Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part I:
Organizational

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Winter 2021/22

Organizational

Lectures: Zoom (in German or English)

Synchronous (key material)

More technical lectures via inverted classroom

Tutorials: One exercise sheet per lecture

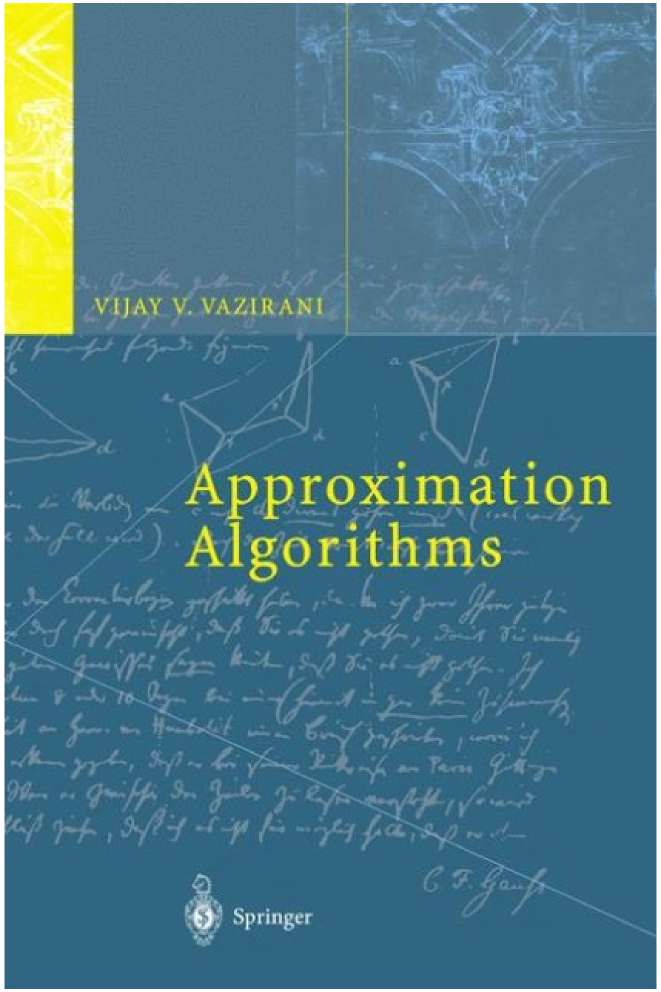
Solving assignments and presenting solutions

Tuesdays 10:15 - 11:45 (SE I)

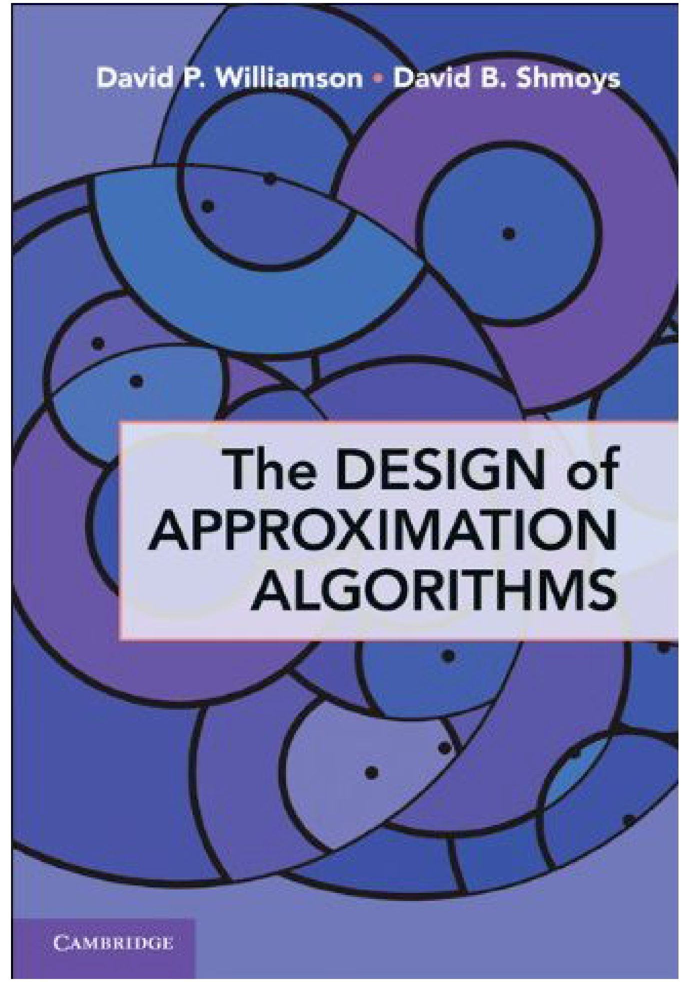
Bonus (+0.3 on final grade) for $\geq$ 50% points

Questions/Tasks during the lecture

Textbooks



Vijay V. Vazirani:
Approximation
Algorithms
Springer-Verlag, 2003.

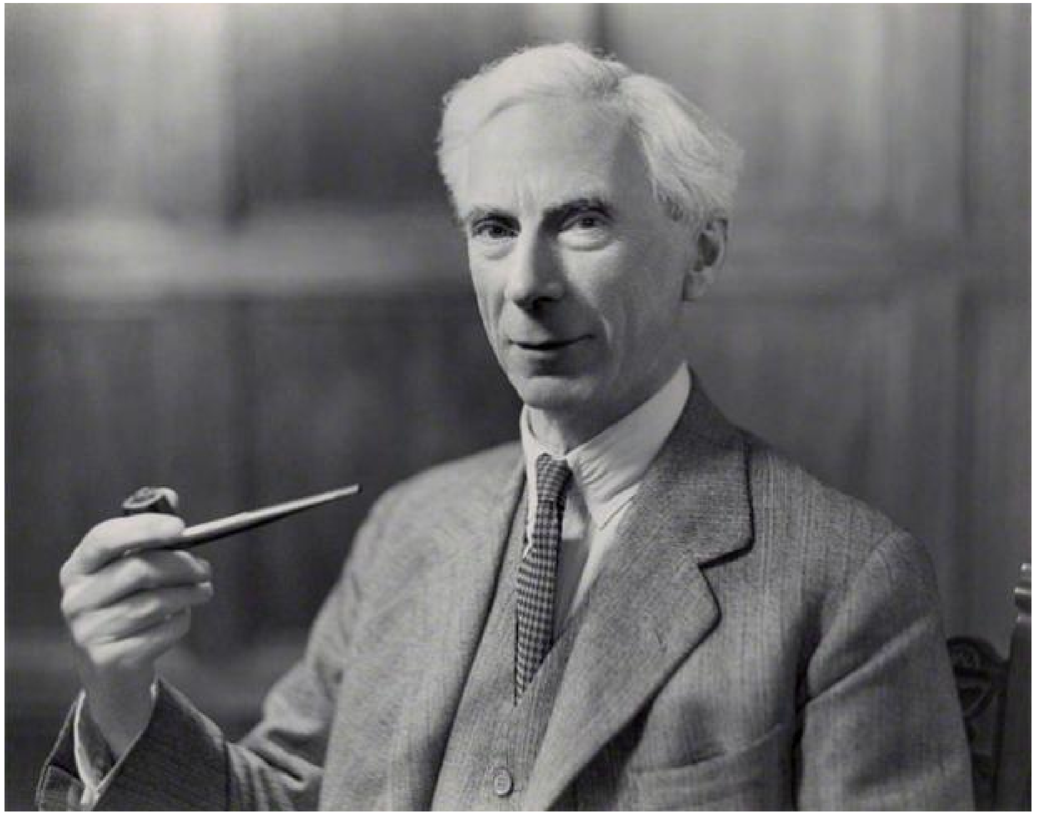


D. P. Williamson & D. B. Shmoys:
The Design of Approximation Algorithms
Cambridge-Verlag, 2011.
http://www.designofapproxalgs.com/

Approximation Algorithms

„All exact science is dominated by the idea of approximation.“

– Bertrand Russell (1872 – 1970)



Approximation Algorithms

- Many optimization problems are NP-hard (e.g. the traveling salesperson problem)
- $\rightsquigarrow$ an optimal solution cannot be efficiently computed unless P=NP.
- However, good approximate solutions can often be found efficiently!
- **Techniques** for the design and analysis of approximation algorithms arise from studying specific optimization problems.

Overview

Combinatorial Algorithms

- Introduction (Vertex Cover)
- Set Cover via Greedy
- Shortest Superstring via reduction to SC
- Steiner Tree via MST
- Multiway Cut via Greedy
- k-Center via param. Pruning
- Min-Deg-Spanning-Tree & local search
- Knapsack via DP & Scaling
- Euclidean TSP via Quadtrees

LP-based Algorithms

- introduction to LP-Duality
- Set Cover via LP Rounding
- Set Cover via Primal-Dual Schema
- Maximum Satisfiability
- Scheduling und Extreme Point Solutions
- Steiner Forest via Primal-Dual

Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part II:
Vertex Cover (card.)

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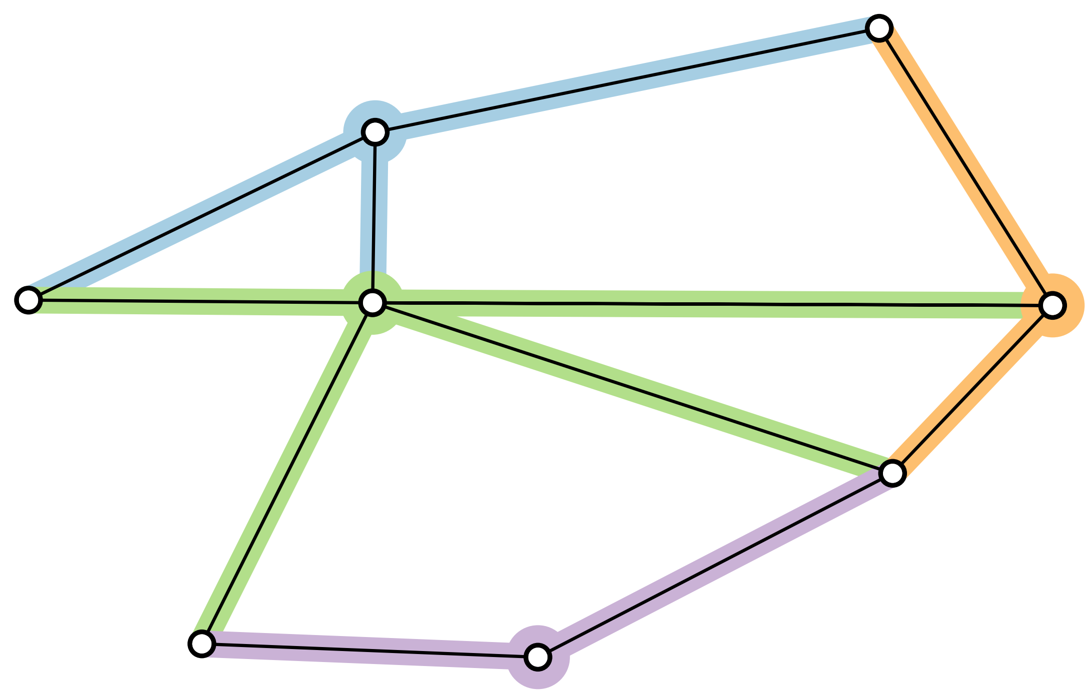
VertexCover (card.)

In: Graph $G = (V, E)$

Out: a minimum **vertex cover**: a minimum vertex set $V' \subseteq V$ such that every edge is **covered** (i.e., for every $uv \in E$, either $u \in V'$ or $v \in V'$).



Optimum (OPT $= 4$) – but in general NP-hard to find :-(

Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part III:
NP-Optimization Problem

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NP-Optimization Problem

An **NP-optimization problem** Π is given by:

- A set D_Π of **instances**.
  We denote the size of an instance $I \in D_\Pi$ by $|I|$.
- For each instance $I \in D_\Pi$ a set $S_\Pi(I) \neq \emptyset$ of **feasible solutions** for I such that:
  - for each solution $s \in S_\Pi(I)$, its size $|s|$ is polynomially bounded in $|I|$, and
  - for each pair (s, I), there is a polynomial time algorithm to decide whether $s \in S_\Pi(I)$.
- A polynomial time computable **objective function** obj_Π which assigns a positive objective value $\mathrm{obj}_\Pi(I, s) \geq 0$ to any given pair (s, I) with $s \in S_\Pi(I)$.
- Π is either a minimization or maximization problem.

VertexCover: NP-Optimization Problem

Task: Fill in the gaps for $\Pi =$ Vertex Cover.

$D_\Pi =$ Set of all graphs

For $I \in D_\Pi$: $\quad |I| =$ Number of vertices $|V|$

$G=(V,E)$ $\quad S_\Pi(I) =$ Set of all vertex covers of G

- Why is $|s| \in \text{poly}(|I|)$ for every $s \in S_\Pi(I)$?

  $s \subseteq V \Rightarrow |s| \leq |V| = |I|$

- For a given pair (s, I), how can we efficiently decide whether $s \in S_\Pi(I)$? Test whether all edges are covered.

$\text{obj}_\Pi(I, s) = |s|$

Π is Minimization problem.

Optimum and optimal objective value

maximization problem

Let Π be a minimization problem and $I \in D_\Pi$ be an instance of Π.
A feasible solution $s^* \in S_\Pi(I)$ is **optimal** if
maximal
$\text{obj}_\Pi(I, s^*)$ is minimal among objective values attained by the feasible solutions of I.

The optimal value $\text{obj}_\Pi(I, s^*)$ of the objective function is also denoted by $\text{OPT}_\Pi(I)$ or simply OPT in context.

Approximation Algorithms

maximization problem $\alpha : \mathbb{N} \to \mathbb{Q}$

Let Π be a minimization problem and ~~$\alpha \in \mathbb{Q}^+$~~.
A factor-α-approximation algorithm for Π is an efficient algorithm which provides for **any** instance $I \in D_\Pi$ a feasible solution $s \in S_\Pi(I)$ such that

$$\frac{\mathrm{obj}_\Pi(I, s)}{\mathrm{OPT}_\Pi(I)} \overset{\geq}{\leq} \cancel{\alpha}.\ \alpha(|I|)$$

Approximation Algorithms

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Part IV: Approximation Algorithm for VertexCover

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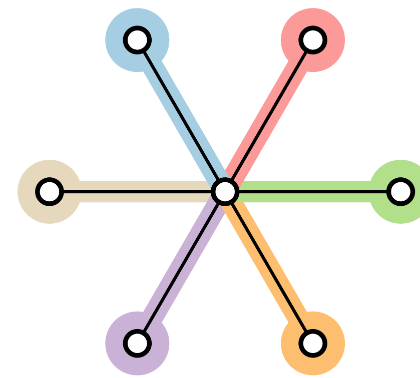
Approximation Alg. for VertexCover

Ideas?

- Edge-Greedy
- Vertex-Greedy



Quality?

Problem: How can we estimate $\mathrm{obj}_\Pi(I, s) / \mathrm{OPT}$, when it is hard to calculate OPT?

Idea: Find a "good" lower bound $L \leq \mathrm{OPT}$ for OPT and compare it to our approximate solution.

$$\frac{\mathrm{obj}_\Pi(I, s)}{\mathrm{OPT}} \leq \frac{\mathrm{obj}_\Pi(I, s)}{L}$$

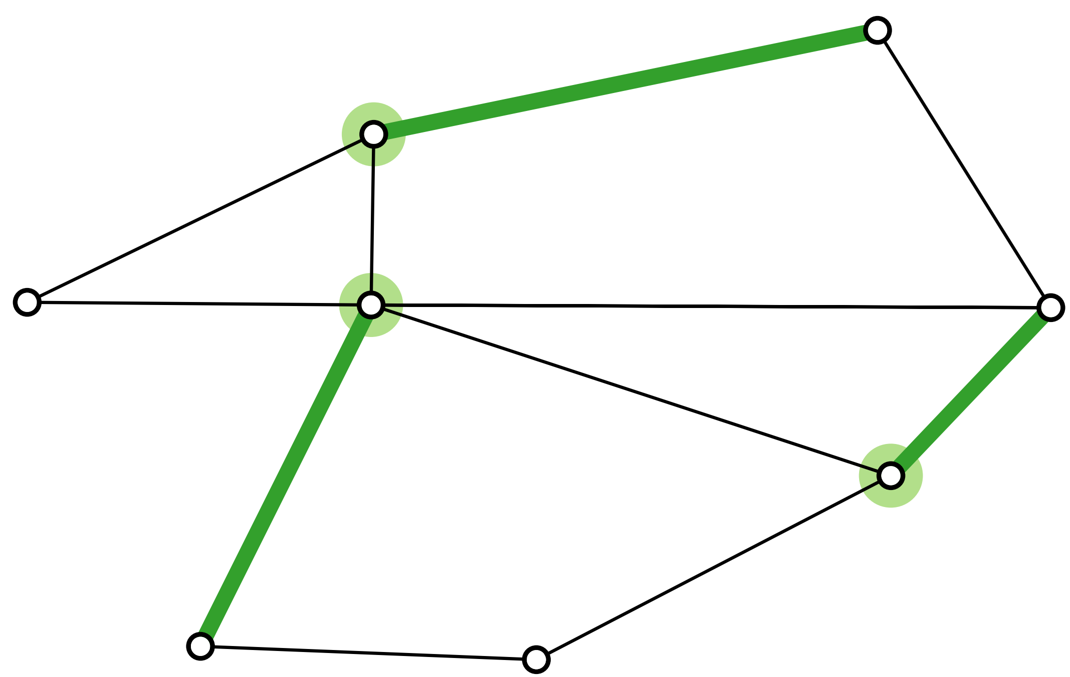
Lower Bound by Matchings

An edge set $M \subseteq E$ of a graph $G = (V, E)$ is a **matching** if no two edges of M are adjacent (i.e., share an end vertex).

M is **maximal** if there is no matching M' with $M' \supsetneq M$.

OPT $\geq |M|$

Vertex cover of M



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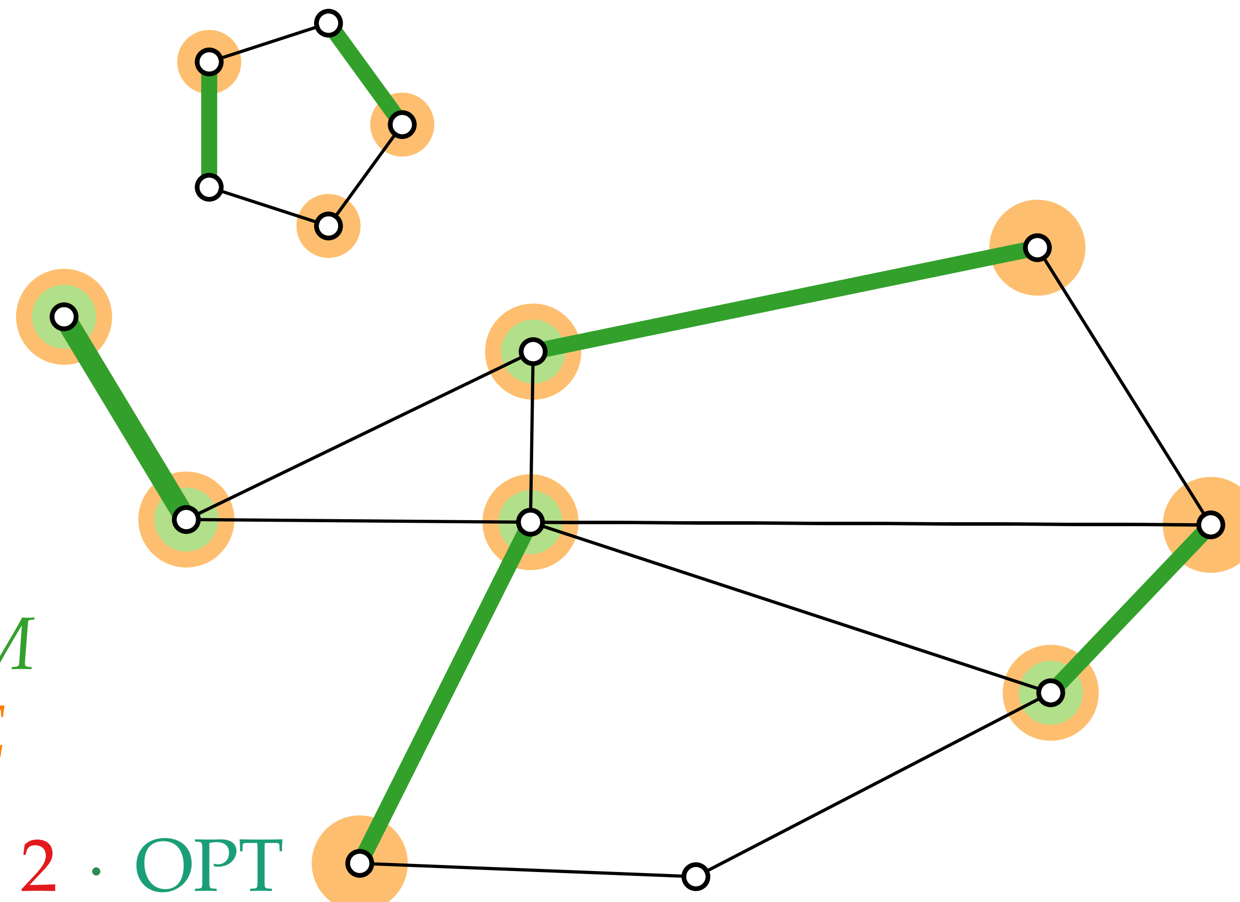

$\mathrm{OPT} \geq |M|$

$\cancel{\mathrm{OPT} = |M|}$?

Vertex cover of M

Vertex cover of E

$\mathrm{ALG} = 2 \cdot |M| \leq 2 \cdot \mathrm{OPT}$

Approximation Alg. for VertexCover

Algorithm VertexCover(G)

$M \leftarrow \emptyset$

foreach $e \in E(G)$ **do**

  if e not adjacent to edge in M **then**

    $M \leftarrow M \cup \{e\}$

return $\{\, u, v \mid uv \in M \,\}$

Theorem. The above algorithm is a factor-2-approximation algorithm for VertexCover.

The best-known approximation factor for VertexCover is $2 - \Theta(1/\sqrt{\log n})$

VertexCover cannot be approximated within factor 1.3606 (unless P=NP)

VertexCover cannot be approximated within factor $2 - \Theta(1)$, if “Unique Games Conjecture” holds.