Approximation Algorithms

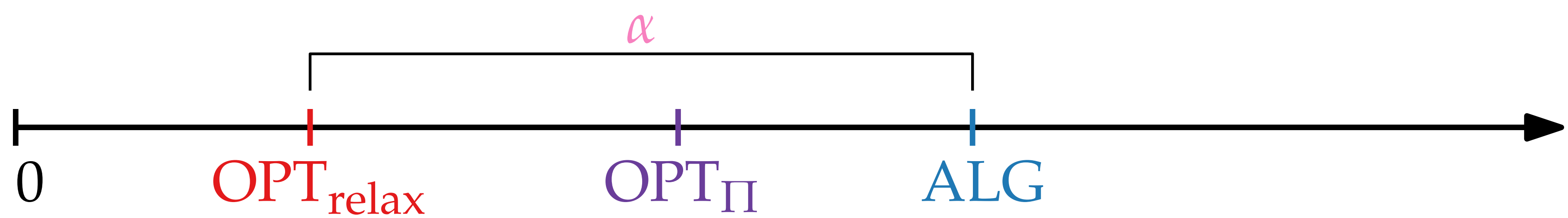
Lecture 5:
LP-based Approximation Algorithms for SetCover

Part I:
LP-based Approximation Techniques

Joachim Spoerhase

Winter 2020/21

I) LP-Rounding



Consider a minimization problem Π in ILP-form.

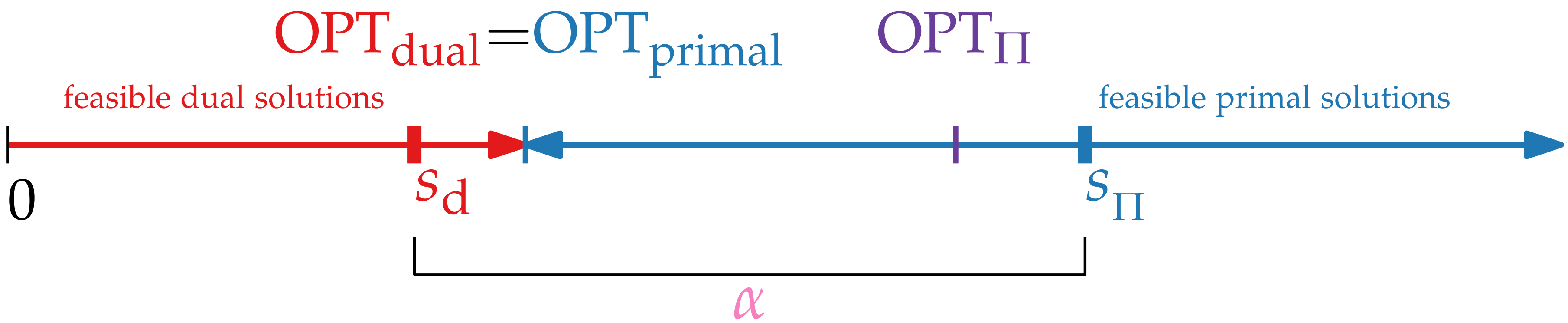
Compute a solution for the LP-relaxation

Round to obtain an integer solution for Π

Difficulty: ensure **feasible** solution of Π

Approximation factor $\mathrm{ALG}/\mathrm{OPT}_\Pi \leq \mathrm{ALG}/\mathrm{OPT}_{\mathrm{relax}}$

II) Primal-Dual Approach



Consider a minimization problem Π in ILP-form.

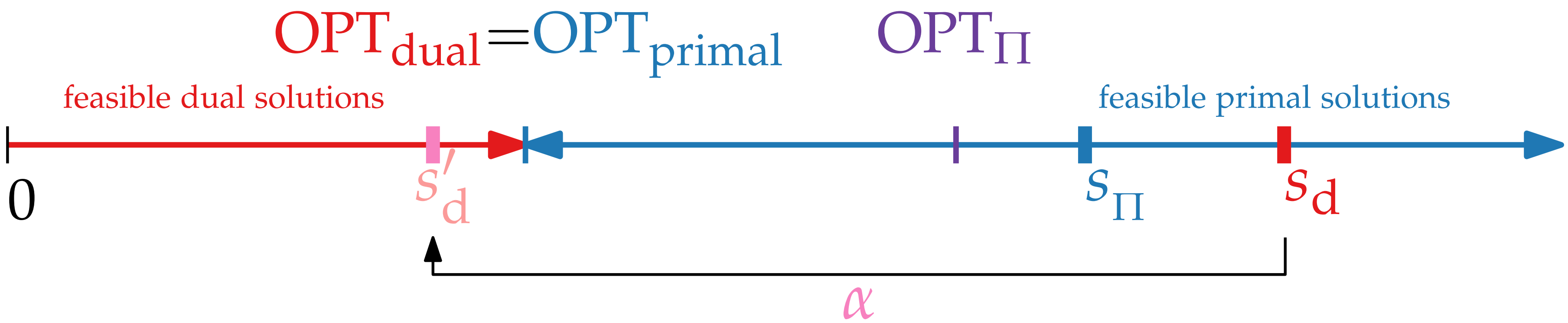
Start with (trivial) feasible dual solution and infeasible primal solution (e.g. all variables $= 0$).

Compute dual solution s_{d} and integral primal solution s_{Π} for Π iteratively:
increase s_{d} according to CS and make s_{Π} "more feasible".

Approximation factor $\leq \mathrm{obj}(s_{\Pi}) / \mathrm{obj}(s_{\mathrm{d}})$

Advantage: don't need LP-"machinery"; possibly faster, more flexible.

III) Dual Fitting



Consider a minimization problem Π in ILP-form.

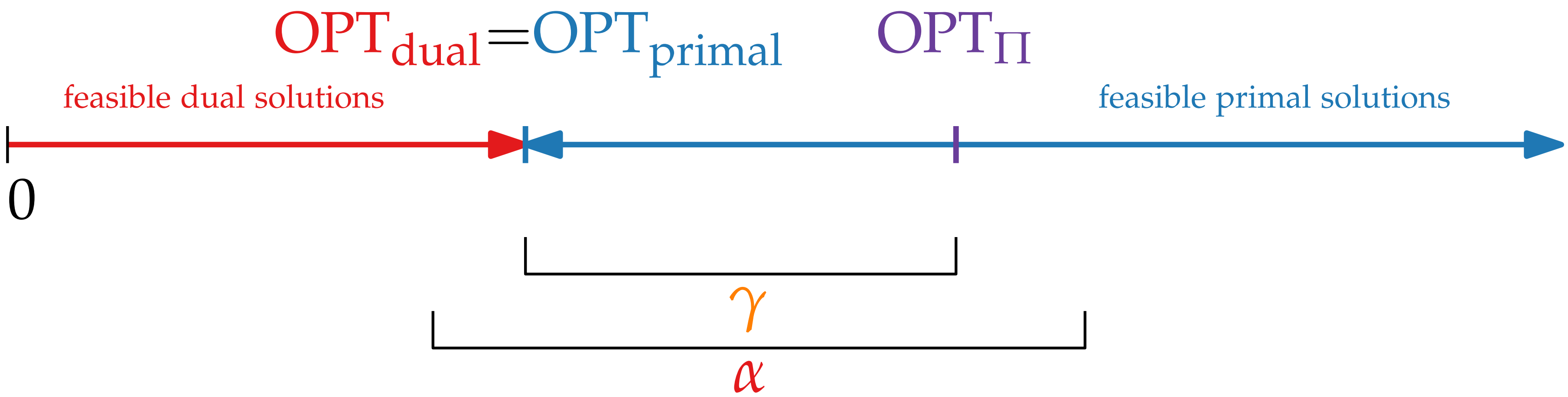
Combinatorial algorithm (e.g., greedy) computes feasible primal solution s_{Π} and infeasible dual solution s_{d} which "completely pays for" s_{Π}, i.e. $\mathrm{obj}(s_{\Pi}) \leq \mathrm{obj}(s_{\mathrm{d}})$.

Scale the dual variables $\rightsquigarrow$ feasible dual solution s'_{d}.

$\Rightarrow \mathrm{obj}(s_{\Pi})/\alpha \leq \mathrm{obj}(s_d)/\alpha = \mathrm{obj}(s'_{\mathrm{d}}) \leq \mathrm{OPT}_{\mathrm{dual}} \leq \mathrm{OPT}_{\Pi}$

$\Rightarrow$ Scaling factor α is approximation factor.

Integrality Gap



Consider a minimization problem Π in ILP-form.

Dual methods (without outside help) are limited by the *Integrality Gap* of the LP-relaxation

$$\alpha \geq \gamma = \sup_{I} \frac{\mathrm{OPT}_{\Pi}(I)}{\mathrm{OPT}_{\mathrm{primal}}(I)}$$

Approximation Algorithms

Lecture 5:
LP-based Approximation Algorithms for SetCover

Part II:
SetCover as an ILP

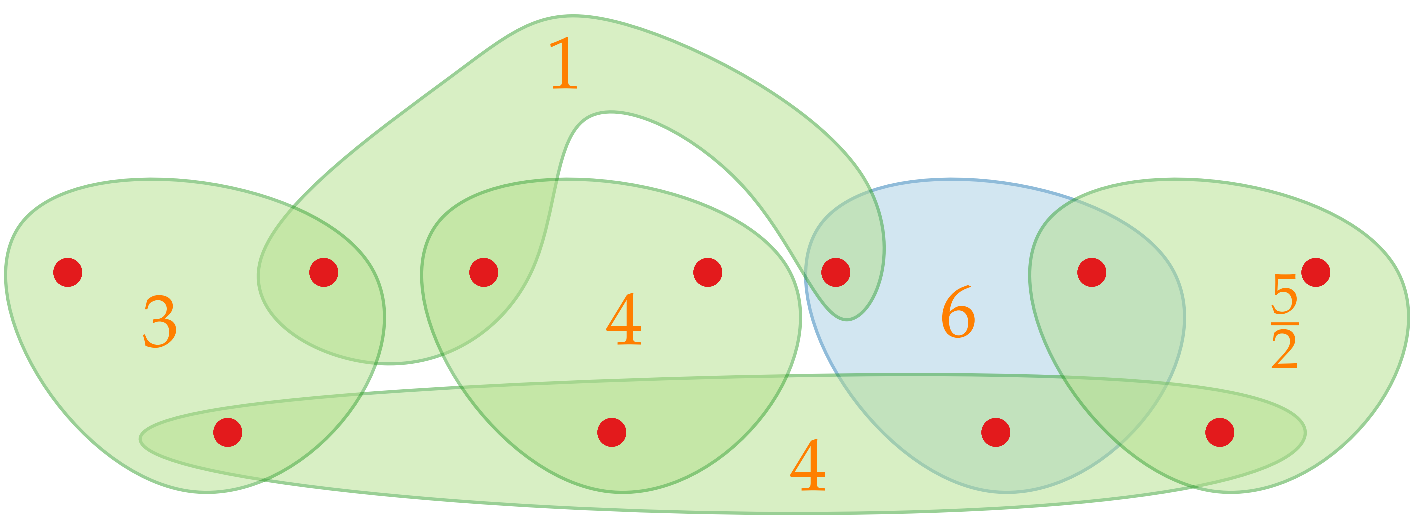
Joachim Spoerhase

Winter 2020/21

SetCover as an ILP

$$\begin{aligned} \textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S \\ \textbf{subject to} \quad & \sum_{S \ni u} x_S \geq 1 \qquad u \in U \\ & x_S \in \{0,1\} \qquad S \in \mathcal{S} \end{aligned}$$

Ground set U
Family $\mathcal{S} \subseteq 2^U$ with $\bigcup \mathcal{S} = U$
Costs $c\colon \mathcal{S} \to \mathbb{Q}^+$

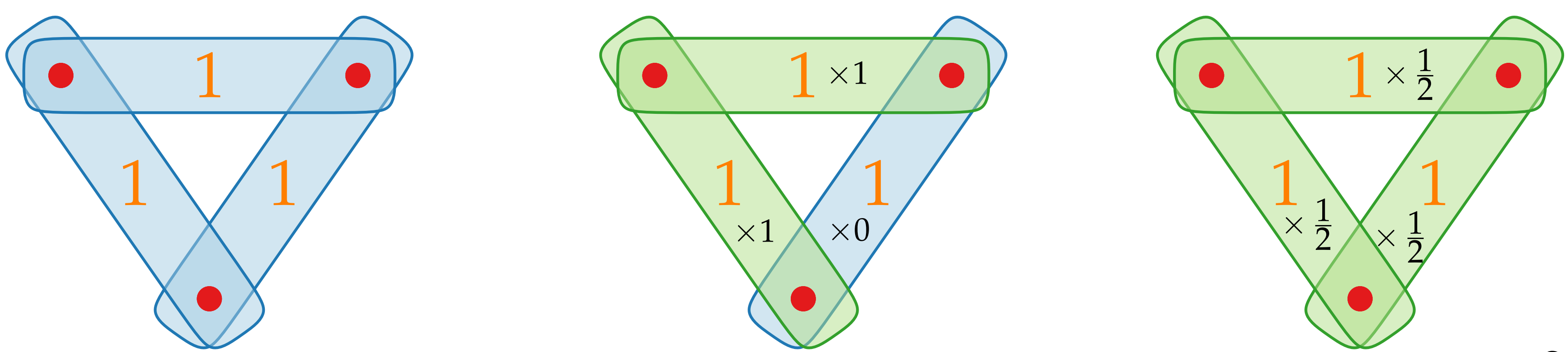


Find cover $\mathcal{S}' \subseteq \mathcal{S}$ of U with minimum cost.

SetCover – LP-Relaxation

$$\begin{array}{lll} \textbf{minimize} & \sum_{S \in \mathcal{S}} c_S x_S & \\ \textbf{subject to} & \sum_{S \ni u} x_S \geq 1 & u \in U \\ & x_S \geq 0 & S \in \mathcal{S} \end{array}$$

Optimal?



integer: 2

fractional: $\frac{3}{2}$

SetCover – Dual LP

$$\begin{aligned} \textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S \\ \textbf{subject to} \quad & \sum_{S \ni u} x_S \geq 1 && u \in U \\ & x_S \geq 0 && S \in \mathcal{S} \end{aligned}$$

$$\begin{aligned} \textbf{maximize} \quad & \sum_{u \in U} y_u \\ \textbf{subject to} \quad & \sum_{u \in S} y_u \leq c_S && S \in \mathcal{S} \\ & y_u \geq 0 && u \in U \end{aligned}$$

Approximation Algorithms

Lecture 5: LP-based Approximation Algorithms for SetCover

Part III: SetCover via LP-Rounding

Joachim Spoerhase

Winter 2020/21

LP-Rounding: Approach I

$$\begin{aligned} \textbf{minimize} \quad & \sum_{S \in \mathcal{S}} c_S x_S \\ \textbf{subject to} \quad & \sum_{S \ni u} x_S \geq 1 \qquad u \in U \\ & x_S \geq 0 \qquad S \in \mathcal{S} \end{aligned}$$

LP-Rounding-One$(U, \mathcal{S}, c)$

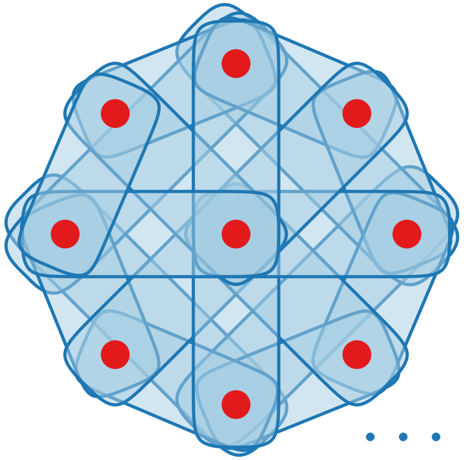
Compute optimal solution x for LP-Relaxation.

Round each x_S with $x_S > 0$ to 1.

Generates a valid solution

Scaling factor arbitrarily large

Use frequency f



LP-Rounding: Approach II

$$\begin{array}{lll} \textbf{minimize} & \sum_{S \in \mathcal{S}} c_S x_S & \\ \textbf{subject to} & \sum_{S \ni u} x_S \geq 1 & u \in U \\ & x_S \geq 0 & S \in \mathcal{S} \end{array}$$

LP-Rounding-Two($U, \mathcal{S}, c$)

Compute optimal solution x for LP-Relaxation.
Round each x_S with $x_S \geq 1/f$ to 1; remaining to 0.

Let f be the frequency of (number of sets containing) the most frequent element.

Theorem. LP-Rounding-Two is a factor-f-approximation algorithm for SetCover.

Approximation Algorithms

Lecture 5:
LP-based Approximation Algorithms for SetCover

Part IV:
SetCover via Primal-Dual Schema

Joachim Spoerhase

Winter 2020/21

Complementary Slackness

$$\begin{array}{llll} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{llll} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Theorem. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be valid solutions for the primal and dual program (resp.). Then x and y are optimal if and only if the following conditions are met:

Primal CS:
For each $j = 1, \ldots, n$: either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
For each $i = 1, \ldots, m$: either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

Relaxing Complementary Slackness

minimize	$c^\top x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\top y$		
subject to	$A^\top y$	$\leq$	c
	y	$\geq$	0

~~Primal CS~~: Relaxed Primal CS

For each $j = 1, \ldots, n$: either $x_j = 0$ or ~~$\sum_{i=1}^m a_{ij} y_i = c_j$~~

$c_j / \alpha \leq \sum_{i=1}^m a_{ij} y_i \leq c_j$

~~Dual CS~~: Relaxed Dual CS

For each $i = 1, \ldots, m$: either $y_i = 0$ or ~~$\sum_{j=1}^n a_{ij} x_j = b_i$~~

$b_i \leq \sum_{j=1}^n a_{ij} x_j \leq \beta \cdot b_i$

~~$\Leftrightarrow \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i y_i$~~ $\Rightarrow \sum_{j=1}^{n} c_j x_j \leq \alpha\beta \sum_{i=1}^{m} b_i y_i \leq \alpha\beta \cdot \mathrm{OPT}_{\mathrm{LP}}$

Primal-Dual Schema

Start with a feasible dual and infeasible primal solution (often trivial).

"Improve" the feasibility of the primal solution...

. . . and simultaneously the obj. value of the dual solution.

Do so until the relaxed CS conditions are met.

Maintain that the primal solution is integer valued.

The feasibility of the primal solution and relaxed CS condition provide an approximation ratio.

Relaxed CS for SetCover

minimize	$\sum_{S \in \mathcal{S}} c_S x_S$	
subject to	$\sum_{S \ni u} x_S \geq 1$	$u \in U$
	$x_S \geq 0$	$S \in \mathcal{S}$

maximize	$\sum_{u \in U} y_u$	
subject to	$\sum_{u \in S} y_u \leq c_S$	$S \in \mathcal{S}$
	$y_u \geq 0$	$u \in U$

critical set

(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

only chooses critical sets

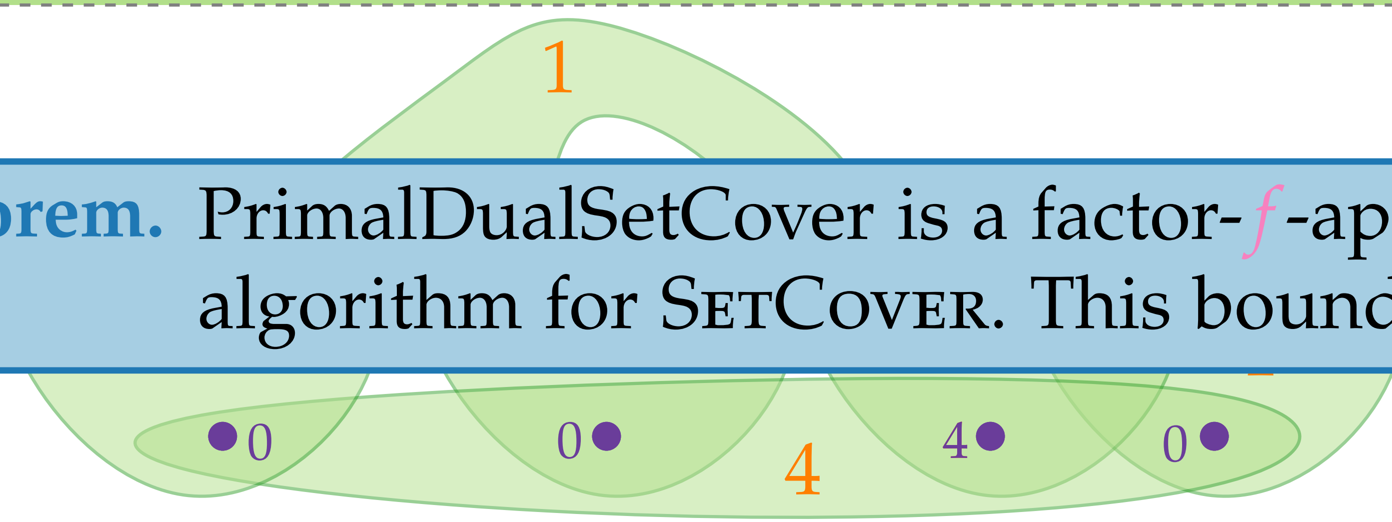
trivial for binary x

Relaxed dual CS: $y_u \neq 0 \Rightarrow 1 \leq \sum_{S \ni u} x_S \leq f \cdot 1$

Primal-Dual-Schema for SetCover

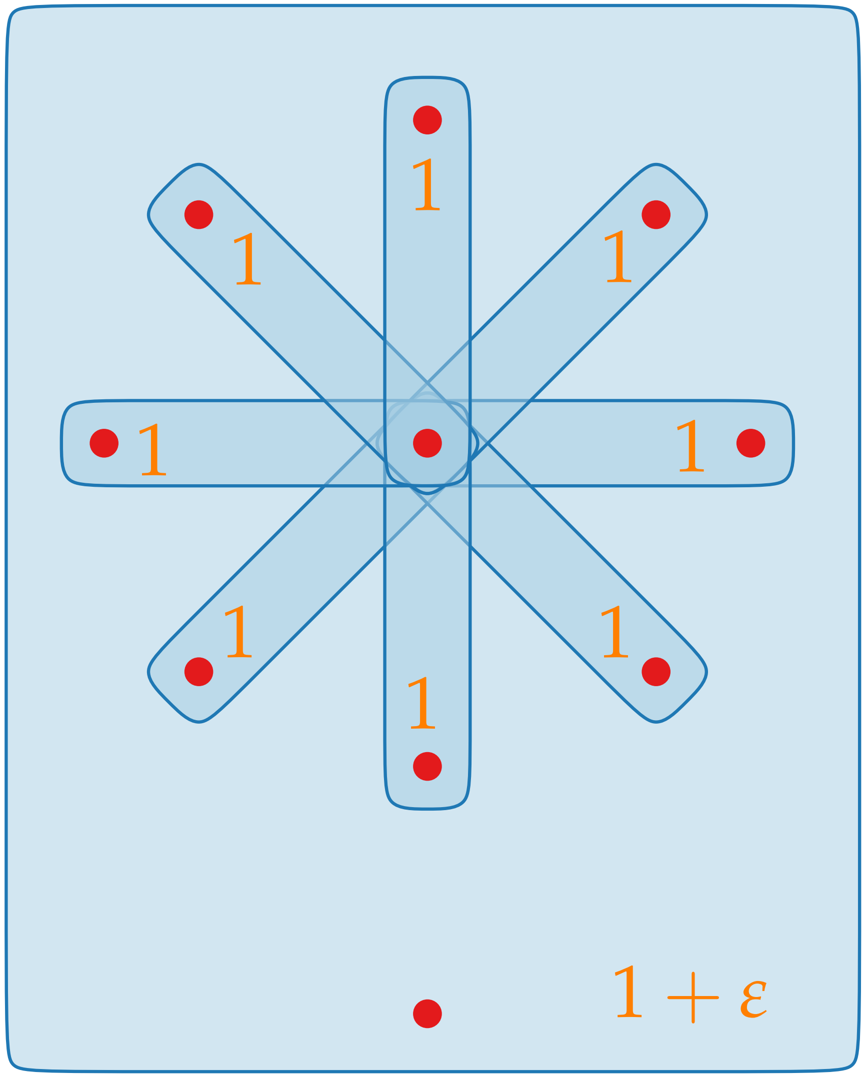
PrimalDualSetCover(U, $\mathcal{S}$, c)

$x \leftarrow 0$, $y \leftarrow 0$
repeat
  Select an uncovered element u.
  Increase y_u until a set S is critical ($\sum_{u' \in S} y_{u'} = c_S$).
  Select all critical sets and update x.
  Mark all elements in these sets as covered.
until all elements are covered.
return x



Theorem. PrimalDualSetCover is a factor-f-approximation algorithm for SetCover. This bound is tight.

Tight Example



Approximation Algorithms

Lecture 5:
LP-based Approximation Algorithms for SetCover

Part V:
SetCover via Dual Fitting

Joachim Spoerhase

Winter 2020/21

Dual Fitting for SetCover

Combinatorial (greedy) algorithm (see Lecture 02):

```
GreedySetCover(U, 𝒮, c)
  C ← ∅
  𝒮' ← ∅
  while C ≠ U do
    S ← Set from 𝒮 that minimizes c(S)/|S \ C|
    foreach u ∈ S \ C do
      price(u) ← c(S)/|S \ C|
    C ← C ∪ S
    𝒮' ← 𝒮' ∪ {S}
  return 𝒮'                                  // Cover of U
```

Reminder: $\sum_{u \in U} \text{price}(u)$ completely pays for $\mathcal{S}'$.

New: LP-based Analysis

Observation. For each $u \in U$, $\text{price}(u)$ is a dual variable
But this dual solution is in general not feasible.

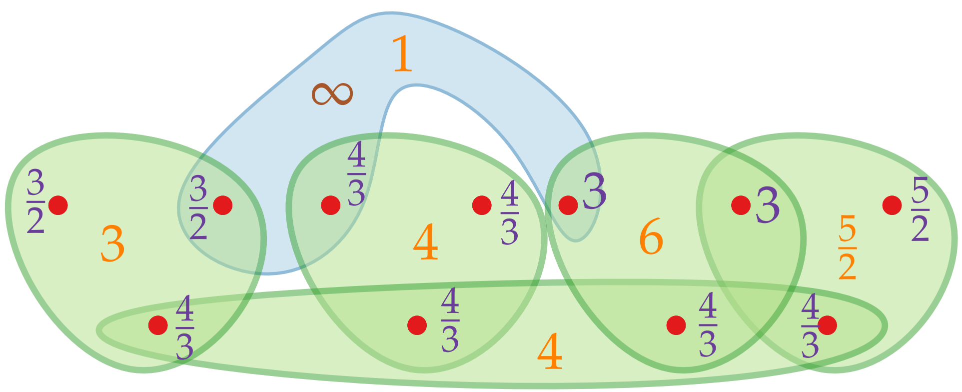
HW: Construct instance where some S are "overpacked" by factor $\approx H_{|S|}$.

Dual-Fitting-Trick:

Scale duale variables such that no set is overpacked.

Take $y_u = \text{price}(u)/\mathcal{H}_k$. $k =$cardinality of largest set in $\mathcal{S}$

The greedy algorithm uses *these* dual variables as lower bound for OPT.



maximize $\sum_{u \in U} y_u$

subject to $\sum_{u \in S} y_u \leq c_S \quad S \in \mathcal{S}$

$y_u \geq 0 \quad u \in U$

Proof. To prove: No set is overpacked by y.

Let $S \in \mathcal{S}$ and $\ell = |S| \leq k$.

Let $u_1, \dots, u_\ell$ be the elements of S –
in the order they are covered by the greedy alg.

Consider the iteration in which u_i is covered

Before that, $\geq \ell - i + 1$ elem. of S are uncovered.

So $\mathrm{price}(u_i) \leq c(S)/(\ell - i + 1)$.

$$\Rightarrow y_{u_i} \leq \frac{c(S)}{\mathcal{H}_k} \cdot \frac{1}{\ell - i + 1} \Rightarrow \sum_{i=1}^{k} y_{u_i} \leq \frac{c(S)}{\mathcal{H}_k} \cdot \overbrace{\left(\frac{1}{\ell} + \cdots + \frac{1}{1}\right)}^{= \mathcal{H}_\ell \leq \mathcal{H}_k}$$

$$\leq c(S)$$

□

Lemma.
The vector $y = (y_u)_{u \in U}$ is a feasible solution for the dual LP.

maximize	$\sum_{u \in U} y_u$	
subject to	$\sum_{u \in S} y_u \leq c_S$	$S \in \mathcal{S}$
	$y_u \geq 0$	$u \in U$

Result for Dual Fitting

Theorem. GreedySetCover is a factor-$\mathcal{H}_k$-approximation algorithm for SetCover, where $k = \max_{S \in \mathcal{S}} |S|$.

Proof.

$$\begin{aligned} \text{ALG} = c(\mathcal{S}') \leq \sum_{u \in U} \text{price}(u) = \mathcal{H}_k \cdot \sum_{u \in U} y_u &\leq \\ &\leq \mathcal{H}_k \cdot \text{OPT}_{\text{relax}} \\ &\leq \mathcal{H}_k \cdot \text{OPT} \qquad \square \end{aligned}$$

Strengthened bound with respect to $\text{OPT}_{\text{relax}} \leq \text{OPT}$.

Dual solution allows a *per-instance* estimation

...which might be stronger than worst-case bound $\mathcal{H}_k$