Approximation Algorithms

Lecture 5:
LP-based Approximation Algorithms for SetCover

Part I:
LP-based Approximation Techniques

Joachim Spoerhase

Winter 2021/22

I) LP-Rounding

Consider a minimization problem Π in ILP-form.

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Approximation factor $\mathrm{ALG}/\mathrm{OPT}_{\Pi} \leq \mathrm{ALG}/\mathrm{OPT}_{\mathrm{relax}}$

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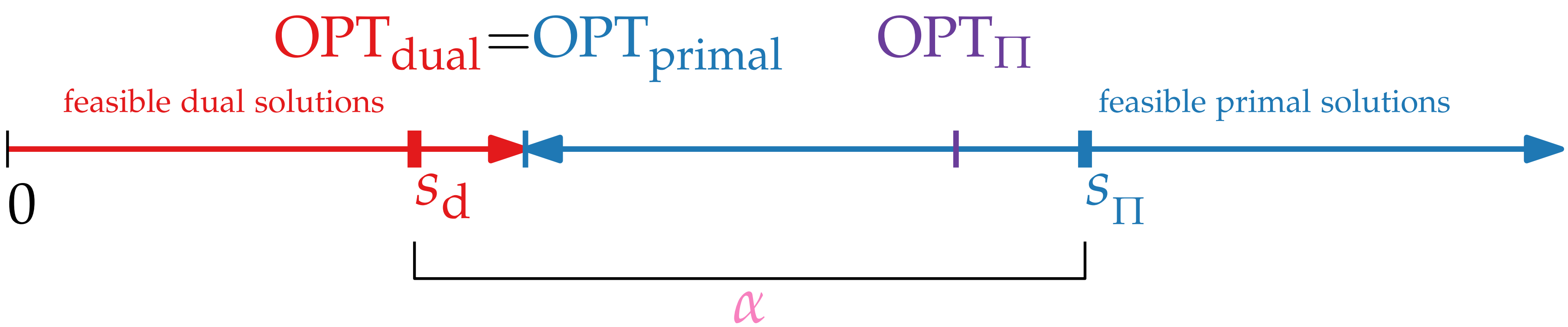

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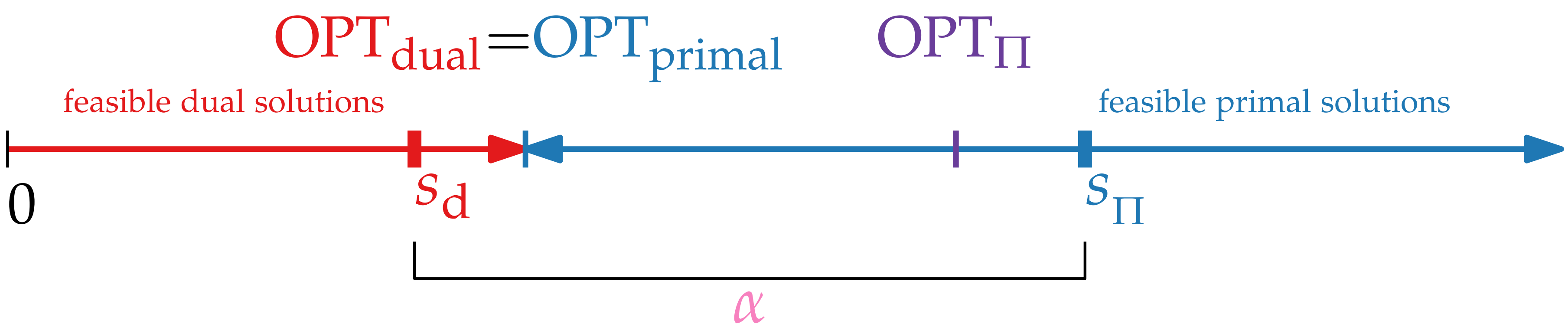

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Approximation factor $\leq \mathrm{obj}(s_\Pi)/\mathrm{obj}(s_\mathrm{d})$

Advantage: don't need LP-"machinery"; possibly faster, more flexible.

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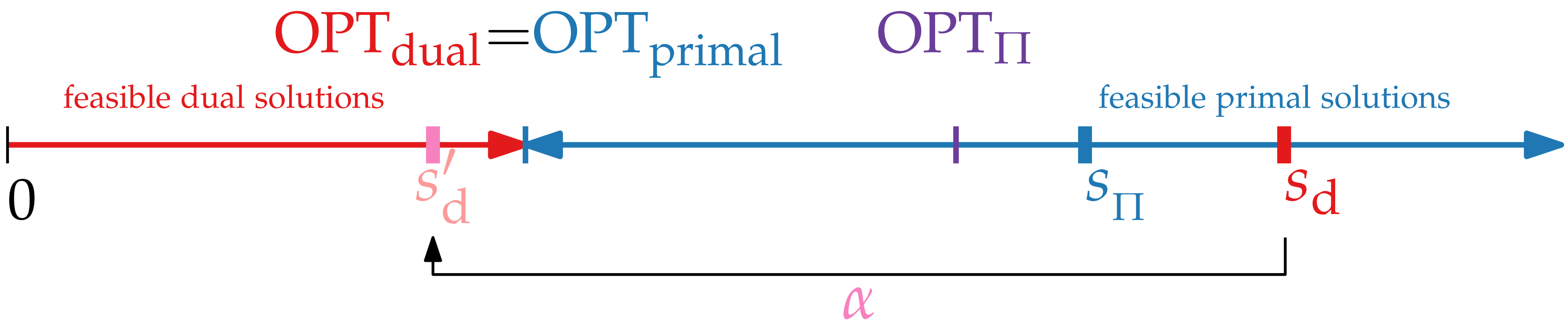

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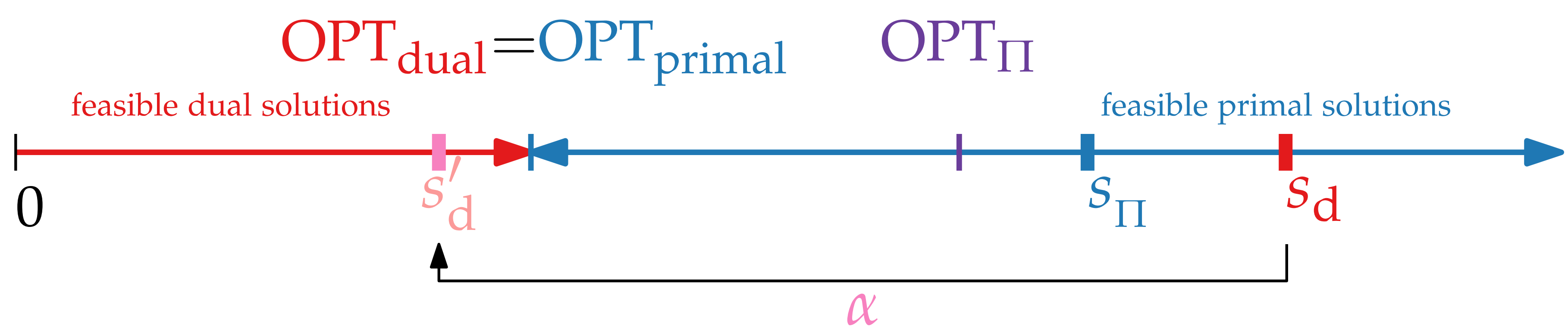

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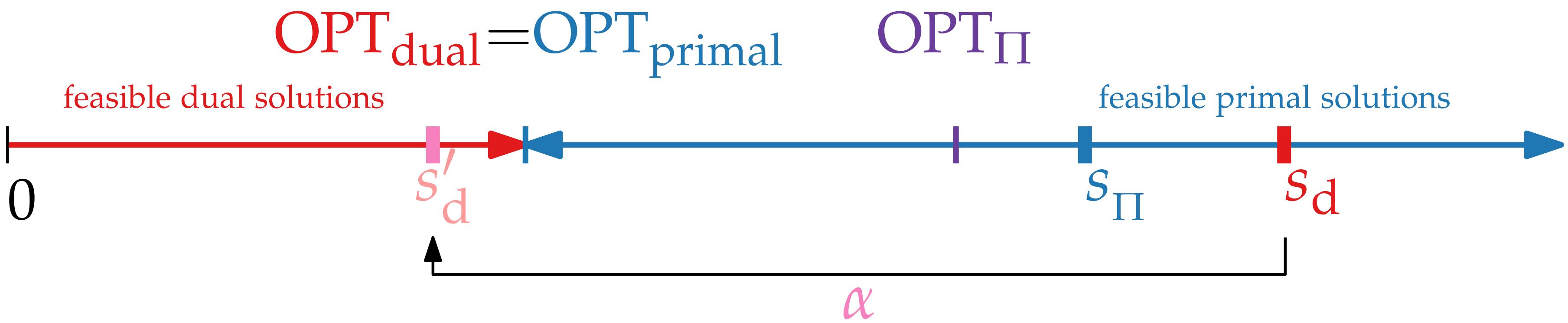

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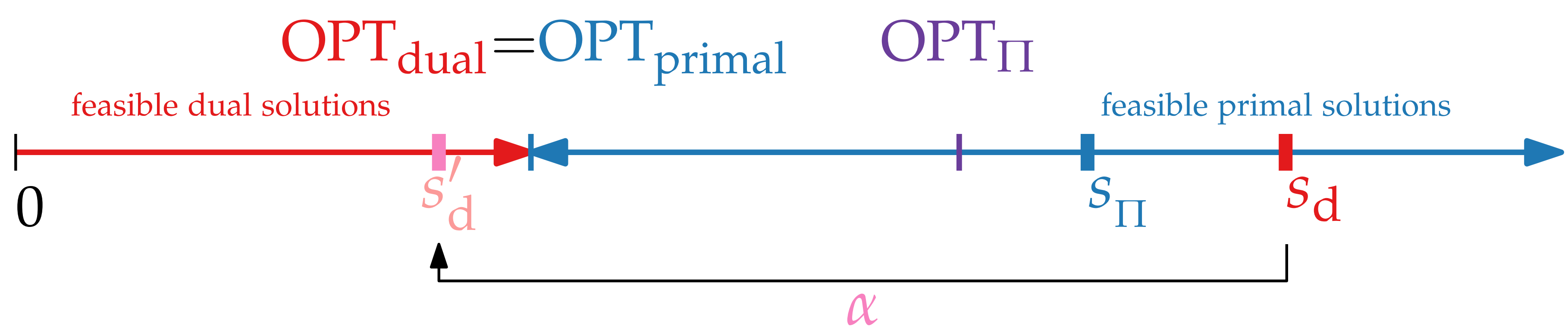

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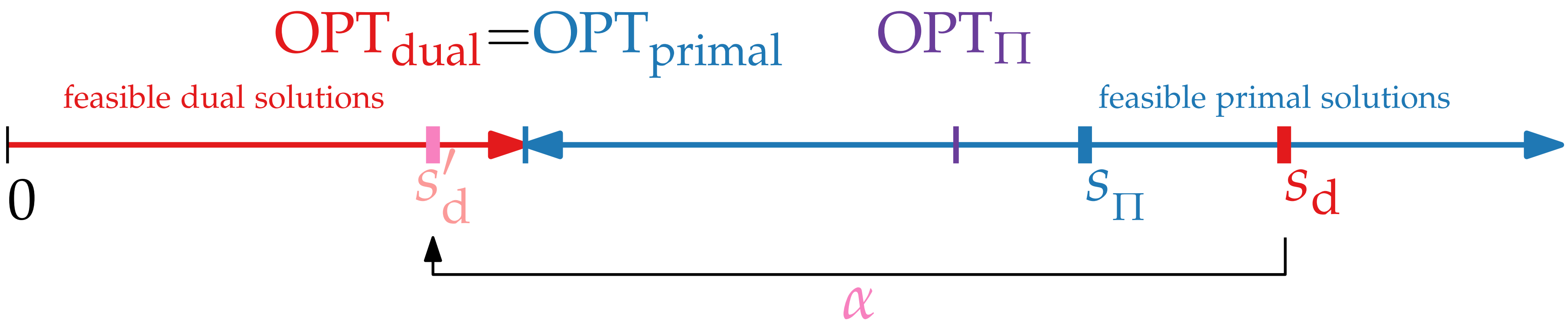

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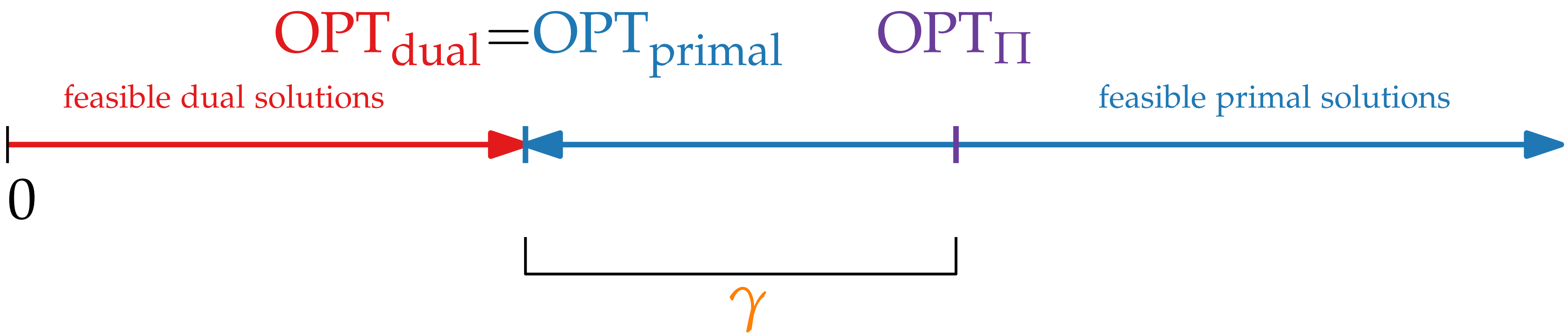
$\Rightarrow$ Scaling factor α is approximation factor.

Integrality Gap



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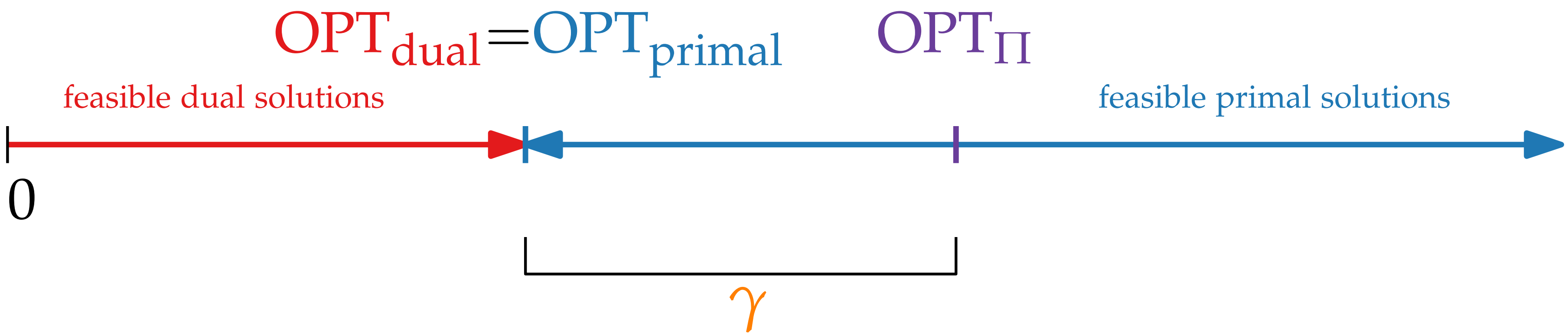

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Dual methods (without outside help) are limited by the *Integrality Gap* of the LP-relaxation

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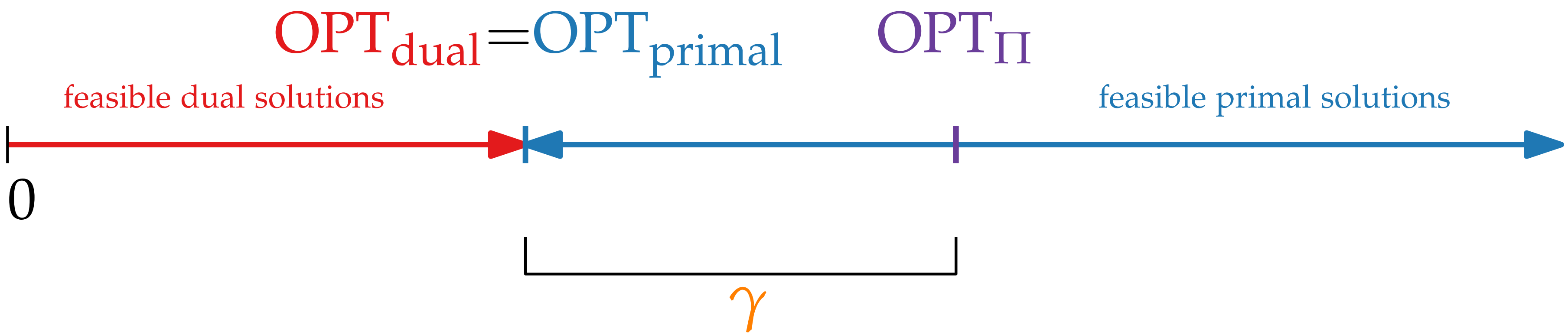

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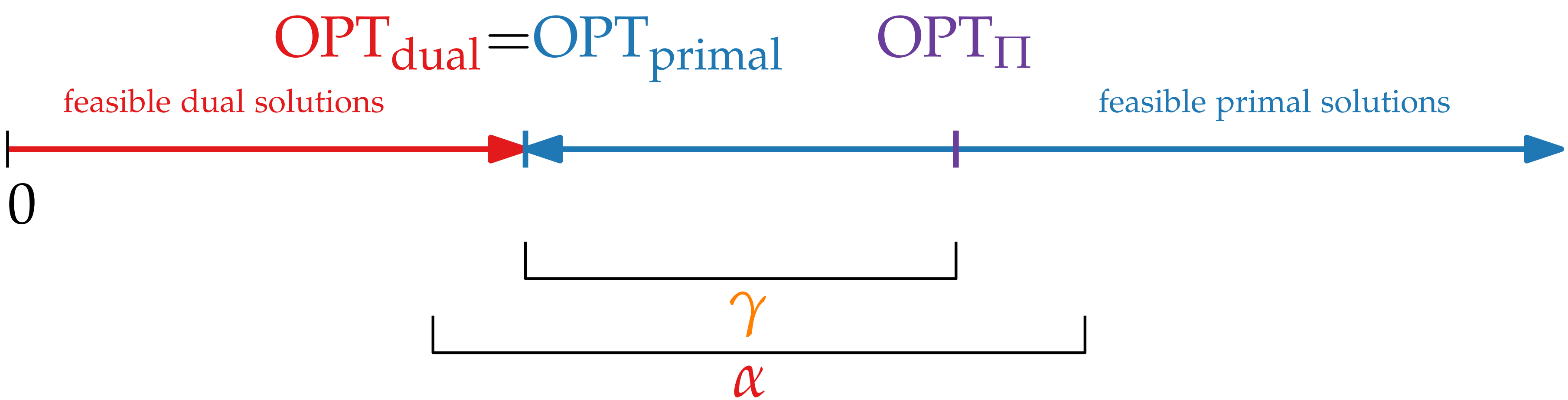

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Lecture 5:
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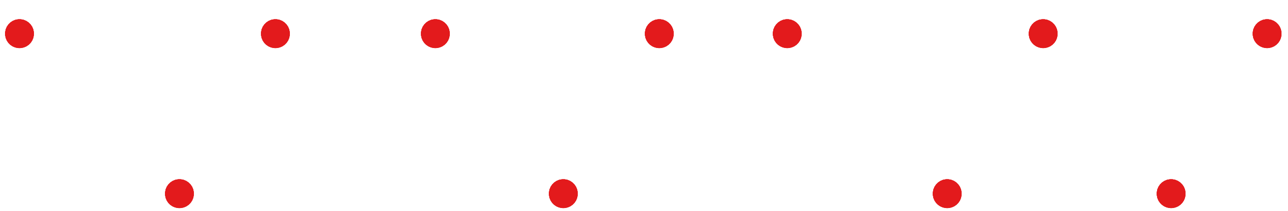
Part II:
SetCover as an ILP

Joachim Spoerhase

Winter 2021/22

SetCover as an ILP

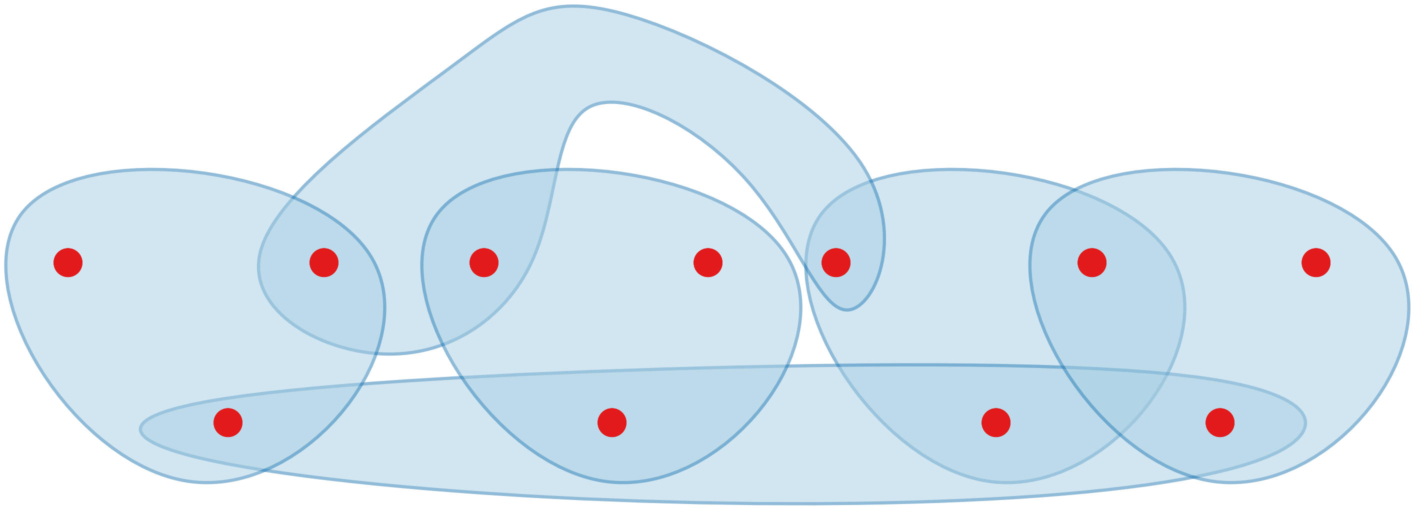
Ground set U



SetCover as an ILP

Ground set U

Family $\mathcal{S} \subseteq 2^U$ with $\bigcup \mathcal{S} = U$



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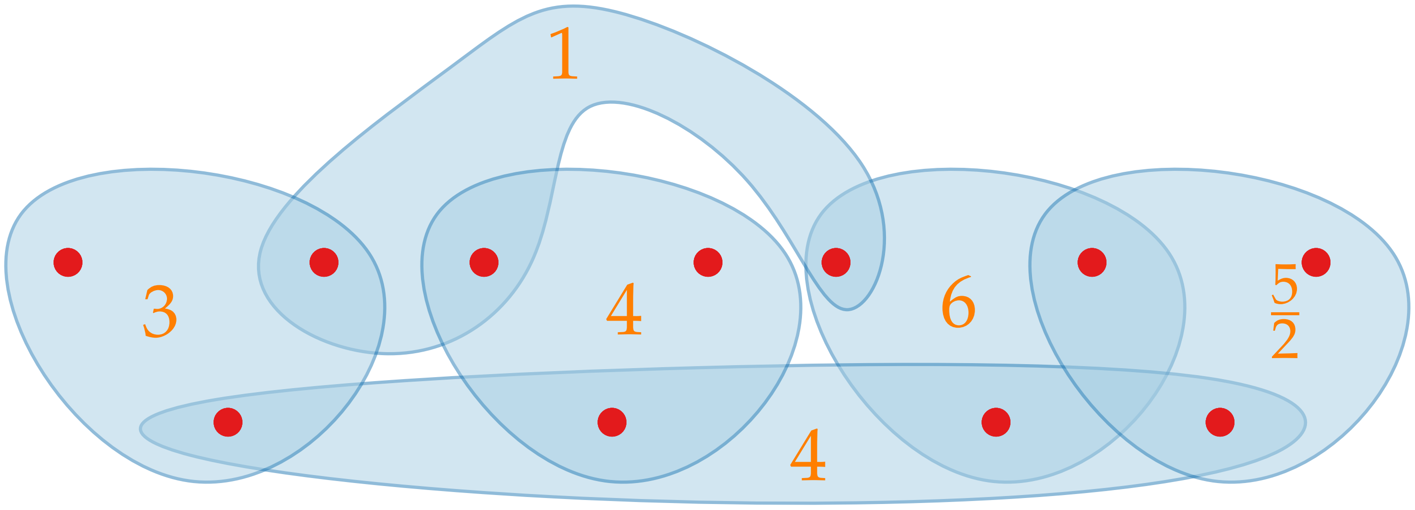

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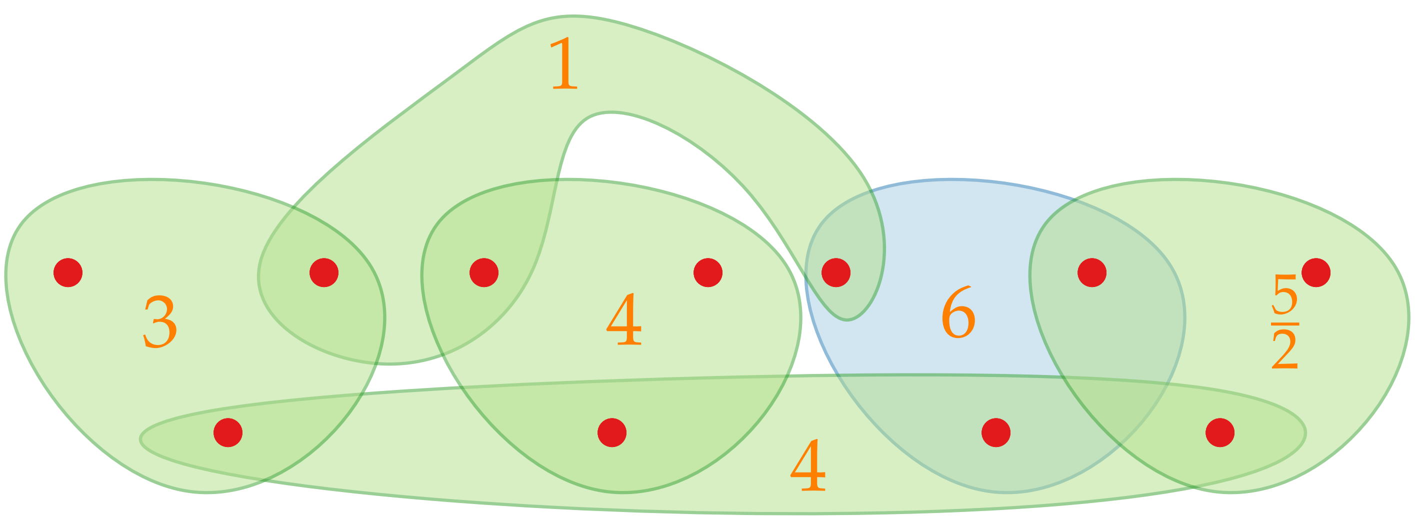

Find cover
$\mathcal{S}' \subseteq \mathcal{S}$ of U with
minimum cost.

SetCover as an ILP

minimize

subject to

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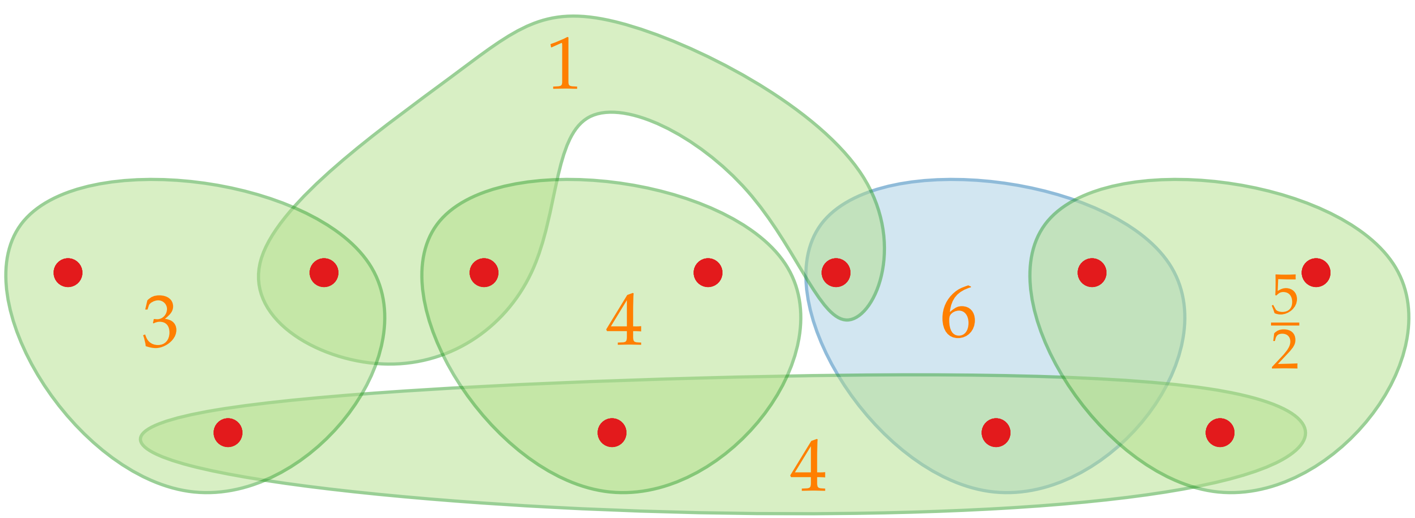

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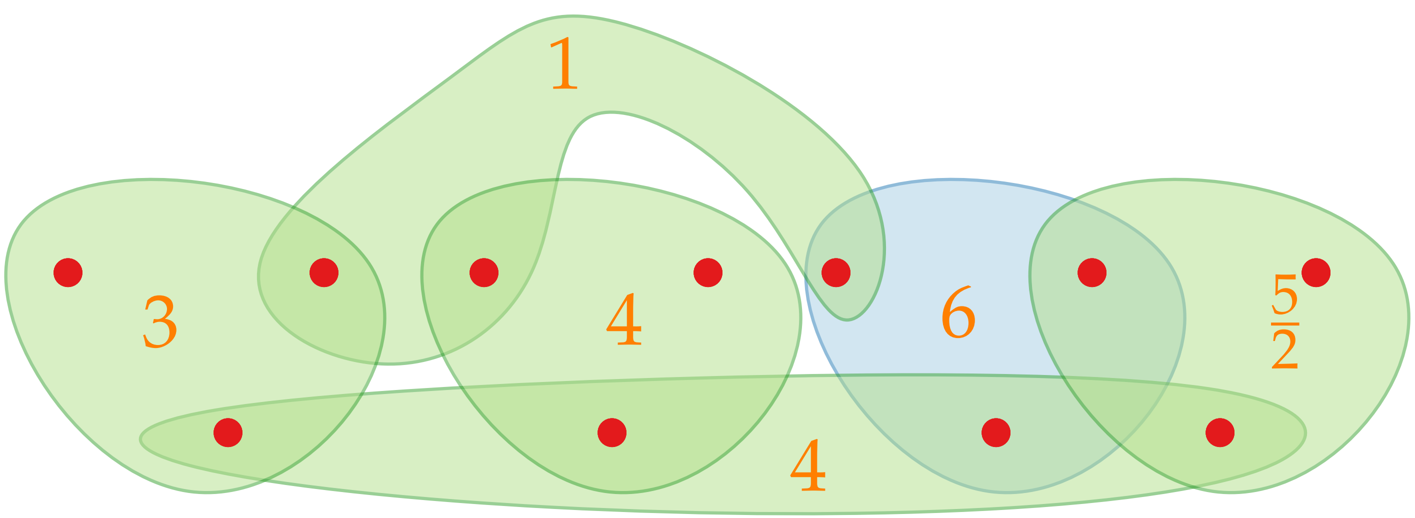

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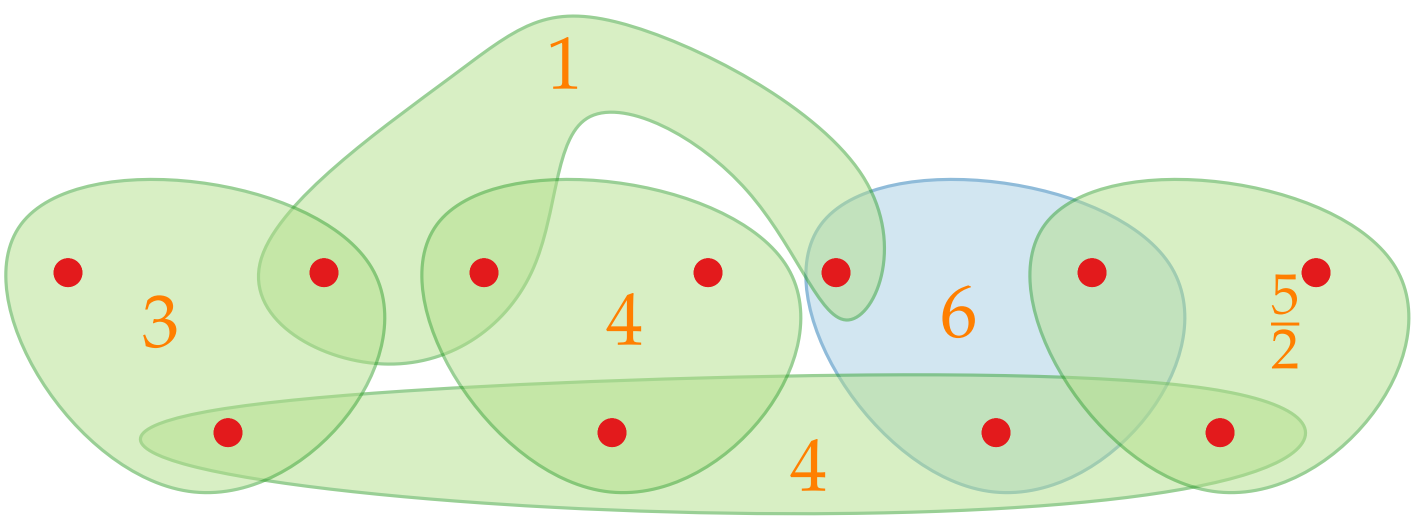

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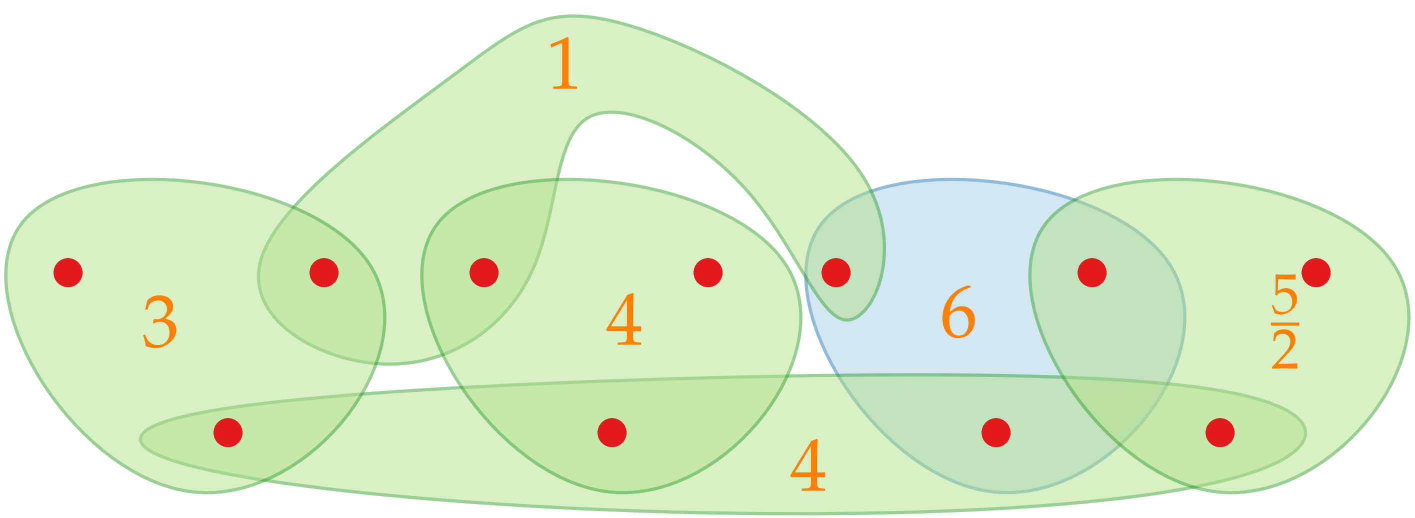

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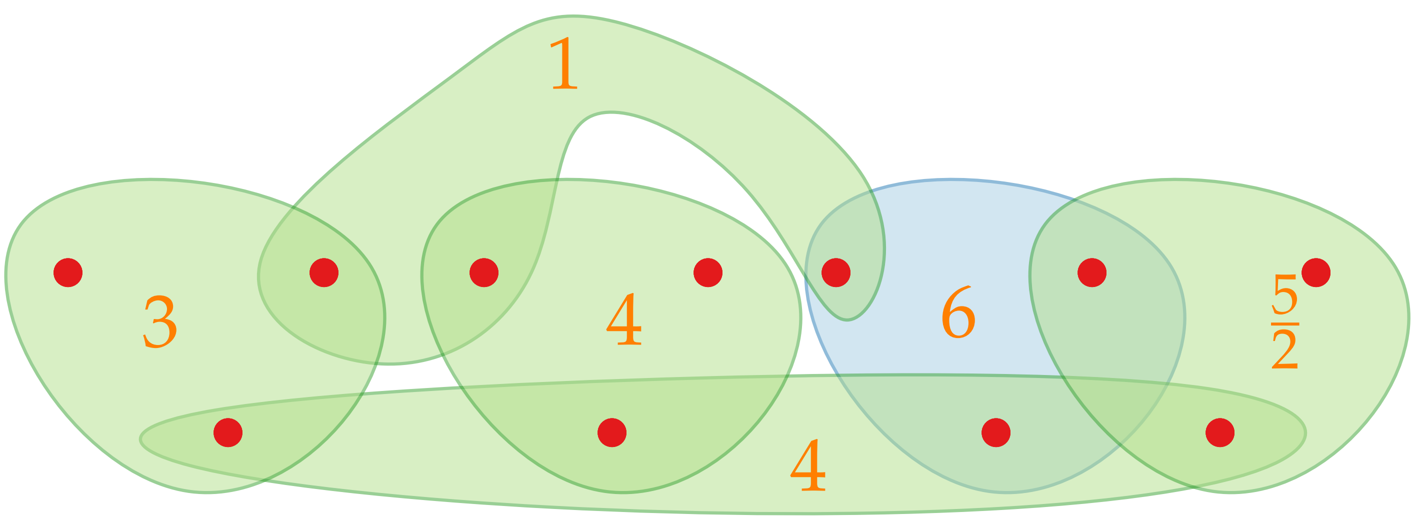

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SetCover – LP-Relaxation

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SetCover – LP-Relaxation

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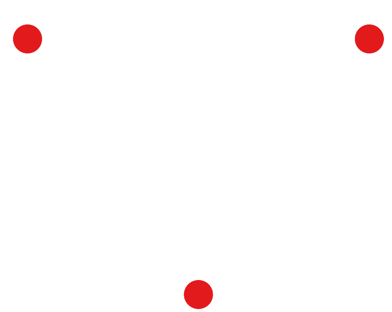

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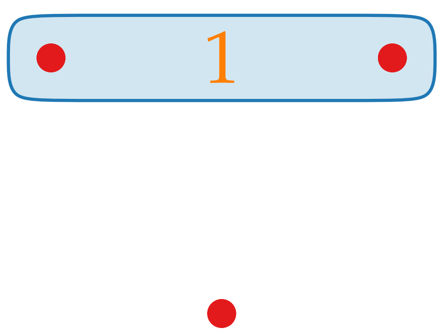

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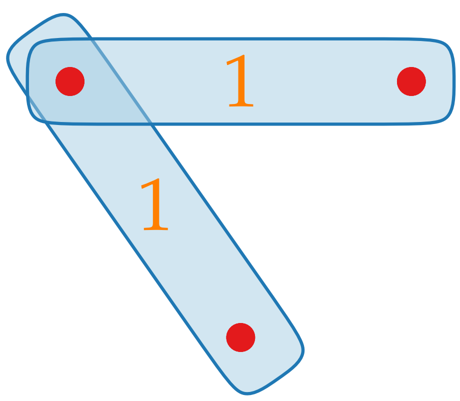

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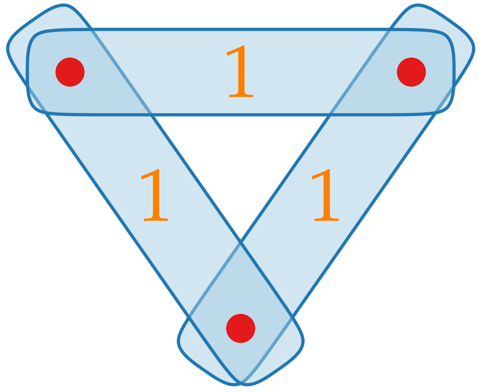

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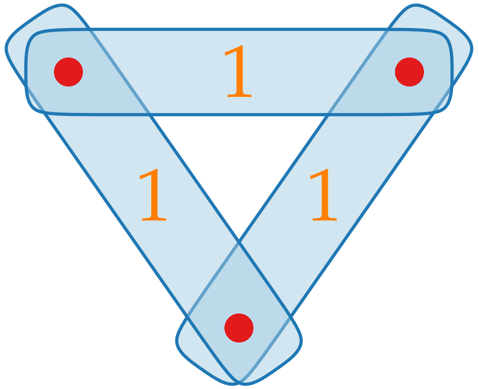

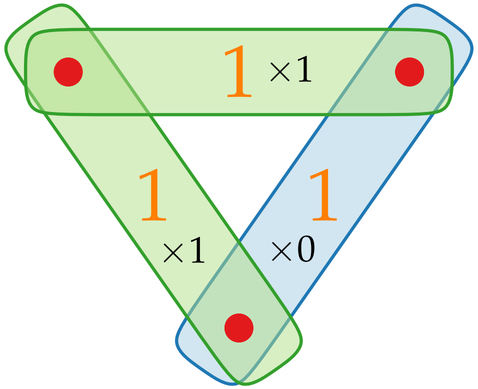

integer: 2

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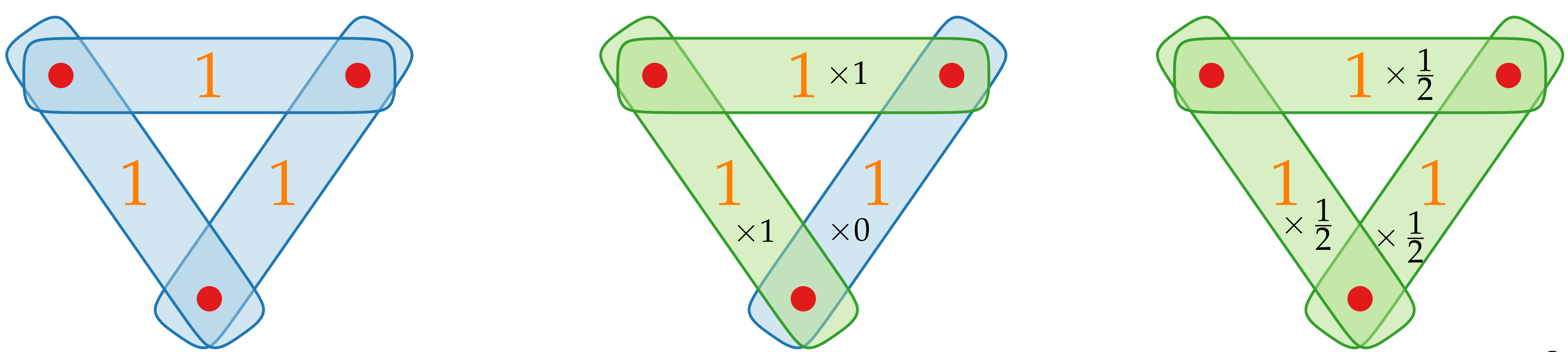

integer: 2

fractional: $\frac{3}{2}$

SetCover – Dual LP

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$$
\begin{array}{lll}
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Approximation Algorithms

Lecture 5:
LP-based Approximation Algorithms for SetCover

Part III:
SetCover via LP-Rounding

Joachim Spoerhase

Winter 2021/22

LP-Rounding: Approach I

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LP-Rounding-One$(U, \mathcal{S}, c)$

LP-Rounding: Approach I

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Round each x_S with $x_S > 0$ to 1.

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Scaling factor arbitrarily large

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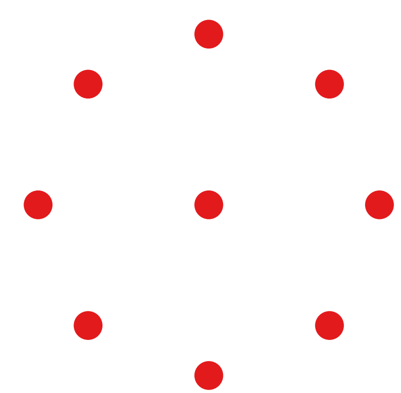
Generates a valid solution

Scaling factor arbitrarily large



LP-Rounding: Approach I

$$\begin{aligned} &\textbf{minimize} && \sum_{S \in \mathcal{S}} c_S x_S \\ &\textbf{subject to} && \sum_{S \ni u} x_S \geq 1 && u \in U \\ &&& x_S \geq 0 && S \in \mathcal{S} \end{aligned}$$

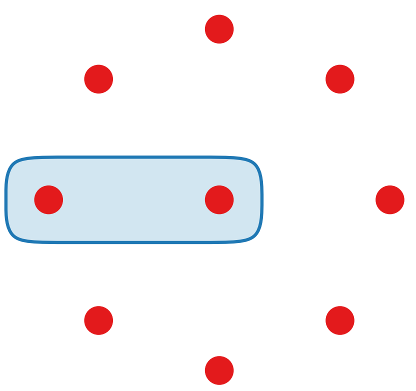
LP-Rounding-One$(U, \mathcal{S}, c)$

Compute optimal solution x for LP-Relaxation.

Round each x_S with $x_S > 0$ to 1.

Generates a valid solution

Scaling factor arbitrarily large



LP-Rounding: Approach I

$$\begin{array}{lll} \textbf{minimize} & \sum_{S \in \mathcal{S}} c_S x_S & \\ \textbf{subject to} & \sum_{S \ni u} x_S \geq 1 & u \in U \\ & x_S \geq 0 & S \in \mathcal{S} \end{array}$$

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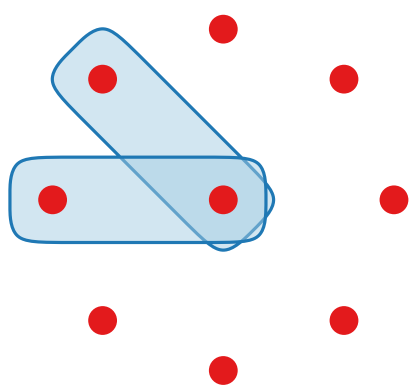

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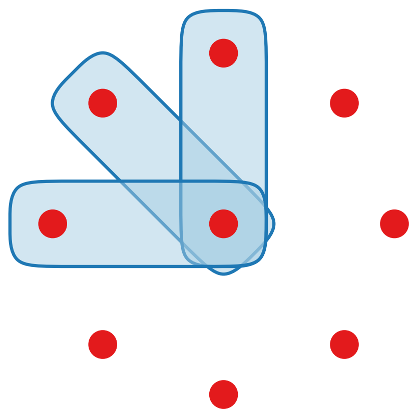
Generates a valid solution

Scaling factor arbitrarily large



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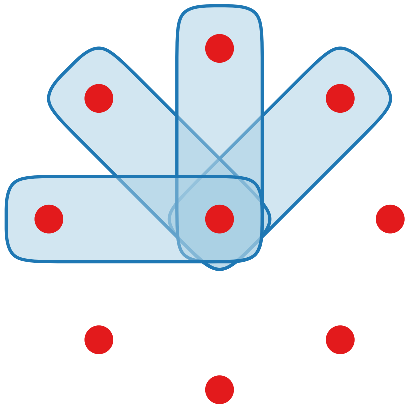

LP-Rounding: Approach I

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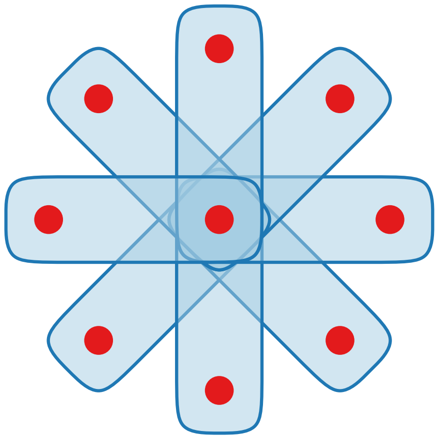
Scaling factor arbitrarily large



LP-Rounding: Approach I

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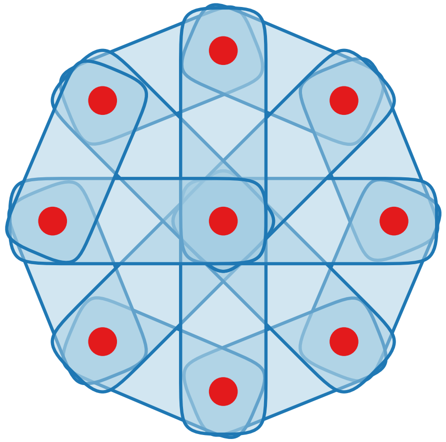
LP-Rounding-One($U, \mathcal{S}, c$)

Compute optimal solution x for LP-Relaxation.

Round each x_S with $x_S > 0$ to 1.

Generates a valid solution

Scaling factor arbitrarily large



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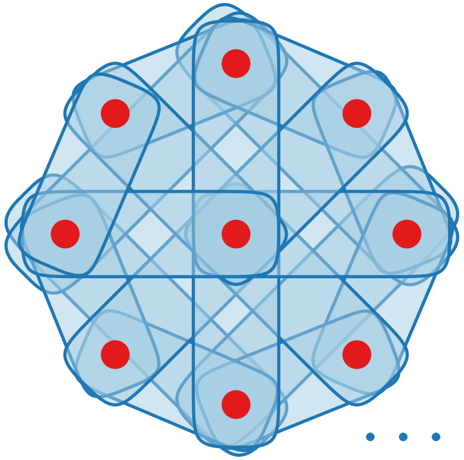

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LP-Rounding-One($U, \mathcal{S}, c$)

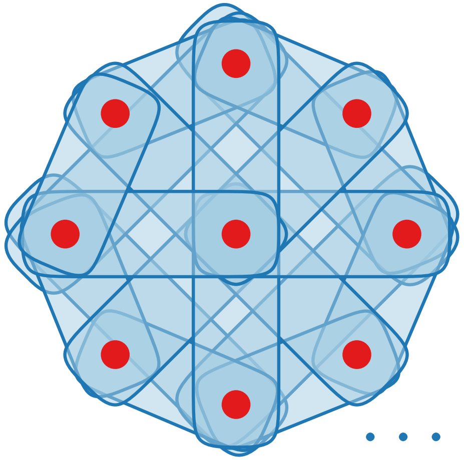
Compute optimal solution x for LP-Relaxation.

Round each x_S with $x_S > 0$ to 1.

Generates a valid solution

Scaling factor arbitrarily large

Use frequency f



LP-Rounding: Approach II

$$\begin{array}{lll} \textbf{minimize} & \sum_{S\in\mathcal{S}} c_S x_S & \\ \textbf{subject to} & \sum_{S\ni u} x_S \geq 1 & u \in U \\ & x_S \geq 0 & S \in \mathcal{S} \end{array}$$

LP-Rounding-Two($U, \mathcal{S}, c$)

Compute optimal solution x for LP-Relaxation.

Round each x_S with $\boxed{x_S \geq 1/f}$ to 1; remaining to 0.

Let f be the frequency of (number of sets containing) the most frequent element.

LP-Rounding: Approach II

$$\begin{array}{lll} \textbf{minimize} & \sum_{S \in \mathcal{S}} c_S x_S & \\ \textbf{subject to} & \sum_{S \ni u} x_S \geq 1 & u \in U \\ & x_S \geq 0 & S \in \mathcal{S} \end{array}$$

LP-Rounding-Two($U, \mathcal{S}, c$)

Compute optimal solution x for LP-Relaxation.
Round each x_S with $\boxed{x_S \geq 1/f}$ to 1; remaining to 0.

Let f be the frequency of (number of sets containing) the most frequent element.

Theorem. LP-Rounding-Two is a factor-f-approximation algorithm for SetCover.

Approximation Algorithms

Lecture 5:
LP-based Approximation Algorithms for SetCover

Part IV:
SetCover via Primal-Dual Schema

Joachim Spoerhase

Winter 2021/22

Complementary Slackness

$$\begin{array}{llll} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{llll} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Theorem. Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be valid solutions for the primal and dual program (resp.). Then x and y are optimal if and only if the following conditions are met:

Primal CS:
For each $j = 1, \ldots, n$: either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
For each $i = 1, \ldots, m$: either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

Relaxing Complementary Slackness

$$\begin{array}{llll} \textbf{minimize} & c^\top x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{llll} \textbf{maximize} & b^\top y & & \\ \textbf{subject to} & A^\top y & \leq & c \\ & y & \geq & 0 \end{array}$$

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For each $i = 1, \ldots, m$: either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

$$\Leftrightarrow \sum_{j=1}^{n} c_j x_j = \sum_{i=1}^{m} b_i y_i$$

Relaxing Complementary Slackness

minimize	$c^\top x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\top y$		
subject to	$A^\top y$	$\leq$	c
	y	$\geq$	0

~~**Primal CS**~~: Relaxed Primal CS

For each $j = 1, \ldots, n$: either $x_j = 0$ or $\cancel{\sum_{i=1}^m a_{ij} y_i = c_j}$

$$c_j / \alpha \leq \sum_{i=1}^m a_{ij} y_i \leq c_j$$

Dual CS:

For each $i = 1, \ldots, m$: either $y_i = 0$ or $\sum_{j=1}^n a_{ij} x_j = b_i$

$$\Leftrightarrow \sum_{j=1}^n c_j x_j = \sum_{i=1}^m b_i y_i$$

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$$\begin{array}{llll} \textbf{minimize} & c^\top x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

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~~**Primal CS**~~: Relaxed Primal CS

For each $j = 1, \ldots, n$: either $x_j = 0$ or $\cancel{\sum_{i=1}^m a_{ij} y_i = c_j}$

$$c_j / \alpha \leq \sum_{i=1}^m a_{ij} y_i \leq c_j$$

~~**Dual CS**~~: Relaxed Dual CS

For each $i = 1, \ldots, m$: either $y_i = 0$ or $\cancel{\sum_{j=1}^n a_{ij} x_j = b_i}$

$$b_i \leq \sum_{j=1}^n a_{ij} x_j \leq \beta \cdot b_i$$

$$\Leftrightarrow \sum_{j=1}^n c_j x_j = \sum_{i=1}^m b_i y_i$$

Relaxing Complementary Slackness

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

~~Primal CS~~: Relaxed Primal CS

For each $j = 1, \ldots, n$: either $x_j = 0$ or $\cancel{\sum_{i=1}^m a_{ij} y_i = c_j}$

$$c_j/\alpha \leq \sum_{i=1}^m a_{ij} y_i \leq c_j$$

~~Dual CS~~: Relaxed Dual CS

For each $i = 1, \ldots, m$: either $y_i = 0$ or $\cancel{\sum_{j=1}^n a_{ij} x_j = b_i}$

$$b_i \leq \sum_{j=1}^n a_{ij} x_j \leq \beta \cdot b_i$$

$$\Leftrightarrow \cancel{\sum_{j=1}^n c_j x_j = \sum_{i=1}^m b_i y_i} \qquad \Rightarrow \sum_{j=1}^n c_j x_j \leq \alpha\beta \sum_{i=1}^m b_i y_i \leq \alpha\beta \cdot \mathrm{OPT}_{\mathrm{LP}}$$

Primal-Dual Schema

Start with a feasible dual and infeasible primal solution (often trivial).

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Do so until the relaxed CS conditions are met.

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Do so until the relaxed CS conditions are met.

Maintain that the primal solution is integer valued.

Primal-Dual Schema

Start with a feasible dual and infeasible primal solution (often trivial).

"Improve" the feasibility of the primal solution...

$\dots$ and simultaneously the obj. value of the dual solution.

Do so until the relaxed CS conditions are met.

Maintain that the primal solution is integer valued.

The feasibility of the primal solution and relaxed CS condition provide an approximation ratio.

Relaxed CS for SetCover

$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{S \in \mathcal{S}} c_S x_S & \\ \textbf{subject to} & \displaystyle\sum_{S \ni u} x_S \geq 1 & u \in U \\ & x_S \geq 0 & S \in \mathcal{S} \end{array}$$

$$\begin{array}{lll} \textbf{maximize} & \displaystyle\sum_{u \in U} y_u & \\ \textbf{subject to} & \displaystyle\sum_{u \in S} y_u \leq c_S & S \in \mathcal{S} \\ & y_u \geq 0 & u \in U \end{array}$$

Relaxed CS for SetCover

minimize	$\displaystyle\sum_{S \in \mathcal{S}} c_S x_S$	
subject to	$\displaystyle\sum_{S \ni u} x_S \geq 1$	$u \in U$
	$x_S \geq 0$	$S \in \mathcal{S}$

maximize	$\displaystyle\sum_{u \in U} y_u$	
subject to	$\displaystyle\sum_{u \in S} y_u \leq c_S$	$S \in \mathcal{S}$
	$y_u \geq 0$	$u \in U$

(Unrelaxed) primal CS:

Relaxed CS for SetCover

minimize	$\sum_{S\in\mathcal{S}} c_S x_S$	
subject to	$\sum_{S\ni u} x_S \geq 1$	$u \in U$
	$x_S \geq 0$	$S \in \mathcal{S}$

maximize	$\sum_{u\in U} y_u$	
subject to	$\sum_{u\in S} y_u \leq c_S$	$S \in \mathcal{S}$
	$y_u \geq 0$	$u \in U$

(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow$

Relaxed CS for SetCover

minimize	$\sum_{S \in \mathcal{S}} c_S x_S$	
subject to	$\sum_{S \ni u} x_S \geq 1$	$u \in U$
	$x_S \geq 0$	$S \in \mathcal{S}$

maximize	$\sum_{u \in U} y_u$	
subject to	$\sum_{u \in S} y_u \leq c_S$	$S \in \mathcal{S}$
	$y_u \geq 0$	$u \in U$

(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

Relaxed CS for SetCover

minimize $\sum_{S \in \mathcal{S}} c_S x_S$	**maximize** $\sum_{u \in U} y_u$
subject to $\sum_{S \ni u} x_S \geq 1 \quad u \in U$	**subject to** $\sum_{u \in S} y_u \leq c_S \quad S \in \mathcal{S}$
$x_S \geq 0 \quad S \in \mathcal{S}$	$y_u \geq 0 \quad u \in U$

critical set

(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

Relaxed CS for SetCover

minimize	$\sum_{S \in \mathcal{S}} c_S x_S$	
subject to	$\sum_{S \ni u} x_S \geq 1$	$u \in U$
	$x_S \geq 0$	$S \in \mathcal{S}$

maximize	$\sum_{u \in U} y_u$	
subject to	$\sum_{u \in S} y_u \leq c_S$	$S \in \mathcal{S}$
	$y_u \geq 0$	$u \in U$

critical set

(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

only chooses critical sets

Relaxed CS for SetCover

$$\begin{array}{lll} \textbf{minimize} & \sum_{S \in \mathcal{S}} c_S x_S & \\ \textbf{subject to} & \sum_{S \ni u} x_S \geq 1 & u \in U \\ & x_S \geq 0 & S \in \mathcal{S} \end{array}$$

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critical set

(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

only chooses critical sets

Relaxed dual CS:

Relaxed CS for SetCover

minimize	$\sum_{S \in \mathcal{S}} c_S x_S$	
subject to	$\sum_{S \ni u} x_S \geq 1$	$u \in U$
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maximize	$\sum_{u \in U} y_u$	
subject to	$\sum_{u \in S} y_u \leq c_S$	$S \in \mathcal{S}$
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Relaxed dual CS: $y_u \neq 0 \Rightarrow$

Relaxed CS for SetCover

minimize	$\sum_{S \in \mathcal{S}} c_S x_S$	
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(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

only chooses critical sets

Relaxed dual CS: $y_u \neq 0 \Rightarrow 1 \leq \sum_{S \ni u} x_S \leq f \cdot$

Relaxed CS for SetCover

minimize	$\sum_{S \in \mathcal{S}} c_S x_S$	
subject to	$\sum_{S \ni u} x_S \geq 1$	$u \in U$
	$x_S \geq 0$	$S \in \mathcal{S}$

maximize	$\sum_{u \in U} y_u$	
subject to	$\sum_{u \in S} y_u \leq c_S$	$S \in \mathcal{S}$
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Relaxed CS for SetCover

$$\begin{array}{lll} \textbf{minimize} & \sum_{S \in \mathcal{S}} c_S x_S & \\ \textbf{subject to} & \sum_{S \ni u} x_S \geq 1 & u \in U \\ & x_S \geq 0 & S \in \mathcal{S} \end{array}$$

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(Unrelaxed) primal CS: $x_S \neq 0 \Rightarrow \sum_{u \in S} y_u = c_S$

only chooses critical sets

trivial for binary x

Relaxed dual CS: $y_u \neq 0 \Rightarrow 1 \leq \sum_{S \ni u} x_S \leq f \cdot 1$

Primal-Dual-Schema for SetCover

PrimalDualSetCover$(U, \mathcal{S}, c)$

$x \leftarrow 0, y \leftarrow 0$

repeat

until all elements are covered.

return x

Primal-Dual-Schema for SetCover

PrimalDualSetCover($U, \mathcal{S}, c$)

$x \leftarrow 0, y \leftarrow 0$
repeat
  Select an uncovered element u.

until all elements are covered.
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Primal-Dual-Schema for SetCover

PrimalDualSetCover($U, \mathcal{S}, c$)

$x \leftarrow 0, y \leftarrow 0$

repeat

Select an uncovered element u.

Increase y_u until a set S is critical ($\sum_{u' \in S} y_{u'} = c_S$).

until all elements are covered.

return x

Primal-Dual-Schema for SetCover

PrimalDualSetCover$(U, \mathcal{S}, c)$

$x \leftarrow 0, y \leftarrow 0$
repeat
  Select an uncovered element u.
  Increase y_u until a set S is critical ($\sum_{u' \in S} y_{u'} = c_S$).
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  Select all critical sets and update x.
  Mark all elements in these sets as covered.
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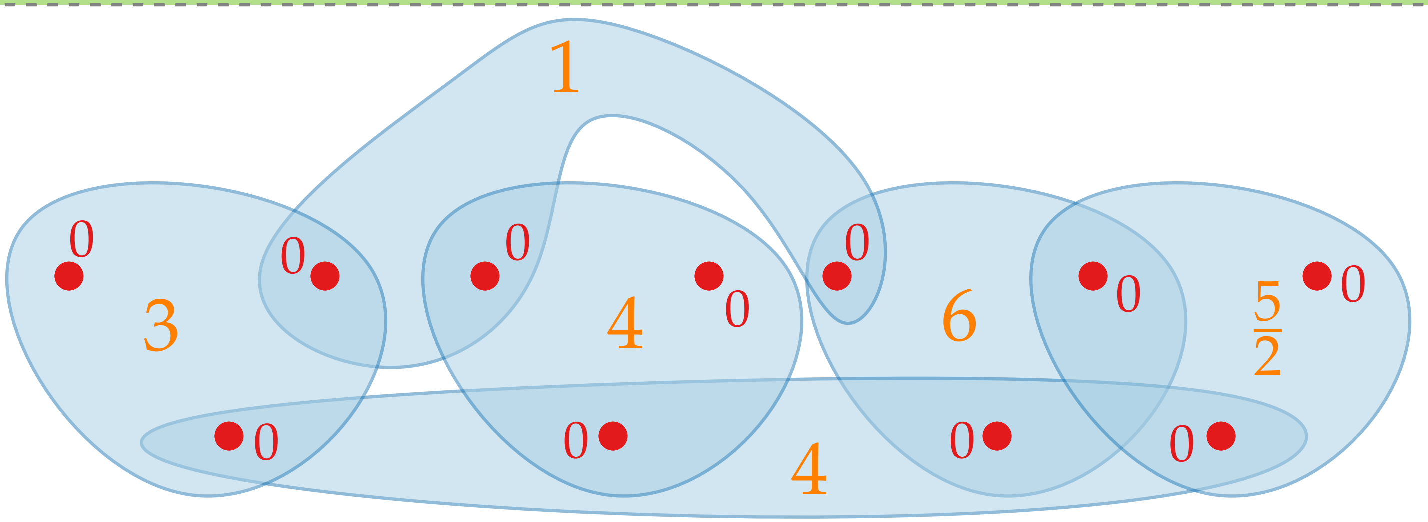

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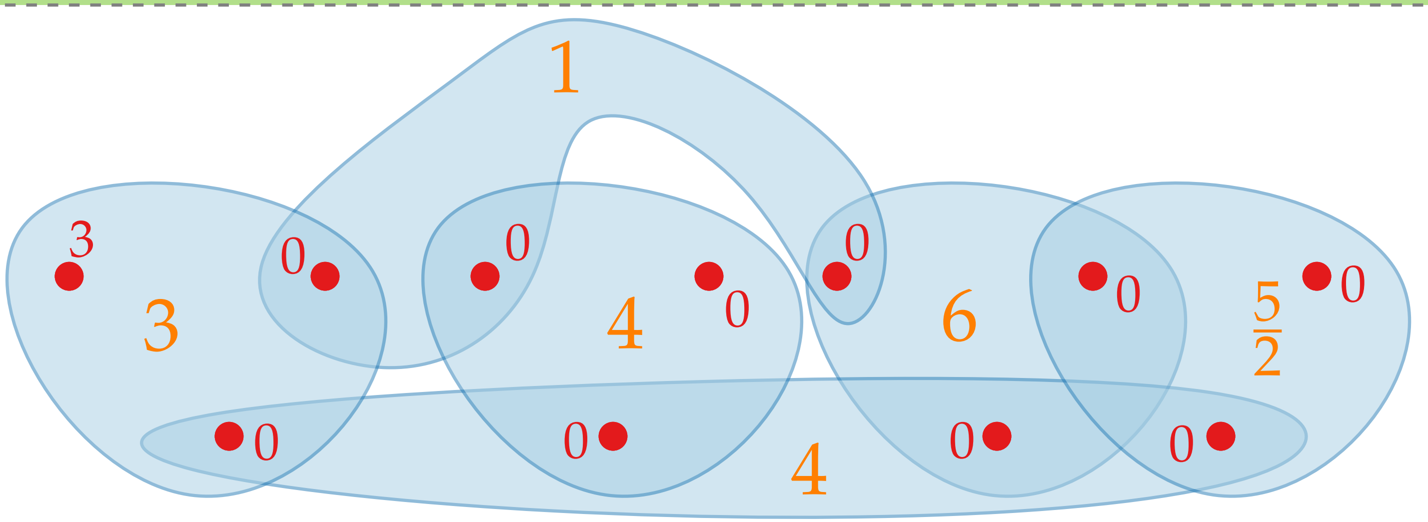

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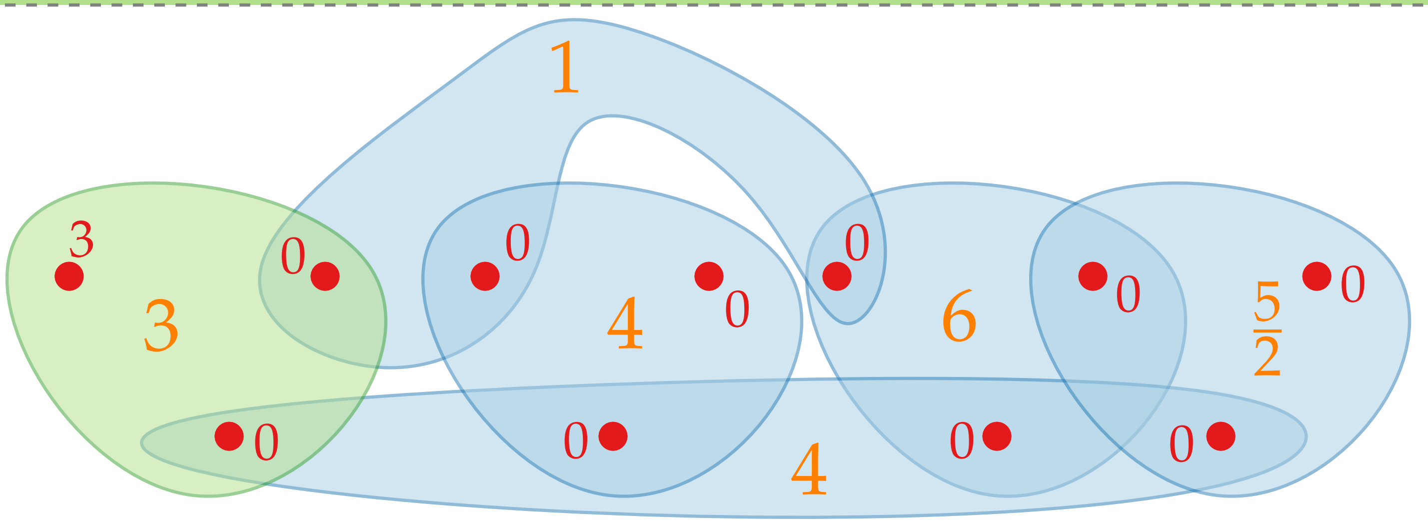

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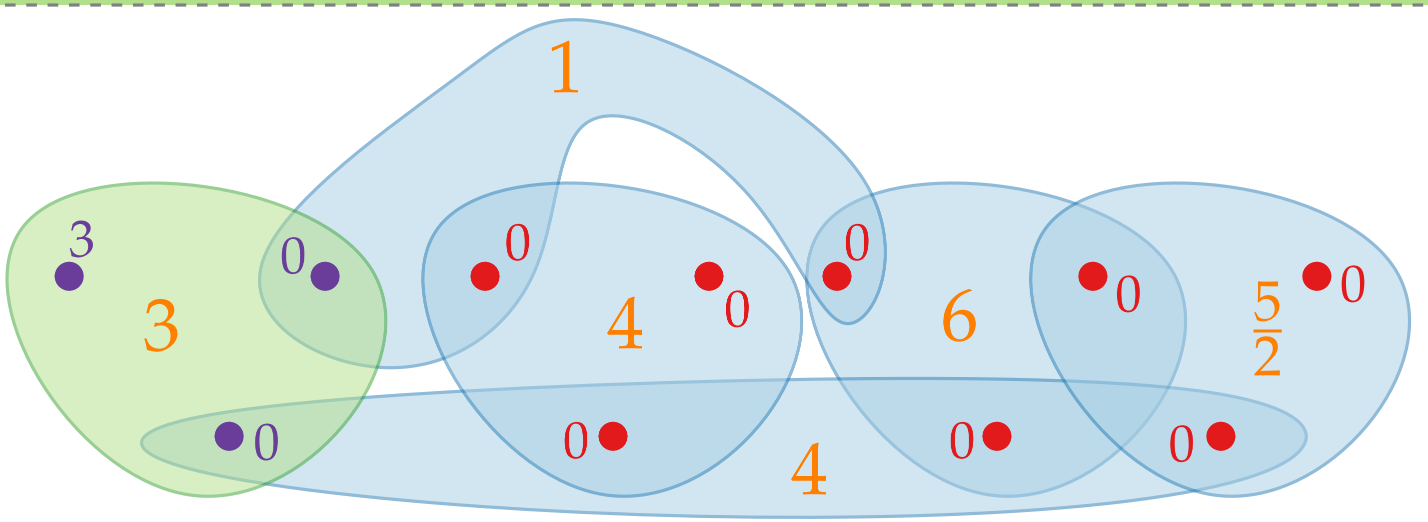

Primal-Dual-Schema for SetCover

PrimalDualSetCover$(U, \mathcal{S}, c)$

$x \leftarrow 0, y \leftarrow 0$

repeat

Select an uncovered element u.

Increase y_u until a set S is critical ($\sum_{u' \in S} y_{u'} = c_S$).

Select all critical sets and update x.

Mark all elements in these sets as covered.

until all elements are covered.

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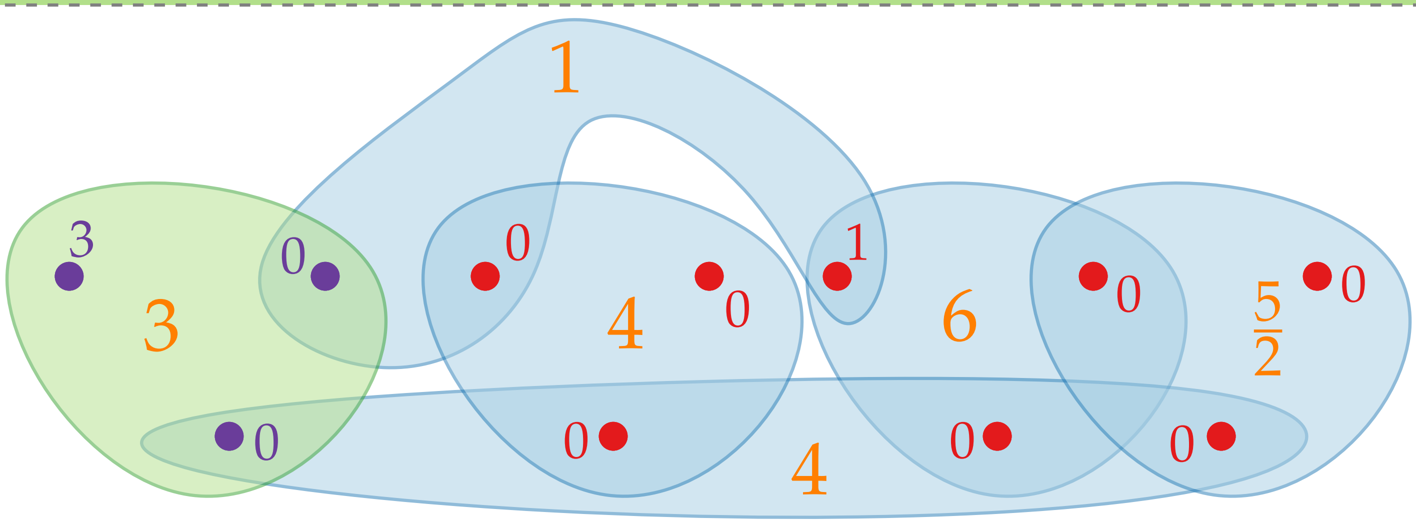

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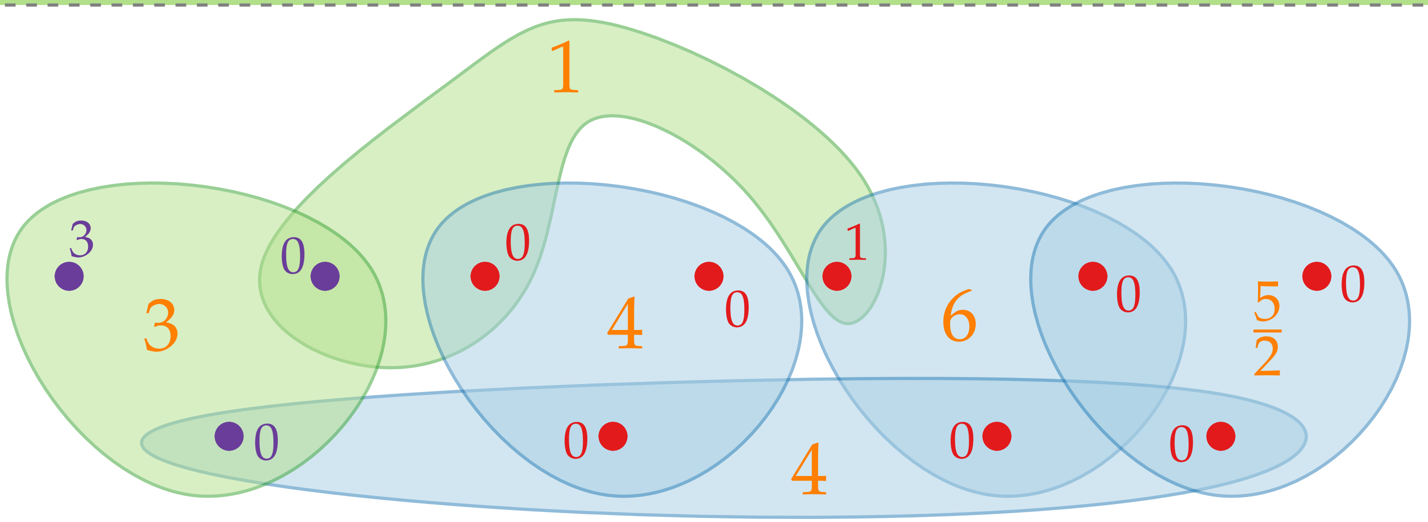

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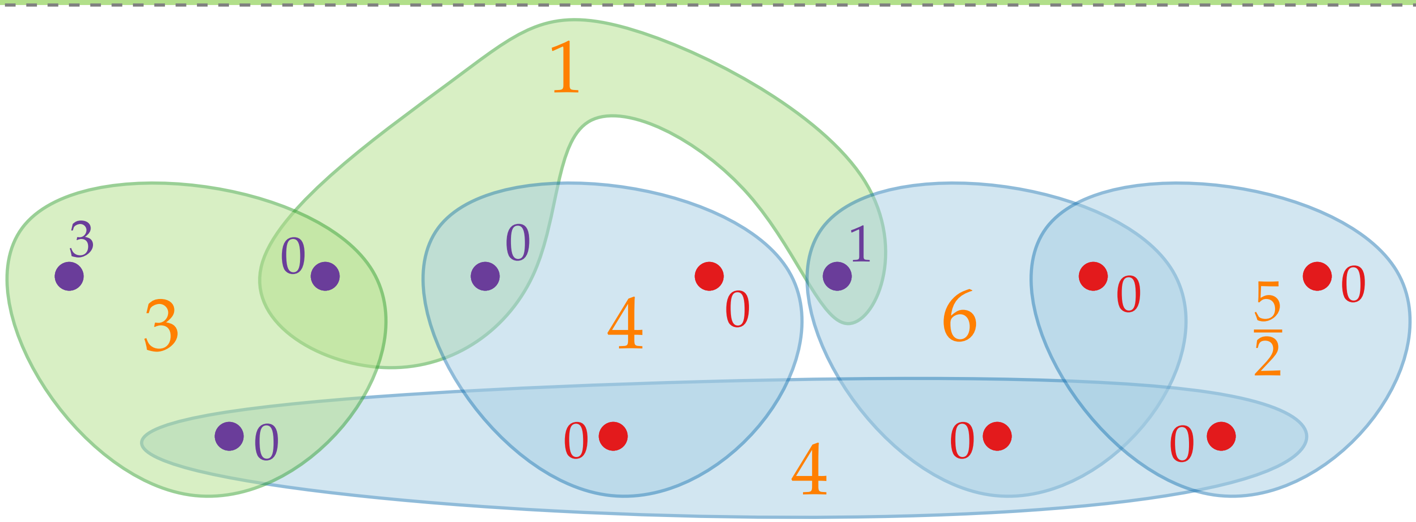

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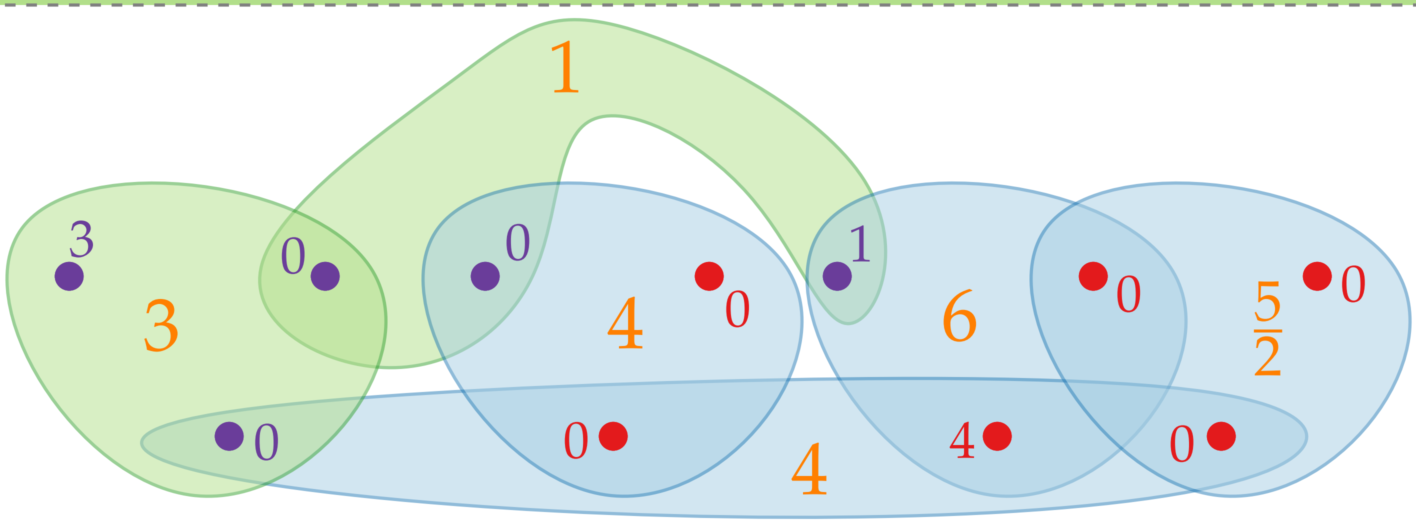

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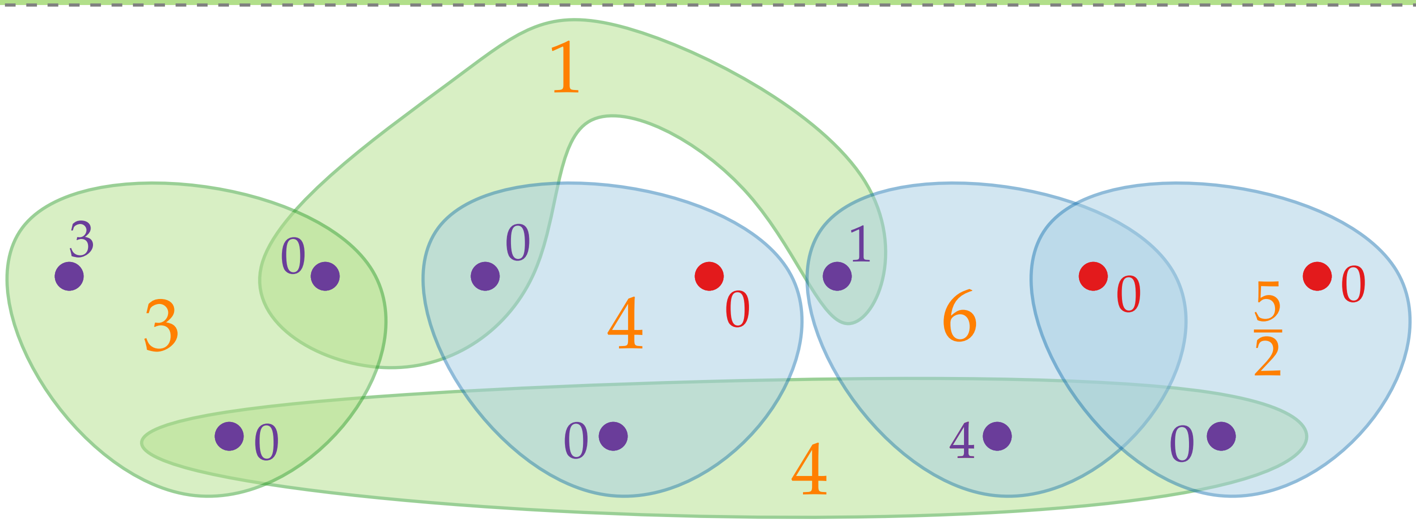

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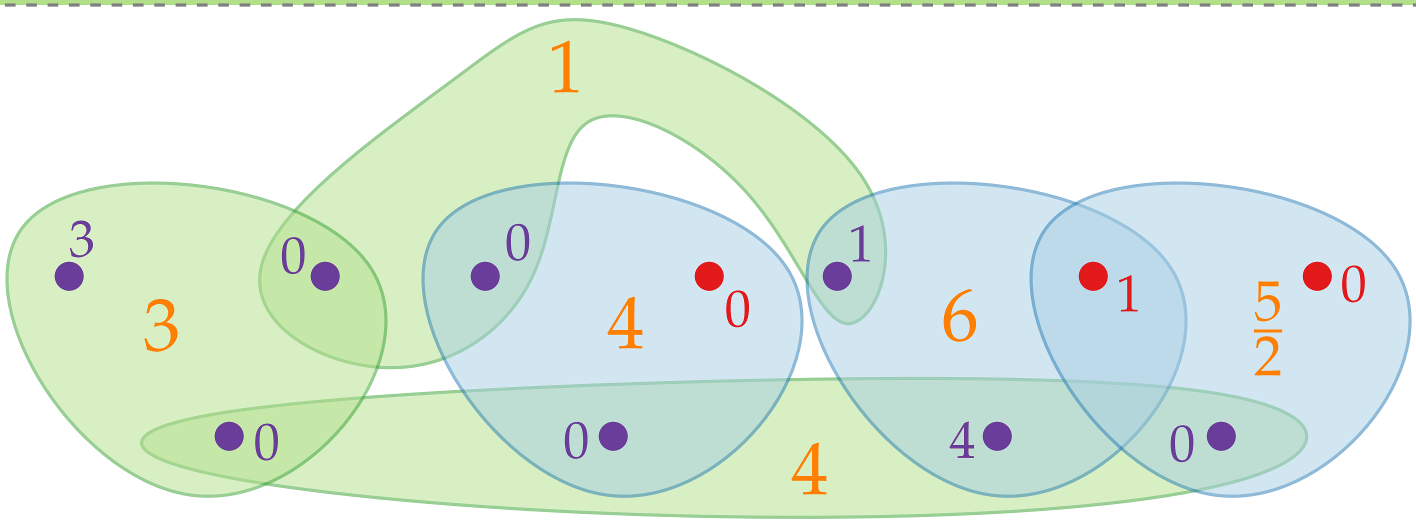

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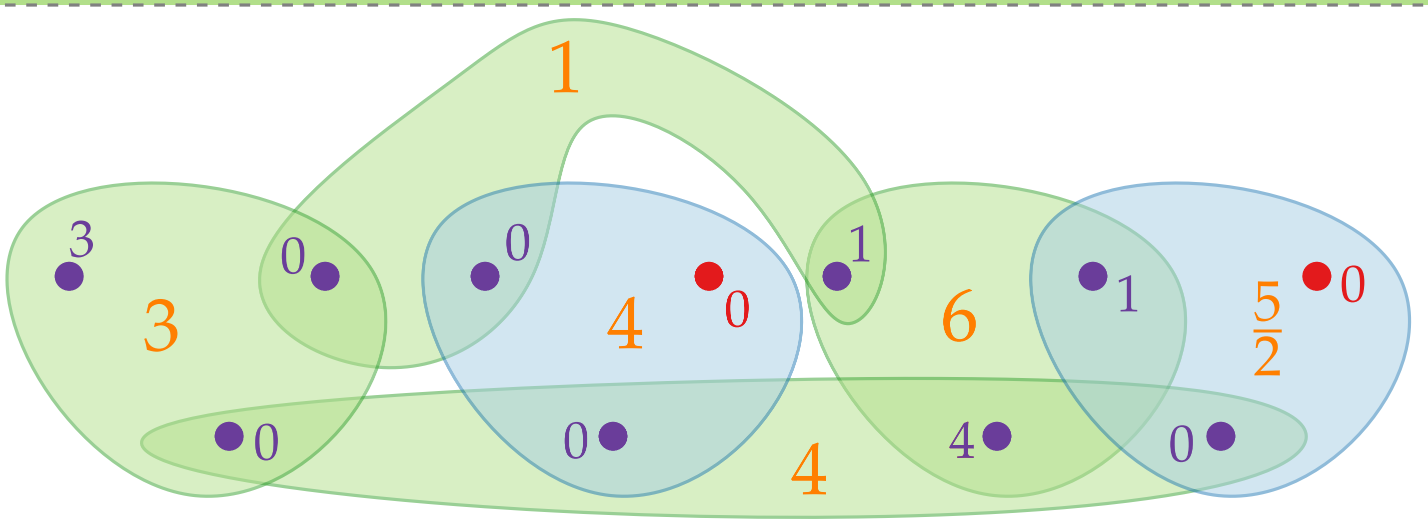

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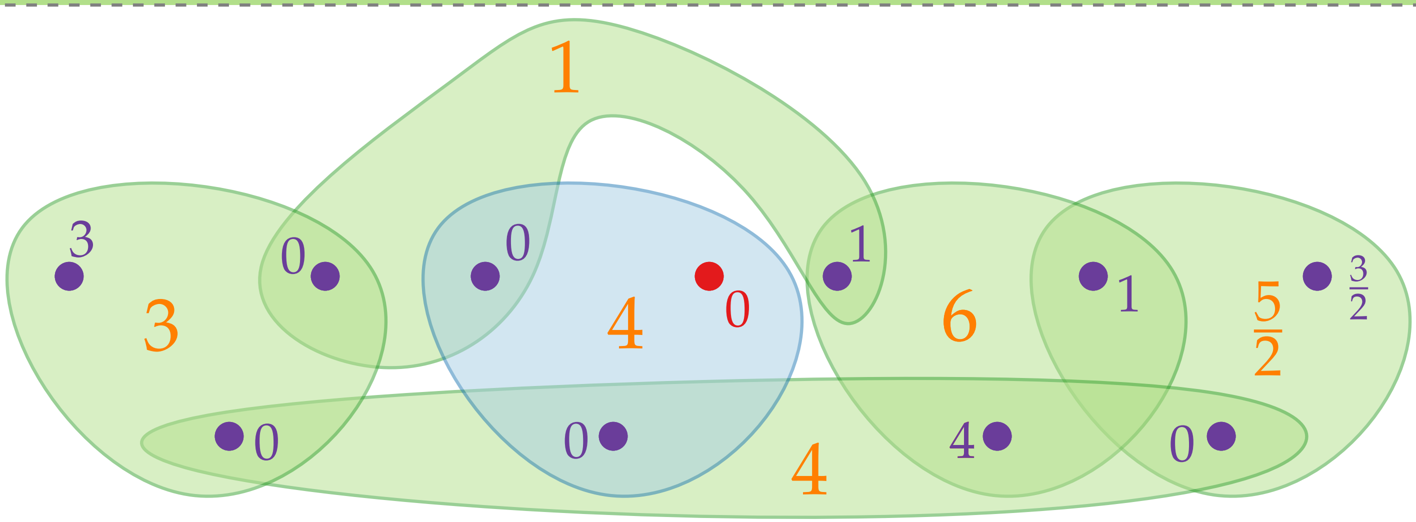

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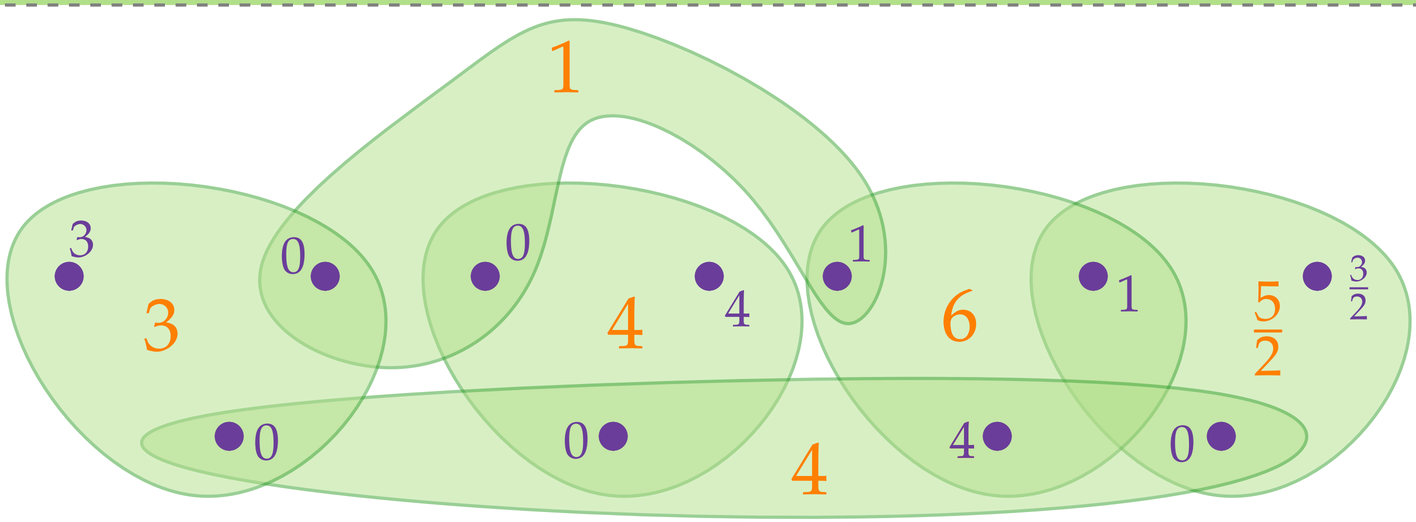

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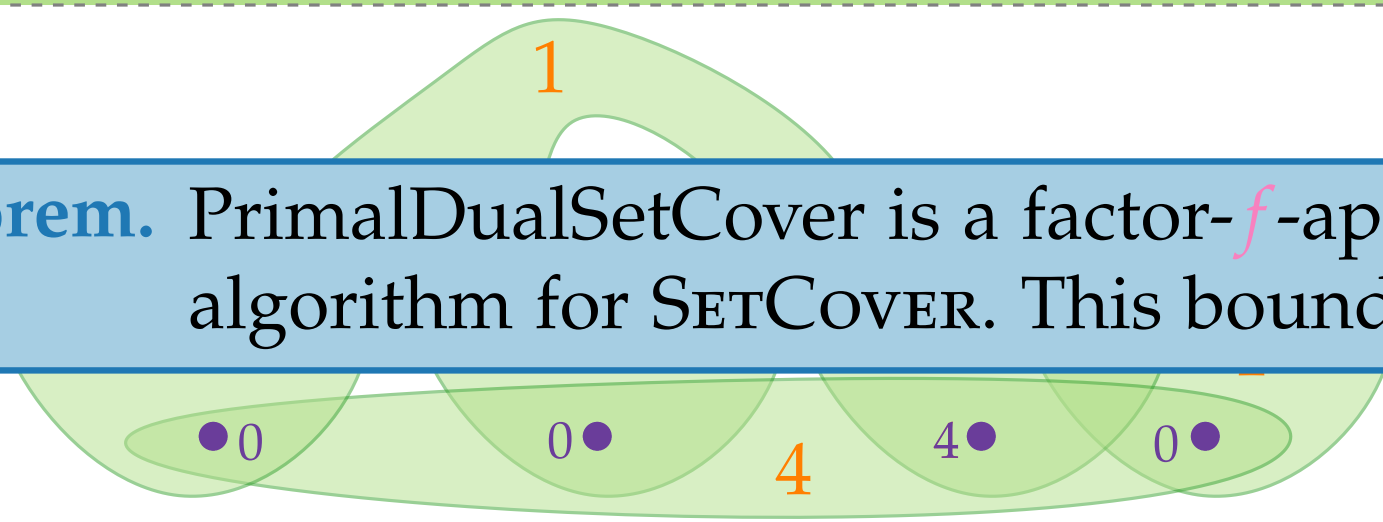

Theorem. PrimalDualSetCover is a factor-f-approximation algorithm for SetCover. This bound is tight.

Tight Example

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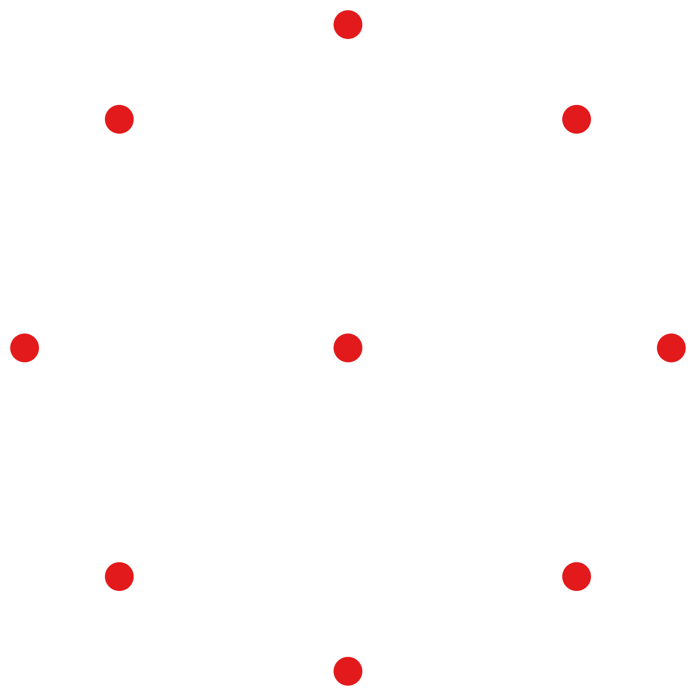

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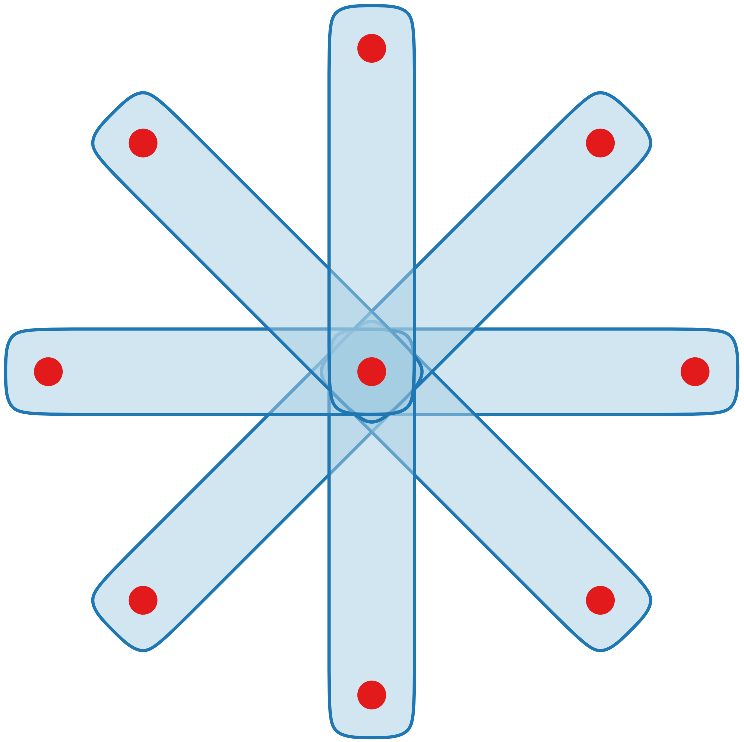

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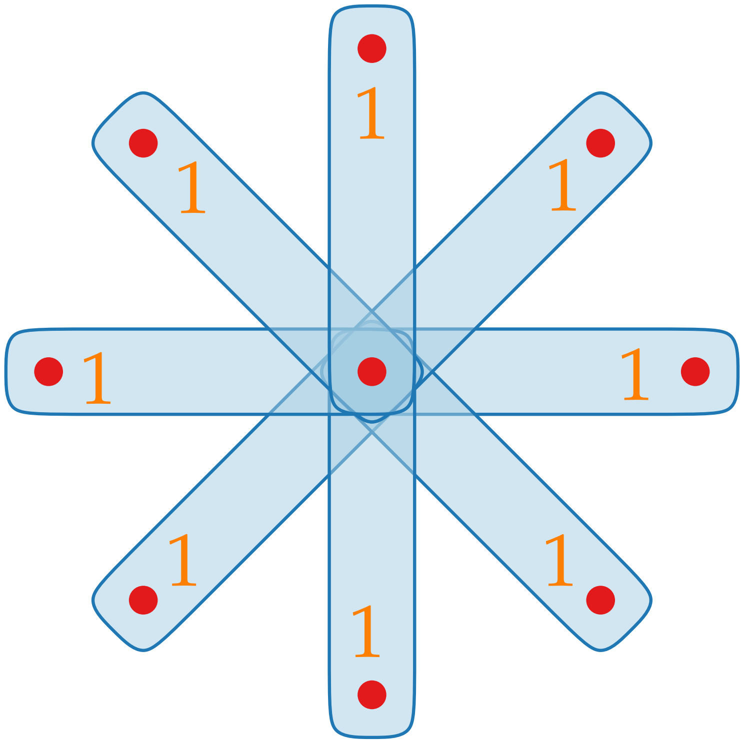

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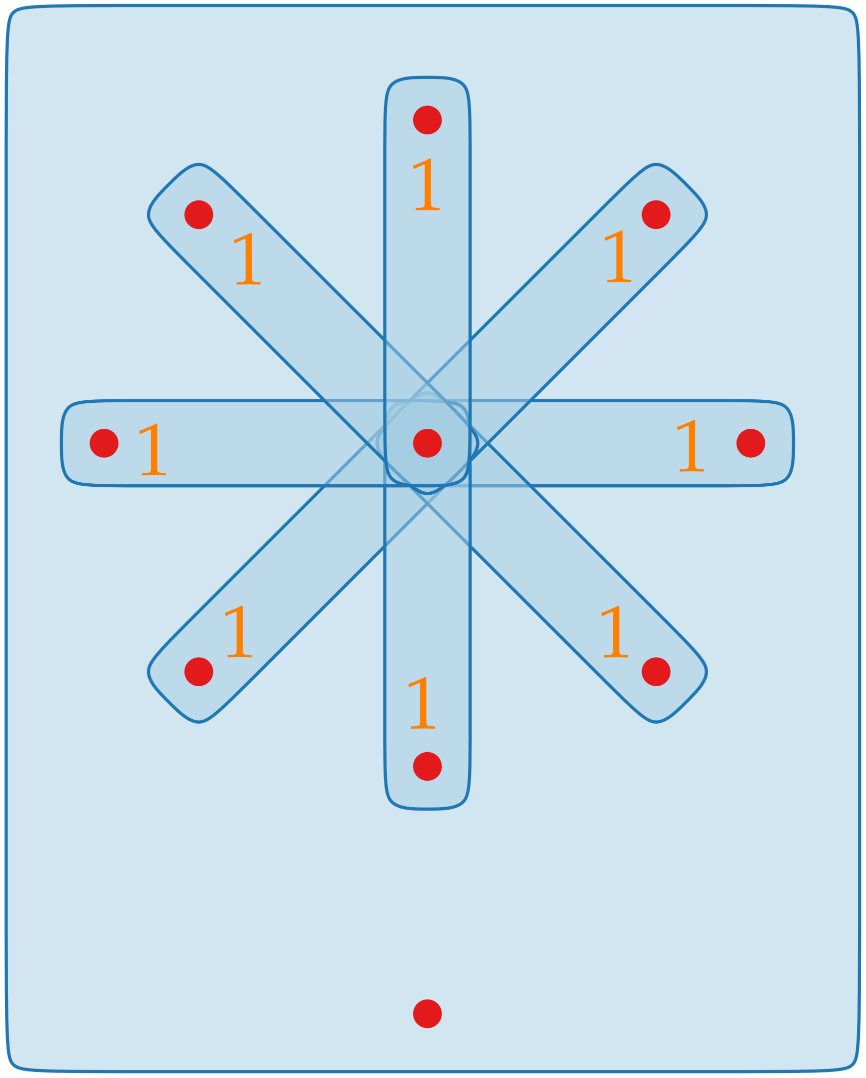

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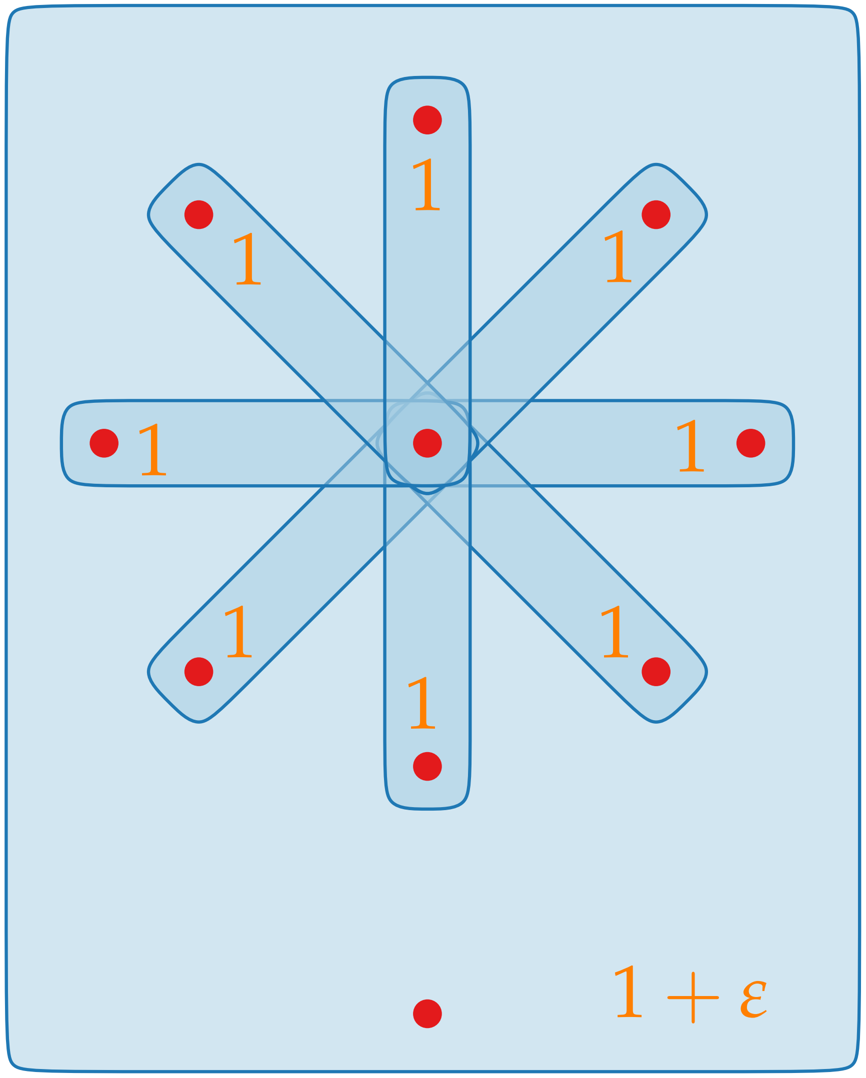

Approximation Algorithms

Lecture 5: LP-based Approximation Algorithms for SetCover

Part V: SetCover via Dual Fitting

Joachim Spoerhase

Winter 2021/22

Dual Fitting for SetCover

Combinatorial (greedy) algorithm (see Lecture 02):

GreedySetCover($U, \mathcal{S}, c$)

$C \leftarrow \emptyset$
$\mathcal{S}' \leftarrow \emptyset$
while $C \neq U$ **do**
  $S \leftarrow$ Set from $\mathcal{S}$ that minimizes $\frac{c(S)}{|S \setminus C|}$
  foreach $u \in S \setminus C$ **do**
    $\text{price}(u) \leftarrow \frac{c(S)}{|S \setminus C|}$
  $C \leftarrow C \cup S$
  $\mathcal{S}' \leftarrow \mathcal{S}' \cup \{S\}$
return $\mathcal{S}'$ // Cover of U

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Reminder: $\sum_{u \in U} \text{price}(u)$ ...

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Reminder: $\sum_{u \in U} \text{price}(u)$ completely pays for $\mathcal{S}'$.

New: LP-based Analysis

Observation. For each $u \in U$, $\text{price}(u)$ is a dual variable

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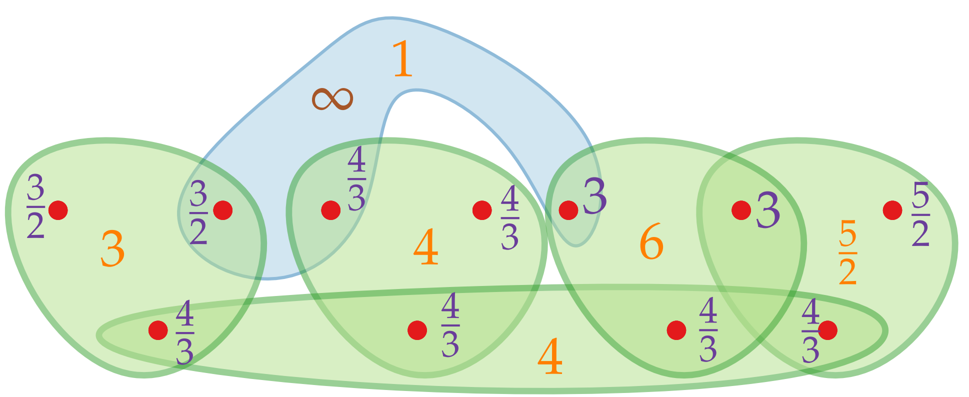

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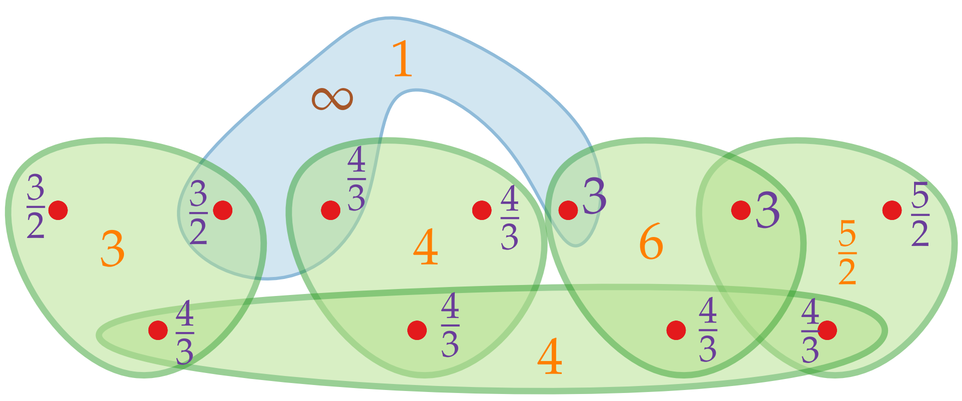

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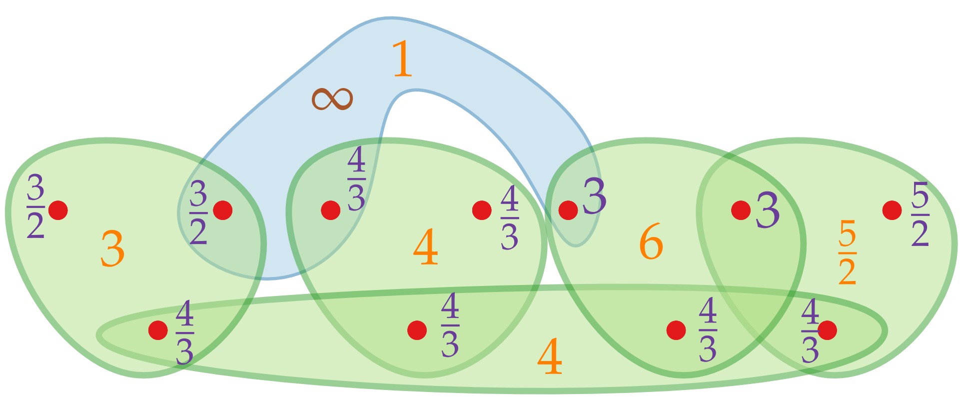

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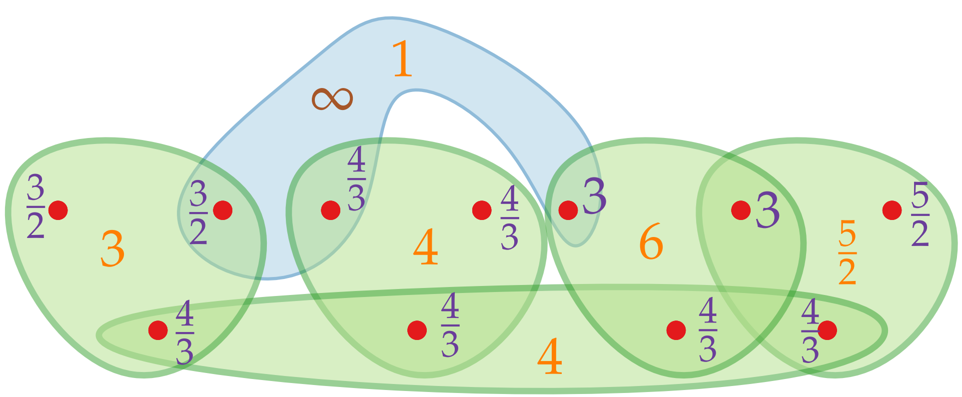
Scale duale variables such that no set is overpacked.



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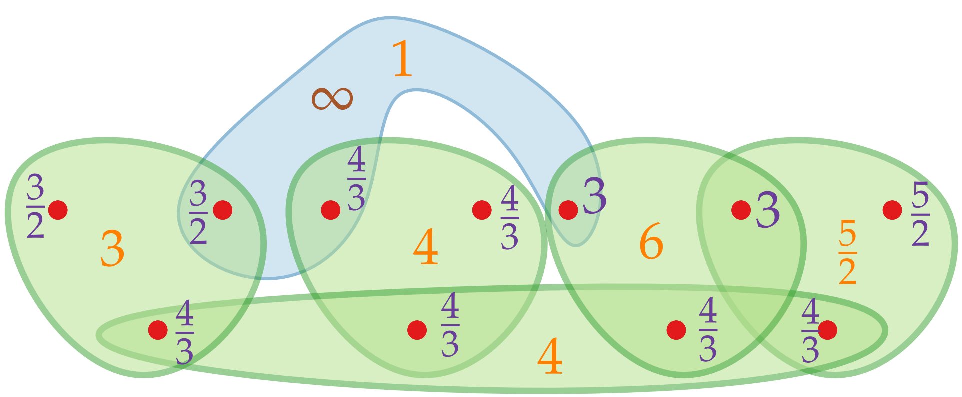

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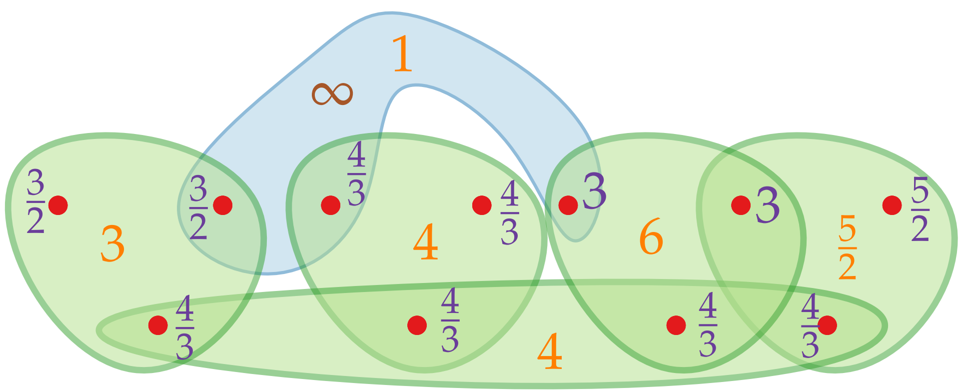

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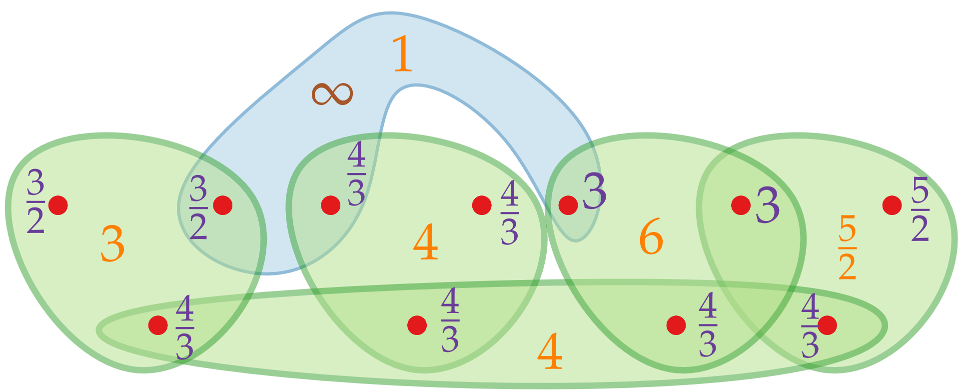

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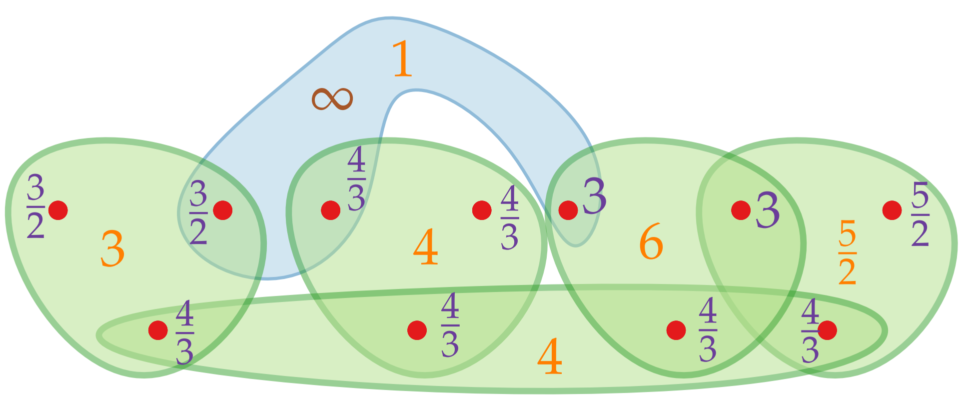

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Proof. To prove: No set is overpacked by y.

Let $S \in \mathcal{S}$ and $\ell = |S| \leq k$.

Let $u_1, \dots, u_\ell$ be the elements of S –
in the order they are covered by the greedy alg.

Consider the iteration in which u_i is covered

Before that, $\geq \ell - i + 1$ elem. of S are uncovered.

So $\mathrm{price}(u_i) \leq c(S)/(\ell - i + 1)$.

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Theorem. GreedySetCover is a factor-$\mathcal{H}_k$-approximation algorithm for SetCover, where $k = \max_{S \in \mathcal{S}} |S|$.

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Strengthened bound with respect to $\text{OPT}_{\text{relax}} \le \text{OPT}$.

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Dual solution allows a *per-instance* estimation

…which might be stronger than worst-case bound $\mathcal{H}_k$