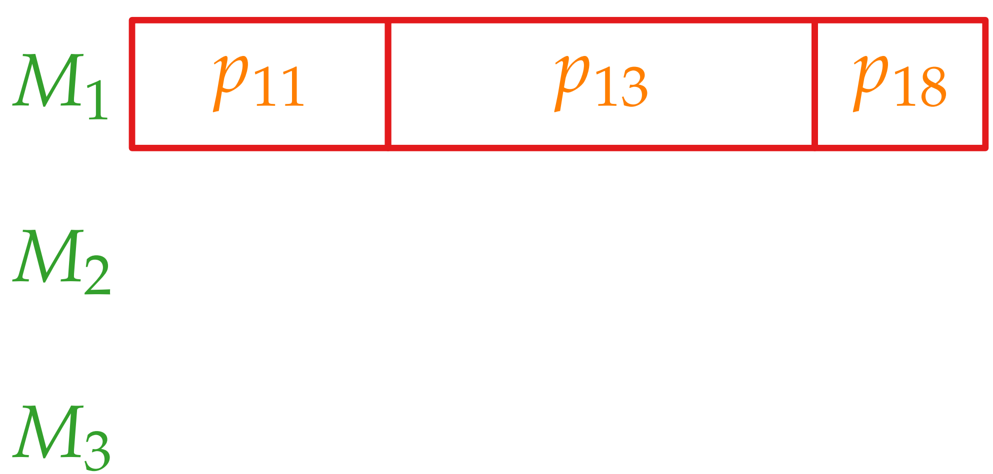

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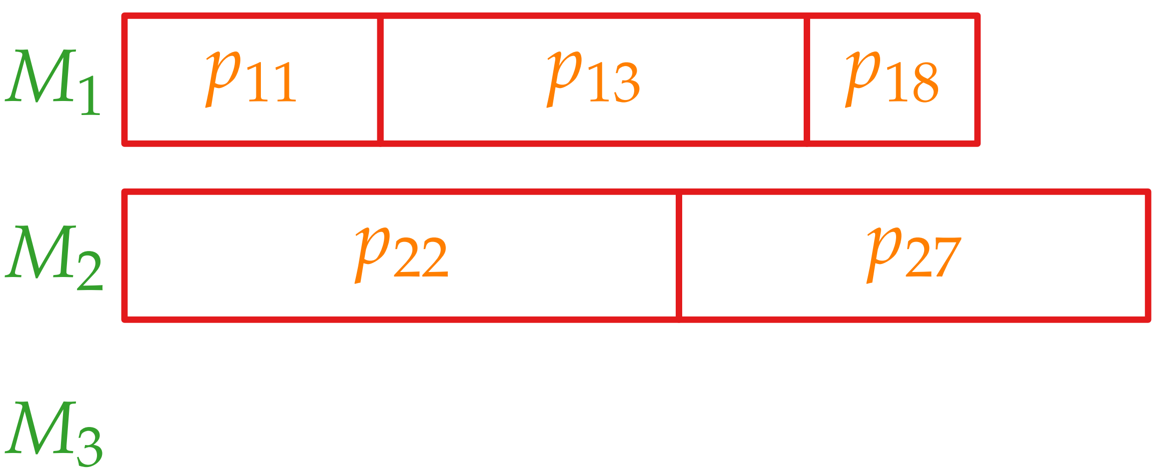

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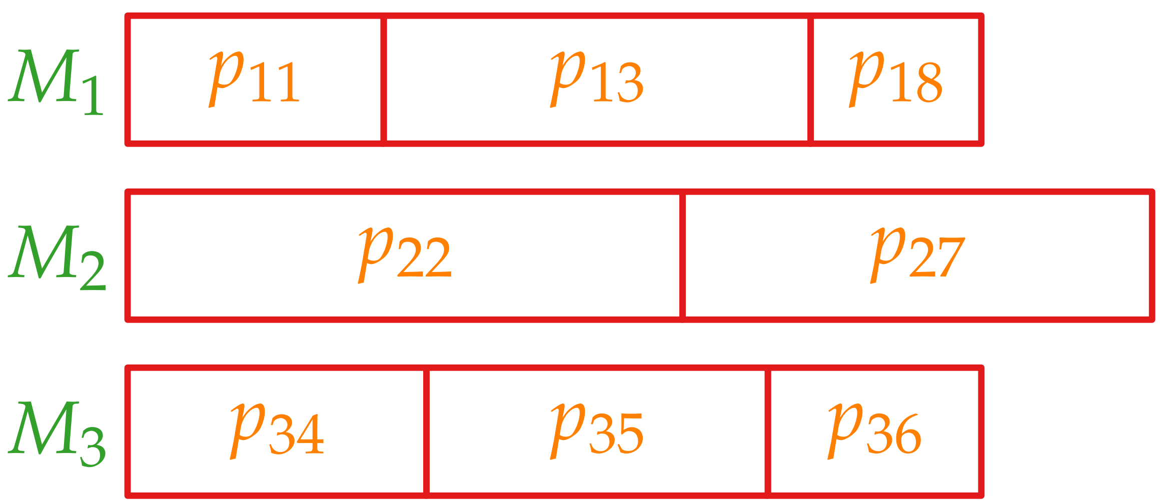

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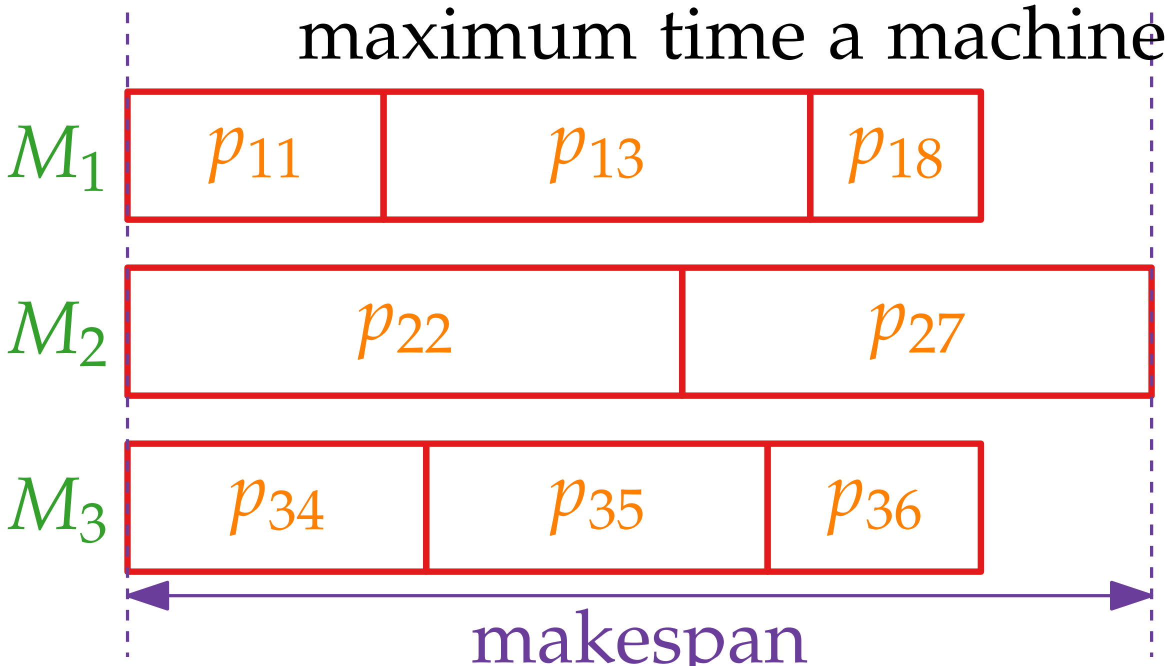

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Formulation as ILP

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Task: Prove that the integrality gap is unbounded!

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$\Rightarrow$ OPT $= m$ and $\text{OPT}_{\text{frac}} = 1$.

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If $p_{ij} > t$, then set $x_{ij} = 0$.

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Define the "pruned" relaxation LP(T):

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But why does this LP give a good integrality gap?

Approximation Algorithms

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Scheduling Jobs on Parallel Machines

Part II:
Properties of Extreme Point Solutions

Joachim Spoerhase

Winter 2020/21

Properties of Extreme Point Solutions

Use binary search to find the smallest T so that LP(T) has a solution.

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Lemma 1.
Each extreme point solution for LP(T) has $\leq |\mathcal{M}| + |\mathcal{J}|$ pos. variables.

Lemma 2.
Any extreme point solution for LP(T) must set $\geq |\mathcal{J}| - |\mathcal{M}|$ jobs integrally.

Lemma 1

$$\sum_{(i,j)\in S_T} x_{ij} = 1, \qquad J_j \in \mathcal{J}$$
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Approximation Algorithms

Lecture 7:
Scheduling Jobs on Parallel Machines

Part III:
An Algorithm

Joachim Spoerhase

Winter 2020/21

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Why is that useful . . . ?

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Approximation Algorithms

Lecture 7:
Scheduling Jobs on Parallel Machines

Part IV:
Pseudo-Trees and -Forests

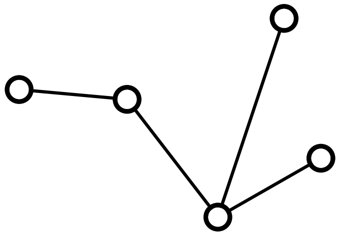
Joachim Spoerhase

Winter 2020/21

Pseudo-Trees and -Forests

Pseudo-Tree: A connected graph $G = (V, E)$ with at most $|V|$ edges.

Pseudo-Trees and -Forests



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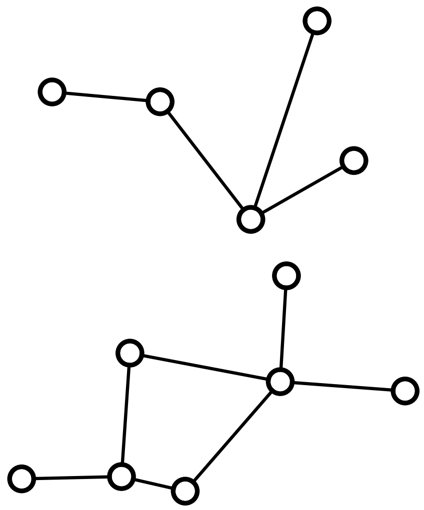

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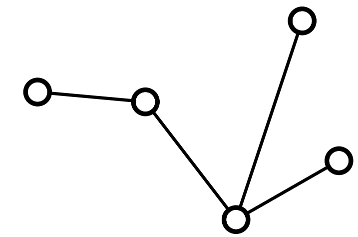

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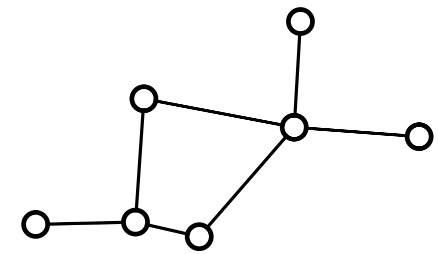

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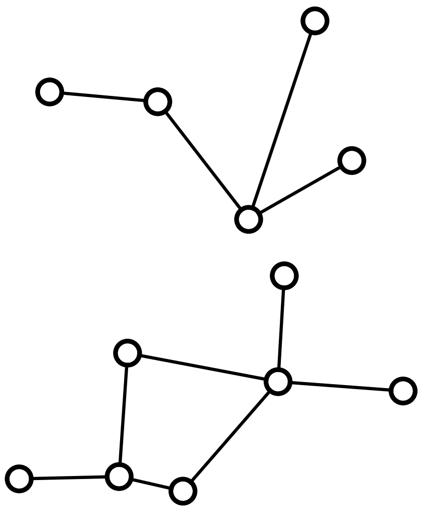
Pseudo-Trees and -Forests



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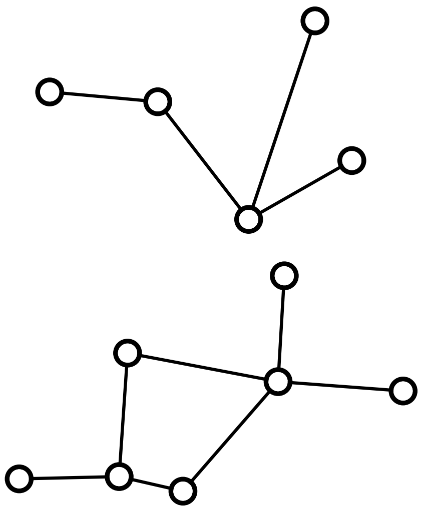

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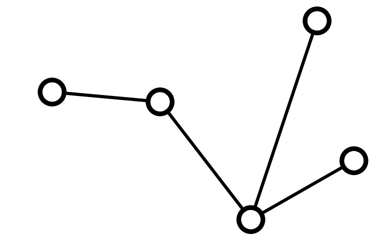

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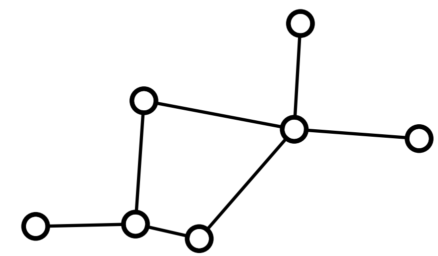

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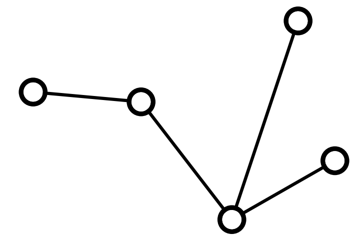

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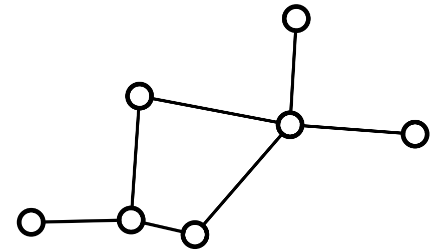

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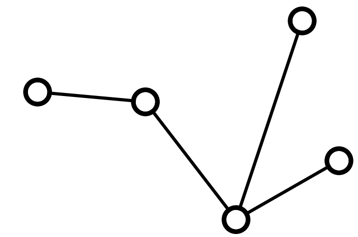

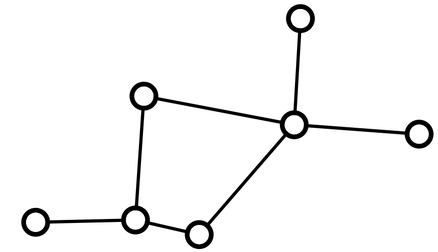

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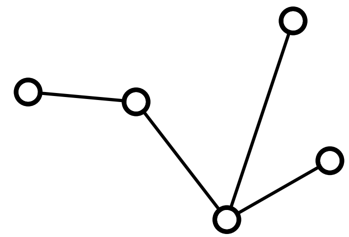
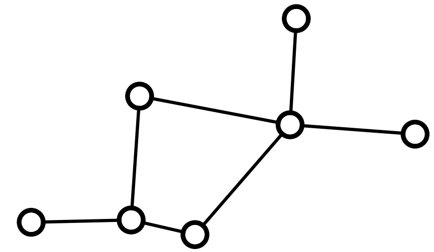
H is also a pseudo-forest: remove 1 edge per $v \in \mathcal{J} \setminus F$

Pseudo-Trees and -Forests





Pseudo-Tree: A connected graph $G = (V, E)$ with at most $|V|$ edges.

A pseudo-tree is either a tree or a tree plus a single edge.

Pseudo-Forest: Collection of disjoint pseudo-trees.

Lemma 3.
The bipartite graph $G = (\mathcal{M} \cup \mathcal{J}, E)$ is a pseudo-forest.

Extreme point solutions have $\leq |\mathcal{M}| + |\mathcal{J}|$ variables > 0 (L1).

Each component of G corresponds to an extreme point solution.

Lemma 4.
The graph H has an F-perfect matching.

H is also a pseudo-forest: remove 1 edge per $v \in \mathcal{J} \setminus F$
Vertices in F have min. degree 2.

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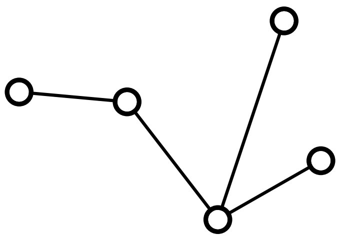

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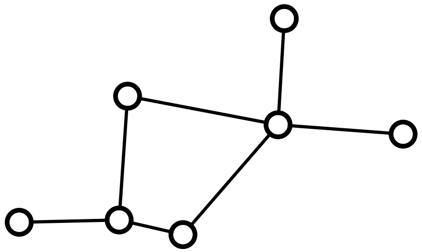

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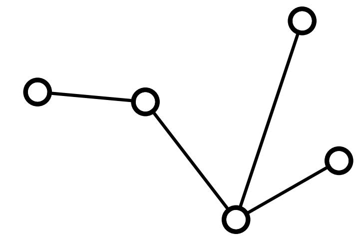

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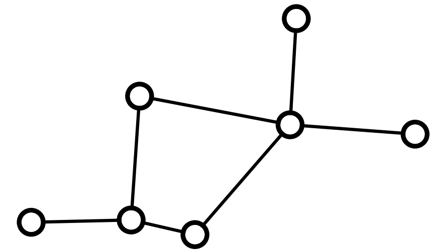

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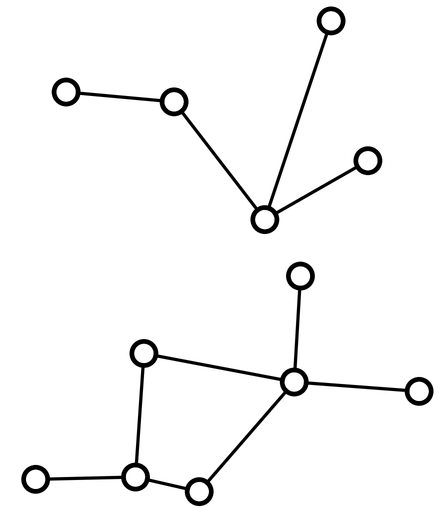

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After iteratively picking all leaves, only even cycles remain.

Scheduling on Parallel Machines

Theorem. There is an LP-based 2-approximation algorithm for the problem of scheduling jobs on unrelated parallel machines.

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(Machines have different speed, but process jobs uniformly.)