Approximation Algorithms

Lecture 6:
k-Center via Parametric Pruning

Part I:
Metric-k-Center

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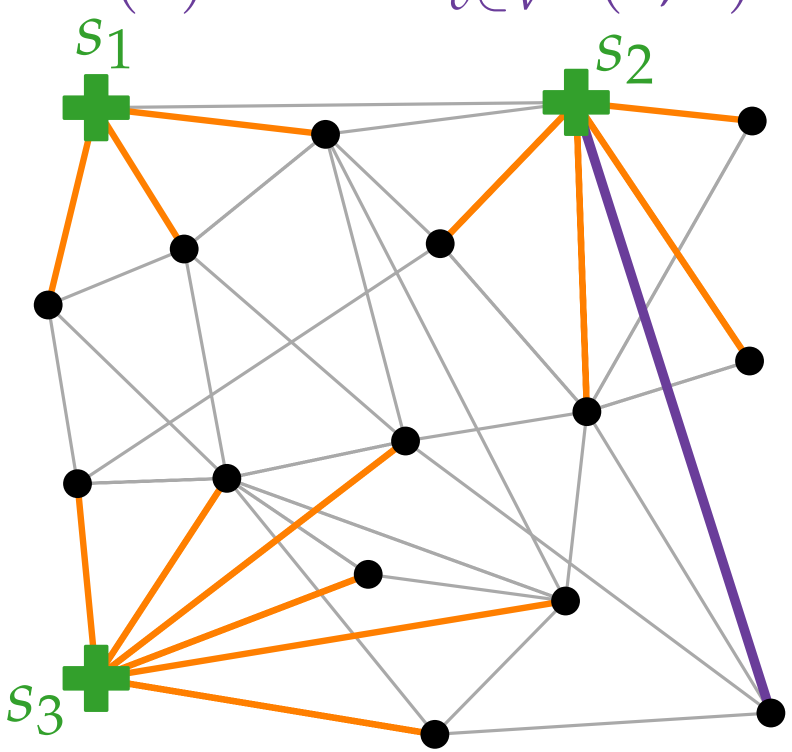
Winter 2021/22

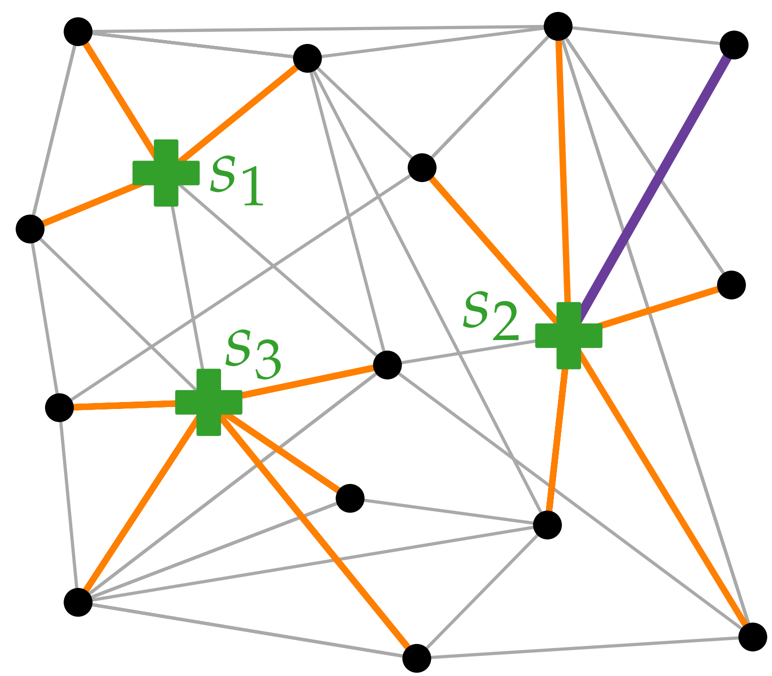
Metric-k-Center

Given: A complete graph $G = (V, E)$ with edge costs $c\colon E \to \mathbb{Q}_{\geq 0}$ satisfying the triangle inequality and a natural number $k \leq |V|$.

For each vertex set $S \subseteq V$, $c(v, S)$ is the cost of the cheapest edge from v to a vertex in S.

Find: A k-element vertex set S, such that $\text{cost}(S) := \max_{v \in V} c(v, S)$ is minimized.





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Part II:
Parametric Pruning

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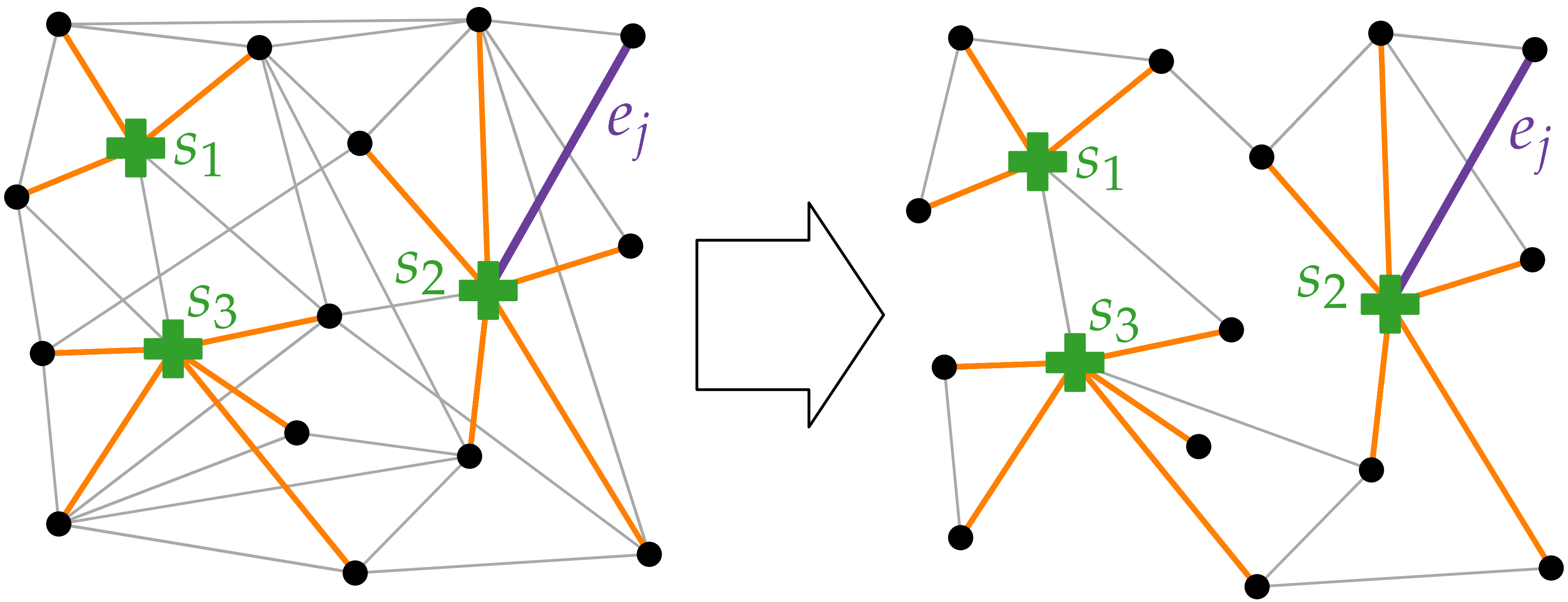
Parametric Pruning

Let $E = \{e_1, \ldots, e_m\}$ with $c(e_1) \leq \ldots \leq c(e_m)$.
Suppose we know that $\text{OPT} = c(e_j)$.

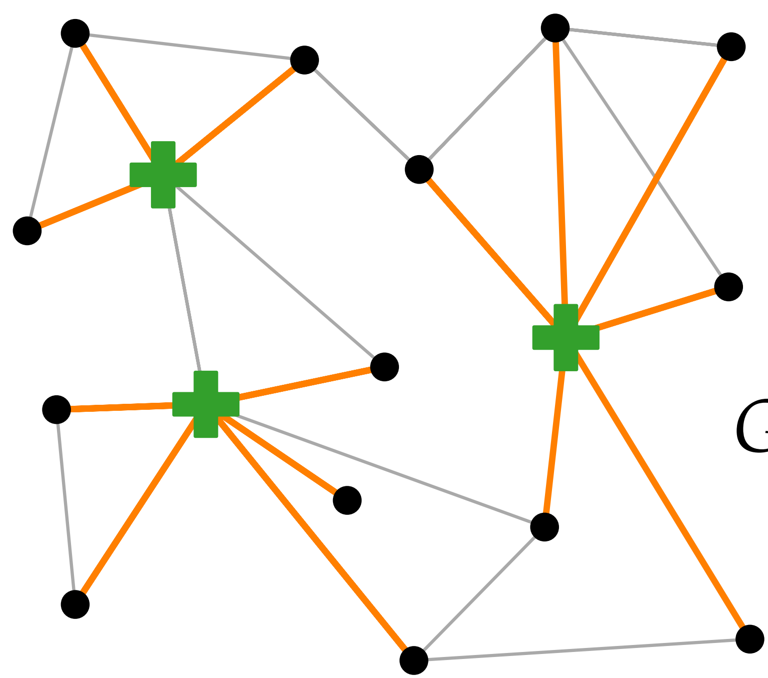


G $\quad$ $G_j := (V, \{e_1, \ldots, e_j\})$

…try each G_j.

$\ldots$ try each G_j.

Def. A vertex set D of a graph H is **dominating** if each vertex is either in D or adjacent to a vertex in D. The cardinality of a smallest dominating set in H is denoted by $\operatorname{dom}(H)$.



$$\operatorname{dom}(G_j) \leq k$$

$$G_j := (V, \{e_1, \ldots, e_j\})$$

$\ldots$ but computing $\operatorname{dom}(H)$ is NP-hard.

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Part III:
Square of a Graph

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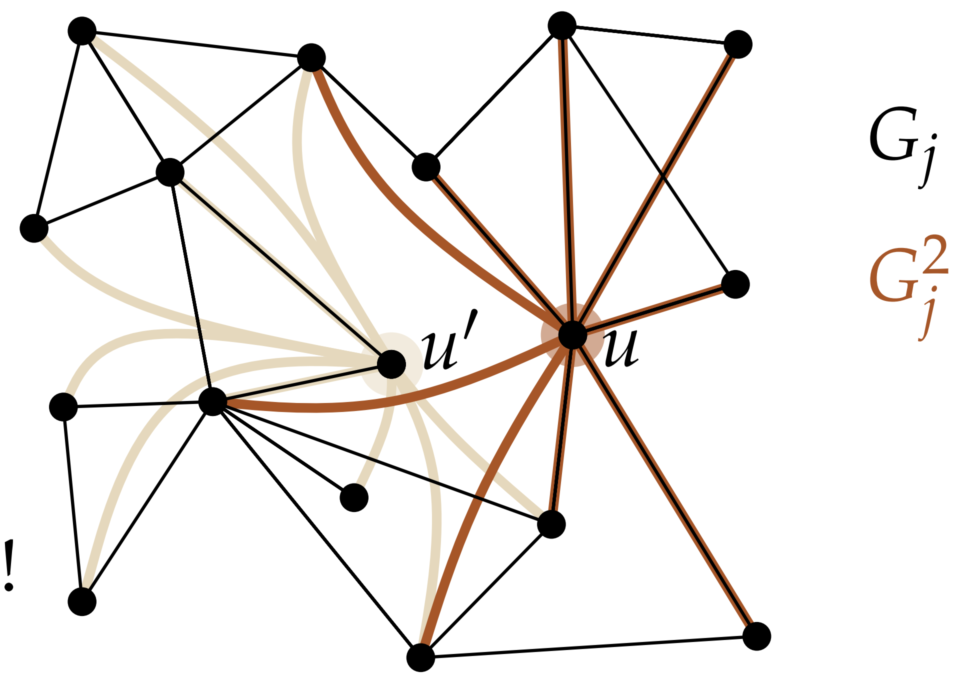
Square of a Graph

Idea: Find a small dominating set in a "coarsened" G_j

Def. The **square** H^2 of a graph H has the same vertex set as H. Additionally, two vertices $u \neq v$ are adjacent in H^2 iff they are within distance at most **two** in H.

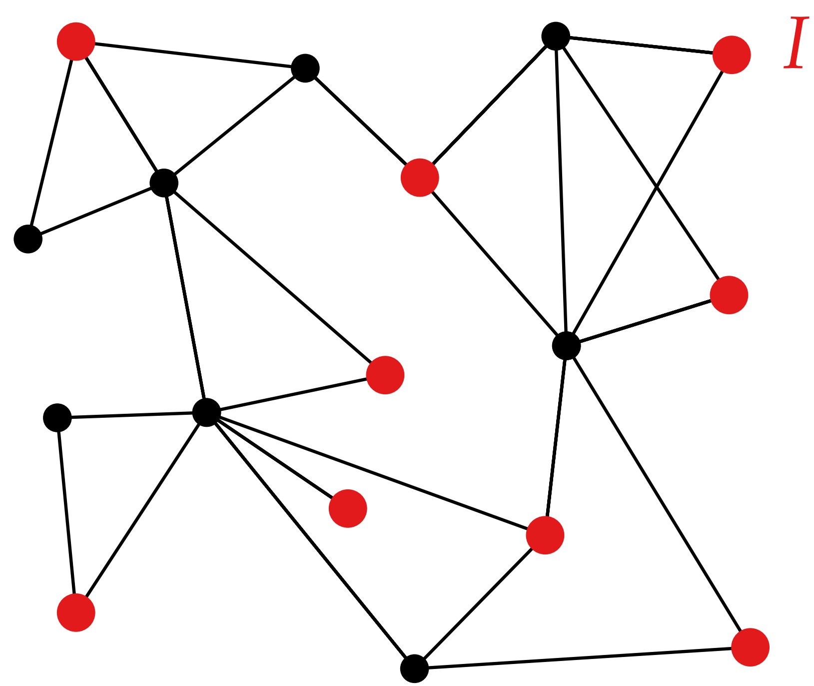
Obs. A dominating set in G_j^2 with $\leq k$ elements is already a 2-approximation.

Why? $\max_{e \in E(G_j)} c(e) = \text{OPT}$!



Independent Sets

Def. A vertex set I in a graph is called **independent** (or **stable**), if no pair of vertices in I form an edge. An independent set is called **maximal** when no superset of it is an independent set.

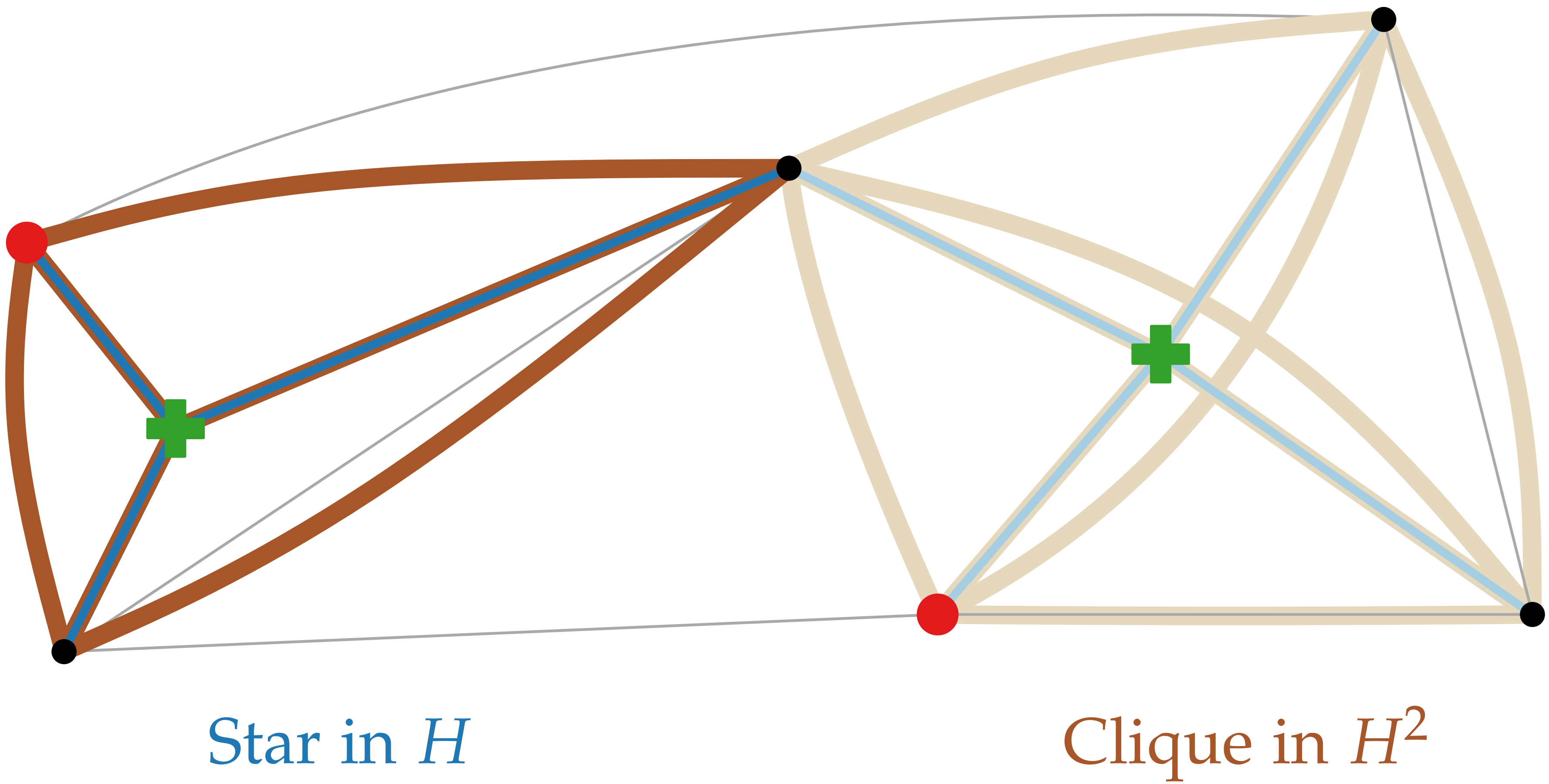


Obs. Maximal independent sets are dominating sets :-)

Independent Sets in H^2

Lemma. For a graph H and an independent set I in H^2, $|I| \leq \mathrm{dom}(H)$.

What does a dominating set of H look like in H^2?



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Part IV:
Factor-2-Approximation for Metric-k-Center

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Factor-2-Approx for Metric-k-Center

Metric-k-Center($G = (V, E; c), k$)

Sort the edges of G by cost: $c(e_1) \leq \ldots \leq c(e_m)$

for $j = 1, \ldots, m$ **do**

Construct G_j^2

Find a maximal independent set I_j in G_j^2

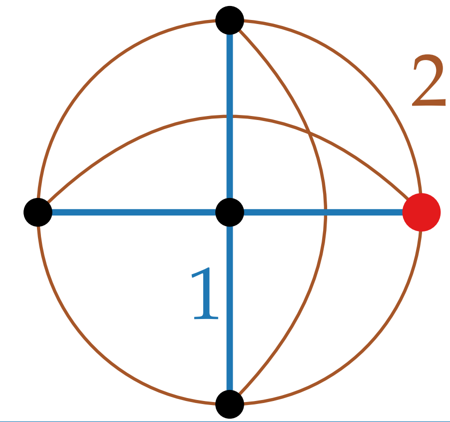
if $|I_j| \leq k$ **then**

return I_j

Lemma. For j provided by the algorithm, we have $c(e_j) \leq \text{OPT}$.

Theorem. The above algorithm is a factor-2-approximation algorithm for Metric-k-Center problem.

Can we do better . . . ?

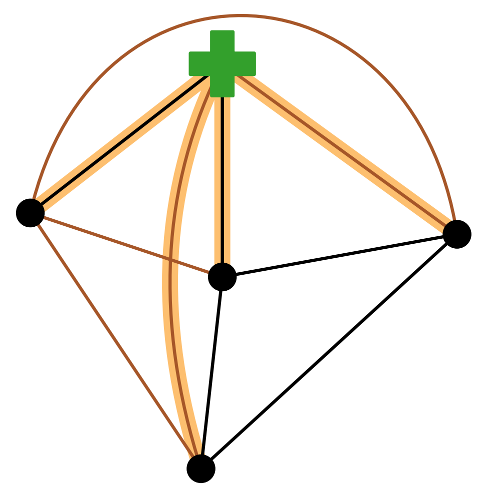


What about a tight example?

Theorem. Assuming P $\neq$ NP, there is no factor-$(2-\varepsilon)$ approximation algorithm for the metric k-Center problem, for any $\varepsilon > 0$.

Proof. Reduce from dominating set to metric k-Center.
Given.: $G = (V, E)$, k
Constr. complete graph $G' = (V, E \cup E')$
with $c(e) = \begin{cases} 1, & \text{if } e \in E \\ 2, & \text{if } e \in E' \end{cases}$

$\triangle$-inequality holds



S: metric k-Center
If $\text{dom}(G) \leq k$, then $\text{cost}(S) = 1$
If $\text{dom}(G) > k$, then $\text{cost}(S) = 2$

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Part V:
Metric-Weighted-Center

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Metric-~~k~~-Center

Weighted

Given: A complete graph $G = (V, E)$ with metric edge costs $c\colon E \to \mathbb{Q}_{\geq 0}$ and ~~a natural number $k \leq |V|$.~~ , vertex weights $w\colon V \to \mathbb{Q}_{\geq 0}$ and a budget $W \in \mathbb{Q}_{+}$

For each vertex set $S \subseteq V$, $c(v, S)$ is the cost of the cheapest edge from v to the a vertex in S.

vertex set S of weight at most W

Find: A ~~k-element vertex set S~~, such that $\mathrm{cost}(S) := \max_{v \in V} c(v, S)$ is minimized.

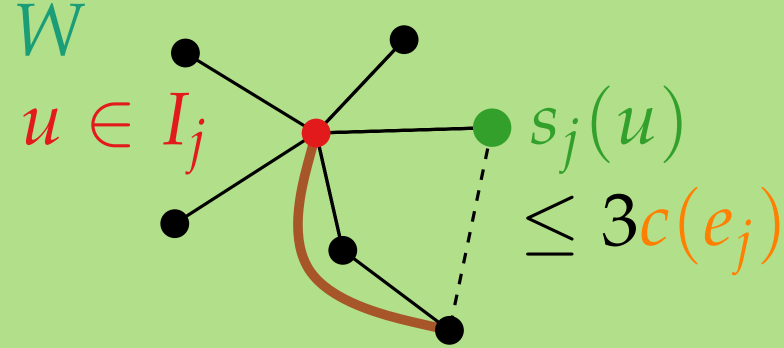
Algorithm for the Weighted Version

Algorithm Metric-Weighted-Center

Sort the edges of G by cost : $c(e_1) \le \ldots \le c(e_m)$

for $j = 1, \ldots, m$ **do**

- Construct G_j^2
- Find a maximal independent set I_j in G_j^2
- Compute $S_j := \{ s_j(u) \mid u \in I_j \}$
- **if** ~~$|I_j| \le k$~~ $w(S_j) \le W$ **then**
  - **return** ~~I_j~~ S_j



$s_j(u) :=$ lightest node in $N_{G_j}(u) \cup \{u\}$

Theorem. The above is a factor-3-approximation algorithm for Metric-Weighted-Center.

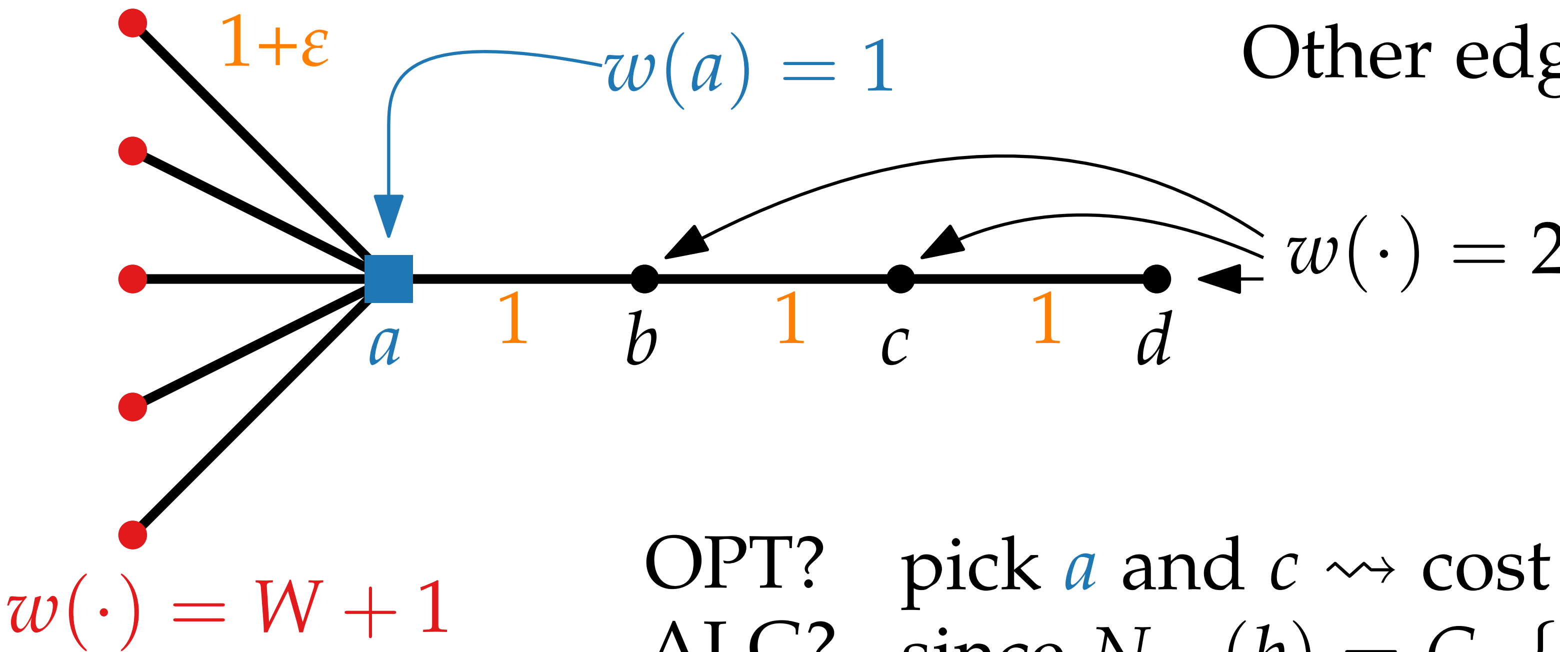
Tight Example... ?

Here, we need to have a budget W, and edge costs satisfying the triangle inequality.

~~Consider $W = 3$~~

Other edge costs?

shortest path



OPT? pick a and $c \rightsquigarrow$ cost $1 + \varepsilon$

ALG? since $N_{G^2}(b) = G$, $\{b\}$ is a maximal independent set in G^2

Thus, alg. picks $a \rightsquigarrow$ cost 3

How can we generalize this to larger W?