Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part I:
Introduction to Linear Programming

Joachim Spoerhase

Winter 2021/22

Maximizing Profits

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Which choice of (x_1, x_2) maximizes the profit?

Solution

Linear constraints:

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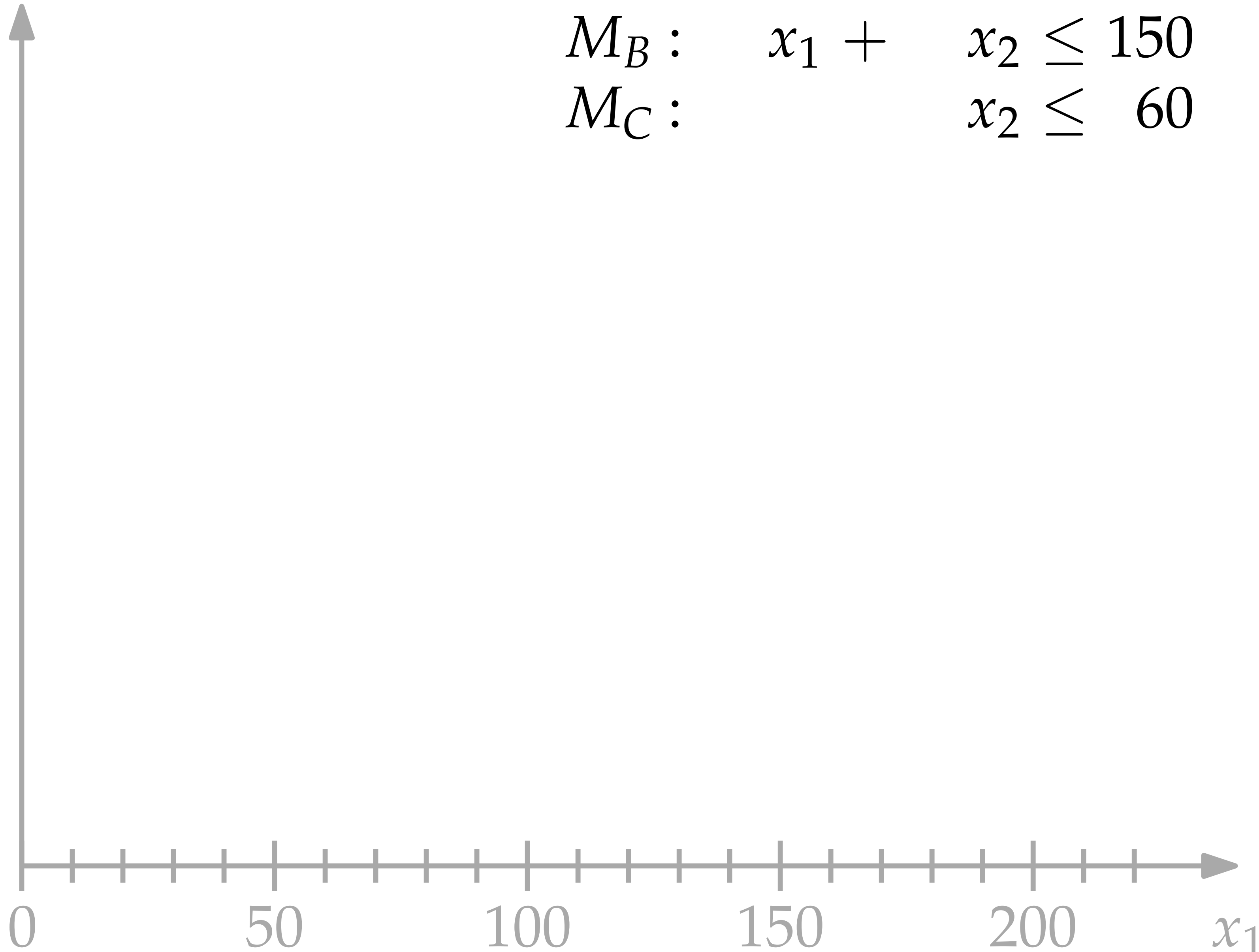

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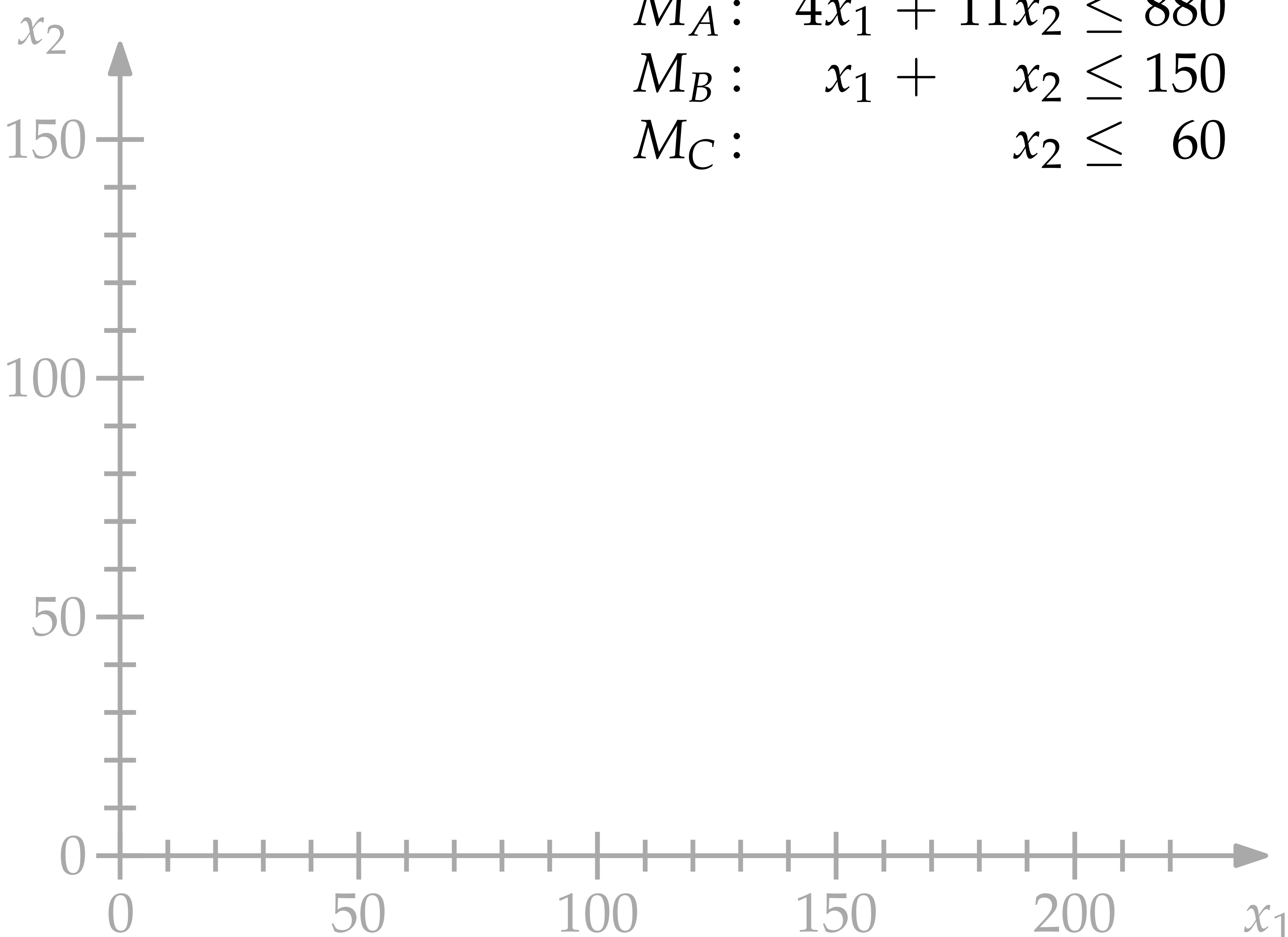

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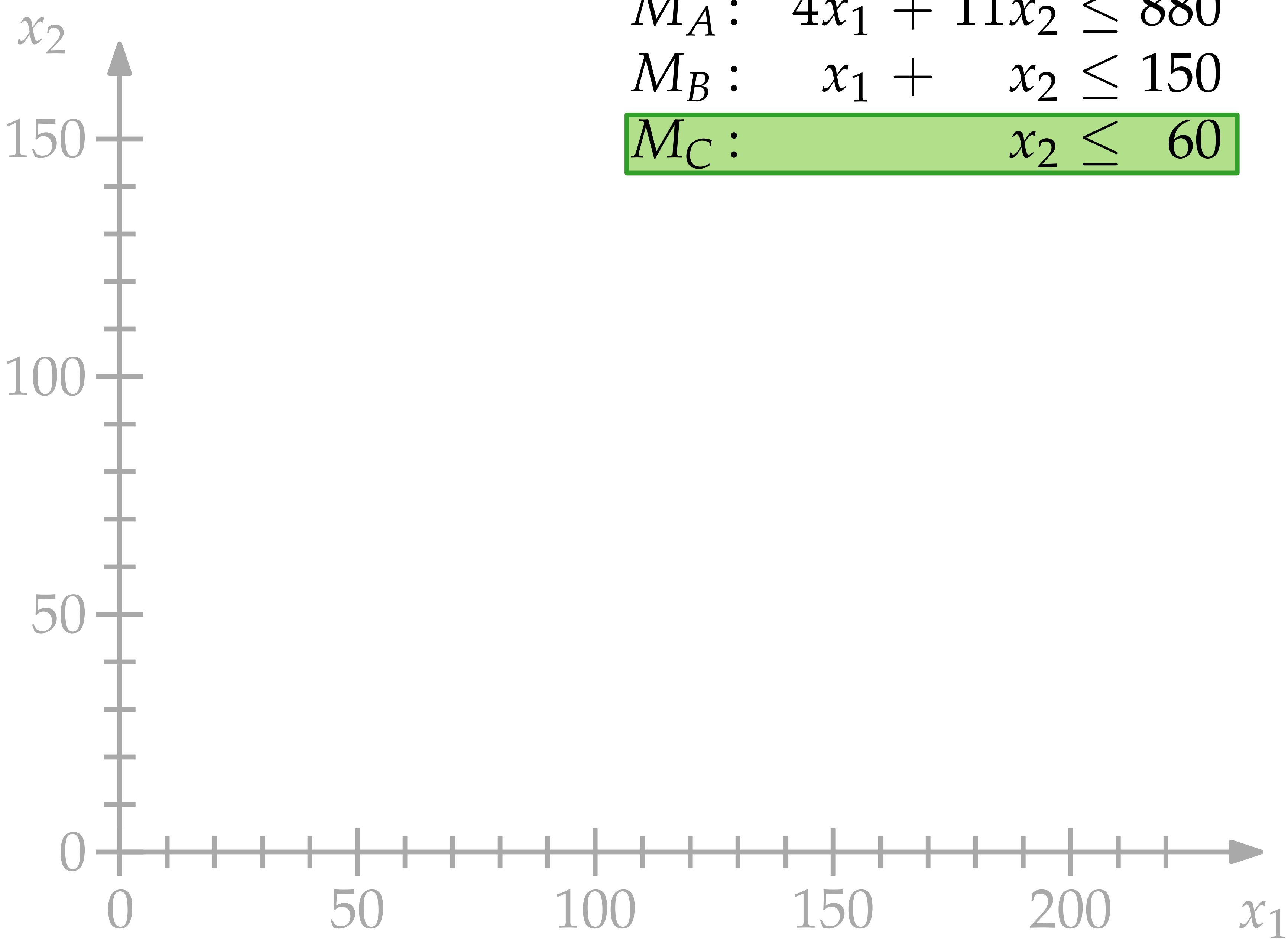

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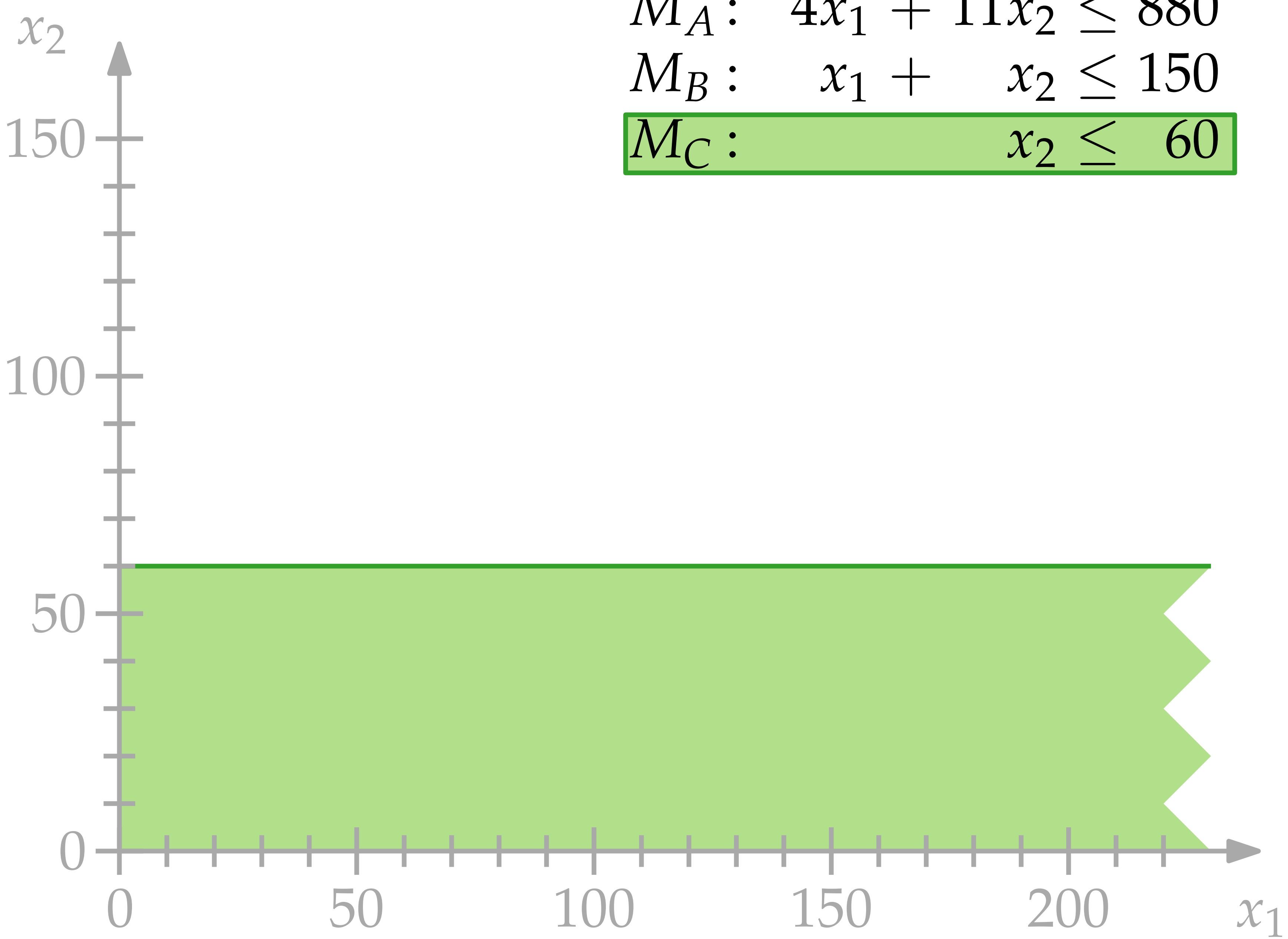

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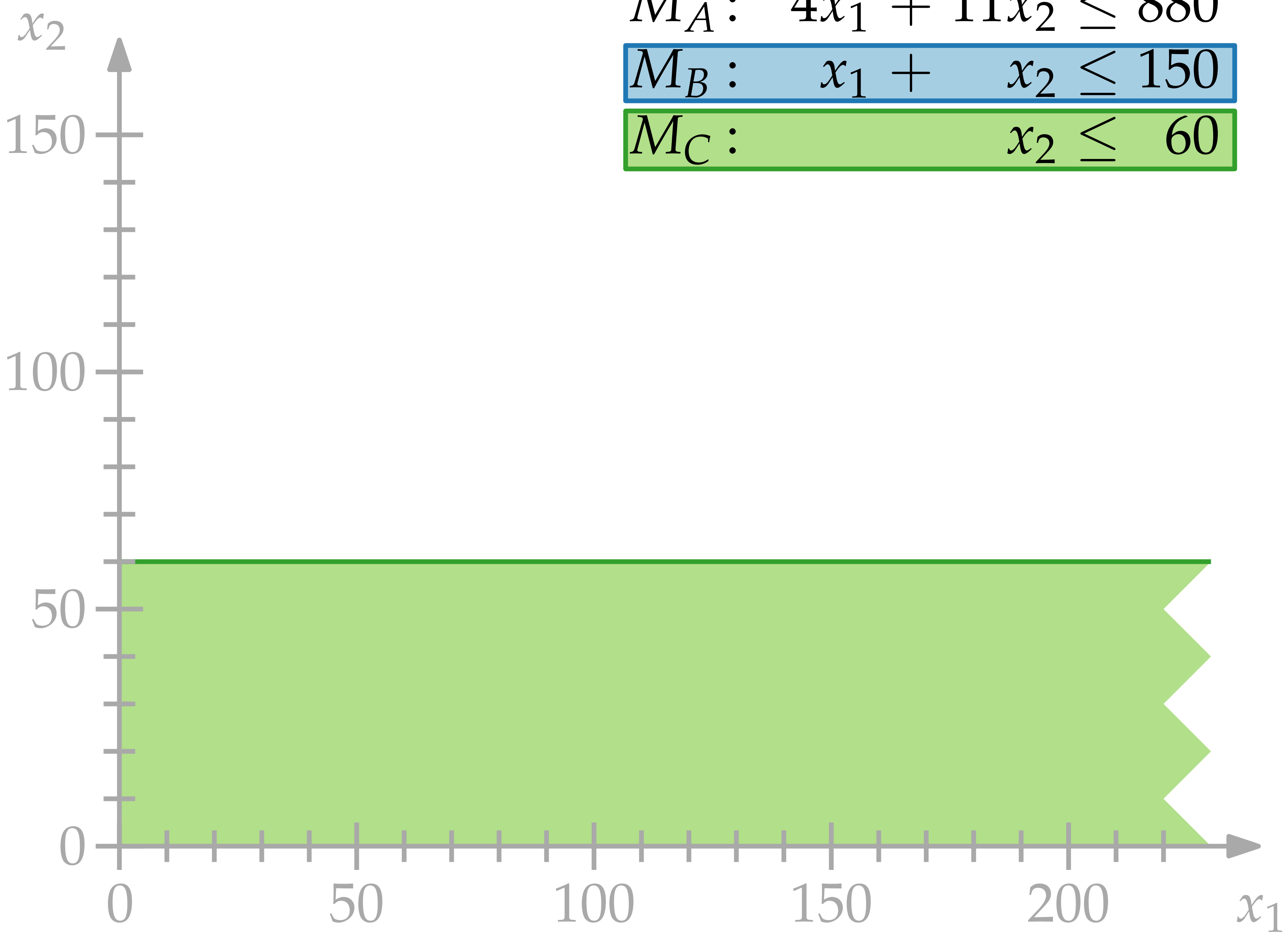

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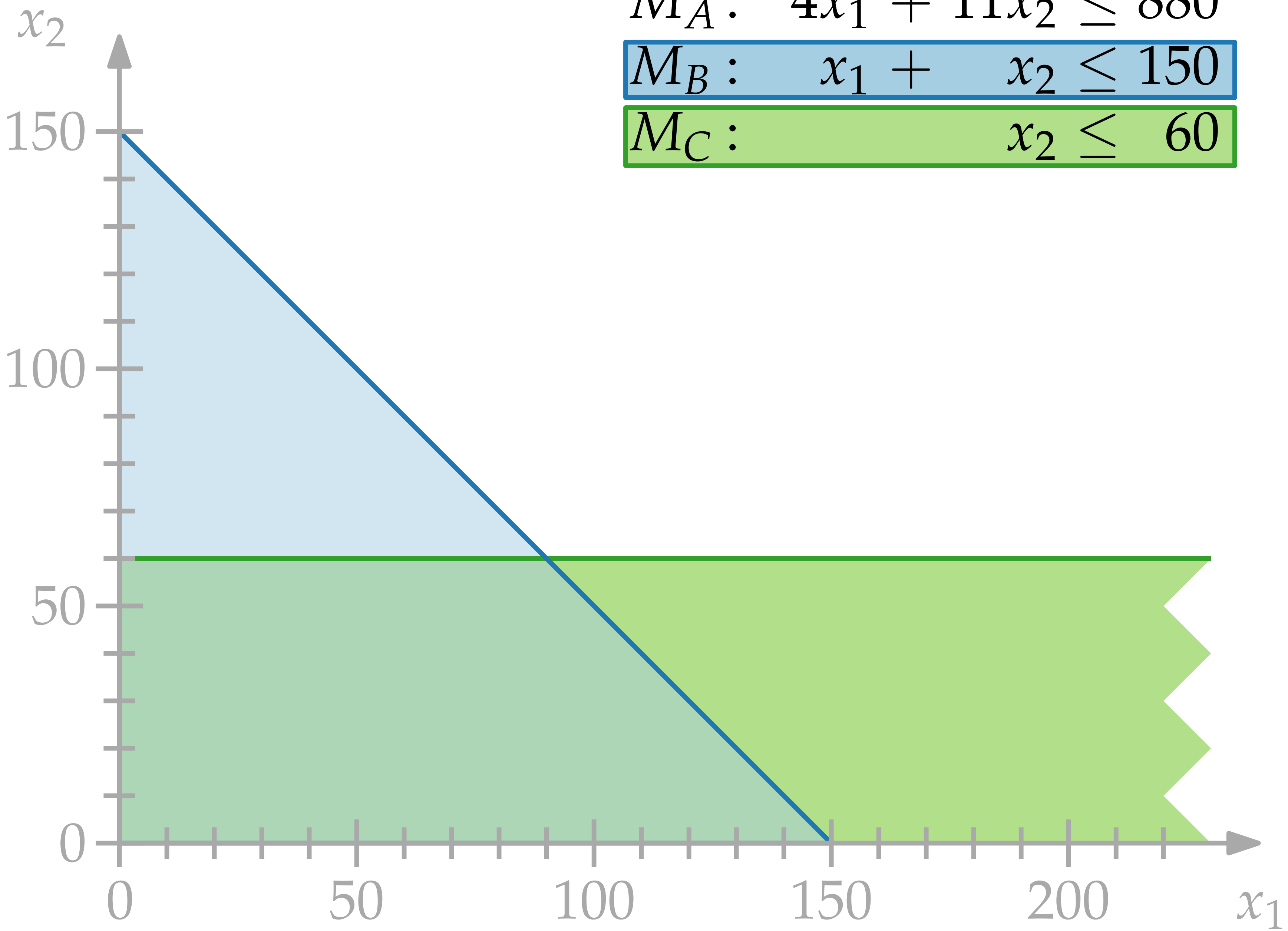

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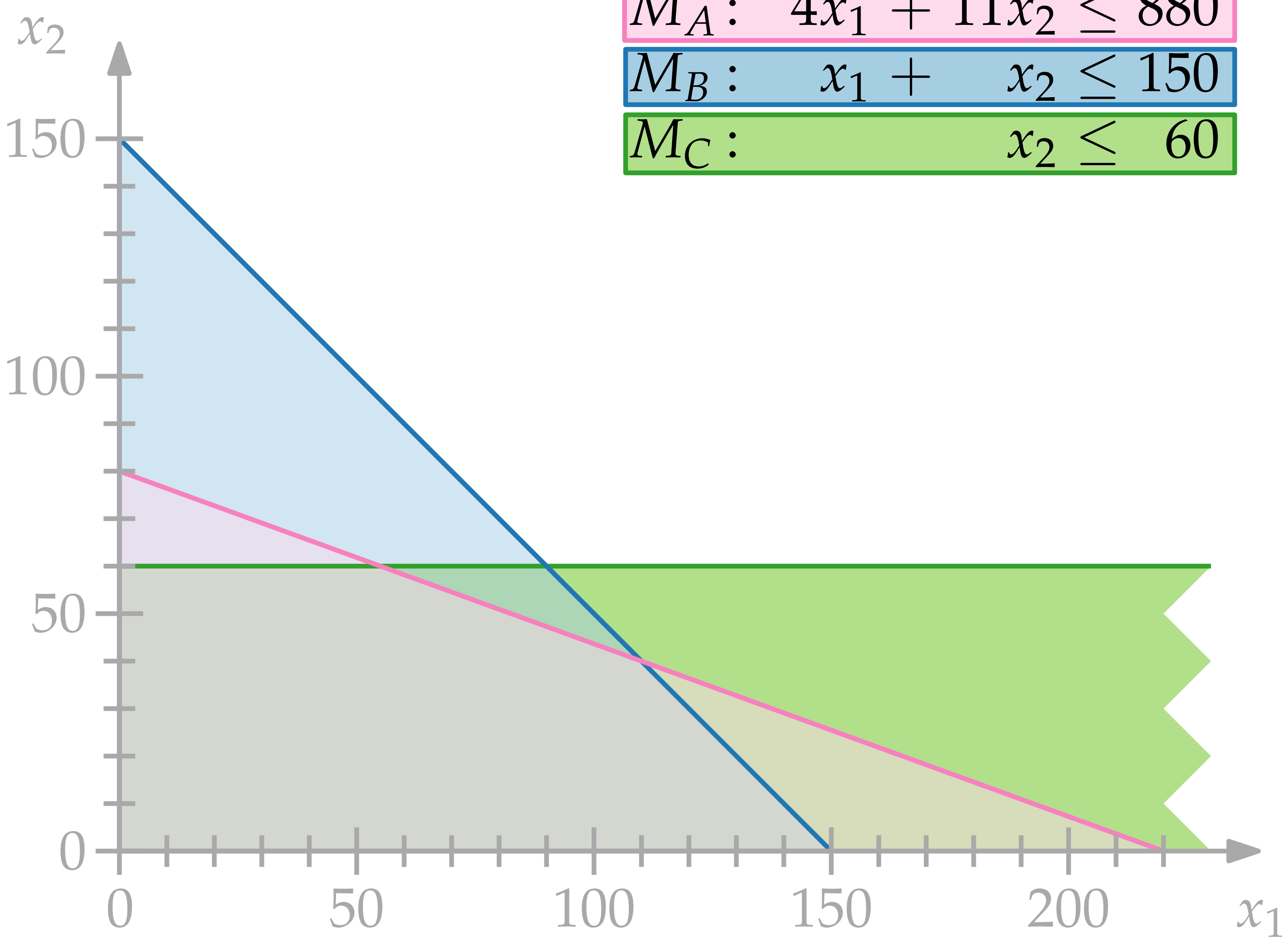

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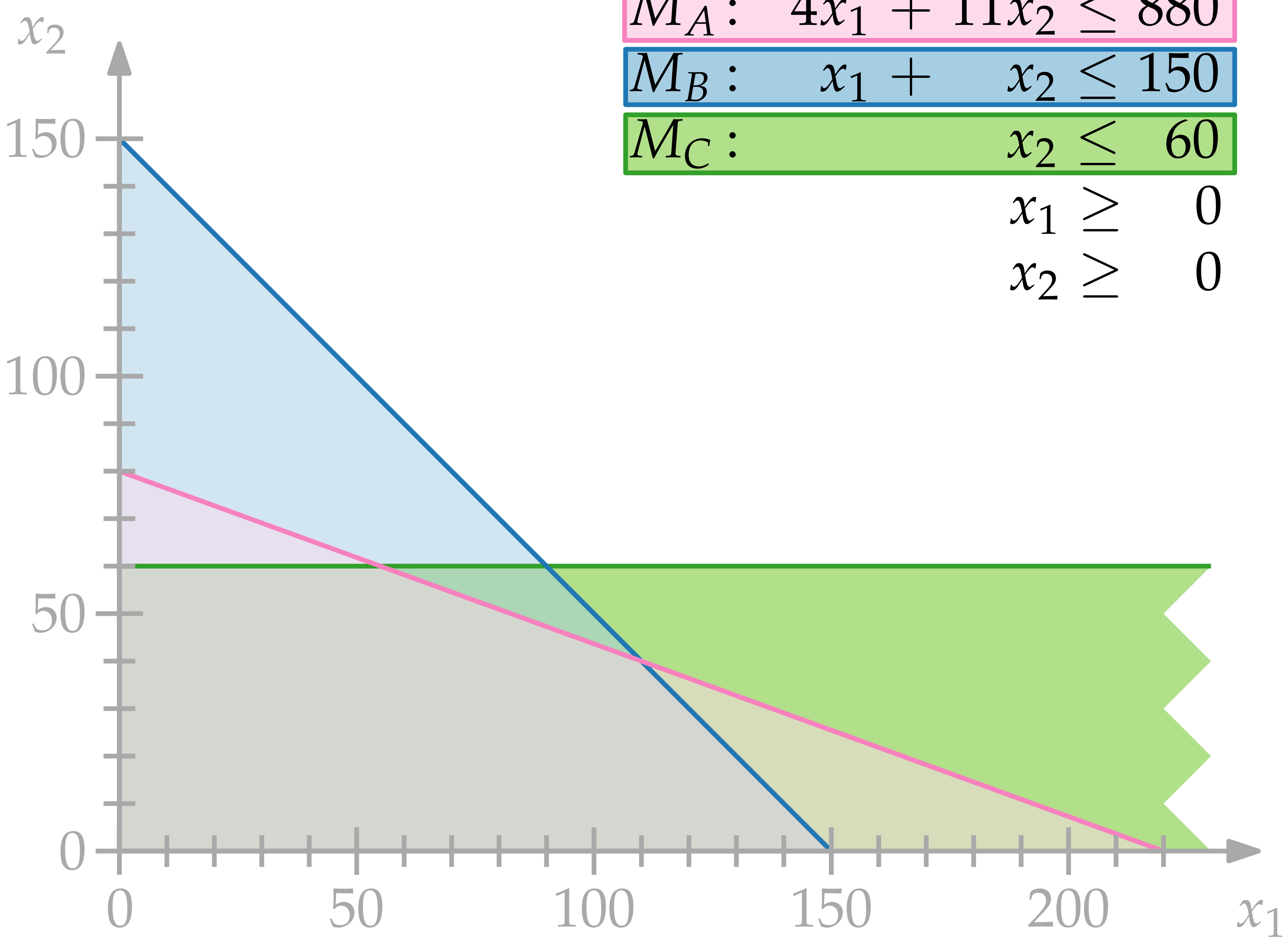

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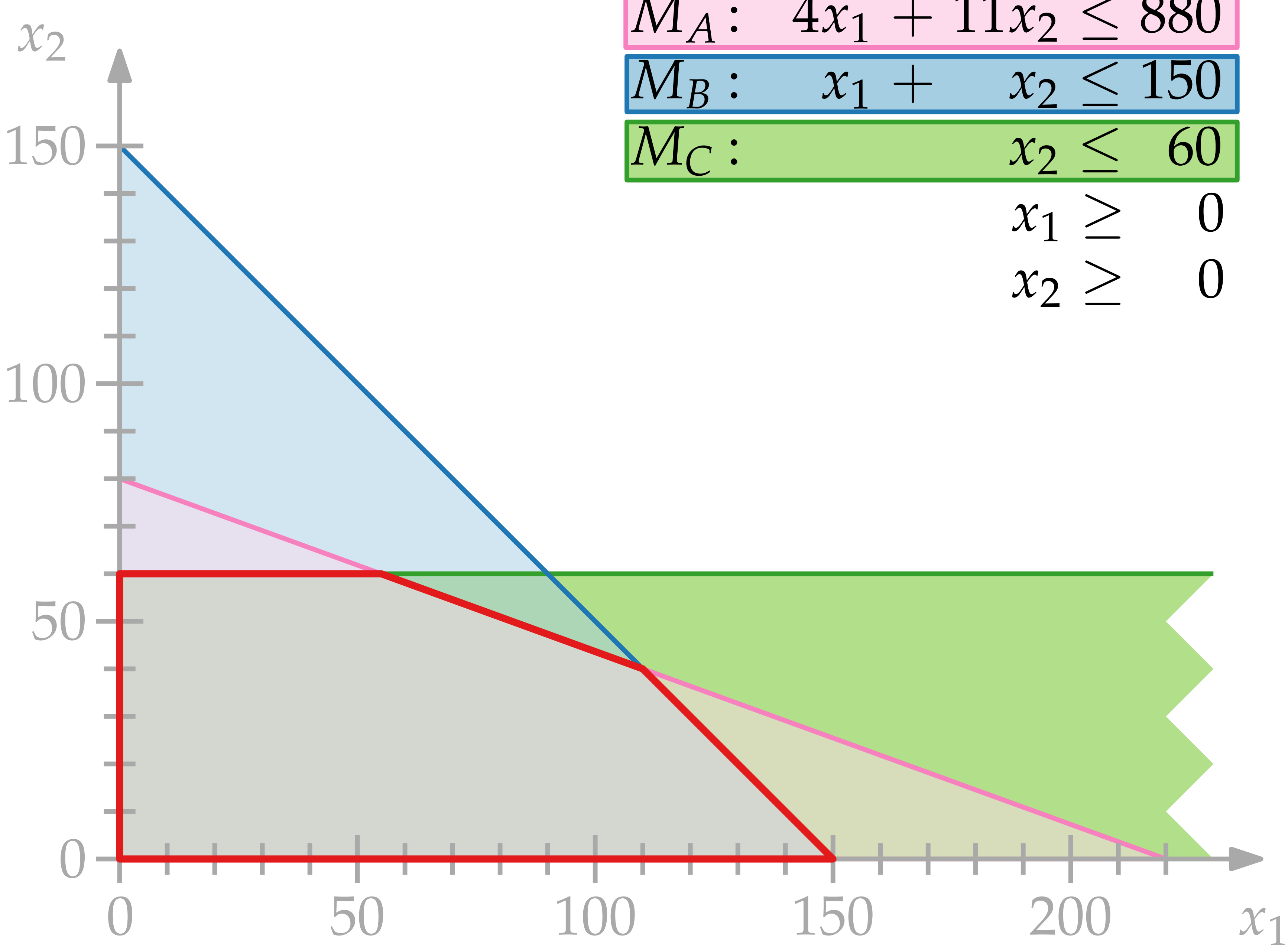

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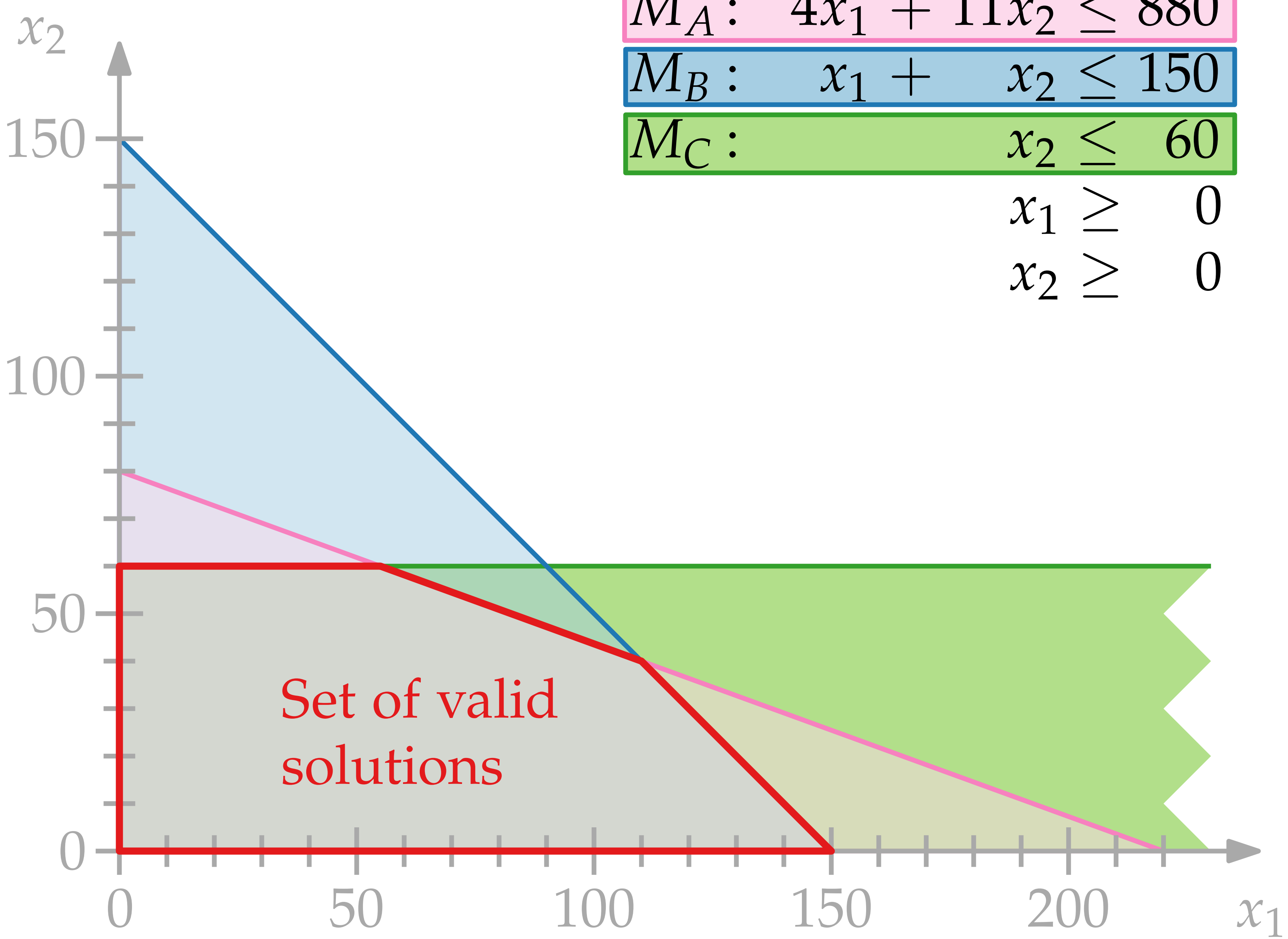

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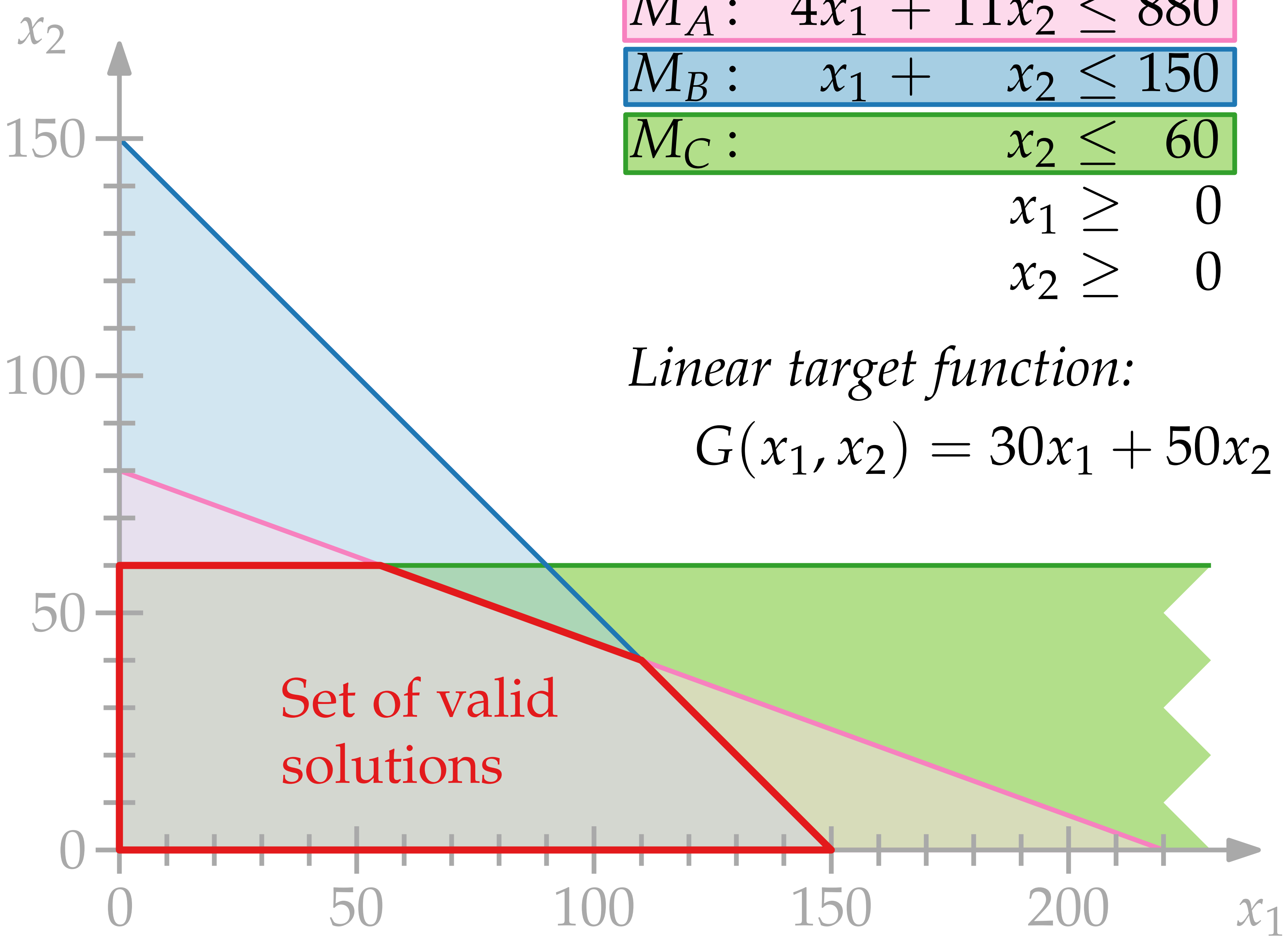

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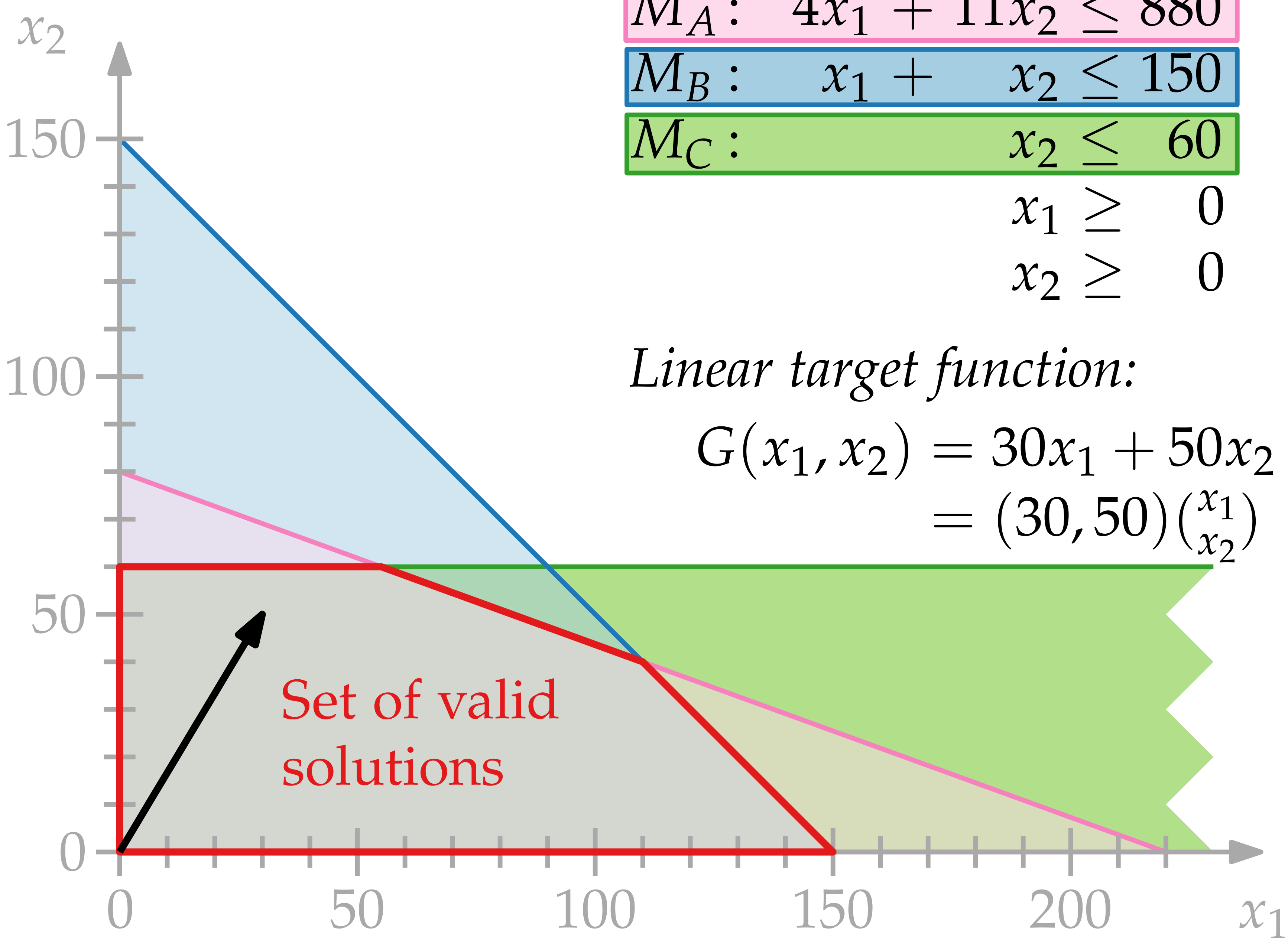

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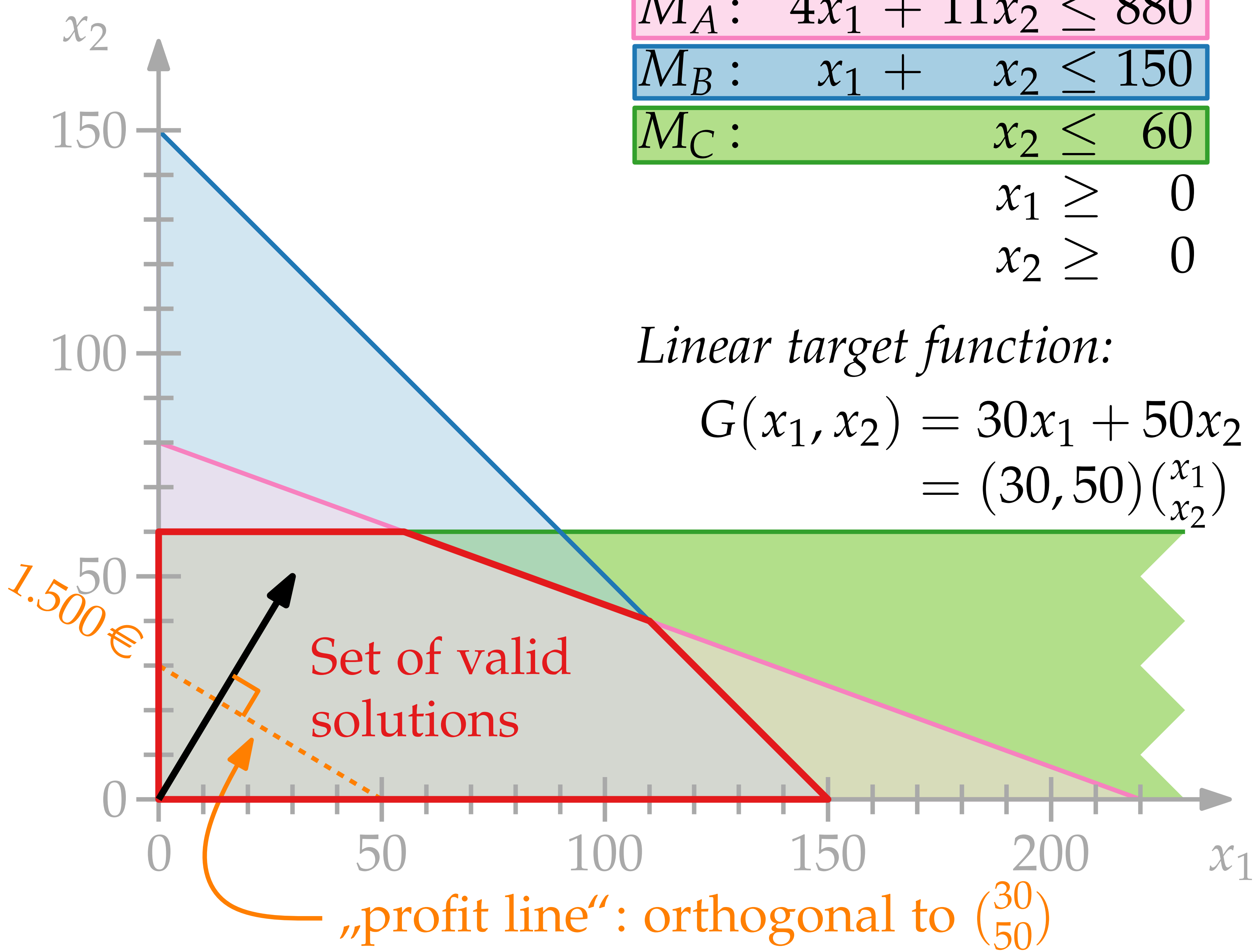

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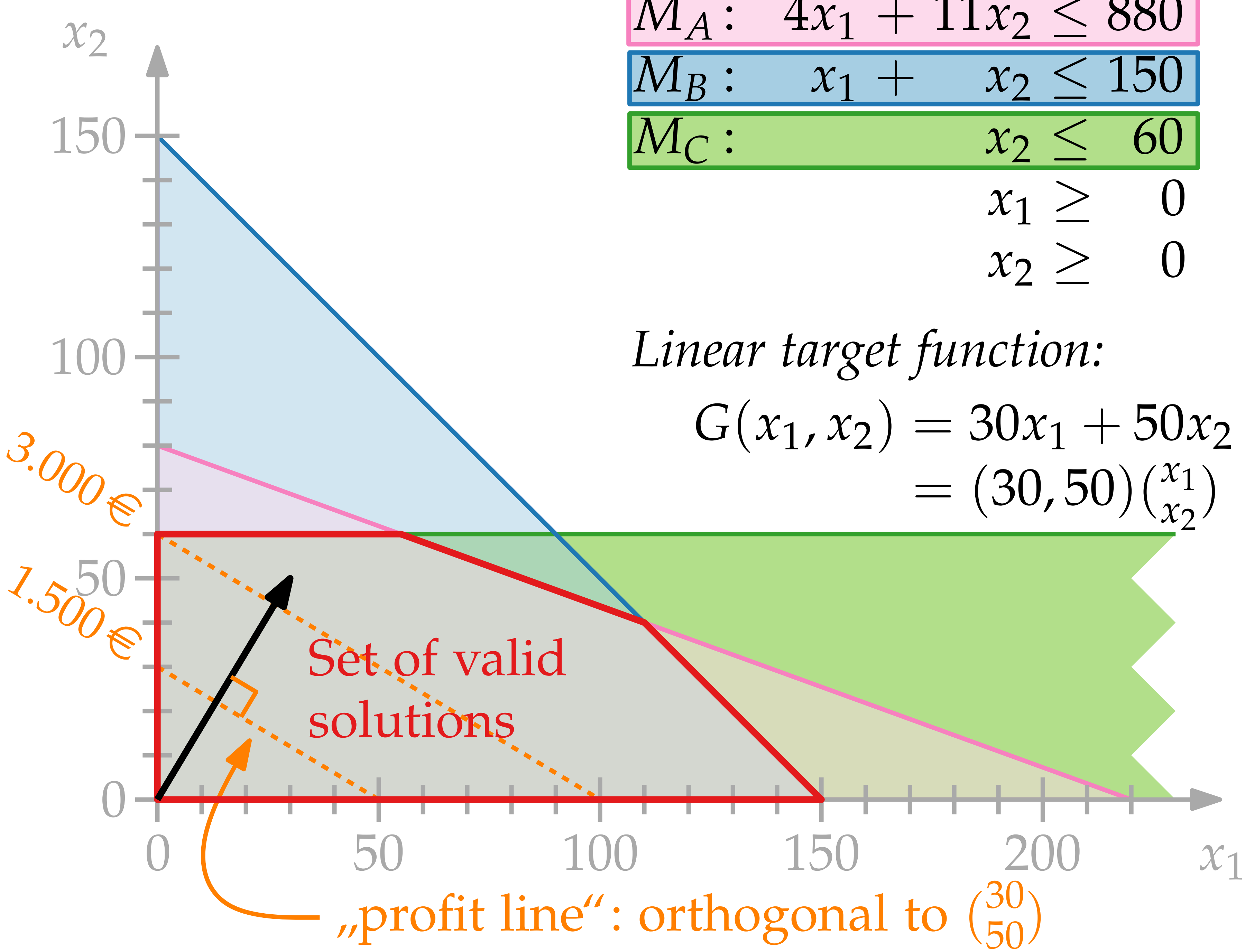

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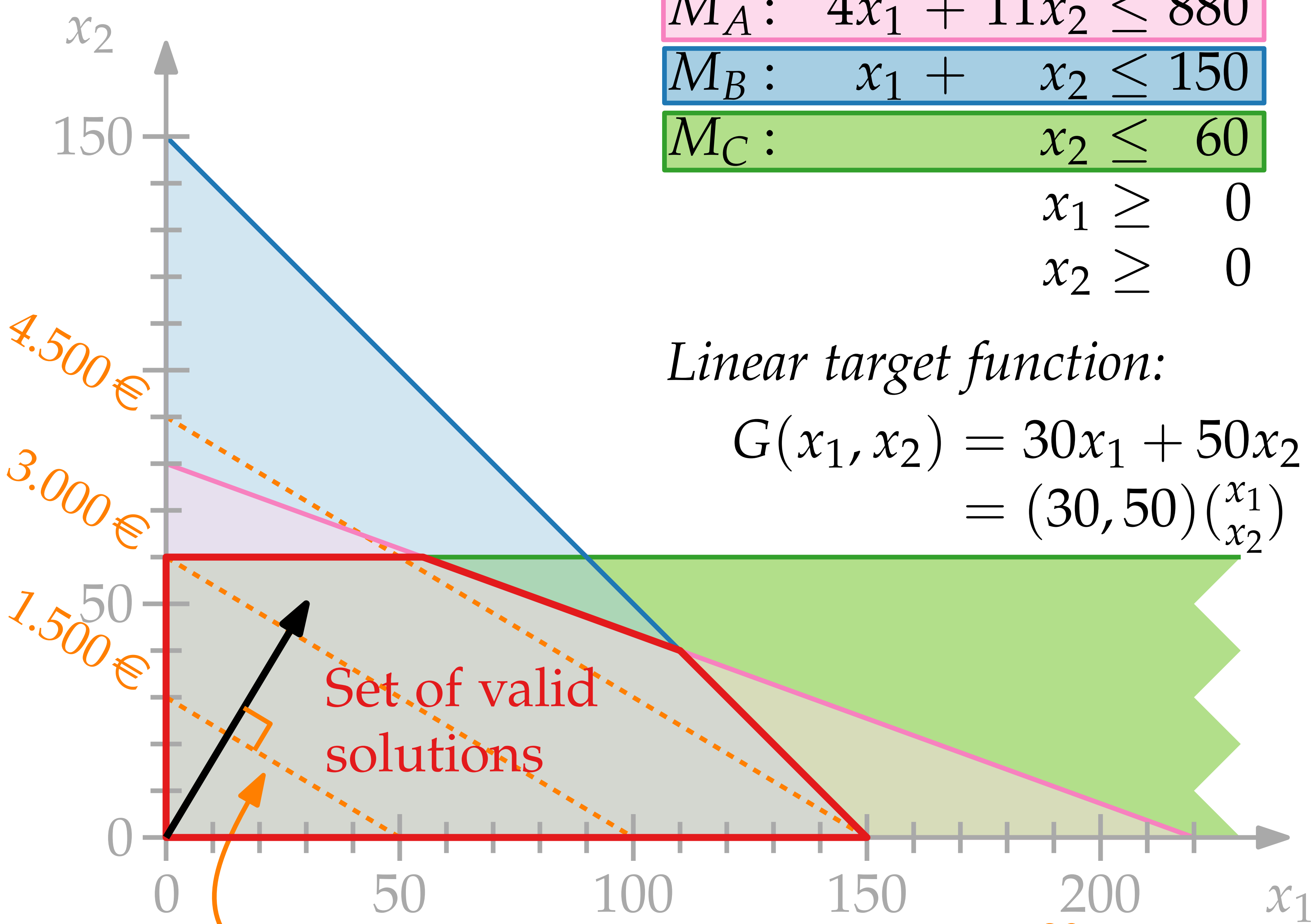

„profit line": orthogonal to $\binom{30}{50}$

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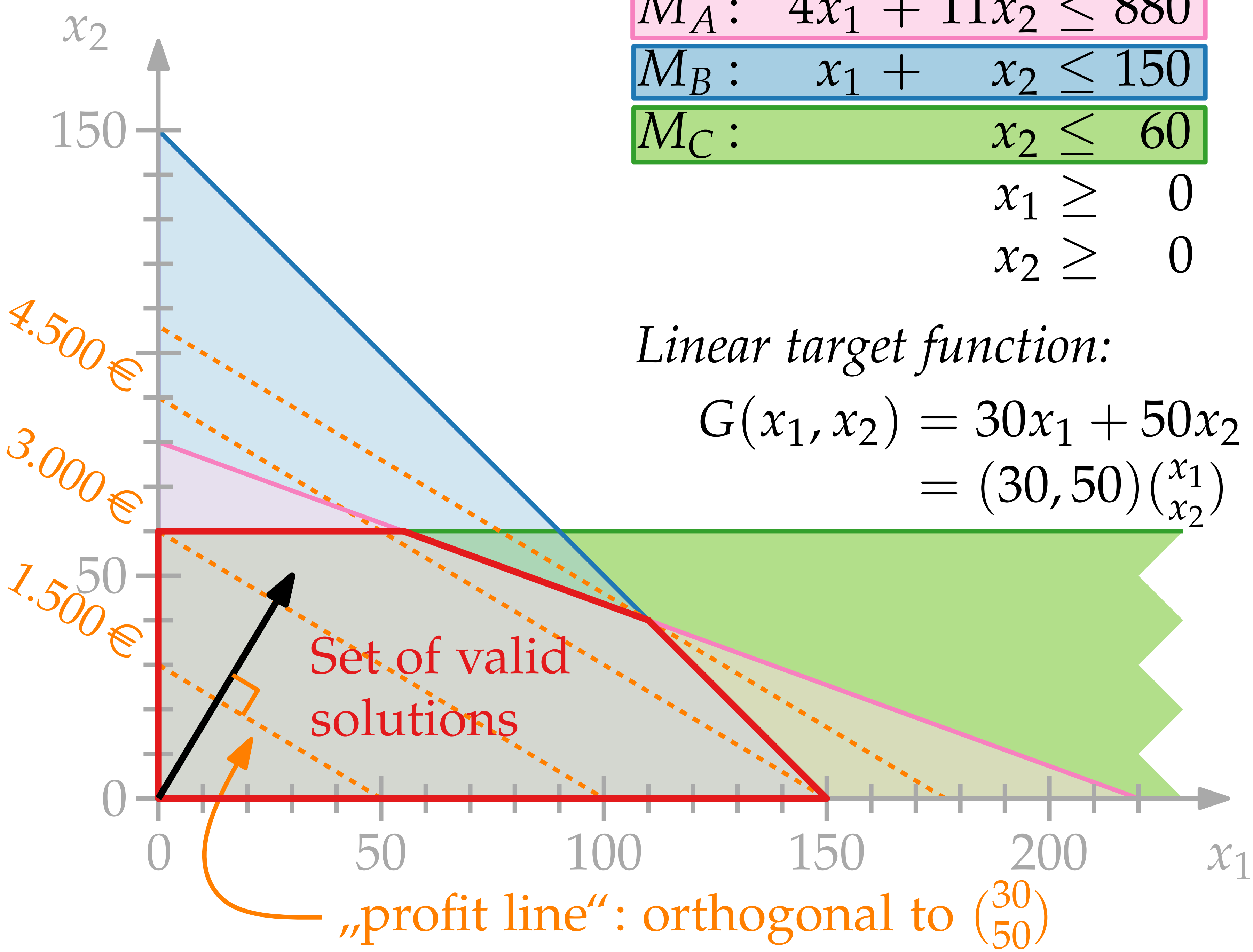

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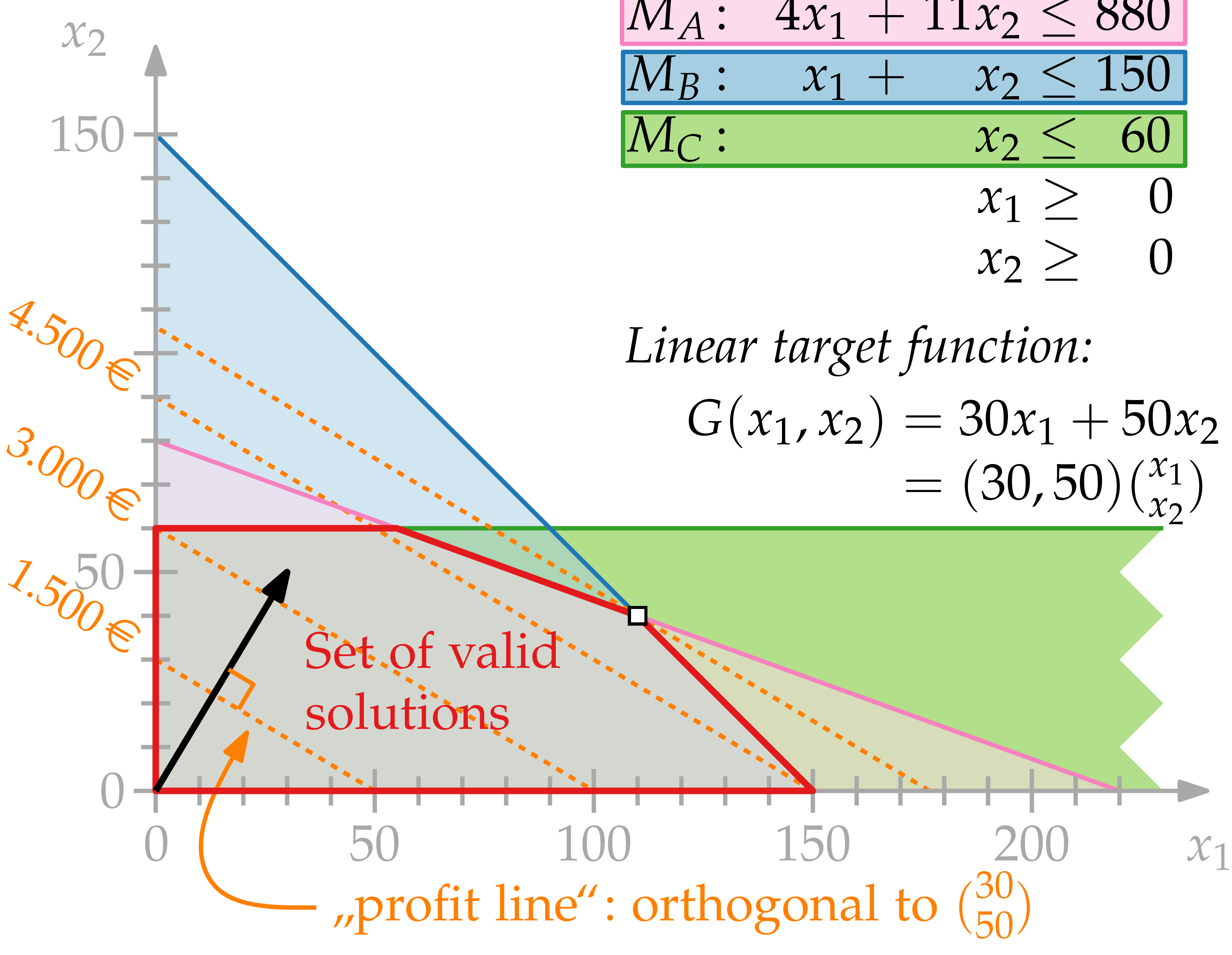

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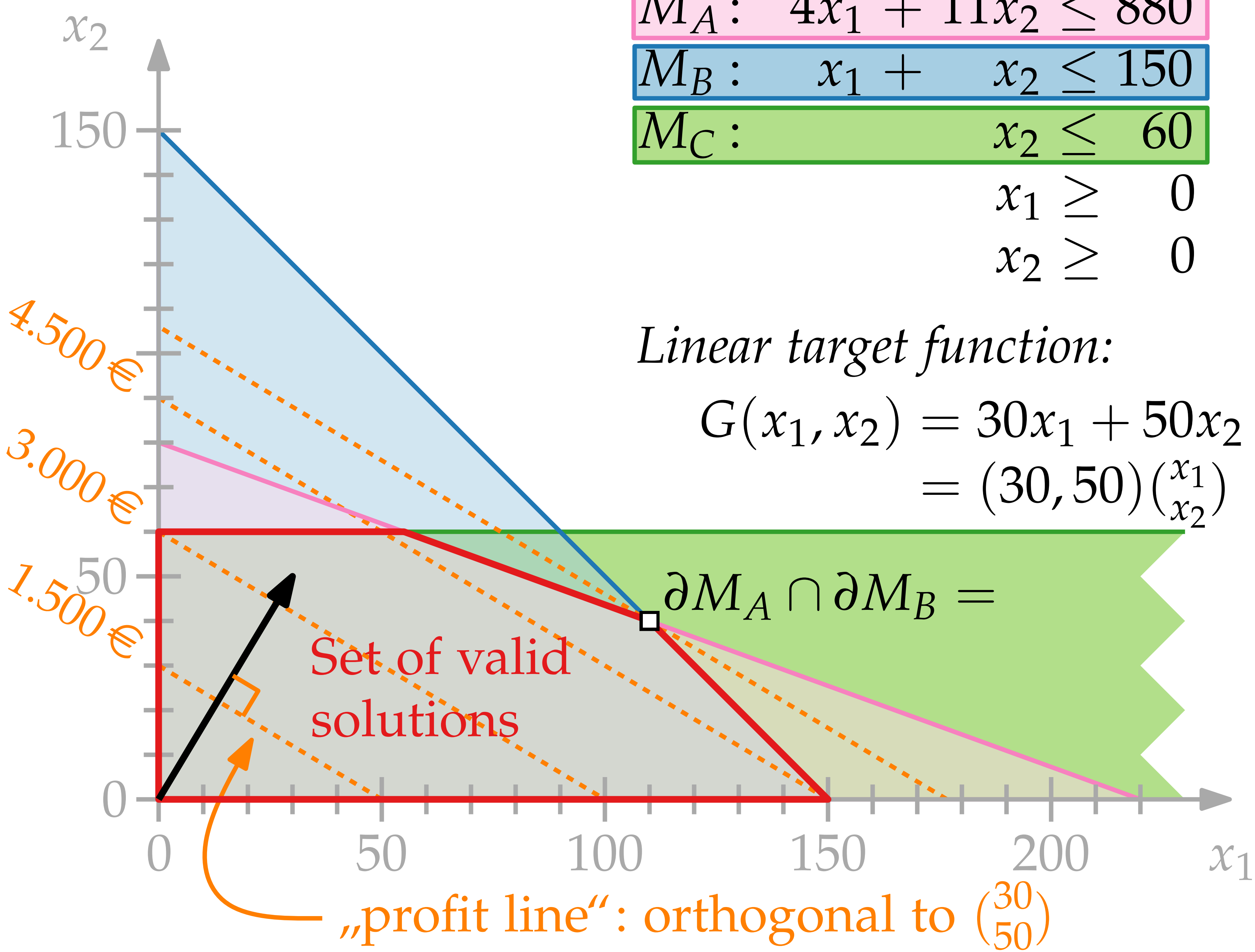

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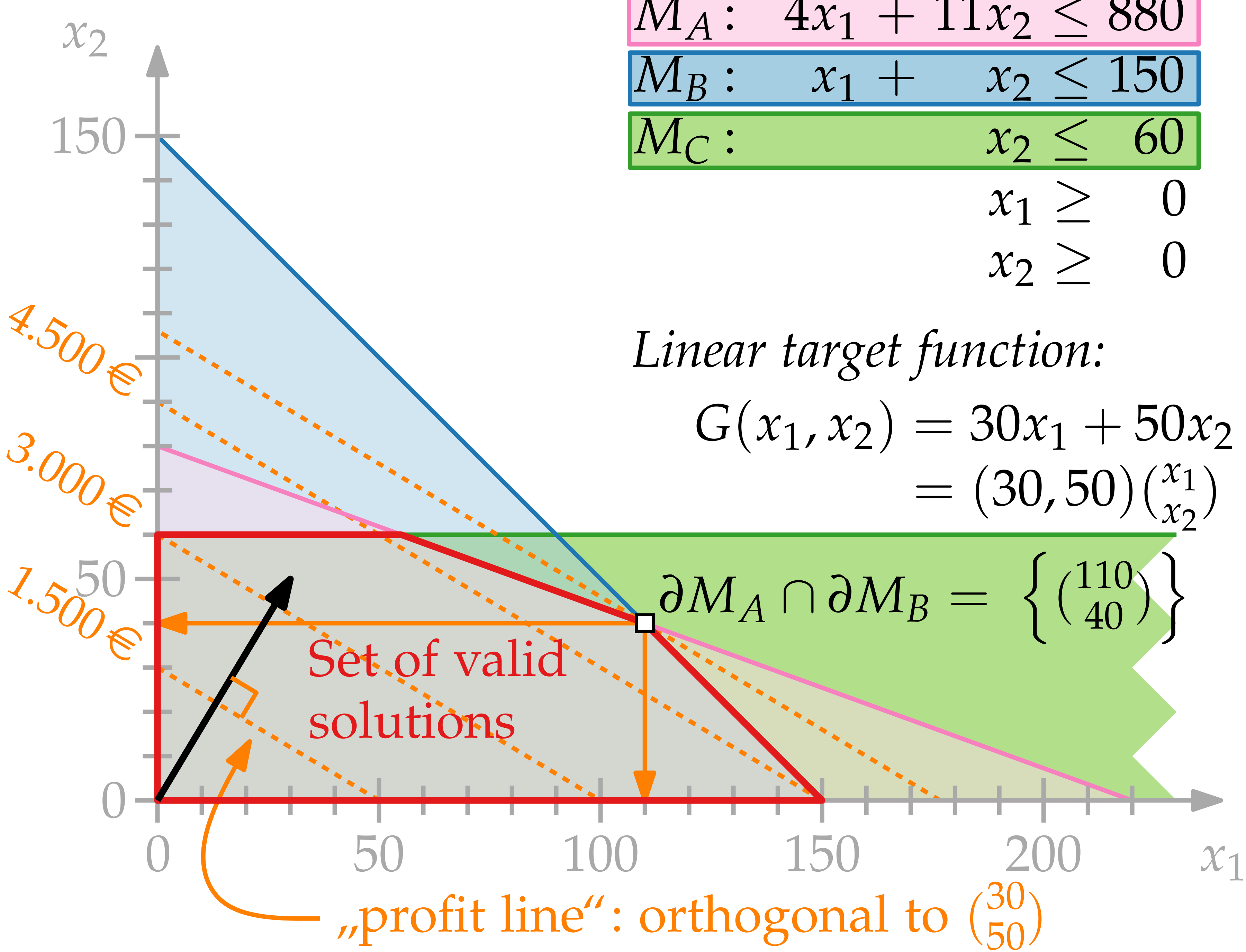

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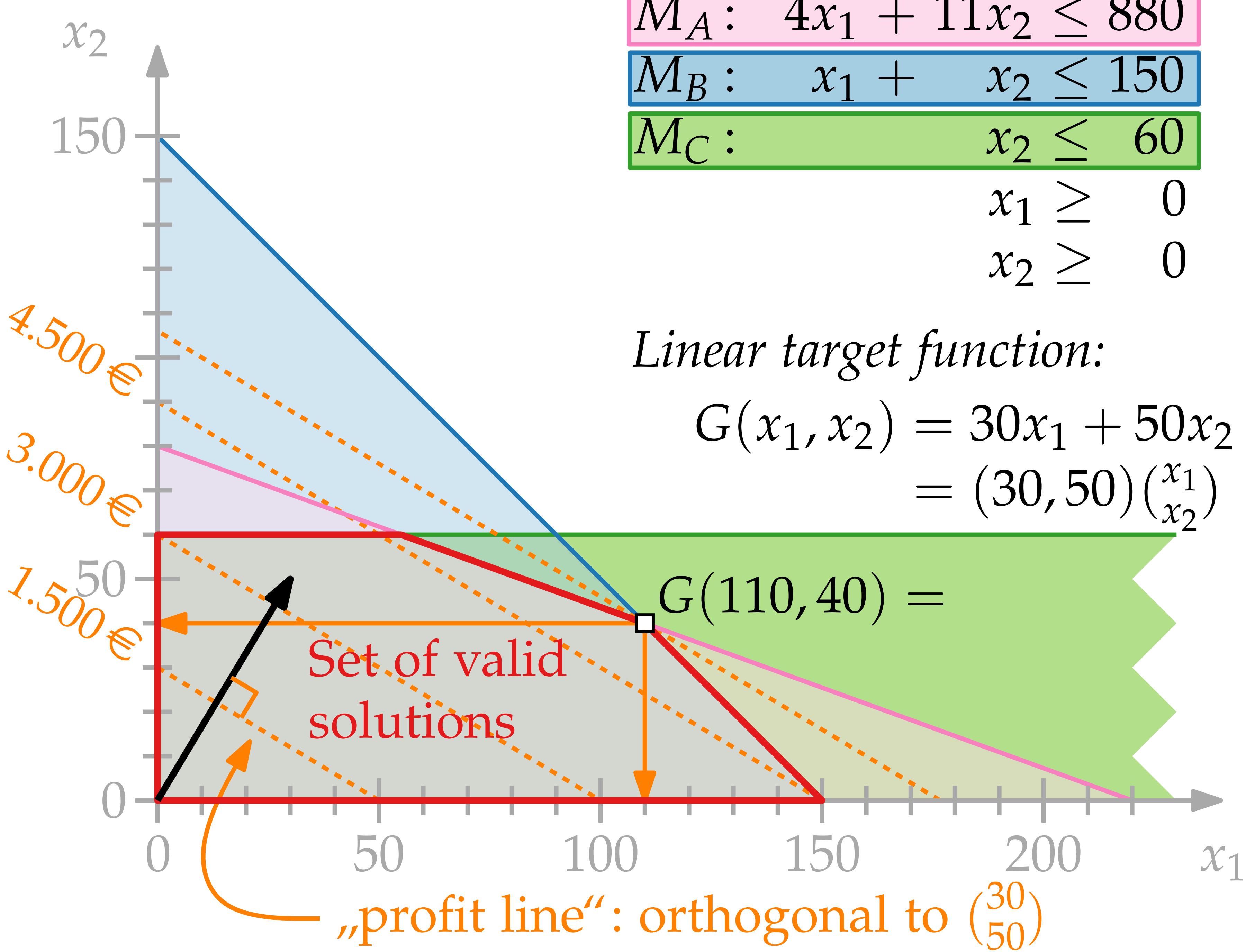

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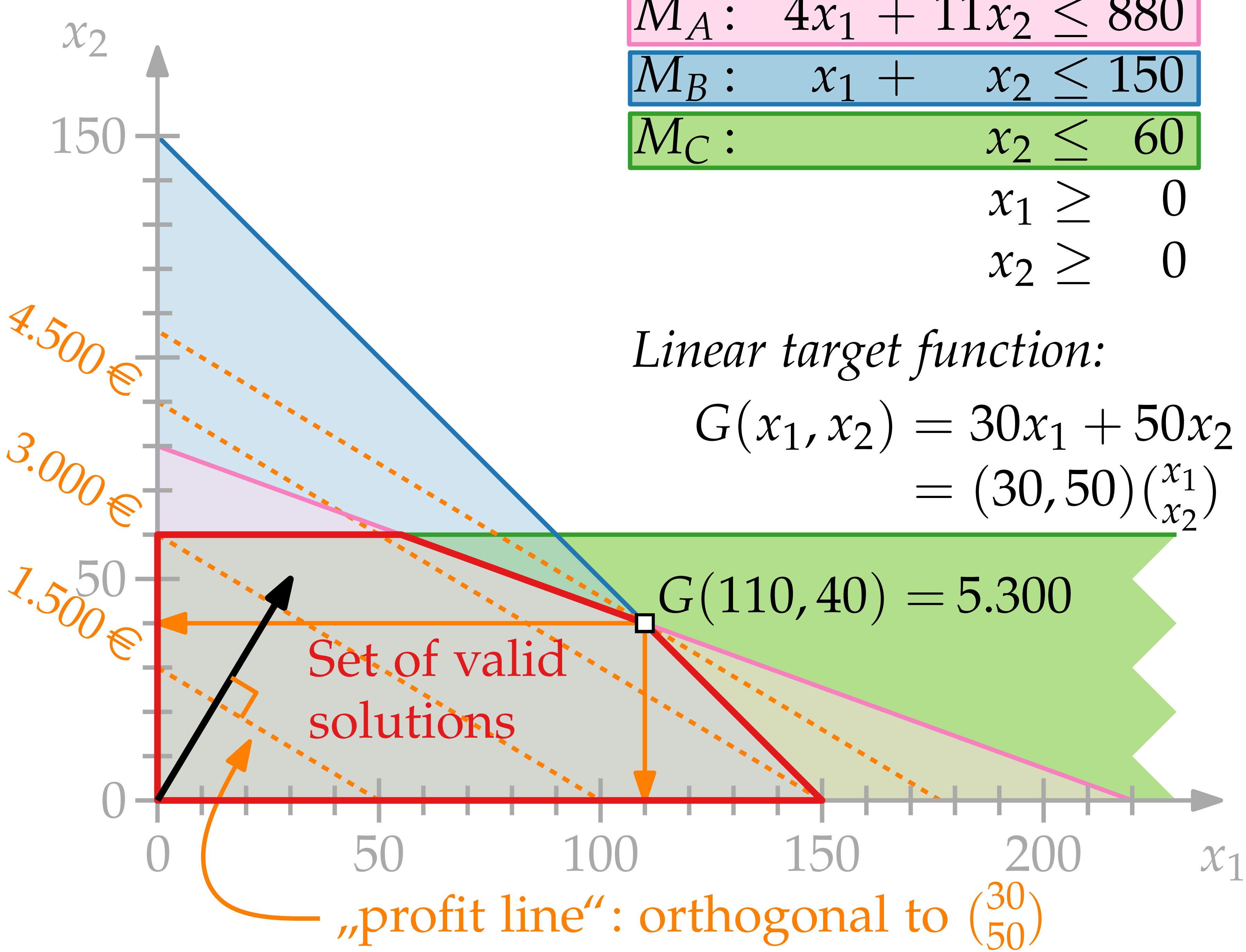

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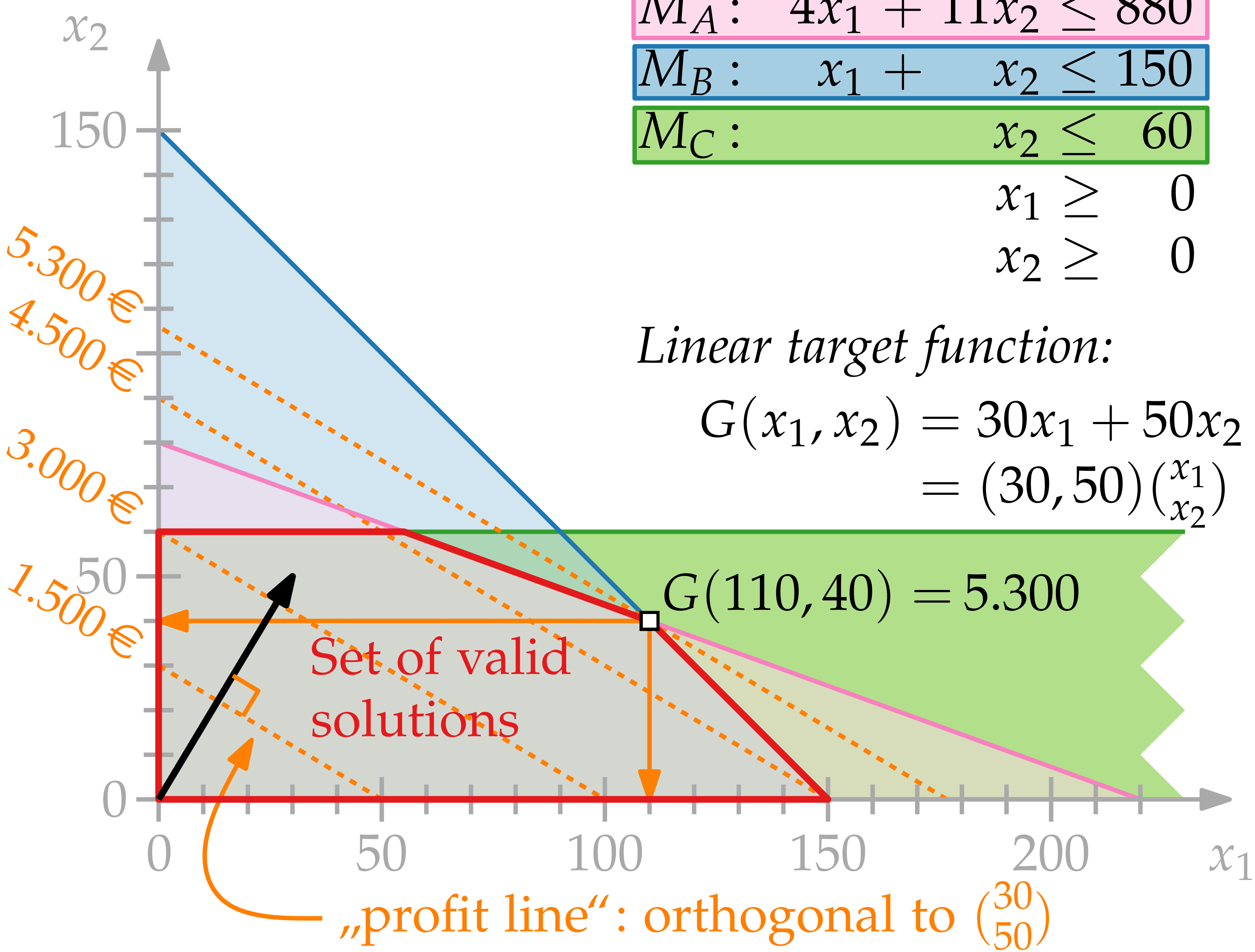

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Part II:
Upper Bounds for LPs

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Consider hard NP-Minimization Problem.

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Examples:

- Vertex Cover: lower bound by matchings
- TSP: lower bound by MST or Cycle Cover

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Example. $c = \begin{pmatrix} 7 \\ 1 \\ 5 \end{pmatrix}$ $A = \begin{pmatrix} 1 & -1 & 3 \\ 5 & 2 & -1 \end{pmatrix}$ $b = \begin{pmatrix} 10 \\ 6 \end{pmatrix}$

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Linear Programming – Upper Bounds

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Valid solution?

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\boxed{
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 2 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 2 & - & x_2 1 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\boxed{
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1\,2 & - & x_2\,1 & + & 3x_3\,9 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 2 & - & x_2 1 & + & 3x_3 9 & \geq & 10 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1\,2 & - & x_2\,1 & + & 3x_3\,9 & \geq & 10\,10 \\
 & 5x_1\,10 & + & 2x_2\,2 & - & x_3\,3 & \geq & 6\,9 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1\,14 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1\,2 & - & x_2\,1 & + & 3x_3\,9 & \geq & 10\,10 \\
& 5x_1\,10 & + & 2x_2\,2 & - & x_3\,3 & \geq & 6\,9 \\
& & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1\,14 & + & x_2\,1 & + & 5x_3 & & \\
\textbf{subject to} & x_1\,2 & - & x_2\,1 & + & 3x_3\,9 & \geq & 10\,10 \\
 & 5x_1\,10 & + & 2x_2\,2 & - & x_3\,3 & \geq & 6\,9 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1\,14 & + & x_2\,1 & + & 5x_3\,15 & & \\
\textbf{subject to} & x_1\,2 & - & x_2\,1 & + & 3x_3\,9 & \geq & 10\,10 \\
 & 5x_1\,10 & + & 2x_2\,2 & - & x_3\,3 & \geq & 6\,9 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1\,14 & + & x_2\,1 & + & 5x_3\,15 & = & 30 \\
\textbf{subject to} & x_1\,2 & - & x_2\,1 & + & 3x_3\,9 & \geq & 10\,10 \\
 & 5x_1\,10 & + & 2x_2\,2 & - & x_3\,3 & \geq & 6\,9 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

Linear Programming – Upper Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcl}
\textbf{minimize} & 7x_1 \, 14 & + & x_2 \, 1 & + & 5x_3 \, 15 & = & 30 \\
\textbf{subject to} & x_1 \, 2 & - & x_2 \, 1 & + & 3x_3 \, 9 & \geq & 10 \, 10 \\
 & 5x_1 \, 10 & + & 2x_2 \, 2 & - & x_3 \, 3 & \geq & 6 \, 9 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Valid solution?

$x = (2, 1, 3)$

$\Rightarrow \text{obj}(x) = 30$ is upper bound for OPT

Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part III:
Lower Bounds for LPs

Joachim Spoerhase

Winter 2021/22

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & \underset{\rotatebox{-90}{$\geq$}}{7x_1} & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\boxed{
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\boxed{
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & x_1, & x_2, x_3 & \geq & 0
\end{array}
}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\boxed{
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
& \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
& 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
& & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
}
$$

$$7x_1 + x_2 + 5x_3 \;\geq\; x_1 - x_2 + 3x_3$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\boxed{
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \mathrm{OPT} \geq 10$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & x_1, & x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \text{OPT} \geq 10$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \geq & & \geq & & \geq & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & + & & + & & & & \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \;\geq\; x_1 - x_2 + 3x_3 \;\Rightarrow\; \mathrm{OPT} \geq 10$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & + & & + & & + & & \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \;\geq x_1 - x_2 + 3x_3 \;\Rightarrow \mathrm{OPT} \geq 10$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \geq & & \geq & & \geq & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & + & & + & & + & & \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \mathrm{OPT} \geq 10$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 10 + 6
\end{aligned}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & x_1 & - & x_2 & + & 3x_3 & \geq & 10 \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \mathrm{OPT} \geq 10$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 10 + 6 \qquad \Rightarrow \mathrm{OPT} \geq 16
\end{aligned}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{llcrcrcl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
& \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \rotatebox{-90}{$\geq$} & & \\
\textbf{subject to} & 2\cdot x_1 & - & 2\cdot x_2 & + & 2\cdot 3x_3 & \geq & 2\cdot 10 \\
& + & & + & & + & & \\
& 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
& & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \mathrm{OPT} \geq 10$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 10 + 6 \qquad \Rightarrow \mathrm{OPT} \geq 16
\end{aligned}
$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$\boxed{\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
& \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \\
\textbf{subject to} & 2\cdot x_1 & - & 2\cdot x_2 & + & 2\cdot 3x_3 & \geq & 2\cdot 10 \\
& + & & + & & + & & \\
& 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
& & & & & x_1, x_2, x_3 & \geq & 0
\end{array}}$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \mathrm{OPT} \geq 10$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 10 + 6 \qquad \Rightarrow \mathrm{OPT} \geq 16
\end{aligned}$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq 2 \cdot (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 2 \cdot 10 + 6
\end{aligned}$$

Linear Programming – Lower Bounds

Optimize (i.e., minimize or maximize) a linear (*objective*) function subject to linear inequalities (*constraints*).

$$
\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
\textbf{subject to} & 2\cdot x_1 & - & 2\cdot x_2 & + & 2\cdot 3x_3 & \geq & 2\cdot 10 \\
 & + & & + & & + & & \\
 & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$7x_1 + x_2 + 5x_3 \geq x_1 - x_2 + 3x_3 \Rightarrow \mathrm{OPT} \geq 10$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 10 + 6 \qquad \Rightarrow \mathrm{OPT} \geq 16
\end{aligned}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq 2 \cdot (x_1 - x_2 + 3x_3) + (5x_1 + 2x_2 - x_3) \\
&\geq 2\cdot 10 + 6 \qquad \Rightarrow \mathrm{OPT} \geq 26
\end{aligned}
$$

Linear Programming – Lower Bounds

$$\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & {\scriptstyle 2\cdot}\, x_1 & - & {\scriptstyle 2\cdot}\, x_2 & + & {\scriptstyle 2\cdot}\, 3x_3 & \geq & {\scriptstyle 2\cdot}\, 10 \\
& 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
& & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq 2 \cdot (x_1 - x_2 + 3x_3) + 1 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq 2 \cdot 10 + 1 \cdot 6 \quad \Rightarrow \text{OPT} \geq 2 \cdot 10 + 1 \cdot 6
\end{aligned}$$

Linear Programming – Lower Bounds

$$
\begin{array}{llrcrcrcr}
\textbf{minimize} & & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & y_1(& x_1 & - & x_2 & + & 3x_3) & \geq & 10\, y_1 \\
& & 5x_1 & + & 2x_2 & - & x_3 & \geq & 6 \\
& & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq 2 \cdot (x_1 - x_2 + 3x_3) + 1 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq 2 \cdot 10 + 1 \cdot 6 \quad \Rightarrow \text{OPT} \geq 2 \cdot 10 + 1 \cdot 6
\end{aligned}
$$

Linear Programming – Lower Bounds

$$\begin{array}{rrcrcrcl} \textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \textbf{subject to} & y_1(\,x_1 & - & x_2 & + & 3x_3\,) & \geq & 10\,y_1 \\ & y_2(\,5x_1 & + & 2x_2 & - & x_3\,) & \geq & 6\,y_2 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

$$7x_1 + x_2 + 5x_3 \geq 2 \cdot (x_1 - x_2 + 3x_3) + 1 \cdot (5x_1 + 2x_2 - x_3)$$

$$\geq 2 \cdot 10 + 1 \cdot 6 \quad \Rightarrow \mathrm{OPT} \geq 2 \cdot 10 + 1 \cdot 6$$

Linear Programming – Lower Bounds

$$\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & y_1(\,x_1 & - & x_2 & + & 3x_3\,) & \geq & 10\,y_1 \\
 & y_2(\,5x_1 & + & 2x_2 & - & x_3\,) & \geq & 6\,y_2 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2
\end{aligned}$$

Linear Programming – Lower Bounds

$$\begin{array}{lrcrcrcl} \textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \textbf{subject to} & y_1(\;x_1 & - & x_2 & + & 3x_3\;) & \geq & 10\,y_1 \\ & y_2(\;5x_1 & + & 2x_2 & - & x_3\;) & \geq & 6\,y_2 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\ &\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2 \end{aligned}$$

$10y_1 + 6y_2$ is lower bound for OPT

Linear Programming – Lower Bounds

$$\begin{array}{rrcrcrcr} \textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\ \textbf{subject to} & y_1(\;x_1 & - & x_2 & + & 3x_3\;) & \geq & 10\,y_1 \\ & y_2(\;5x_1 & + & 2x_2 & - & x_3\;) & \geq & 6\,y_2 \\ & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

$$\begin{aligned} 7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\ &\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2 \end{aligned}$$

Bounds for y_1, y_2:

Linear Programming – Lower Bounds

$$\begin{array}{lrcrcrcl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
& \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \\
\textbf{subject to} \; y_1(& x_1 & - & x_2 & + & 3x_3\,) & \geq & 10\,y_1 \\
& + & & + & & + & & \\
y_2(& 5x_1 & + & 2x_2 & - & x_3\,) & \geq & 6\,y_2 \\
& & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2
\end{aligned}$$

Bounds for y_1, y_2:

Linear Programming – Lower Bounds

$$
\begin{array}{lrcrcrll}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} \ y_1(& x_1 & - & x_2 & + & 3x_3) & \geq & 10\, y_1 \\
y_2(& 5x_1 & + & 2x_2 & - & x_3) & \geq & 6\, y_2 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2
\end{aligned}
$$

$$y_1 \quad + \quad 5y_2 \quad \leq \quad 7$$

Bounds for y_1, y_2:

Linear Programming – Lower Bounds

$$\boxed{\begin{array}{llcrcrcl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & y_1(\;x_1 & - & x_2 & + & 3x_3\;) & \geq & 10\,y_1 \\
 & y_2(\;5x_1 & + & 2x_2 & - & x_3\;) & \geq & 6\,y_2 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}}$$

$$7x_1 + x_2 + 5x_3 \geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3)$$

$$\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2$$

Bounds for y_1, y_2:

$$\begin{array}{rcrcl}
y_1 & + & 5y_2 & \leq & 7 \\
-y_1 & + & 2y_2 & \leq & 1
\end{array}$$

Linear Programming – Lower Bounds

$$
\begin{array}{llcrcrcrl}
\textbf{minimize} & & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \\
\textbf{subject to} & y_1(& x_1 & - & x_2 & + & 3x_3 &) \geq & 10\, y_1 \\
 & & + & & + & & + & & \\
 & y_2(& 5x_1 & + & 2x_2 & - & x_3 &) \geq & 6\, y_2 \\
 & & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2
\end{aligned}
$$

Bounds for y_1, y_2:

$$
\begin{array}{rcrcl}
y_1 & + & 5y_2 & \leq & 7 \\
-y_1 & + & 2y_2 & \leq & 1 \\
3y_1 & - & y_2 & \leq & 5
\end{array}
$$

Linear Programming – Lower Bounds

$$
\begin{array}{lrcrcrcl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \rotatebox{90}{$\le$} & & \\
\textbf{subject to} \; y_1(& x_1 & - & x_2 & + & 3x_3\,) & \geq & 10\,y_1 \\
 & + & & + & & + & & \\
y_2(& 5x_1 & + & 2x_2 & - & x_3\,) & \geq & 6\,y_2 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2
\end{aligned}
$$

Bounds for y_1, y_2:

$$
\begin{array}{rcrcl}
y_1 & + & 5y_2 & \leq & 7 \\
-y_1 & + & 2y_2 & \leq & 1 \\
3y_1 & - & y_2 & \leq & 5 \\
 & & y_1, y_2 & \geq & 0
\end{array}
$$

Linear Programming – Lower Bounds

$$\begin{array}{lrcrcrcr}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & y_1(\;x_1 & - & x_2 & + & 3x_3\;) & \geq & 10\,y_1 \\
 & y_2(\;5x_1 & + & 2x_2 & - & x_3\;) & \geq & 6\,y_2 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

$$7x_1 + x_2 + 5x_3 \geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3)$$

$$\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2$$

$$\begin{array}{lrcrcr}
\textbf{maximize} & 10y_1 & + & 6y_2 & & \\
\textbf{subject to} & y_1 & + & 5y_2 & \leq & 7 \\
 & -y_1 & + & 2y_2 & \leq & 1 \\
 & 3y_1 & - & y_2 & \leq & 5 \\
 & & & y_1, y_2 & \geq & 0
\end{array}$$

Linear Programming – Lower Bounds

Primal

$$\begin{array}{llcccccc} \textbf{minimize} & & 7x_1 & + & x_2 & + & 5x_3 & & \\ & & \text{IV} & & \text{IV} & & \text{IV} & & \\ \textbf{subject to} & y_1(& x_1 & - & x_2 & + & 3x_3 &) \geq & 10\,y_1 \\ & & + & & + & & + & & \\ & y_2(& 5x_1 & + & 2x_2 & - & x_3 &) \geq & 6\,y_2 \\ & & & & & & x_1, x_2, x_3 & \geq & 0 \end{array}$$

$$7x_1 + x_2 + 5x_3 \geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3)$$

$$\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \text{OPT} \geq 10y_1 + 6y_2$$

$$\begin{array}{lccccc} \textbf{maximize} & 10y_1 & + & 6y_2 & & \\ \textbf{subject to} & y_1 & + & 5y_2 & \leq & 7 \\ & -y_1 & + & 2y_2 & \leq & 1 \\ & 3y_1 & - & y_2 & \leq & 5 \\ & & & y_1, y_2 & \geq & 0 \end{array}$$

Linear Programming – Lower Bounds

Primal

$$\begin{array}{lrcrcrcl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
 & \vee\!| & & \vee\!| & & \vee\!| & & \\
\textbf{subject to} & y_1(\;x_1 & - & x_2 & + & 3x_3\;) & \geq & 10\,y_1 \\
 & + & & + & & + & & \\
 & y_2(\;5x_1 & + & 2x_2 & - & x_3\;) & \geq & 6\,y_2 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}$$

$$\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2
\end{aligned}$$

Dual

$$\begin{array}{lrcrcl}
\textbf{maximize} & 10y_1 & + & 6y_2 & & \\
\textbf{subject to} & y_1 & + & 5y_2 & \leq & 7 \\
 & -y_1 & + & 2y_2 & \leq & 1 \\
 & 3y_1 & - & y_2 & \leq & 5 \\
 & & & y_1, y_2 & \geq & 0
\end{array}$$

Linear Programming – Lower Bounds

Primal

$$
\begin{array}{lrcrcrcl}
\textbf{minimize} & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & y_1\,(\,x_1 & - & x_2 & + & 3x_3\,) & \geq & 10\,y_1 \\
 & y_2\,(\,5x_1 & + & 2x_2 & - & x_3\,) & \geq & 6\,y_2 \\
 & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2
\end{aligned}
$$

Dual

$$
\begin{array}{lrcrcl}
\textbf{maximize} & 10y_1 & + & 6y_2 & & \\
\textbf{subject to} & y_1 & + & 5y_2 & \leq & 7 \\
 & -y_1 & + & 2y_2 & \leq & 1 \\
 & 3y_1 & - & y_2 & \leq & 5 \\
 & & & y_1, y_2 & \geq & 0
\end{array}
$$

Any feasible solution to the **dual** program provides a lower bound for the optimum of the **primal** program.

Linear Programming – Lower Bounds

Primal

$$
\begin{array}{llcrcrcll}
\textbf{minimize} & & 7x_1 & + & x_2 & + & 5x_3 & & \\
\textbf{subject to} & y_1(& x_1 & - & x_2 & + & 3x_3) & \geq & 10\,y_1 \\
 & y_2(& 5x_1 & + & 2x_2 & - & x_3) & \geq & 6\,y_2 \\
 & & & & & & x_1, x_2, x_3 & \geq & 0
\end{array}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2
\end{aligned}
$$

Dual

$$
\begin{array}{lrcrcl}
\textbf{maximize} & 10y_1 & + & 6y_2 & & \\
\textbf{subject to} & y_1 & + & 5y_2 & \leq & 7 \\
 & -y_1 & + & 2y_2 & \leq & 1 \\
 & 3y_1 & - & y_2 & \leq & 5 \\
 & & & y_1, y_2 & \geq & 0
\end{array}
$$

Any feasible solution to the **dual** program provides a lower bound for the optimum of the **primal** program.

$x = (7/4, 0, 11/4)$ both $y = (2, 1)$ provide objective value 26.

Linear Programming – Lower Bounds

$$
\begin{array}{llcccccccl}
\textbf{minimize} & & 7x_1 & + & x_2 & + & 5x_3 & & & \text{Primal} \\
\textbf{subject to} & y_1(& x_1 & - & x_2 & + & 3x_3 &) \geq & 10\,y_1 & \\
 & y_2(& 5x_1 & + & 2x_2 & - & x_3 &) \geq & 6\,y_2 & \\
 & & & & & & x_1, x_2, x_3 & \geq & 0 &
\end{array}
$$

$$
\begin{aligned}
7x_1 + x_2 + 5x_3 &\geq y_1 \cdot (x_1 - x_2 + 3x_3) + y_2 \cdot (5x_1 + 2x_2 - x_3) \\
&\geq y_1 \cdot 10 + y_2 \cdot 6 \Rightarrow \mathrm{OPT} \geq 10y_1 + 6y_2
\end{aligned}
$$

$$
\begin{array}{lcccccl}
\textbf{maximize} & 10y_1 & + & 6y_2 & & & \text{Dual} \\
\textbf{subject to} & y_1 & + & 5y_2 & \leq & 7 & \\
 & -y_1 & + & 2y_2 & \leq & 1 & \\
 & 3y_1 & - & y_2 & \leq & 5 & \\
 & & & y_1, y_2 & \geq & 0 &
\end{array}
$$

Any feasible solution to the **dual** program provides a lower bound for the optimum of the **primal** program.

$x = (7/4, 0, 11/4)$ both $y = (2, 1)$ provide objective value $\overset{\mathrm{OPT}}{\overset{\|}{26}}$.

Primal – Dual

Primal Program

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Primal – Dual

Primal Program

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Dual Program

$$\begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Primal – Dual

Primal Program

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Dual Program

$$\begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Dual Program of the Dual Program

Primal – Dual

Primal Program

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Dual Program

$$\begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Dual Program of the Dual Program

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part IV:
LP-Duality and Complementary Slackness

Joachim Spoerhase

Winter 2021/22

LP-Duality

Primal

$$\begin{array}{llcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Dual

$$\begin{array}{llcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

LP-Duality

Primal

$$\begin{array}{llll} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Dual

$$\begin{array}{llll} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Theorem. The primal program has a finite optimum
$\Leftrightarrow$ the dual program has a finite optimum.

LP-Duality

Primal

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Dual

$$\begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Theorem. The primal program has a finite optimum $\Leftrightarrow$ the dual program has a finite optimum. Moreover, if $x^* = (x_1^*, \ldots, x_n^*)$ and $y^* = (y_1^*, \ldots, y_m^*)$ are *optimal* solutions for the primal and dual program (resp.), then

LP-Duality

Primal

$$\begin{array}{lrcl} \textbf{minimize} & c^\top x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

Dual

$$\begin{array}{lrcl} \textbf{maximize} & b^\top y & & \\ \textbf{subject to} & A^\top y & \leq & c \\ & y & \geq & 0 \end{array}$$

Theorem. The primal program has a finite optimum $\Leftrightarrow$ the dual program has a finite optimum. Moreover, if $x^* = (x_1^*, \ldots, x_n^*)$ and $y^* = (y_1^*, \ldots, y_m^*)$ are *optimal* solutions for the primal and dual program (resp.), then

$$\sum_{j=1}^{n} c_j x_j^* = \sum_{i=1}^{m} b_i y_i^* \, .$$

Weak LP-Duality

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\intercal y$		
subject to	$A^\intercal y$	$\leq$	c
	y	$\geq$	0

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program (resp.), then

Weak LP-Duality

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\intercal y$		
subject to	$A^\intercal y$	$\leq$	c
	y	$\geq$	0

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program (resp.), then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Weak LP-Duality

$$\begin{array}{llll} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array}$$

$$\begin{array}{llll} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & A^\intercal y & \leq & c \\ & y & \geq & 0 \end{array}$$

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program (resp.), then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

Weak LP-Duality

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\intercal y$		
subject to	$A^\intercal y$	$\leq$	c
	y	$\geq$	0

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program (resp.), then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} c_j x_j \geq \qquad\qquad \geq \sum_{i=1}^{m} b_i y_i \,.$$

Weak LP-Duality

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	x	$\geq$	0

maximize	$b^\intercal y$		
subject to	$A^\intercal y$	$\leq$	c
	y	$\geq$	0

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program (resp.), then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \, .$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \geq \qquad \geq \sum_{i=1}^{m} b_i y_i \, .$$

Weak LP-Duality

$$\begin{array}{lrcl} \textbf{minimize} & c^\top x & & \\ \textbf{subject to} & Ax & \geq & b \\ & x & \geq & 0 \end{array} \qquad \begin{array}{lrcl} \textbf{maximize} & b^\top y & & \\ \textbf{subject to} & \boxed{A^\top y \leq c} & & \\ & y & \geq & 0 \end{array}$$

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program (resp.), then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \geq \qquad \geq \sum_{i=1}^{m} b_i y_i \,.$$

Weak LP-Duality

minimize	$c^\intercal x$		
subject to	Ax	$\geq$	b
	$\boxed{x}$	$\boxed{\geq}$	$\boxed{0}$

maximize	$b^\intercal y$		
subject to	$\boxed{A^\intercal y}$	$\boxed{\leq}$	$\boxed{c}$
	y	$\geq$	0

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program (resp.), then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \geq \qquad \geq \sum_{i=1}^{m} b_i y_i \,.$$

Weak LP-Duality

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & \boxed{x \geq 0} & & \end{array} \qquad \begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & \boxed{A^\intercal y \leq c} & & \\ & y & \geq & 0 \end{array}$$

Theorem. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ are *valid* solutions for the primal and dual program (resp.), then
$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \geq \sum_{j=1}^{n} \boxed{\left(\sum_{i=1}^{m} a_{ij} y_i \right)} x_j = \qquad \geq \sum_{i=1}^{m} b_i y_i \,.$$

Weak LP-Duality

$$\begin{array}{lrcl} \textbf{minimize} & c^\top x & & \\ \textbf{subject to} & Ax & \geq & b \\ & \boxed{x \geq 0} & & \end{array} \qquad \begin{array}{lrcl} \textbf{maximize} & b^\top y & & \\ \textbf{subject to} & \boxed{A^\top y \leq c} & & \\ & y & \geq & 0 \end{array}$$

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program (resp.), then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \overset{\geq}{\geq} \sum_{j=1}^{n} \boxed{\left(\sum_{i=1}^{m} a_{ij} y_i \right)} x_j = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} a_{ij} y_i x_j \right) \geq \sum_{i=1}^{m} b_i y_i \,.$$

Weak LP-Duality

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & \boxed{x \geq 0} & & \end{array}$$

$$\begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & \boxed{A^\intercal y \leq c} & & \\ & y & \geq & 0 \end{array}$$

Theorem. If $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$ are *valid* solutions for the primal and dual program (resp.), then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \geq \sum_{j=1}^{n} \boxed{\left(\sum_{i=1}^{m} a_{ij} y_i \right)} x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} y_i x_j \right) \geq \sum_{i=1}^{m} b_i y_i \,.$$

Weak LP-Duality

$$\begin{array}{lrcl} \textbf{minimize} & c^\top x & & \\ \textbf{subject to} & Ax & \geq & b \\ & \boxed{x} & \boxed{\geq} & \boxed{0} \end{array} \qquad \begin{array}{lrcl} \textbf{maximize} & b^\top y & & \\ \textbf{subject to} & \boxed{A^\top y} & \boxed{\leq} & \boxed{c} \\ & y & \geq & 0 \end{array}$$

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program (resp.), then
$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \geq \sum_{j=1}^{n} \boxed{\left(\sum_{i=1}^{m} a_{ij} y_i \right)} x_j = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) y_i \geq \sum_{i=1}^{m} b_i y_i \,.$$

Weak LP-Duality

$$\begin{array}{lrcl} \textbf{minimize} & c^\intercal x & & \\ \textbf{subject to} & Ax & \geq & b \\ & \boxed{x} & \boxed{\geq} & \boxed{0} \end{array}$$

$$\begin{array}{lrcl} \textbf{maximize} & b^\intercal y & & \\ \textbf{subject to} & \boxed{A^\intercal y} & \boxed{\leq} & \boxed{c} \\ & y & \geq & 0 \end{array}$$

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program (resp.), then

$$\sum_{j=1}^{n} c_j x_j \geq \sum_{i=1}^{m} b_i y_i \,.$$

Proof.

$$\sum_{j=1}^{n} \boxed{c_j} x_j \overset{\geq}{\geq} \sum_{j=1}^{n} \boxed{\left(\sum_{i=1}^{m} a_{ij} y_i \right)} x_j = \sum_{i=1}^{m} \boxed{\left(\sum_{j=1}^{n} a_{ij} x_j \right)} y_i \geq \sum_{i=1}^{m} b_i y_i \,.$$

Weak LP-Duality

minimize $c^\intercal x$ **subject to** $\boxed{Ax \geq b}$ $\boxed{x \geq 0}$	**maximize** $b^\intercal y$ **subject to** $\boxed{A^\intercal y \leq c}$ $y \geq 0$

Theorem. If $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_m)$ are *valid* solutions for the primal and dual program (resp.), then

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	x	$\geq$	0

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LPs and Convex Polytopes

The feasible solutions of an LP with n variables from a **convex polytope** in $\mathbb{R}^n$ (intersection of halfspaces).

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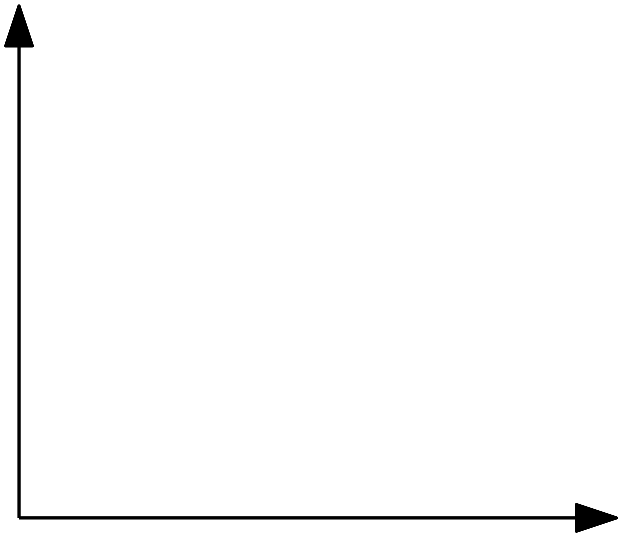

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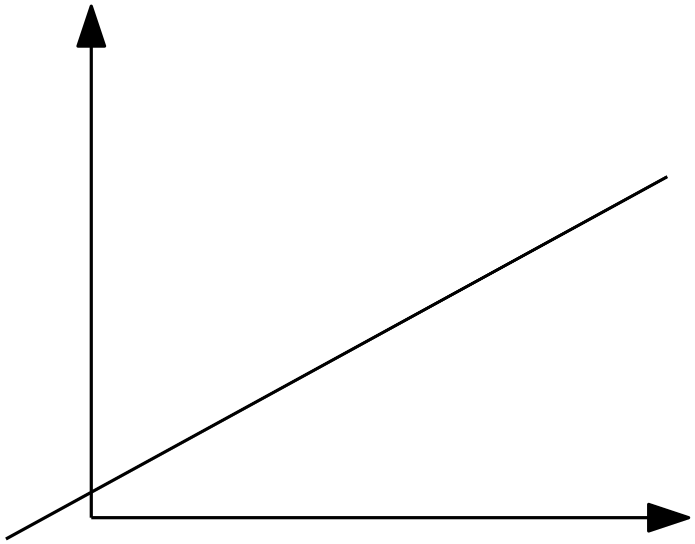

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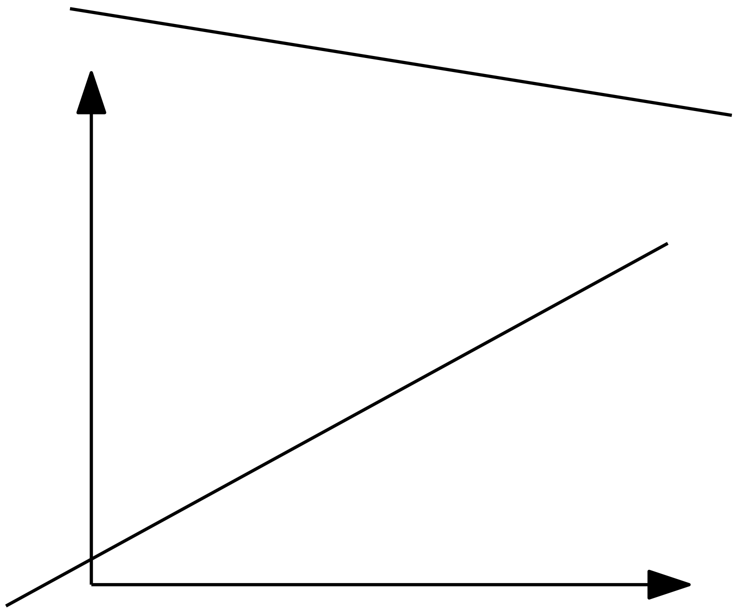

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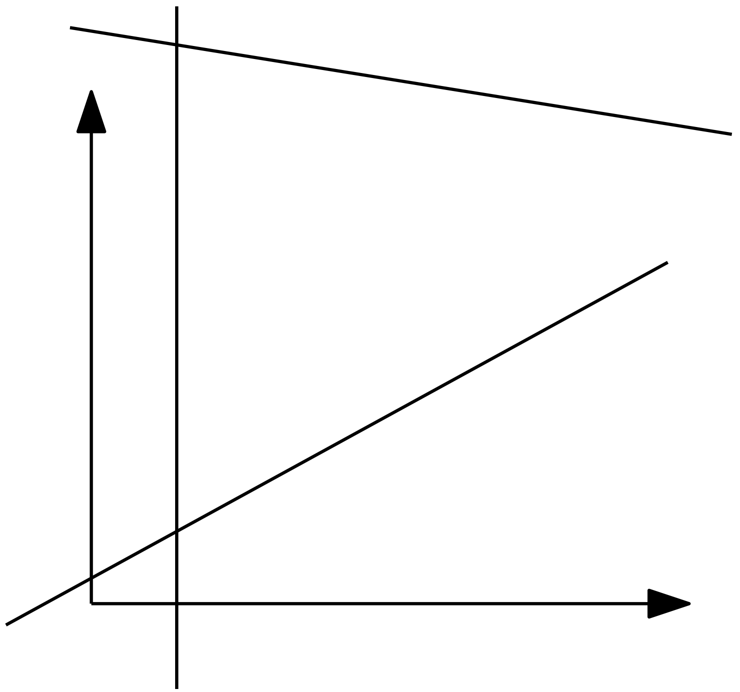

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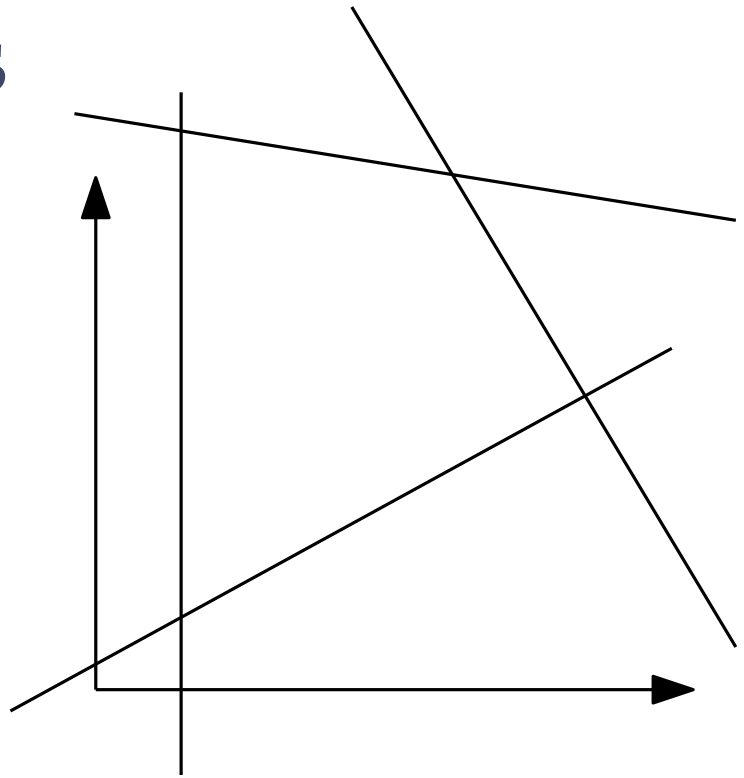

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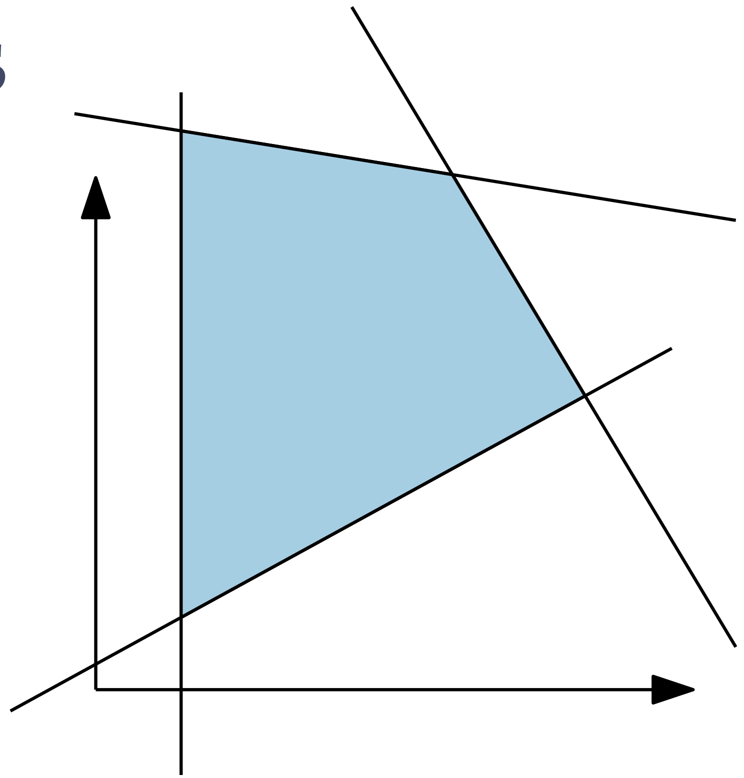

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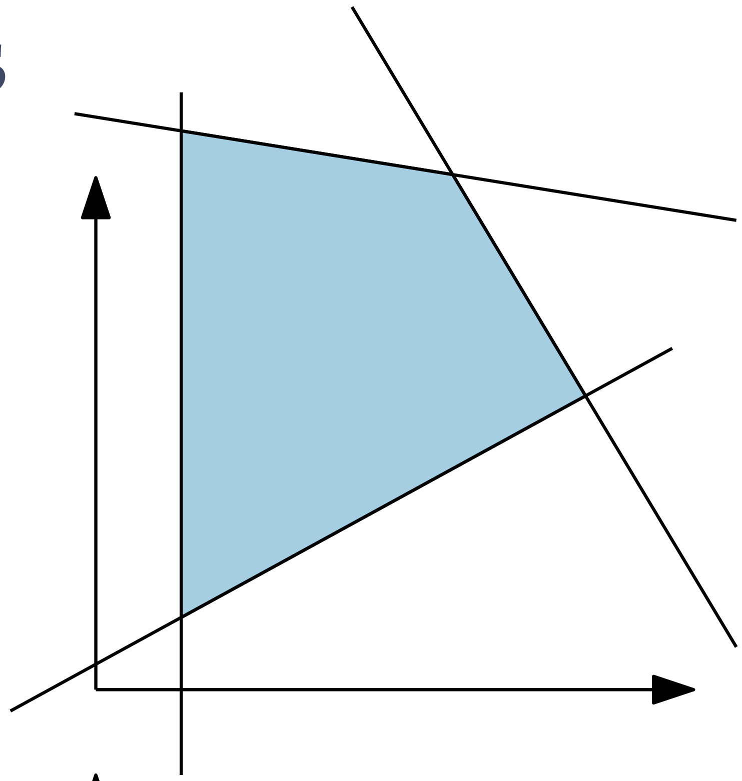

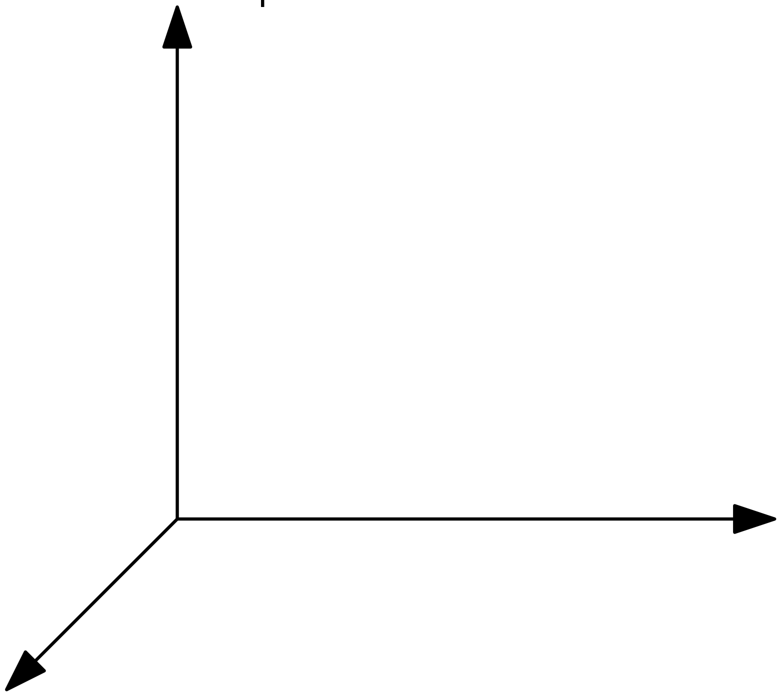

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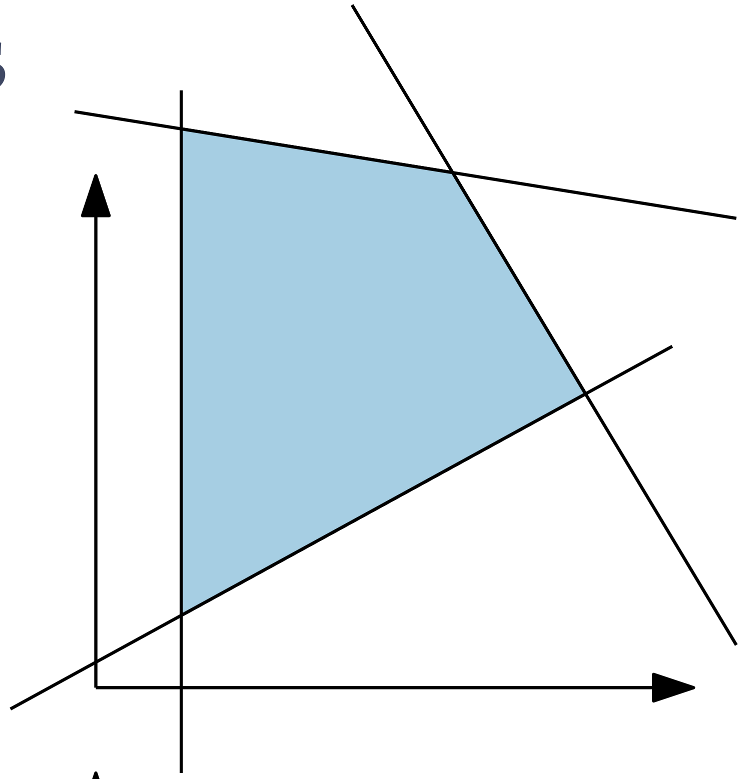

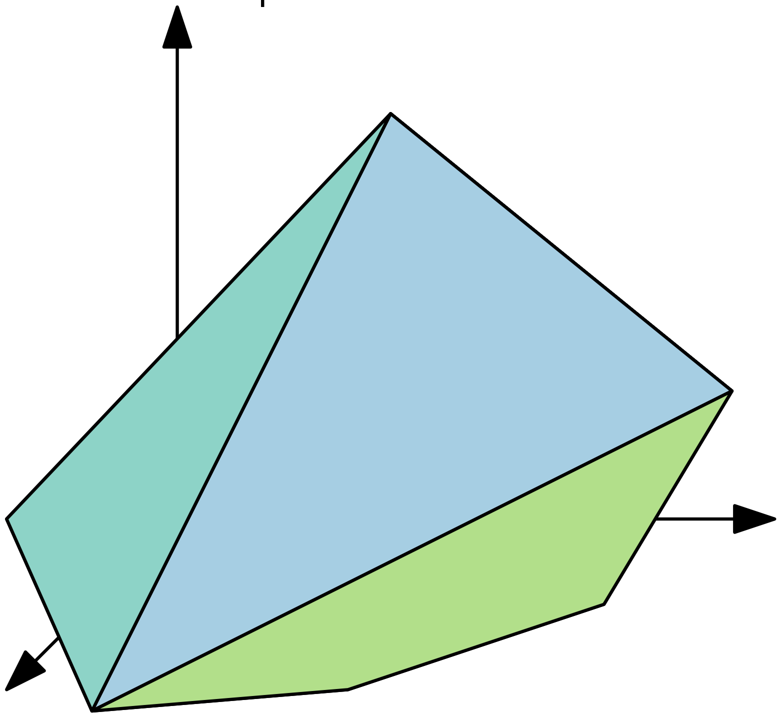

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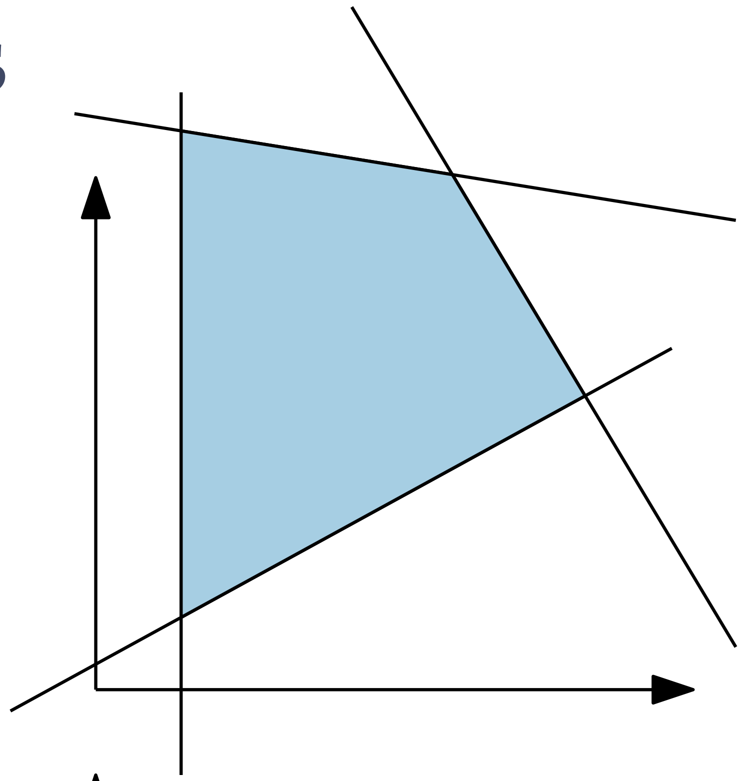
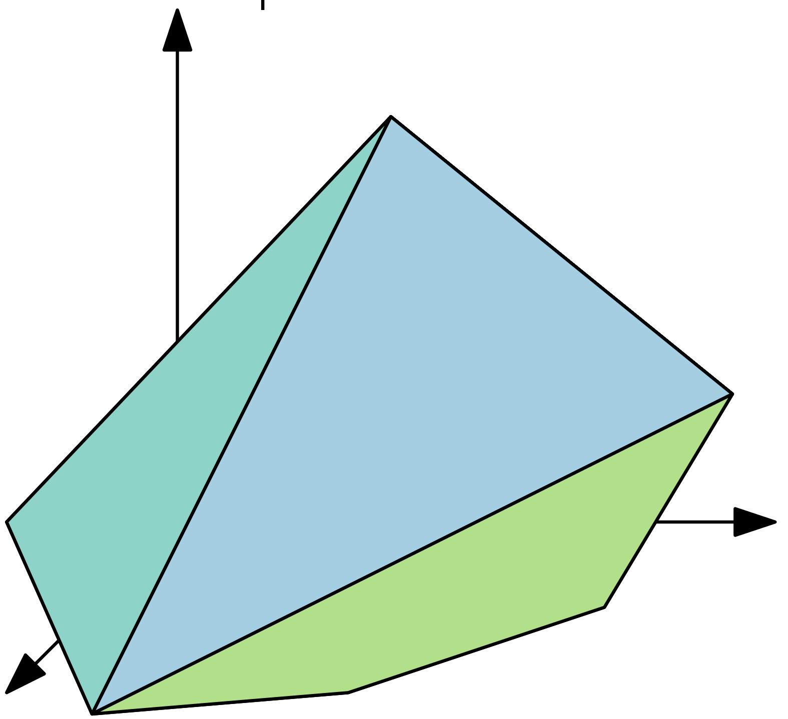


Corners of the polytope are called **extreme point solutions** $\Leftrightarrow$ n linearly independent inequalities (constraints) are satisfied with equality.



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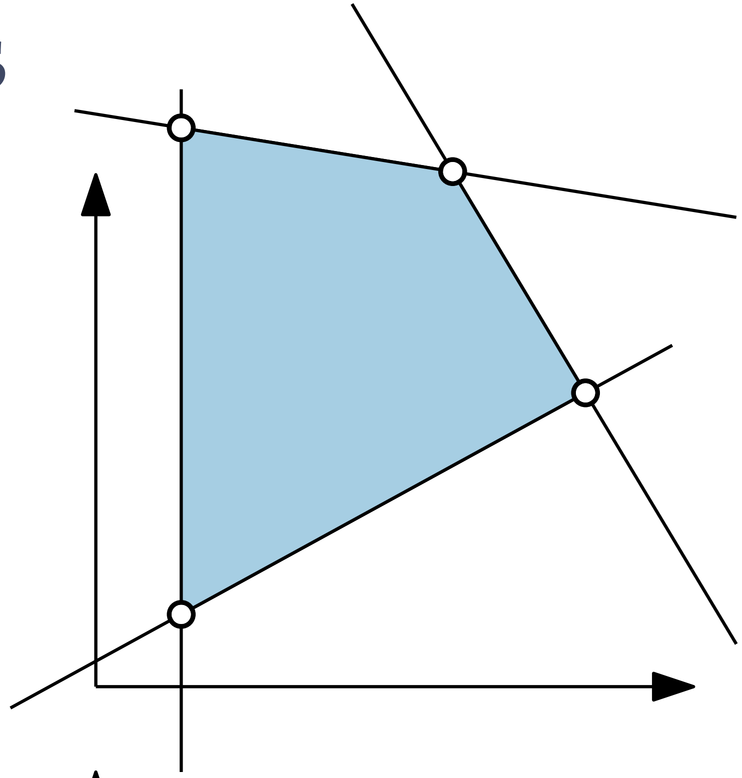

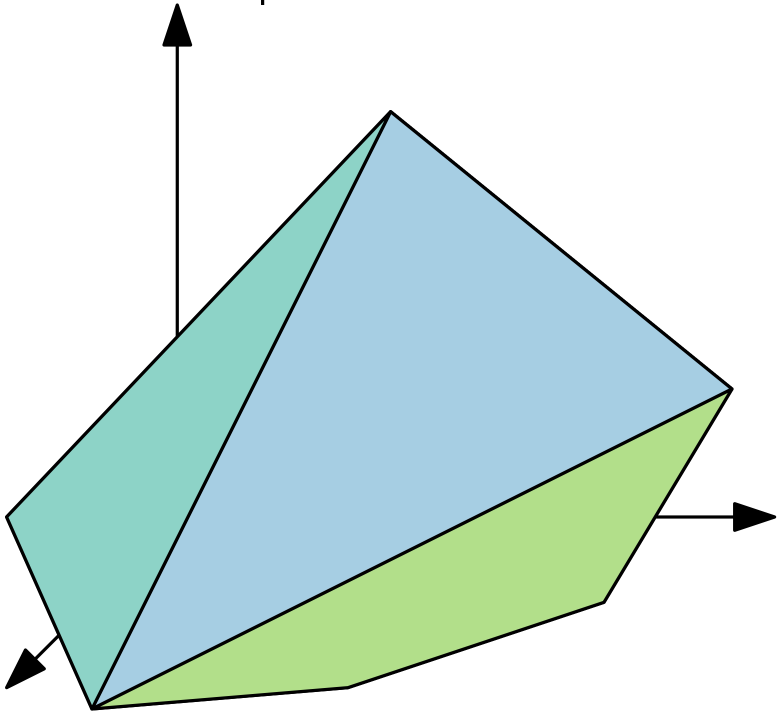

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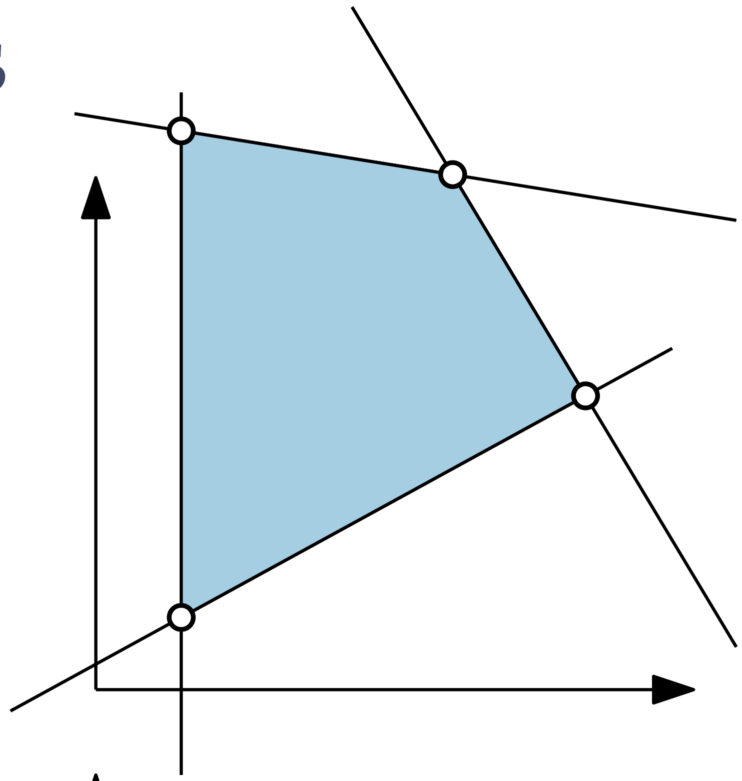

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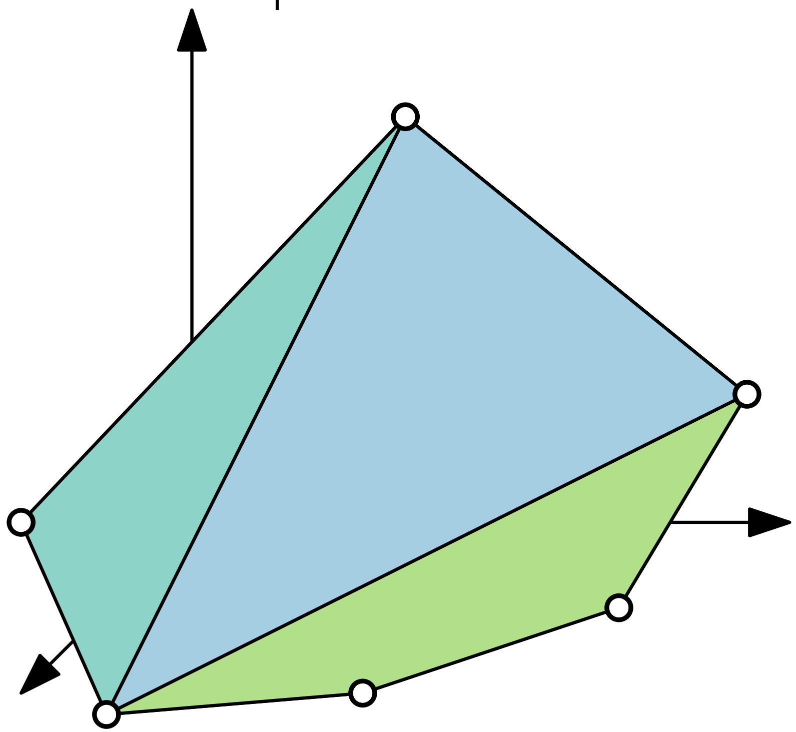

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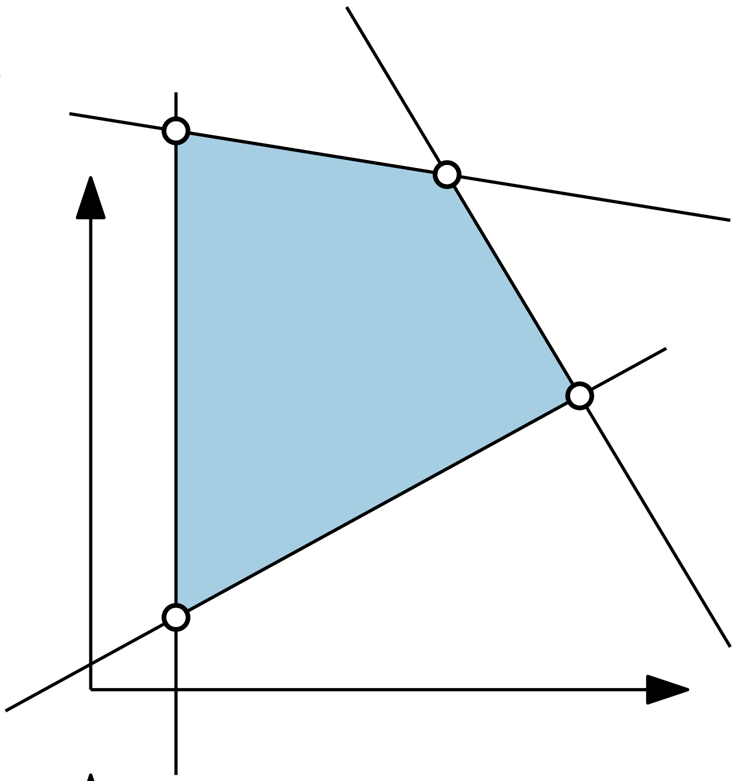

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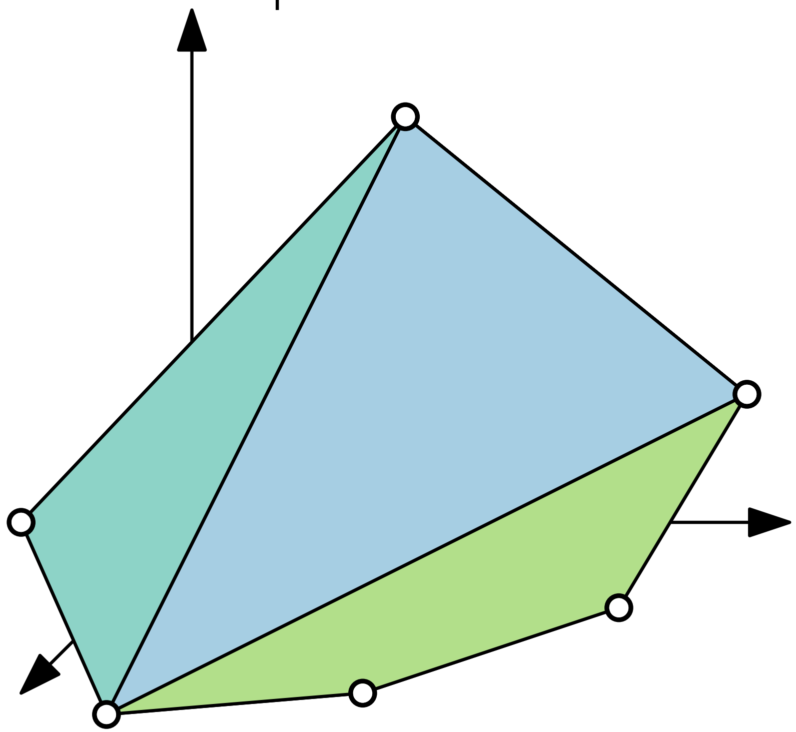

When an optimal solution exists, some extreme point will also be optimal.

Integer Linear Programs (ILPs)

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LP-Relaxation provides lower bound: $\mathrm{OPT}_{\mathrm{LP}} \leq \mathrm{OPT}_{\mathrm{ILP}}$

Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part V:
Min-Max-Relationships

Joachim Spoerhase

Winter 2021/22

Max-Flow-Problem

Given: A directed graph $G = (V, E)$ with edge capacities $c \colon E \to \mathbb{Q}_+$ and two special vertices: the source s and sink t.

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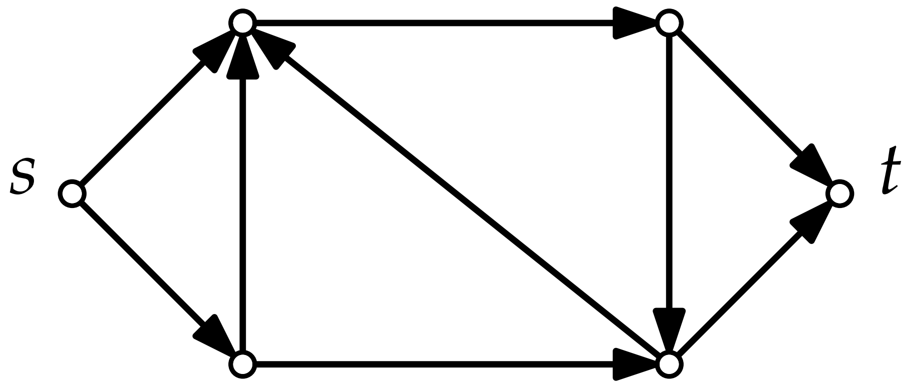

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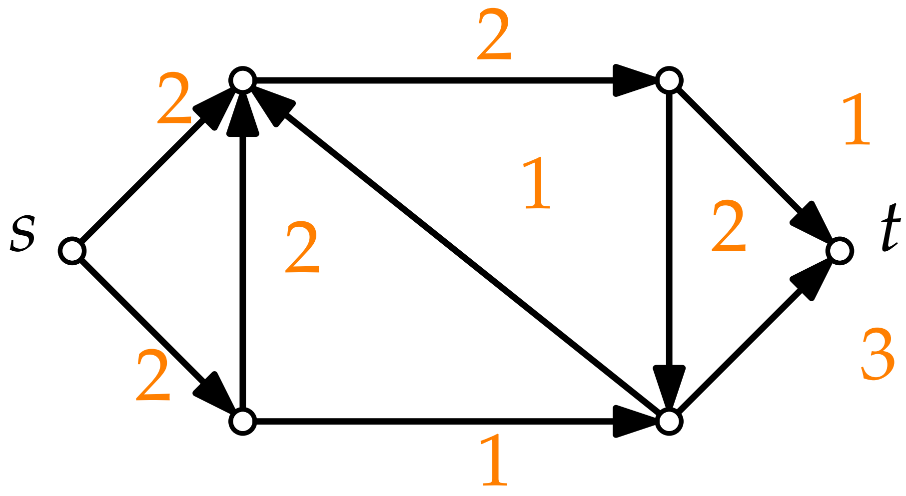

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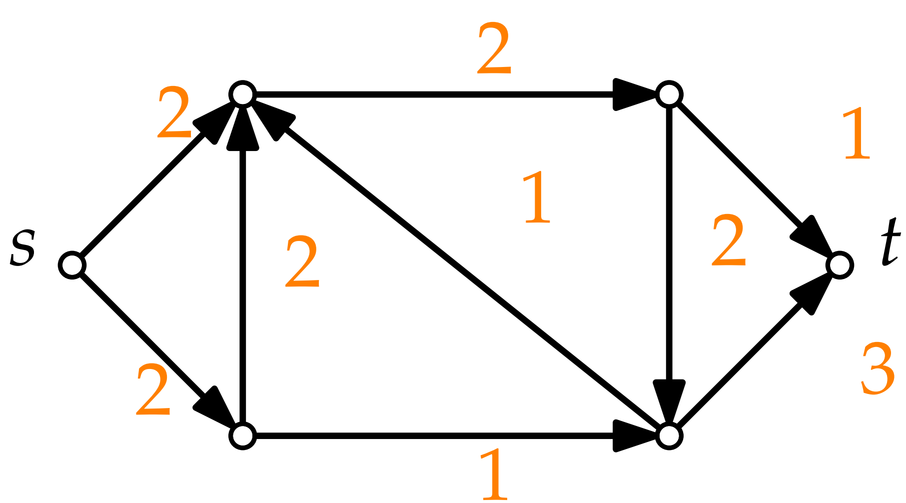

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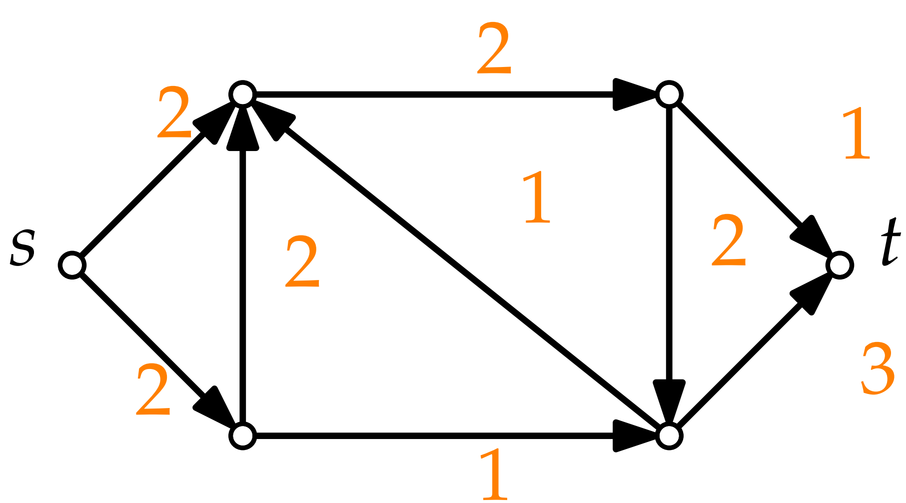

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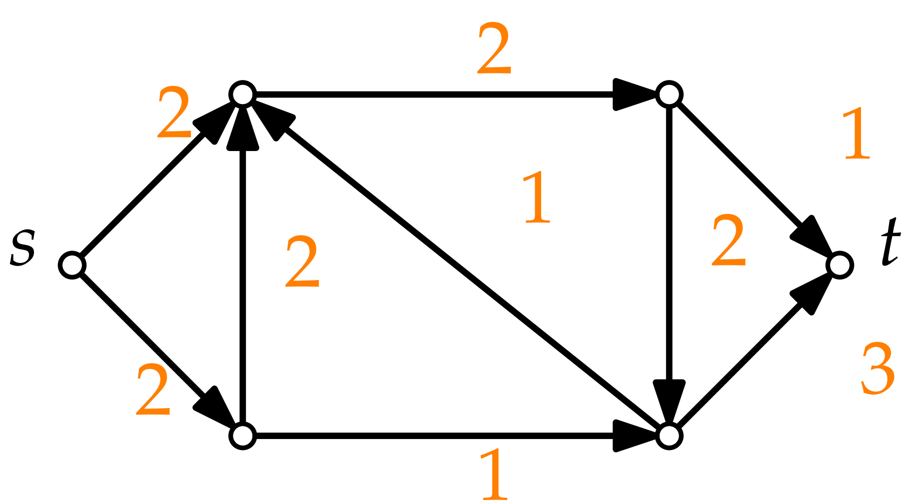

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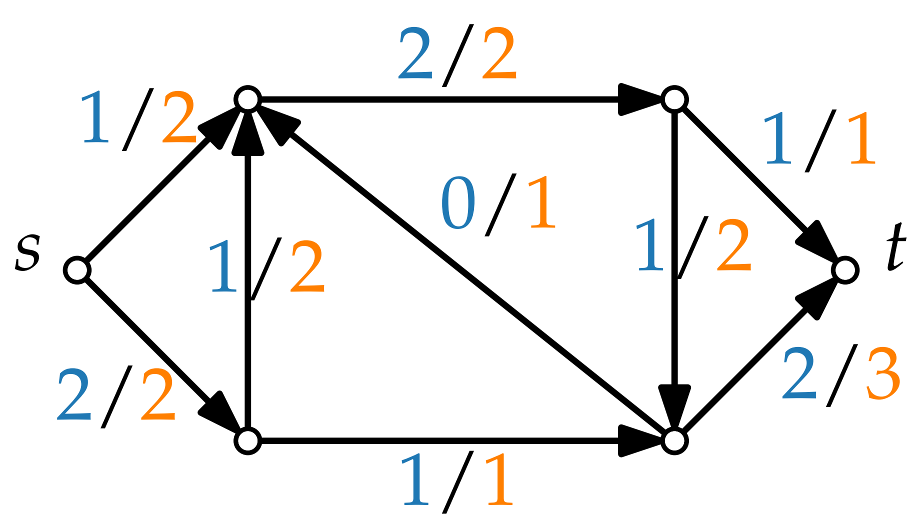
- $f(u, v) \leq c(u, v)$ for each edge $(u, v) \in E$
- $\sum_{(u,v) \in E} f(u, v) = \sum_{(v,z) \in E} f(v, z)$ for each vertex $v \in V - \{s, t\}$



Max-Flow-Problem

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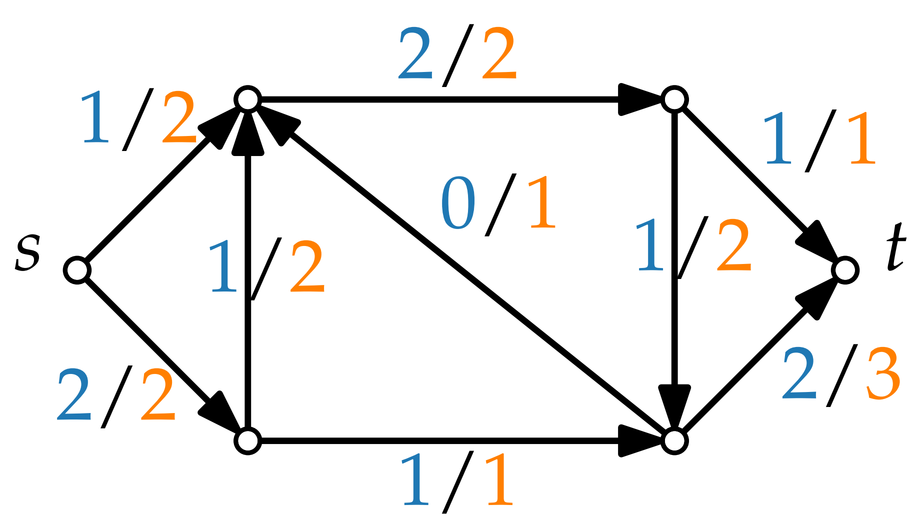
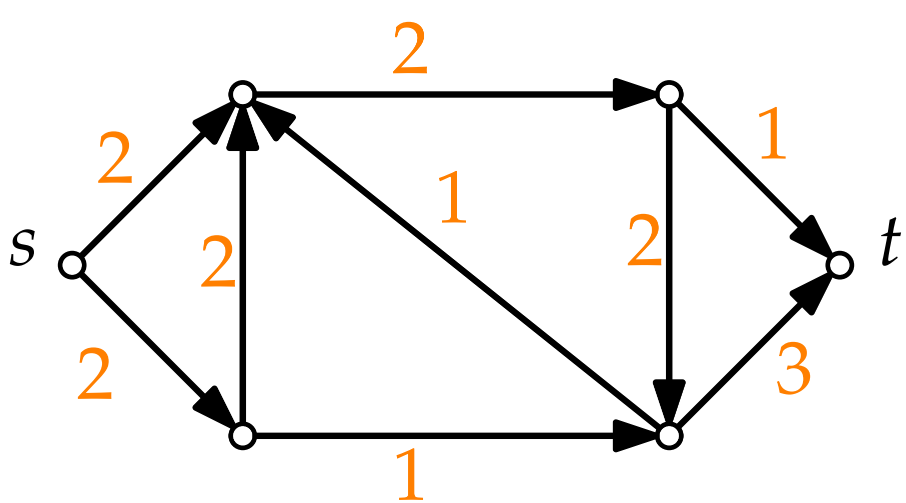
The **flow value** is the inflow to t minus the outflow from t.



Min-Cut-Problem

Given: A directed graph $G = (V, E)$ with edge capacities $c\colon E \to \mathbb{Q}_+$ and two special vertices: the source s and sink t.

Find: An s–t-cut, i.e., a vertex set X with $s \in X$ and $t \in \overline{X}$, such that the total weight $c(X, \overline{X})$ of the edges from X to $\overline{X}$ is minimum.



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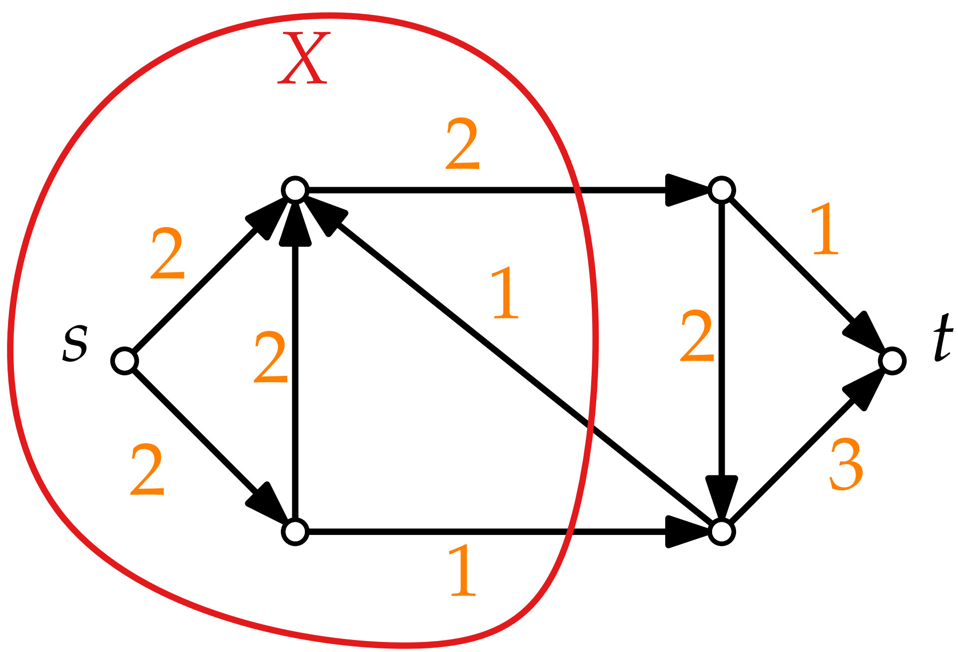

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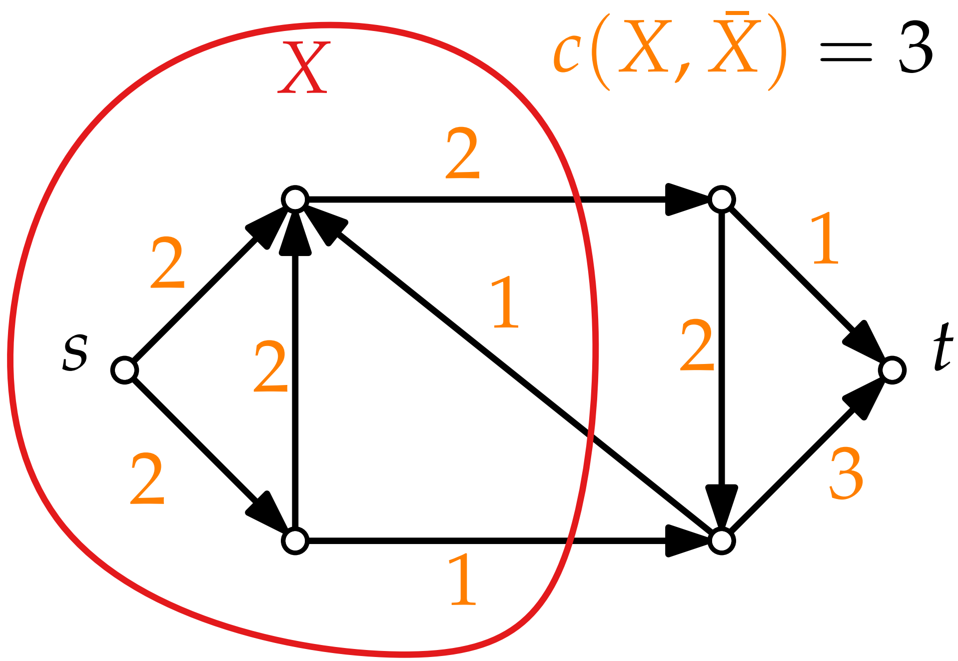

Max-Flow-Min-Cut-Theorem

Theorem. The value of a maximum s–t-flow and the weight of a minimum s–t-cut are the same.

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Proof. Special case of LP-Duality …

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maximize
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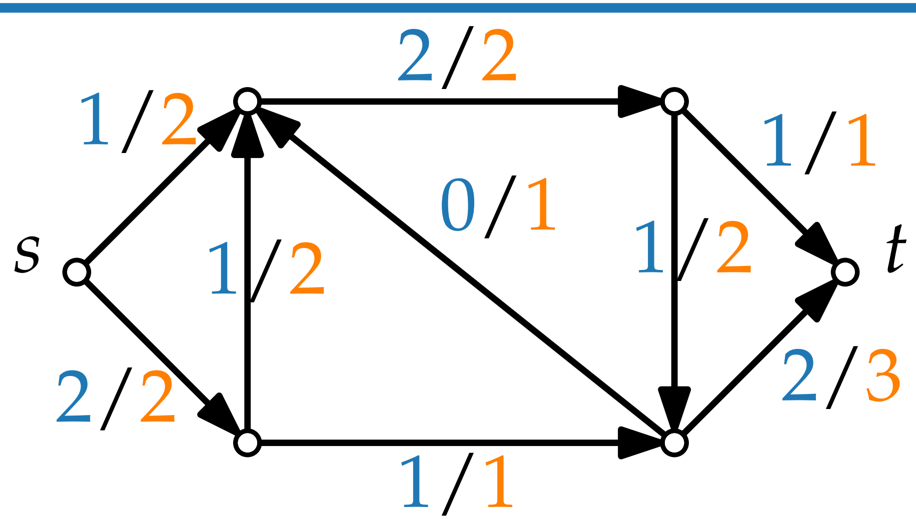

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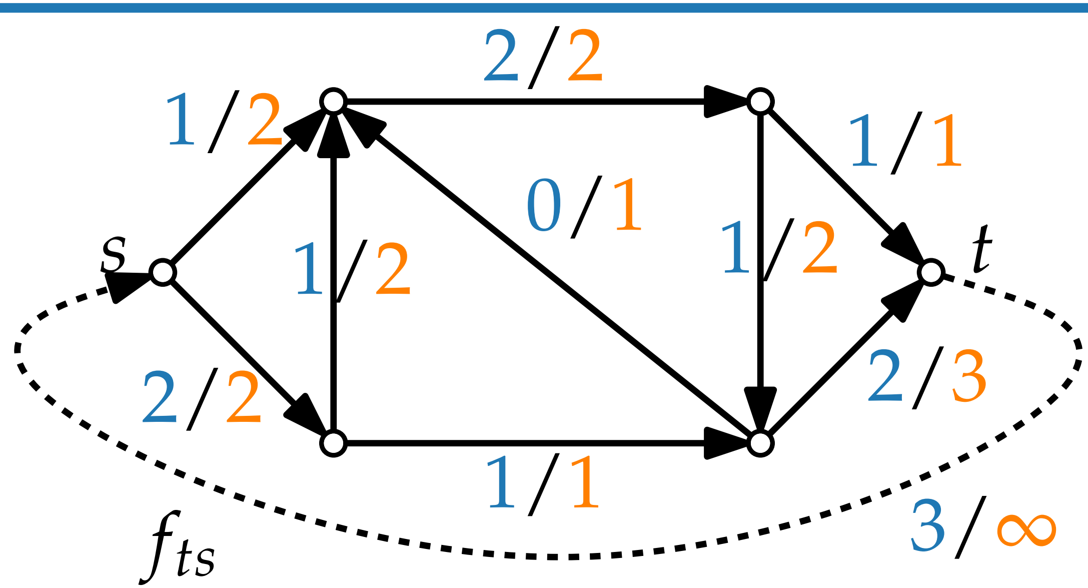

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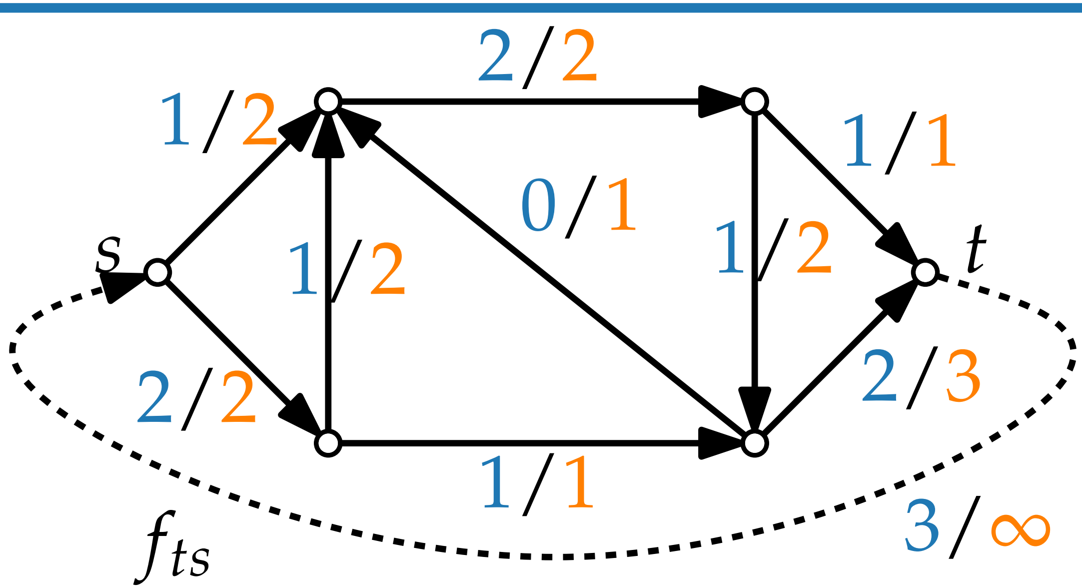

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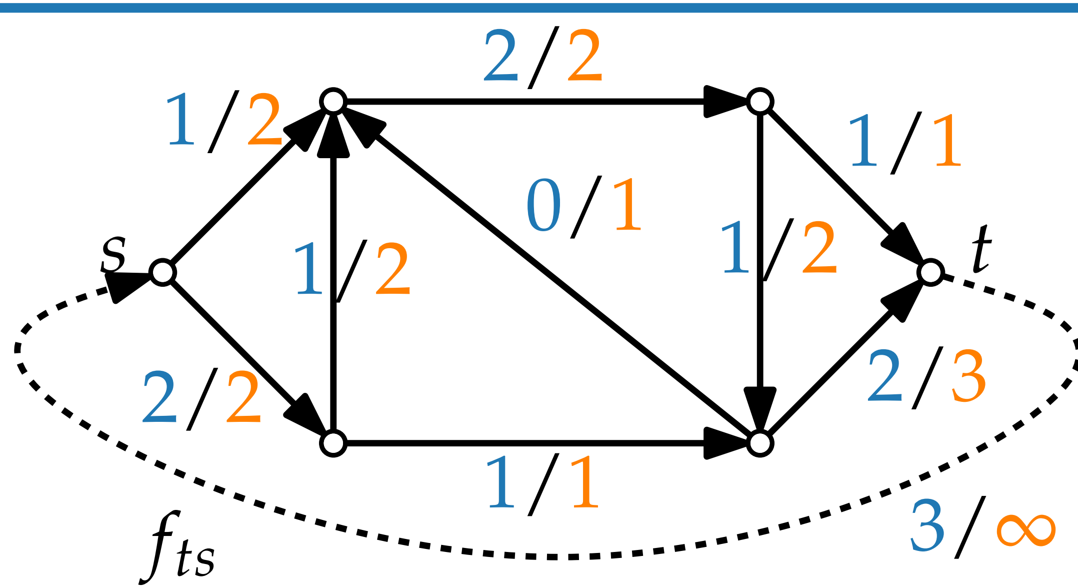
$$\sum_{u\colon (u,v)\in E} f_{uv} - \sum_{z\colon (v,z)\in E} f_{vz} \leq 0 \qquad \forall v \in V$$



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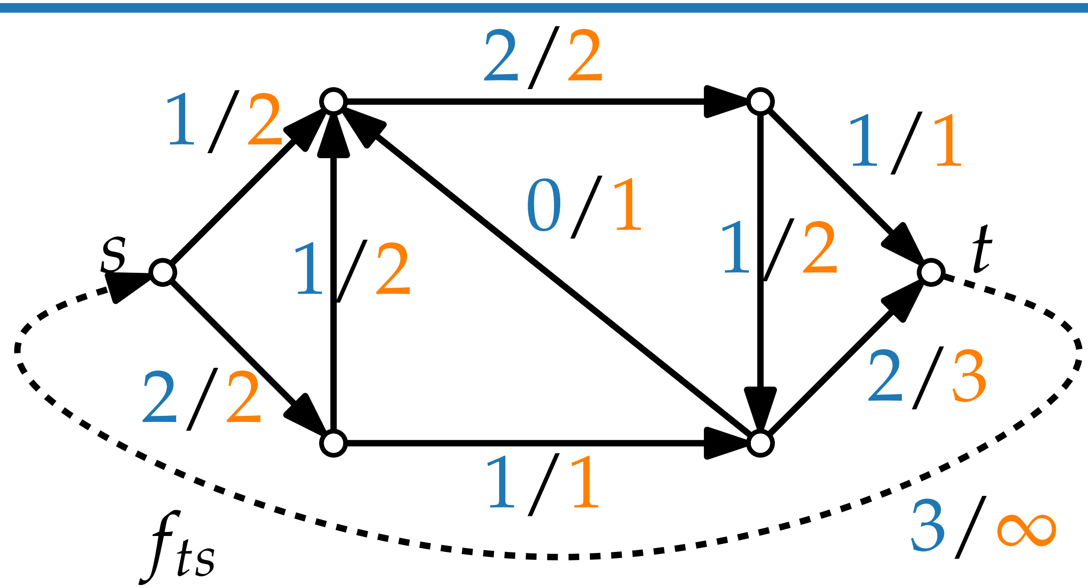

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maximize $c^\intercal x$

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Which constraints contain $f_{uv} \neq f_{ts}$?

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Which constraints contain f_{ts}?

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Which constraints contain $f_{uv} \neq f_{ts}$? $\qquad d_{uv}, p_u, p_v$

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$$
\begin{array}{lll}
\textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v) \in E \setminus \{(t,s)\} \\
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\textbf{minimize} & \sum\limits_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} & \\
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\end{array}$$

Approximation Algorithms

Lecture 4:
Linear Programming and LP-Duality

Part VI:
Dual LP of Max Flow

Joachim Spoerhase

Winter 2020/21

Dual LP – Interpretation as ILP

$$
\begin{array}{lll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & \\
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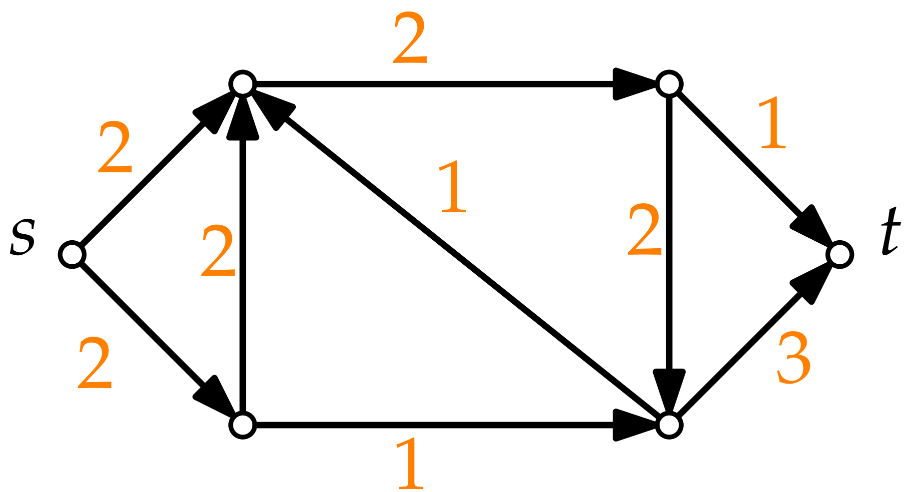

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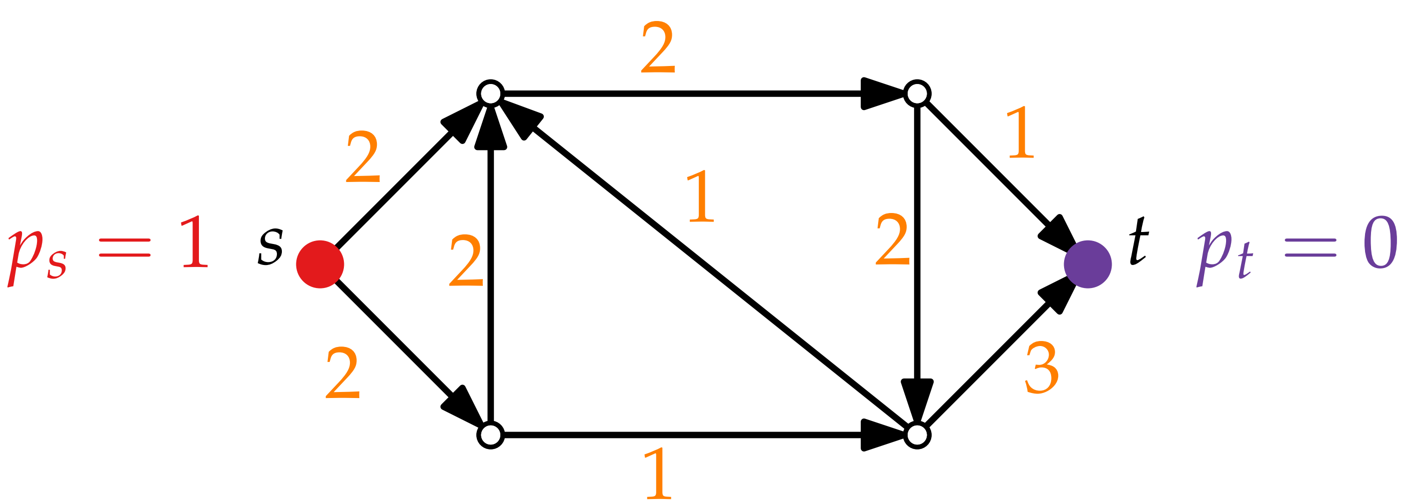

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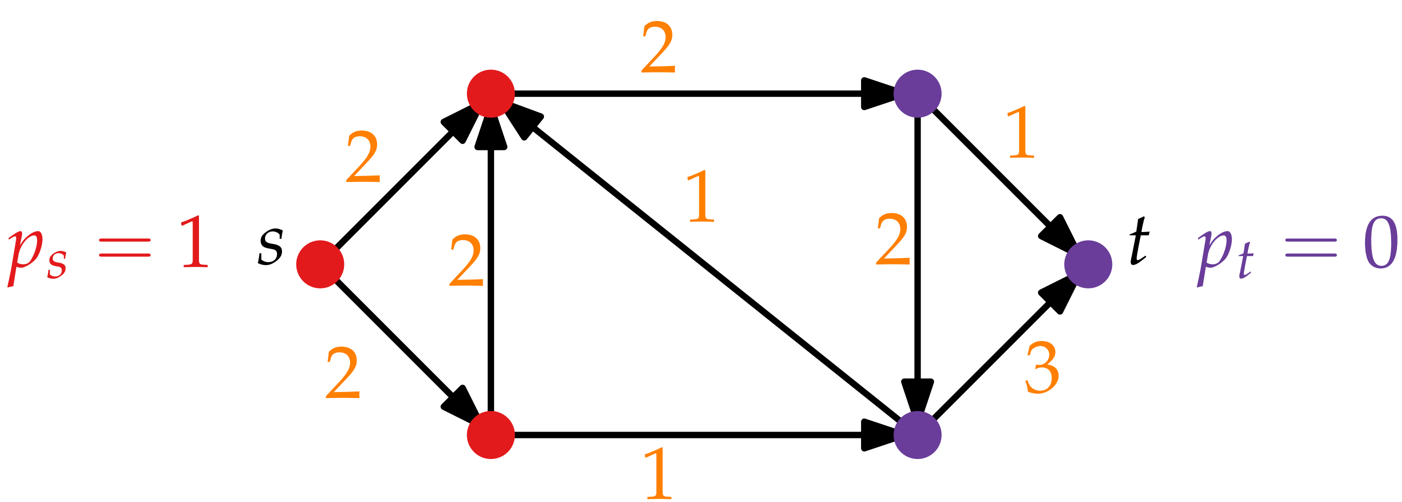
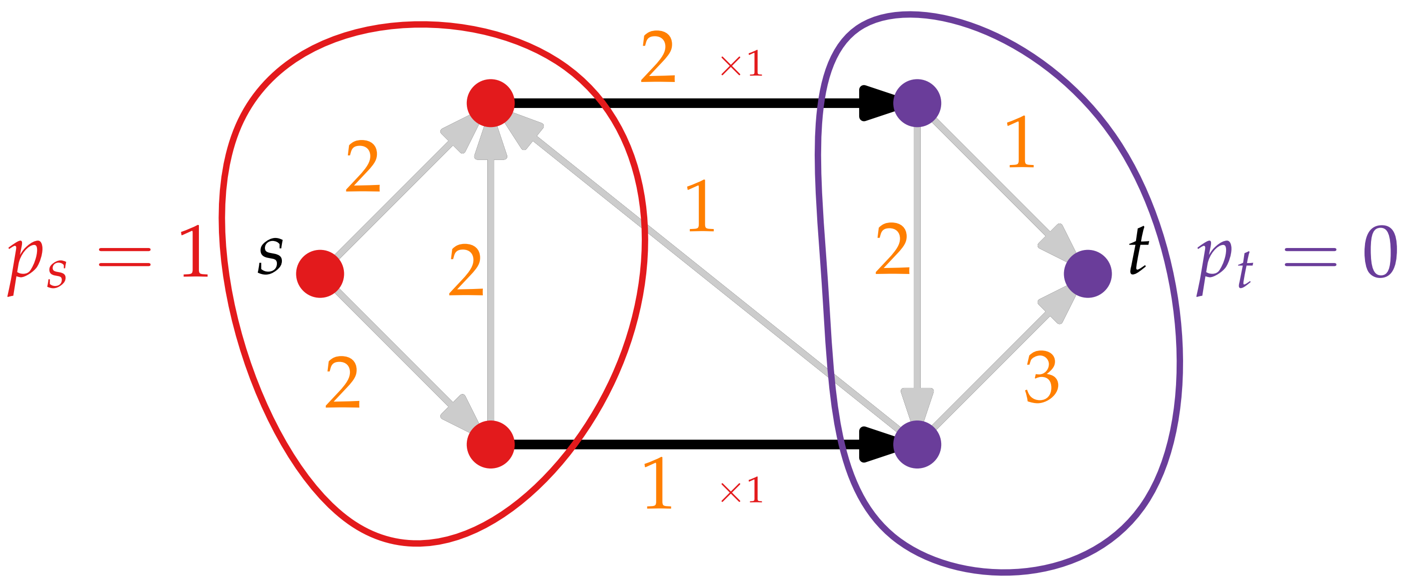
\begin{array}{lll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & \\
\textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v)\in E\setminus\{(t,s)\} \\
 & p_s - p_t \geq 1 & \\
 & d_{uv} \cancel{\geq 0} \in \{0,1\} & \forall (u,v)\in E \\
 & p_u \cancel{\geq 0} \in \{0,1\} & \forall u\in V
\end{array}
$$



Dual LP – Interpretation as ILP

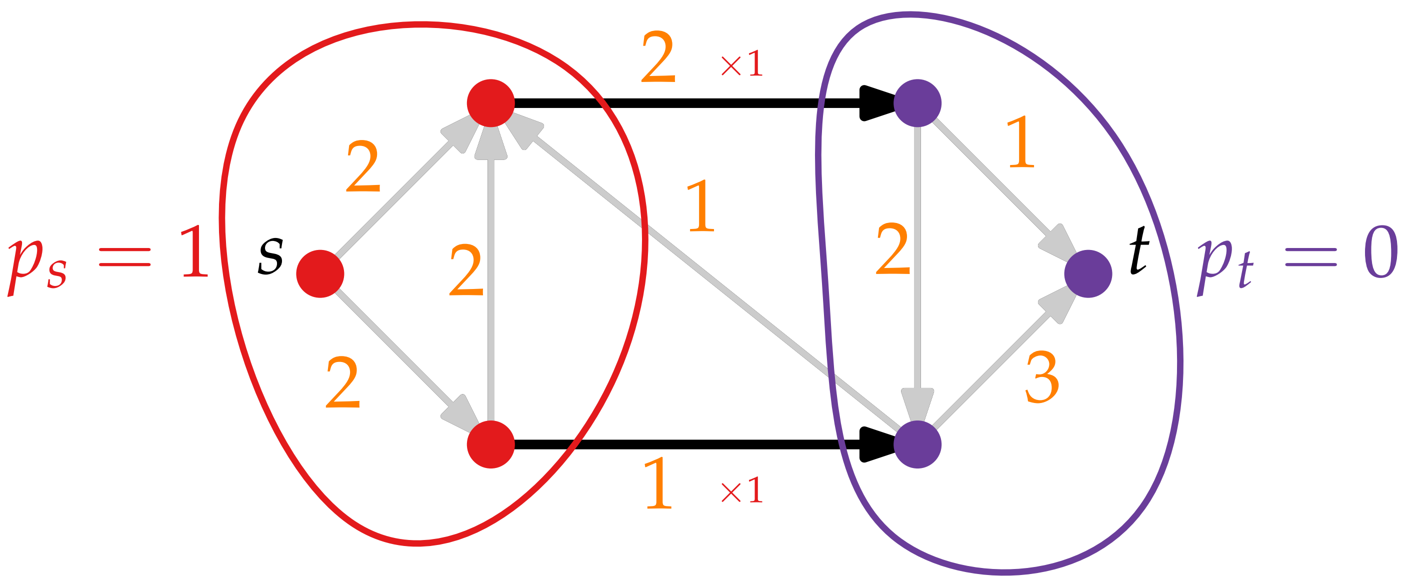
$$\begin{array}{lll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & \\
\textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v)\in E\setminus\{(t,s)\} \\
& p_s - p_t \geq 1 & \\
& d_{uv} \cancel{\geq 0} \in \{0,1\} & \forall (u,v)\in E \\
& p_u \cancel{\geq 0} \in \{0,1\} & \forall u \in V
\end{array}$$



Dual LP – Interpretation as ILP

$$
\begin{array}{lll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & \\
\textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v) \in E \setminus \{(t,s)\} \\
 & p_s - p_t \geq 1 & \\
 & d_{uv} \cancel{\geq 0} \in \{0,1\} & \forall (u,v) \in E \\
 & p_u \cancel{\geq 0} \in \{0,1\} & \forall u \in V
\end{array}
$$

equivalent to Min-Cut!

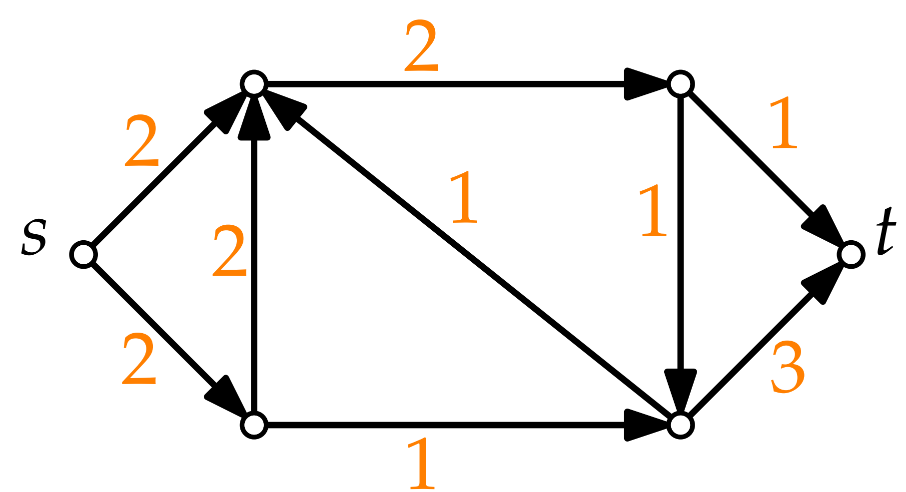


Dual LP – Factional Cuts

$$
\begin{array}{lll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} \equiv \text{LP-Relaxation of the ILP} & \\
\textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v) \in E \setminus \{(t,s)\} \\
& p_s - p_t \geq 1 & \\
& d_{uv} \geq 0 & \forall (u,v) \in E \\
& p_u \geq 0 & \forall u \in V
\end{array}
$$

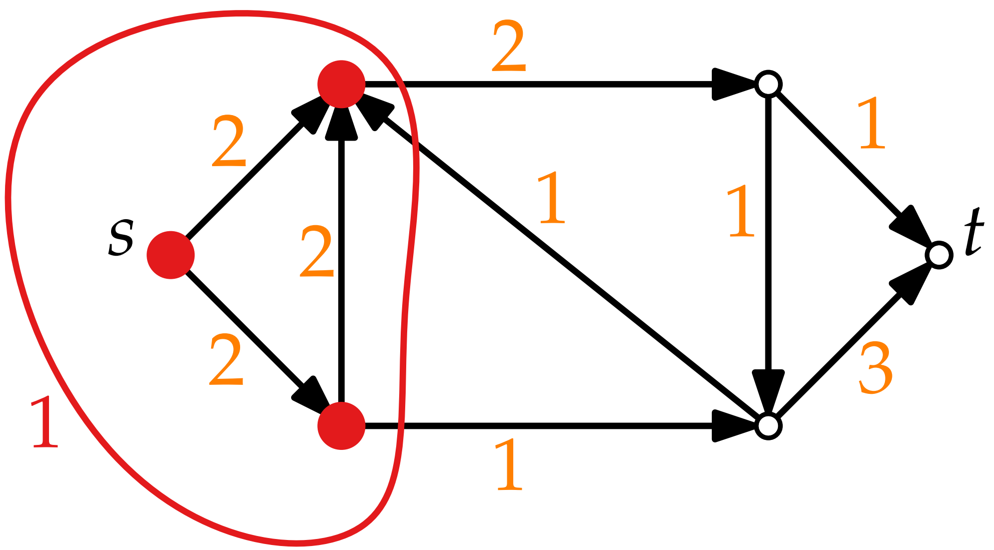
Dual LP – Factional Cuts

$$\begin{aligned}
\textbf{minimize} \quad & \sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} \equiv \text{LP-Relaxation of the ILP} \\
\textbf{subject to} \quad & d_{uv} - p_u + p_v \geq 0 && \forall (u,v) \in E \setminus \{(t,s)\} \\
& p_s - p_t \geq 1 \\
& d_{uv} \geq 0 && \forall (u,v) \in E \\
& p_u \geq 0 && \forall u \in V
\end{aligned}$$



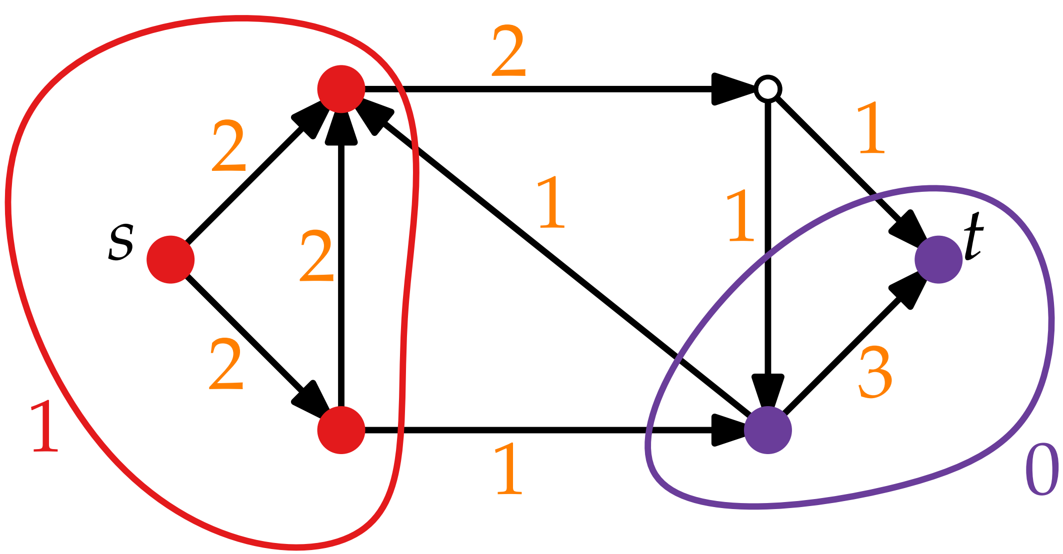
Dual LP – Factional Cuts

$$\begin{array}{lll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & \equiv \text{LP-Relaxation of the ILP} \\
\textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v) \in E \setminus \{(t,s)\} \\
& p_s - p_t \geq 1 & \\
& d_{uv} \geq 0 & \forall (u,v) \in E \\
& p_u \geq 0 & \forall u \in V
\end{array}$$



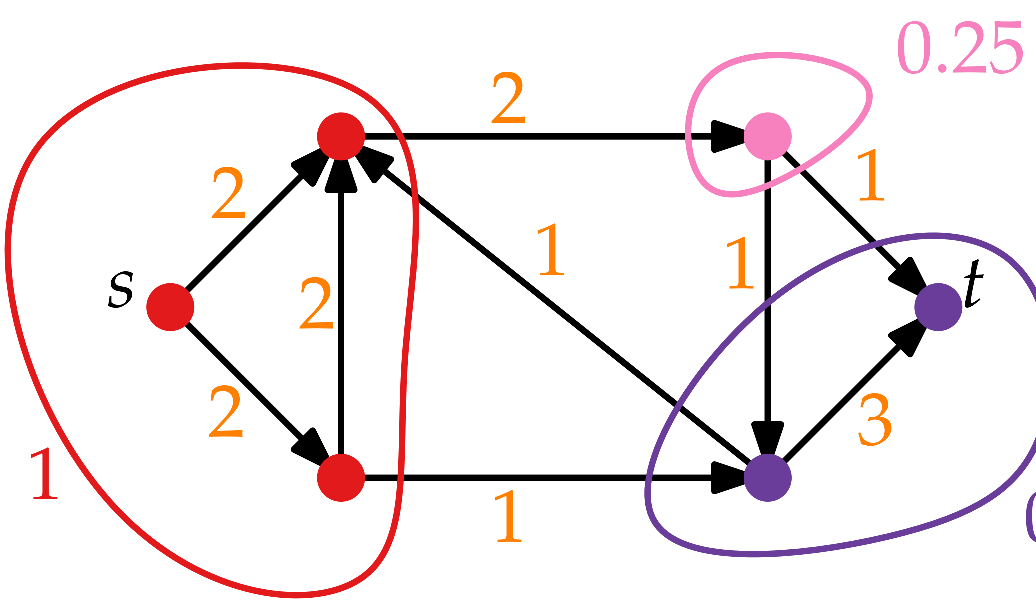
Dual LP – Factional Cuts

$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} \equiv \text{LP-Relaxation of the ILP} & \\ \textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v)\in E\setminus\{(t,s)\} \\ & p_s - p_t \geq 1 & \\ & d_{uv} \geq 0 & \forall (u,v)\in E \\ & p_u \geq 0 & \forall u\in V \end{array}$$



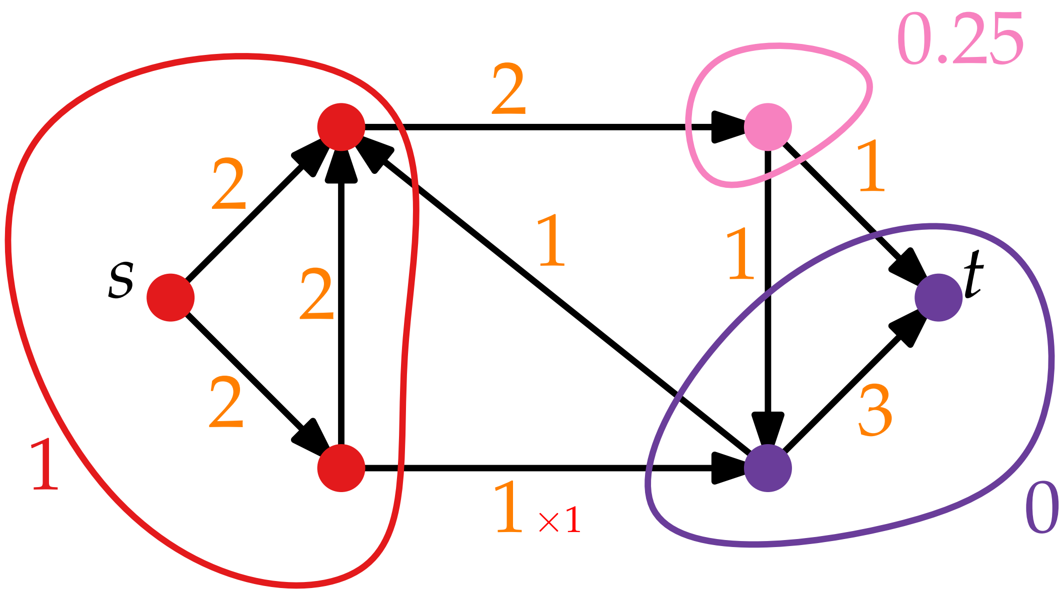
Dual LP – Factional Cuts

minimize	$\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} \equiv$ LP-Relaxation of the ILP	
subject to	$d_{uv} - p_u + p_v \geq 0$	$\forall (u,v) \in E \setminus \{(t,s)\}$
	$p_s - p_t \geq 1$	
	$d_{uv} \geq 0$	$\forall (u,v) \in E$
	$p_u \geq 0$	$\forall u \in V$



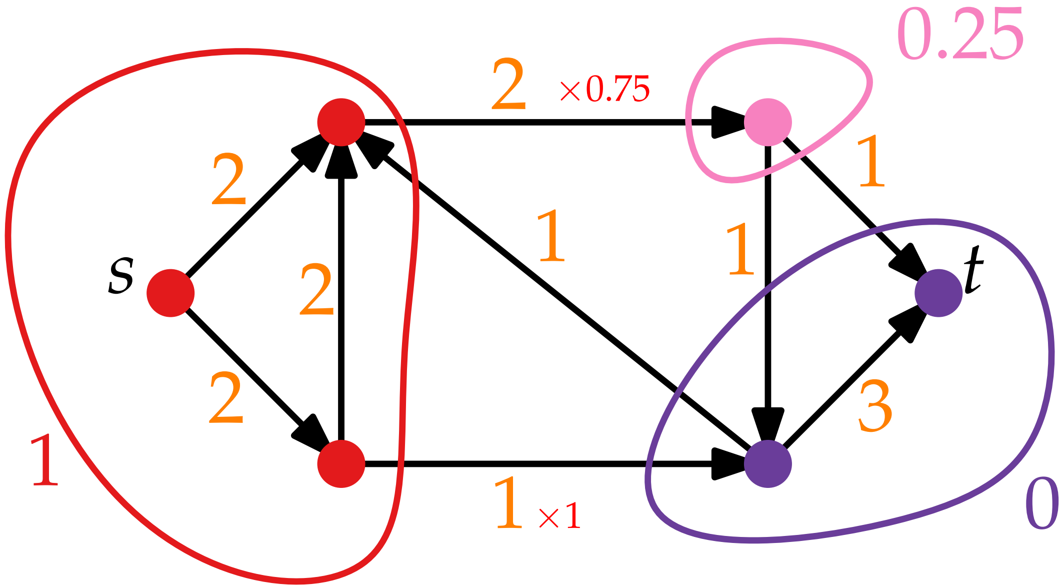
Dual LP – Factional Cuts

$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & \equiv \text{LP-Relaxation of the ILP} \\ \textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v)\in E\setminus\{(t,s)\} \\ & p_s - p_t \geq 1 & \\ & d_{uv} \geq 0 & \forall (u,v)\in E \\ & p_u \geq 0 & \forall u \in V \end{array}$$



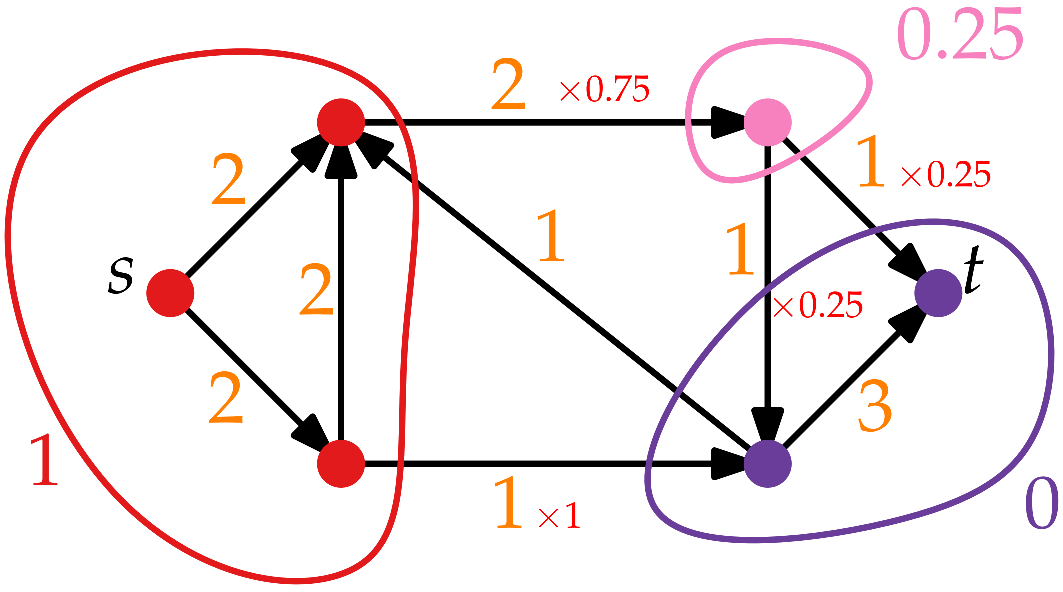
Dual LP – Factional Cuts

$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} \equiv \text{LP-Relaxation of the ILP} & \\ \textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v)\in E\setminus\{(t,s)\} \\ & p_s - p_t \geq 1 & \\ & d_{uv} \geq 0 & \forall (u,v)\in E \\ & p_u \geq 0 & \forall u \in V \end{array}$$



Dual LP – Factional Cuts

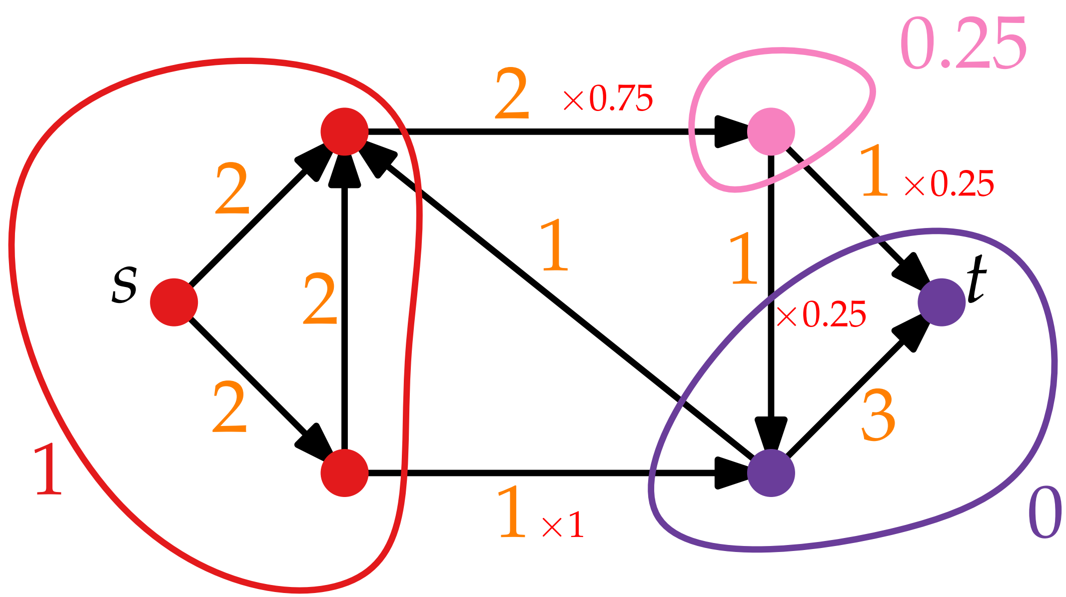
$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} \equiv \text{LP-Relaxation of the ILP} & \\ \textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v)\in E\setminus\{(t,s)\} \\ & p_s - p_t \geq 1 & \\ & d_{uv} \geq 0 & \forall (u,v)\in E \\ & p_u \geq 0 & \forall u \in V \end{array}$$



Dual LP – Factional Cuts

$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} \equiv \text{LP-Relaxation of the ILP} & \\ \textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v) \in E \setminus \{(t,s)\} \\ & p_s - p_t \geq 1 & \\ & d_{uv} \geq 0 & \forall (u,v) \in E \\ & p_u \geq 0 & \forall u \in V \end{array}$$

Each s–t-path $s = v_0, \ldots, v_k = t$ has length ≥ 1 w.r.t. d:

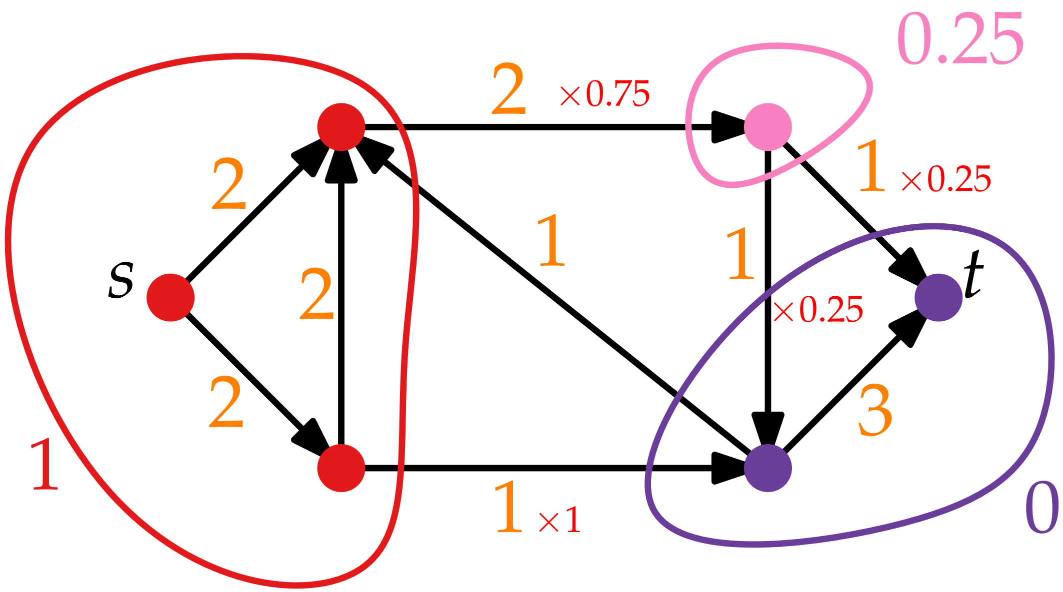


Dual LP – Factional Cuts

$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & \equiv \text{LP-Relaxation of the ILP} \\ \textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v)\in E\setminus\{(t,s)\} \\ & p_s - p_t \geq 1 & \\ & d_{uv} \geq 0 & \forall (u,v)\in E \\ & p_u \geq 0 & \forall u \in V \end{array}$$

Each s–t-path $s = v_0, \ldots, v_k = t$ has length ≥ 1 w.r.t. d:

$$\sum_{i=0}^{k-1} d_{i,i+1} \geq \sum_{i=0}^{k-1}(p_i - p_{i+1})$$
$$= p_s - p_t$$



Dual LP – Factional Cuts

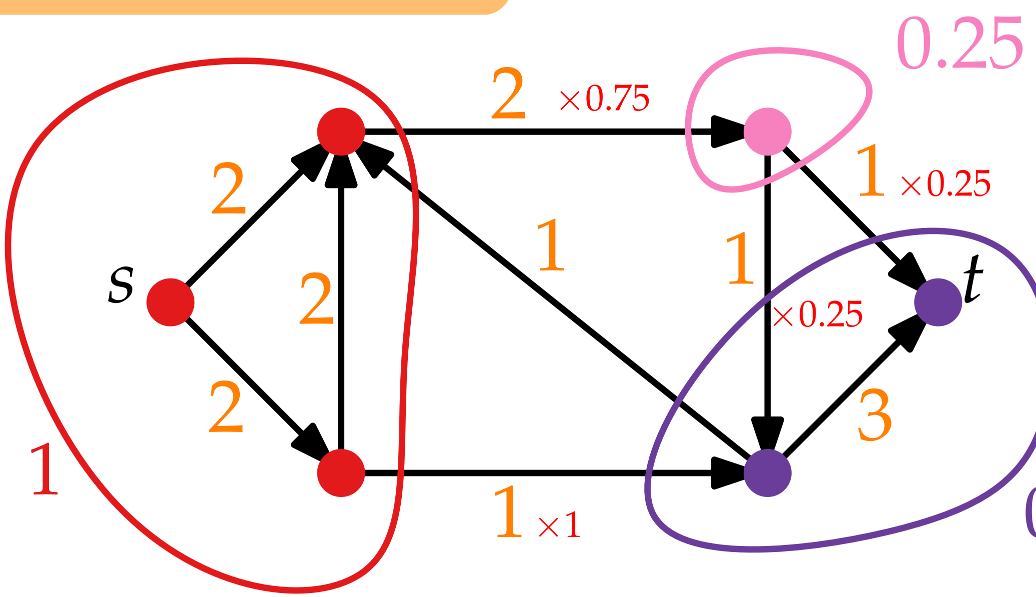
$$\begin{array}{lll} \textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv}\cdot d_{uv} & \\ \textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v)\in E\setminus\{(t,s)\} \\ & p_s - p_t \geq 1 & \\ & d_{uv} \geq 0 & \forall (u,v)\in E \\ & p_u \geq 0 & \forall u\in V \end{array}$$

≡ LP-Relaxation of the ILP

Each extreme-point solution is **integral!** (HW)

Each s–t-path $s = v_0, \dots, v_k = t$ has length ≥ 1 w.r.t. d:

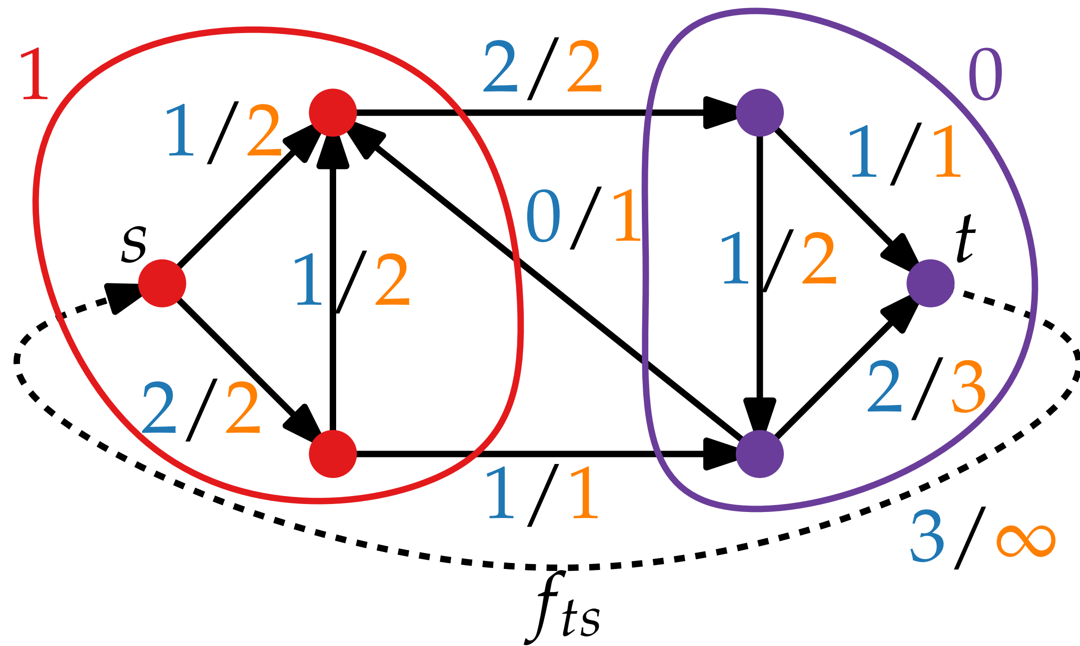
$$\sum_{i=0}^{k-1} d_{i,i+1} \geq \sum_{i=0}^{k-1} (p_i - p_{i+1}) = p_s - p_t$$



Dual LP – Complementary Slackness

$$\begin{array}{lll}
\textbf{maximize} & f_{ts} & \\
\textbf{subject to} & f_{uv} \leq c_{uv} & \forall (u,v) \in E \setminus \{(t,s)\} \\
& \displaystyle\sum_{u\colon (u,v)\in E} f_{uv} - \sum_{z\colon (v,z)\in E} f_{vz} \leq 0 & \forall v \in V \\
& f_{uv} \geq 0 & \forall (u,v) \in E
\end{array}$$

$$\begin{array}{lll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} & \\
\textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v) \in E \setminus \{(t,s)\} \\
& p_s - p_t \geq 1 & \\
& d_{uv} \geq 0 & \forall (u,v) \in E \\
& p_u \geq 0 & \forall u \in V
\end{array}$$

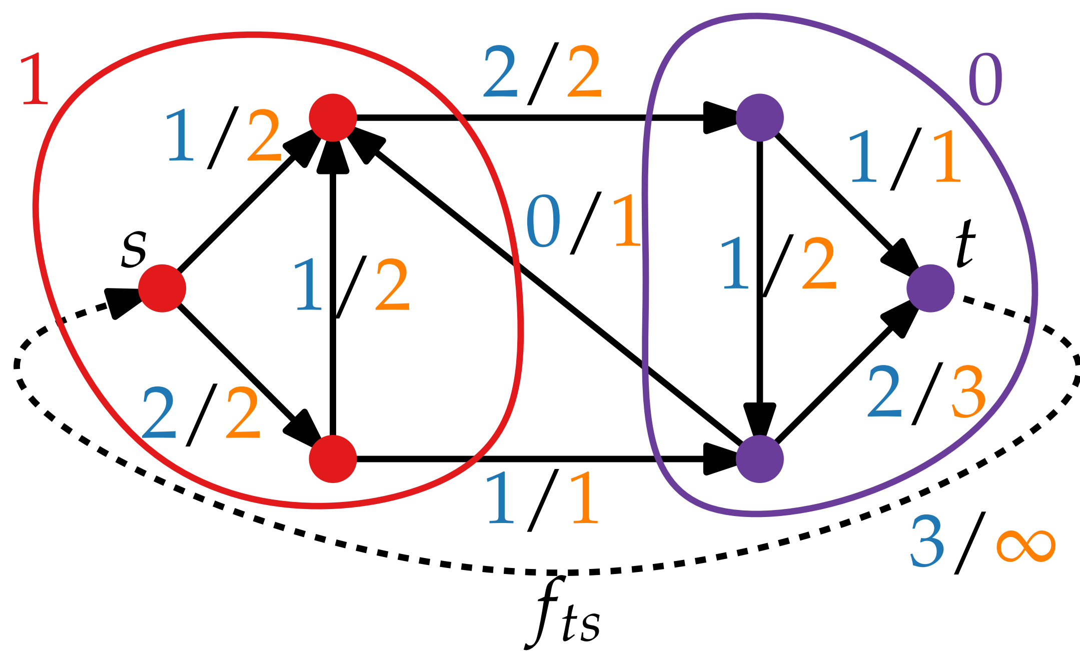


Dual LP – Complementary Slackness

$$
\begin{array}{lll}
\textbf{maximize} & f_{ts} & \\
\textbf{subject to} & f_{uv} \leq c_{uv} & \forall (u,v) \in E \setminus \{(t,s)\} \\
& \displaystyle\sum_{u\colon (u,v)\in E} f_{uv} - \sum_{z\colon (v,z)\in E} f_{vz} \leq 0 & \forall v \in V \\
& f_{uv} \geq 0 & \forall (u,v) \in E
\end{array}
$$

$$
\begin{array}{lll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} & \\
\textbf{subject to} & d_{uv} - p_u + p_v \geq 0 & \forall (u,v) \in E \setminus \{(t,s)\} \\
& p_s - p_t \geq 1 & \\
& d_{uv} \geq 0 & \forall (u,v) \in E \\
& p_u \geq 0 & \forall u \in V
\end{array}
$$

For a max flow and min cut:



Dual LP – Complementary Slackness

maximize f_{ts}

subject to

$$f_{uv} \leq c_{uv} \qquad \forall (u,v) \in E \setminus \{(t,s)\}$$

$$\sum_{u:\,(u,v)\in E} f_{uv} - \sum_{z:\,(v,z)\in E} f_{vz} \leq 0 \qquad \forall v \in V$$

$$f_{uv} \geq 0 \qquad \forall (u,v) \in E$$

minimize $\displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$

subject to

$$d_{uv} - p_u + p_v \geq 0 \qquad \forall (u,v) \in E \setminus \{(t,s)\}$$

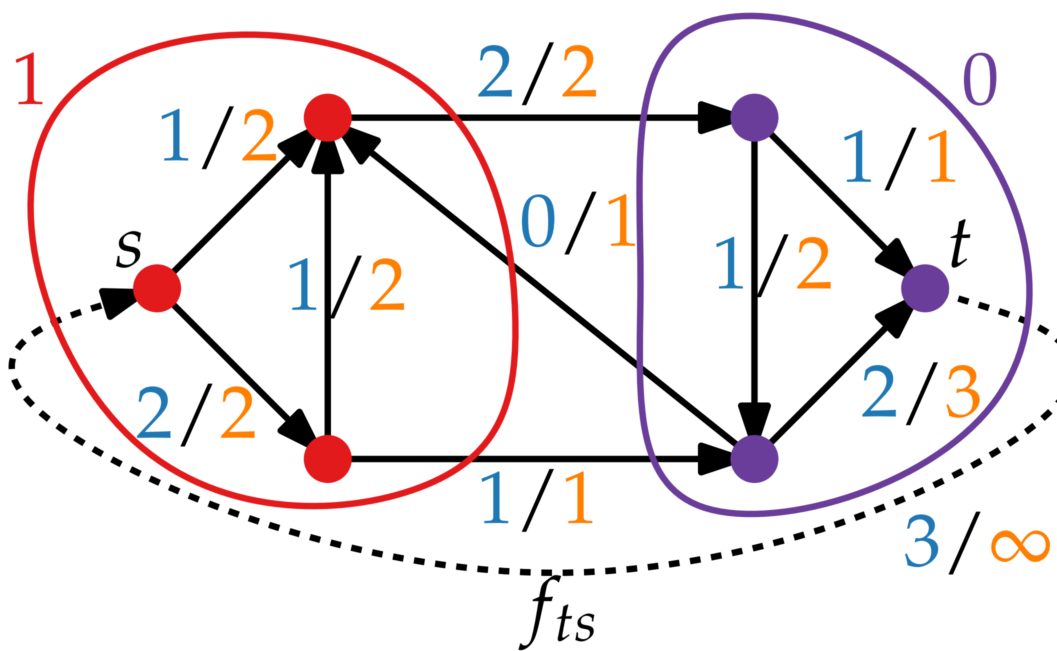
$$p_s - p_t \geq 1$$

$$d_{uv} \geq 0 \qquad \forall (u,v) \in E$$

$$p_u \geq 0 \qquad \forall u \in V$$

For a max flow and min cut:

- For each forward edge (u,v) of the cut: $f_{uv} = c_{uv}$.



Dual LP – Complementary Slackness

maximize f_{ts}

subject to

$$f_{uv} \leq c_{uv} \qquad \forall (u,v) \in E \setminus \{(t,s)\}$$

$$\sum_{u\colon (u,v)\in E} f_{uv} - \sum_{z\colon (v,z)\in E} f_{vz} \leq 0 \qquad \forall v \in V$$

$$f_{uv} \geq 0 \qquad \forall (u,v) \in E$$

minimize $\displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$

subject to

$$d_{uv} - p_u + p_v \geq 0 \qquad \forall (u,v) \in E \setminus \{(t,s)\}$$

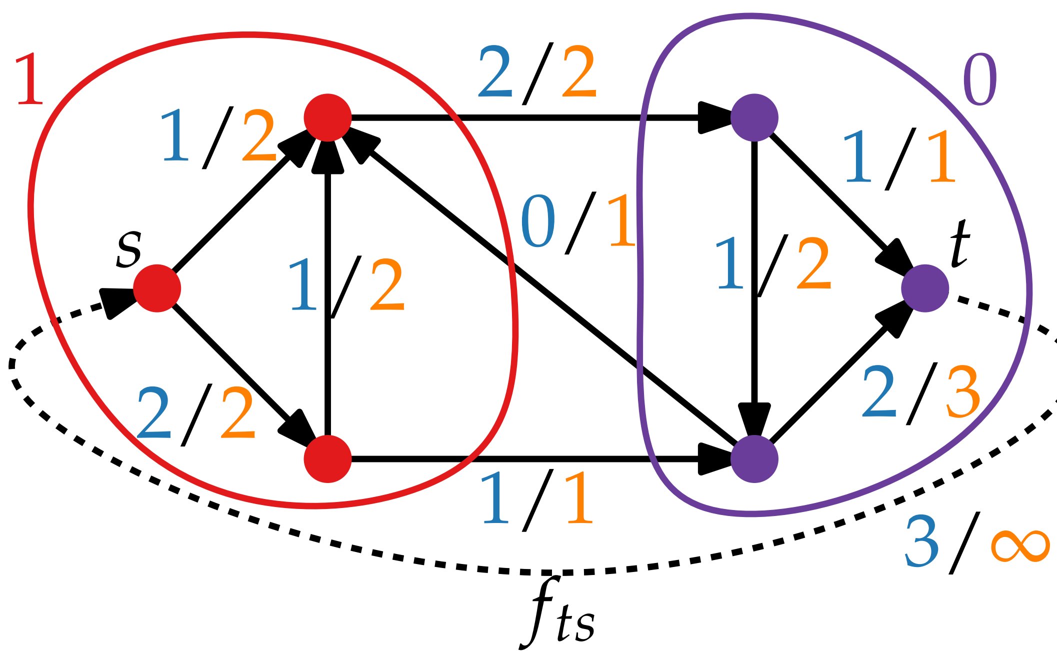
$$p_s - p_t \geq 1$$

$$d_{uv} \geq 0 \qquad \forall (u,v) \in E$$

$$p_u \geq 0 \qquad \forall u \in V$$

For a max flow and min cut:

- For each forward edge (u,v) of the cut: $f_{uv} = c_{uv}$.
  ($d_{uv} = 1$, so by dual CS: $f_{uv} = c_{uv}$.)



Dual LP – Complementary Slackness

maximize f_{ts}

subject to
$$f_{uv} \le c_{uv} \qquad \forall (u,v) \in E \setminus \{(t,s)\}$$
$$\sum_{u\colon (u,v)\in E} f_{uv} - \sum_{z\colon (v,z)\in E} f_{vz} \le 0 \qquad \forall v \in V$$
$$f_{uv} \ge 0 \qquad \forall (u,v) \in E$$

minimize $\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$

subject to
$$d_{uv} - p_u + p_v \ge 0$$
$$p_s - p_t \ge 1$$
$$d_{uv} \ge 0$$
$$p_u \ge 0$$

Primal CS:
$\forall j$: Either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
$\forall i$: Either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

For a max flow and min cut:

- For each forward edge (u,v) of the cut: $f_{uv} = c_{uv}$.
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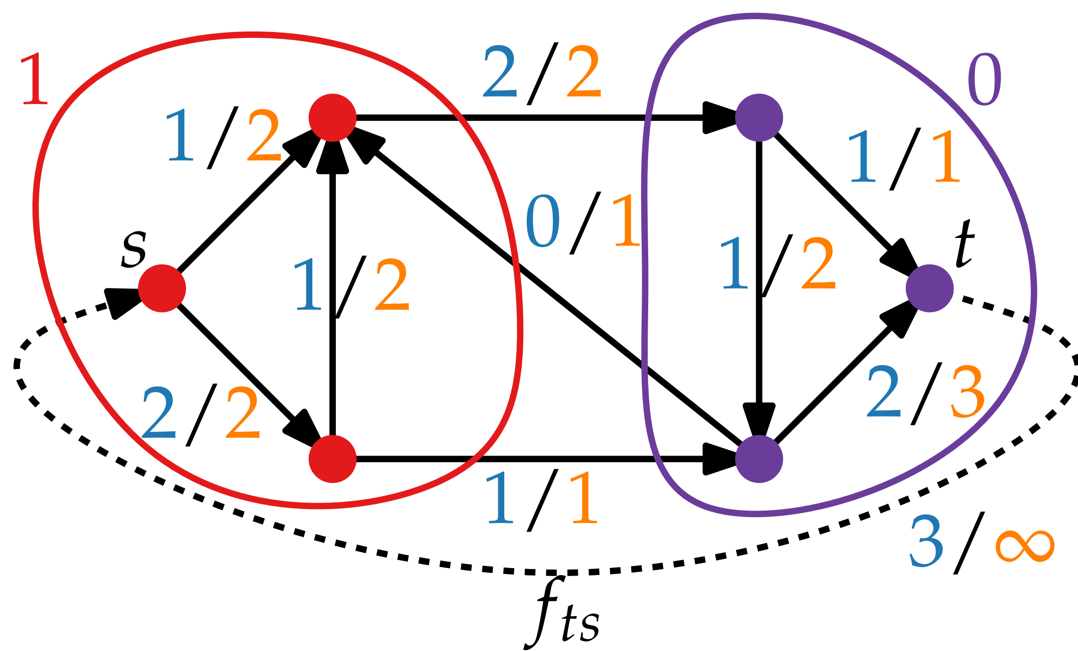


Dual LP – Complementary Slackness

maximize f_{ts}

subject to

$$f_{uv} \leq c_{uv} \qquad \forall (u,v) \in E \setminus \{(t,s)\}$$

$$\sum_{u:\,(u,v)\in E} f_{uv} - \sum_{z:\,(v,z)\in E} f_{vz} \leq 0 \qquad \forall v \in V$$

$$f_{uv} \geq 0 \qquad \forall (u,v) \in E$$

minimize $\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv}$

subject to

$$d_{uv} - p_u + p_v \geq 0$$

$$p_s - p_t \geq 1$$

$$d_{uv} \geq 0$$

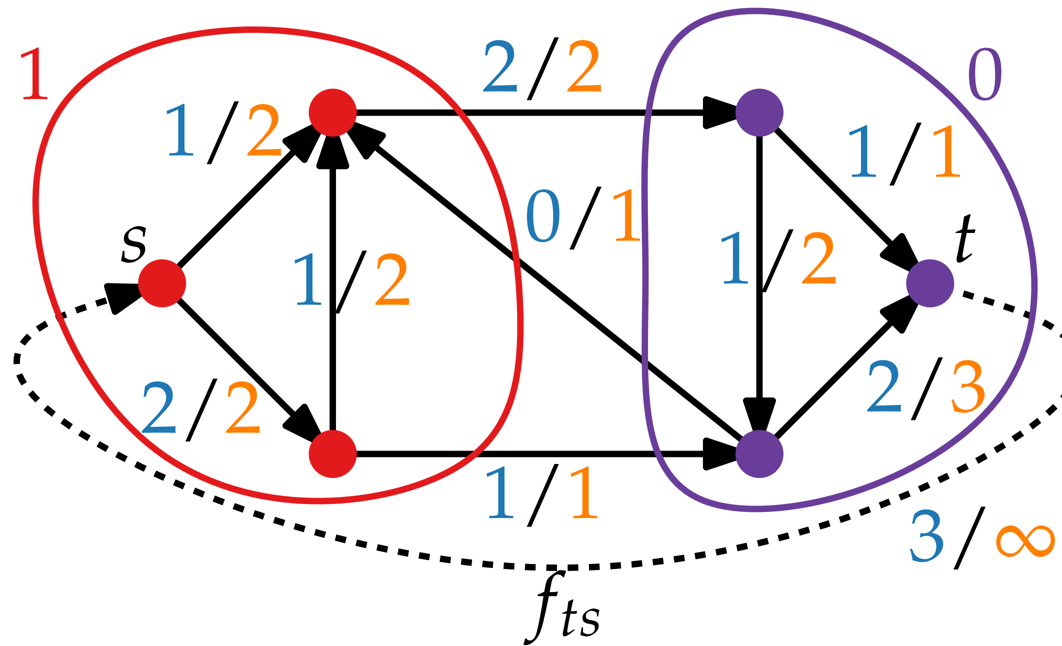
$$p_u \geq 0$$

Primal CS:
$\forall j$: Either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
$\forall i$: Either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

For a max flow and min cut:

- For each forward edge (u,v) of the cut: $f_{uv} = c_{uv}$.
  ($d_{uv} = 1$, so by dual CS: $f_{uv} = c_{uv}$.)
- For each backward edge (u,v) of the cut: $f_{uv} = 0$.



Dual LP – Complementary Slackness

$$
\begin{array}{lll}
\textbf{maximize} & f_{ts} & \\
\textbf{subject to} & f_{uv} \leq c_{uv} & \forall (u,v) \in E \setminus \{(t,s)\} \\
 & \displaystyle\sum_{u\colon (u,v)\in E} f_{uv} - \sum_{z\colon (v,z)\in E} f_{vz} \leq 0 & \forall v \in V \\
 & f_{uv} \geq 0 & \forall (u,v) \in E
\end{array}
$$

$$
\begin{array}{ll}
\textbf{minimize} & \displaystyle\sum_{(u,v)\in E\setminus\{(t,s)\}} c_{uv} \cdot d_{uv} \\
\textbf{subject to} & d_{uv} - p_u + p_v \geq 0 \\
 & p_s - p_t \geq 1 \\
 & d_{uv} \geq 0 \\
 & p_u \geq 0
\end{array}
$$

Primal CS:
$\forall j$: Either $x_j = 0$ or $\sum_{i=1}^{m} a_{ij} y_i = c_j$

Dual CS:
$\forall i$: Either $y_i = 0$ or $\sum_{j=1}^{n} a_{ij} x_j = b_i$

For a max flow and min cut:

- For each forward edge (u,v) of the cut: $f_{uv} = c_{uv}$.
  ($d_{uv} = 1$, so by dual CS: $f_{uv} = c_{uv}$.)
- For each backward edge (u,v) of the cut: $f_{uv} = 0$.
  (Otherwise, by primal CS: $d_{uv} - 0 + 1 = 0$.)