Approximation Algorithms

Lecture 3: SteinerTree and MultiwayCut

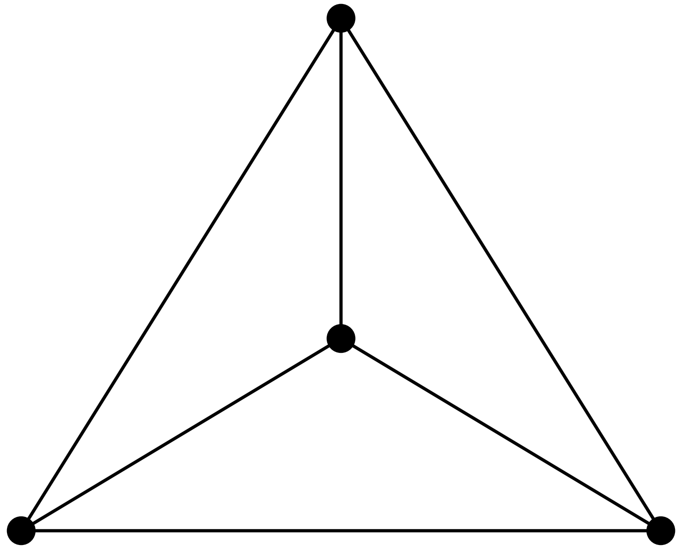
Part I: SteinerTree

Joachim Spoerhase

Winter 2021/22

SteinerTree

Given: A graph $G = (V, E)$



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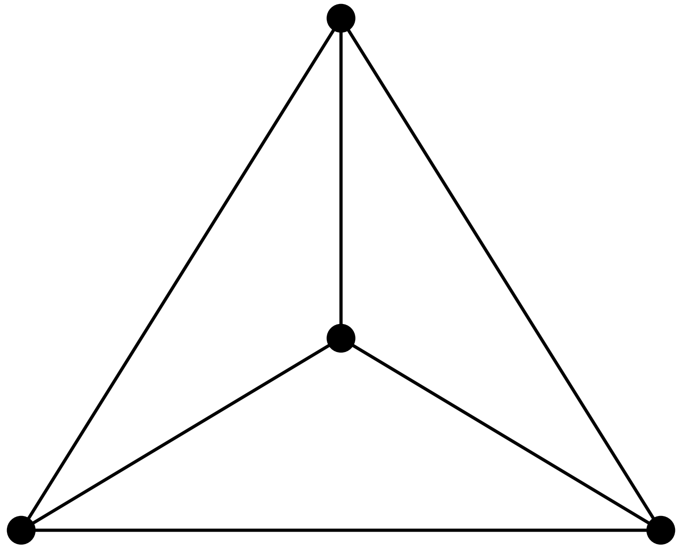

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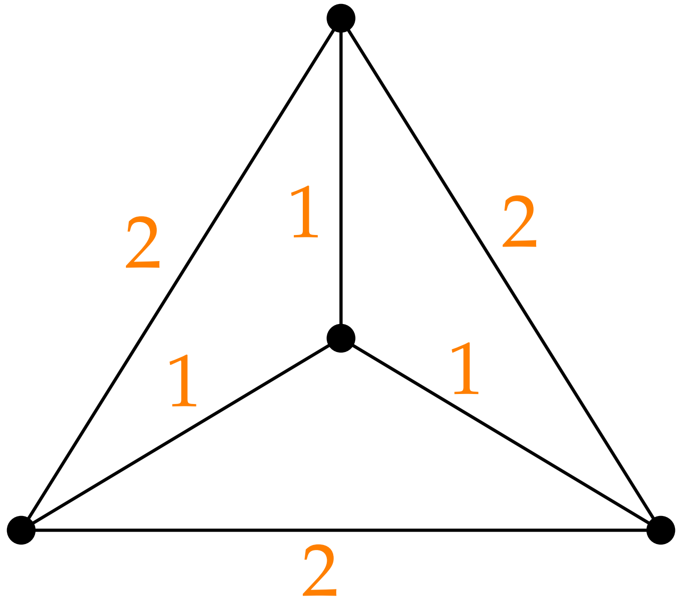

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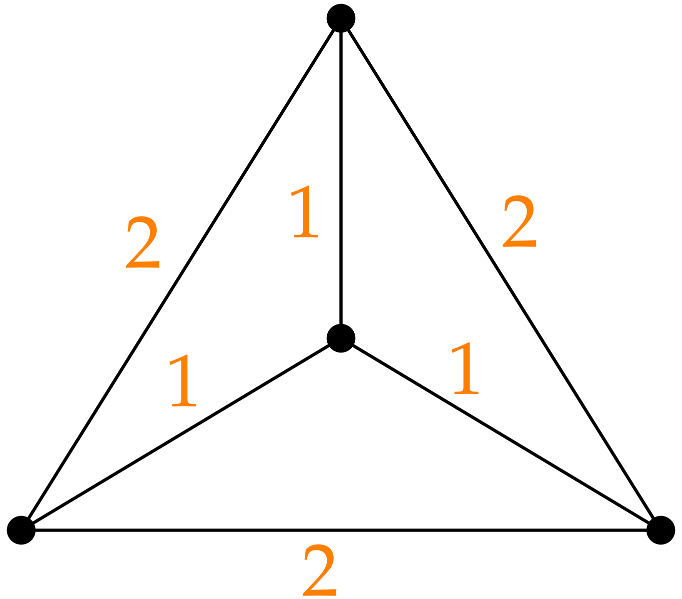

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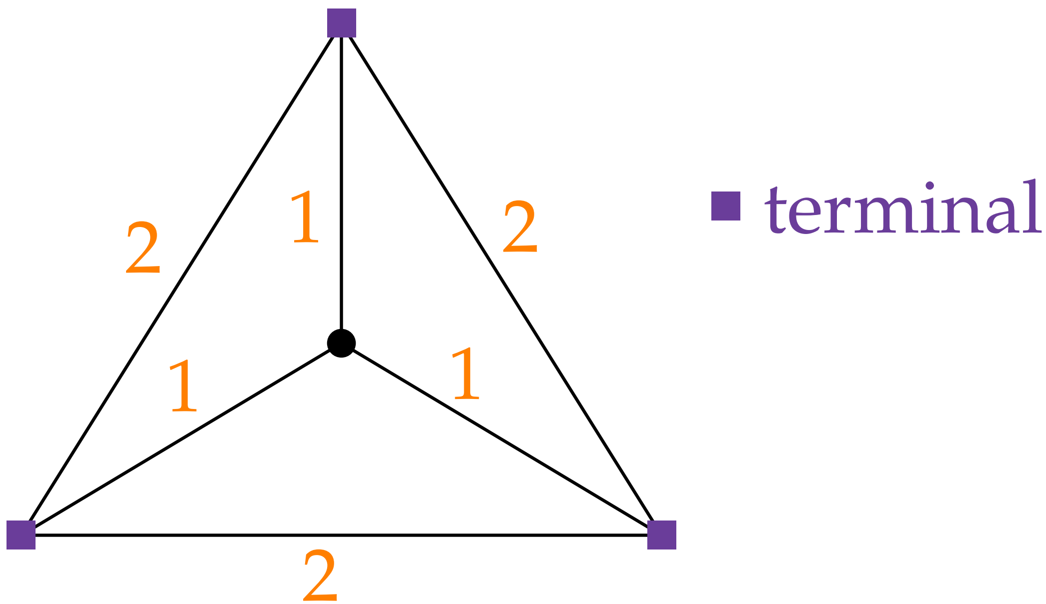

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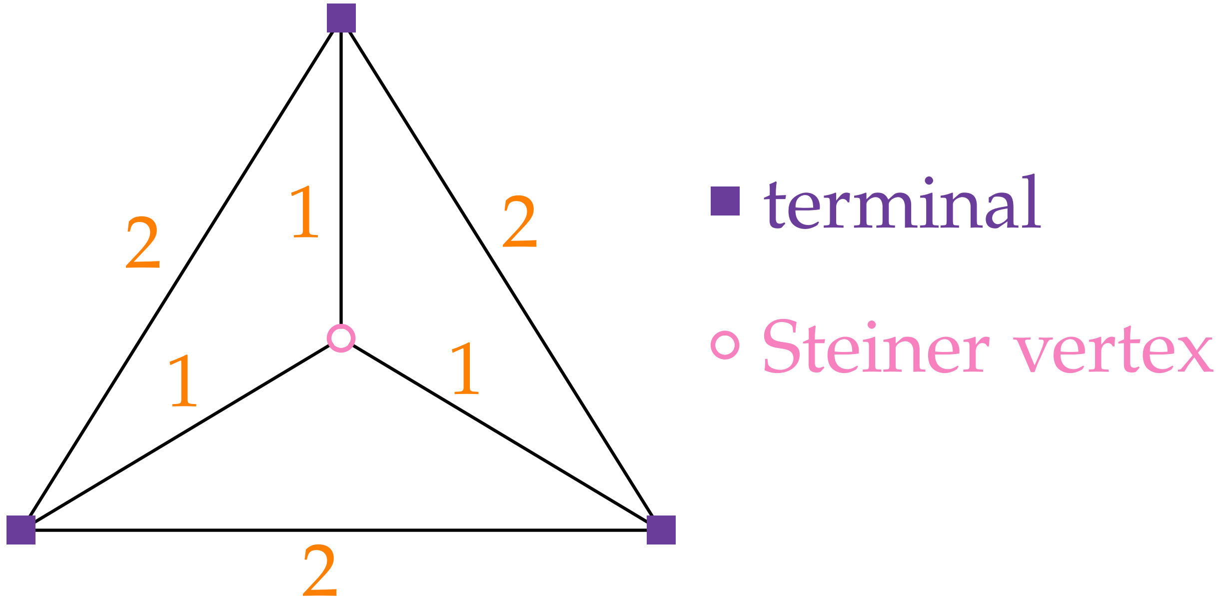

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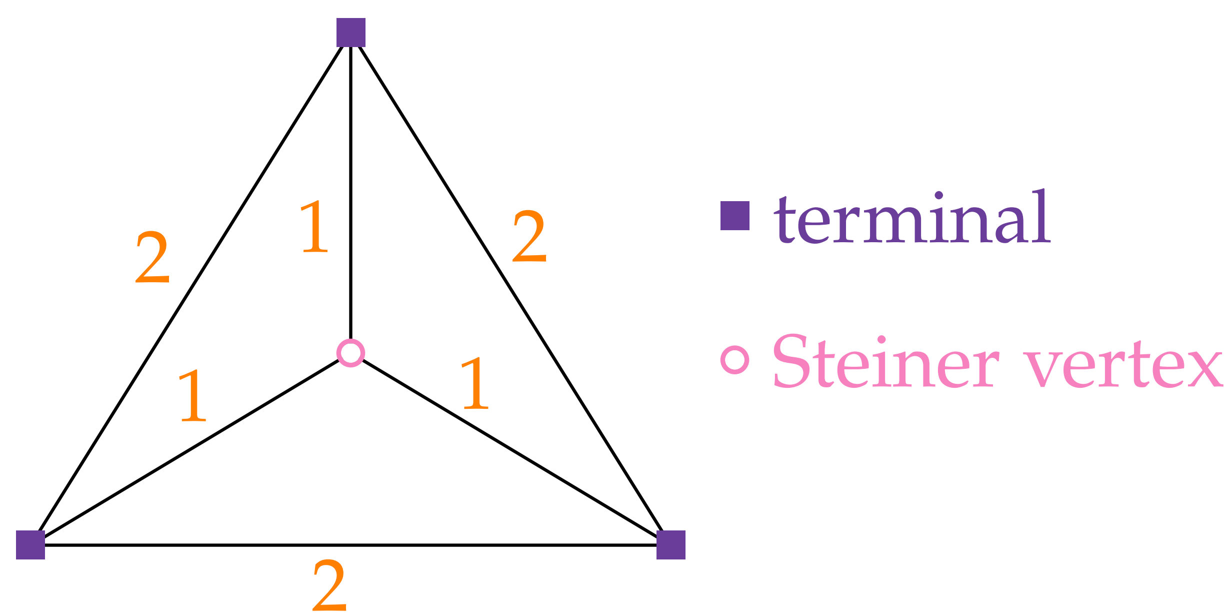

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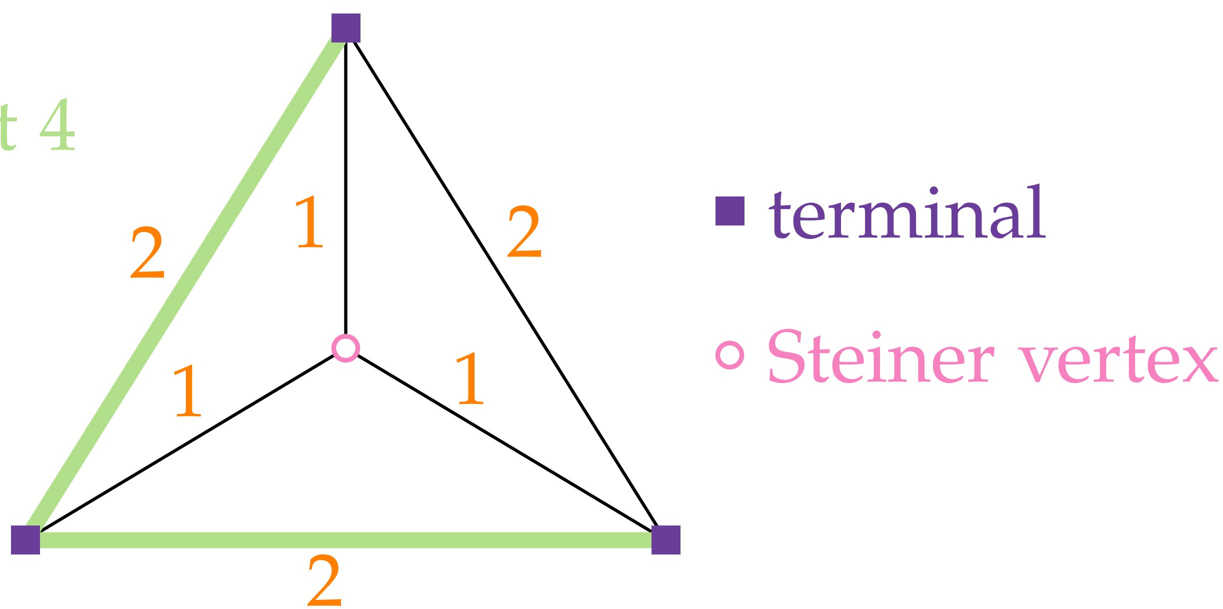

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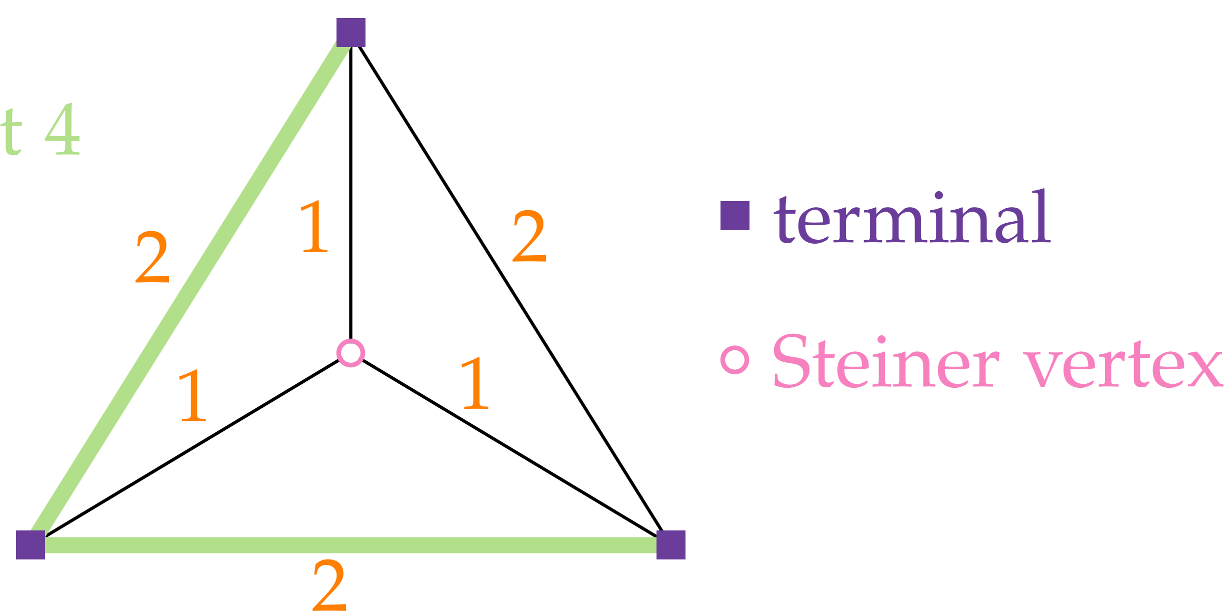

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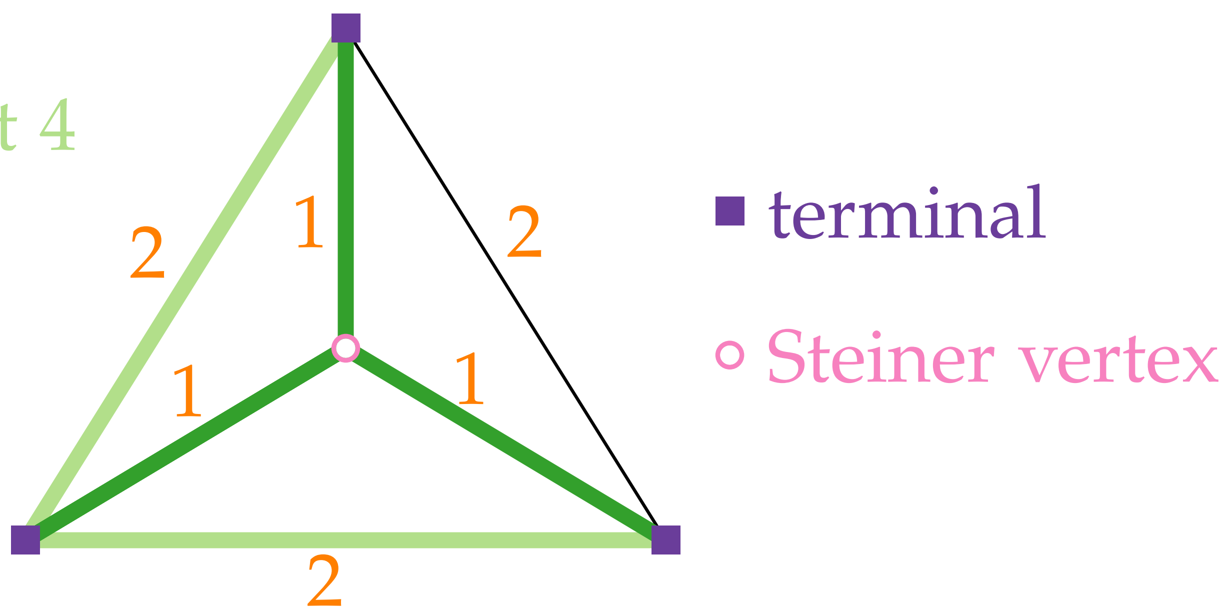

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Restriction of SteinerTree where the graph G is complete and the cost function is **metric**

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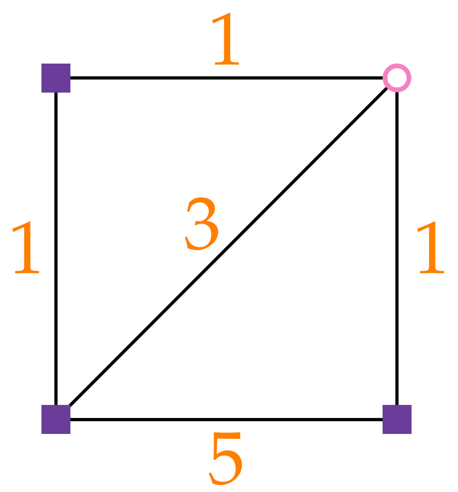

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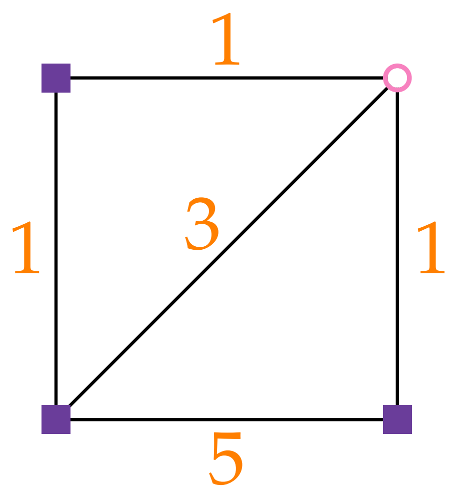

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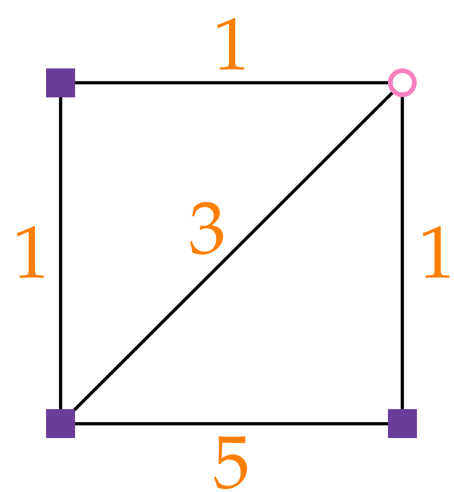

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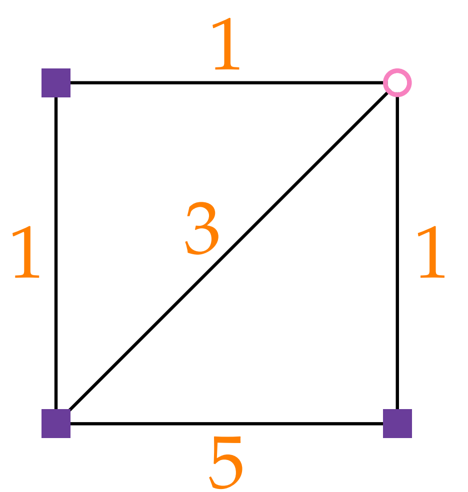

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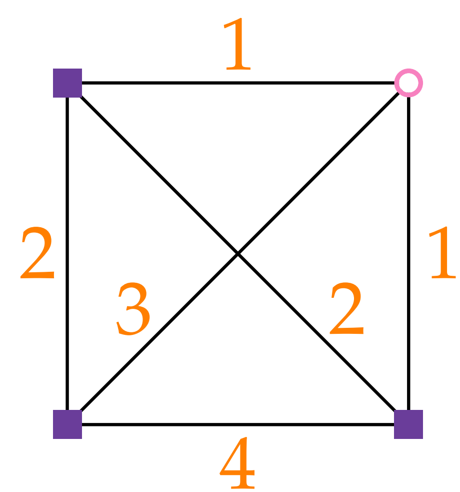

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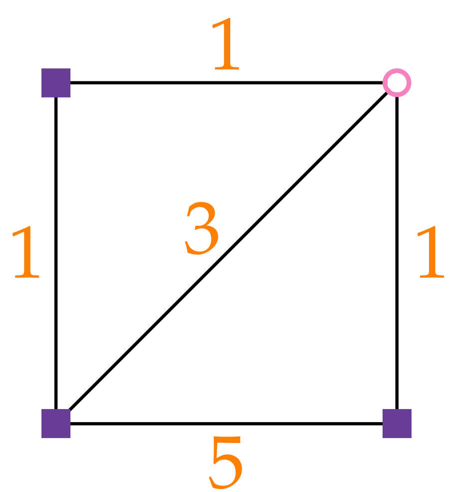

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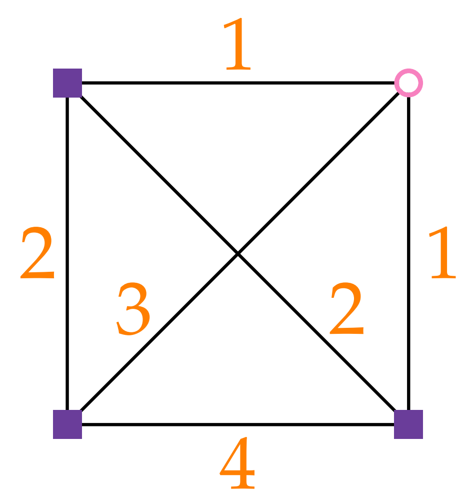

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Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part II:
Approximation Preserving Reduction

Joachim Spoerhase

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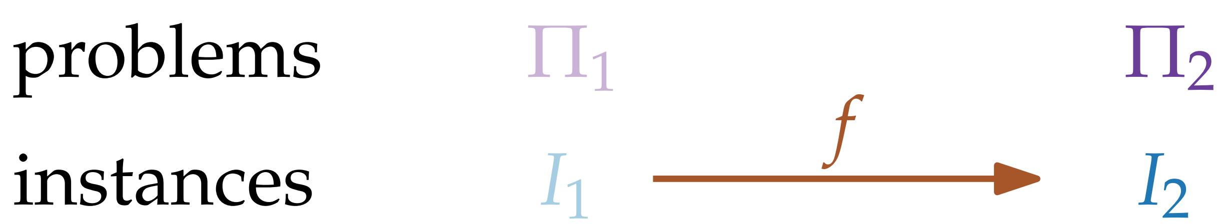

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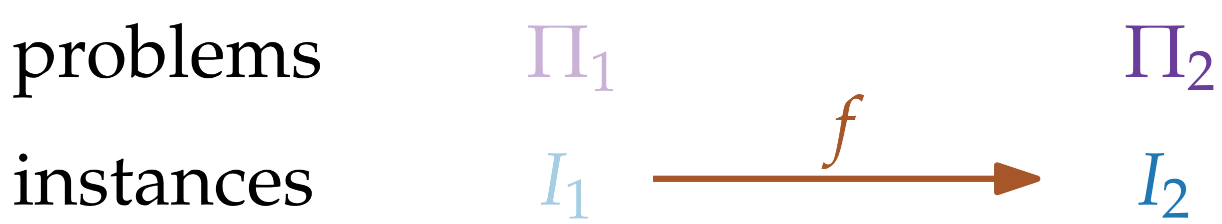

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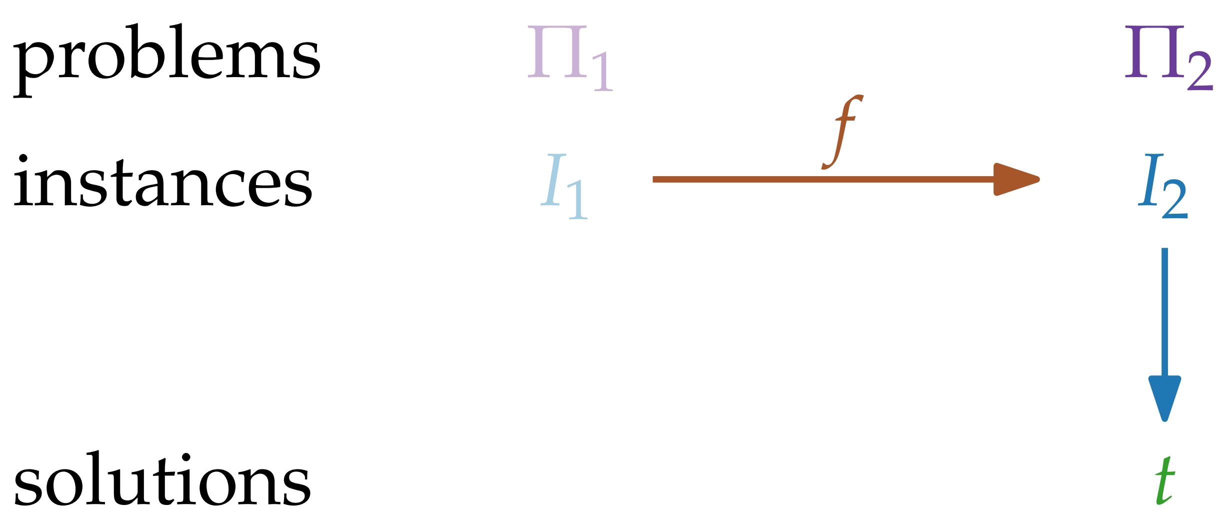

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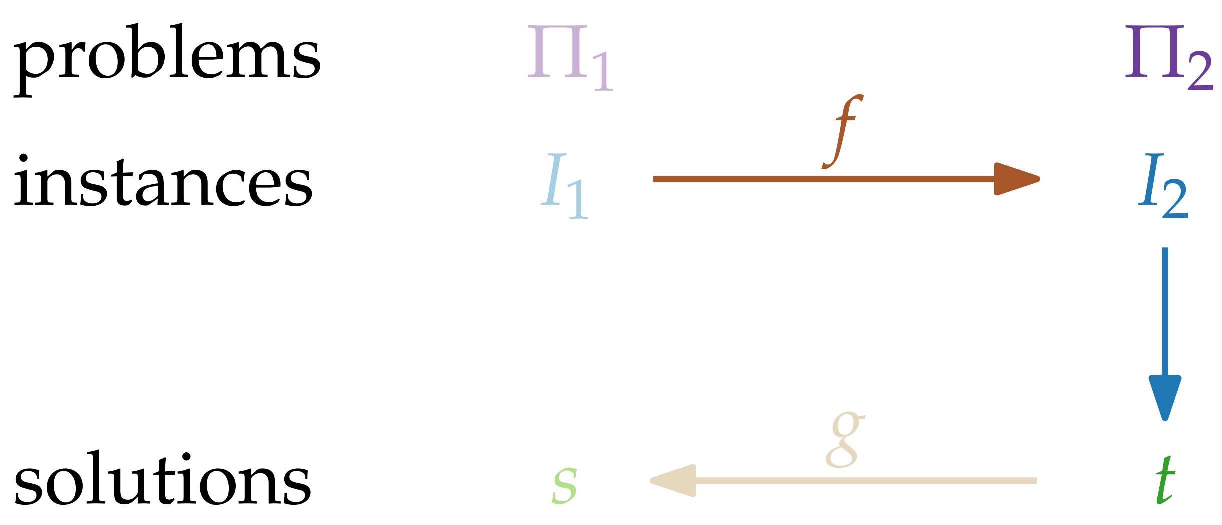

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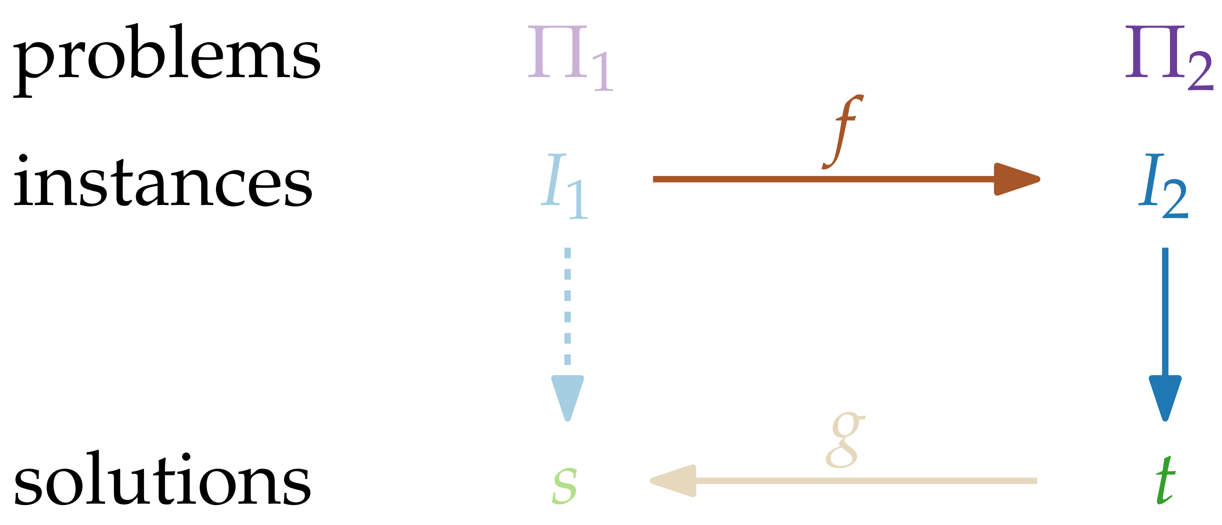

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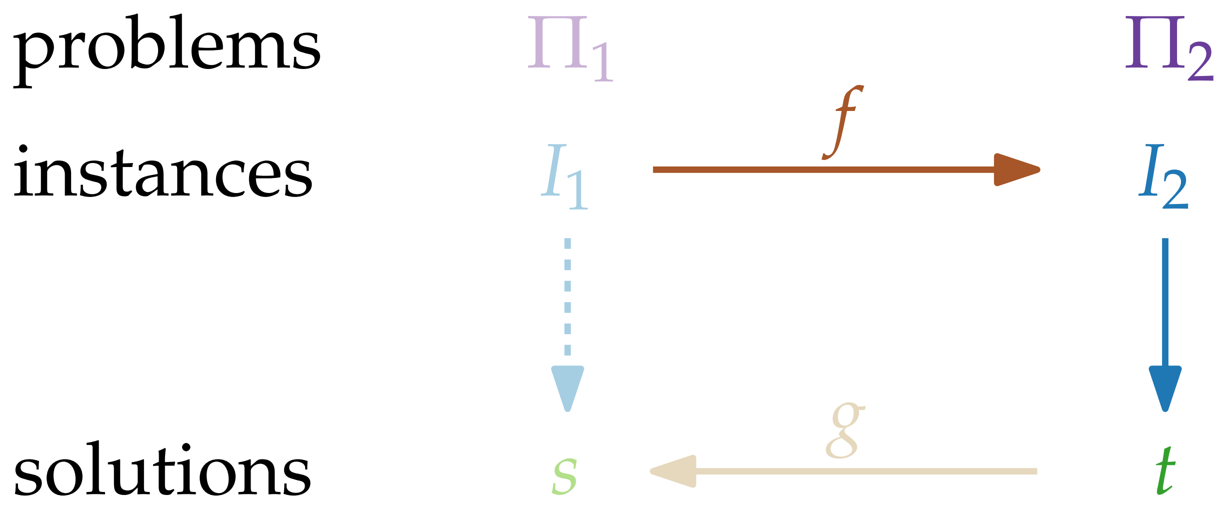

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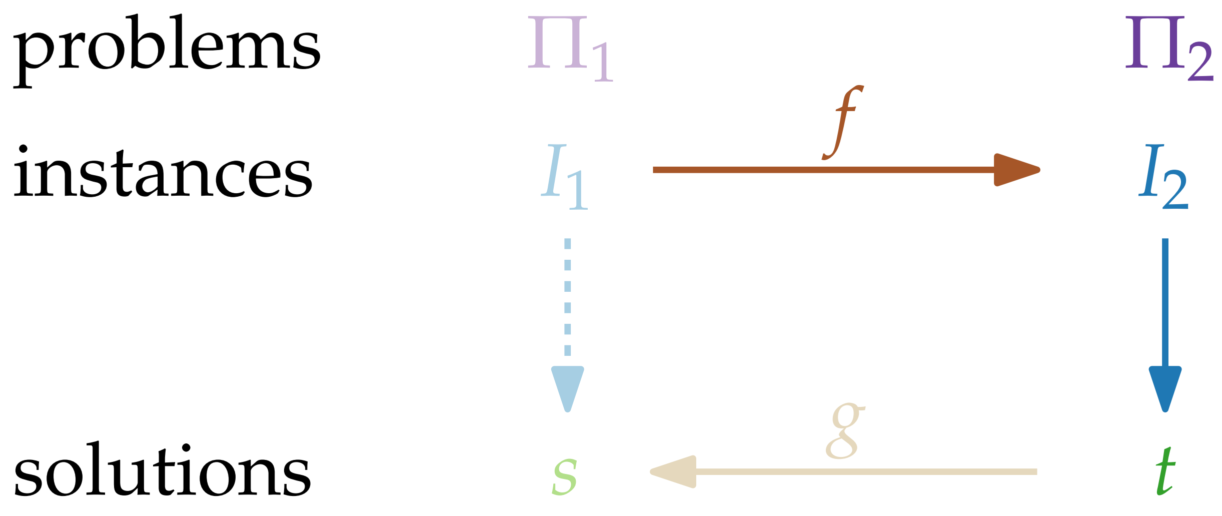

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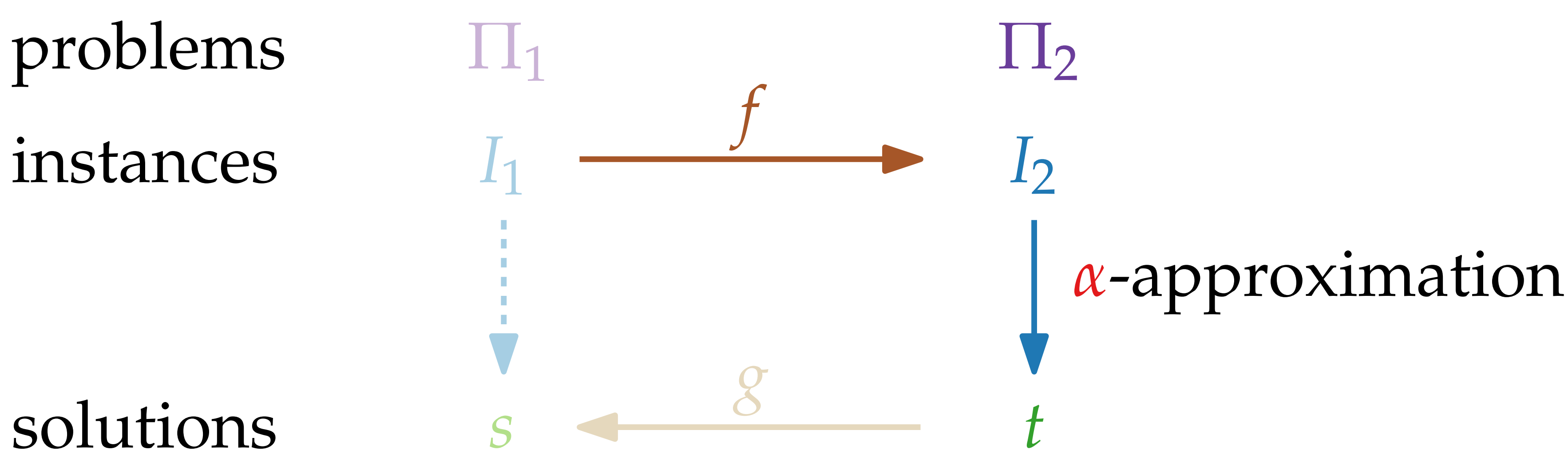

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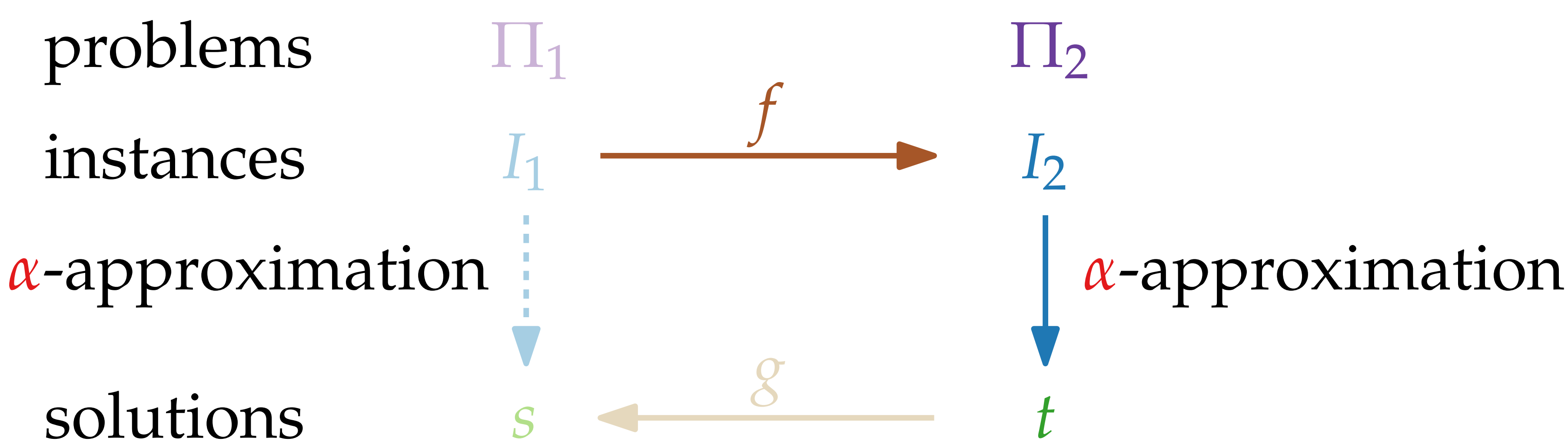

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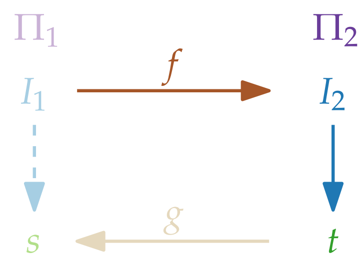

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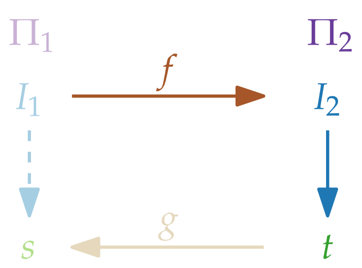

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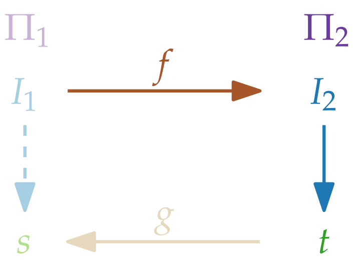

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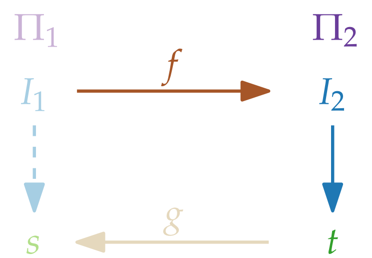

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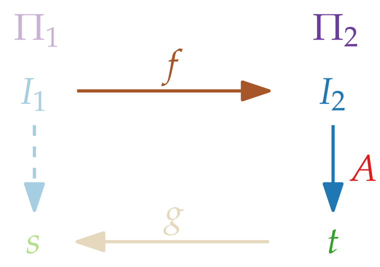

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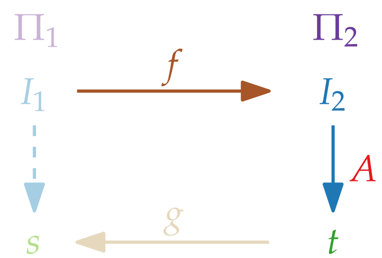

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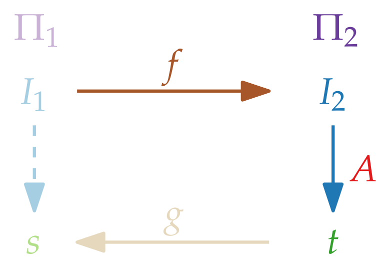

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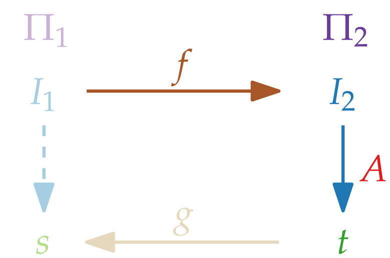

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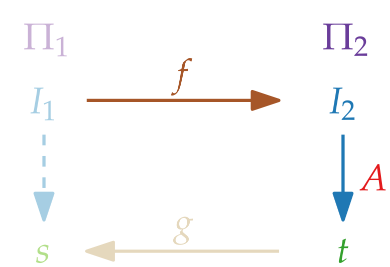

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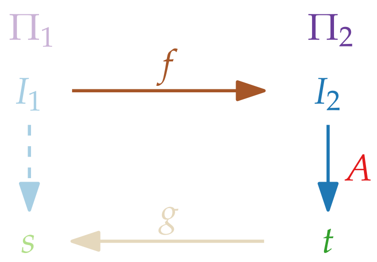

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Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part III:
Reduction to MetricSteinerTree

Joachim Spoerhase

Winter 2021/22

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Theorem. There is an approximation preserving reduction from SteinerTree to MetricSteinerTree.

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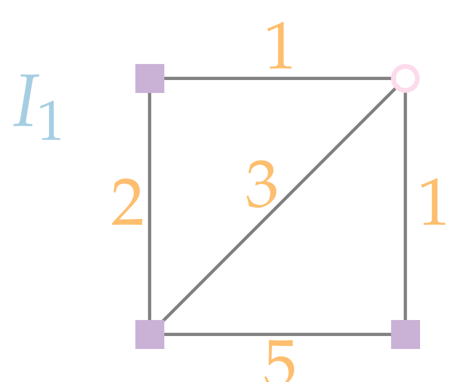

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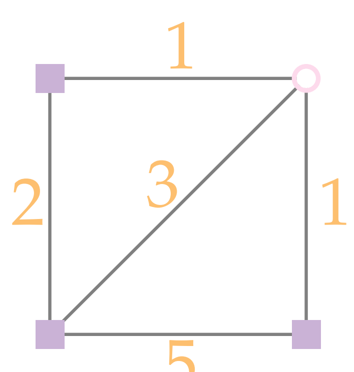

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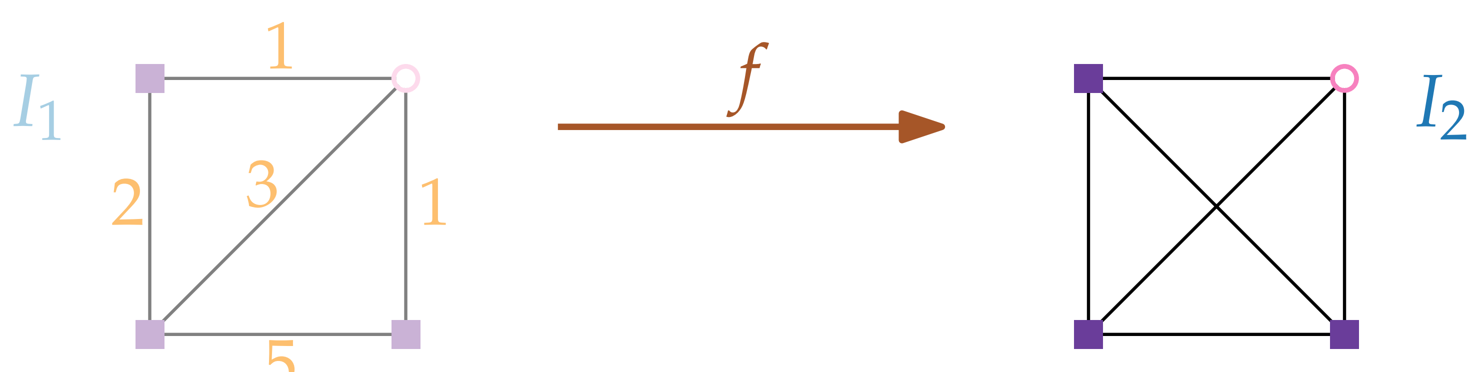

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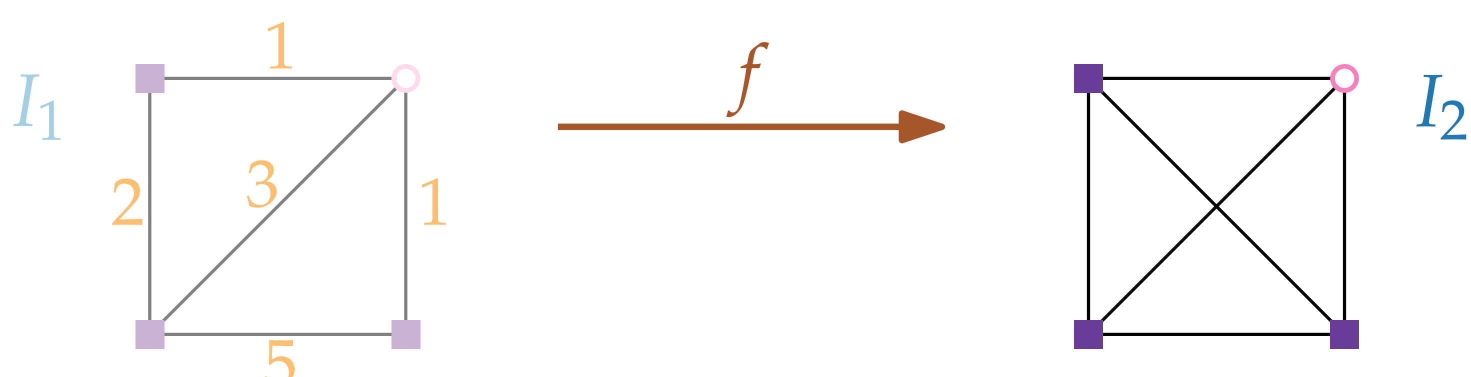

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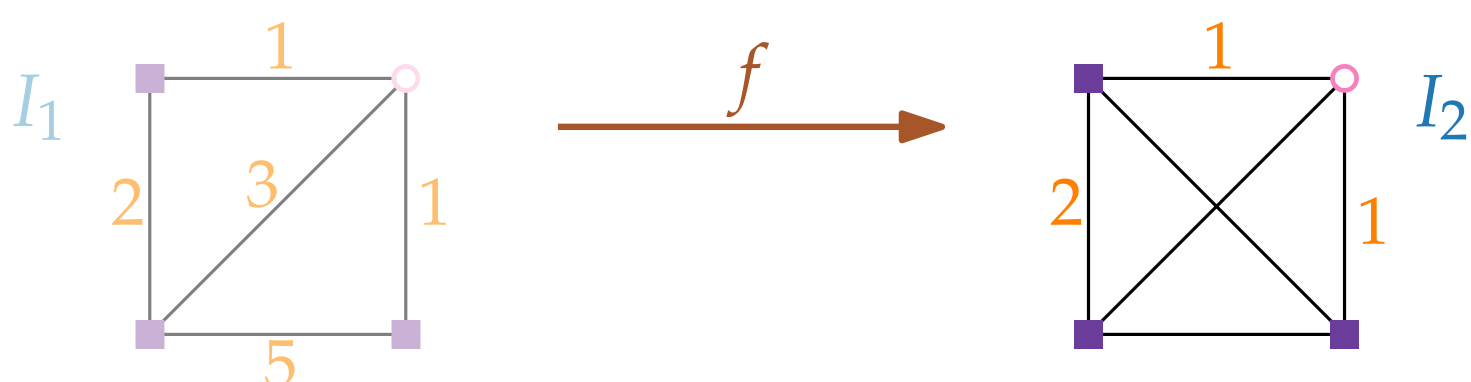

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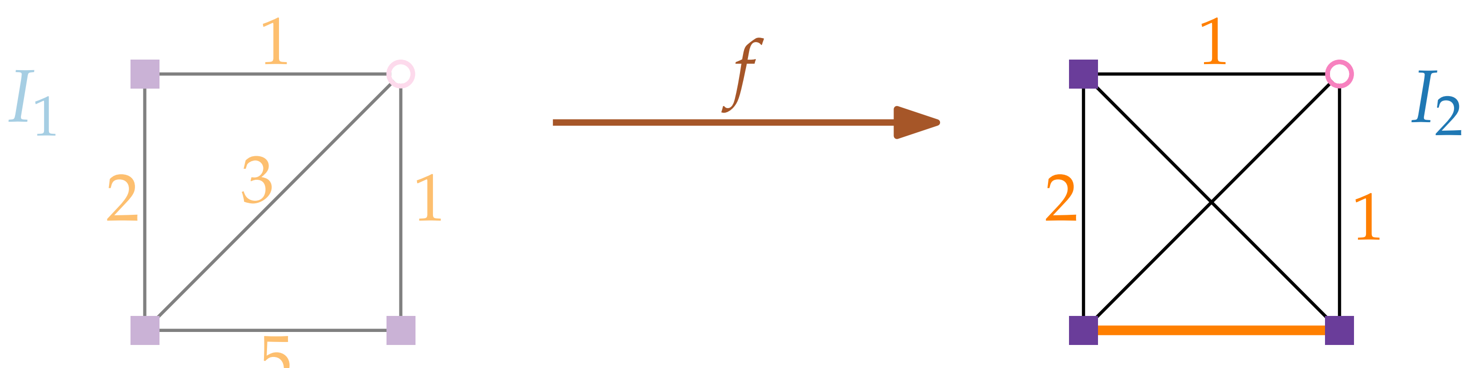

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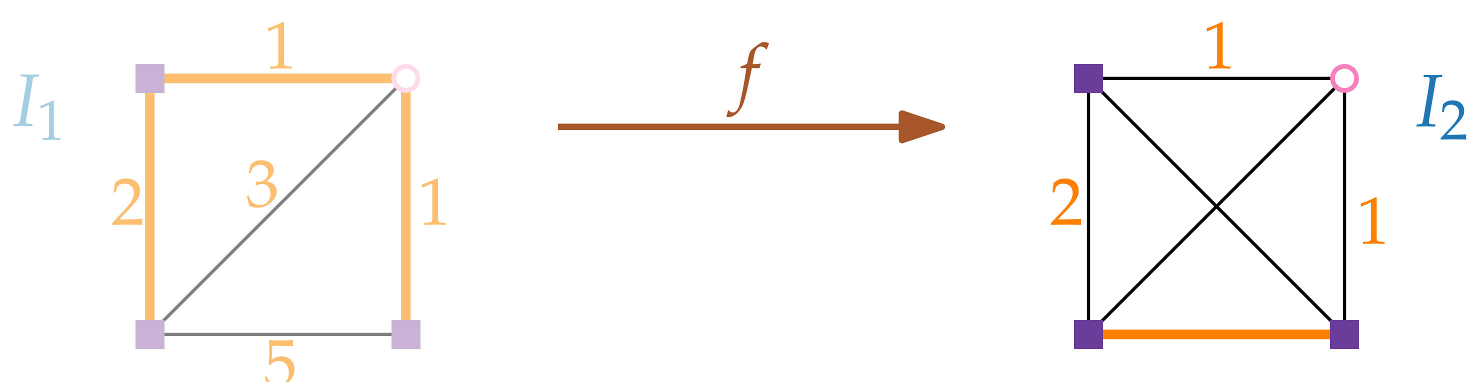

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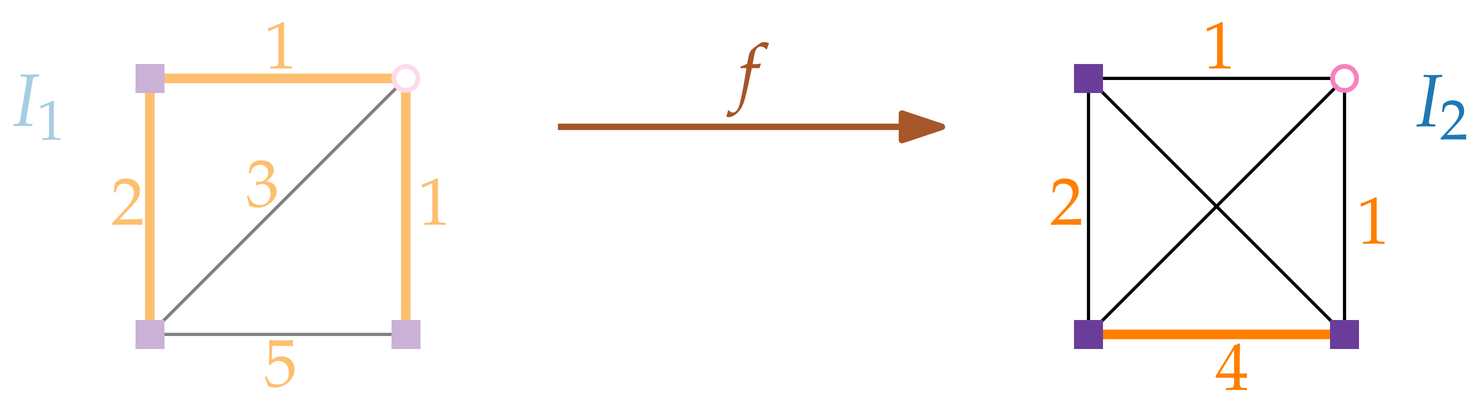

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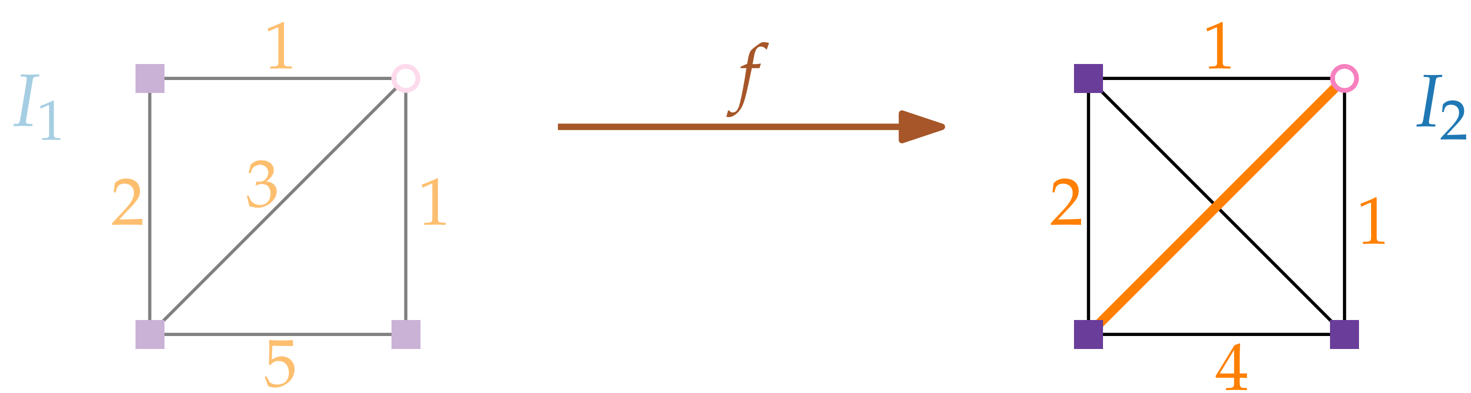

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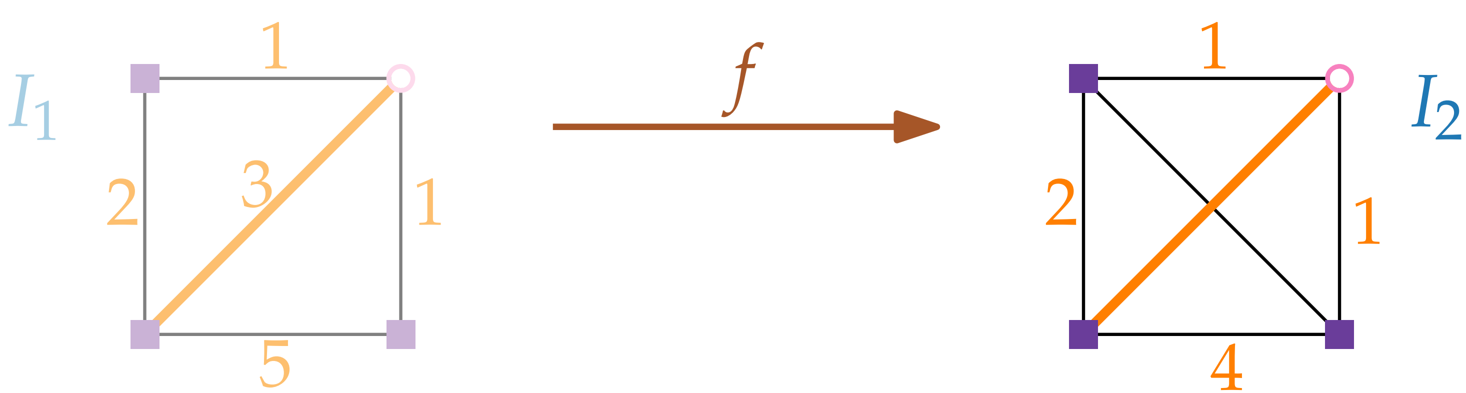

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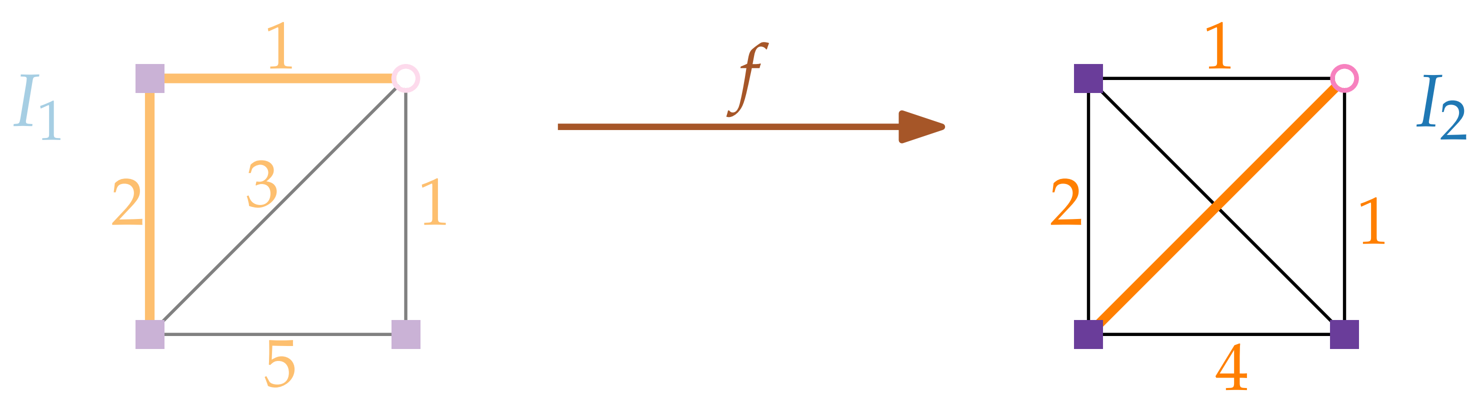

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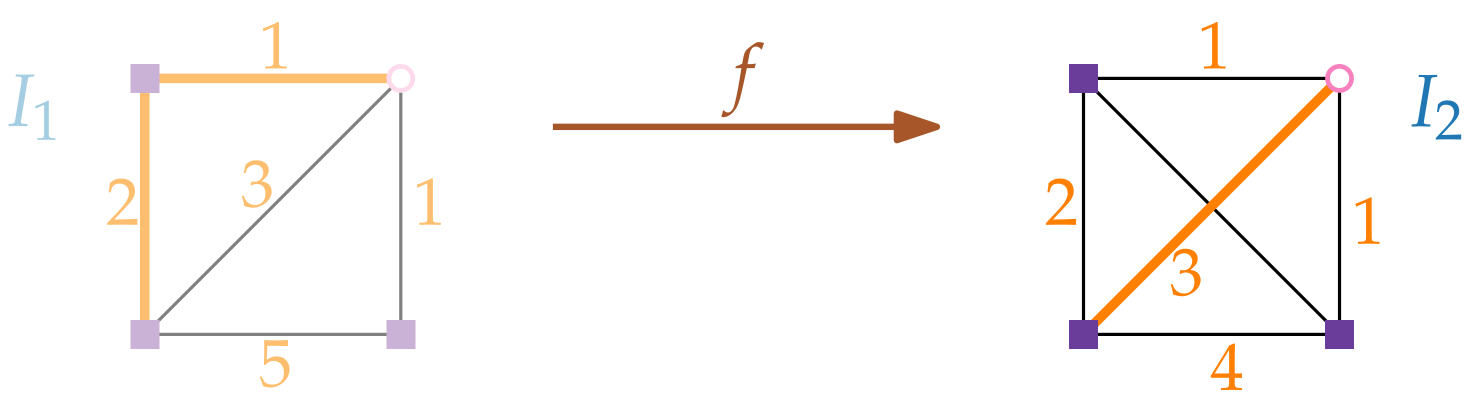

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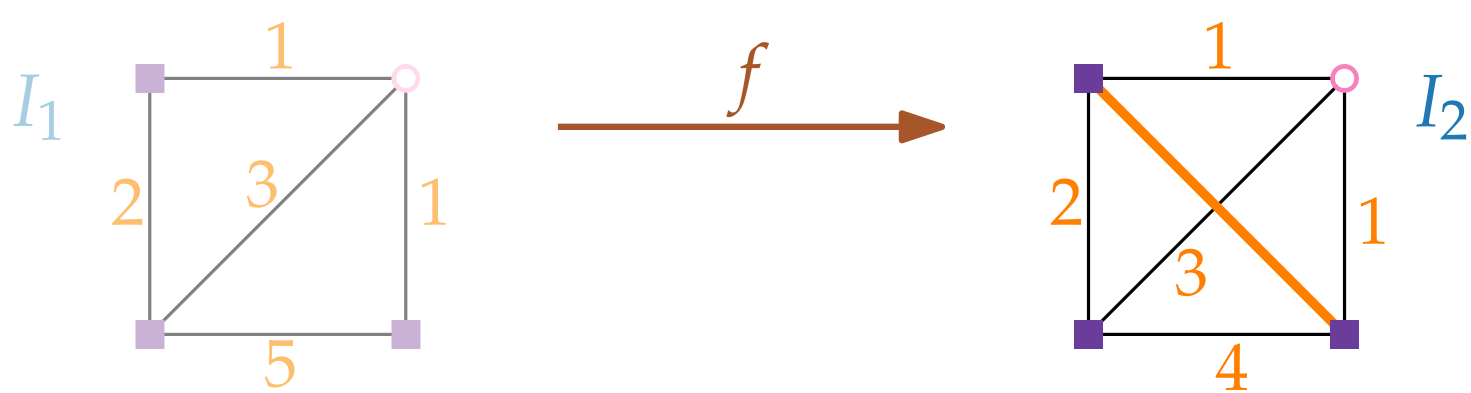

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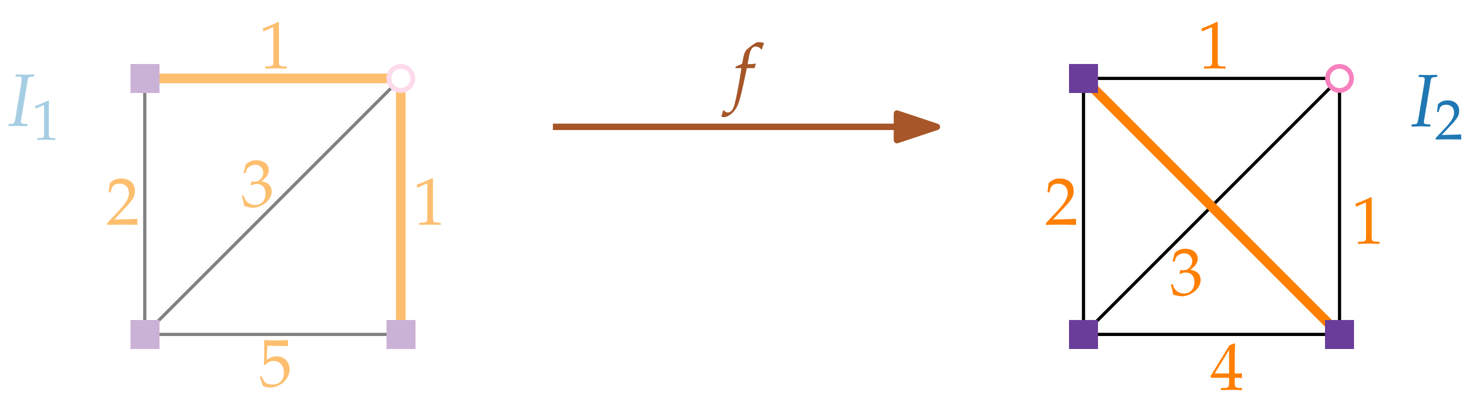

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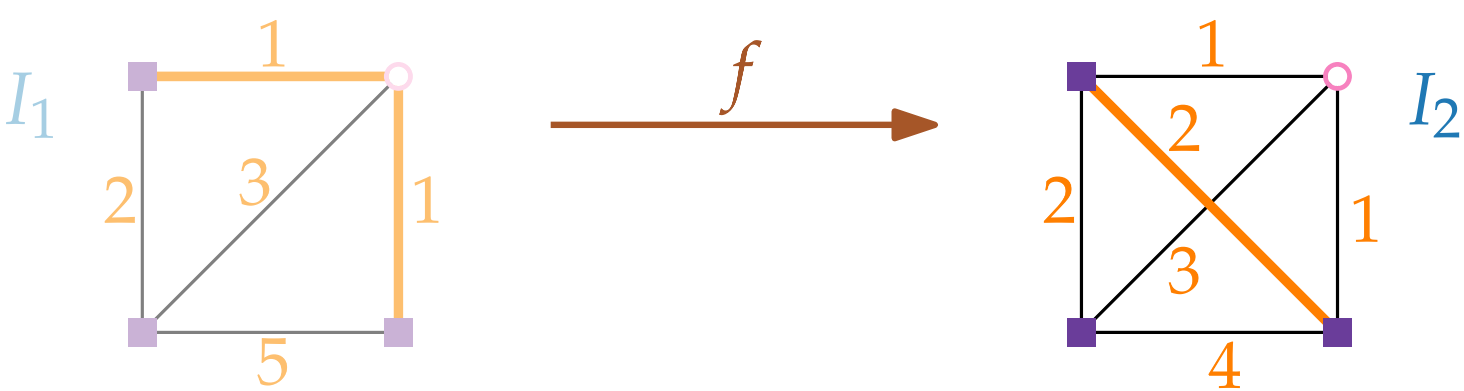

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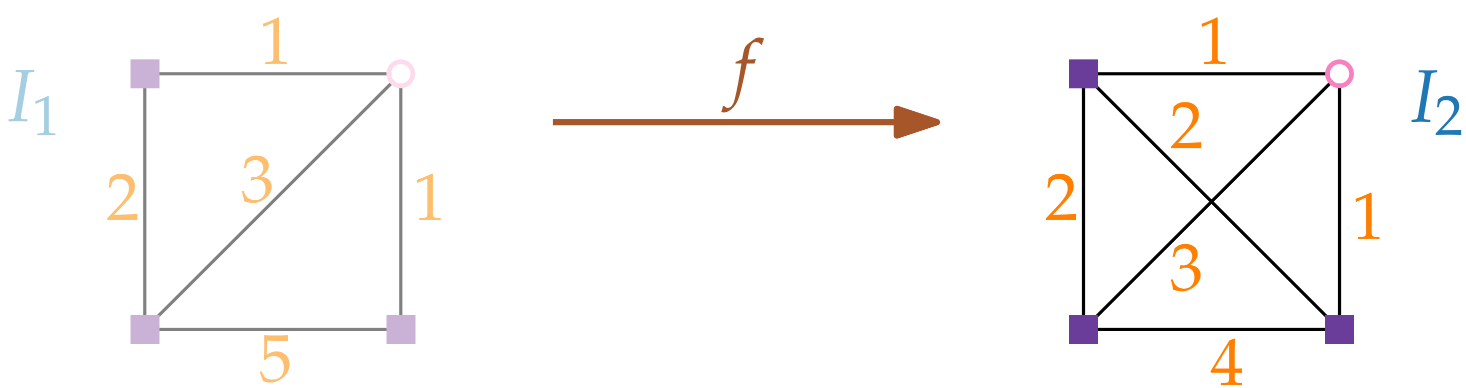

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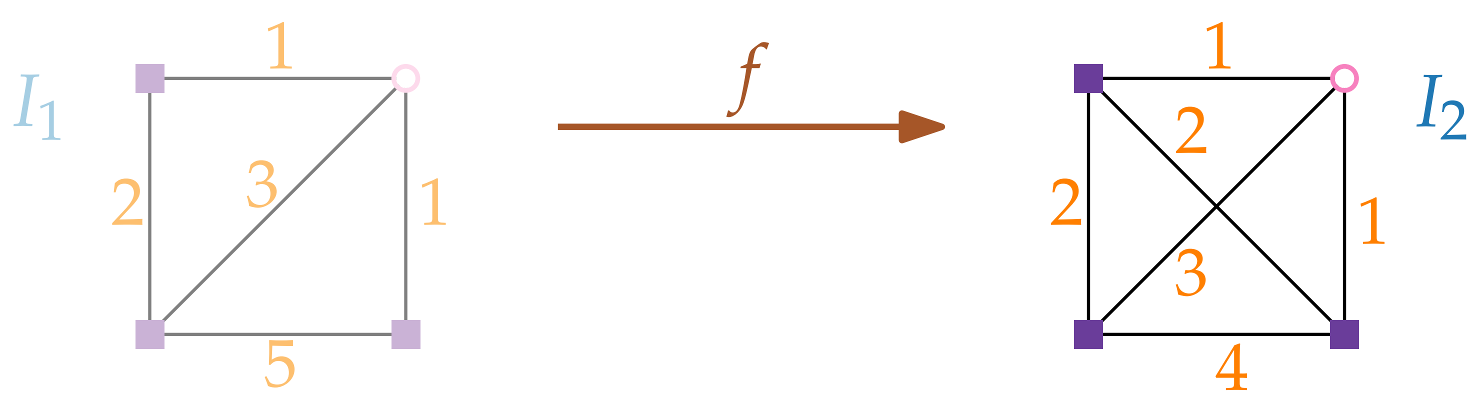
$c_2(u, v) \leq c_1(u, v)$ for all $(u, v) \in E$



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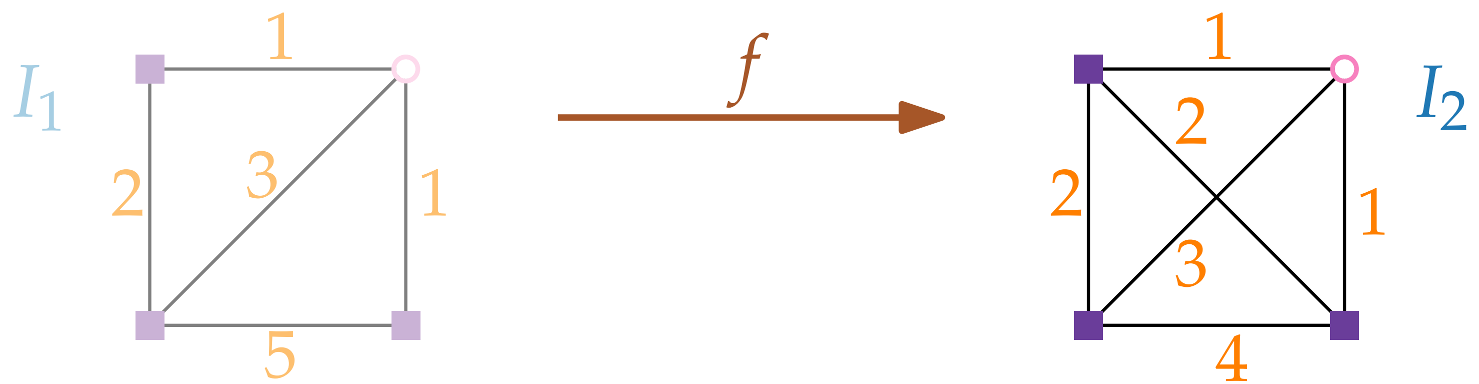

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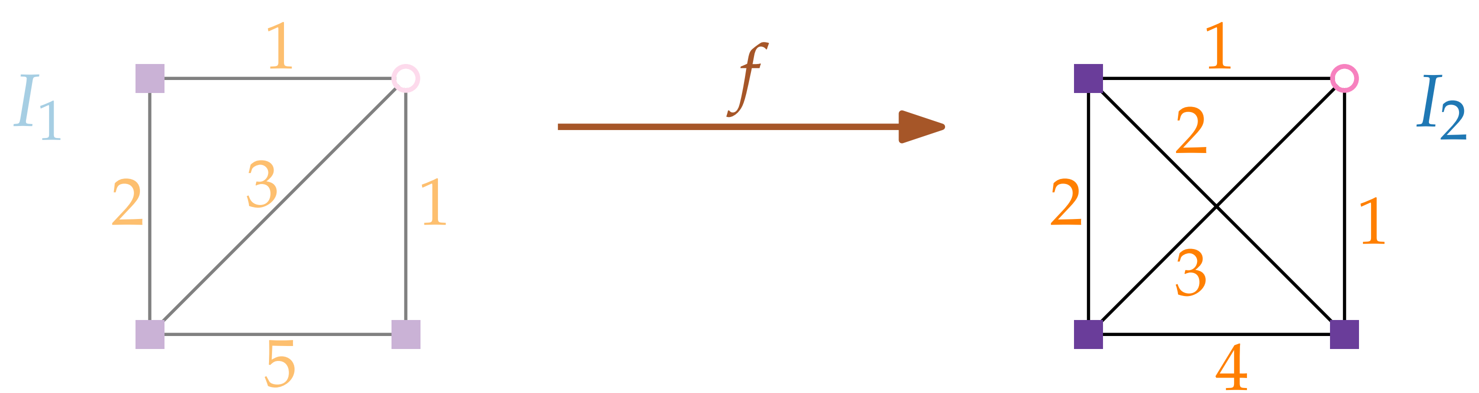

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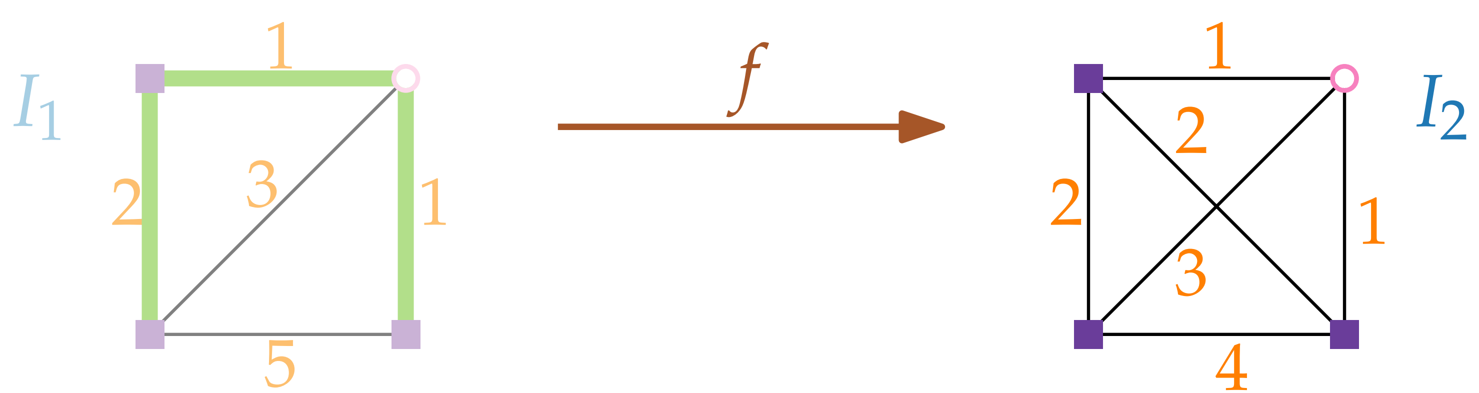

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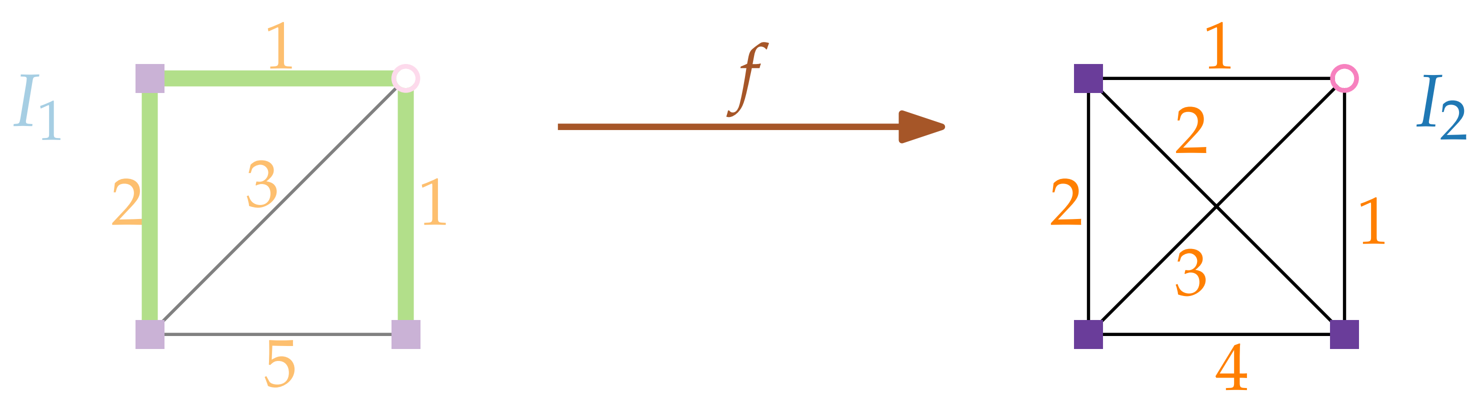

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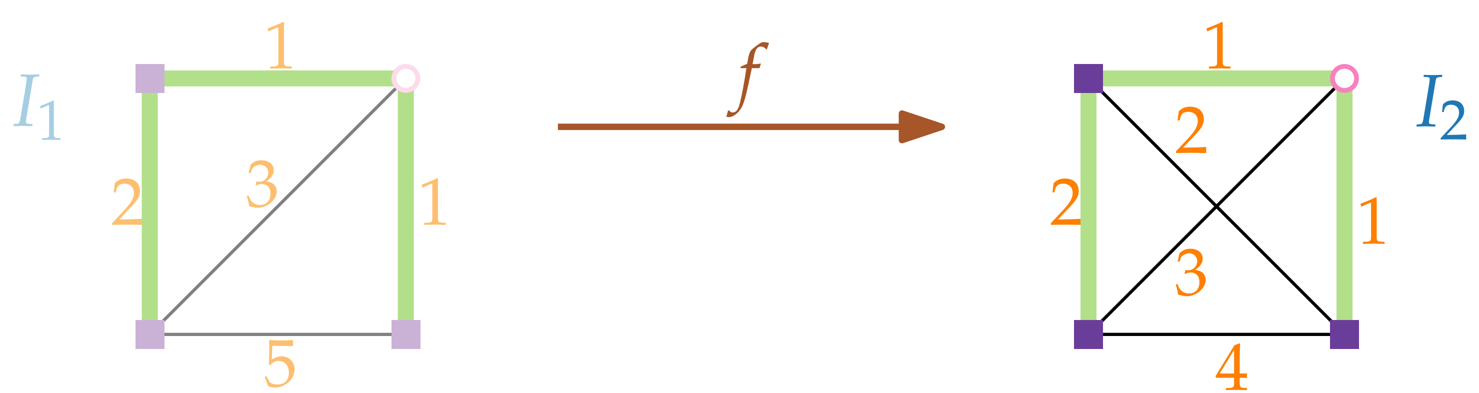

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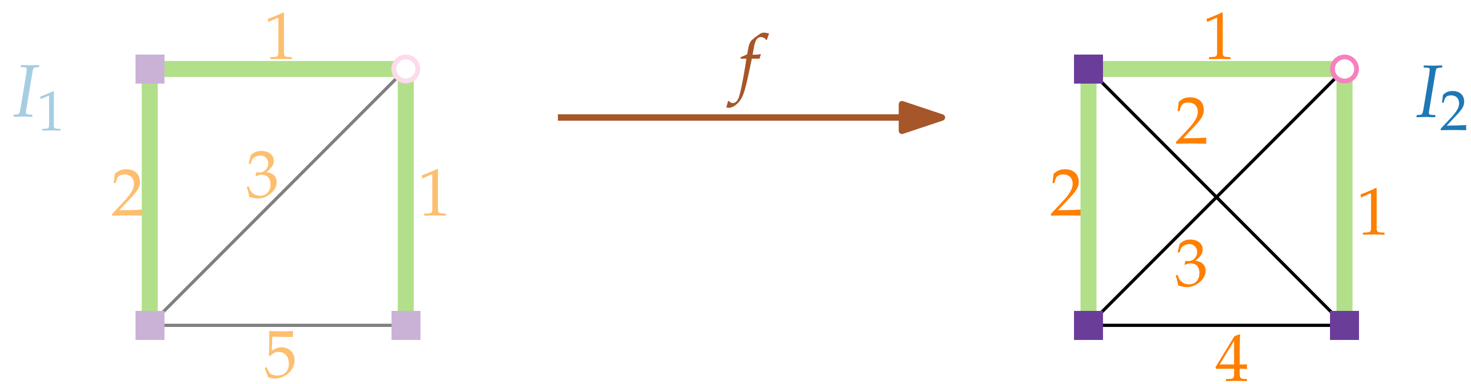

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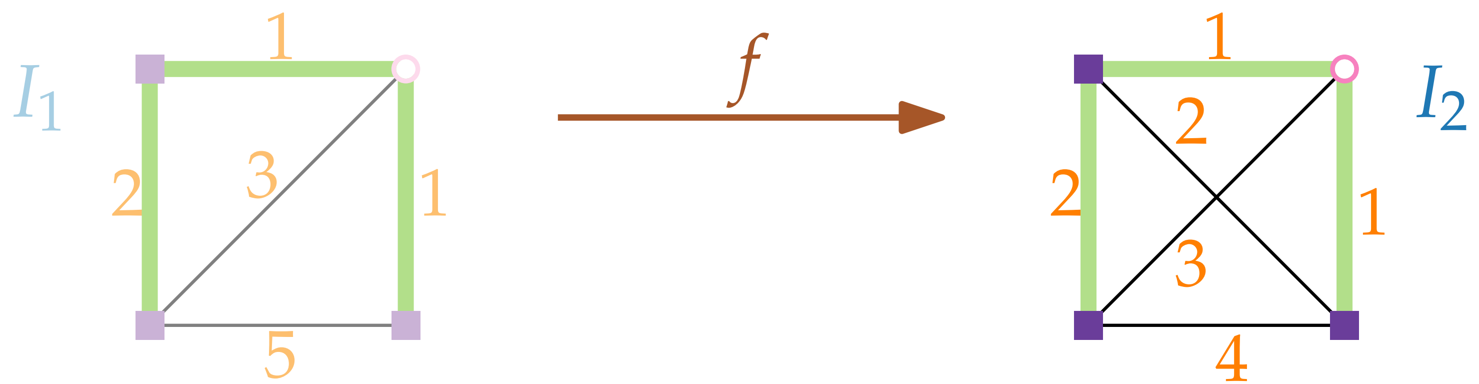

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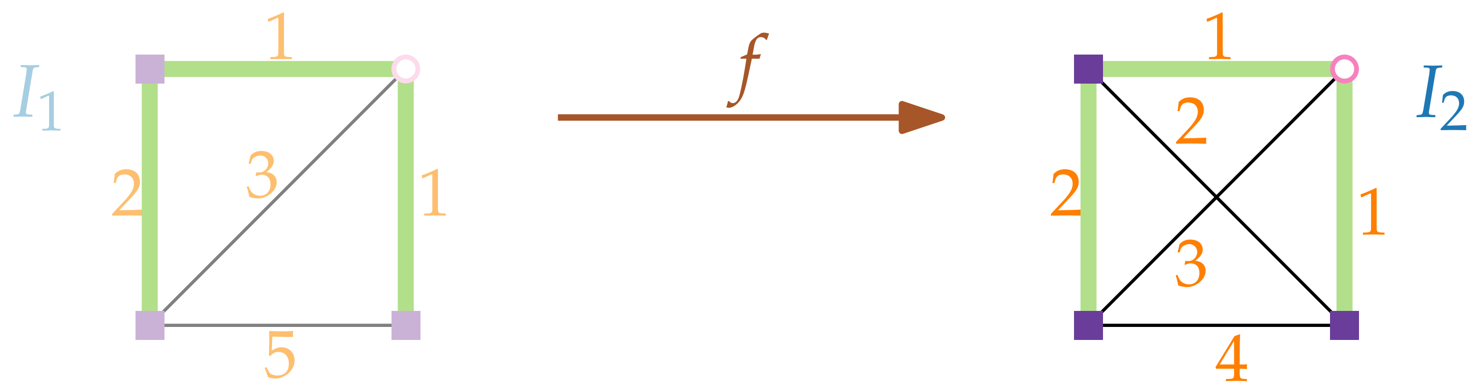

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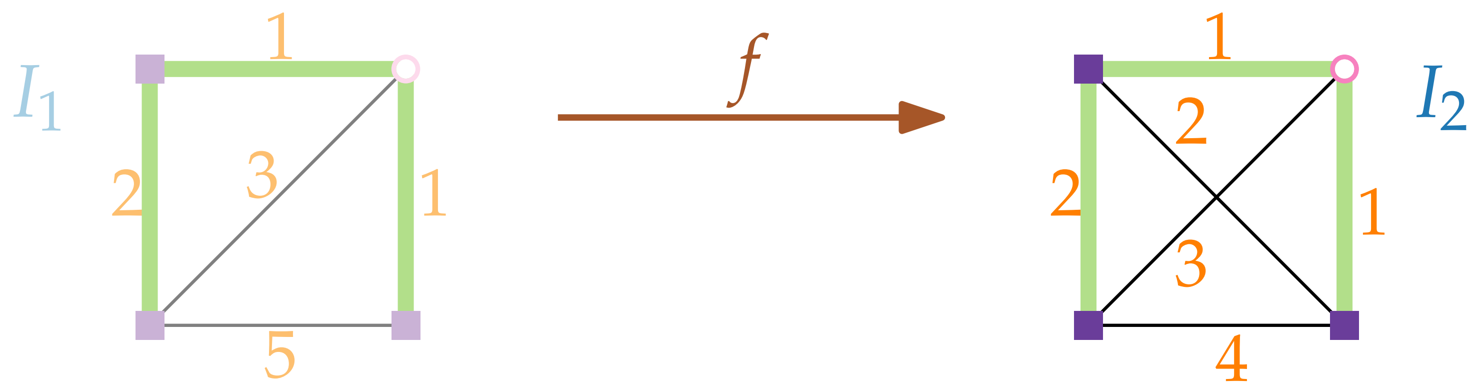

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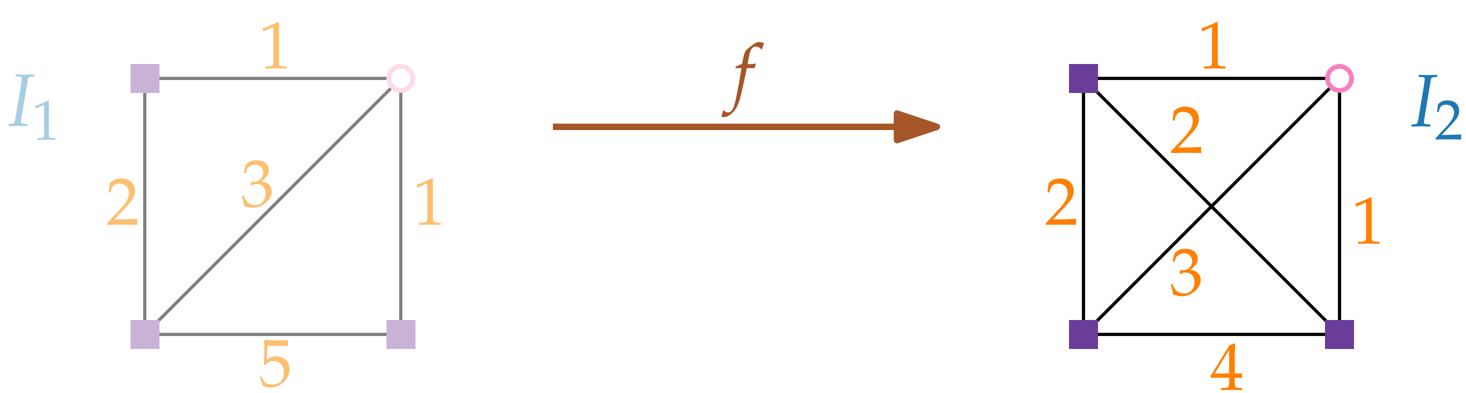

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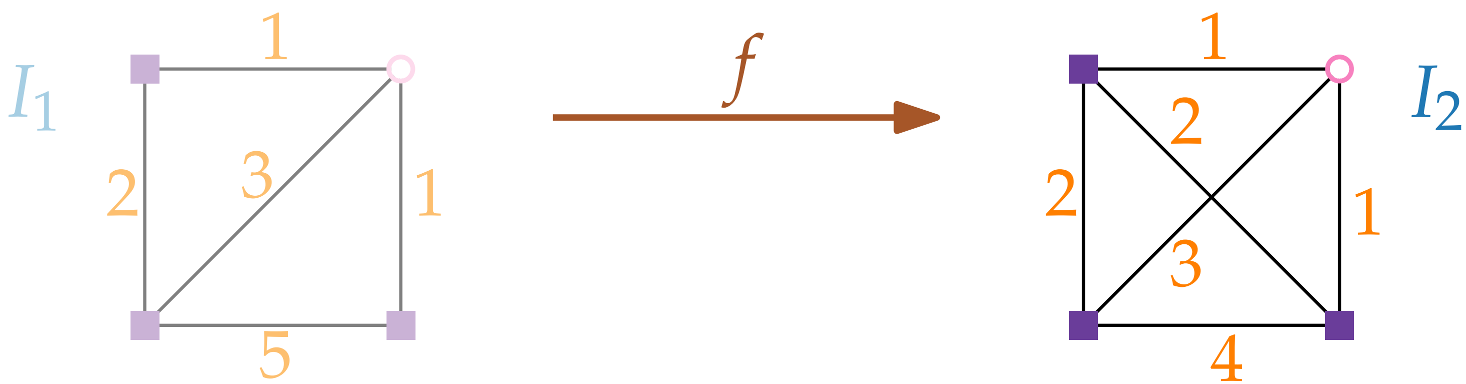

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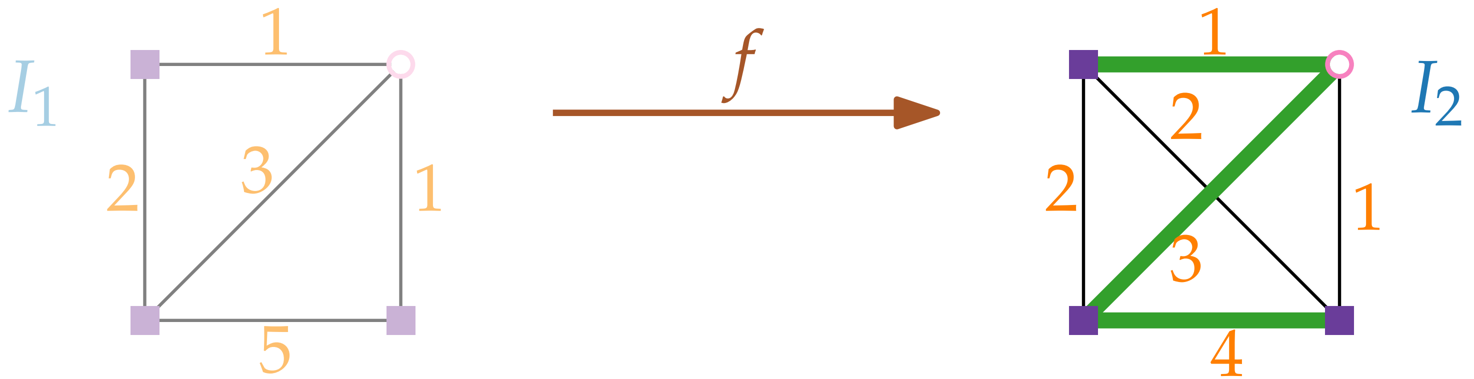

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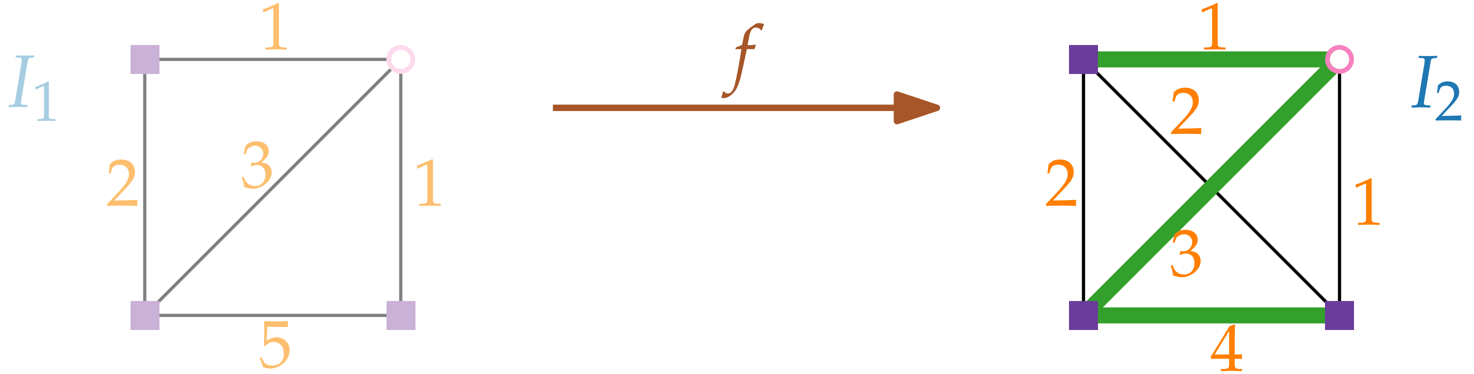

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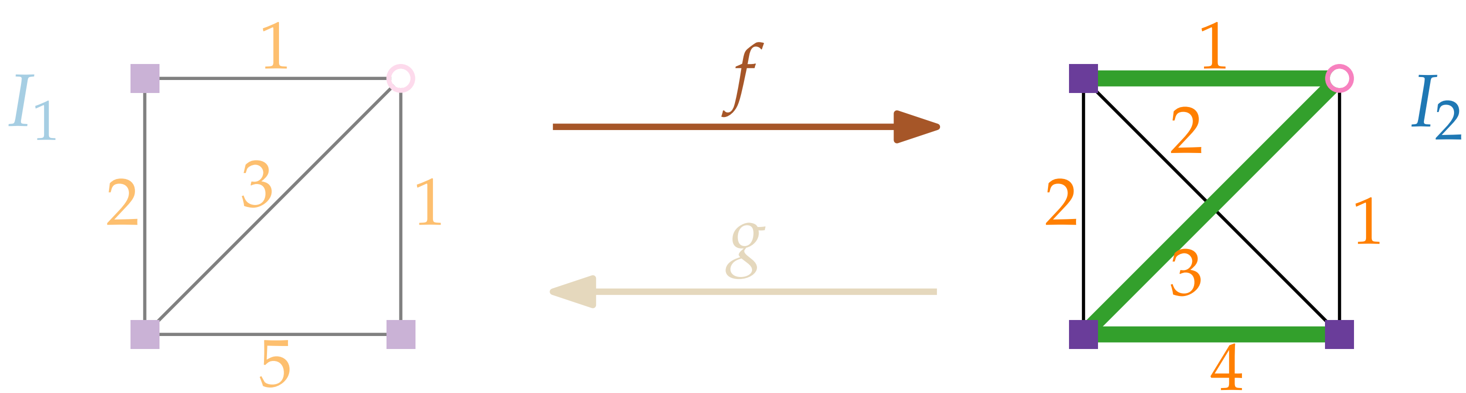

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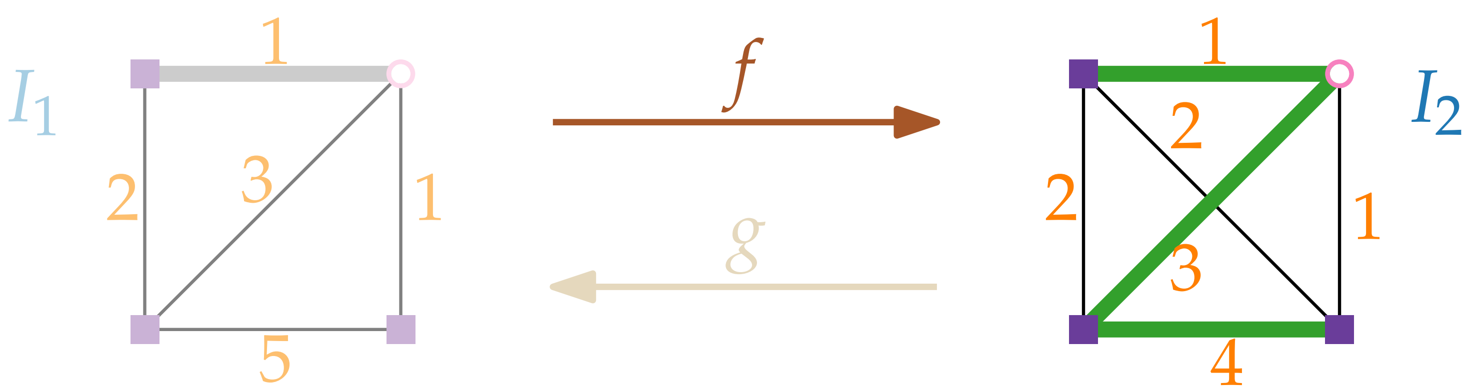
Construct $G'_1 \subseteq G_1$ from B_2 by replacing each edge (u, v) of B_2 by a shortest u–v-path in G_1.



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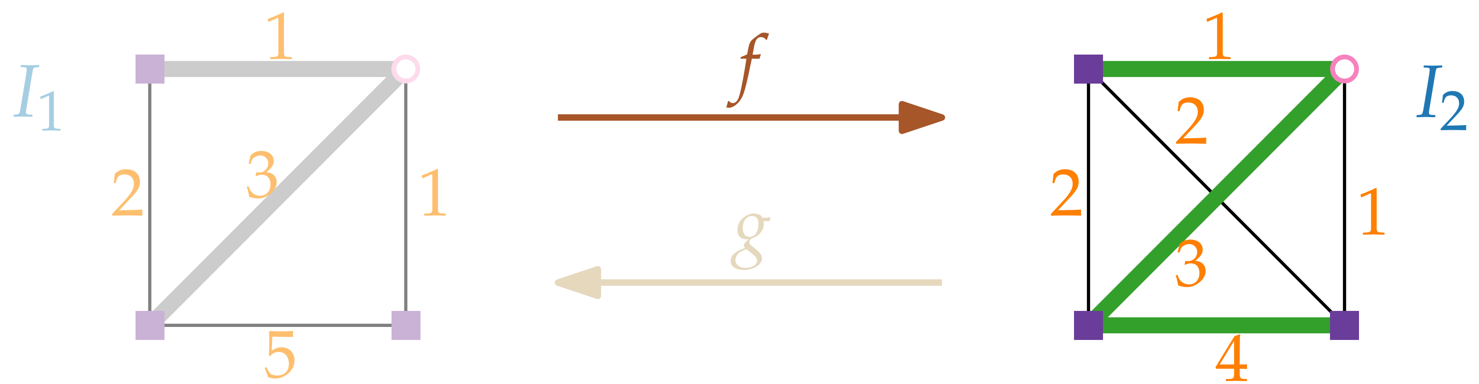

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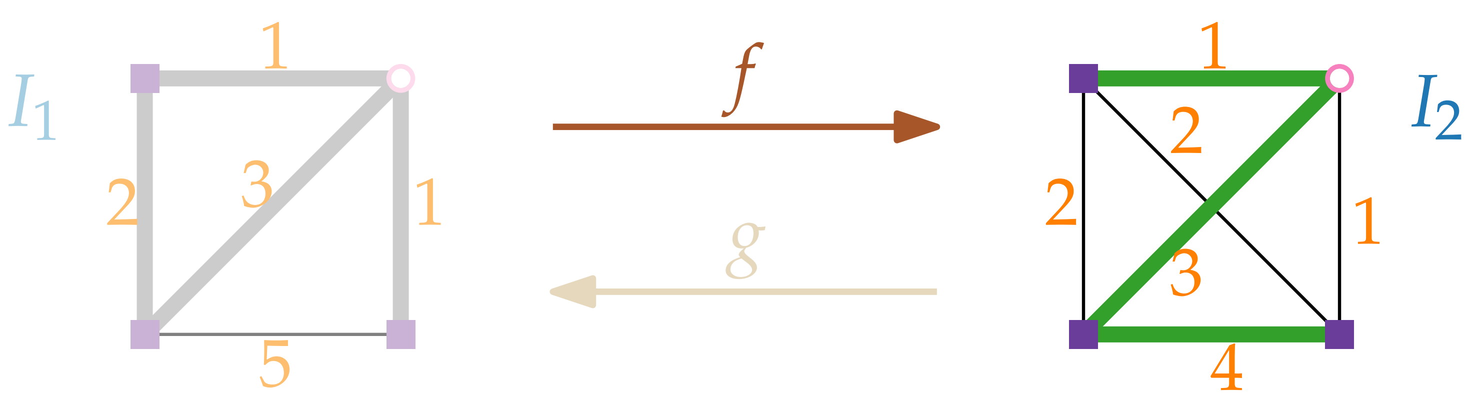

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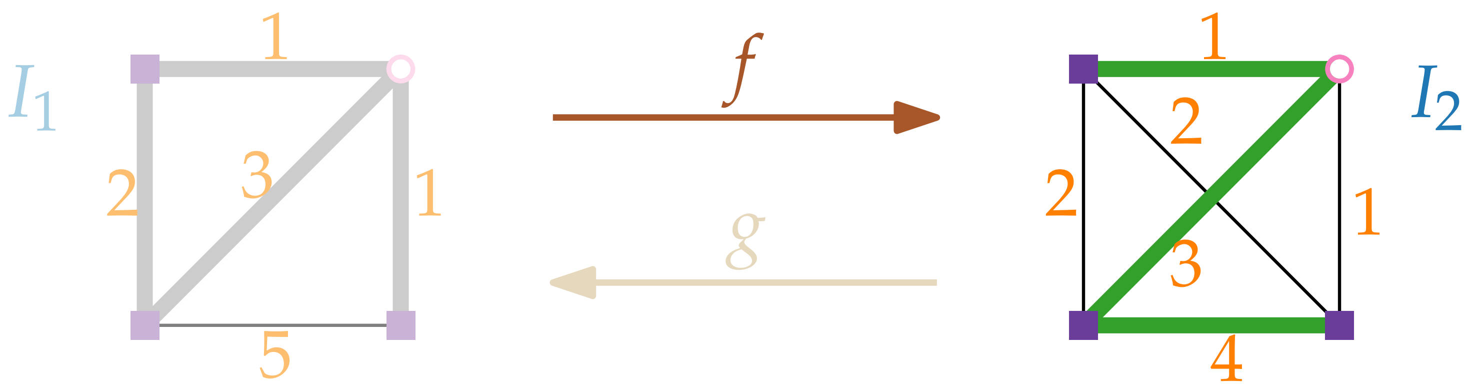

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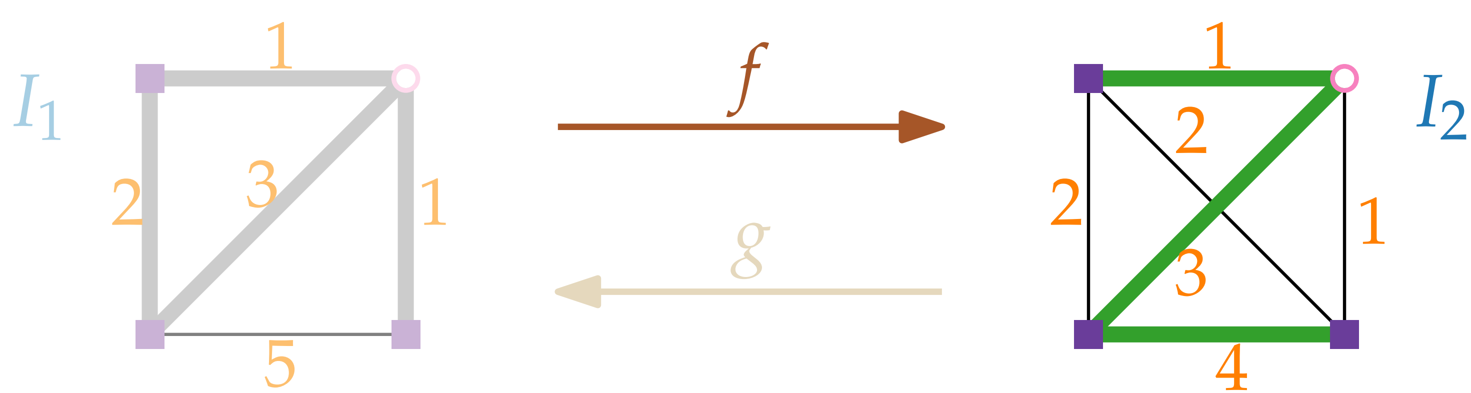
$c_1(G_1') \leq c_2(B_2)$; G_1' connects all terminals



MetricSteinerTree

Theorem. There is an approximation preserving reduction from SteinerTree to MetricSteinerTree.

Proof. (3) Mapping g $s \xleftarrow{g} t$

Let B_2 be Steiner tree of G_2

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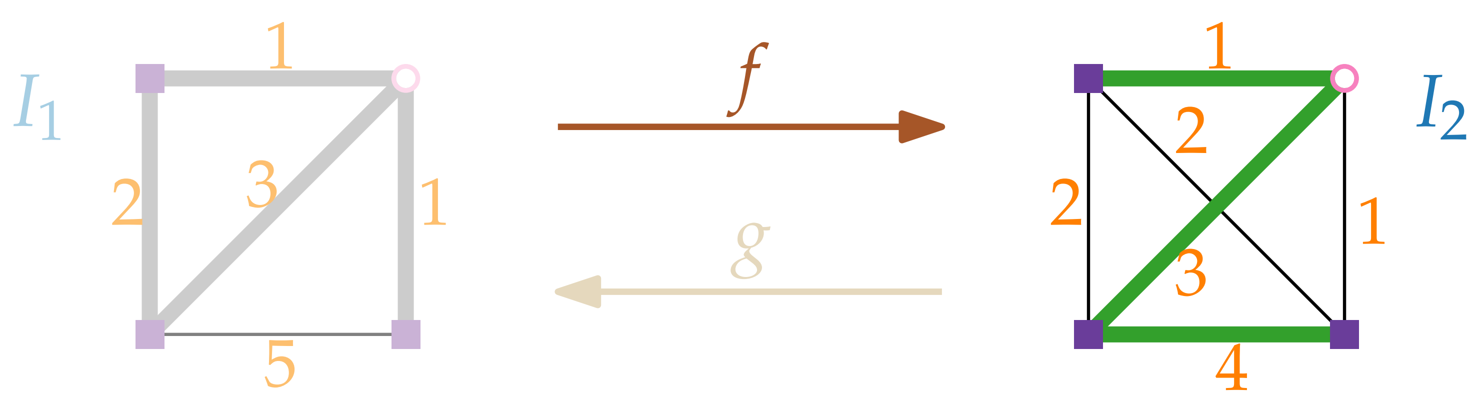

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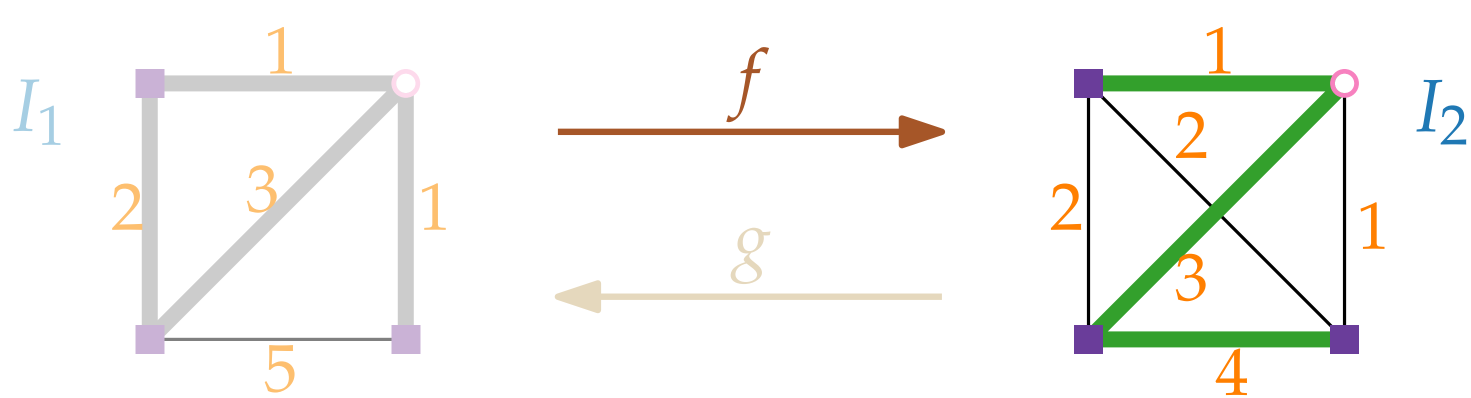

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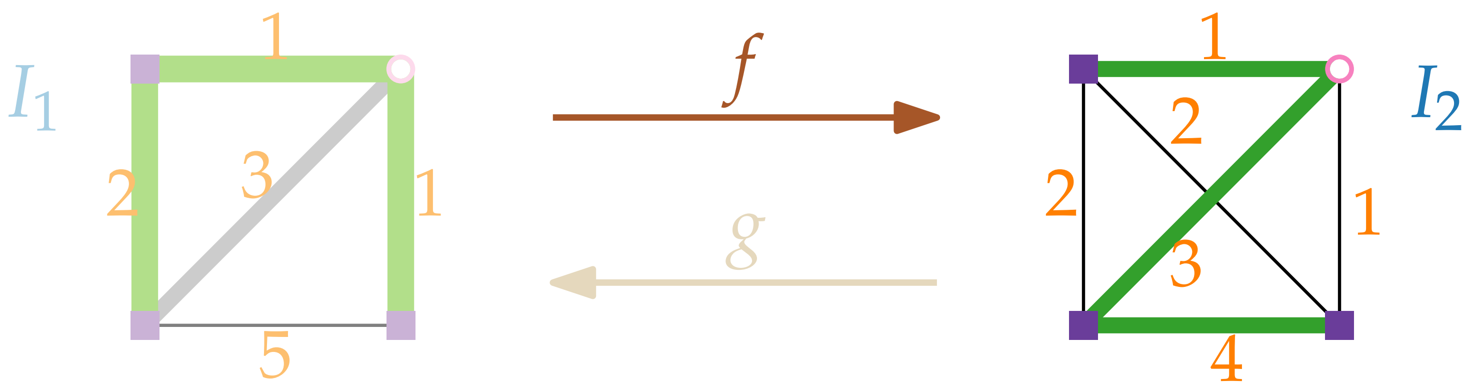

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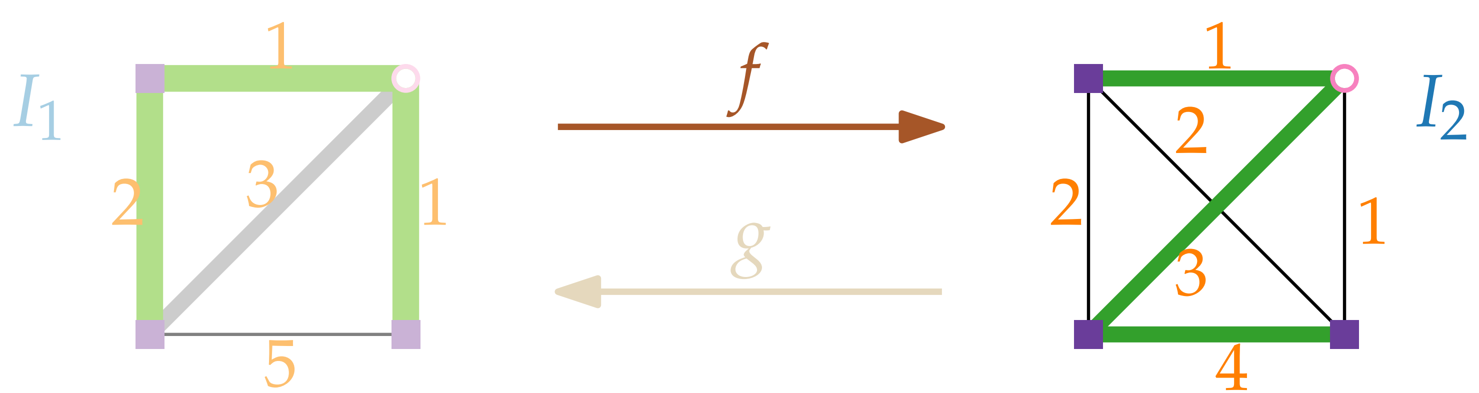

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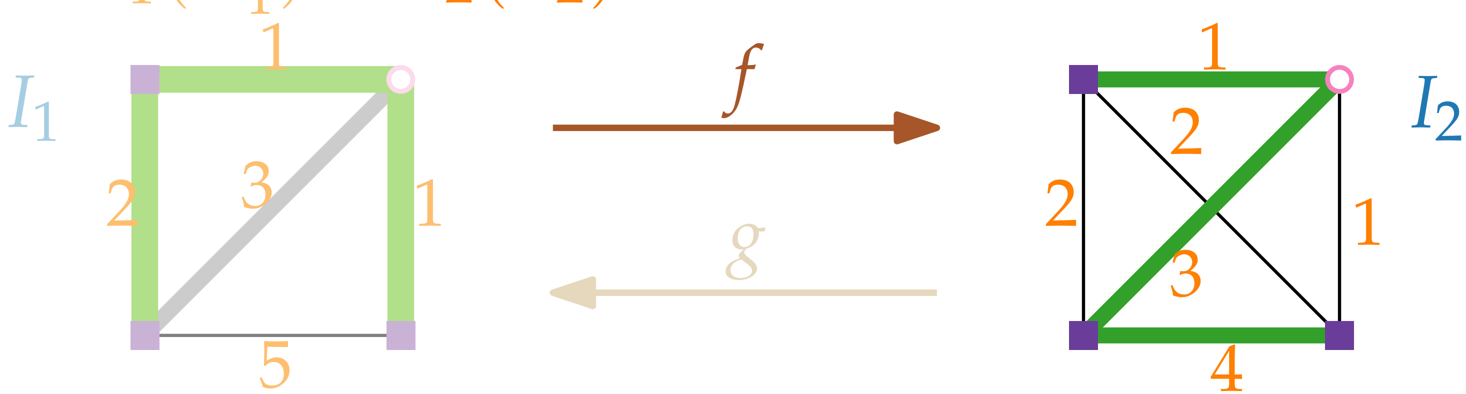

Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part IV:
2-Approximation for SteinerTree

Joachim Spoerhase

Winter 2021/22

2-Approximation for SteinerTree

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Theorem. For an instance of MetricSteinerTree, let B be a minimum spanning tree (MST) of the subgraph $G[T]$ induced by the terminal set T. Then $c(B) \leq 2 \cdot \mathrm{OPT}$.

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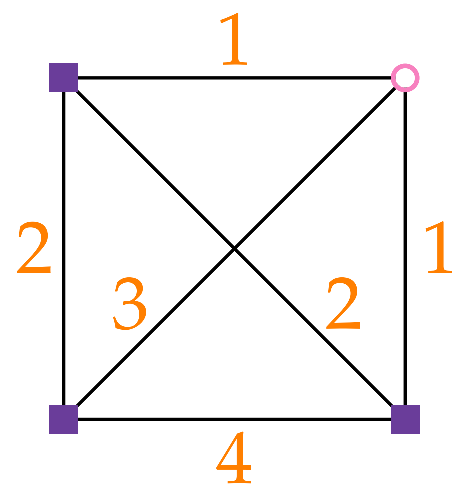

G

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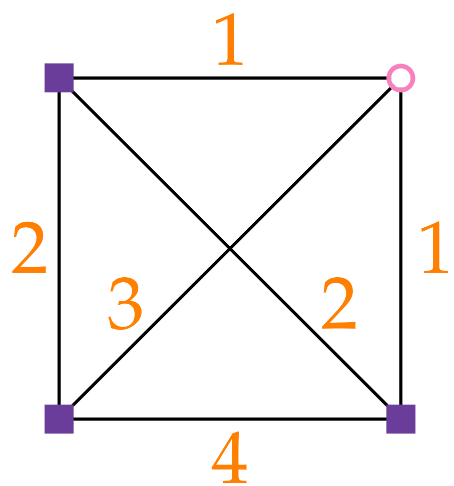

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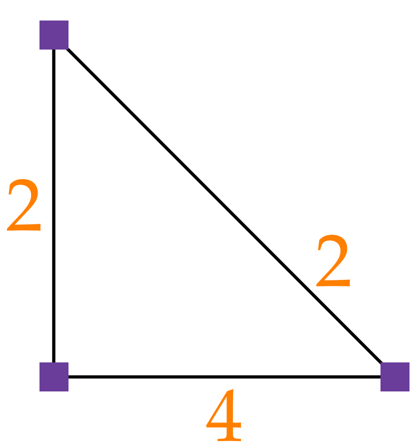

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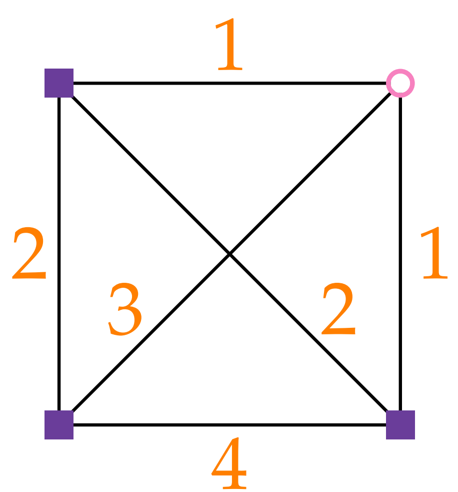

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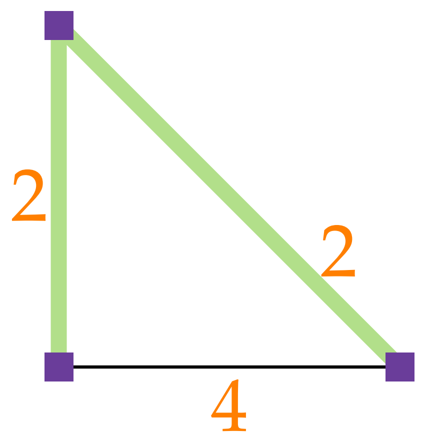

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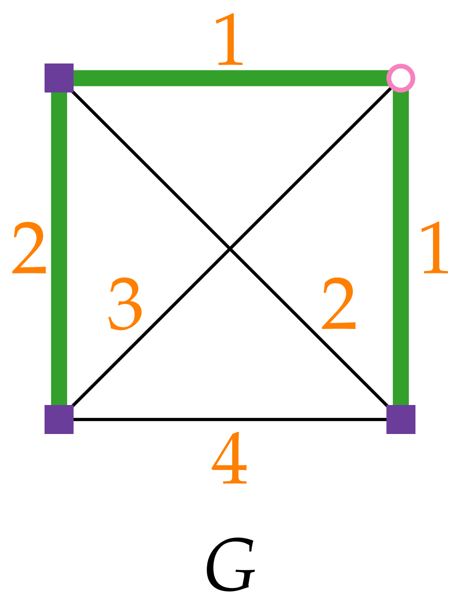

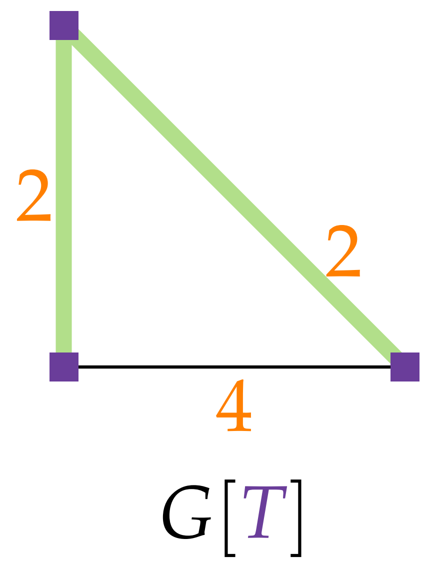

Proof of Approximation Factor

Consider optimal Steiner tree B^*

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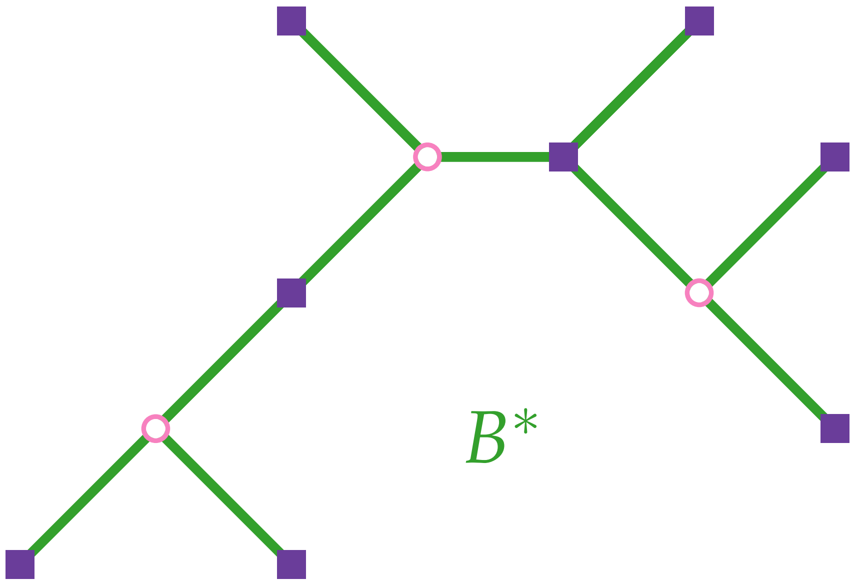

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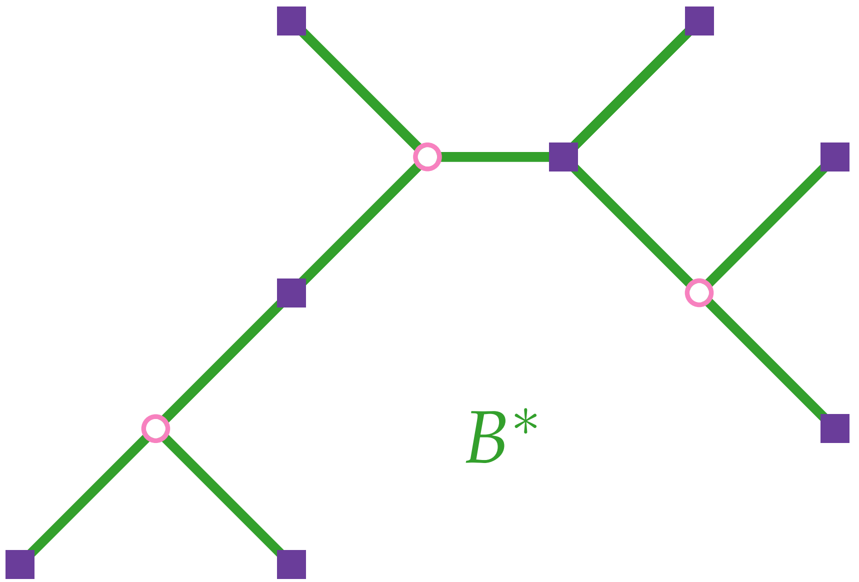

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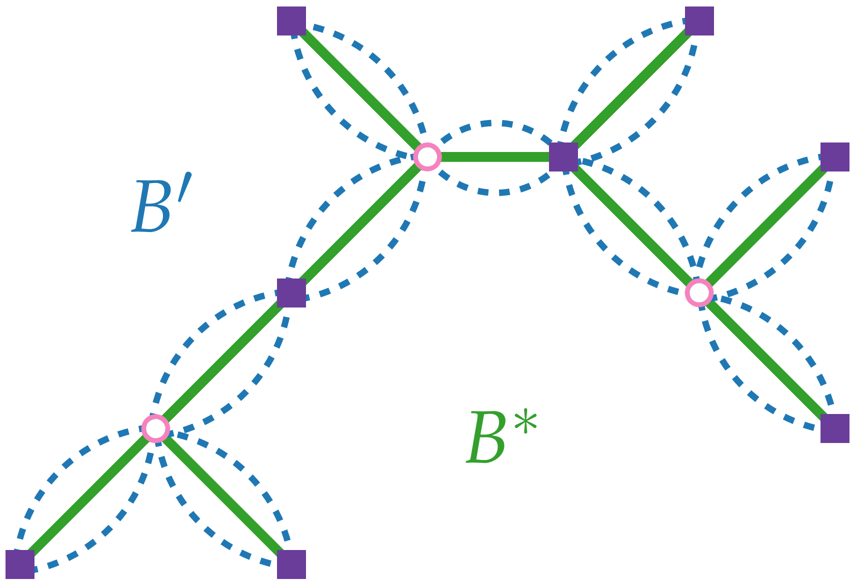

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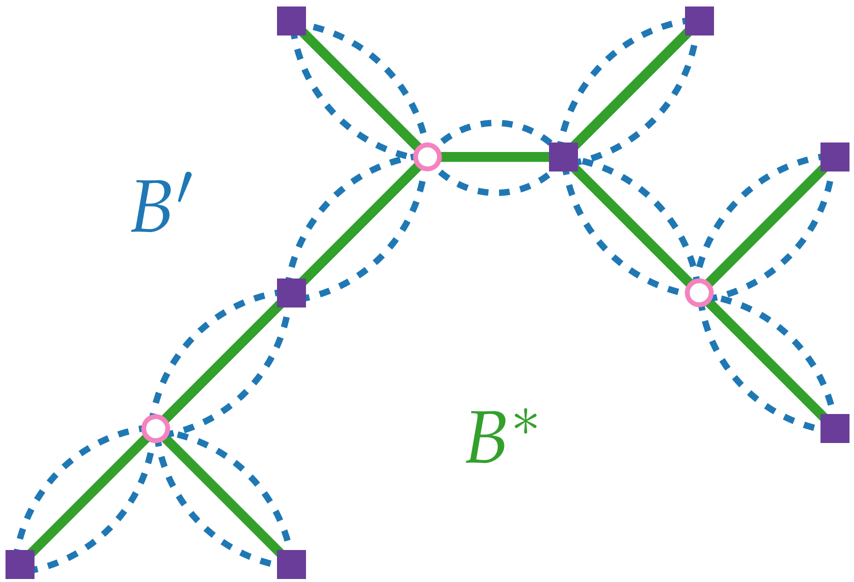

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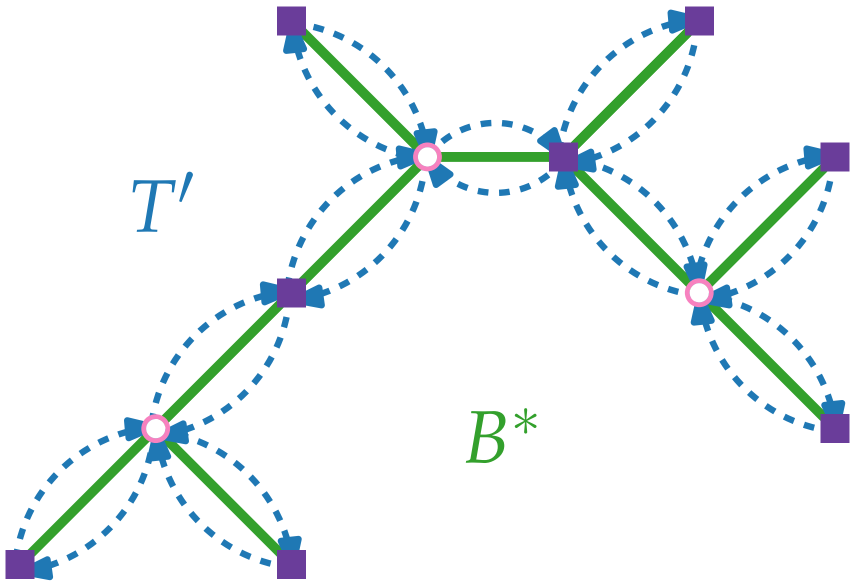

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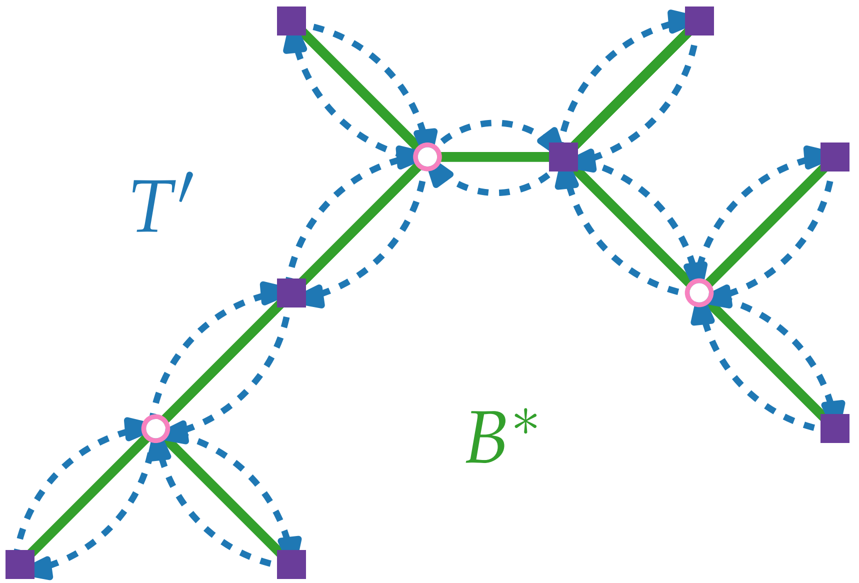

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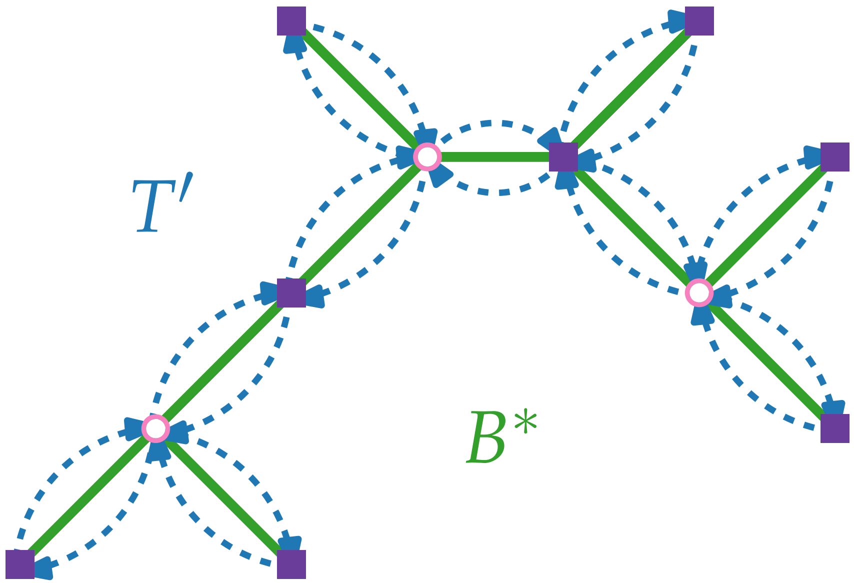

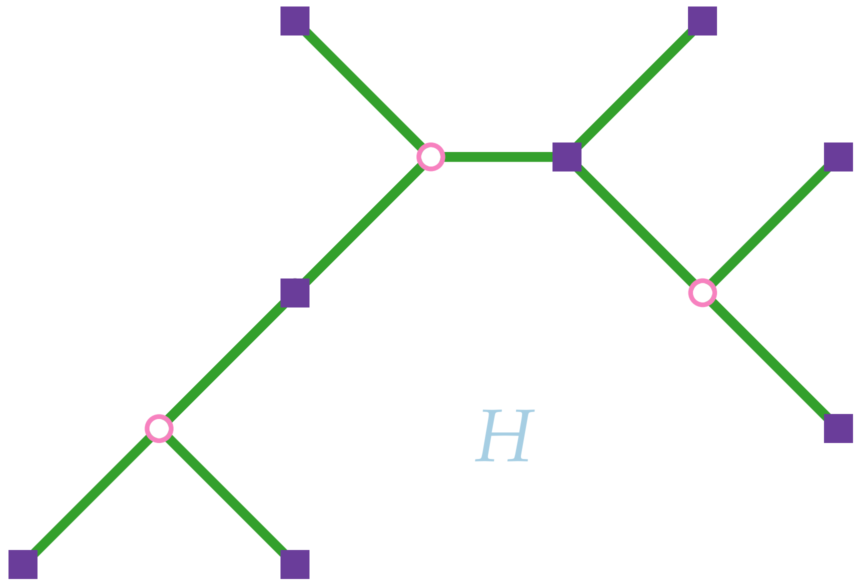

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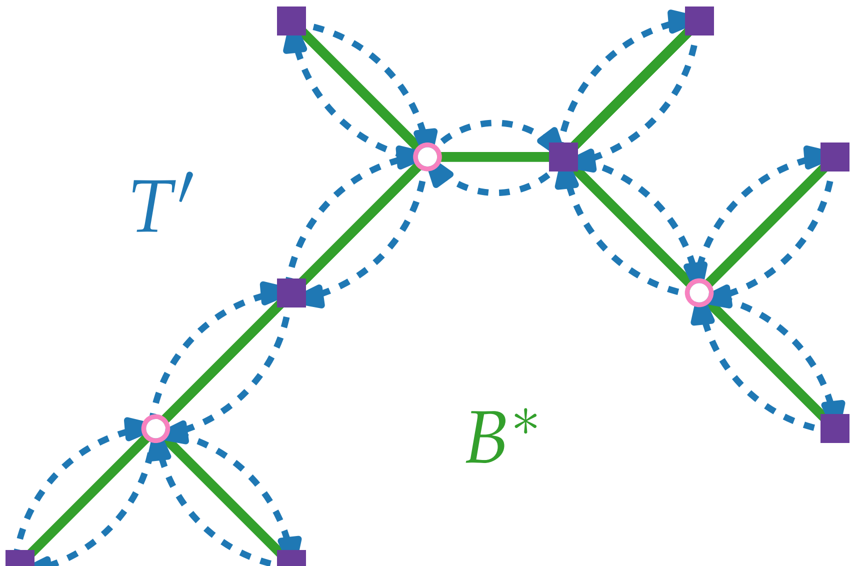

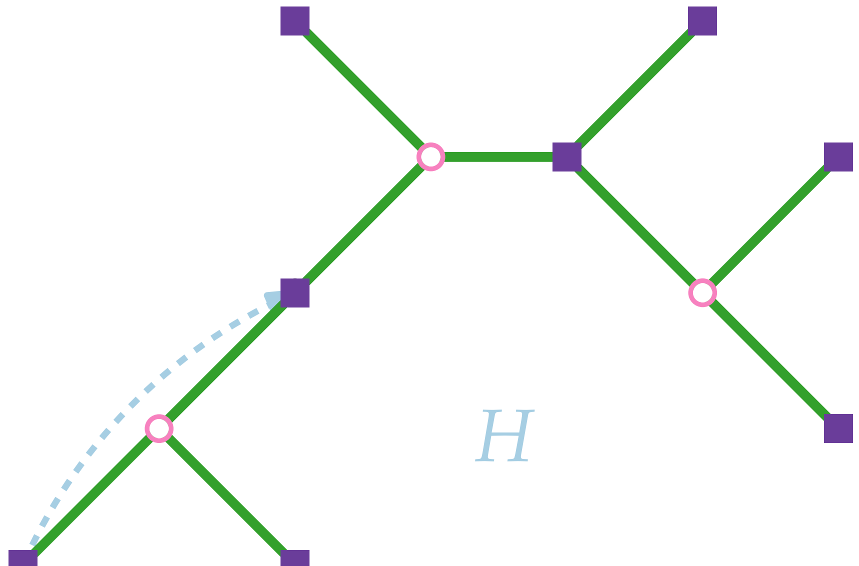

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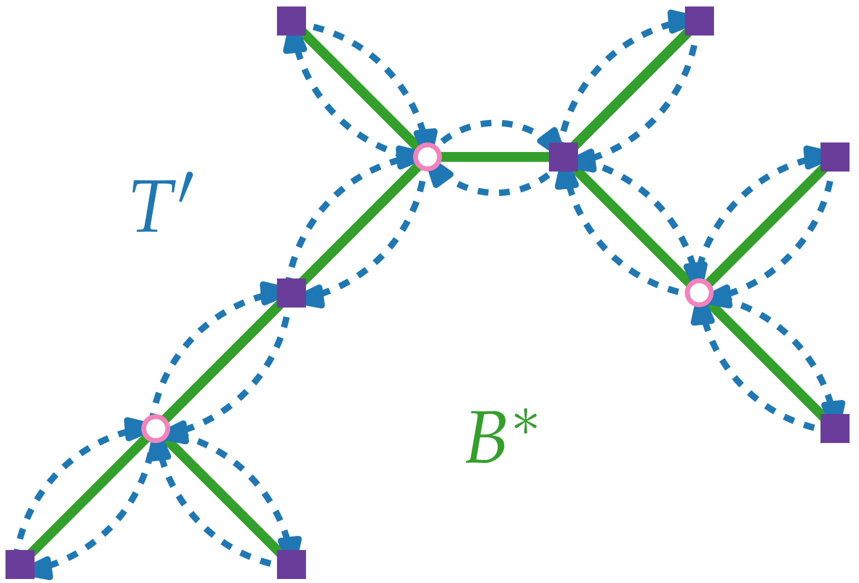

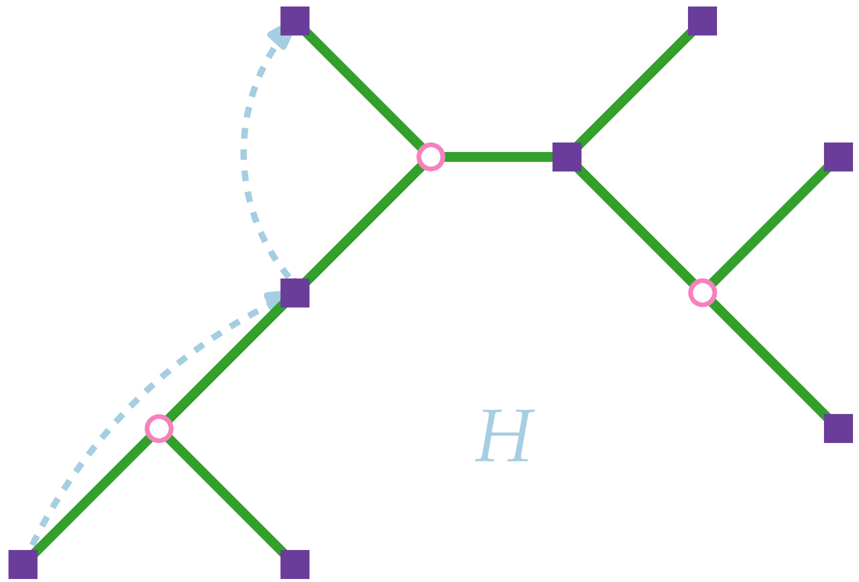

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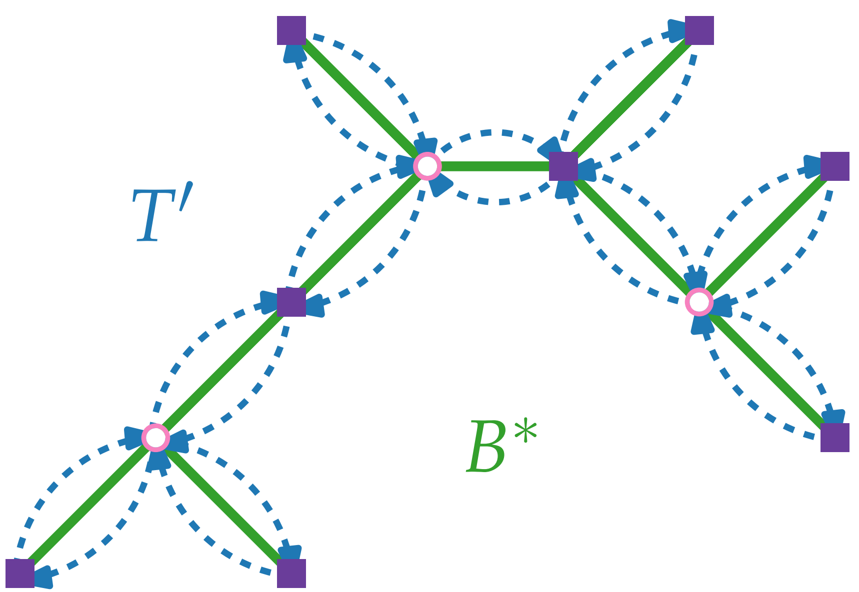

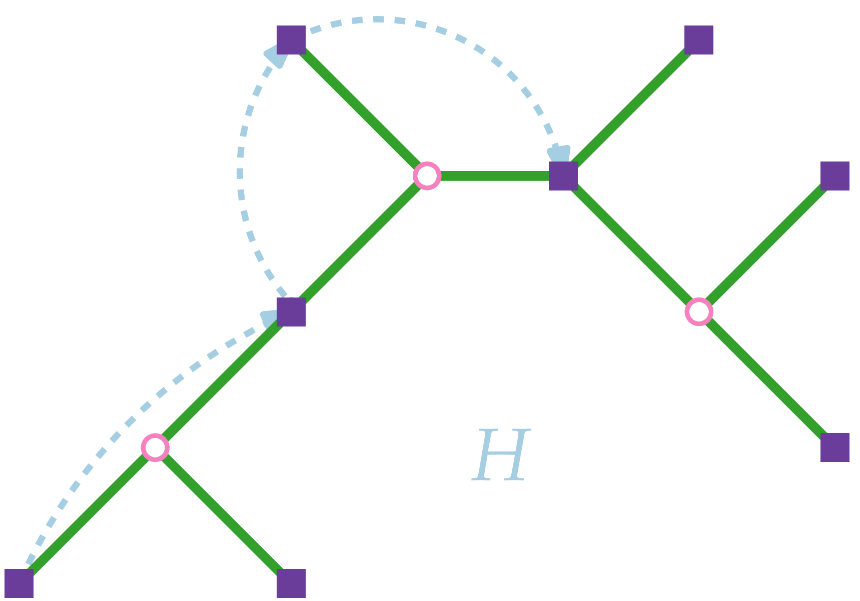

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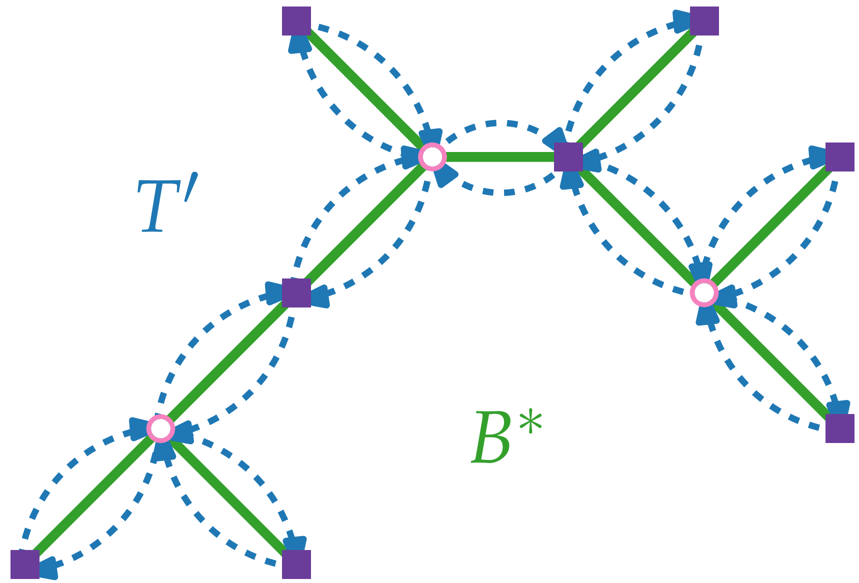

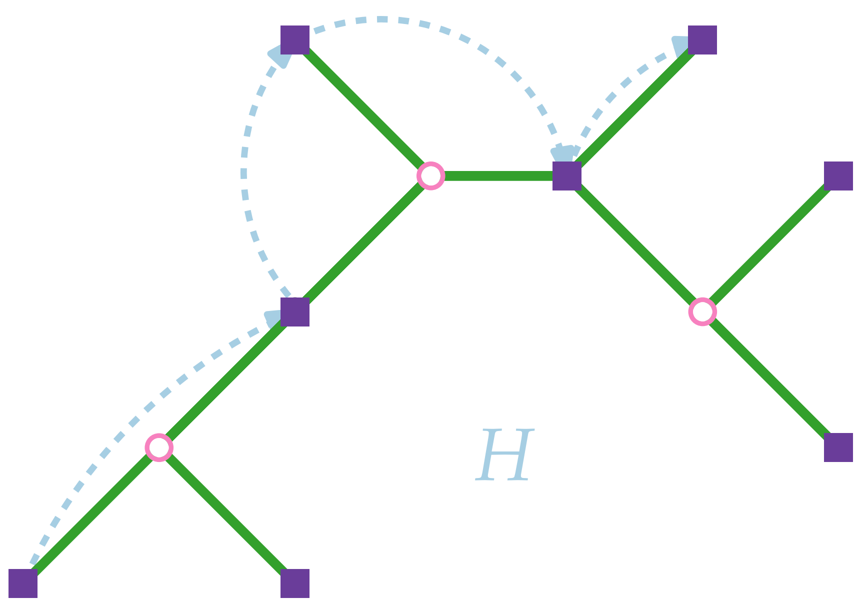

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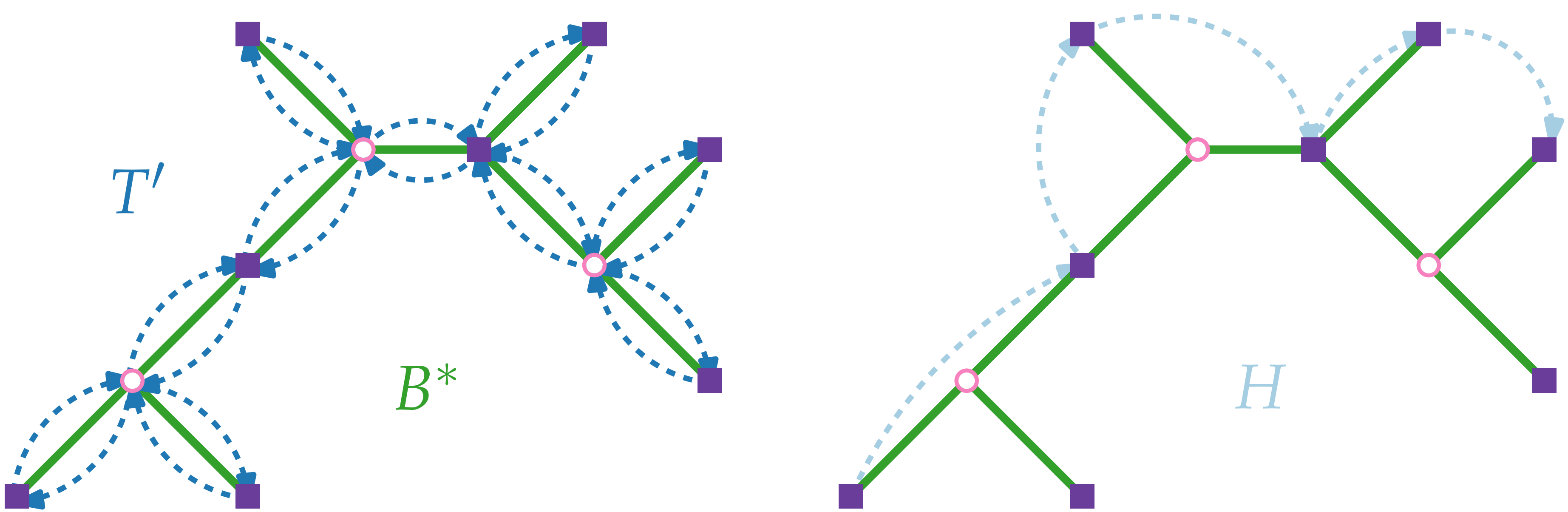

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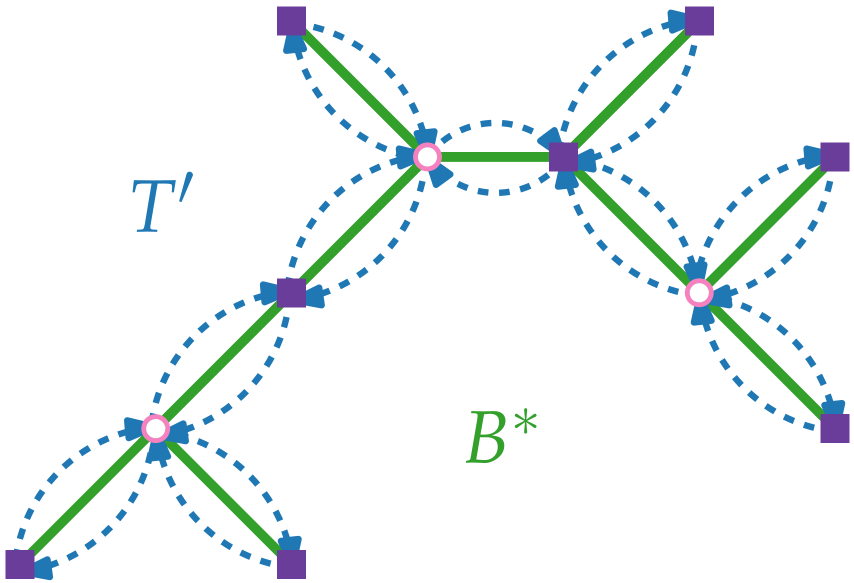

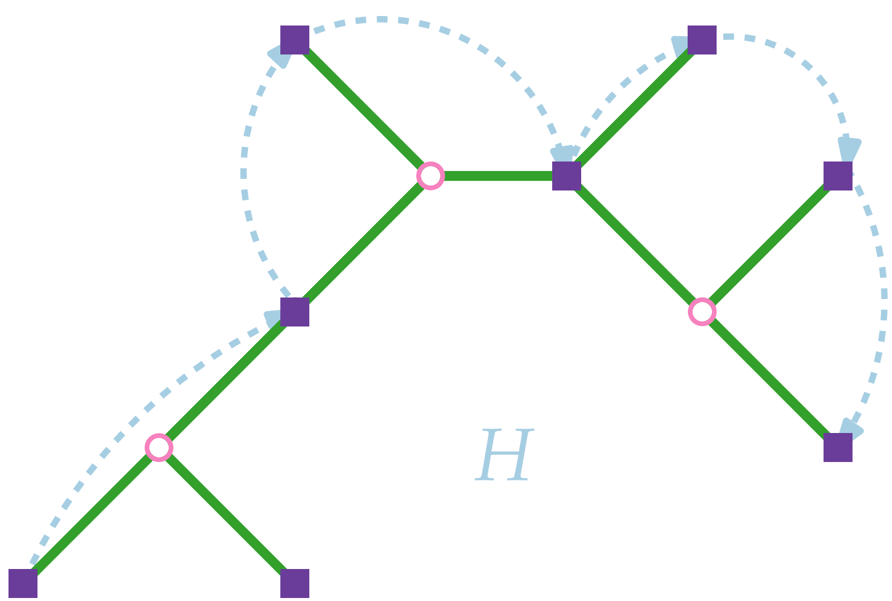

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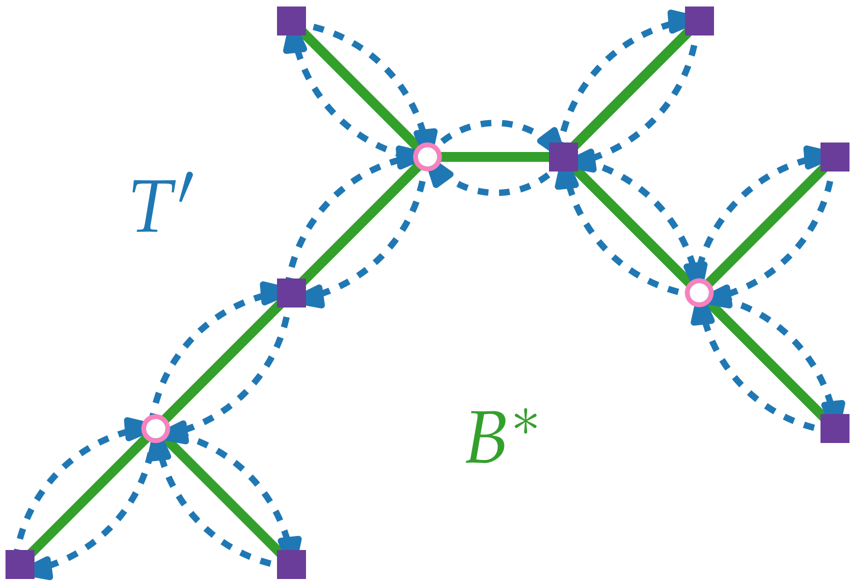

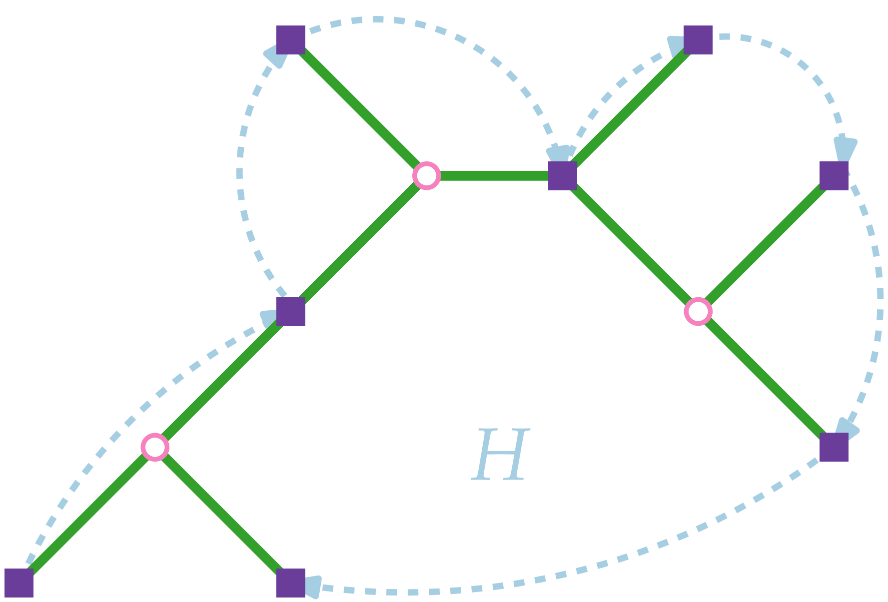

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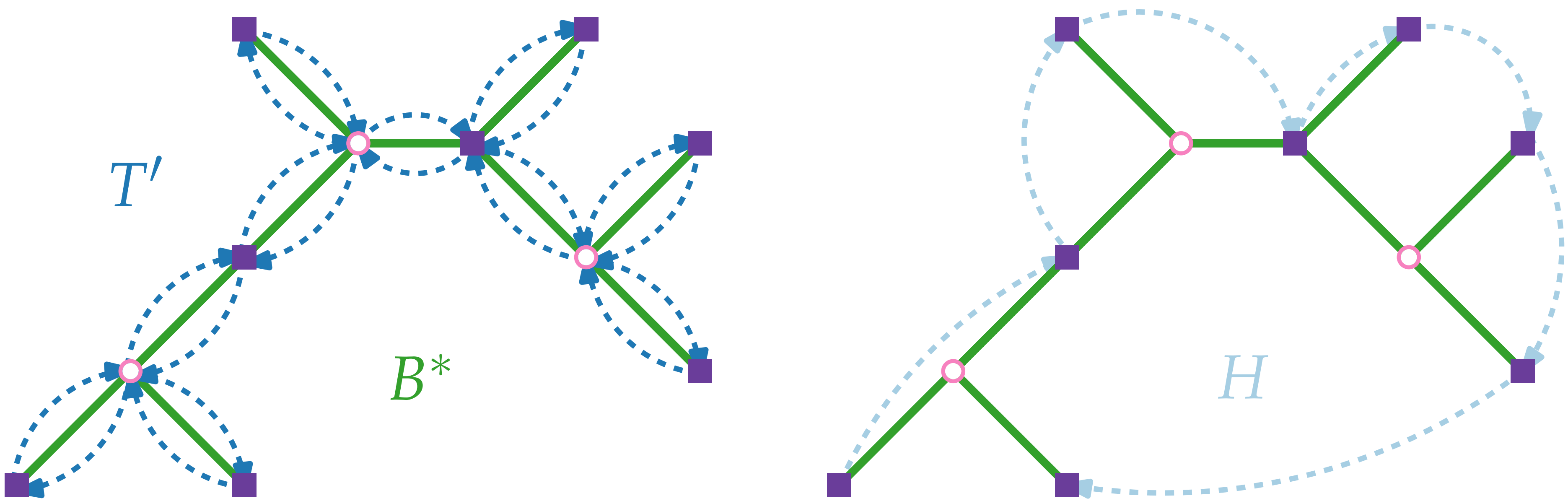

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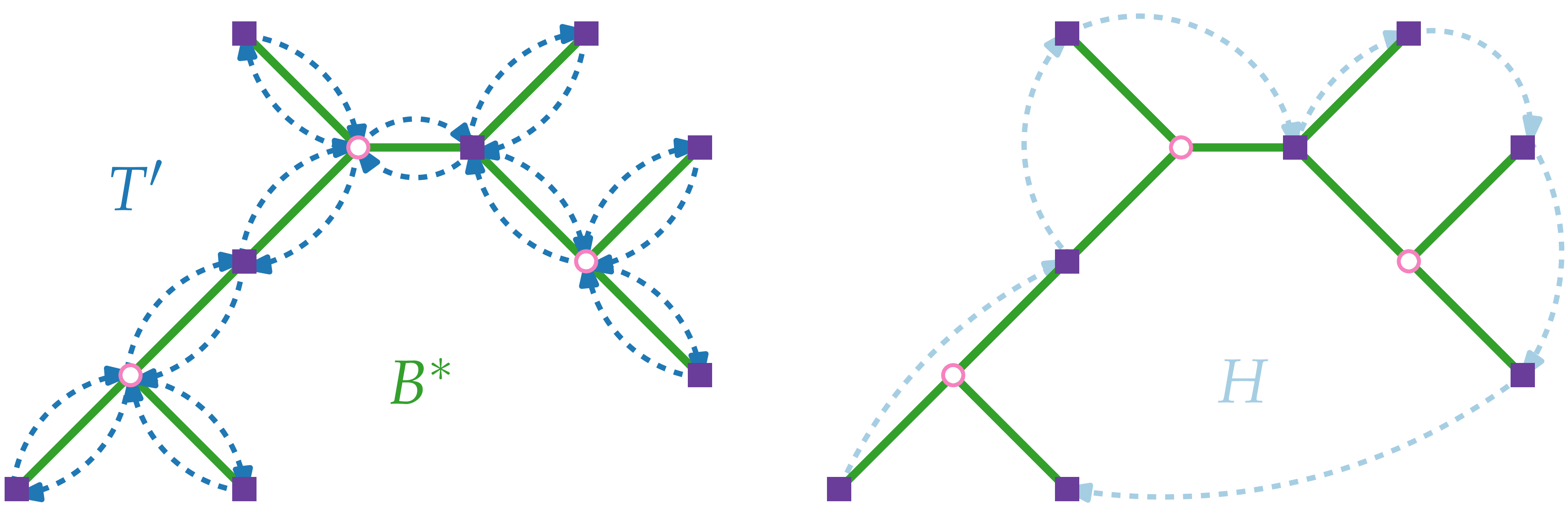

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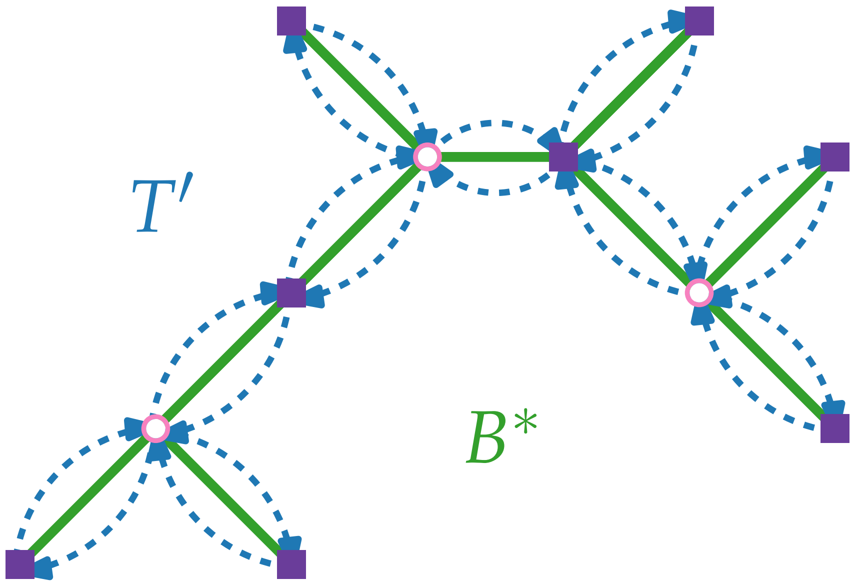

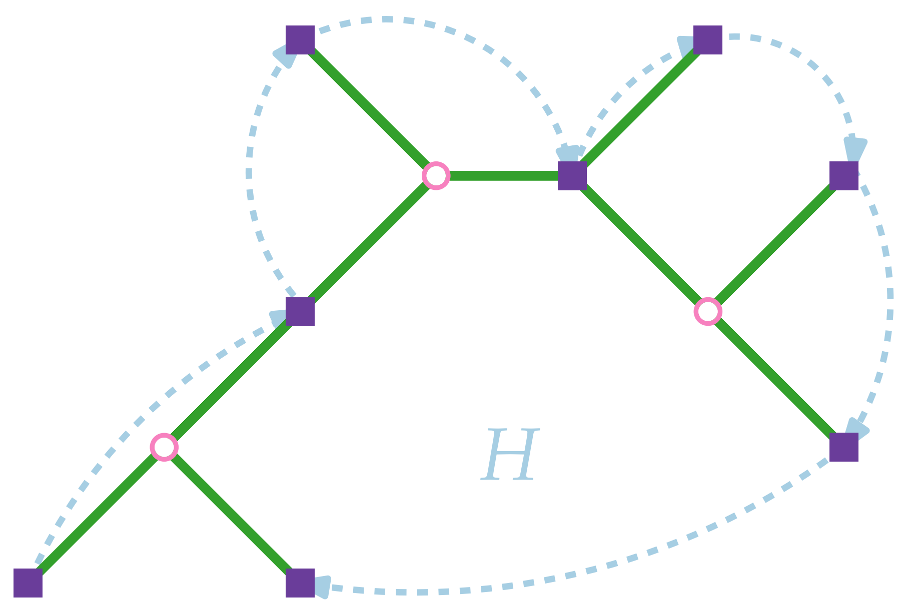

Analysis Sharp?

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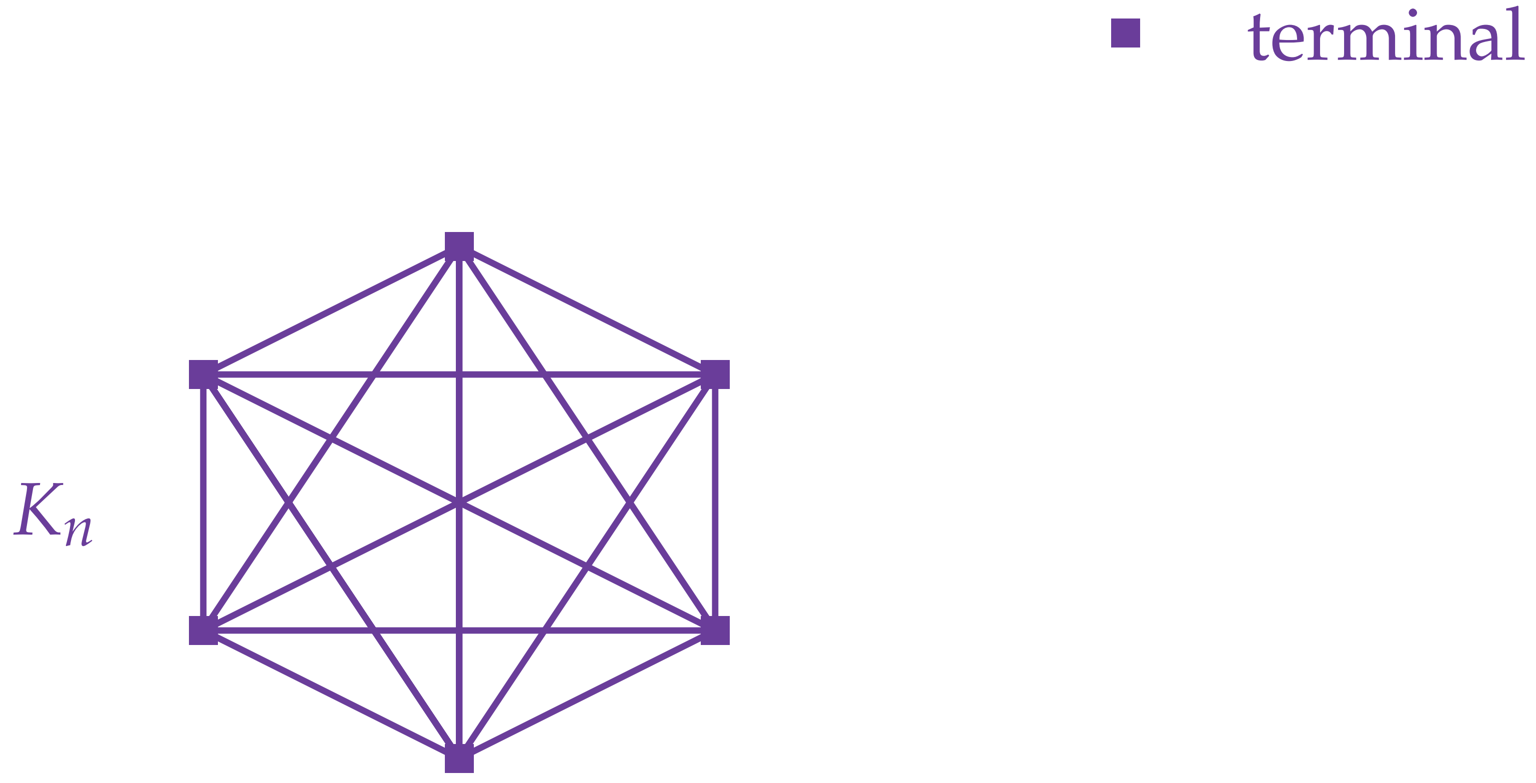

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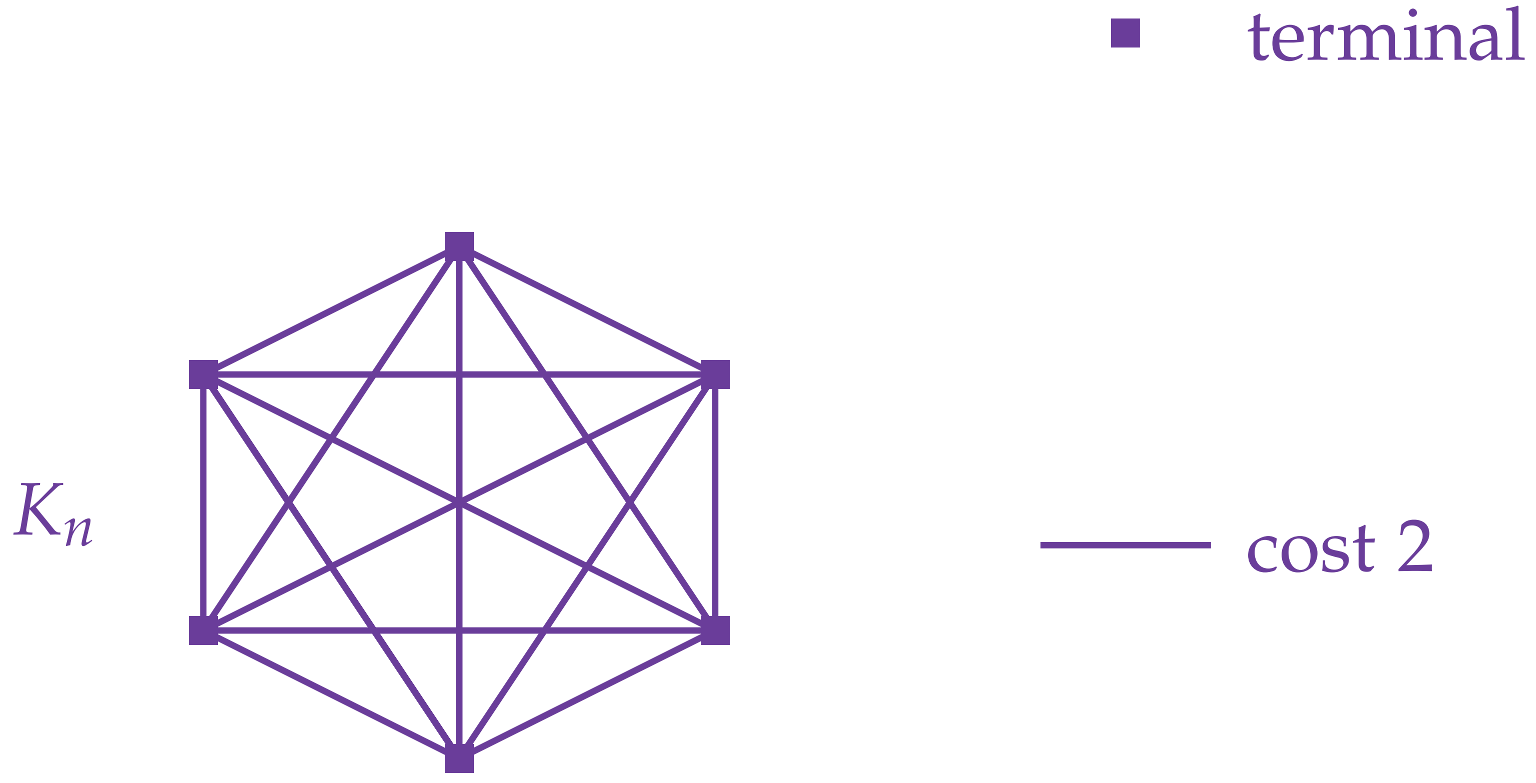

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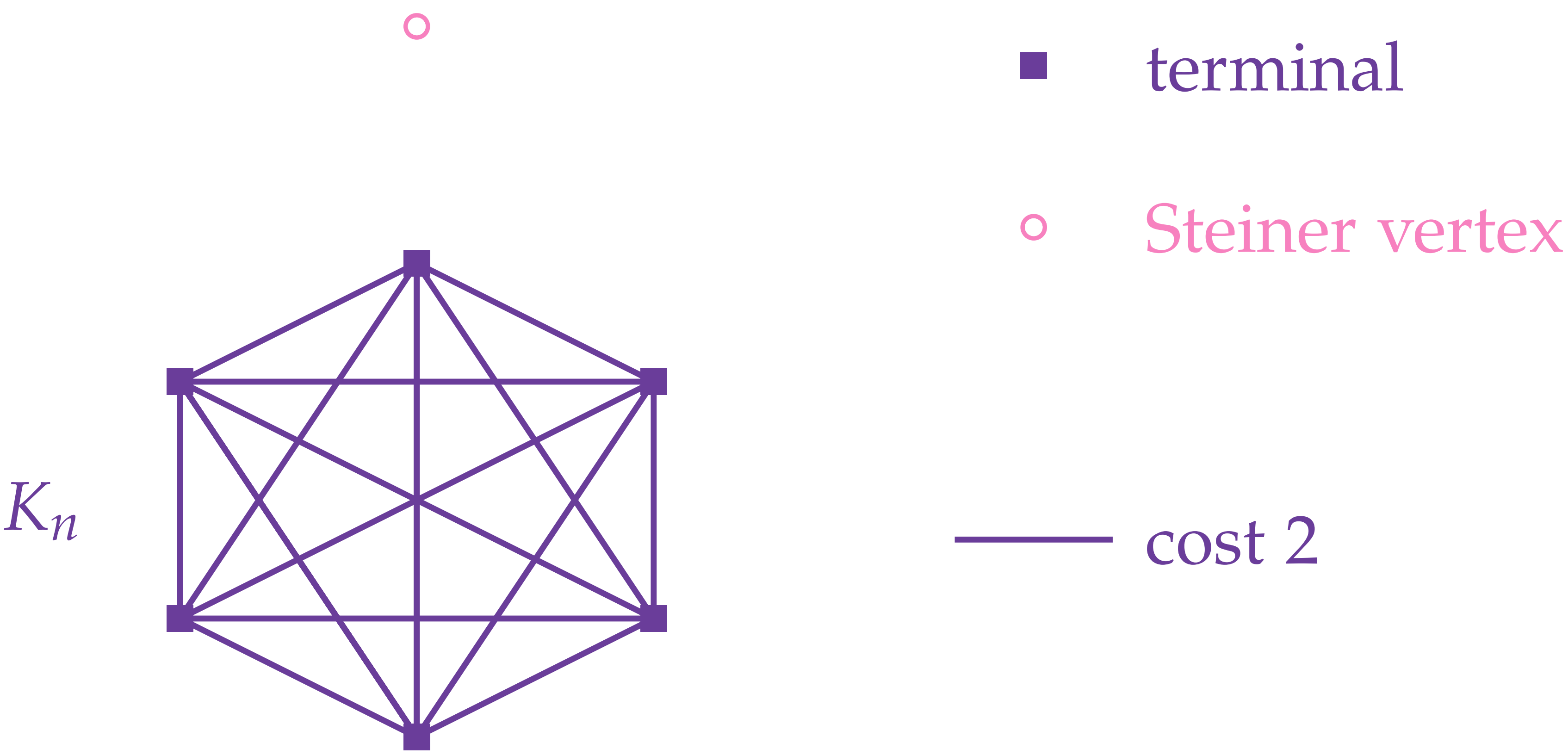

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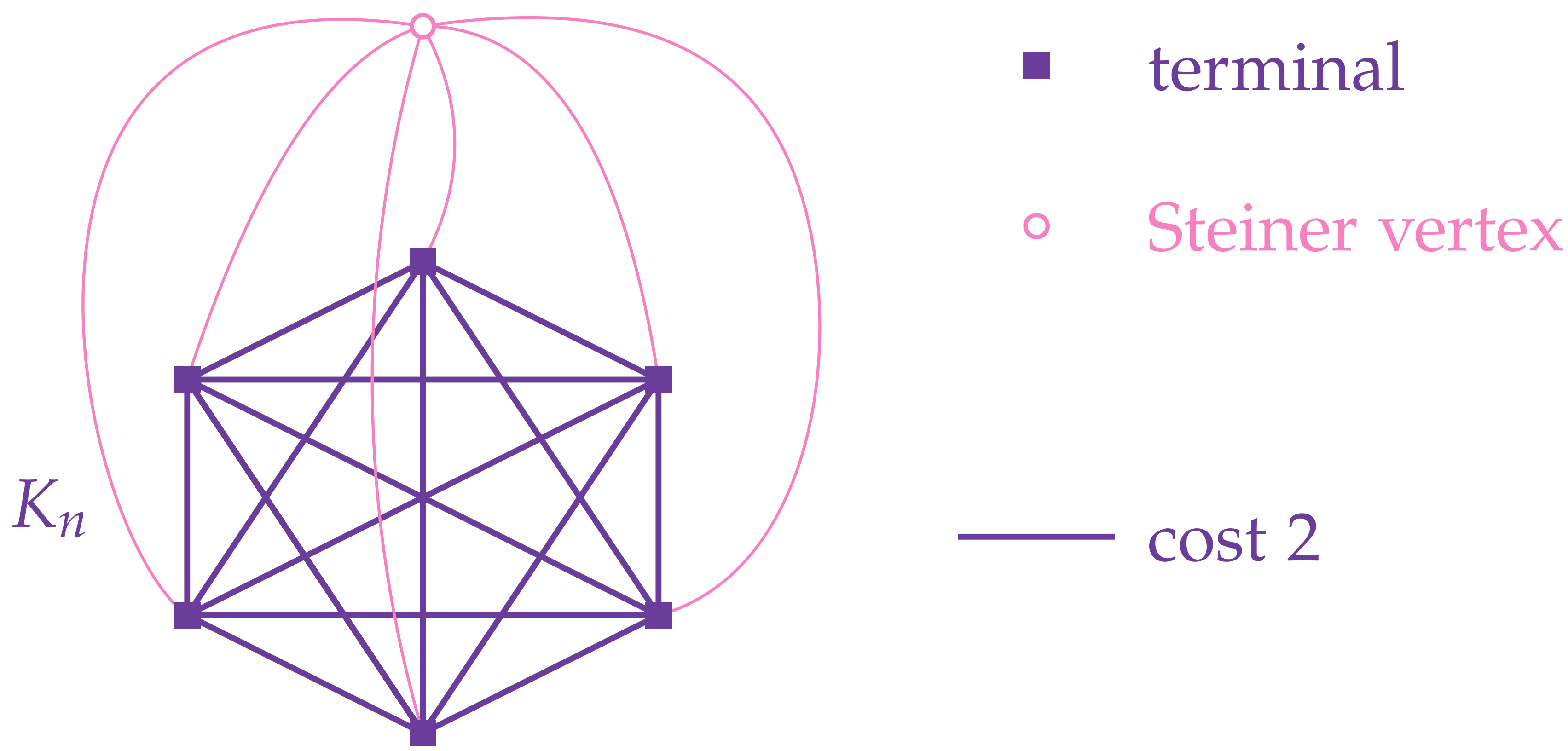

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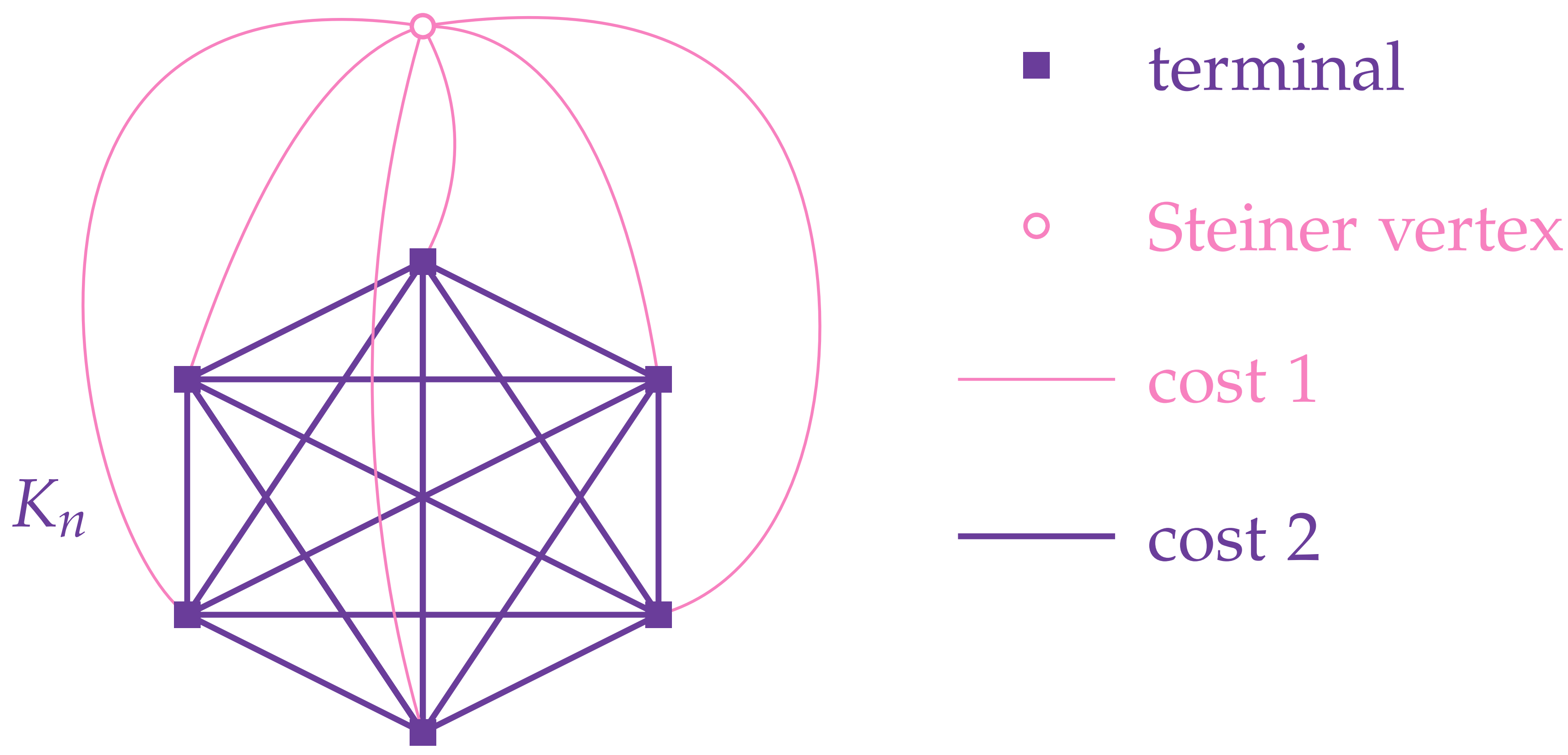

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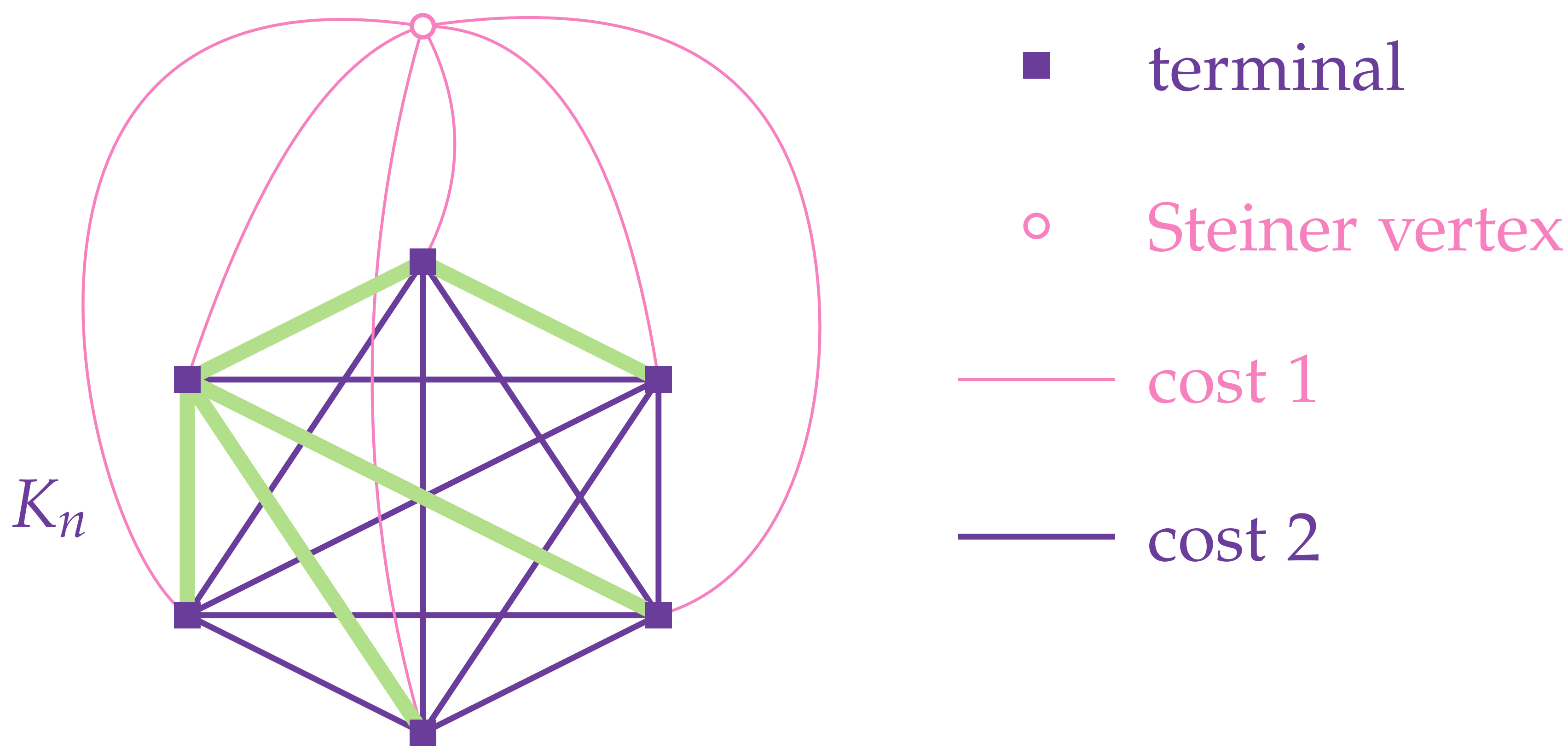
MST of $G[T]$ with cost $2(n-1)$



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MST of $G[T]$ with cost $2(n-1)$
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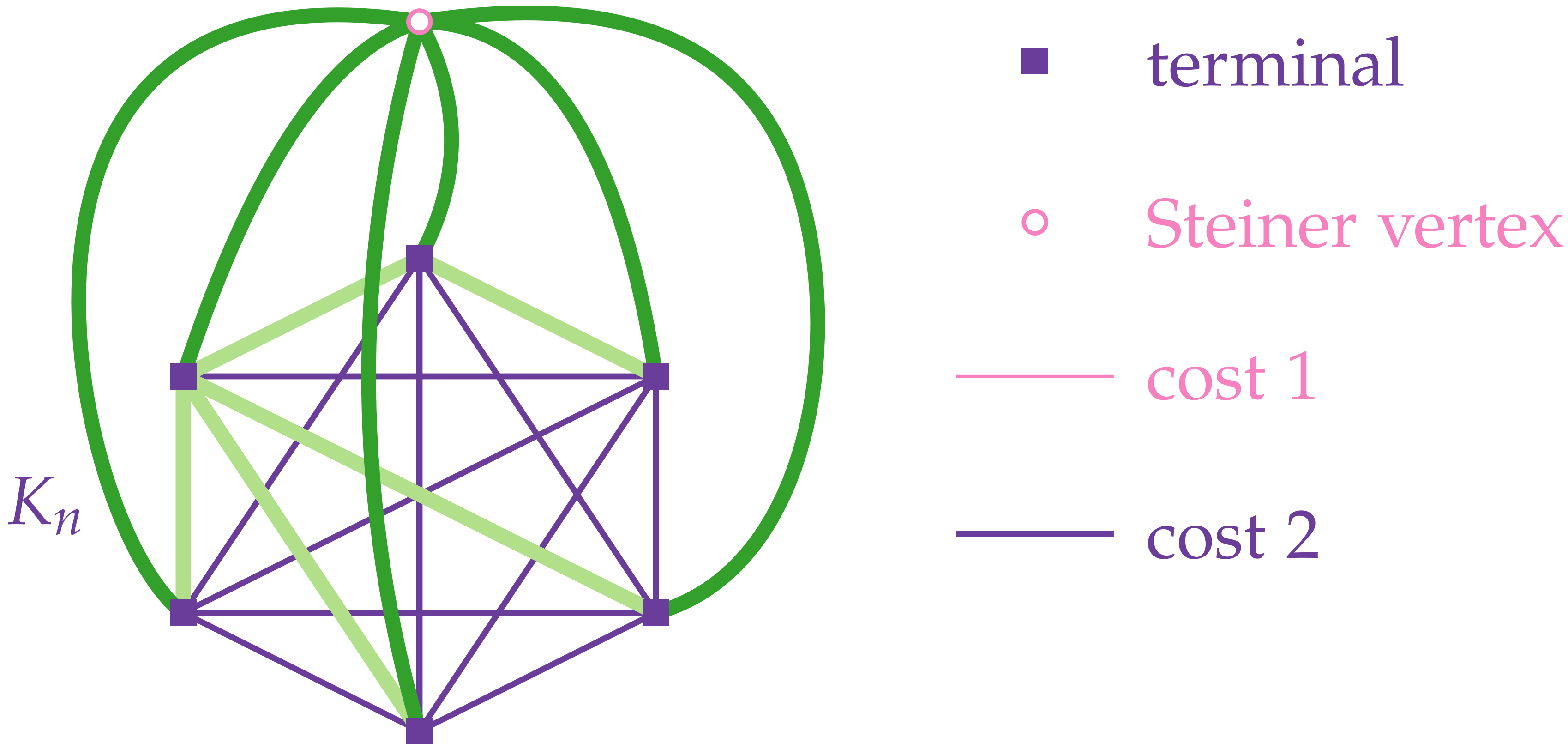

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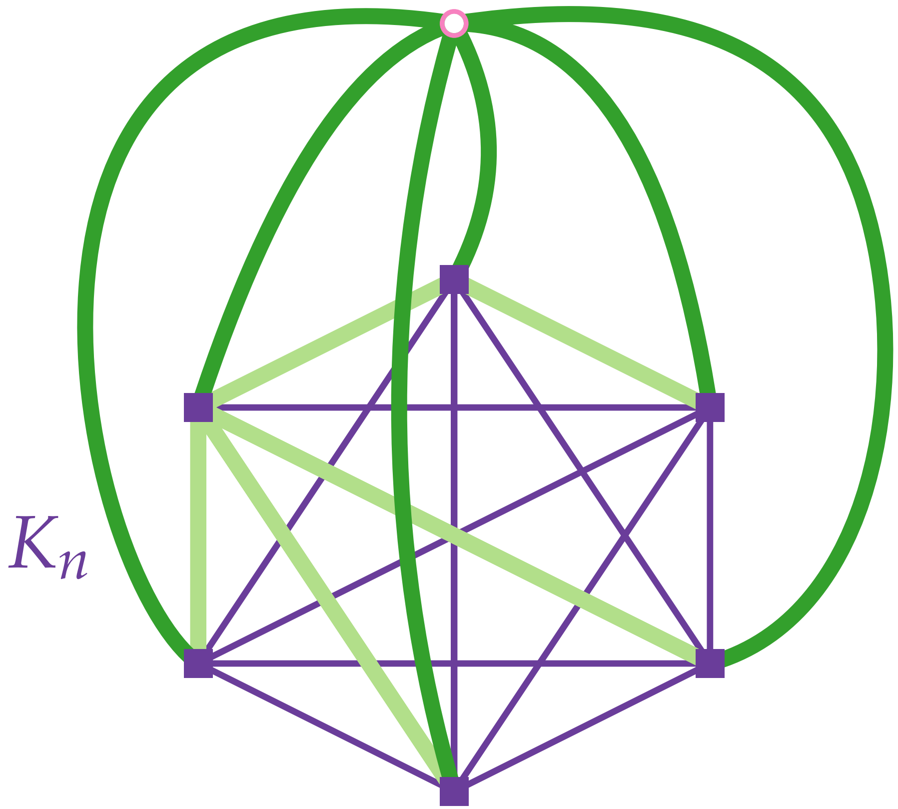
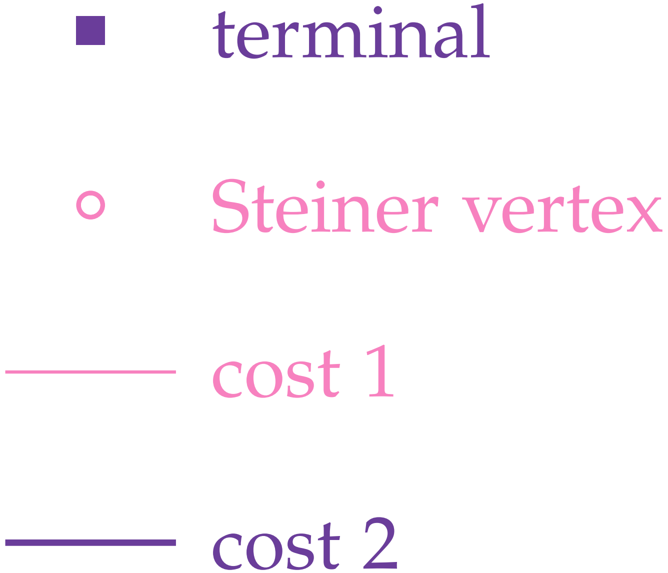
$$\frac{2(n-1)}{n} \to 2$$





Analysis Sharp?

MST of $G[T]$ with cost $2(n-1)$

Optimal solution with cost n

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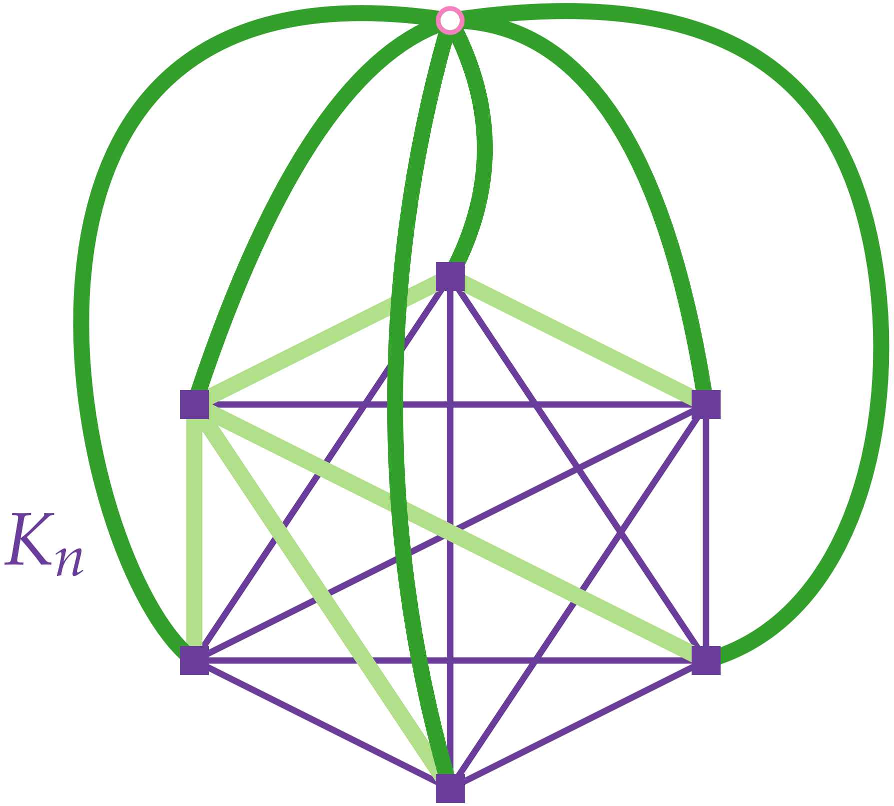

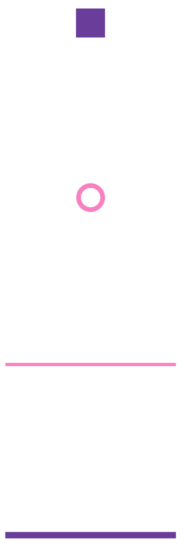

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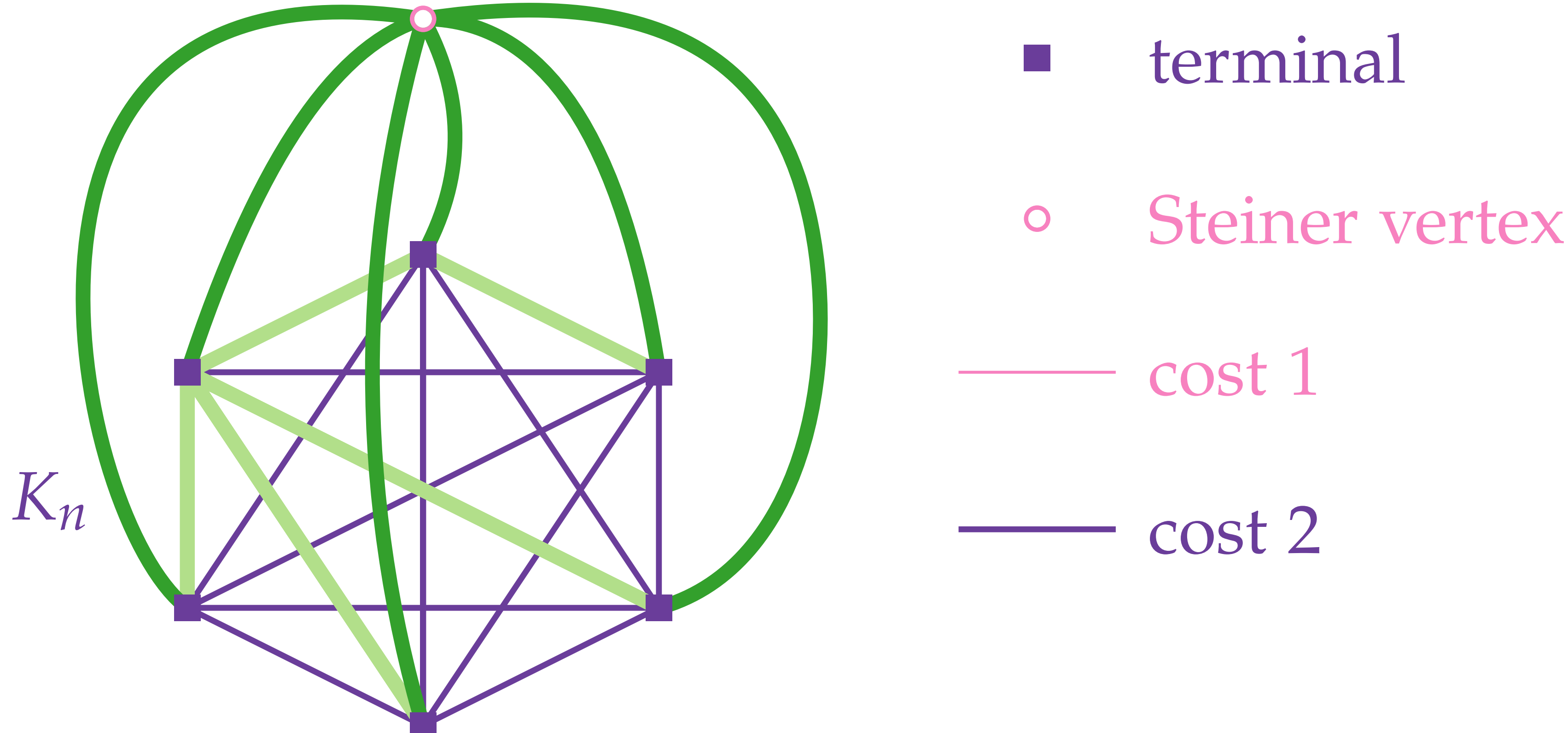

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The best-known approximation factor for SteinerTree is $\ln(4) + \varepsilon \approx 1.39$

[Byrka, Grandoni, Rothvoß & Sanita '10]

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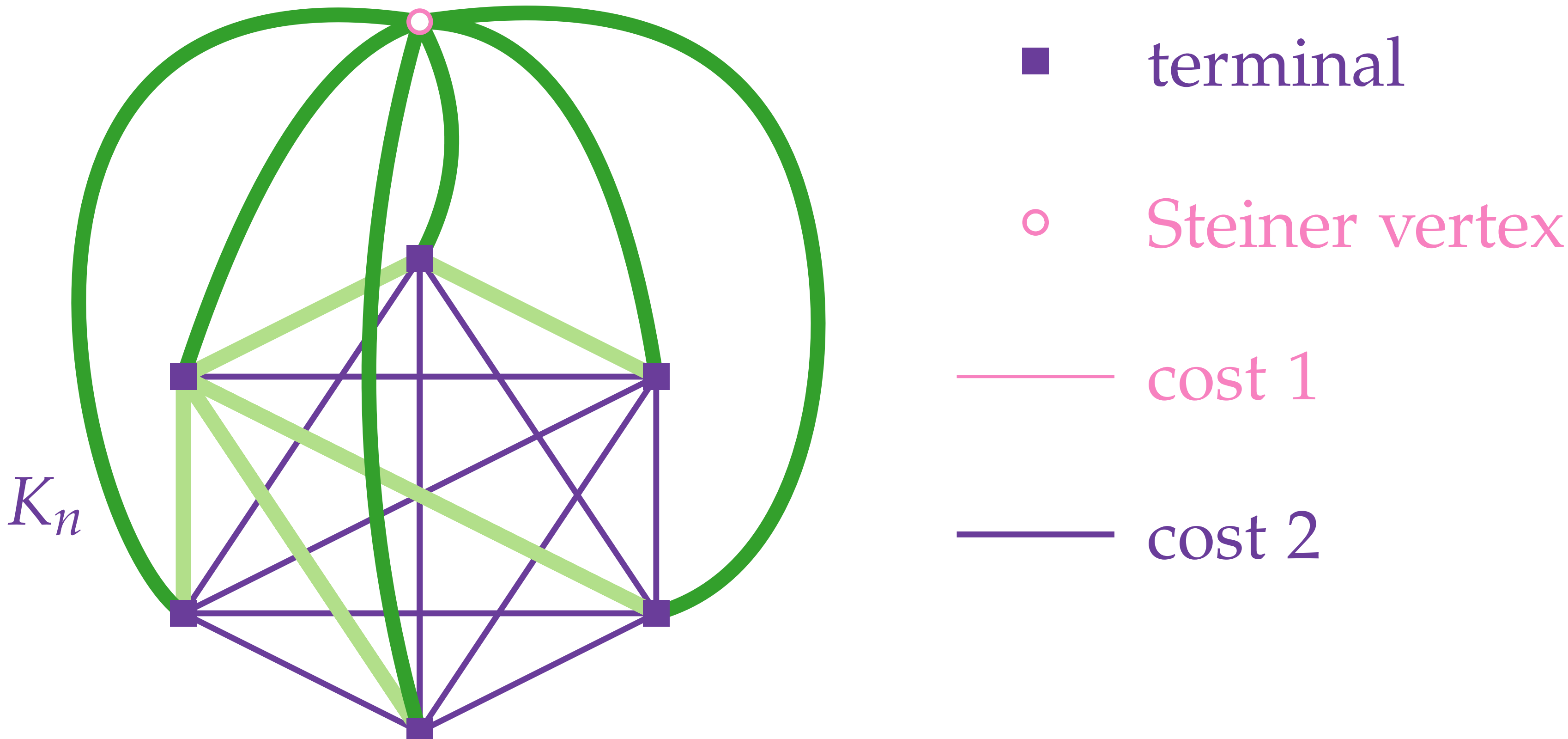

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SteinerTree cannot be approximated within factor $\frac{96}{95} \approx 1.0105$ (unless P=NP) [Chlebik & Chlebikova '08]

Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part V:
MultiwayCut

Joachim Spoerhase

Winter 2021/22

MultiwayCut

Given: A connected graph $G = (V, E)$

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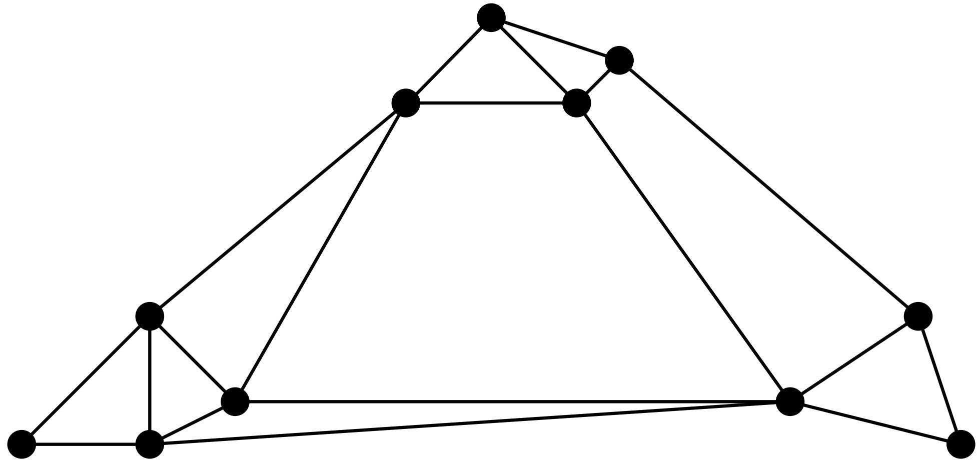

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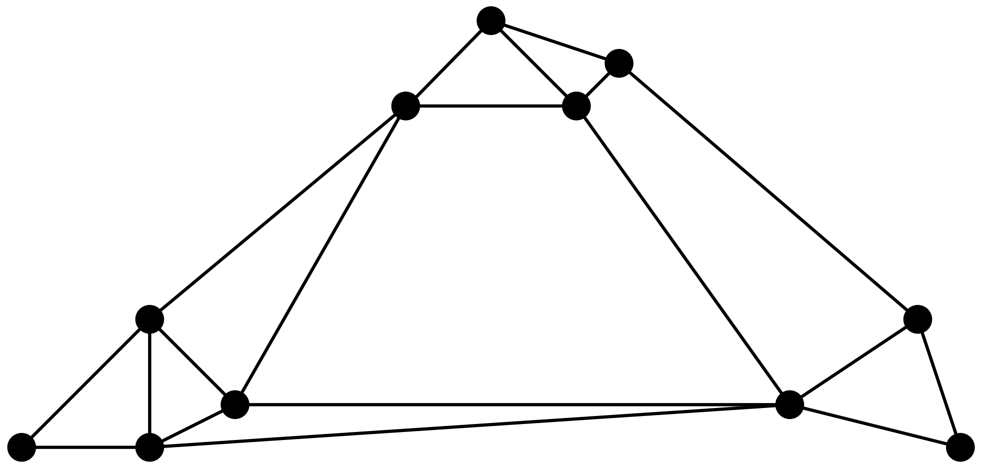

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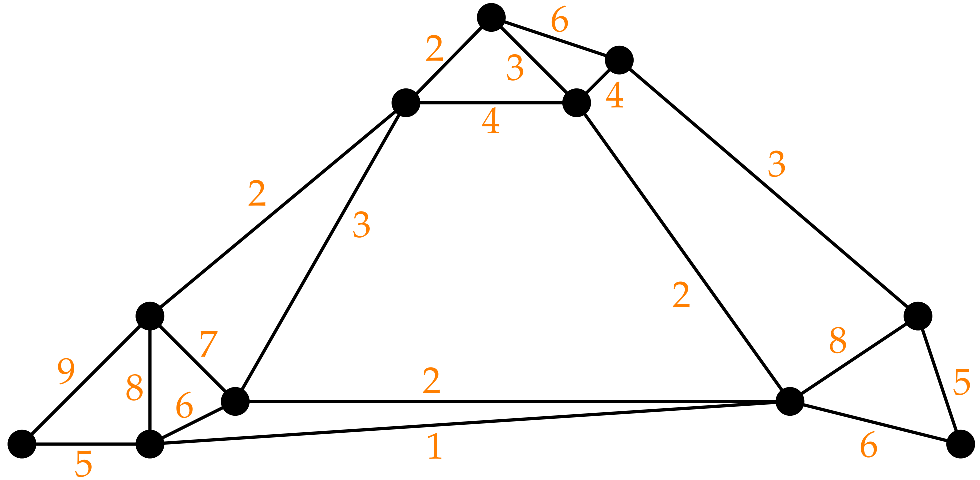

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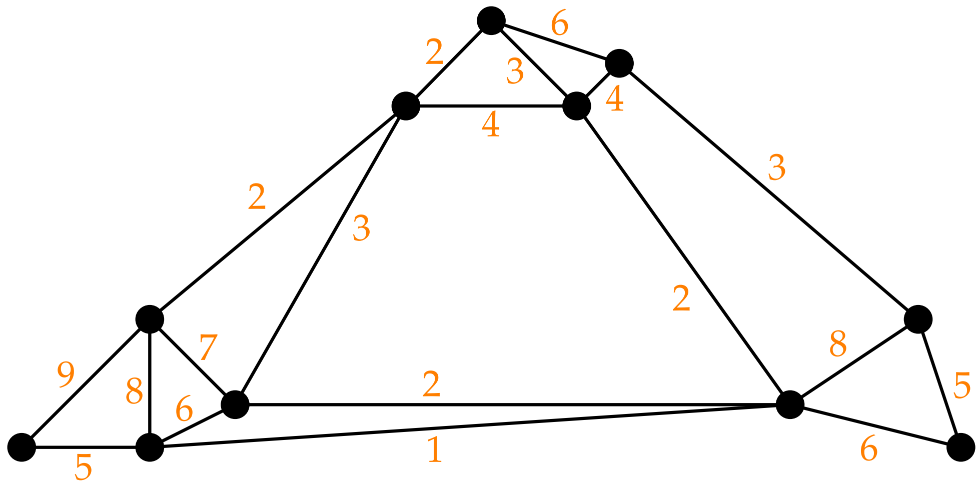

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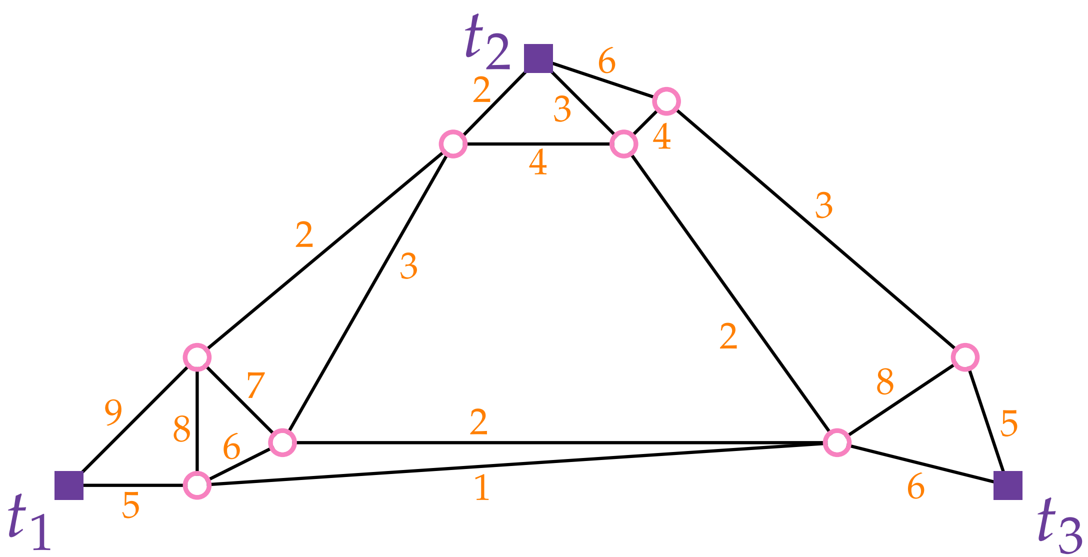

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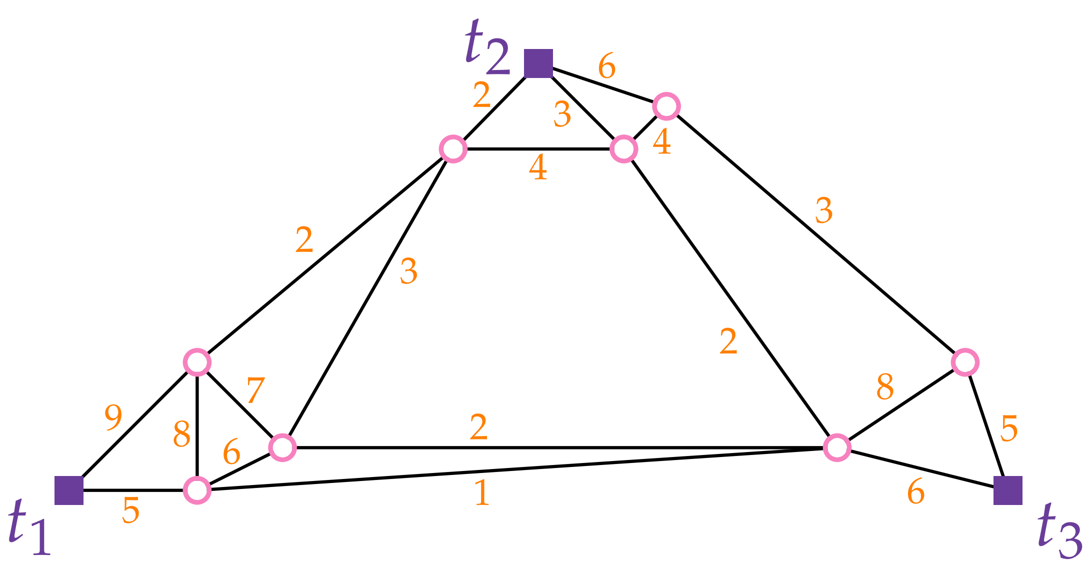

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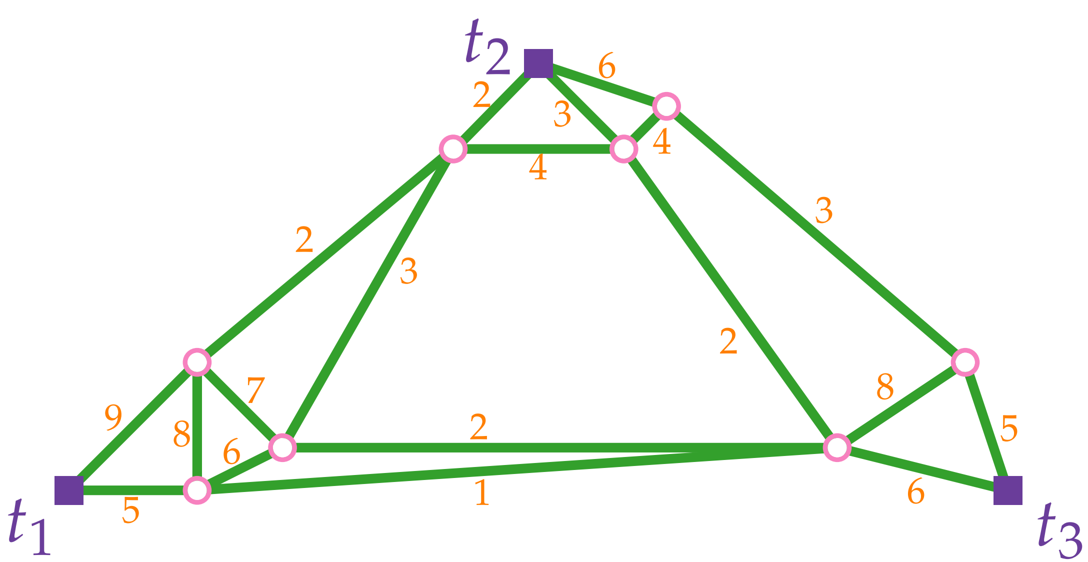

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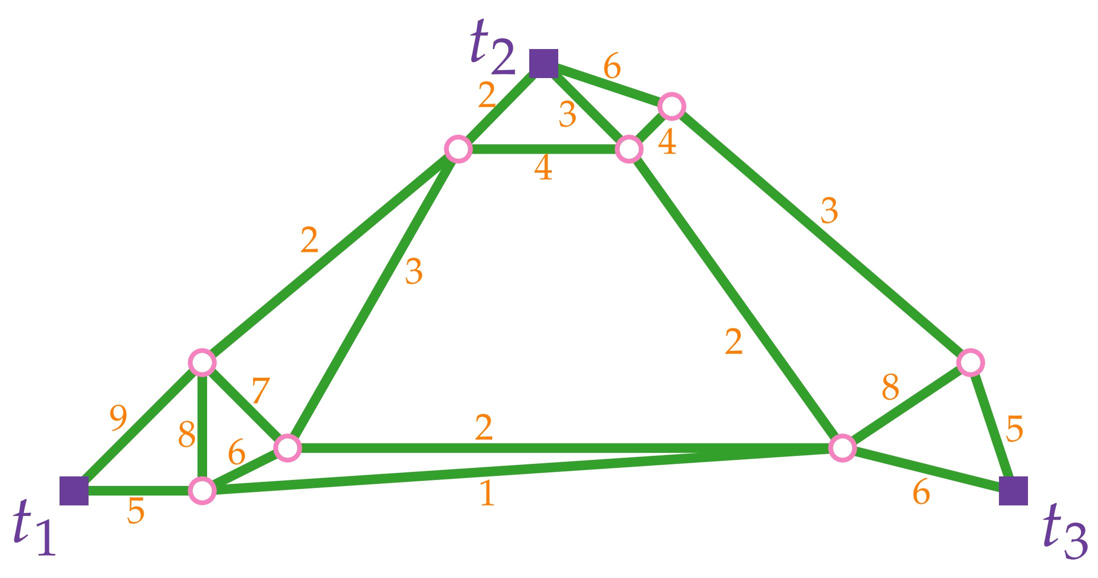

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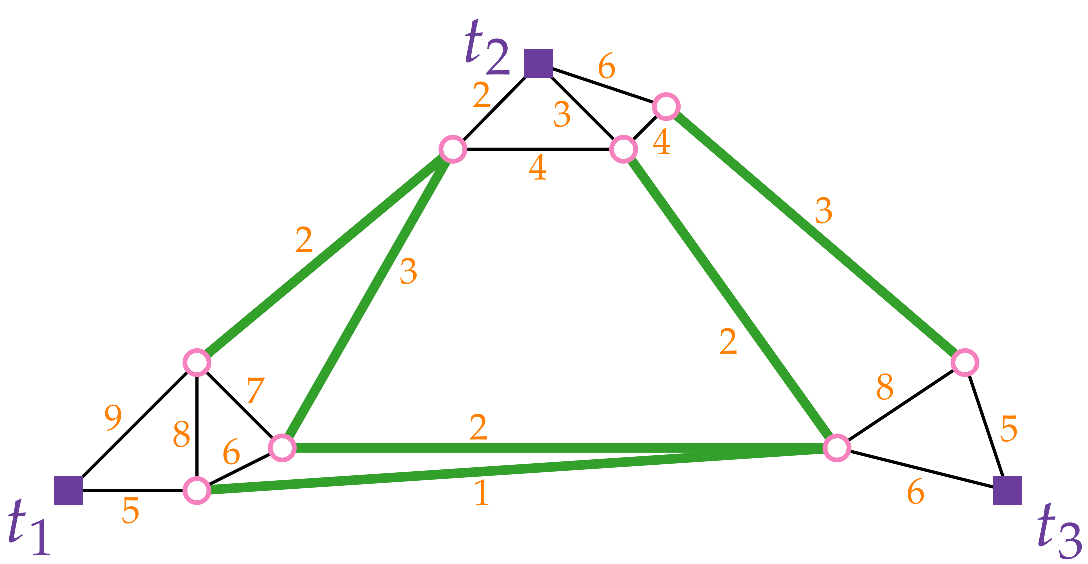

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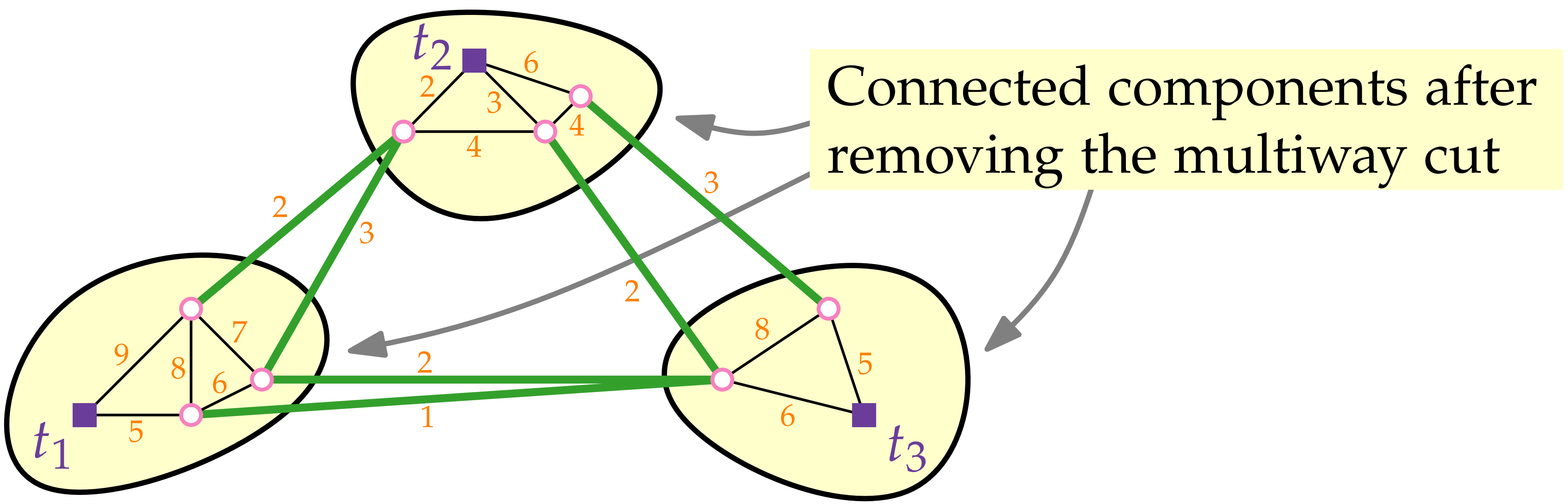

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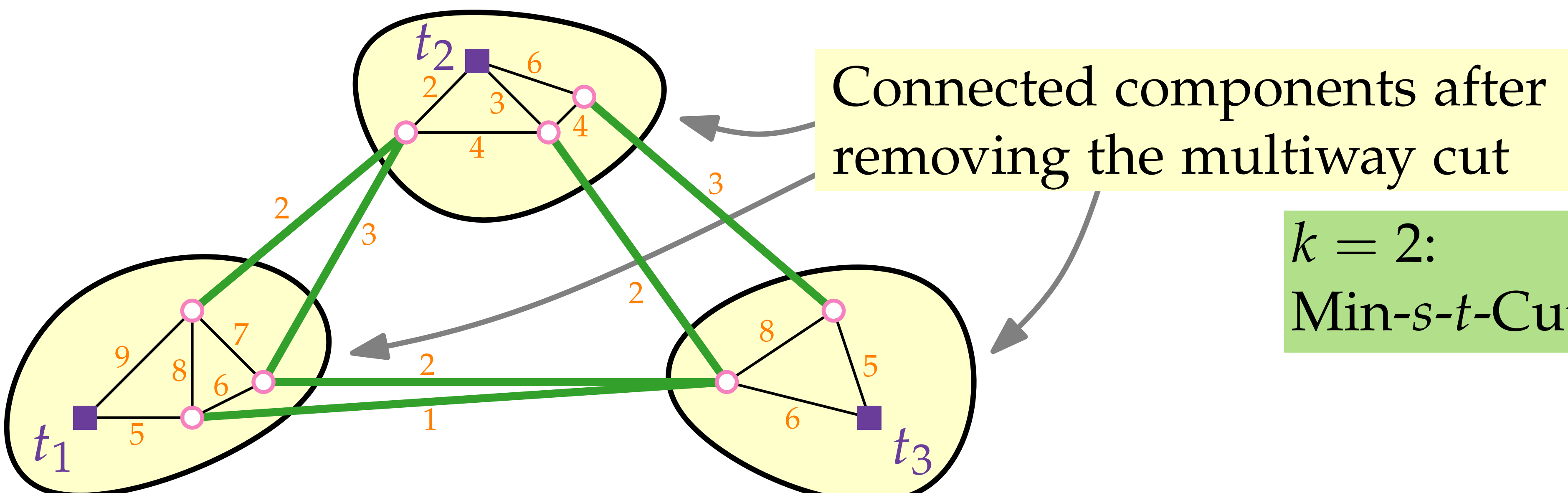

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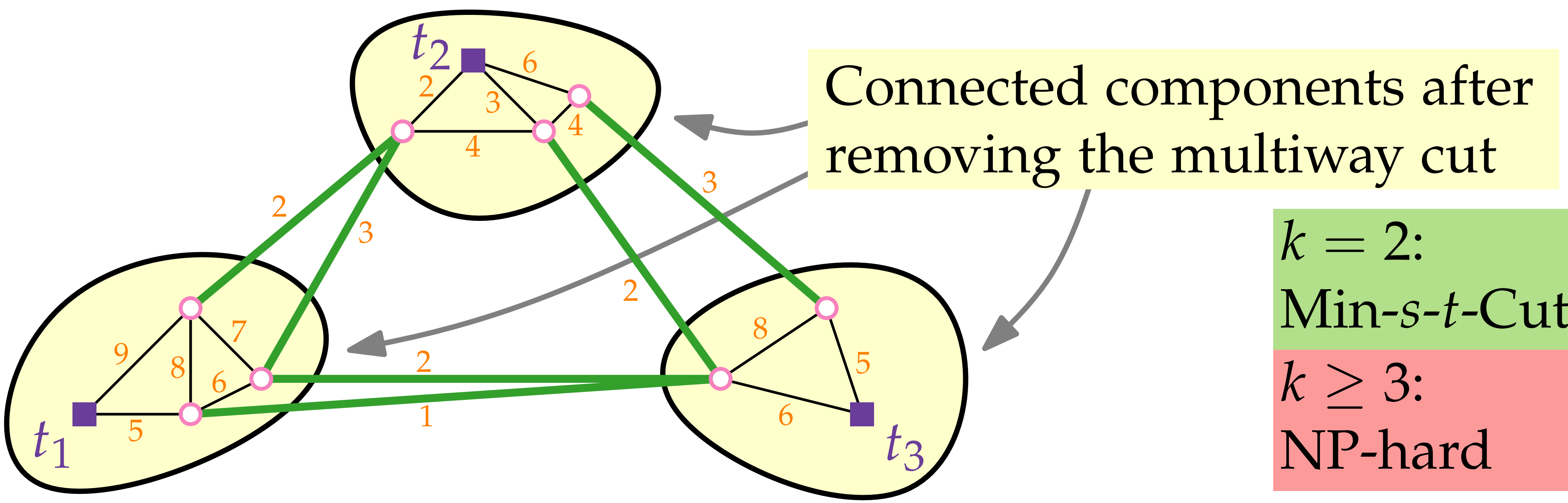

$k = 2$: Min-s-t-Cut

$k \geq 3$: NP-hard

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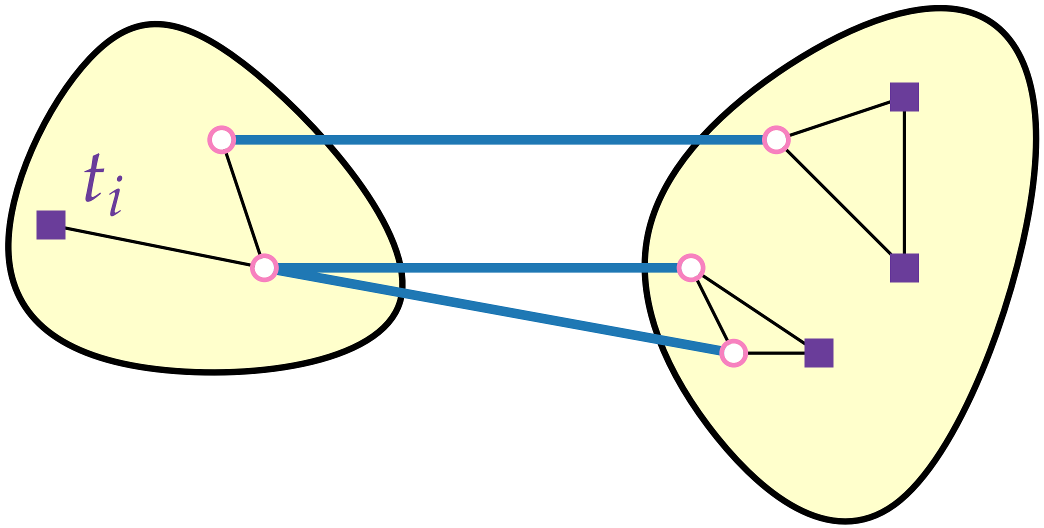

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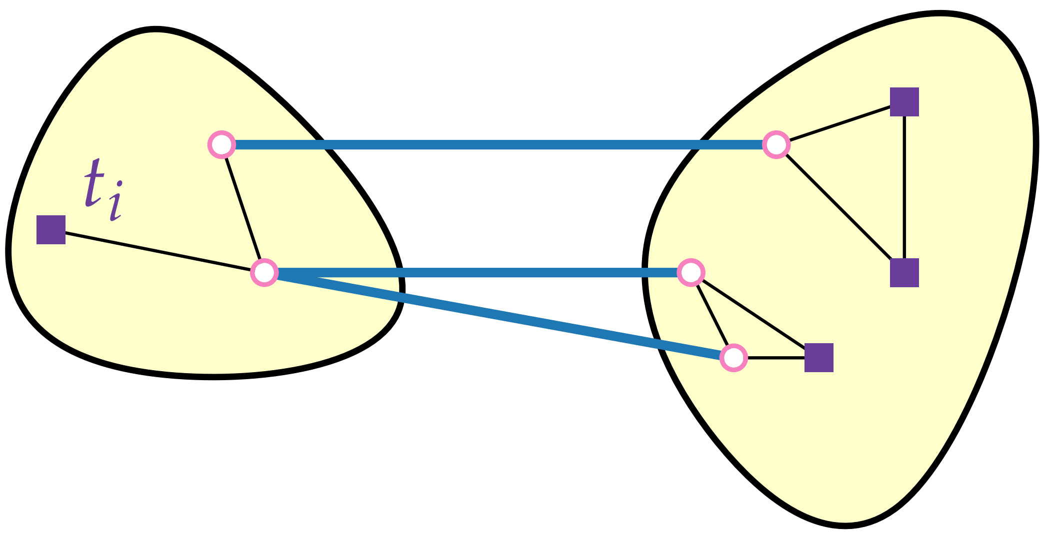

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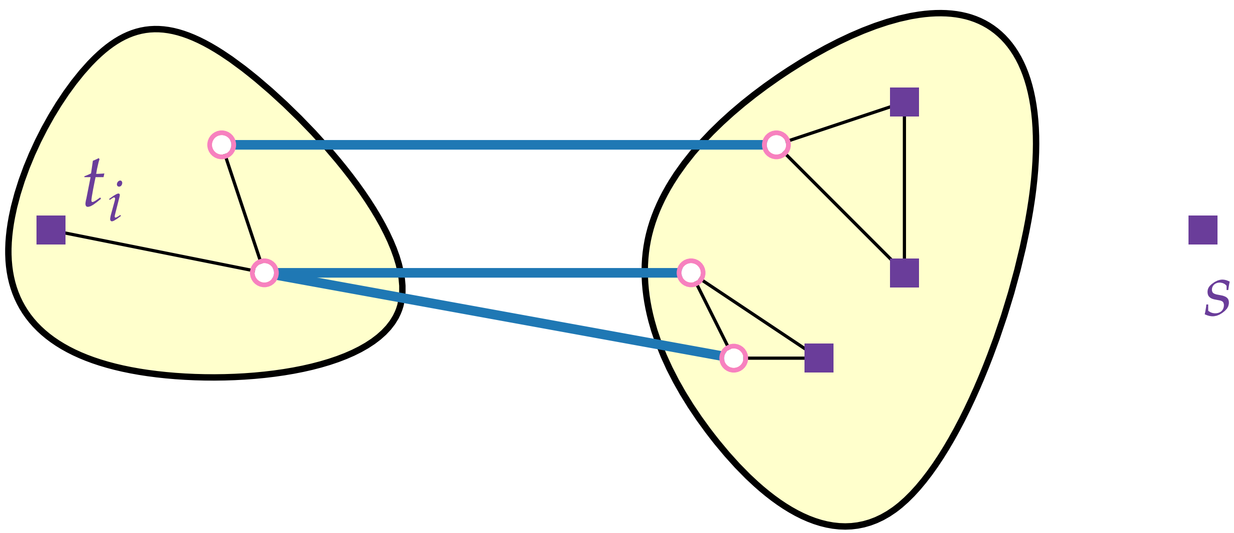

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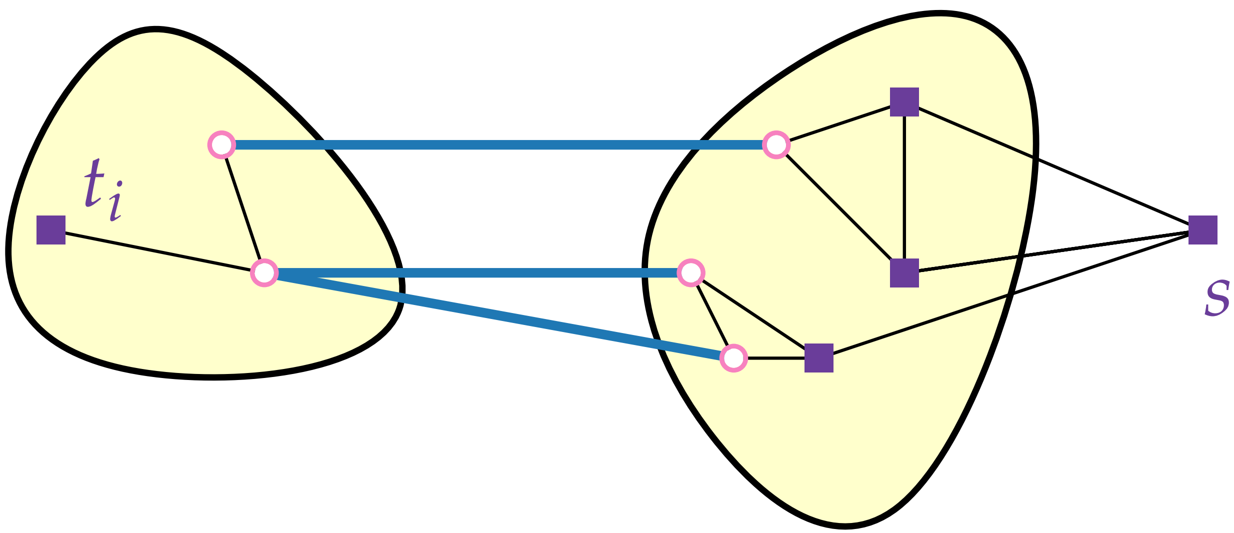

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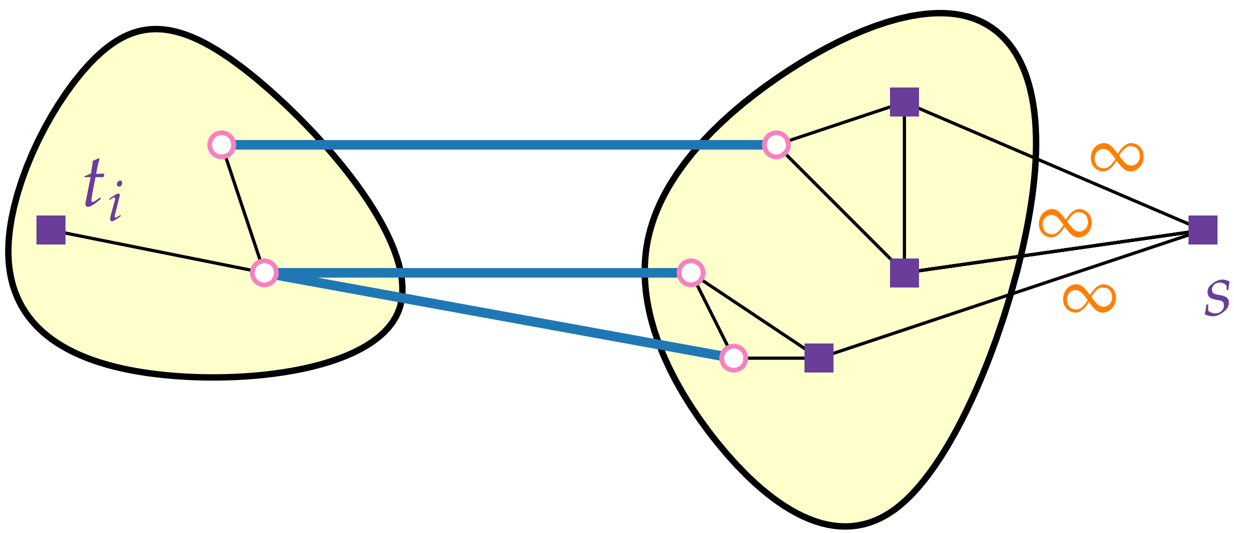

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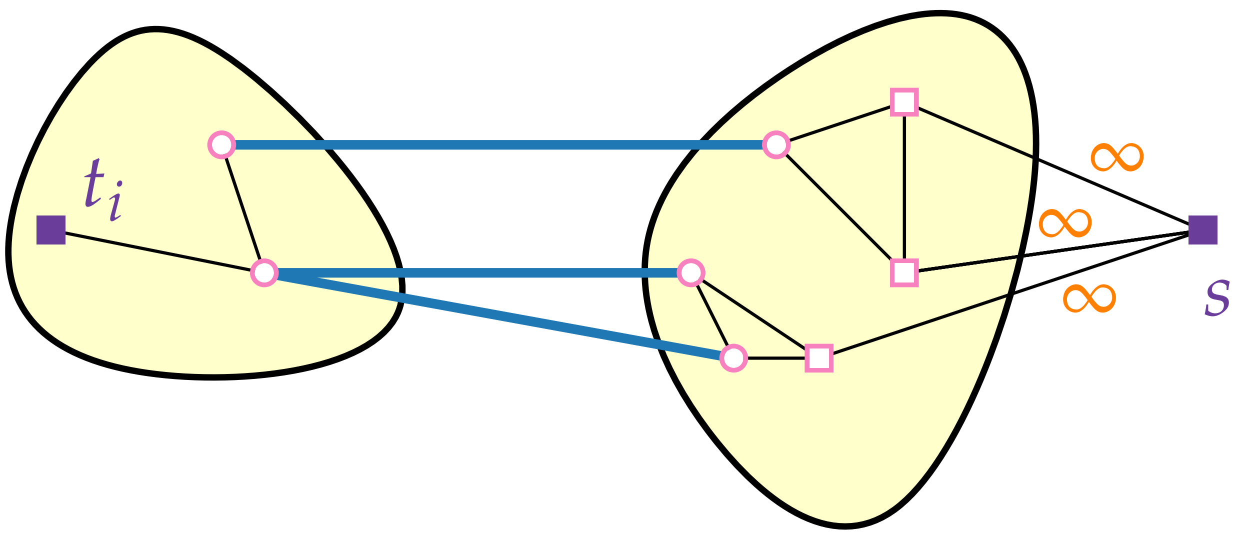

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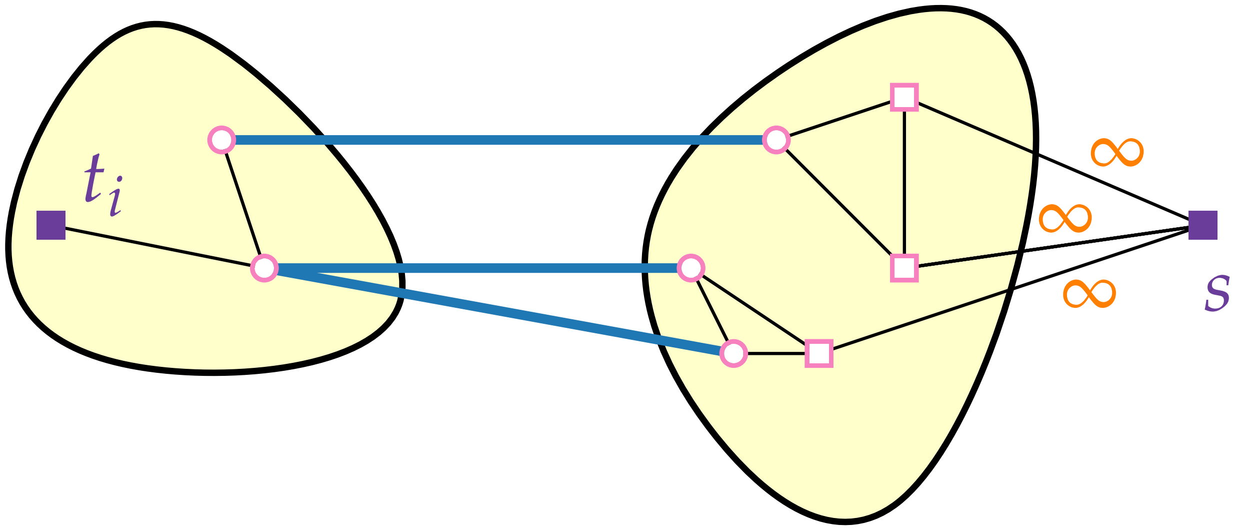

Add dummy terminal s and find minimum cost s-t_i-cut.

Approximation Algorithms

Lecture 3:
SteinerTree and MultiwayCut

Part VI:
Algorithm for MultiwayCut

Joachim Spoerhase

Winter 2021/22

Algorithm MultiwayCut

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Compute a minimum cost isolating cut C_i for t_i.

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Theorem. This algorithm is a factor-( )-approximation algorithm for MultiwayCut.

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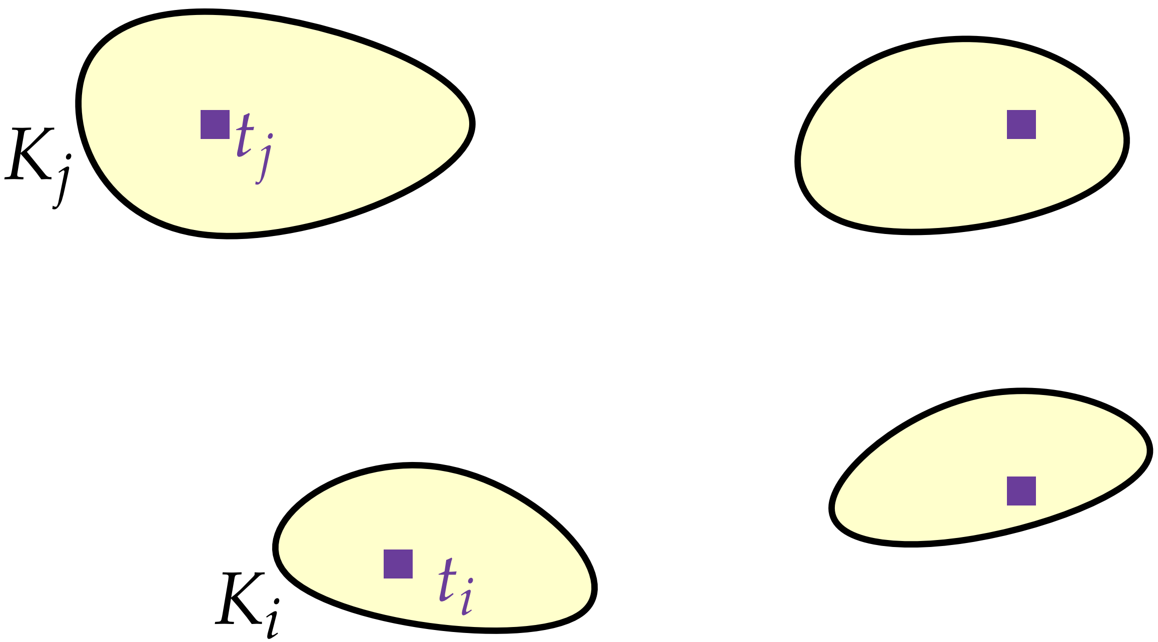

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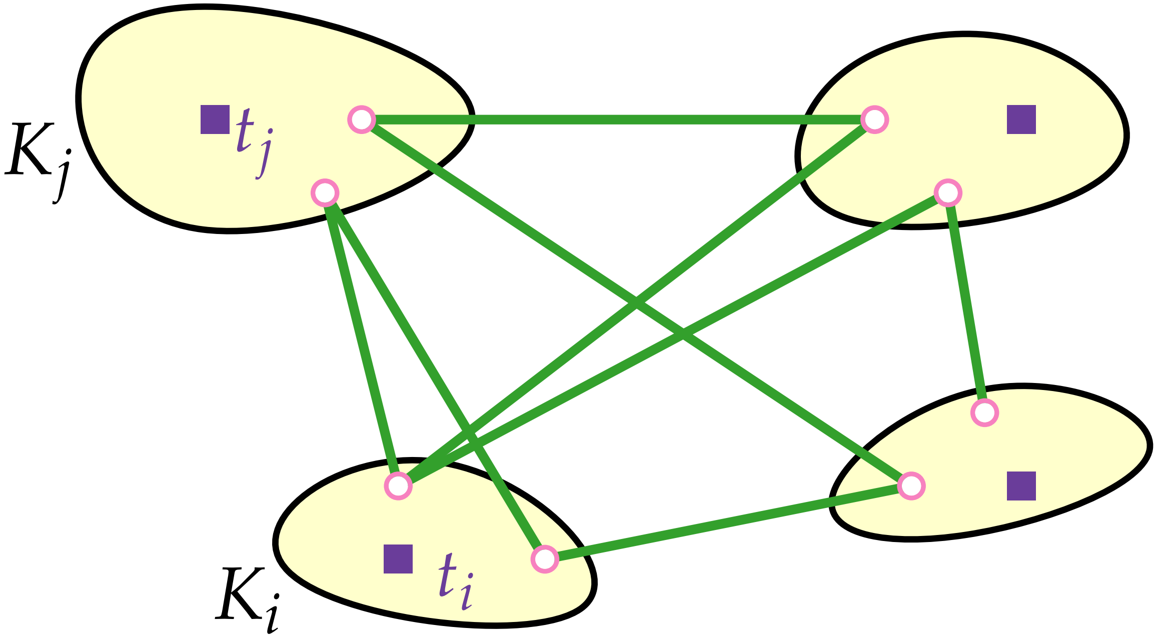

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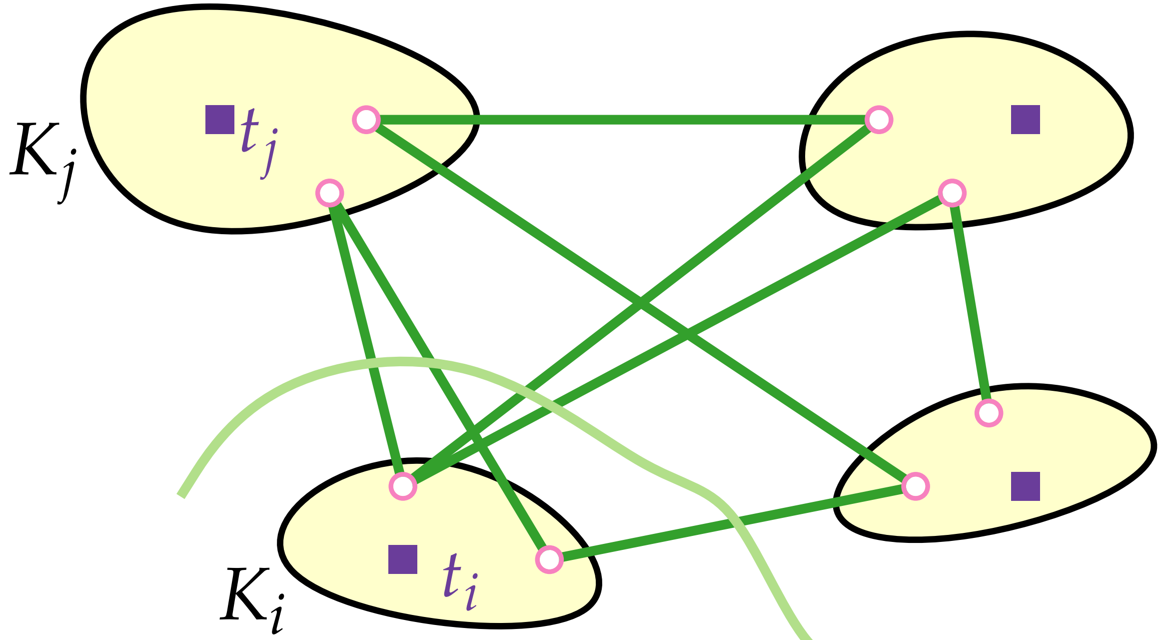

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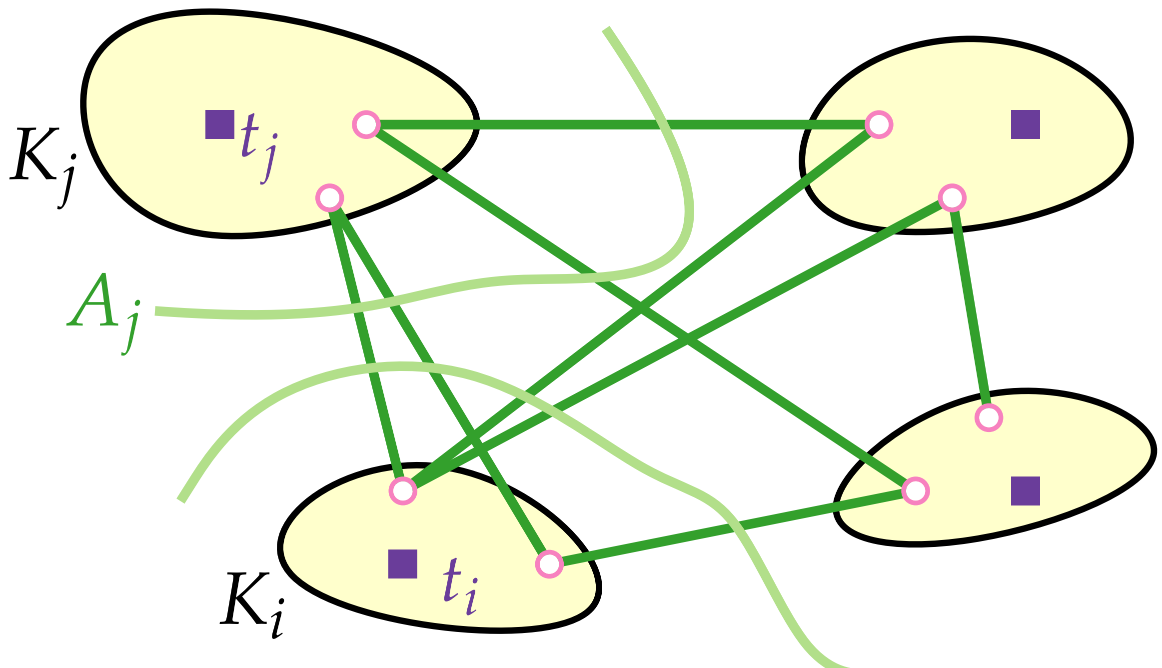

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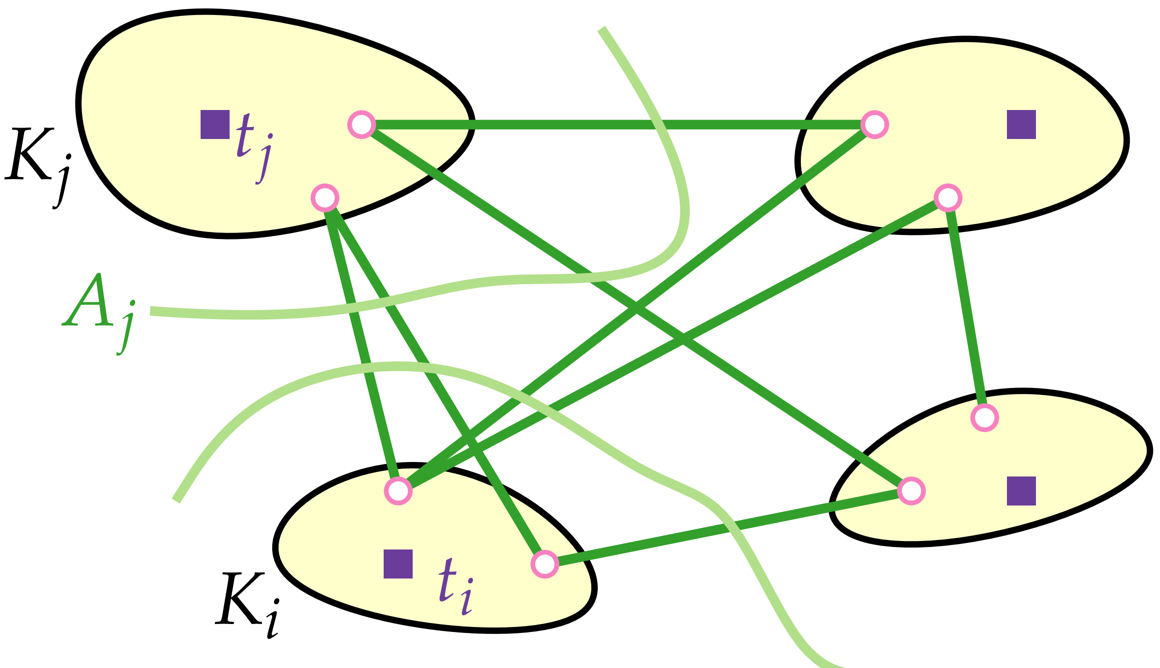

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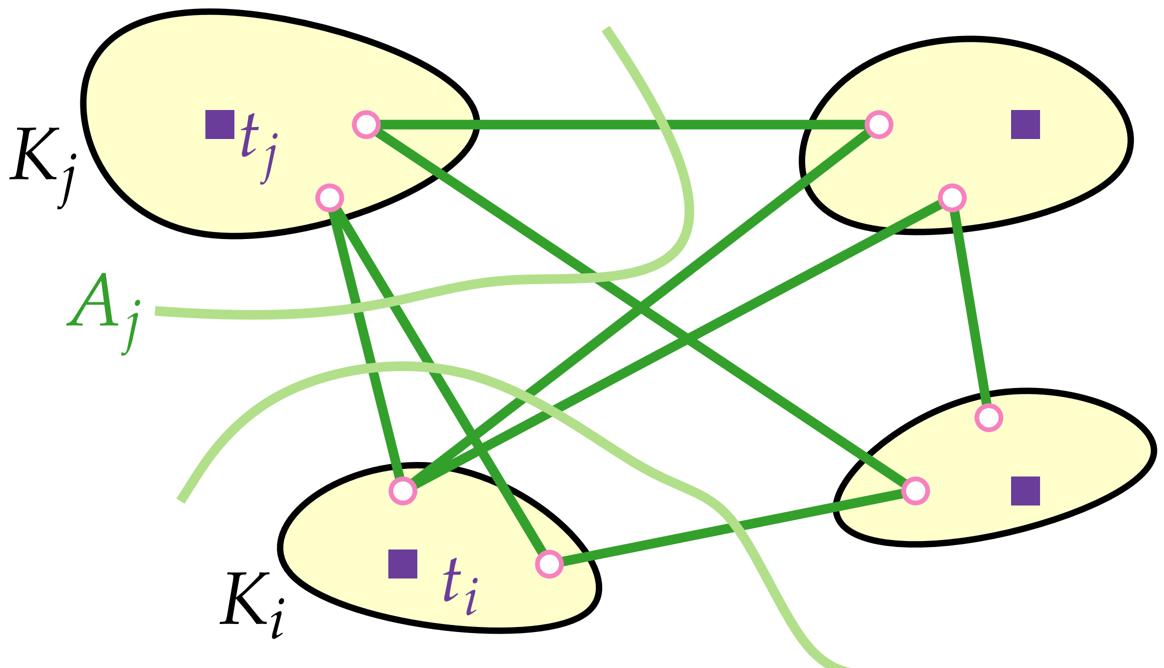

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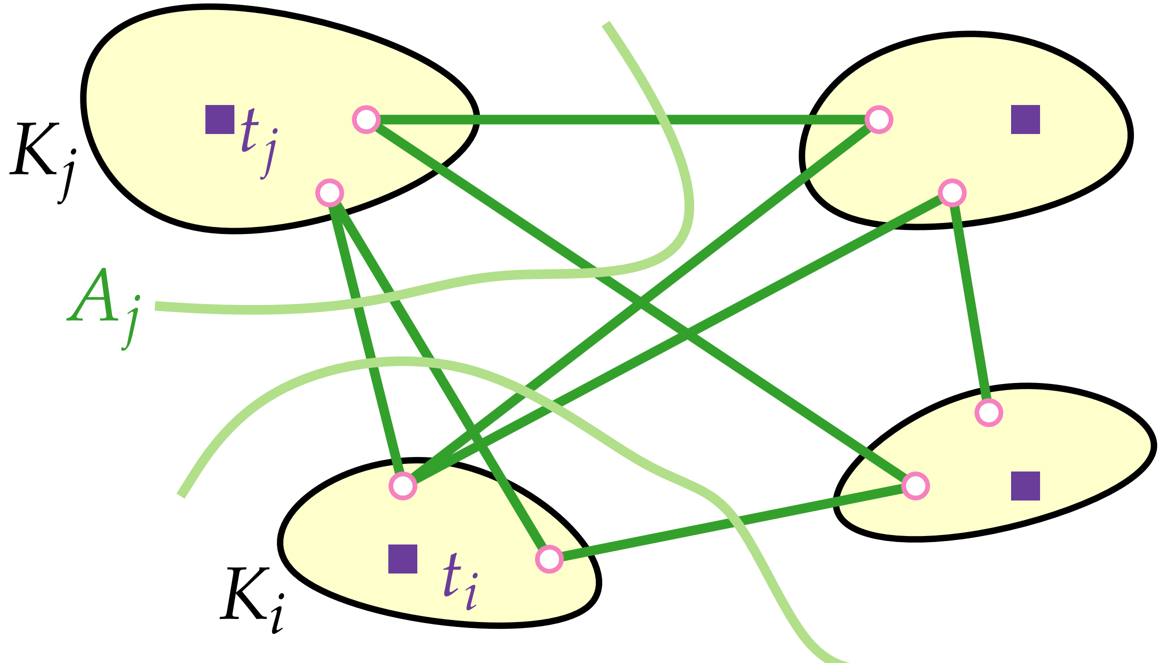

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Observation. $A = \bigcup_{i=1}^{k} A_i$ and $\sum_{i=1}^{k} c(A_i) \leq 2 \cdot c(A) = 2 \cdot \mathrm{OPT}$

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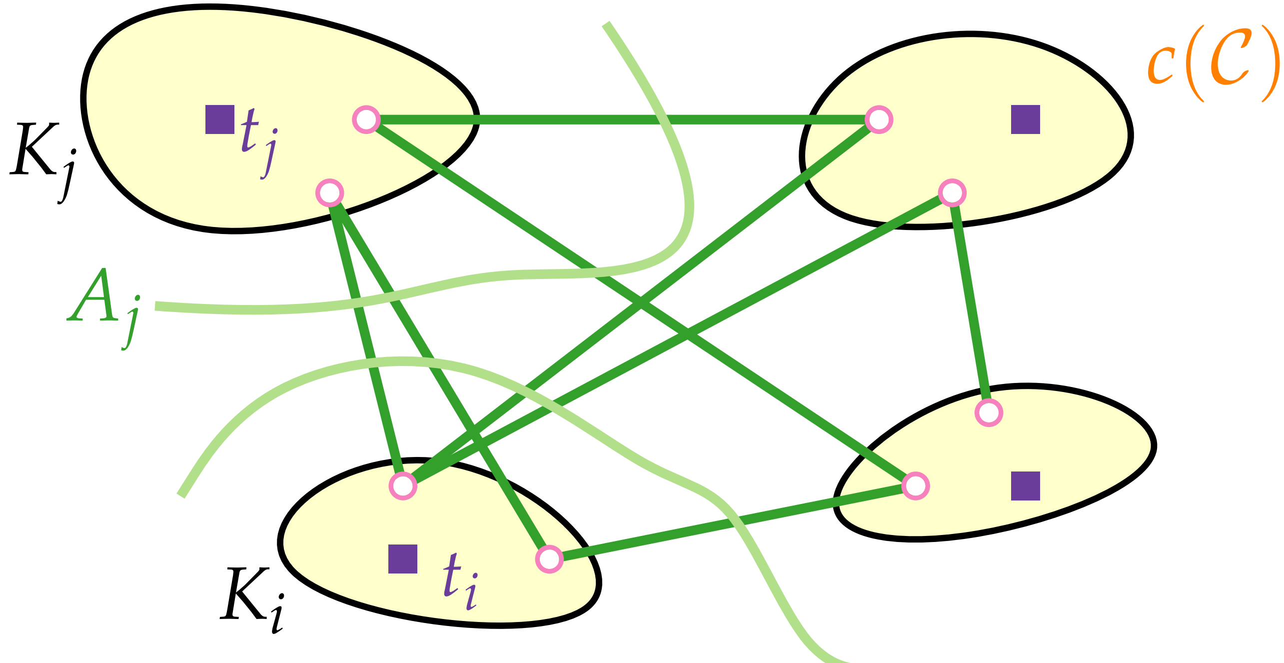

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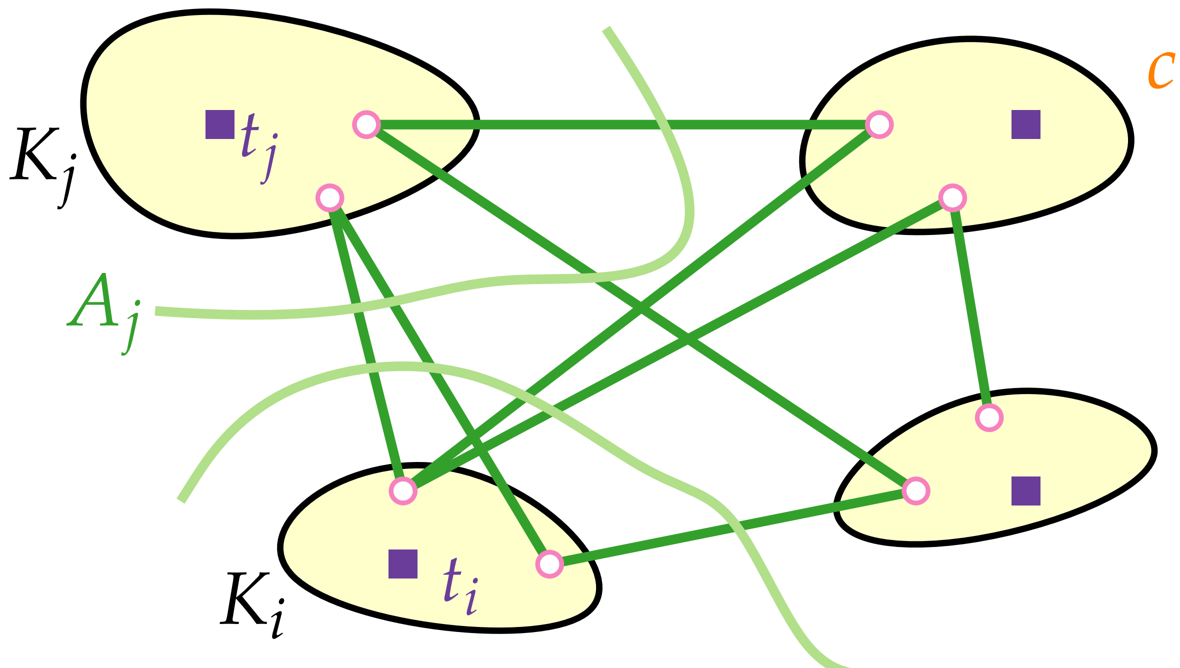

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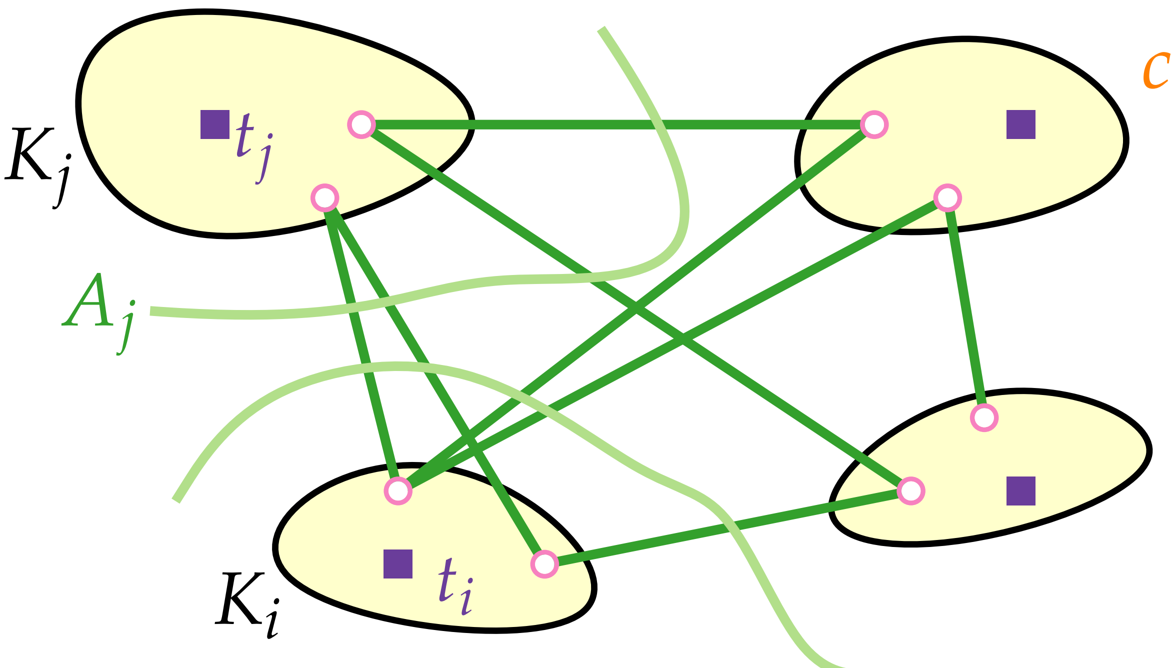

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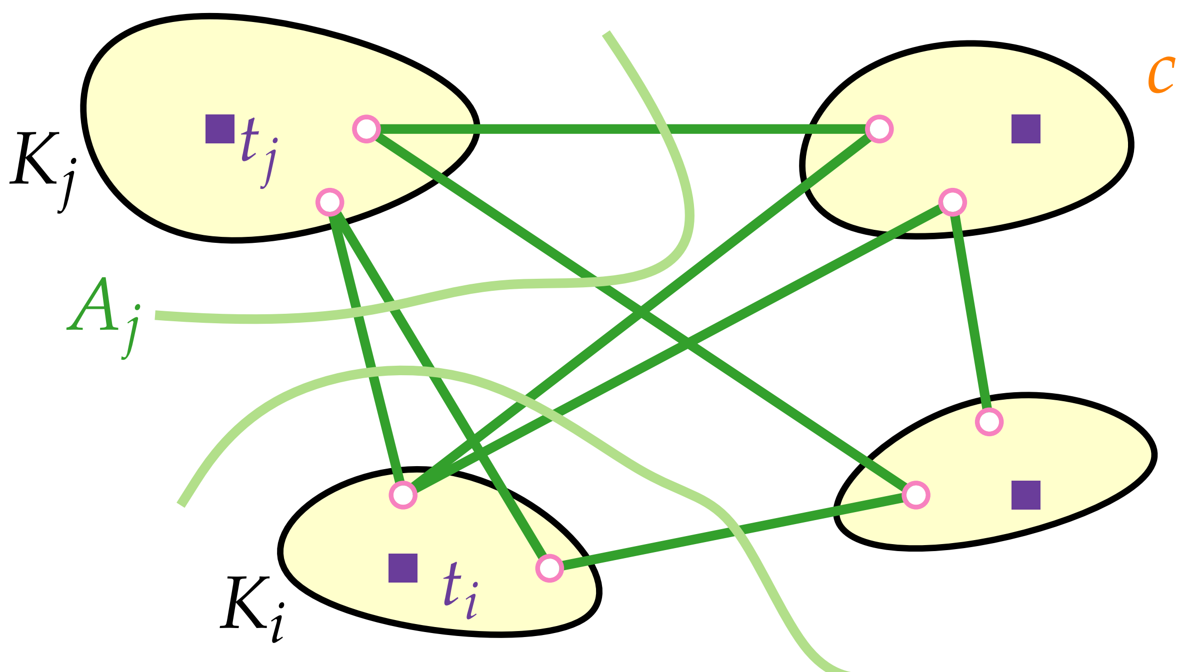

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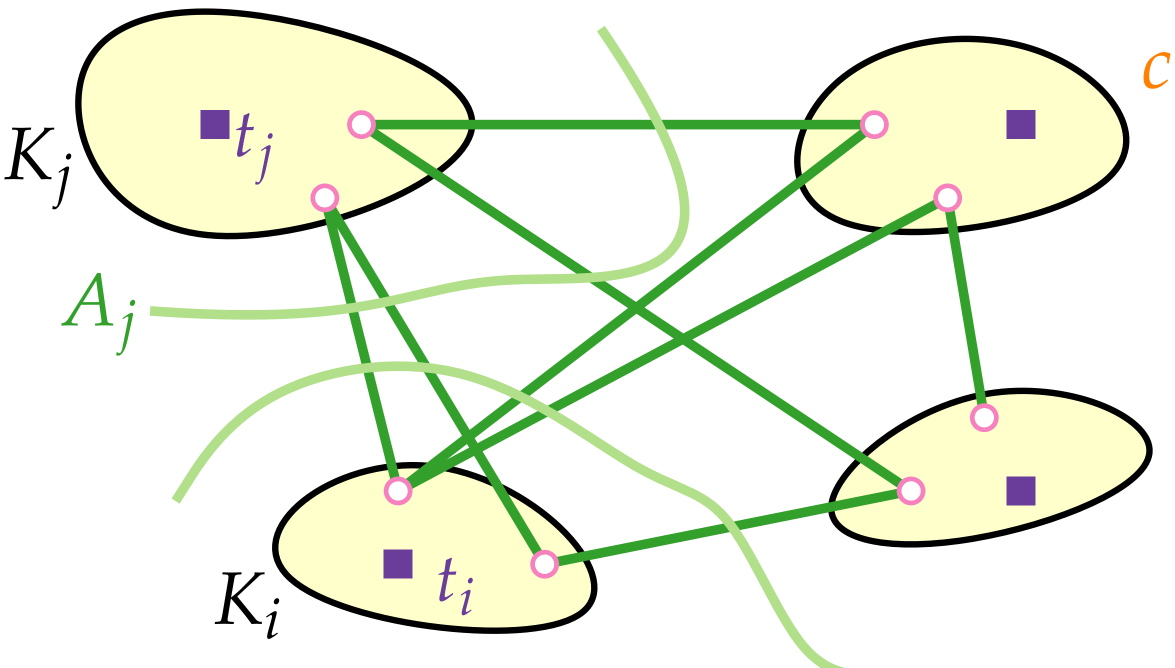

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$$\leq \left(2 - \frac{2}{k}\right) \mathrm{OPT}$$

$$A_i = \{uv \in A : u \in K_i, v \notin K_i\}$$

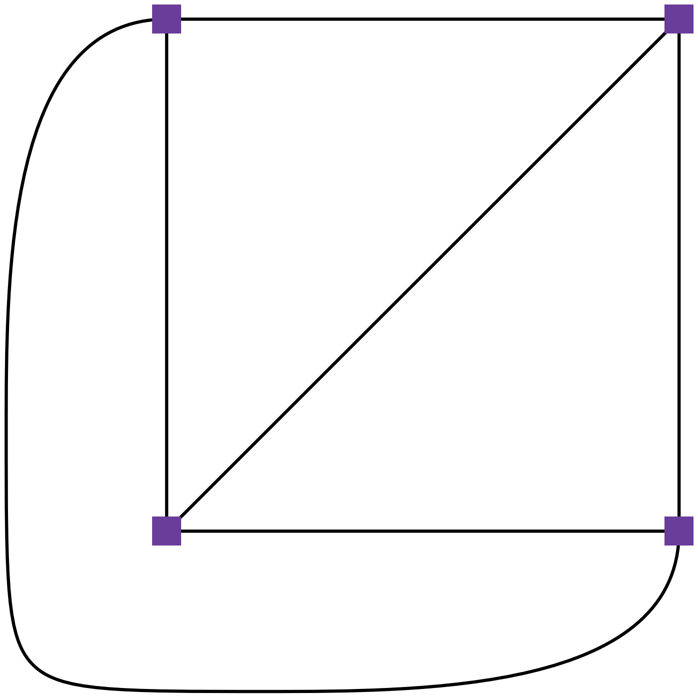
Observation. $A = \bigcup_{i=1}^{k} A_i$ and $\sum_{i=1}^{k} c(A_i) \leq 2 \cdot c(A) = 2 \cdot \mathrm{OPT}$

Analysis Sharp?

K_k

Analysis Sharp?

K_k



Analysis Sharp?

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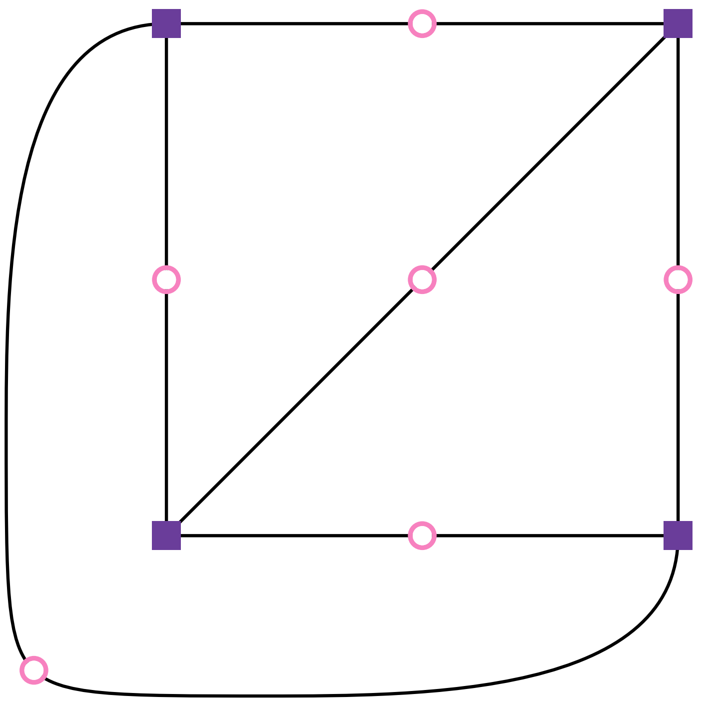

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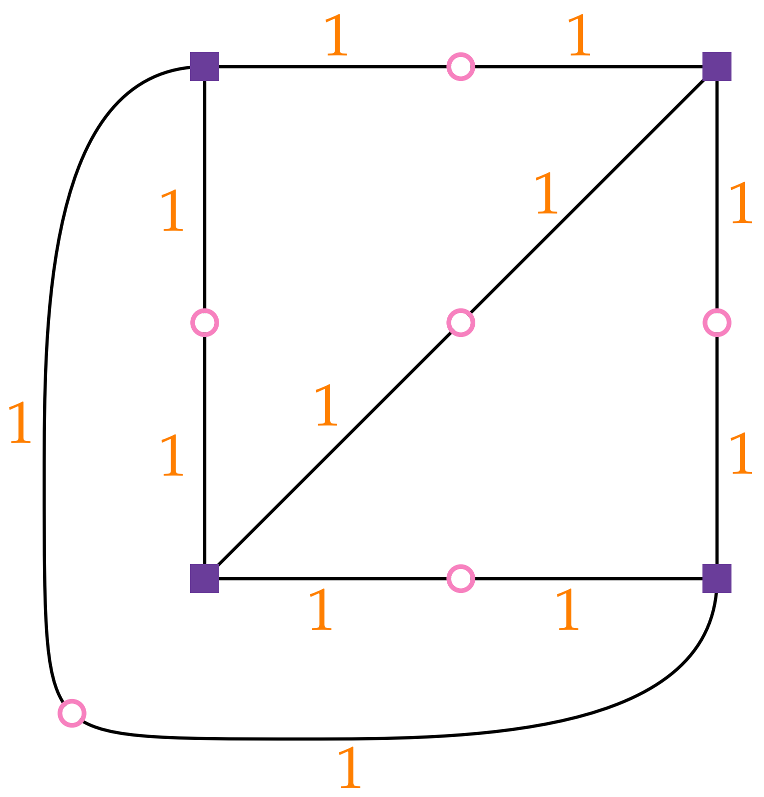

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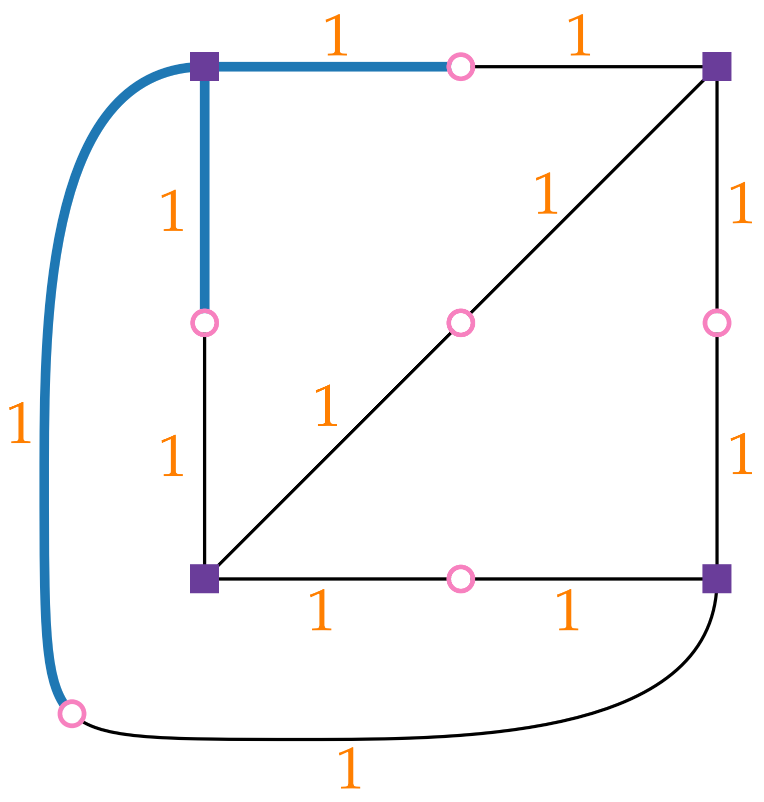

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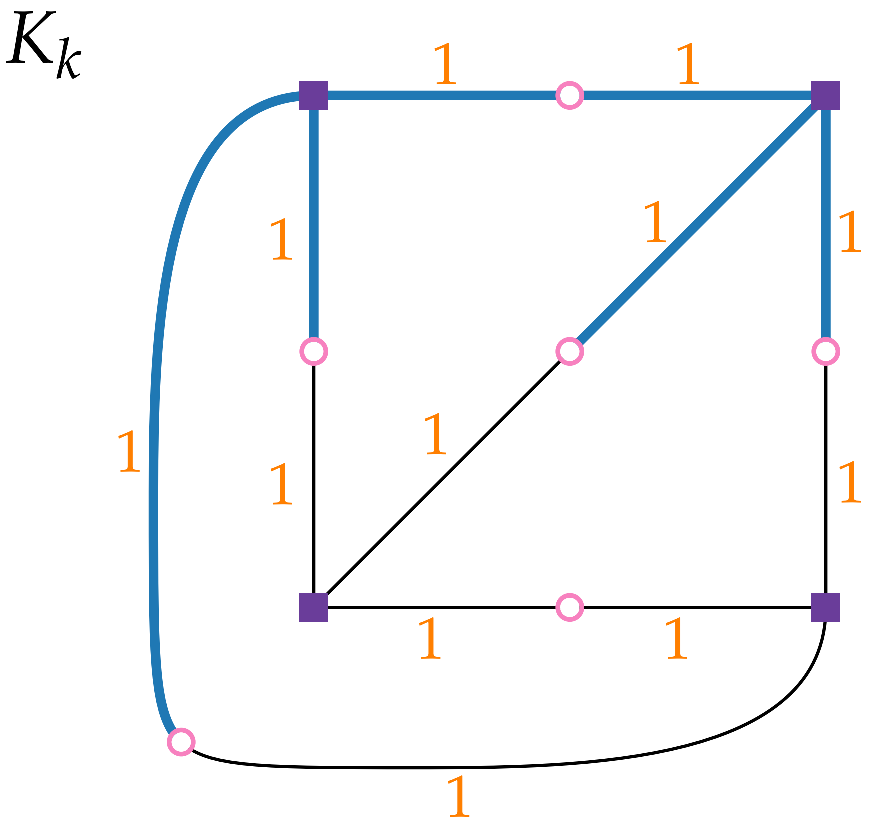

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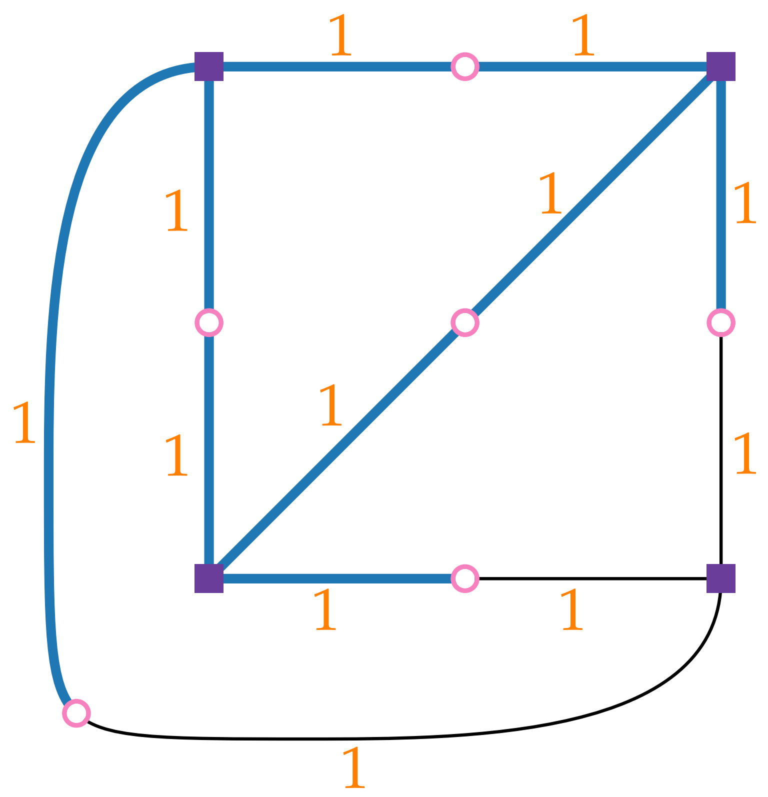

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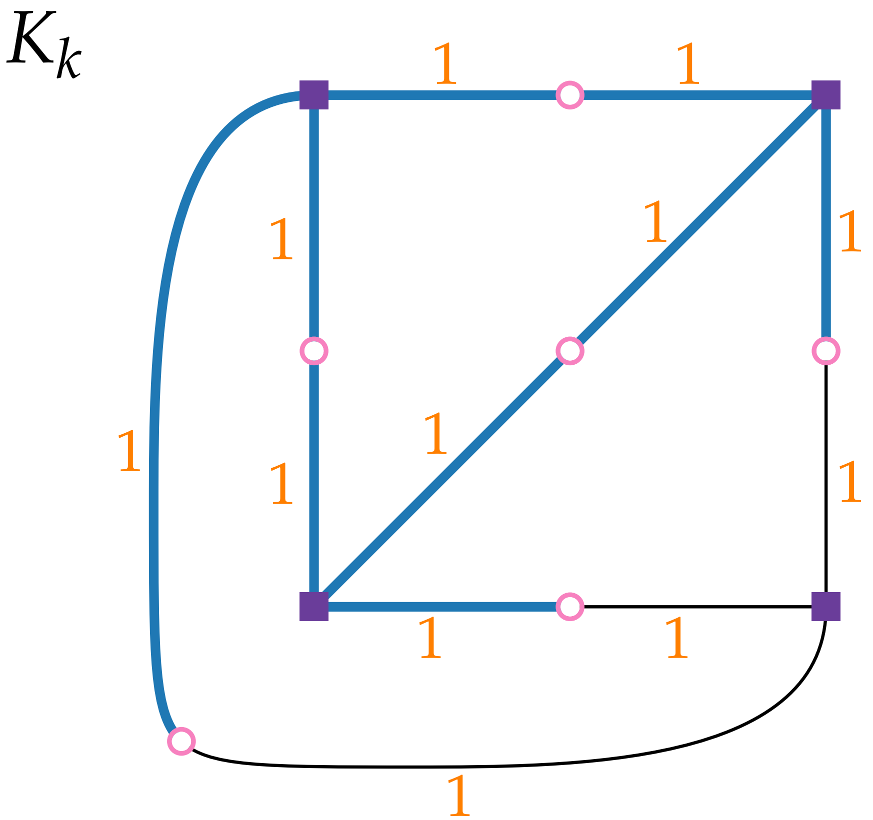
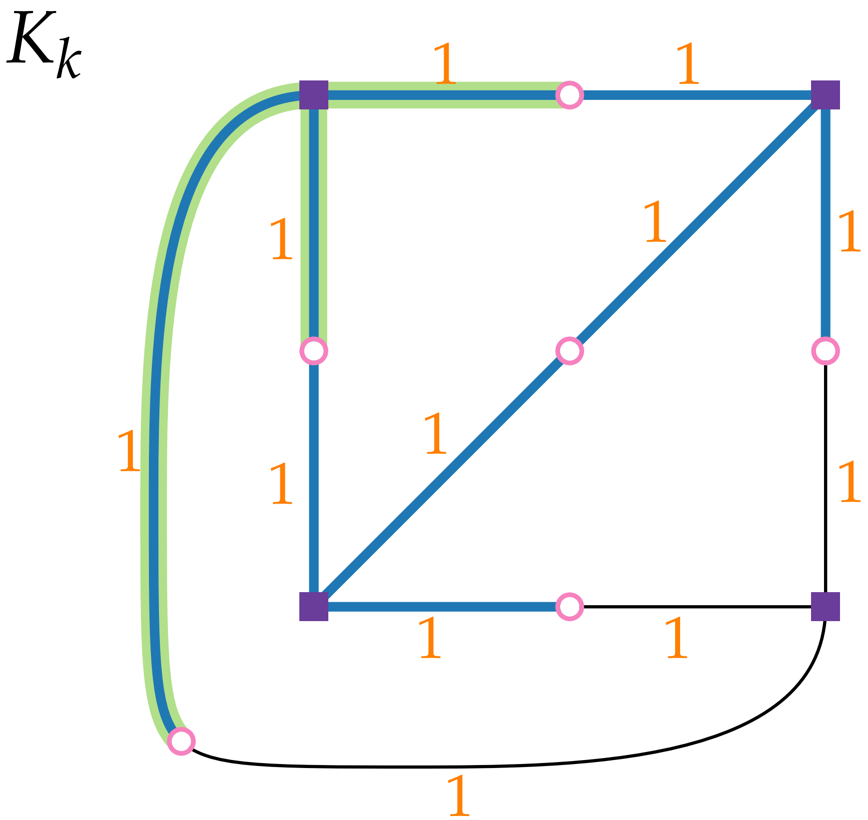


$\text{ALG} = (k-1)(k-1)$

Analysis Sharp?



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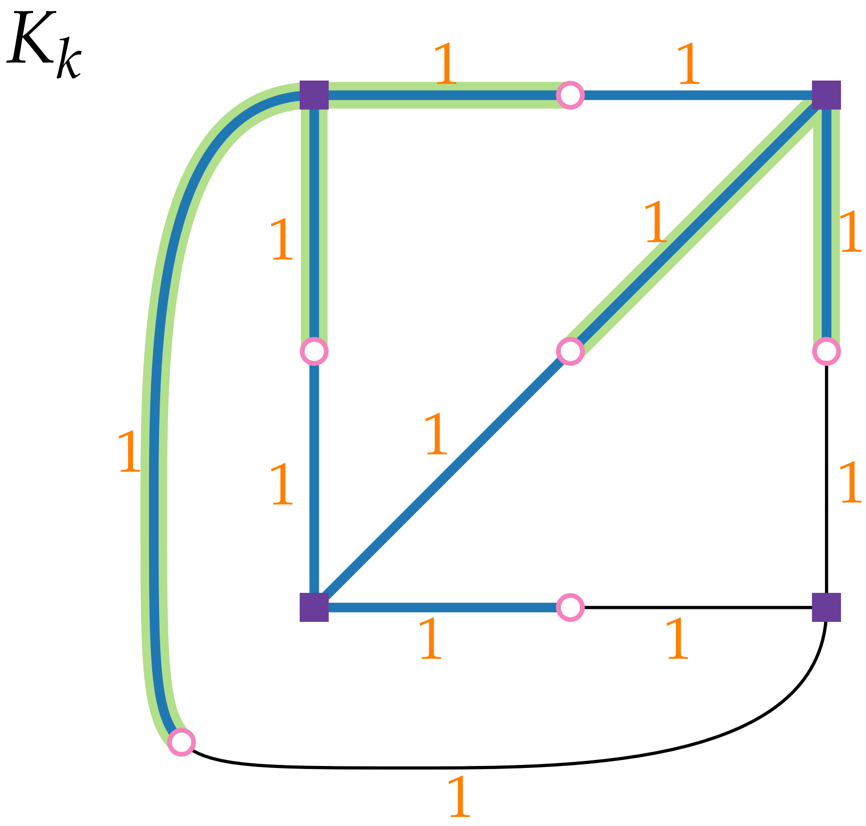

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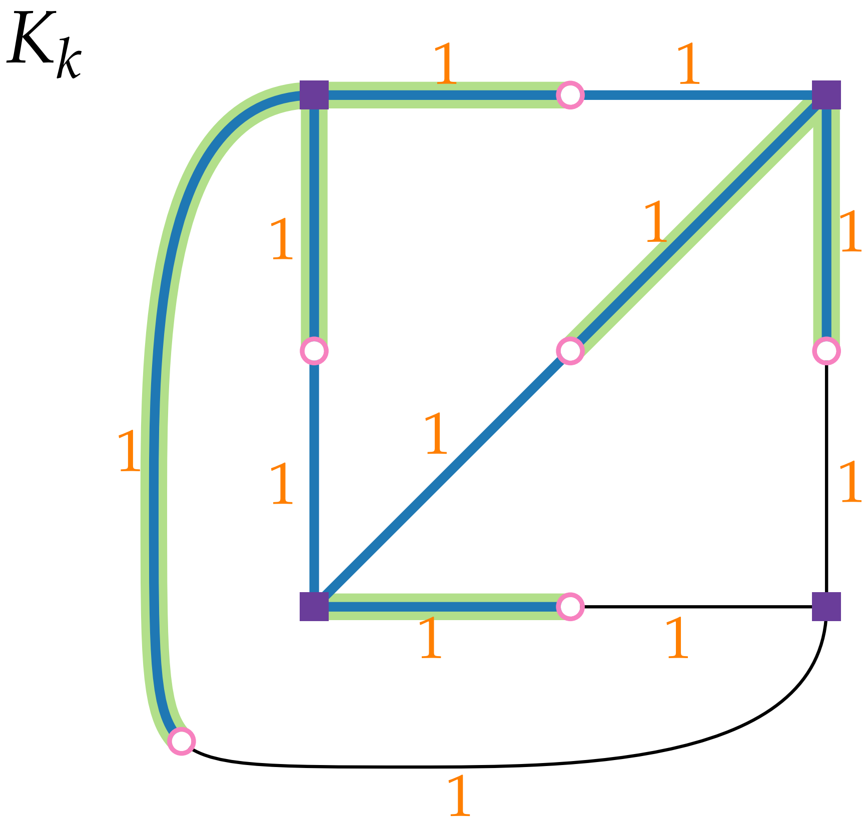

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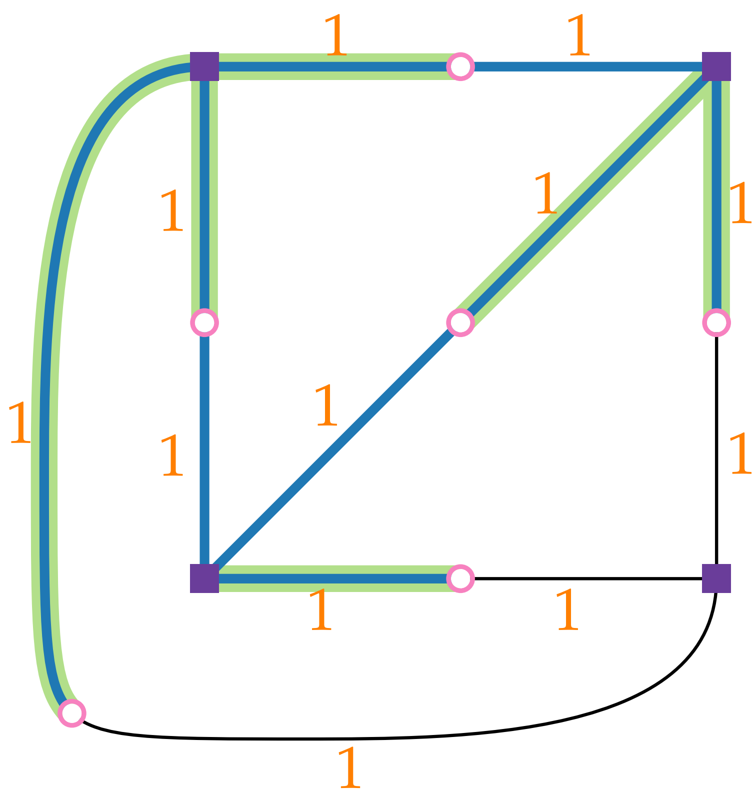
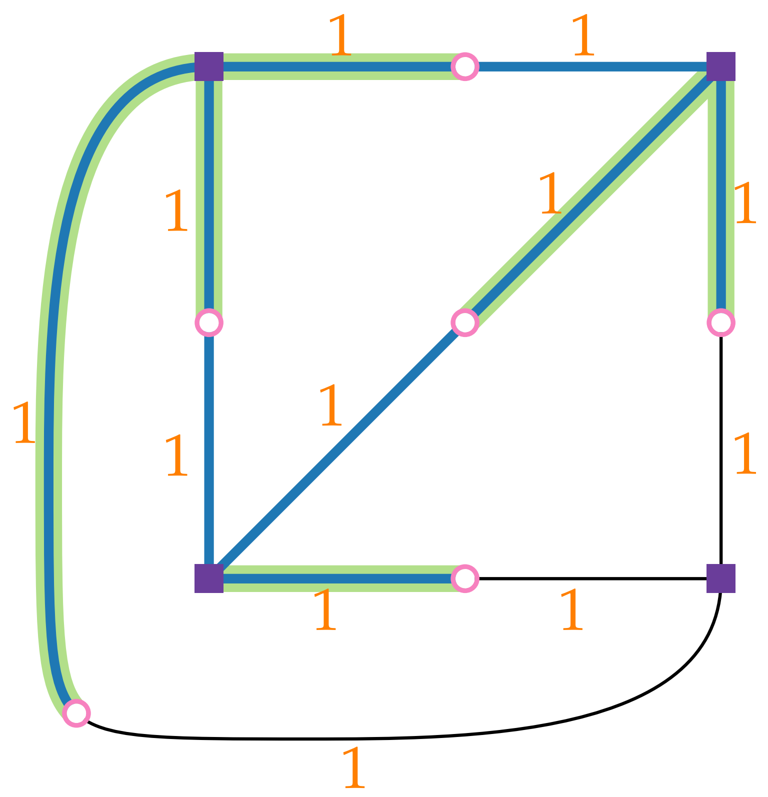


$\text{ALG} = (k-1)(k-1)$

$\text{OPT} = \sum_{i=1}^{k-1} i =$

Analysis Sharp?

K_k

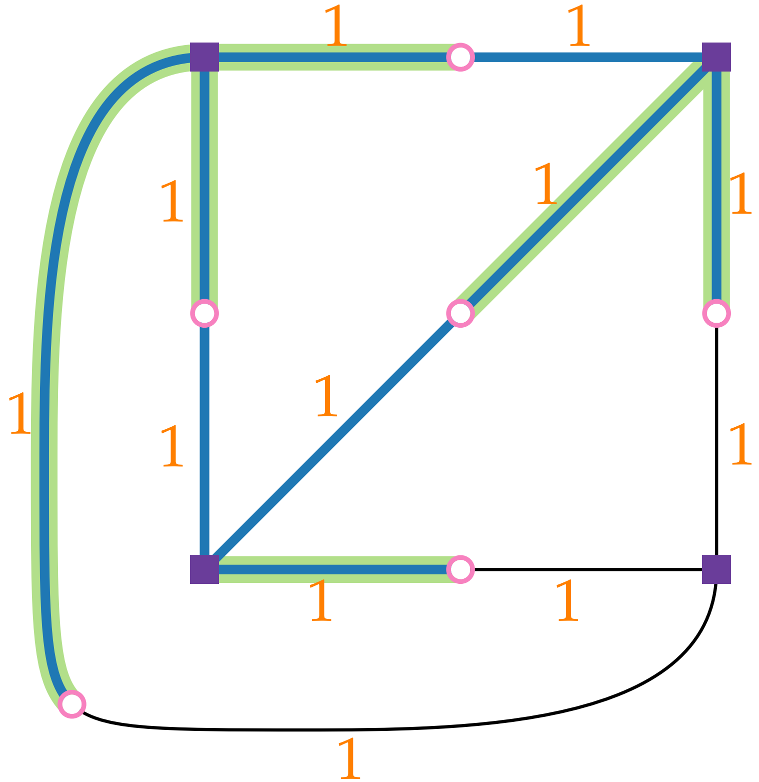


$\text{ALG} = (k-1)(k-1)$

$\text{OPT} = \sum_{i=1}^{k-1} i = \frac{k \cdot (k-1)}{2}$

Analysis Sharp?

K_k



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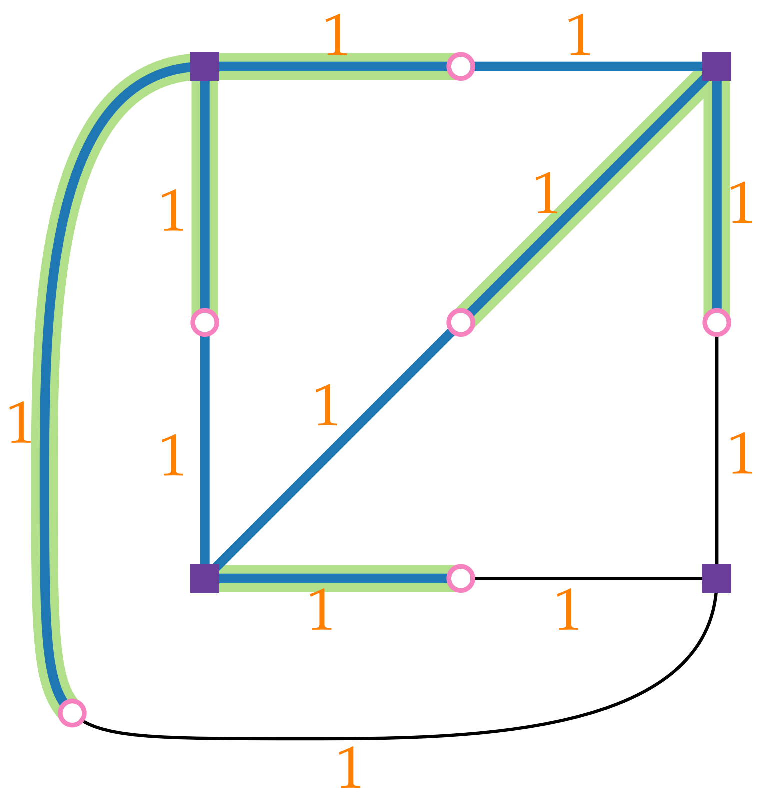
$\text{ALG}/\text{OPT} =$

Analysis Sharp?

K_k



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$\text{OPT} = \sum_{i=1}^{k-1} i = \frac{k\cdot(k-1)}{2}$

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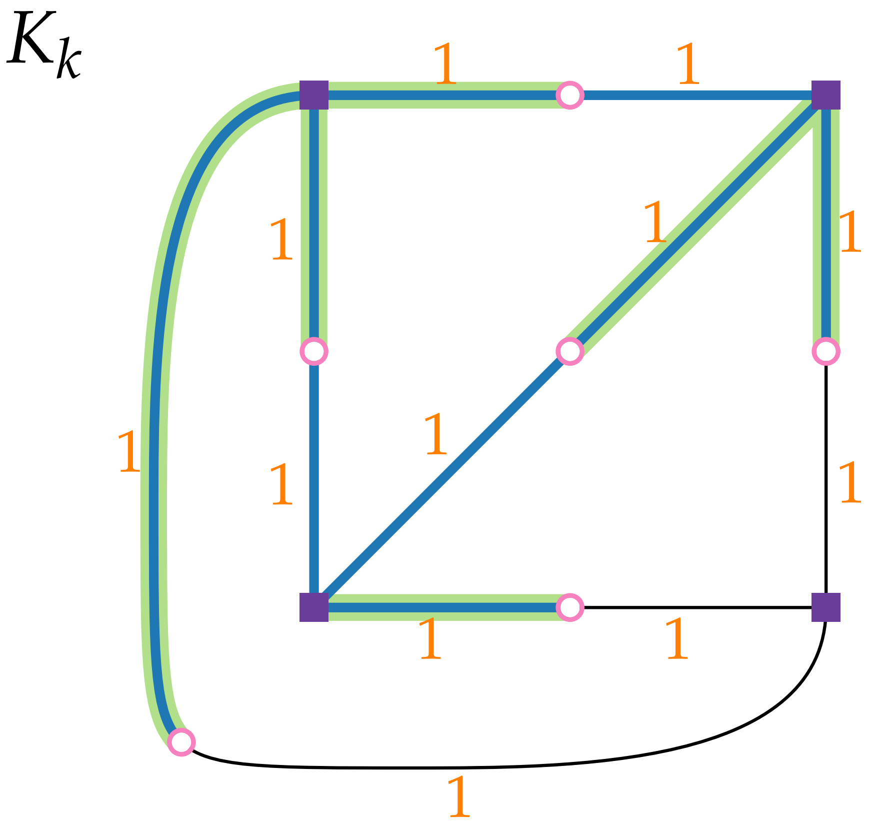

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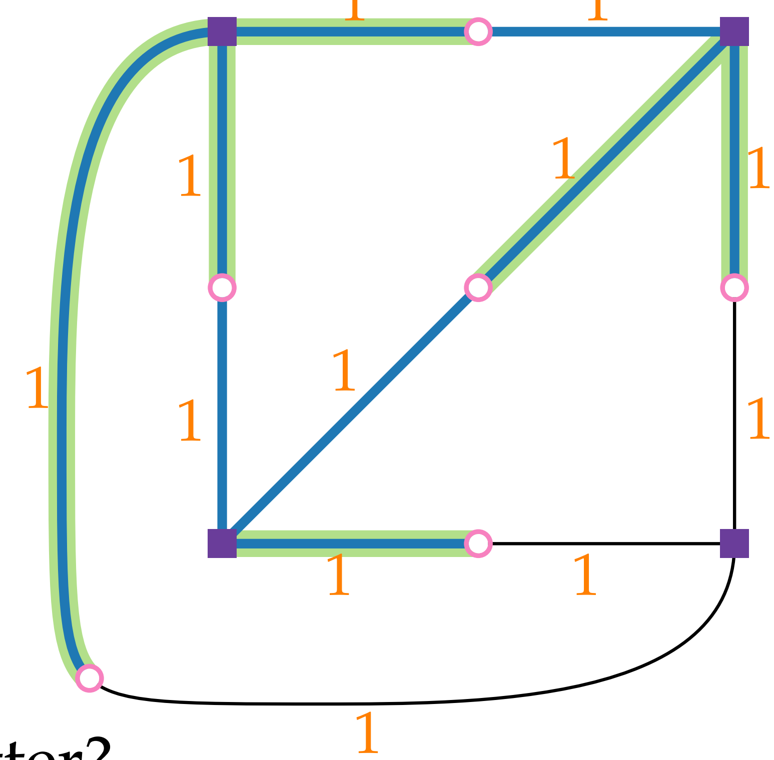
$\text{OPT} = \sum_{i=1}^{k-1} i = \frac{k \cdot (k-1)}{2}$

$\text{ALG}/\text{OPT} = \frac{2k-2}{k} = 2 - \frac{2}{k}$

Analysis Sharp?

K_k



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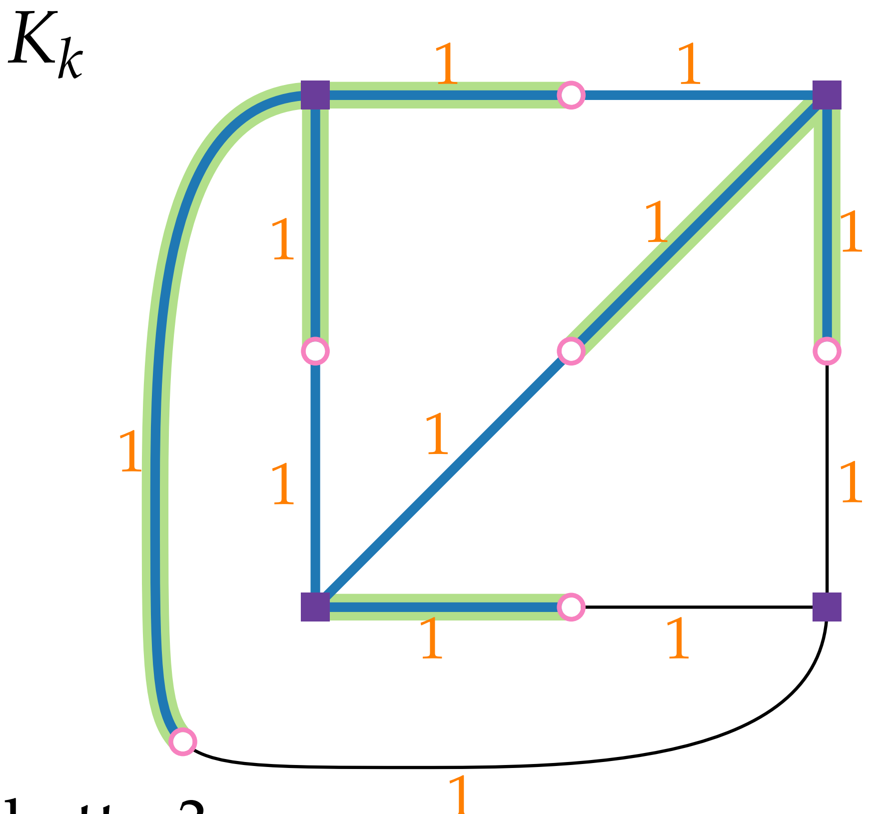
better?

Analysis Sharp?



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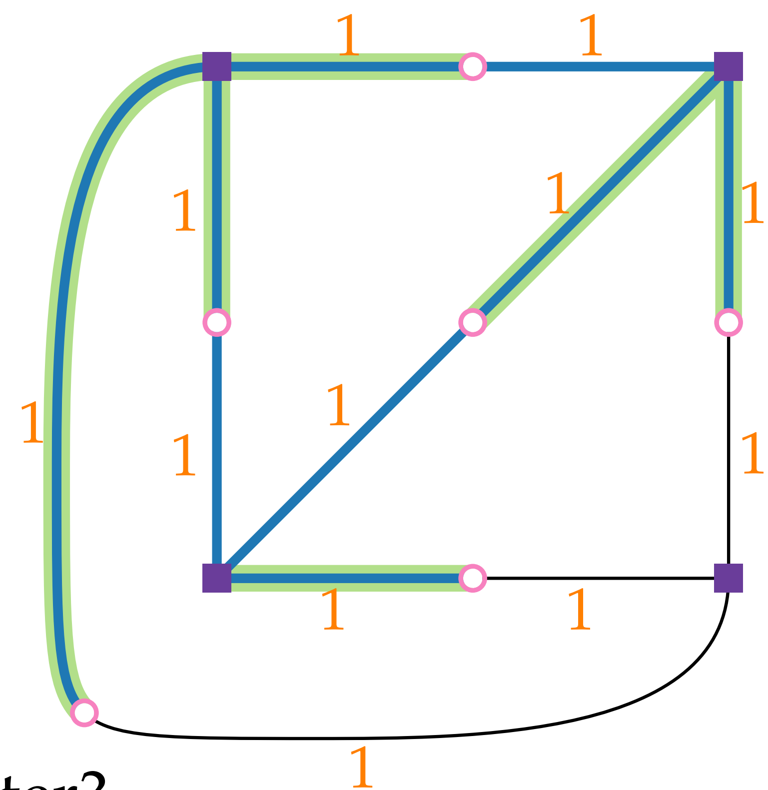
better?

The best known approximation factor for MultiwayCut is $1.2965 - \frac{1}{k}$. [Sharma & Vondrák '14]

Analysis Sharp?

K_k



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better?

The best known approximation factor for MultiwayCut is $1.2965 - \frac{1}{k}$. [Sharma & Vondrák '14]

MultiwayCut cannot be approximated within factor $1.20016 - O(1/k)$ (unless P=NP). [Bérczi, Chandrasekaran, Király & Madan '18]