# Approximation Algorithms

## Lecture 4:
## Linear Programming and LP-Duality

### Part I:
### Introduction to Linear Programming

Joachim Spoerhase

Winter 2020/21

# Maximizing Profits

You're the boss of a small company that produces two products $P_1$ and $P_2$. For the production of $x_1$ units of $P_1$ and $x_2$ units of $P_2$, you're profit in € is:

$$G(x_1, x_2) = 30x_1 + 50x_2$$

Three machines $M_A$, $M_B$ and $M_C$ produce the required components $A$, $B$ and $C$ for the products. The components are used in different quantities for the products, and each machine requires some time for the production.

$$\begin{aligned} M_A: &\quad 4x_1 + 11x_2 \leq 880 \\ M_B: &\quad x_1 + x_2 \leq 150 \\ M_C: &\quad x_2 \leq 60 \end{aligned}$$

Which choice of $(x_1, x_2)$ maximizes the profit?

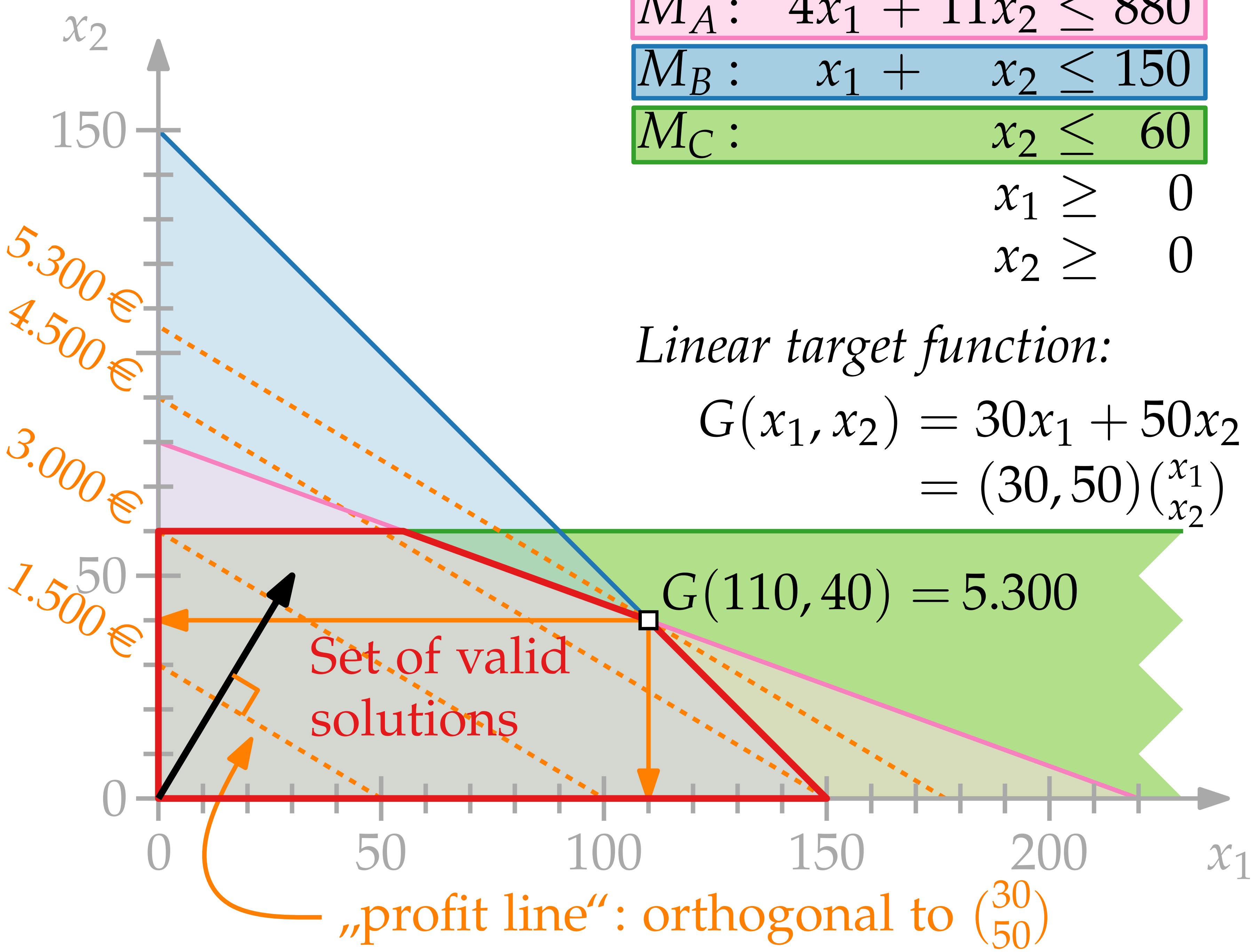
# Solution

*Linear constraints:*

$$
\begin{aligned}
M_A: &\quad 4x_1 + 11x_2 \leq 880 \\
M_B: &\quad x_1 + x_2 \leq 150 \\
M_C: &\quad x_2 \leq 60 \\
&\quad x_1 \geq 0 \\
&\quad x_2 \geq 0
\end{aligned}
$$

*Linear target function:*

$$
\begin{aligned}
G(x_1, x_2) &= 30x_1 + 50x_2 \\
&= (30, 50)\binom{x_1}{x_2}
\end{aligned}
$$



# Approximation Algorithms

## Lecture 4:
## Linear Programming and LP-Duality

### Part II:
### Upper Bounds for LPs

Joachim Spoerhase

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# Motivation: Upper and Lower Bounds

Consider hard NP-Minimization Problem.

Decision Problem:
For given $S$, is $\mathrm{obj}(S)$ an **upper bound** for OPT?

Efficiently verifiable "Yes"-certificates.

**Lower bounds** / "no"-certificates?
$\rightsquigarrow$ probably not! (conjecture: NP $\neq$ coNP)

Need lower bound $\mathrm{obj}(S) \leq \mathrm{OPT}/\alpha$
(approximate "no"-certificates)
for approximation algorithms!

Examples:

- Vertex Cover: lower bound by matchings
- TSP: lower bound by MST or Cycle Cover

# Linear Programming

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

Standard form (HA)

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

**Example.** $c = \begin{pmatrix} 7 \\ 1 \\ 5 \end{pmatrix}$ $A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & 2 & -1 \end{pmatrix}$ $b = \begin{pmatrix} 10 \\ 6 \end{pmatrix}$

$$\begin{array}{lrcrcrcr} \textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\ & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

# Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$\begin{array}{llcllcllcl} \textbf{minimize} & 7x_1 \, 14 & + & x_2 \, 1 & + & 5x_3 \, 15 & = & 30 \\ \textbf{subject to} & x_1 \, 2 & - & x_2 \, 1 & + & 3x_3 \, 9 & \geq & 10 \, 10 \\ & 5x_1 \, 10 & + & 2x_2 \, 2 & - & x_3 \, 3 & \geq & 6 \, 9 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

Valid solution?

$x = (2, 1, 3)$

$\Rightarrow$ obj$(x) = 30$ is upper bound for OPT

# Approximation Algorithms

## Lecture 4:
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### Part III:
### Lower Bounds for LPs

Joachim Spoerhase

Winter 2020/21

# Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \text{\tiny IV} & & \text{\tiny IV} & & \text{\tiny IV} & & \\
\textbf{subject to} & 2\cdot x_1 & - & 2\cdot x_2 & + & 2\cdot 3x_3 & \geq & 2\cdot 10 \\
 & + & & + & & + & & \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \text{OPT} \geq 10$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 10 + 6 \qquad \Rightarrow \text{OPT} \geq 16
\end{aligned}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq 2 \cdot (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 2 \cdot 10 + 6 \qquad \Rightarrow \text{OPT} \geq 26
\end{aligned}
$$

# Linear Programming – Lower Bounds

**Primal**

$$\begin{array}{llcl}
\textbf{minimize} & 7x_1 + x_2 + 5x_3 & & \\
\textbf{subject to} & y_1(x_1 - x_2 + 3x_3) & \geq & 10\,y_1 \\
 & y_2(5x_1 + 2x_2 - x_3) & \geq & 6\,y_2 \\
 & x_1, x_2, x_3 & \geq & 0
\end{array}$$

$$7x_1 + x_2 + 5x_3 \geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3)$$
$$\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2$$

**Dual**

$$\begin{array}{llcl}
\textbf{maximize} & 10y_1 + 6y_2 & & \\
\textbf{subject to} & y_1 + 5y_2 & \leq & 7 \\
 & -y_1 + 2y_2 & \leq & 1 \\
 & 3y_1 - y_2 & \leq & 5 \\
 & y_1, y_2 & \geq & 0
\end{array}$$

Any feasible solution to the **dual** program provides a lower bound for the optimum of the **primal** program.

$x = (7/4, 0, 11/4)$ both $y = (2, 1)$ provide objective value 26. (26 = OPT)

# Primal – Dual

Primal Program

$$\begin{array}{llll} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Dual Program

$$\begin{array}{llll} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Dual Program of the Dual Program

$$\begin{array}{llll} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

# Approximation Algorithms

## Lecture 4:
## Linear Programming and LP-Duality

### Part IV:
### LP-Duality and Complementary Slackness

Joachim Spoerhase

Winter 2020/21

# LP-Duality

Primal

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Dual

$$\begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

**Theorem.** The primal program has a finite optimum $\Leftrightarrow$ the dual program has a finite optimum. Moreover, if $x^* = (x_1^*, \ldots, x_n^*)$ and $y^* = (y_1^*, \ldots, y_m^*)$ are *optimal* solutions for the primal and dual program (resp.), then

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^* \,.$$

# Weak LP-Duality

| | | |
|---|---|---|
| **minimize** | $c^\intercal x$ | |
| **subject to** | $Ax \geq b$ | |
| | $x \geq 0$ | |

| | | |
|---|---|---|
| **maximize** | $b^\intercal y$ | |
| **subject to** | $A^\intercal y \leq c$ | |
| | $y \geq 0$ | |

**Theorem.** If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program (resp.), then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

**Proof.**

$$\sum_{j=1}^{n} c_j x_j \overset{\geq}{\geq} \sum_{j=1}^{n} \left( \sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} a_{ij} x_j \right) y_i \overset{\geq}{\geq} \sum_{i=1}^{m} b_i y_i \,.$$

# Complementary Slackness

$$\begin{array}{ll} \textbf{minimize} & c^\intercal x \\ \textbf{subject to} & Ax \geq b \\ & x \geq 0 \end{array}$$

$$\begin{array}{ll} \textbf{maximize} & b^\intercal y \\ \textbf{subject to} & A^\intercal y \leq c \\ & y \geq 0 \end{array}$$

**Theorem.** Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ be valid solutions for the primal and dual program (resp.). Then $x$ and $y$ are optimal if and only if the following conditions are met:

**Primal CS**:
For each $j = 1, \ldots, n$: either $x_j = 0$ or $\sum_{i=1}^m a_{ij} y_i = c_j$

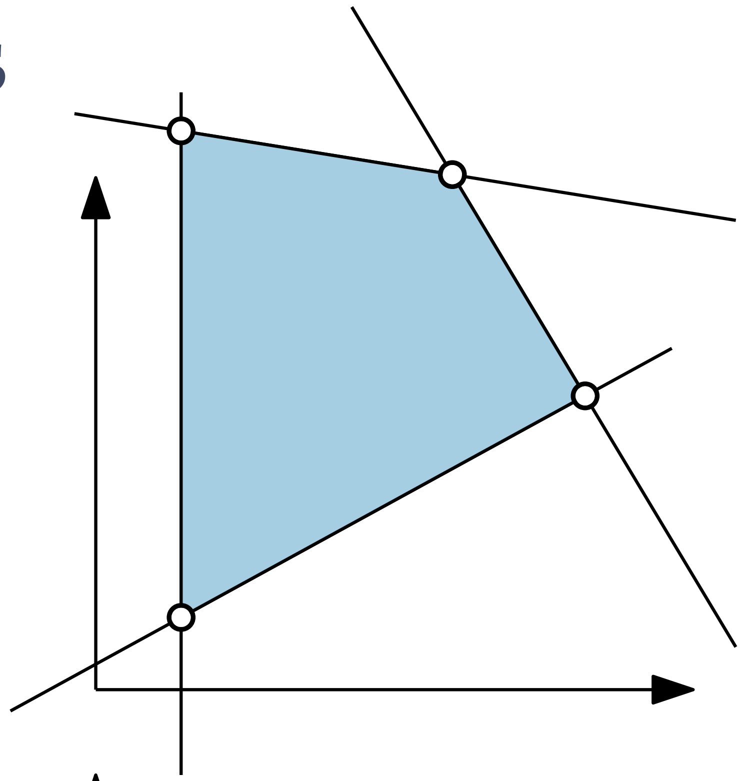
**Dual CS**:
For each $i = 1, \ldots, m$: either $y_i = 0$ or $\sum_{j=1}^n a_{ij} x_j = b_i$

**Proof.** Follows from LP-Duality:

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{j=1}^{n} \left( \sum_{i=1}^{m} a_{ij} y_i \right) x_j = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} a_{ij} x_j \right) y_i \geq \sum_{i=1}^{m} b_i y_i \,.$$

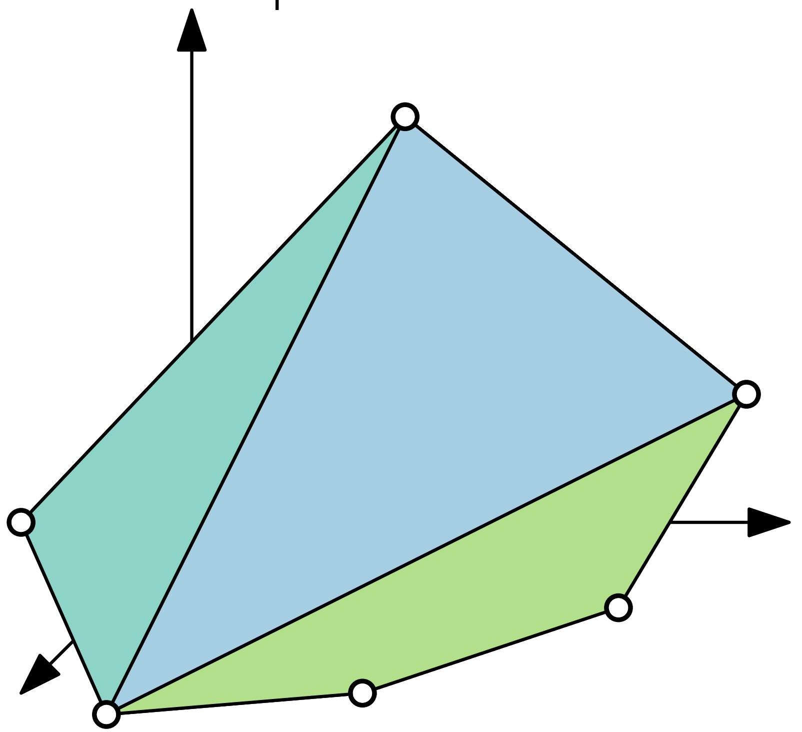
$=$ $=$

# LPs and Convex Polytopes



The feasible solutions of an LP with $n$ variables from a **convex polytope** in $\mathbb{R}^n$ (intersection of halfspaces).

Corners of the polytope are called **extreme point solutions** $\Leftrightarrow$ $n$ linearly independent inequalities (constraints) are satisfied with equality.



When an optimal solution exists, some extreme point will also be optimal.

# Integer Linear Programs (ILPs)

$$\begin{array}{lrcl} \textbf{minimize} & c^\top x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{lrcl} \textbf{minimize} & c^\top x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \in & \mathbb{N} \end{array}$$

Many NP-optimization problems can be formulated as ILPs; thus ILPs are NP-hard to solve.

LP-Relaxation provides lower bound: $\mathrm{OPT}_{\mathrm{LP}} \leq \mathrm{OPT}_{\mathrm{ILP}}$

# Approximation Algorithms

## Lecture 4:
## Linear Programming and LP-Duality

### Part V:
### Min-Max-Relationships

Joachim Spoerhase

Winter 2020/21

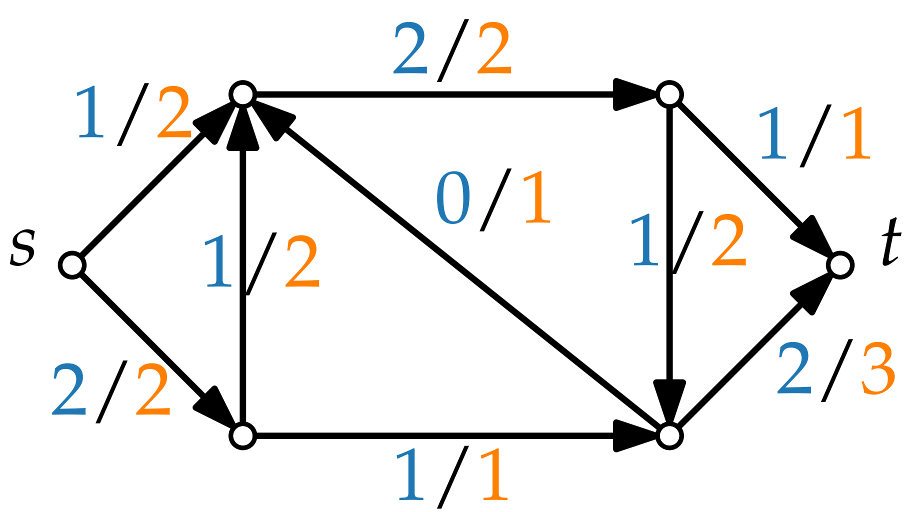
# Max-Flow-Problem

**Given**: A directed graph $G = (V, E)$ with edge capacities $c\colon E \to \mathbb{Q}_+$ and two special vertices: the source $s$ and sink $t$.

**Find**: A maximum $s$–$t$-flow (i.e., non-negative edge weights $f$), such that

- $f(u,v) \leq c(u,v)$ for each edge $(u,v) \in E$
- $\sum_{(u,v)\in E} f(u,v) = \sum_{(v,z)\in E} f(v,z)$ for each vertex $v \in V - \{s,t\}$

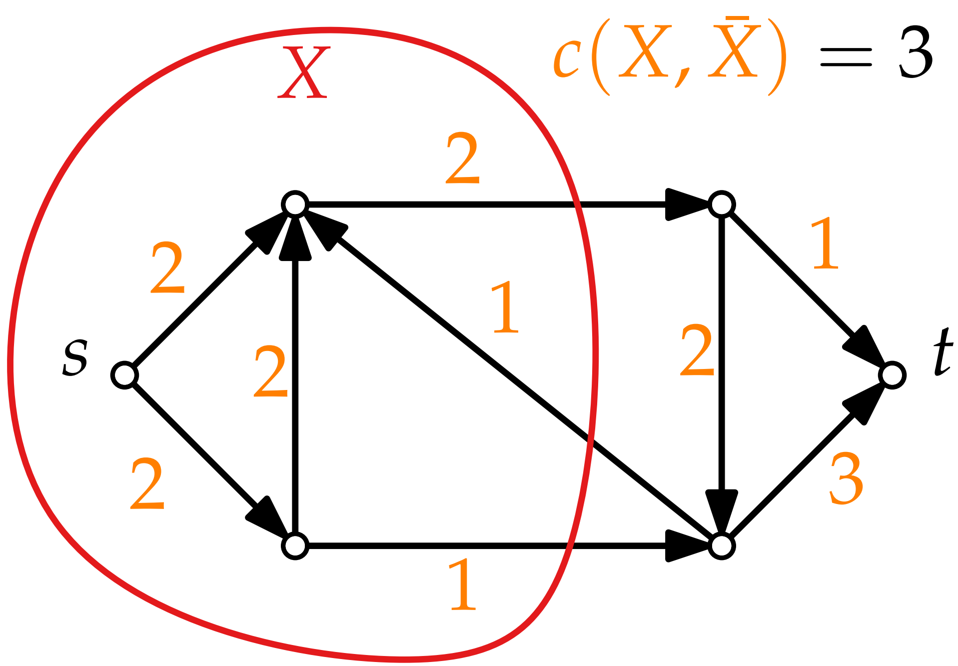
The **flow value** is the inflow to $t$ minus the outflow from $t$.



# Min-Cut-Problem

**Given**: A directed graph $G = (V, E)$ with edge capacities $c\colon E \to \mathbb{Q}_+$ and two special vertices: the source $s$ and sink $t$.

**Find**: An $s$–$t$-cut, i.e., a vertex set $X$ with $s \in X$ and $t \in \overline{X}$, such that the total weight $c(X, \overline{X})$ of the edges from $X$ to $\overline{X}$ is minimum.

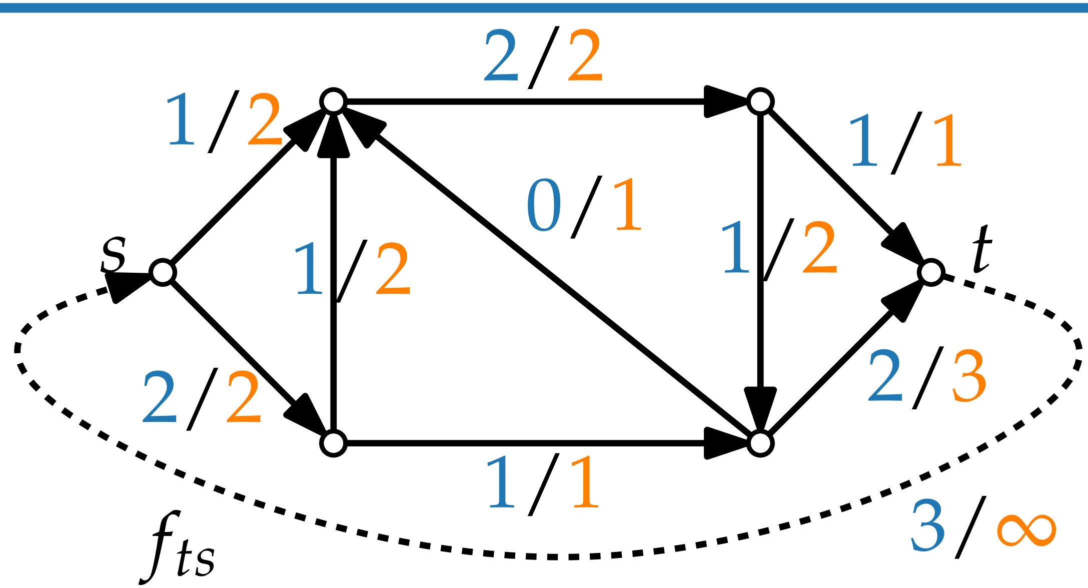


# Max-Flow-Min-Cut-Theorem

**Theorem.** The value of a maximum $s$–$t$-flow and the weight of a minimum $s$–$t$-cut are the same.

**Proof.** Special case of LP-Duality …

$$
\begin{array}{lll}
\textbf{maximize} & f_{ts} & \\
\textbf{subject to} & f_{uv} \leq c_{uv} & \forall (u,v) \in E \setminus \{(t,s)\} \\
& \displaystyle\sum_{u\colon (u,v)\in E} f_{uv} - \sum_{z\colon (v,z)\in E} f_{vz} \leq 0 & \forall v \in V \\
& f_{uv} \geq 0 & \forall (u,v) \in E
\end{array}
$$



# Max-Flow-Min-Cut-Theorem

**Theorem.** The value of a maximum $s$–$t$-flow and the weight of a minimum $s$–$t$-cut are the same.

**Proof.** Special case of LP-Duality …

$$\begin{array}{lllr}
\textbf{maximize} & f_{ts} & & \\
\textbf{subject to} & f_{uv} \le c_{uv} & \forall (u,v) \in E \setminus \{(t,s)\} & d_{uv} \\
& \sum\limits_{u\colon (u,v)\in E} f_{uv} - \sum\limits_{z\colon (v,z)\in E} f_{vz} \le 0 & \forall v \in V & p_v \\
& f_{uv} \ge 0 & \forall (u,v) \in E &
\end{array}$$

**maximize** $c^\intercal x \;=\; \sum_{(u,v)\in E} (0 \cdot f_{uv}) \;+1 \cdot f_{ts} \;\Rightarrow c = (0,\dots,0,1)$

Which constraints contain $f_{uv} \neq f_{ts}$? $d_{uv}, p_u, p_v$

$$\Rightarrow d_{uv} - p_u + p_v \ge 0$$

Which constraints contain $f_{ts}$? $p_s, p_t$

$$\Rightarrow p_s - p_t \ge 1$$

# Max-Flow-Min-Cut-Theorem

**Theorem.** The value of a maximum $s$–$t$-flow and the weight of a minimum $s$–$t$-cut are the same.

**Proof.** Special case of LP-Duality …

$$\begin{array}{lll l}
\textbf{maximize} & f_{ts} & & \\
\textbf{subject to} & f_{uv} \leq c_{uv} & \forall (u,v) \in E \setminus \{(t,s)\} & d_{uv} \\
& \sum_{u\colon (u,v)\in E} f_{uv} - \sum_{z\colon (v,z)\in E} f_{vz} \leq 0 & \forall v \in V & p_v \\
& f_{uv} \geq 0 & \forall (u,v) \in E &
\end{array}$$

$$\begin{array}{lll}
\textbf{minimize} & \sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} & \\
\textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v) \in E \setminus \{(t,s)\} \\
& p_s - p_t \geq 1 & \\
& d_{uv} \geq 0 & \forall (u,v) \in E \\
& p_u \geq 0 & \forall u \in V
\end{array}$$

# Approximation Algorithms

## Lecture 4:
## Linear Programming and LP-Duality

### Part VI:
### Dual LP of Max Flow

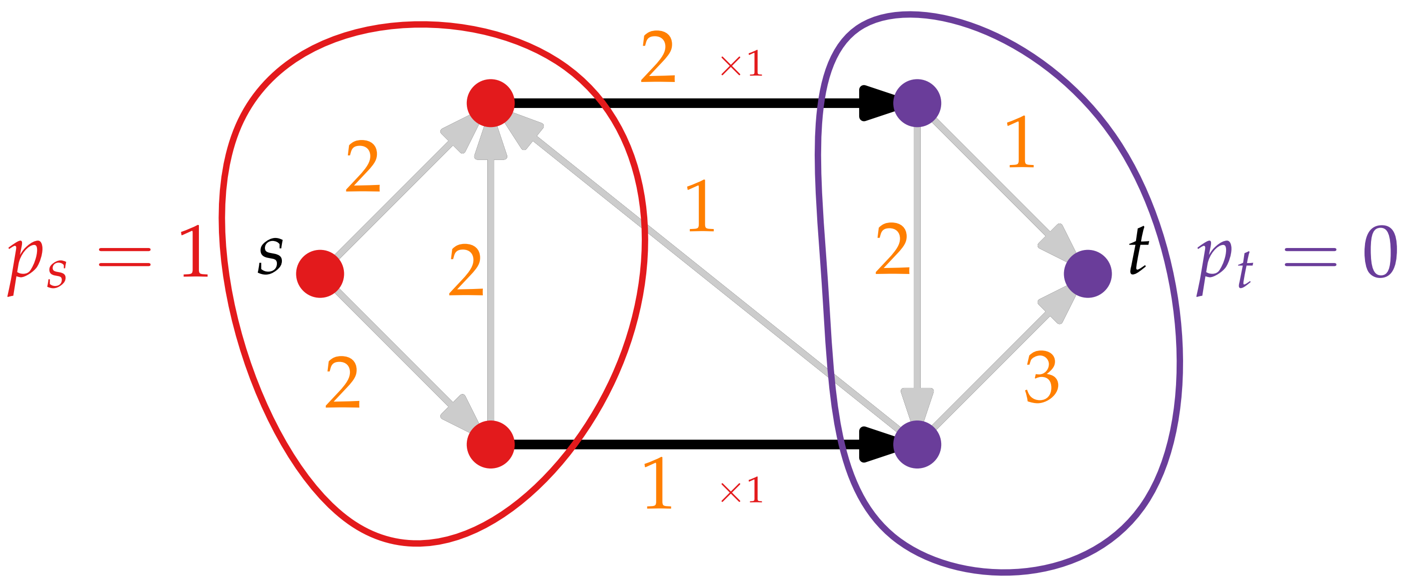
Philipp Kindermann

Summer Semester 2020

# Dual LP – Interpretation as ILP

$$\begin{array}{lll} \textbf{minimize} & \sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} & \\ \textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v) \in E \setminus \{(t,s)\} \\ & p_s - p_t \geq 1 & \\ & d_{uv} \not\geq 0 \ \in \{0,1\} & \forall (u,v) \in E \\ & p_u \not\geq 0 \ \in \{0,1\} & \forall u \in V \end{array}$$

equivalent to Min-Cut!



# Dual LP – Factional Cuts

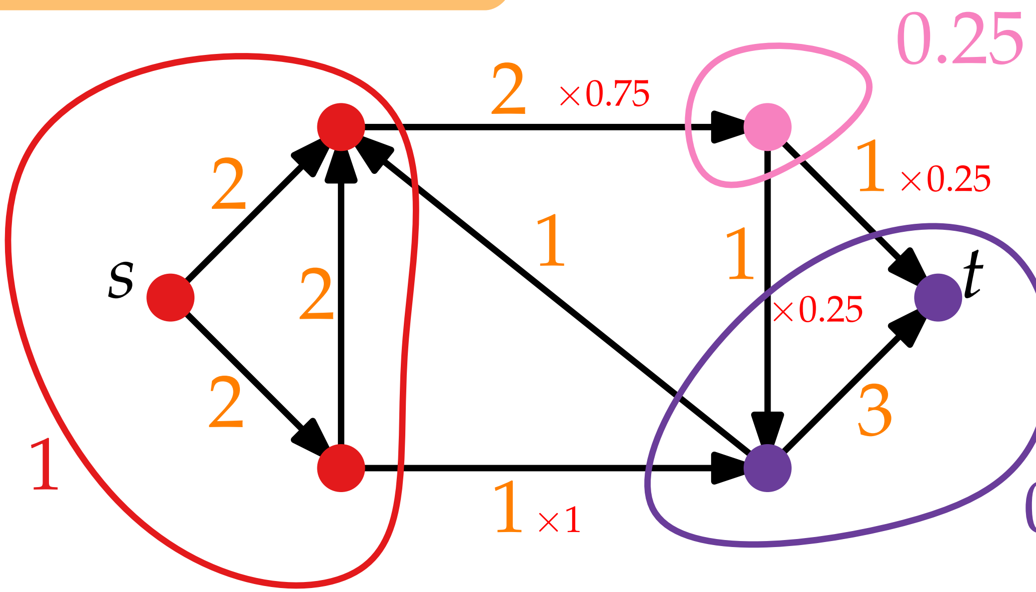
| | | |
|---|---|---|
| **minimize** | $\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$ ≡ LP-Relaxation of the ILP | |
| **subject to** | $d_{uv} - p_u + p_v \geq 0$ | $\forall (u,v) \in E \setminus \{(t,s)\}$ |
| | $p_s - p_t \geq 1$ | |
| | $d_{uv} \geq 0$ | $\forall (u,v) \in E$ |
| | $p_u \geq 0$ | $\forall u \in V$ |

Each extreme-point solution is **integral!** (HA)

Each $s$–$t$-path $s = v_0, \ldots, v_k = t$ has length $\geq 1$ w.r.t. $d$:

$$\sum_{i=0}^{k-1} d_{i,i+1} \geq \sum_{i=0}^{k-1} (p_i - p_{i+1})$$

$$= p_s - p_t$$



# Dual LP – Complementary Slackness

**maximize** $f_{ts}$

**subject to**

$f_{uv} \leq c_{uv} \quad \forall (u,v) \in E \setminus \{(t,s)\}$

$\sum_{u:\,(u,v)\in E} f_{uv} - \sum_{z:\,(v,z)\in E} f_{vz} \leq 0 \quad \forall v \in V$

$f_{uv} \geq 0 \quad \forall (u,v) \in E$

**minimize** $\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$

**subject to**

$d_{uv} - p_u + p_v \geq 0$

$p_s - p_t \geq 1$

$d_{uv} \geq 0$

$p_u \geq 0$

**Primal CS**:
$\forall j$: Either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

**Dual CS**:
$\forall i$: Either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

For a max flow and min cut:

- For each forward edge $(u,v)$ of the cut: $f_{uv} = c_{uv}$.
  ($d_{uv} = 1$, so by dual CS: $f_{uv} = c_{uv}$.)
- For each backward edge $(u,v)$ of the cut: $f_{uv} = 0$.
  (Otherwise, by primal CS: $d_{uv} - 0 + 1 = 0$.)