Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part I:
Organizational

Joachim Spoerhase

Winter 2021/22

Organizational

Lectures: Zoom (in German or English)

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Synchronous (key material)

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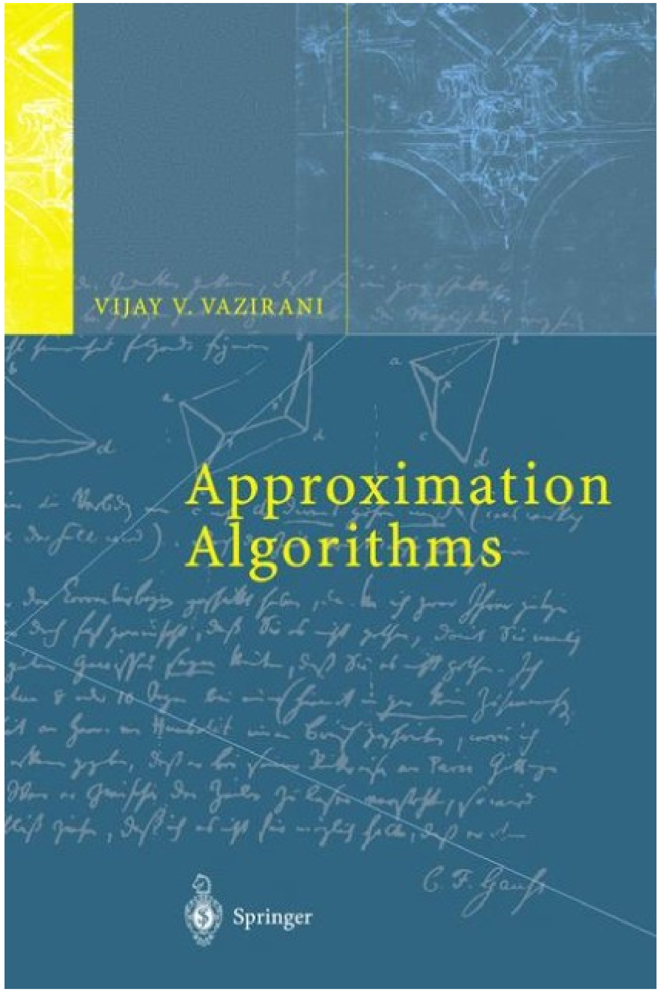
Questions/Tasks during the lecture

Textbooks



Vijay V. Vazirani:
Approximation
Algorithms
Springer-Verlag, 2003.

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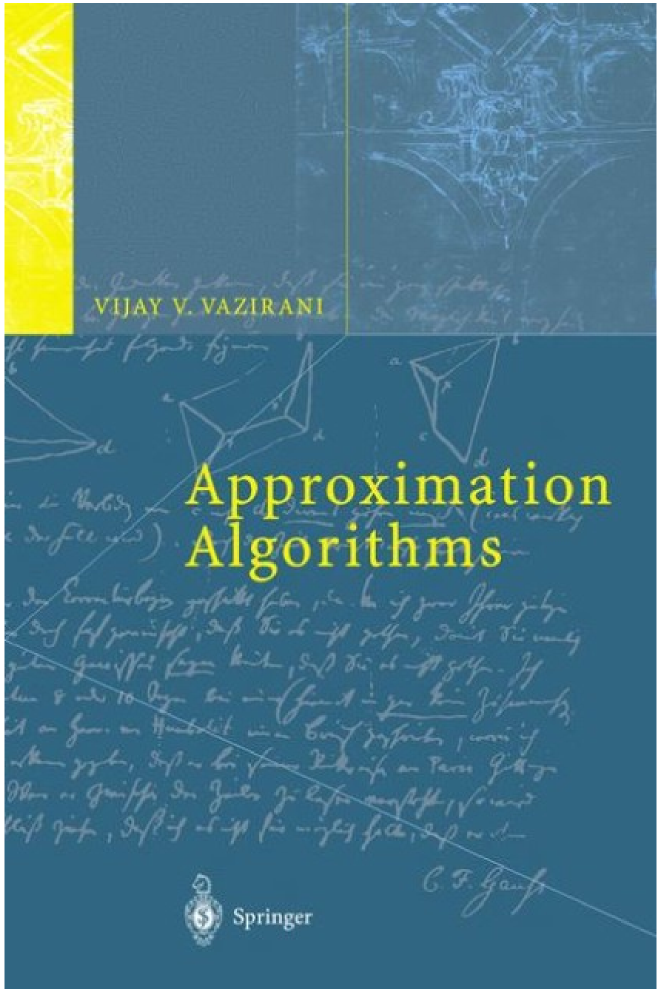

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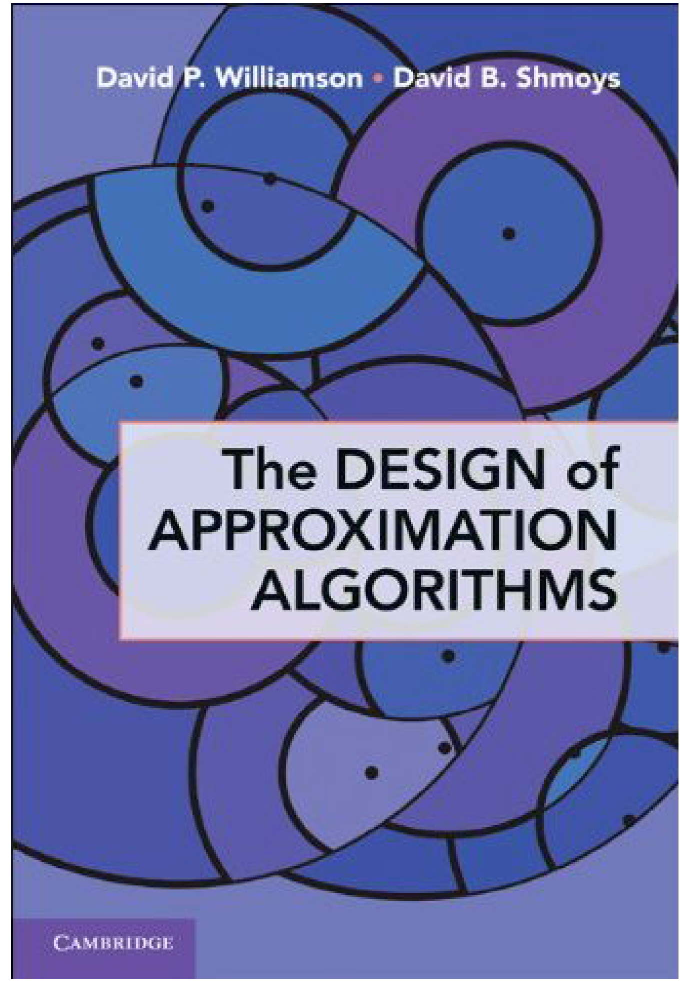
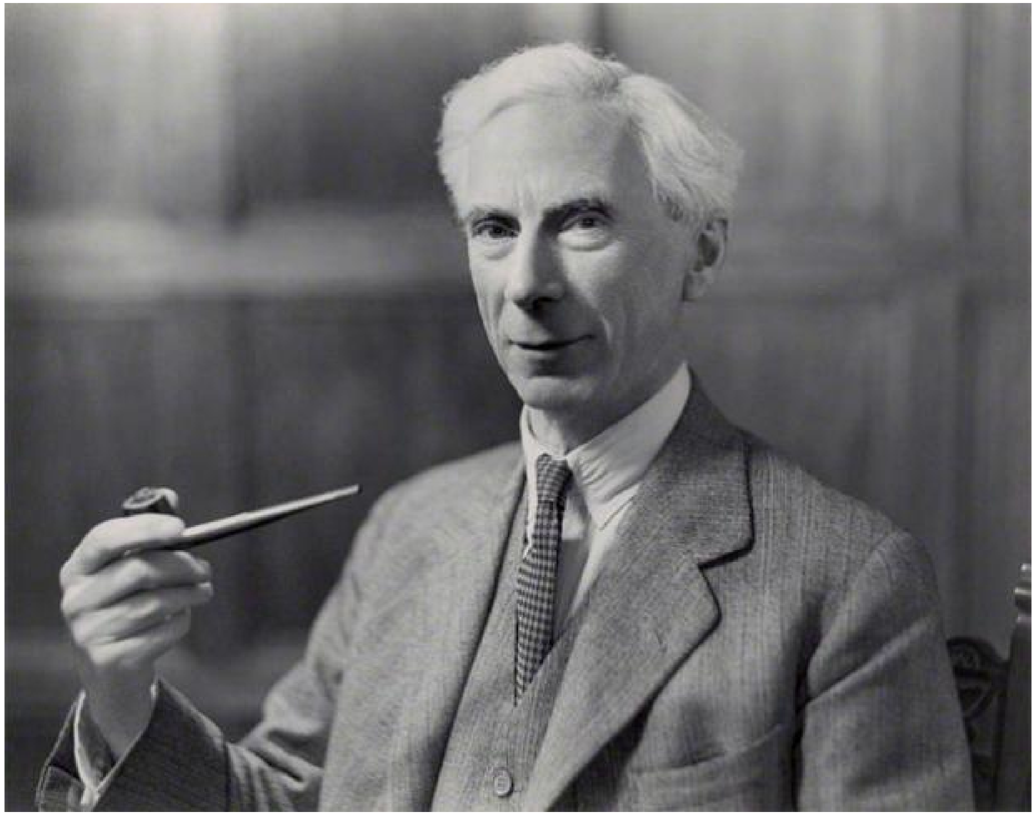


D. P. Williamson & D. B. Shmoys:
The Design of Approximation Algorithms
Cambridge-Verlag, 2011.
http://www.designofapproxalgs.com/

Approximation Algorithms

„All exact science is dominated by the idea of approximation.“

– Bertrand Russell
(1872 – 1970)



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- $\rightsquigarrow$ an optimal solution cannot be efficiently computed unless P=NP.
- However, good approximate solutions can often be found efficiently!
- **Techniques** for the design and analysis of approximation algorithms arise from studying specific optimization problems.

Overview

Combinatorial Algorithms

- Introduction (Vertex Cover)
- Set Cover via Greedy
- Shortest Superstring via reduction to SC
- Steiner Tree via MST
- Multiway Cut via Greedy
- k-Center via param. Pruning
- Min-Deg-Spanning-Tree & local search
- Knapsack via DP & Scaling
- Euclidean TSP via Quadtrees

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LP-based Algorithms

- introduction to LP-Duality
- Set Cover via LP Rounding
- Set Cover via Primal-Dual Schema
- Maximum Satisfiability
- Scheduling und Extreme Point Solutions
- Steiner Forest via Primal-Dual

Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part II:
Vertex Cover (card.)

Joachim Spoerhase

Winter 2021/22

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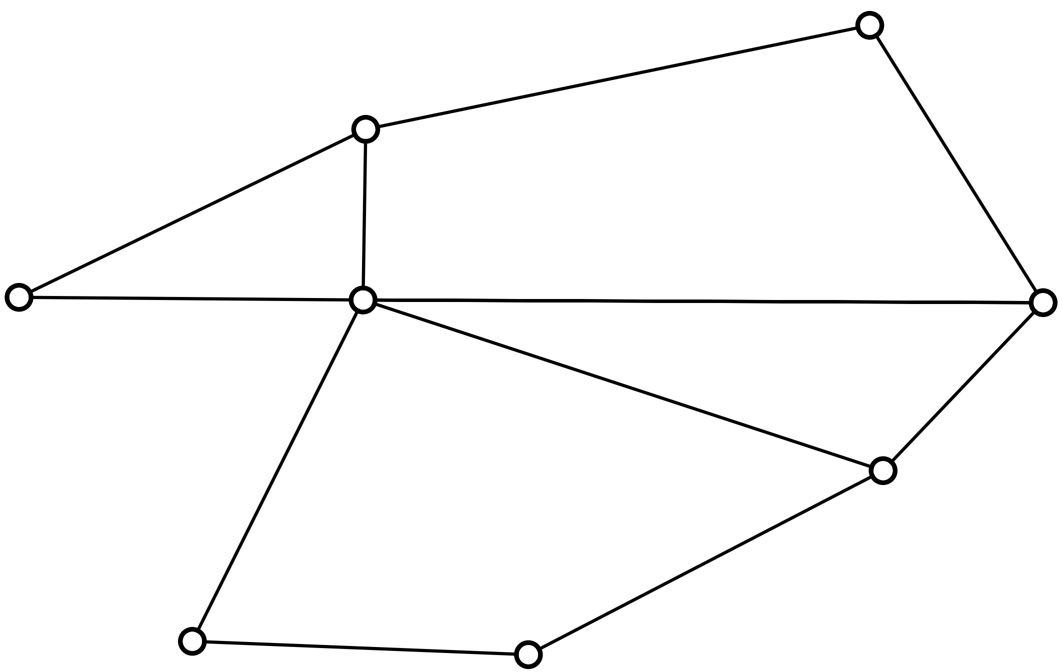

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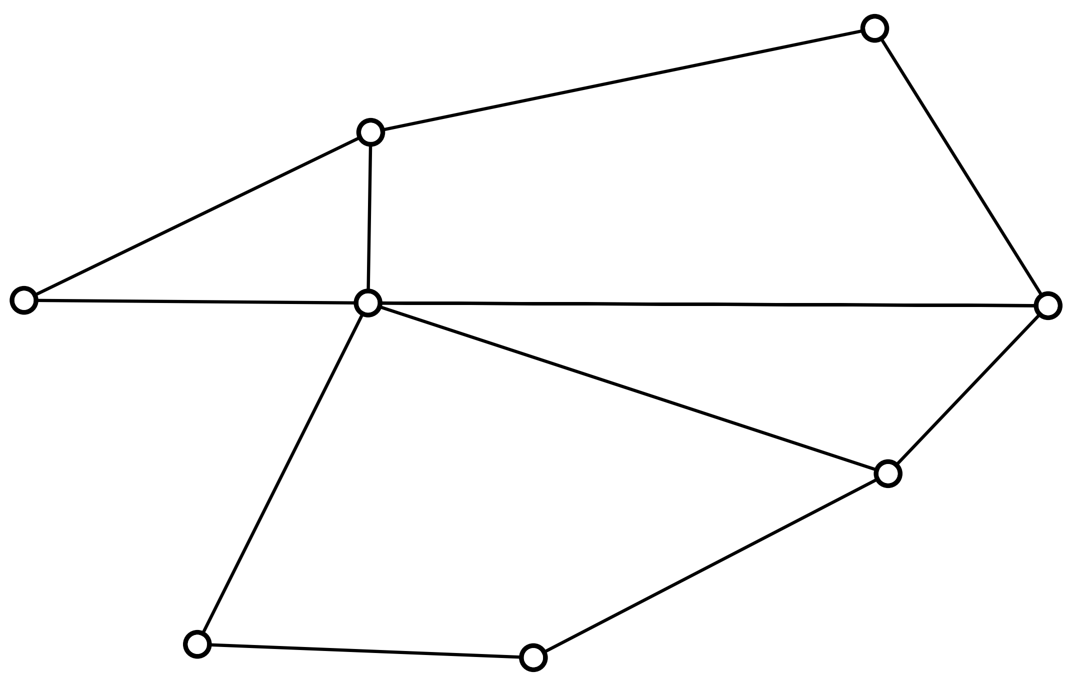

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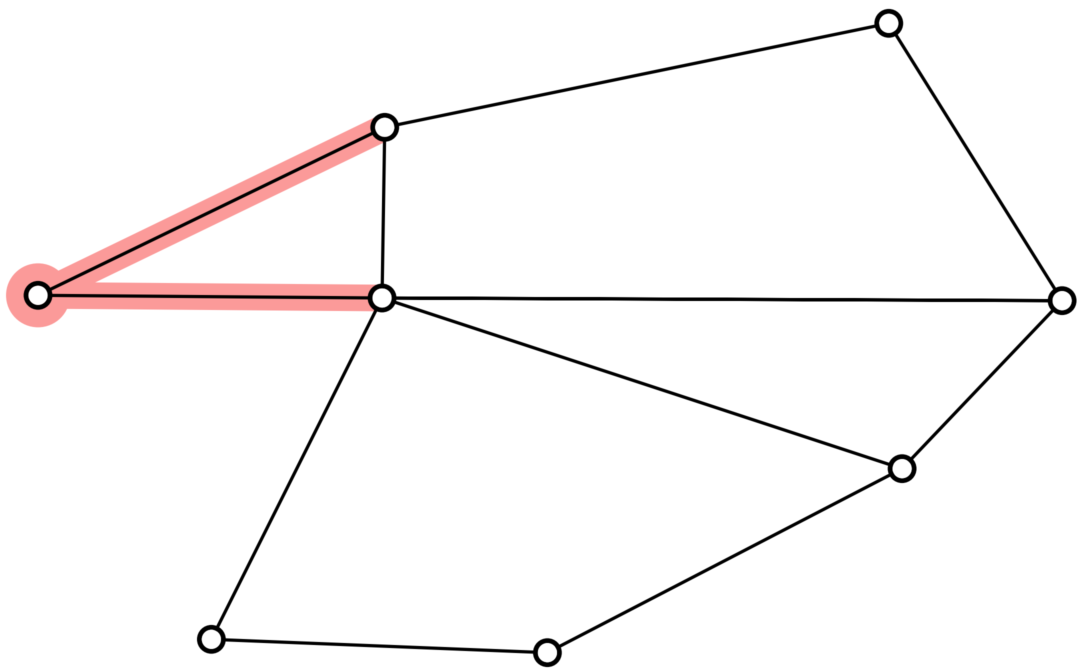

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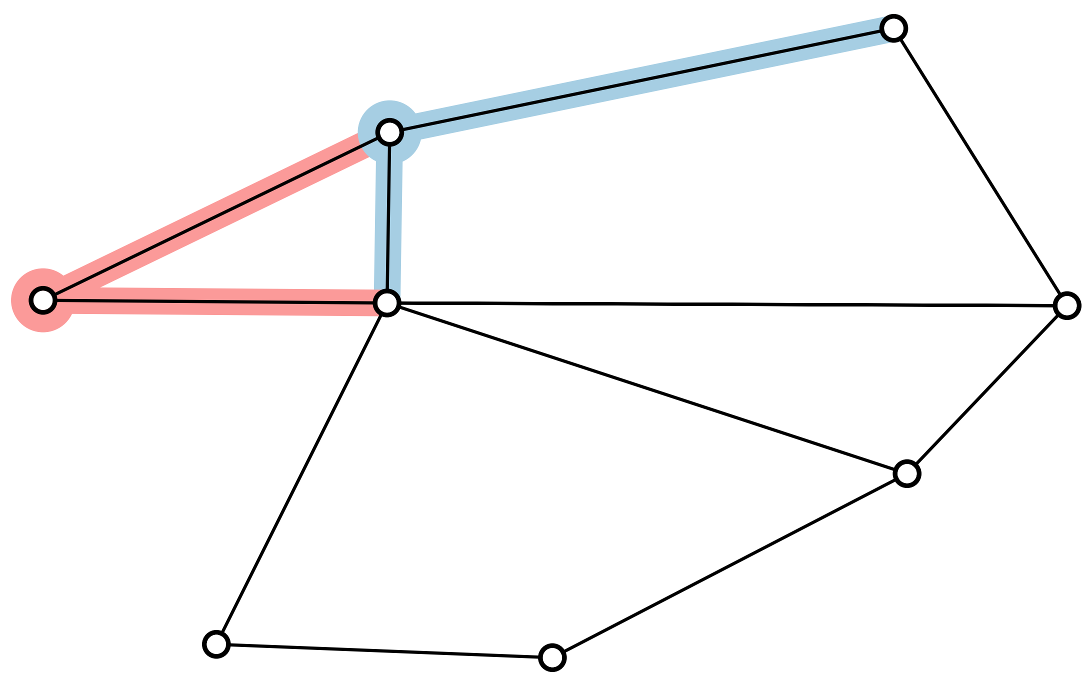

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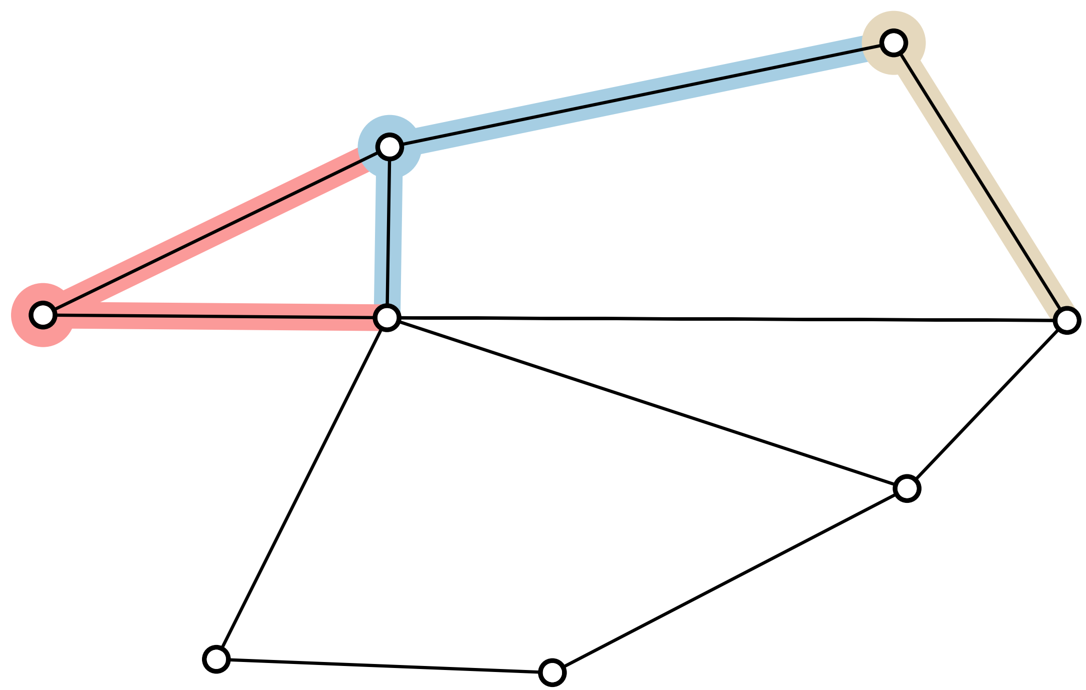

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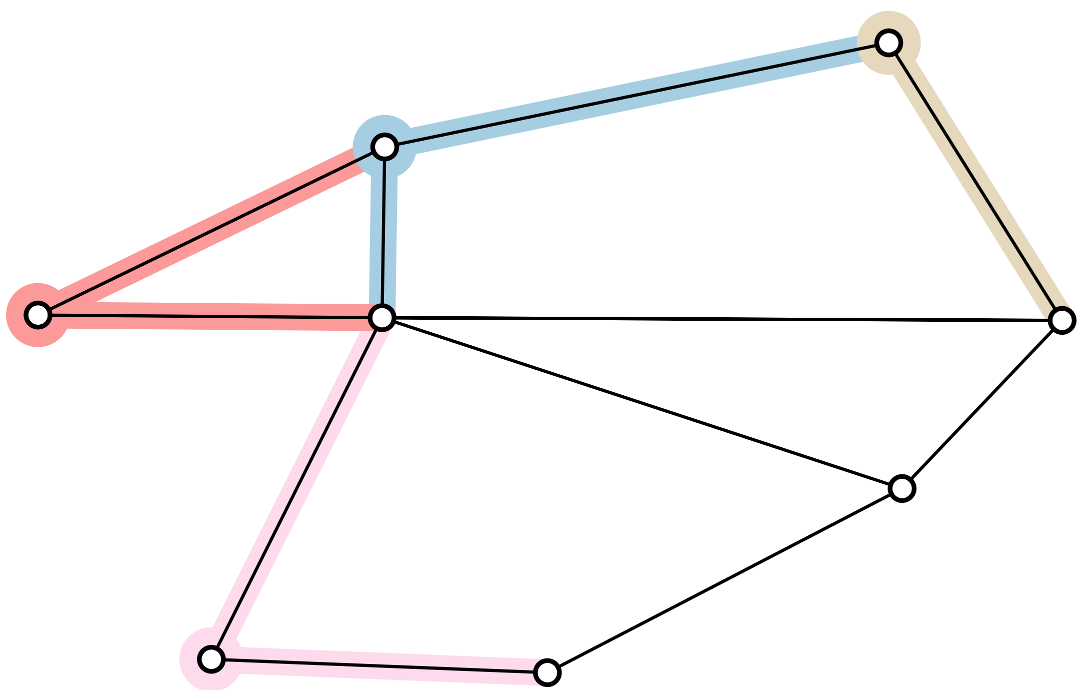

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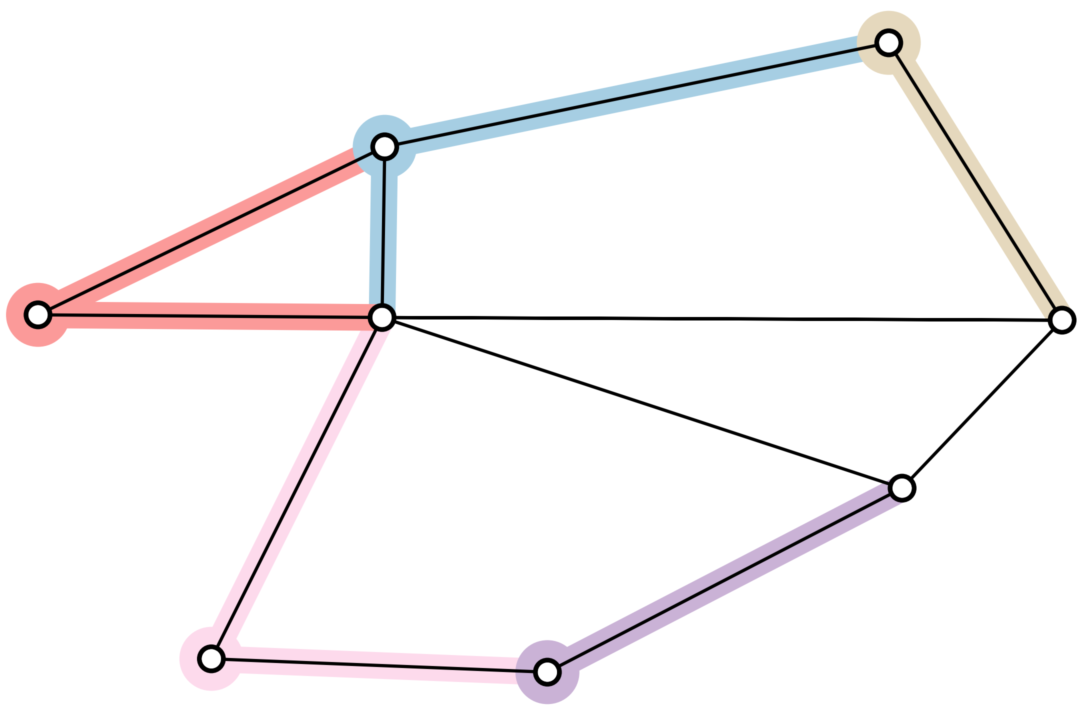

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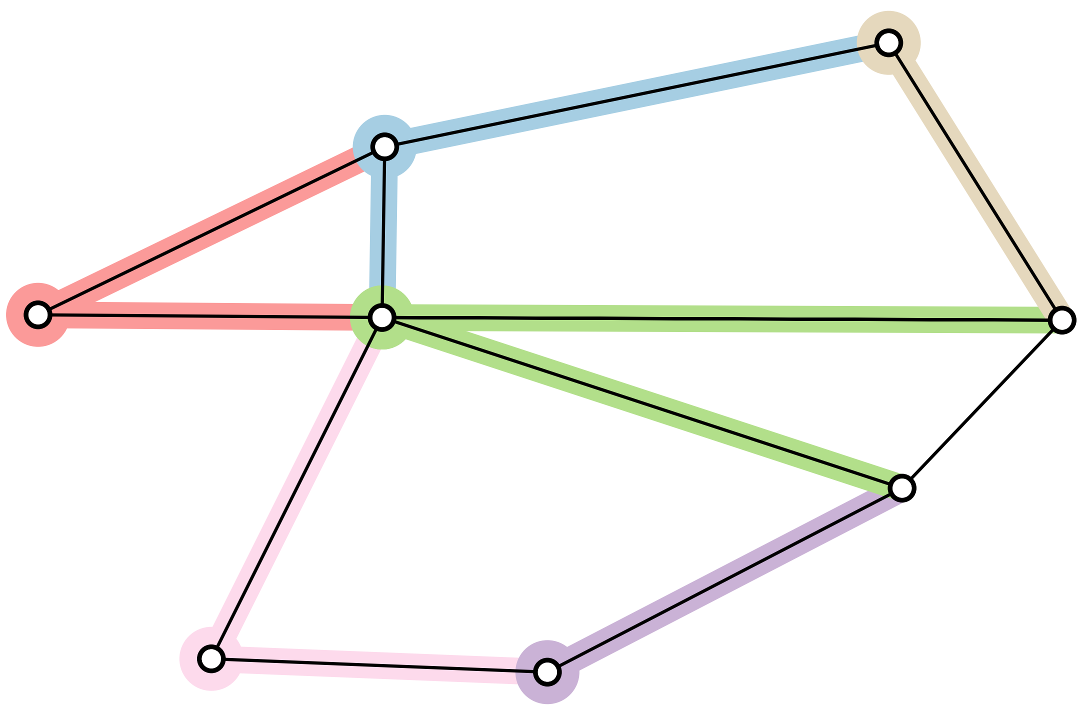

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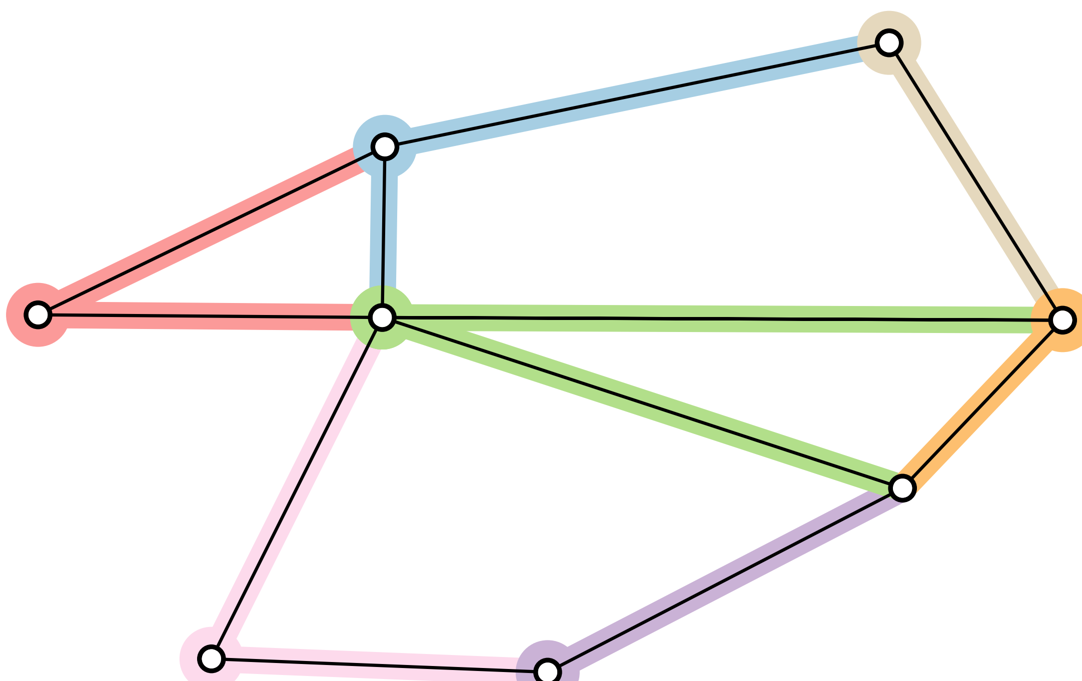

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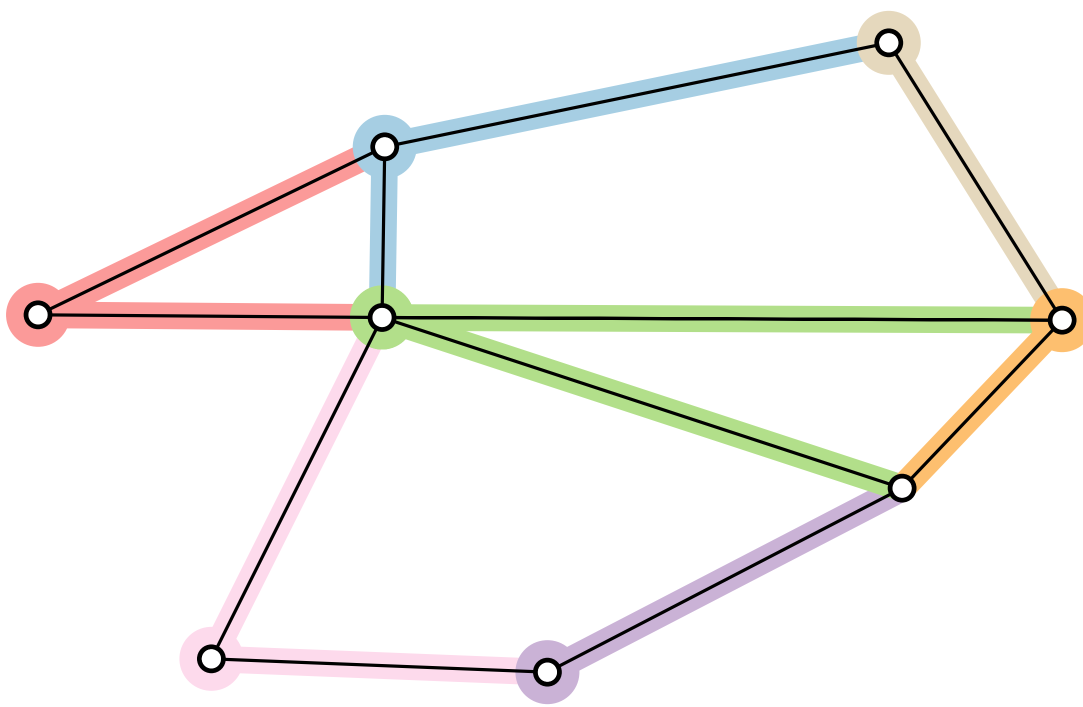

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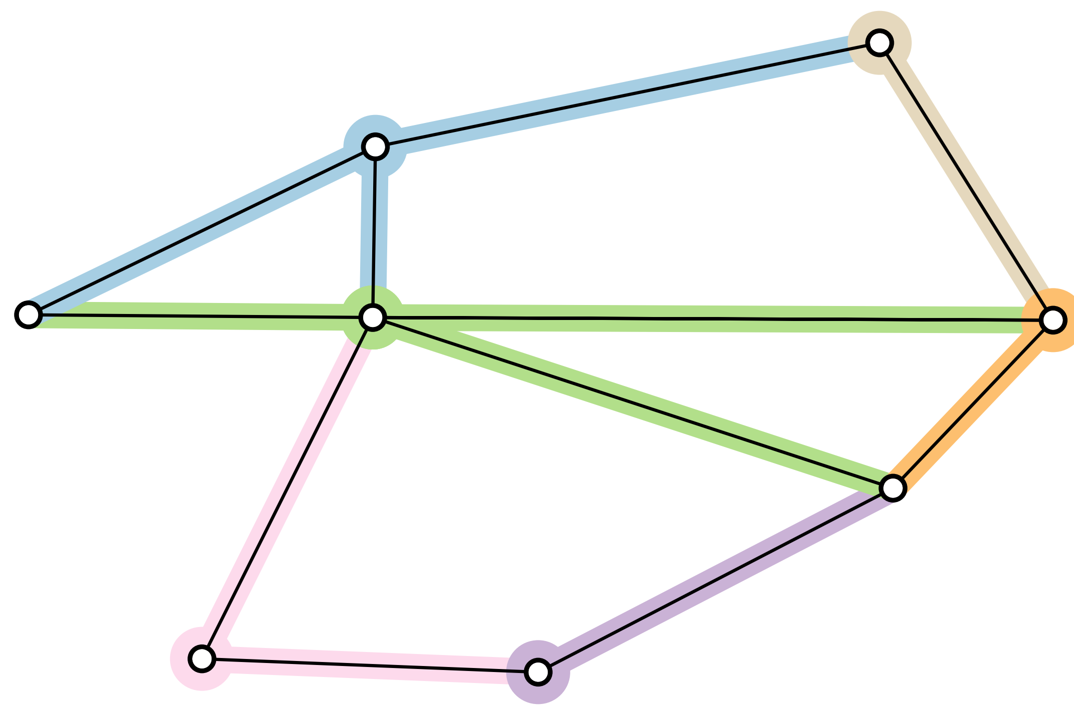

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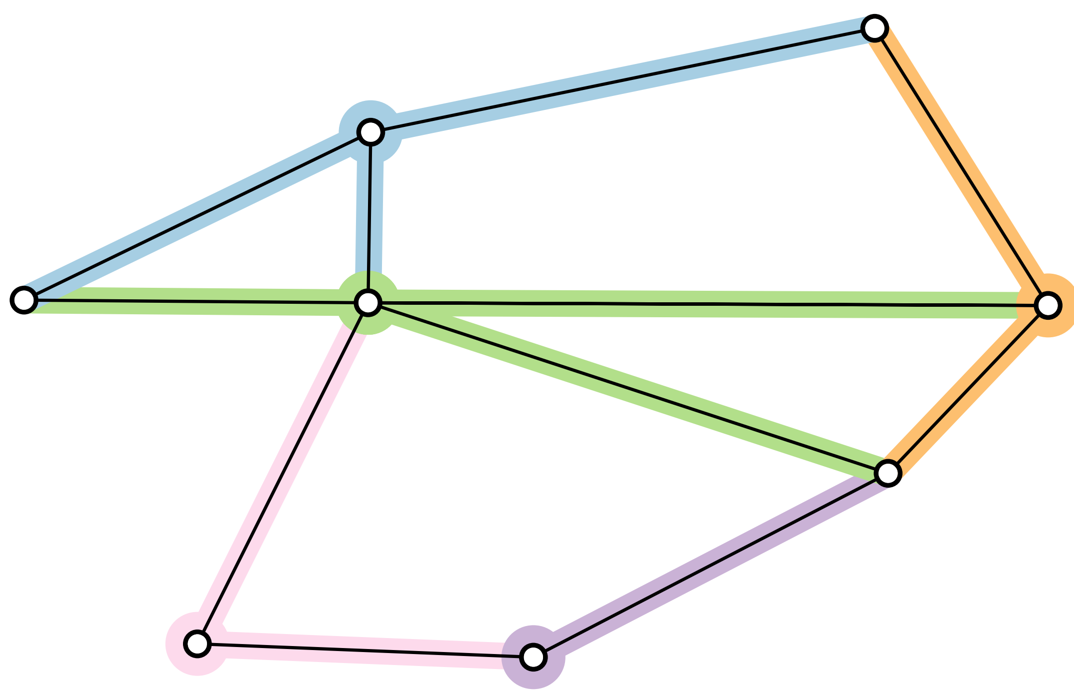

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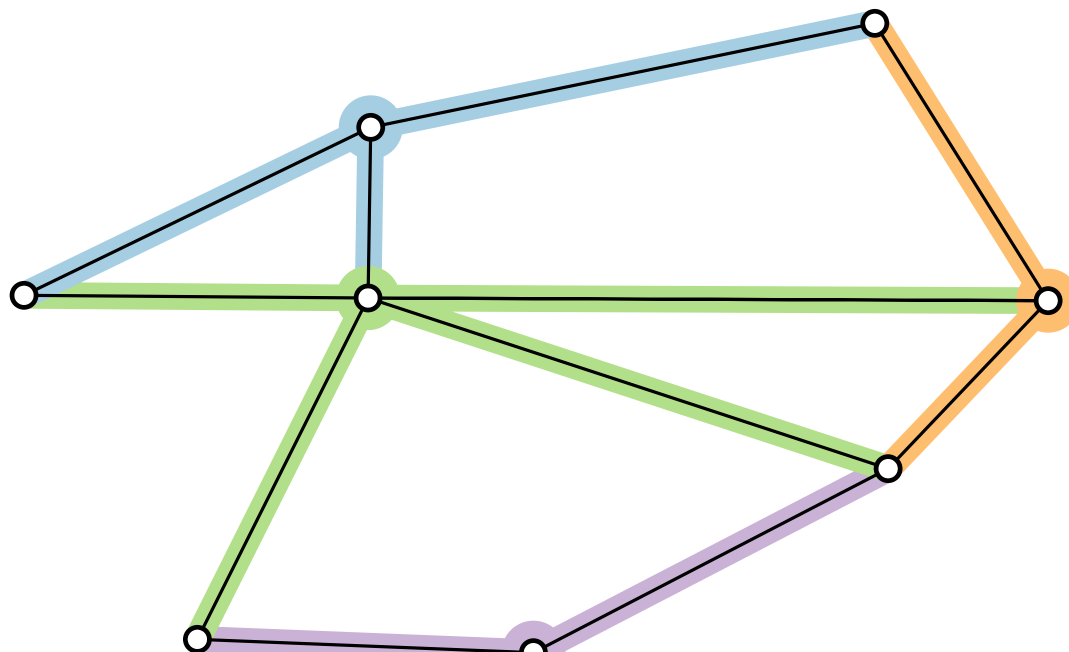

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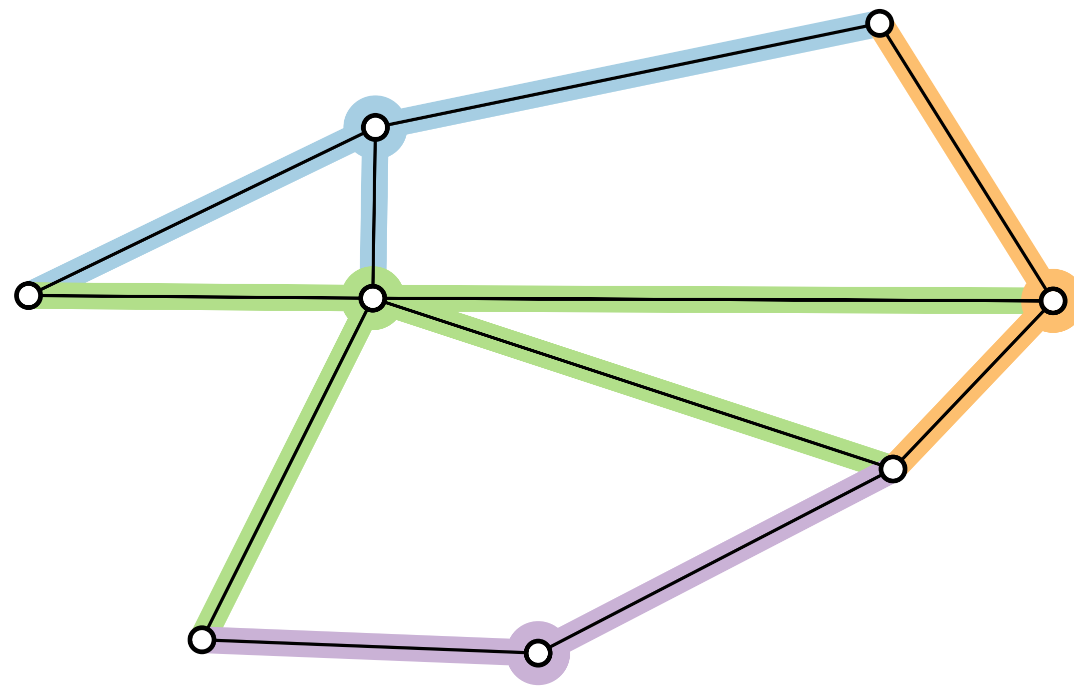

Optimum ($\mathrm{OPT} = 4$)

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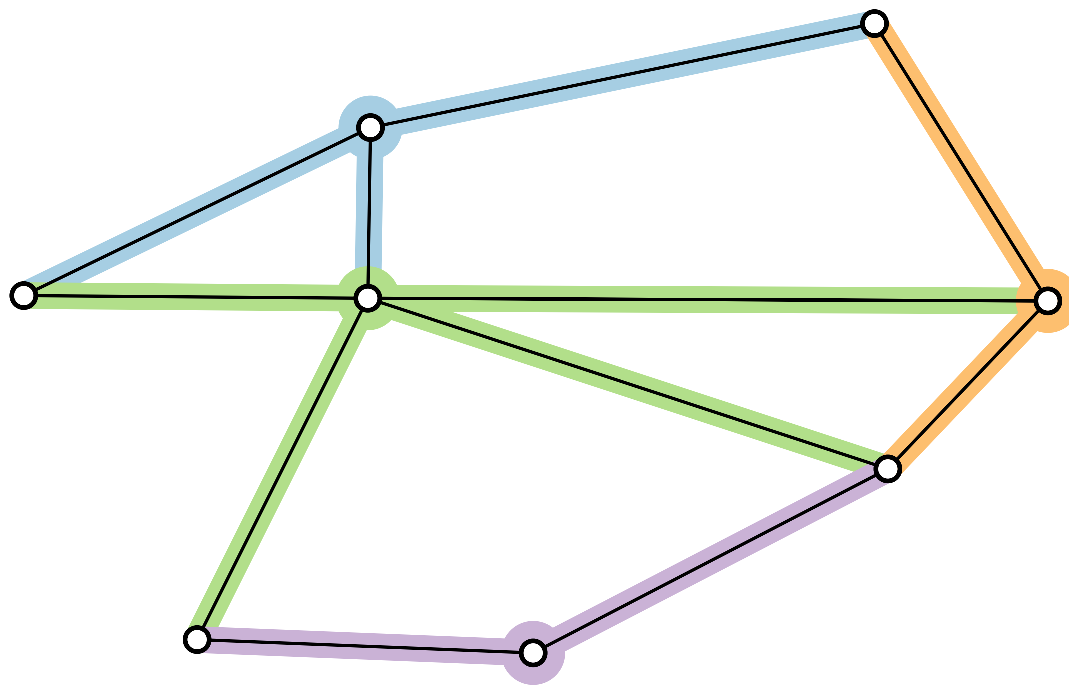

Optimum (OPT $= 4$) – but in general NP-hard to find :-(

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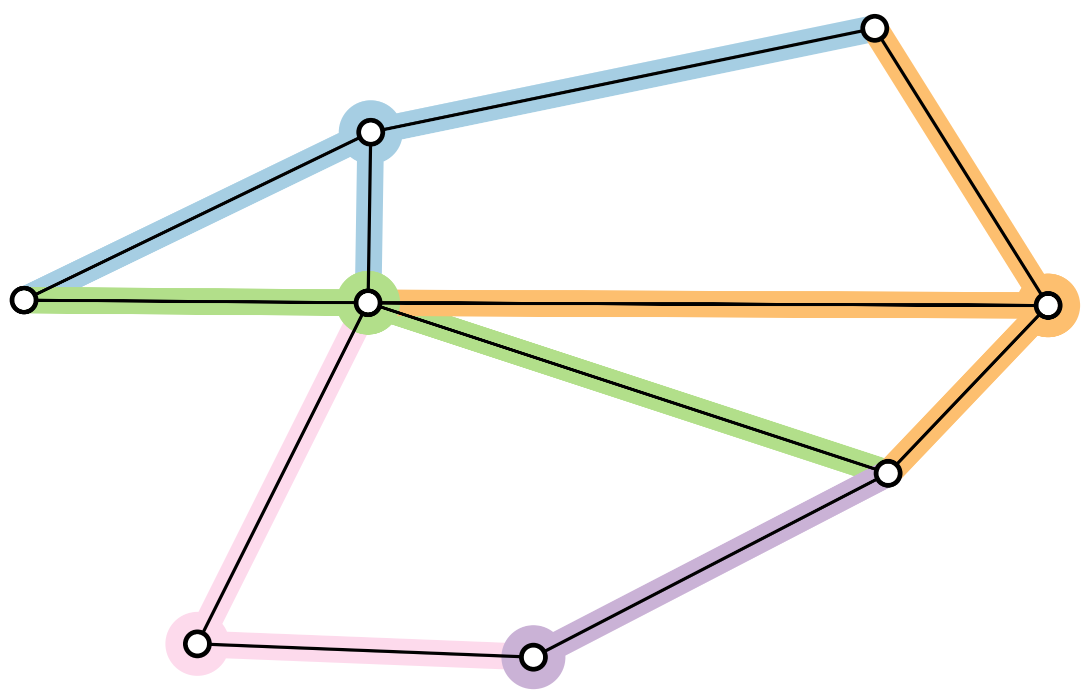

"good" approximate solution (5/4-approximation)

Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part III:
NP-Optimization Problem

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Winter 2021/22

NP-Optimization Problem

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- Π is either a minimization or maximization problem.

VertexCover: NP-Optimization Problem

Task: Fill in the gaps for $\Pi =$ Vertex Cover.

$D_\Pi =$

For $I \in D_\Pi$: $|I| =$

$S_\Pi(I) =$

- Why is $|s| \in \text{poly}(|I|)$ for every $s \in S_\Pi(I)$?

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Π is Minimization problem.

Optimum and optimal objective value

Let Π be a minimization problem and $I \in D_{\Pi}$ be an instance of Π.

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The optimal value $\mathrm{obj}_\Pi(I, s^*)$ of the objective function is also denoted by $\mathrm{OPT}_\Pi(I)$ or simply OPT in context.

Approximation Algorithms

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Approximation Algorithms

$\alpha : \mathbb{N} \to \mathbb{Q}$

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Approximation Algorithms

Lecture 1:
Introduction and Vertex Cover

Part IV:
Approximation Algorithm for VertexCover

Joachim Spoerhase

Winter 2021/22

Approximation Alg. for VertexCover

Ideas?

Approximation Alg. for VertexCover

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- Edge-Greedy

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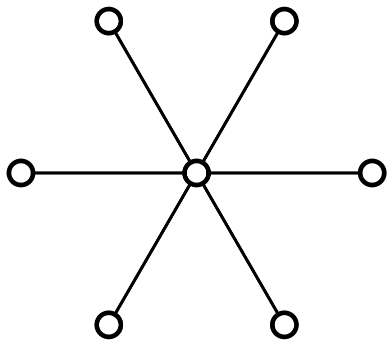

Approximation Alg. for VertexCover

Ideas?

- Edge-Greedy
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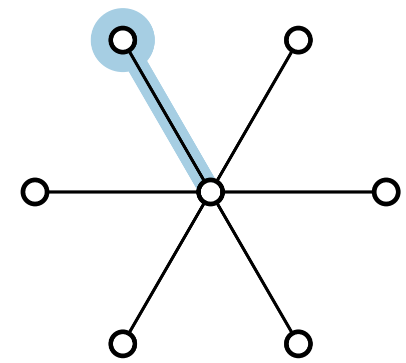

Approximation Alg. for VertexCover

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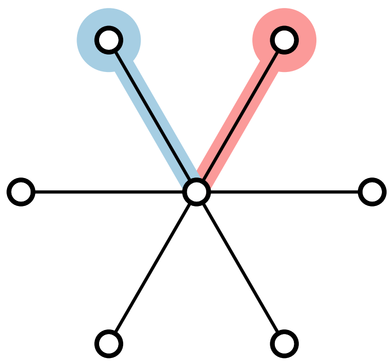

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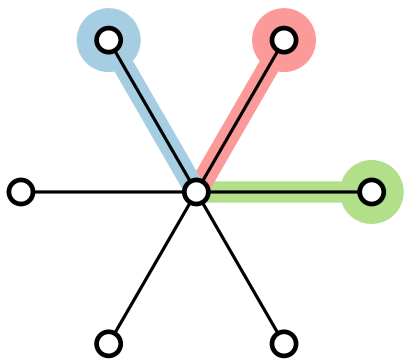

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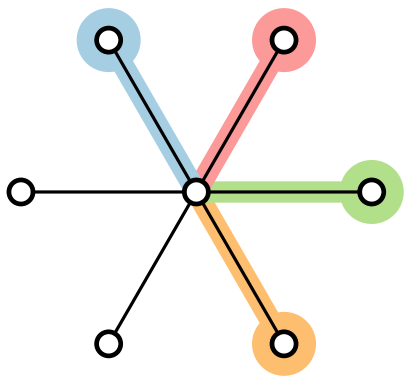

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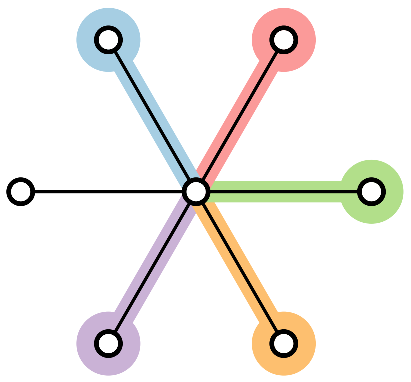

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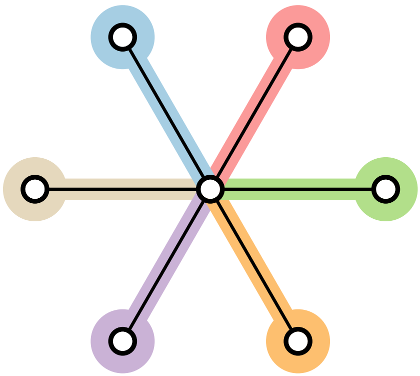

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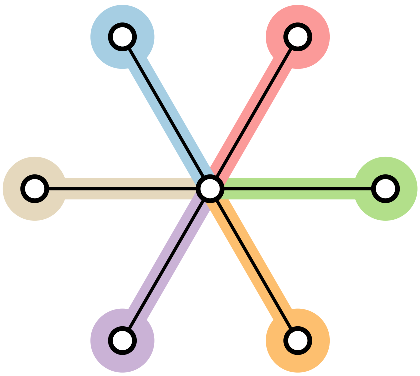

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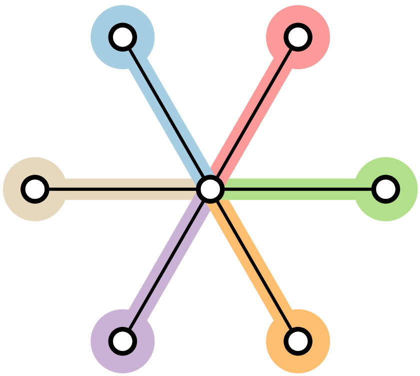

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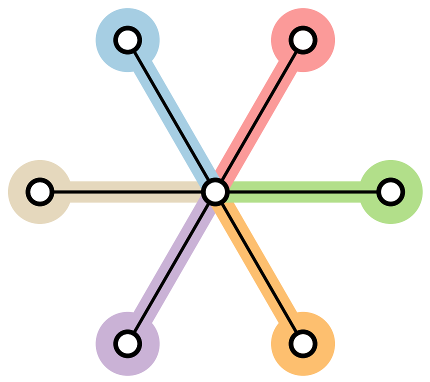

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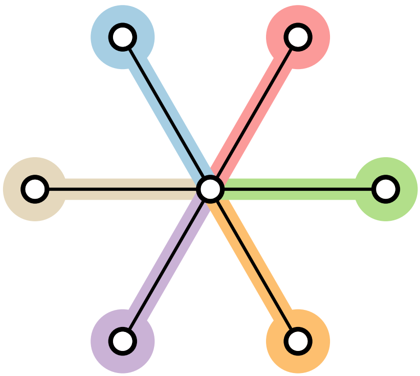

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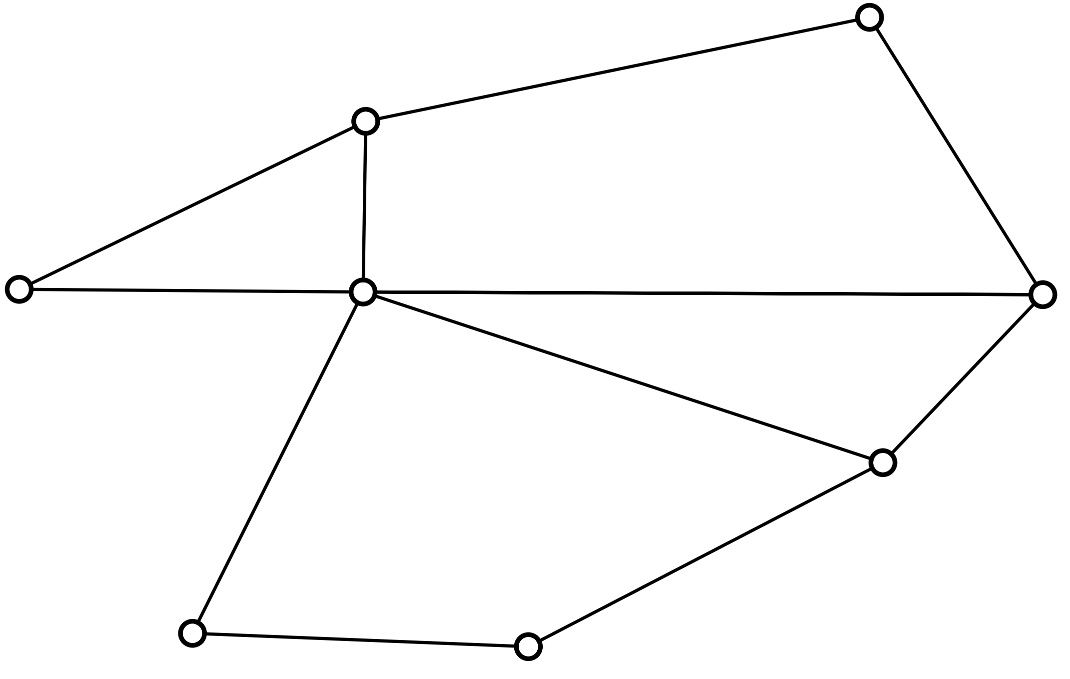
$$\frac{\mathrm{obj}_\Pi(I,s)}{\mathrm{OPT}} \leq \frac{\mathrm{obj}_\Pi(I,s)}{L}$$

Lower Bound by Matchings



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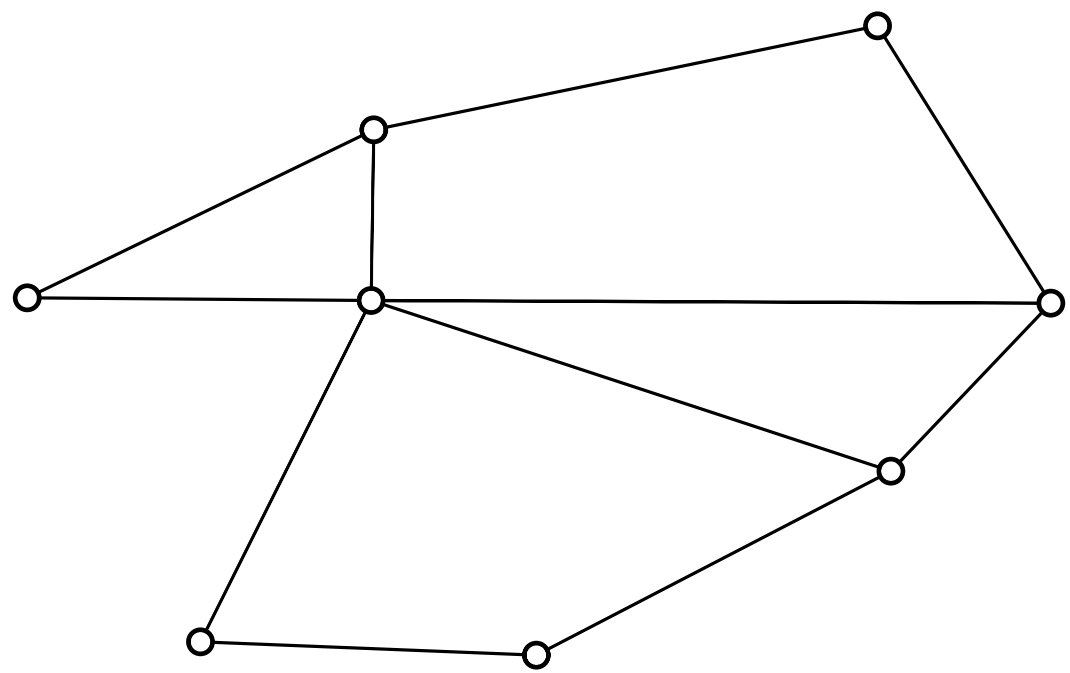

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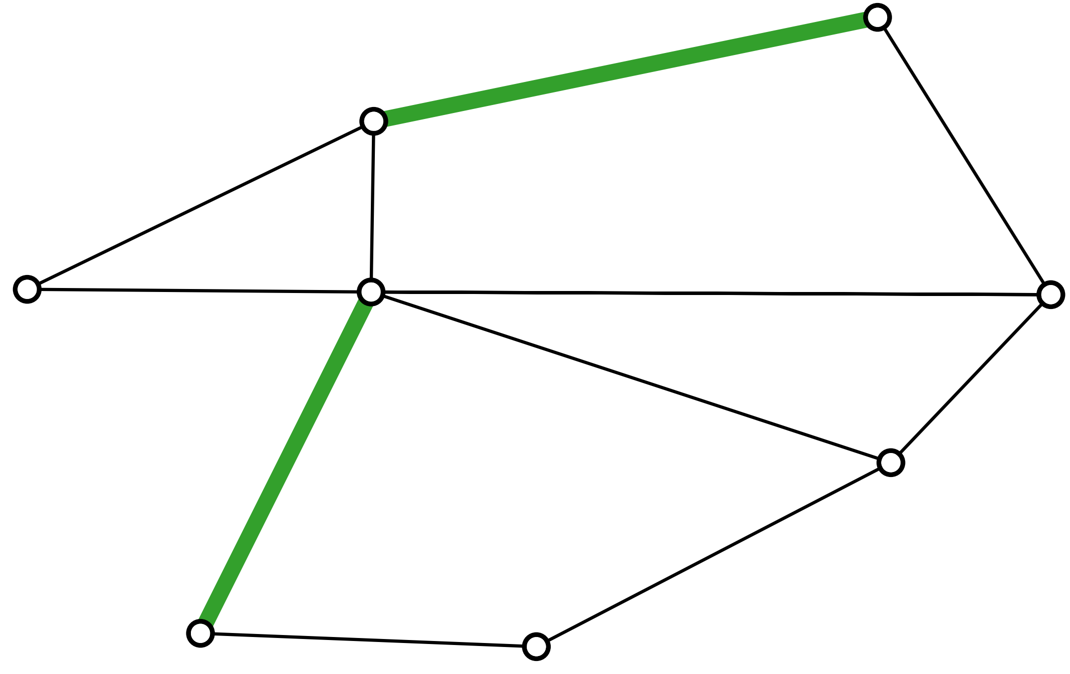

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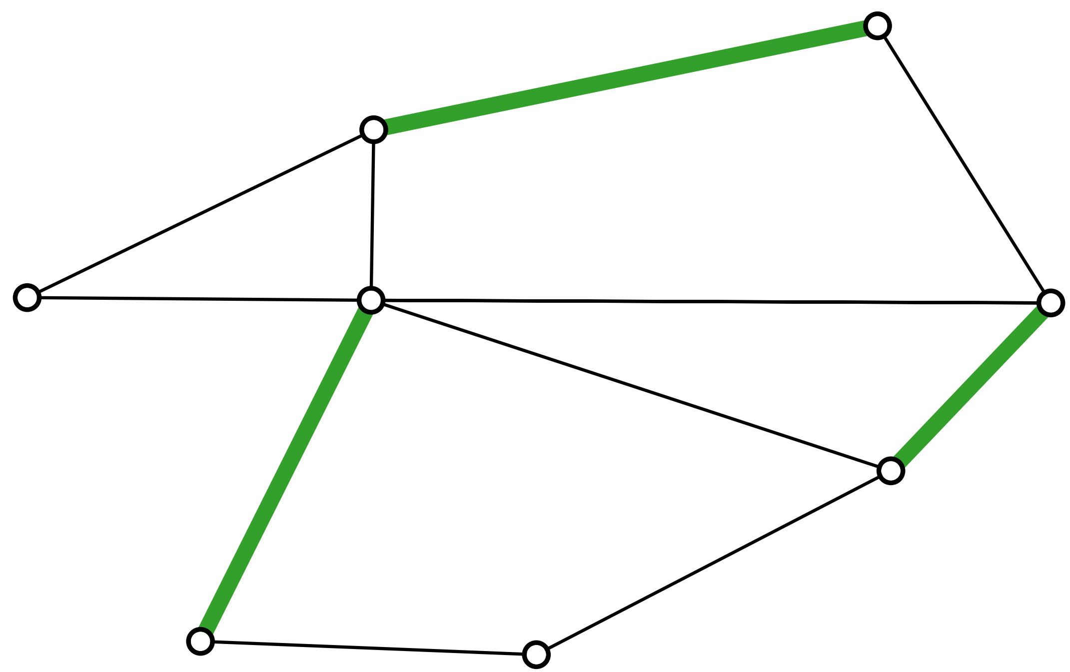

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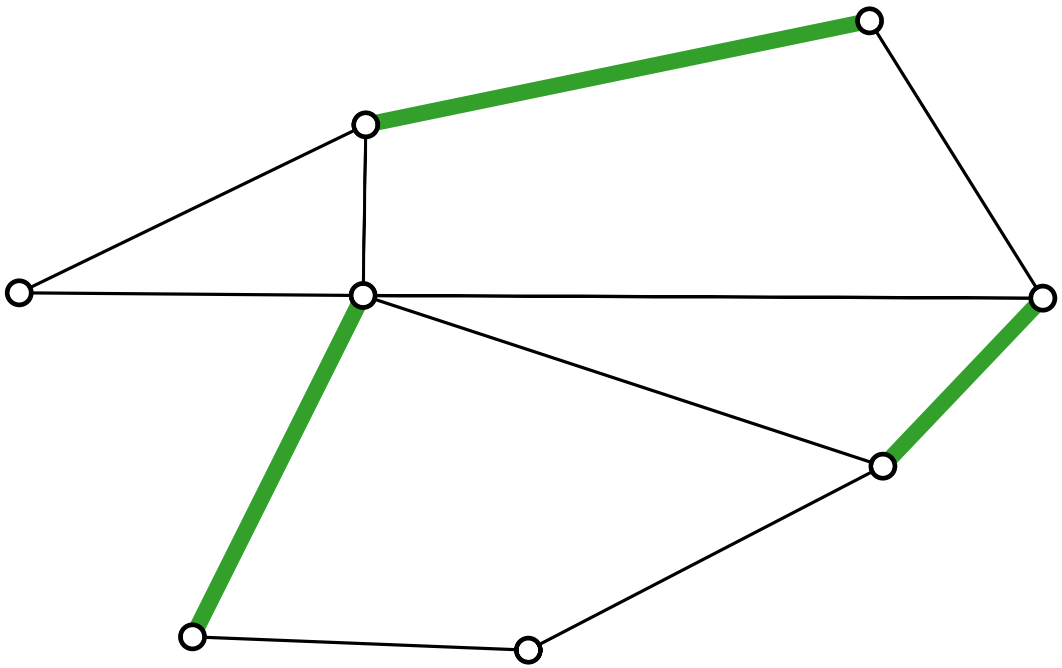

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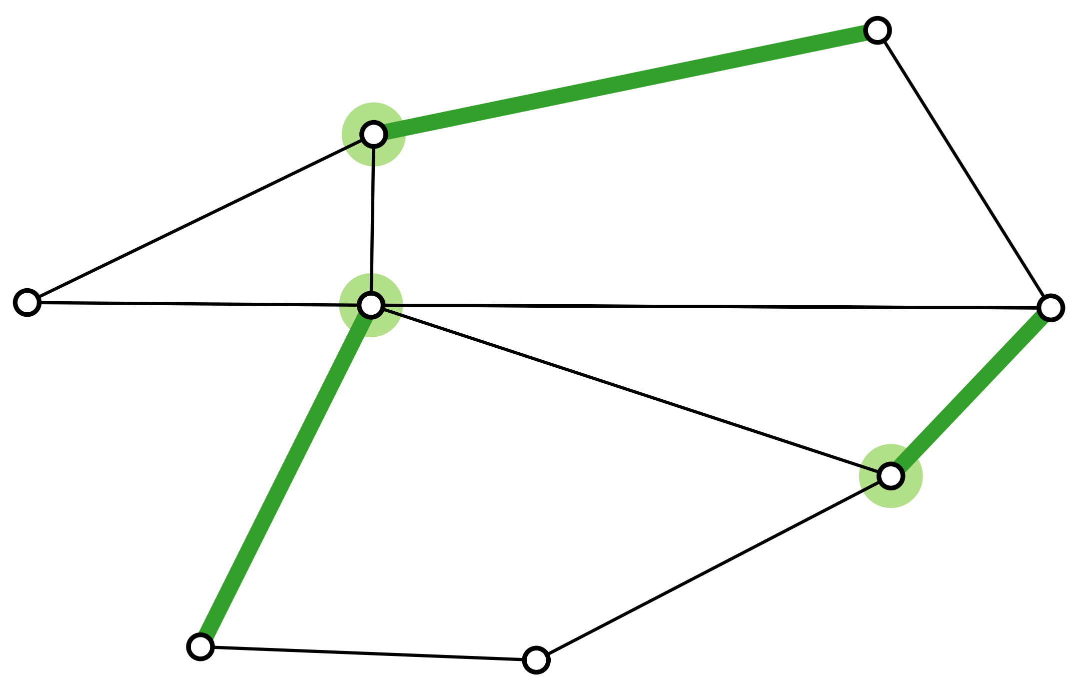
Vertex cover of M



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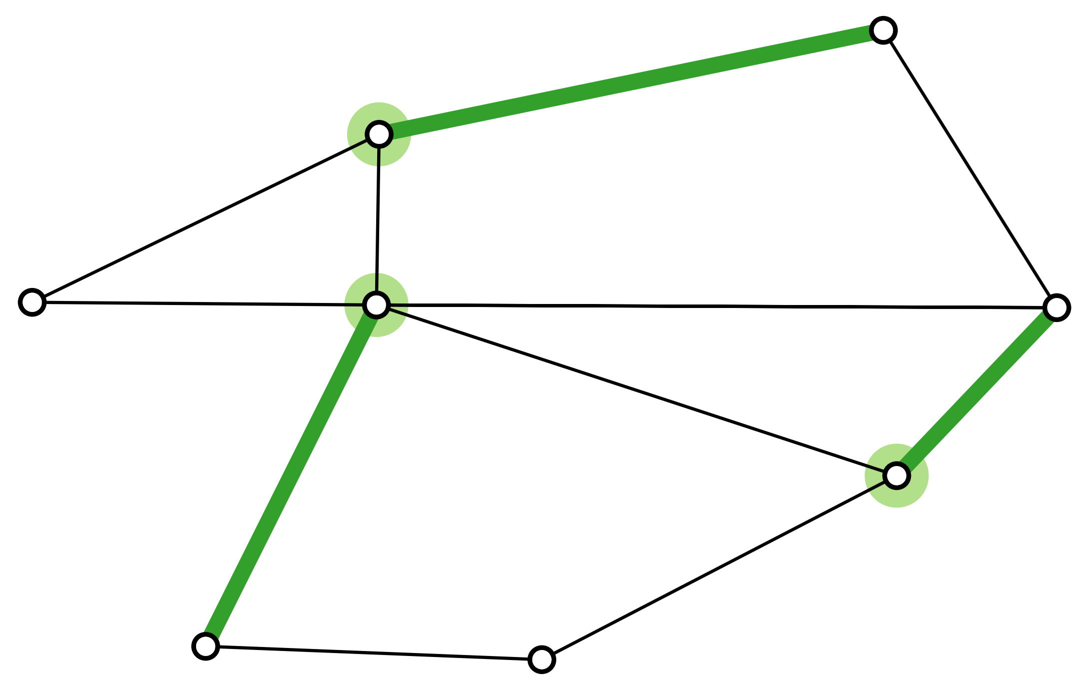

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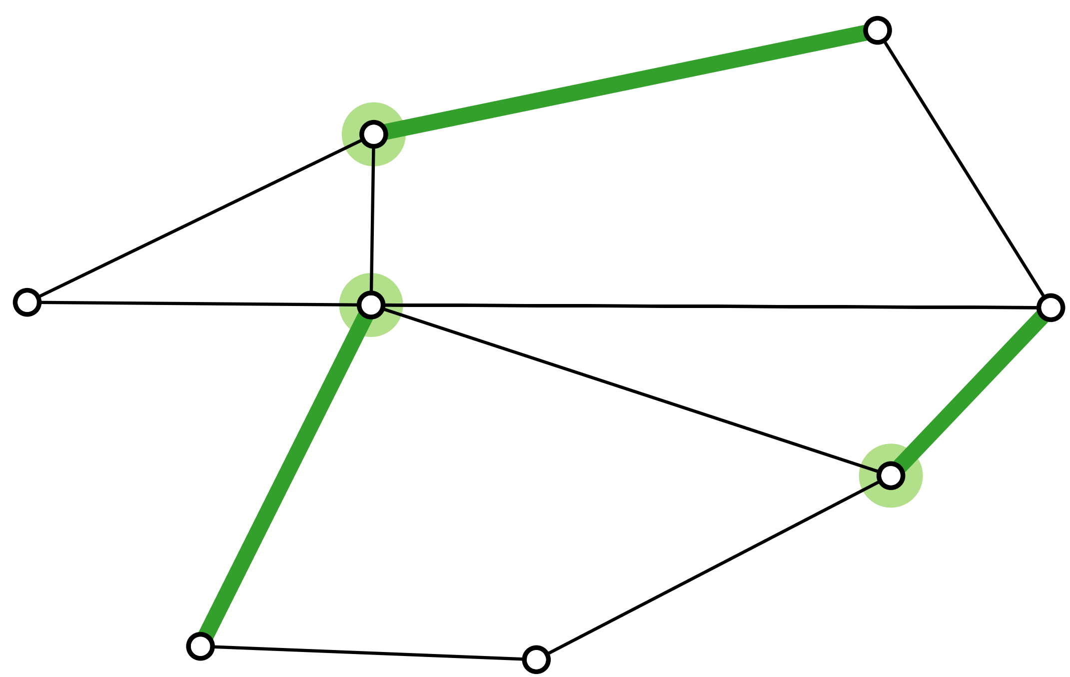

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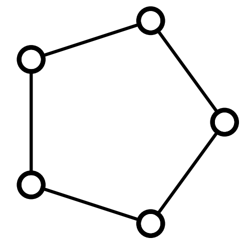

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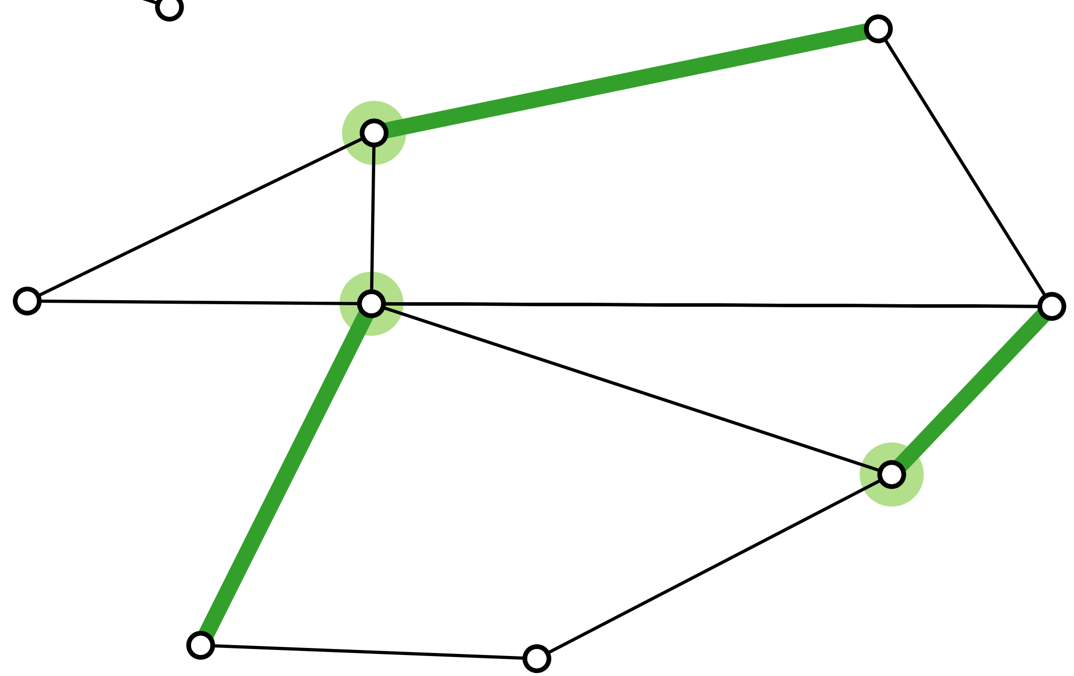

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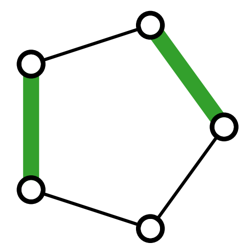

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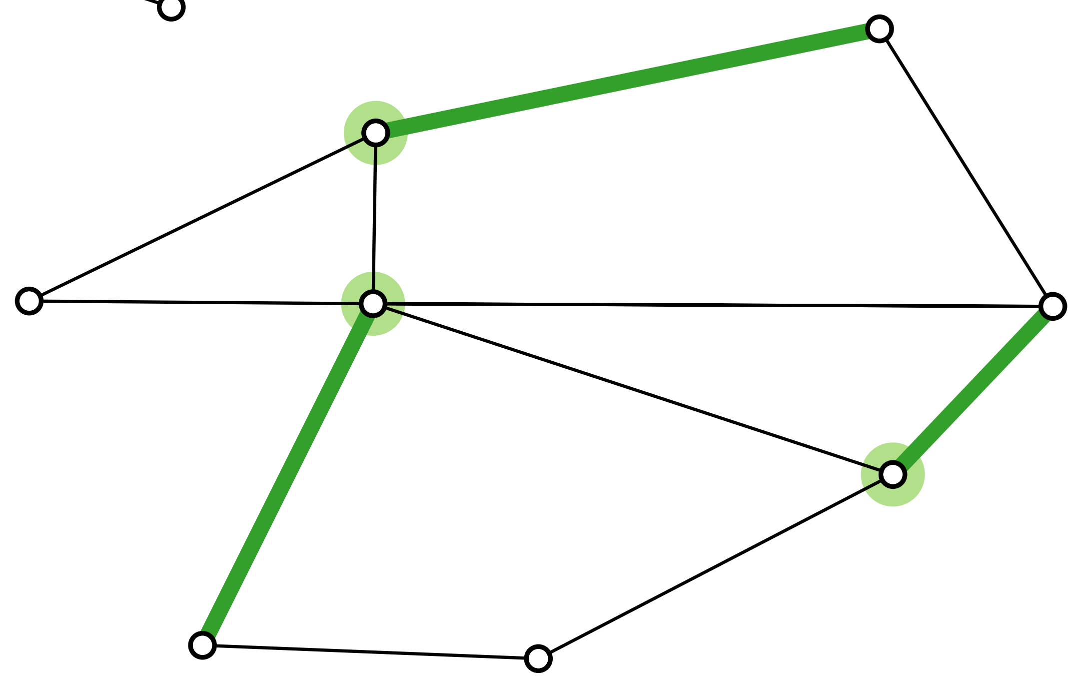

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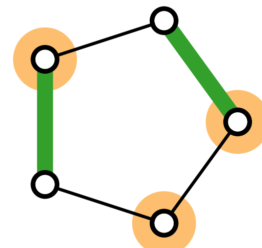

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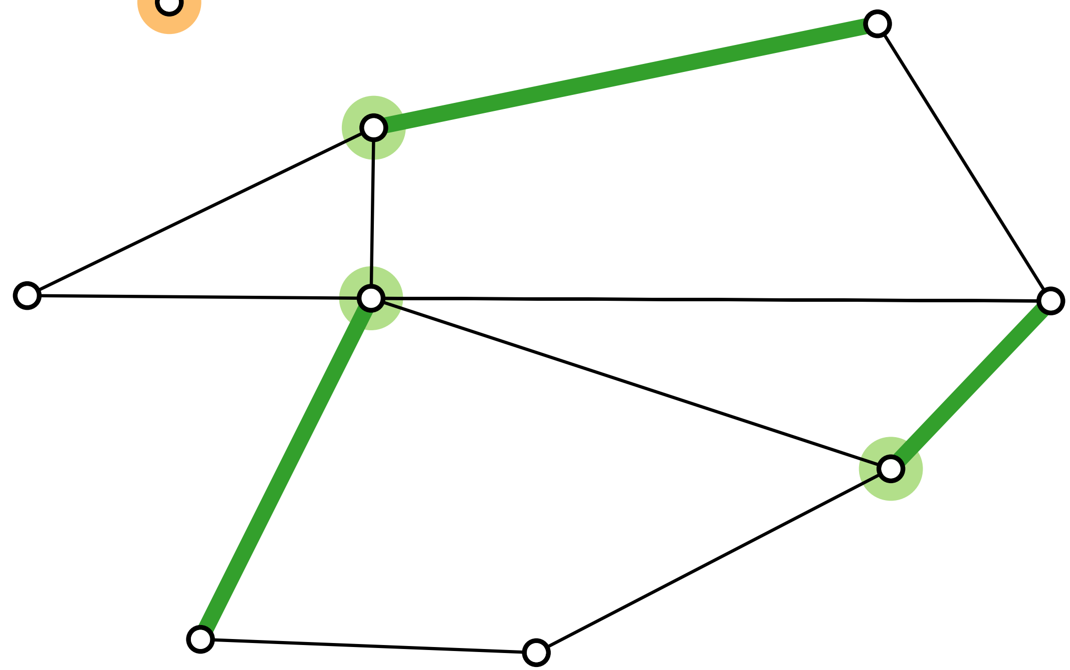

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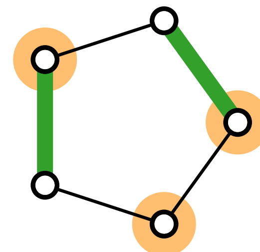

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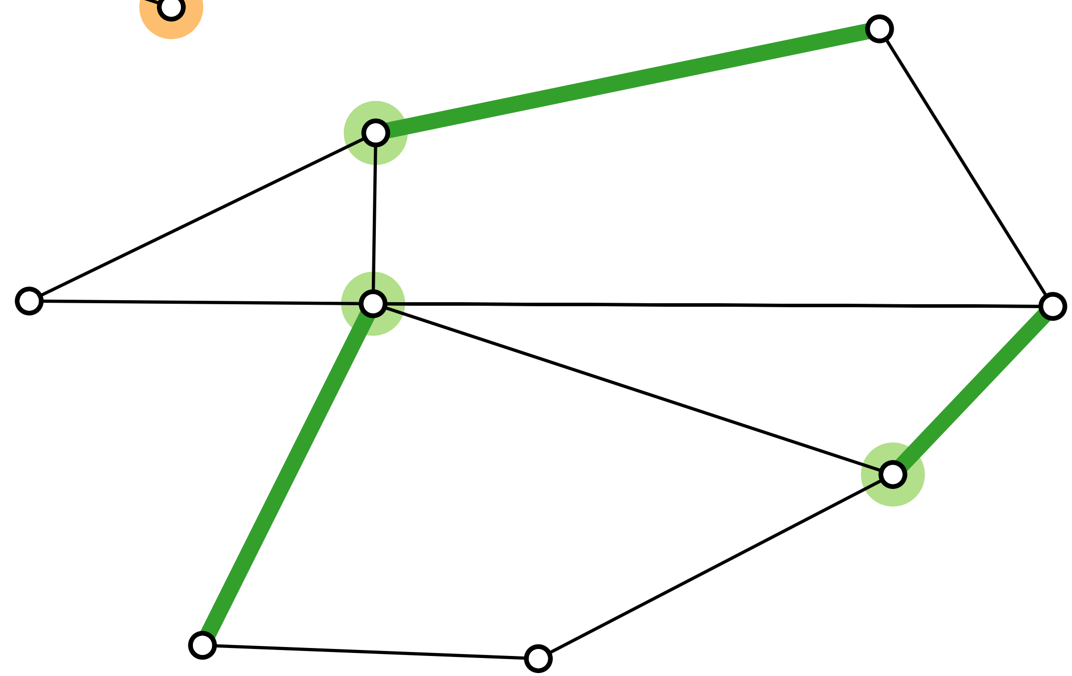

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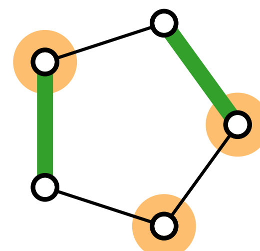

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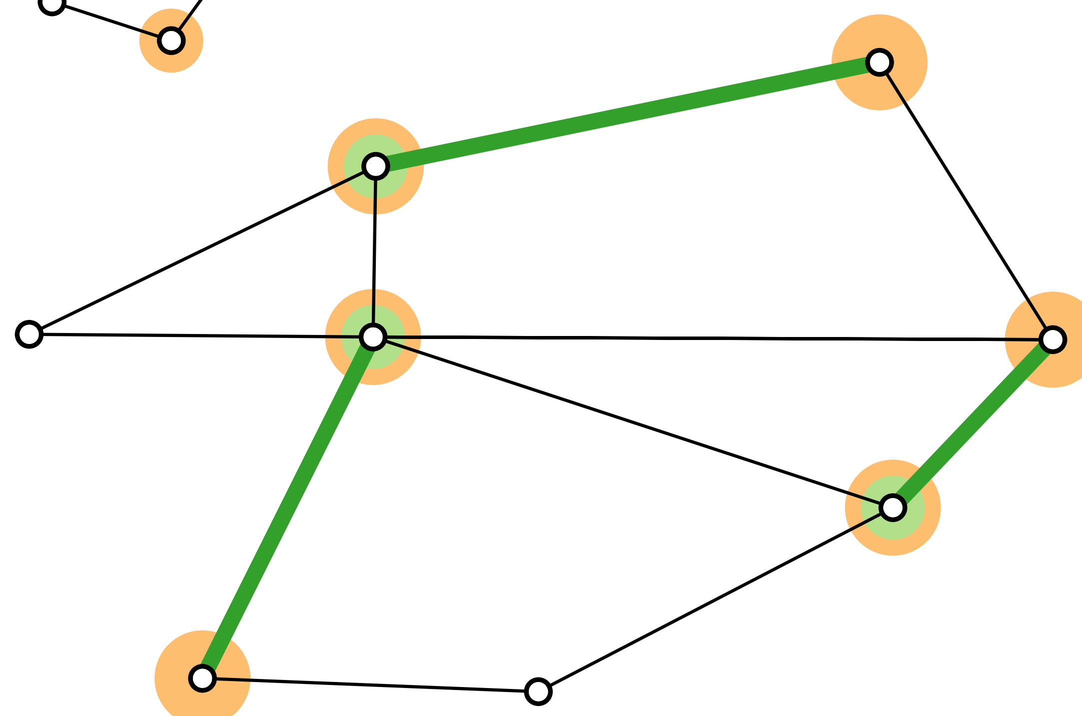

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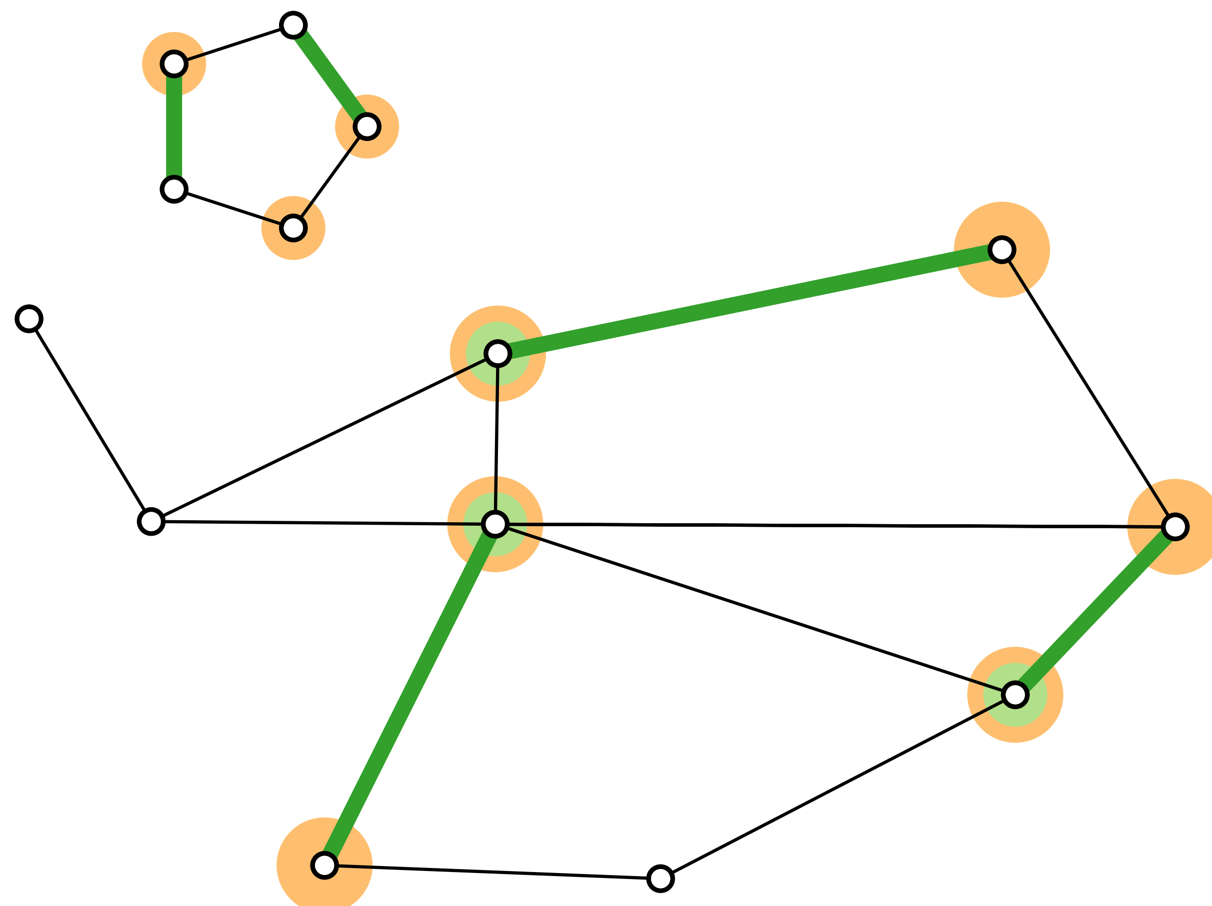

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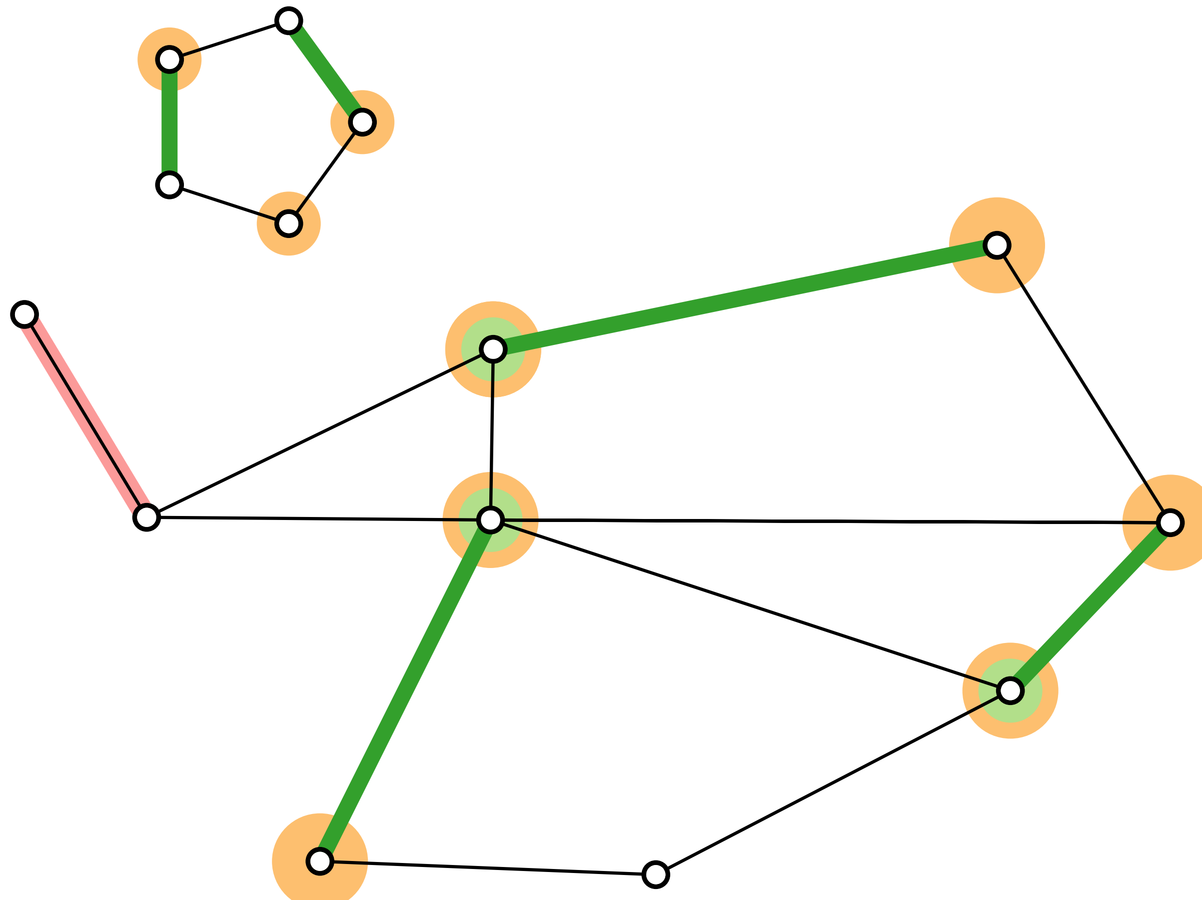

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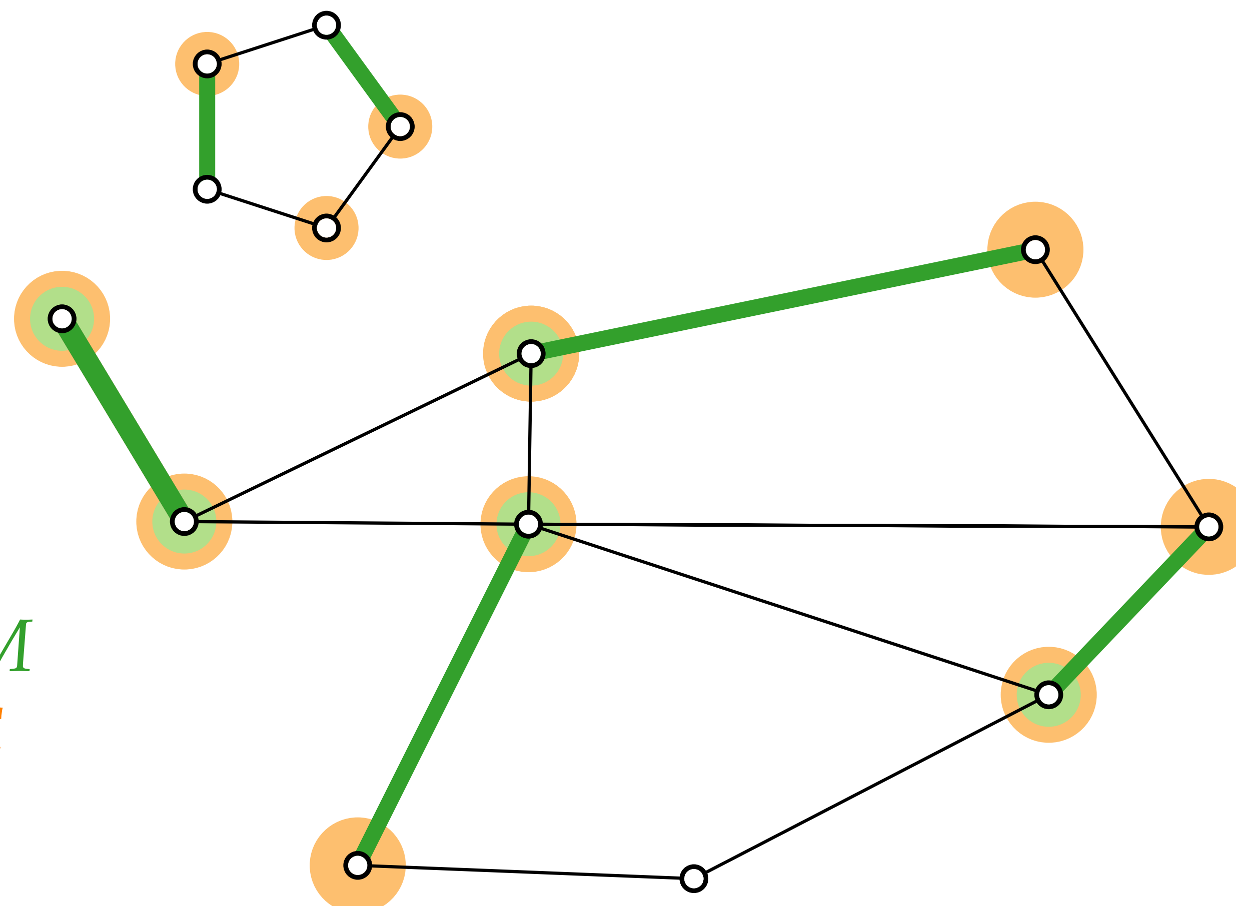

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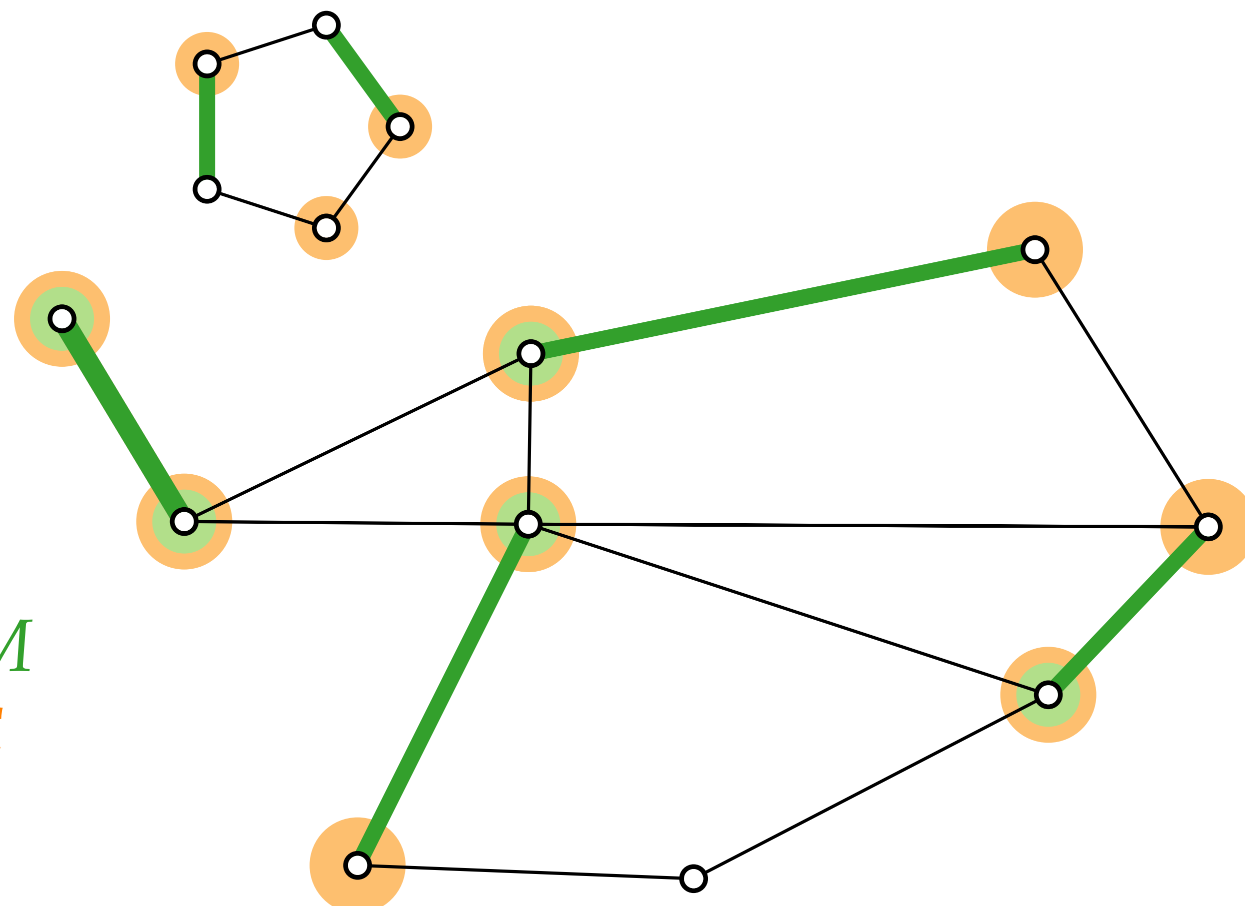

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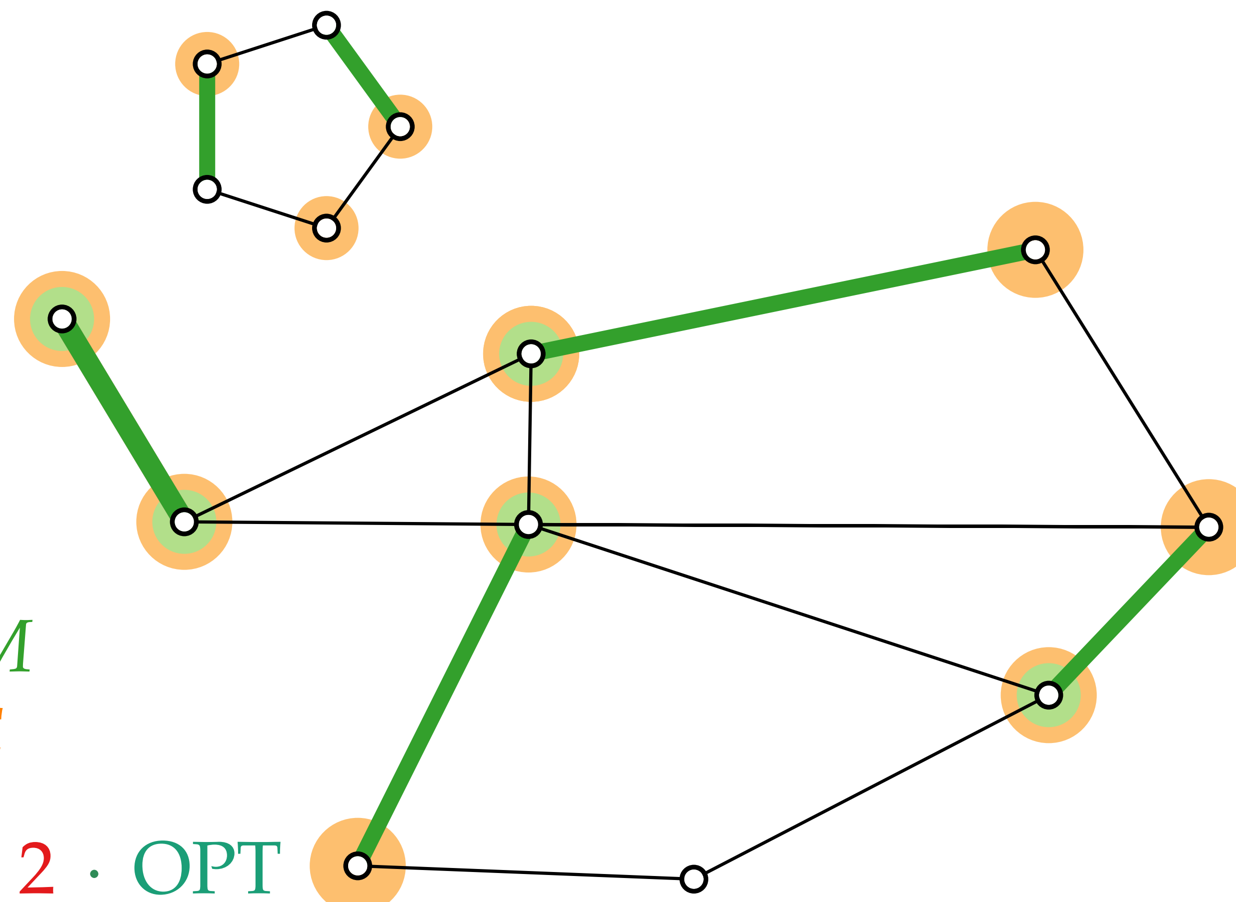

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