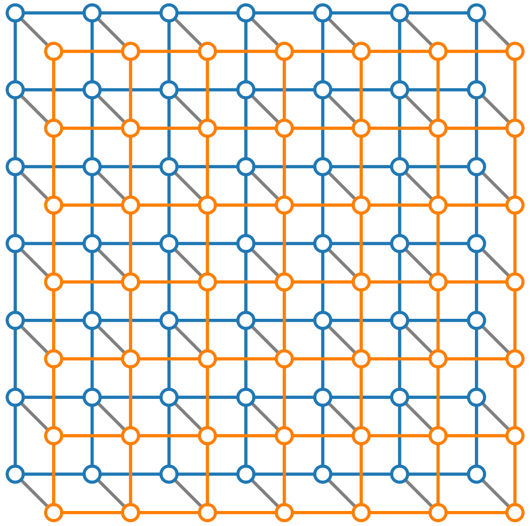
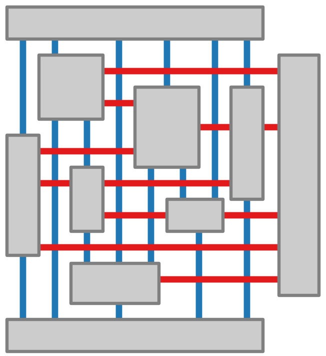
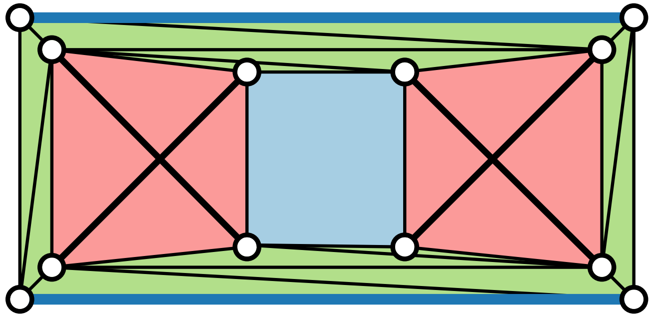
# Visualization of Graphs

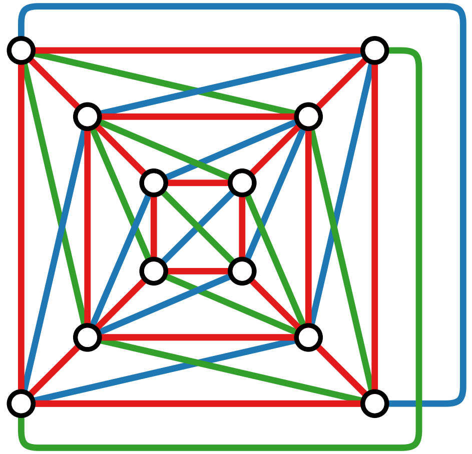




## Lecture 11:
## Beyond Planarity
### Drawing Graphs with Crossings

### Part I:
### Graph Classes and Drawing Styles





Jonathan Klawitter

Partially based on slides by Fabrizio Montecchini, Michalis Bekos, and Walter Didimo.

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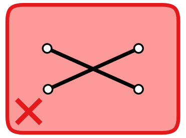

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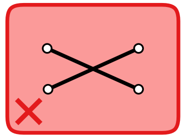

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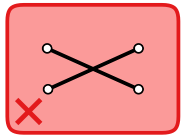

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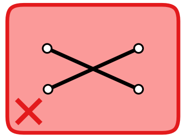

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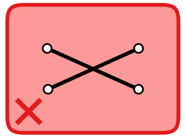

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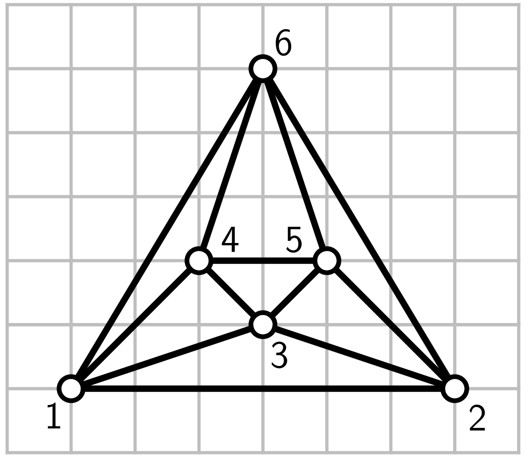


straight-line drawing

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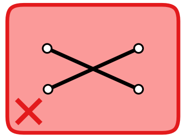

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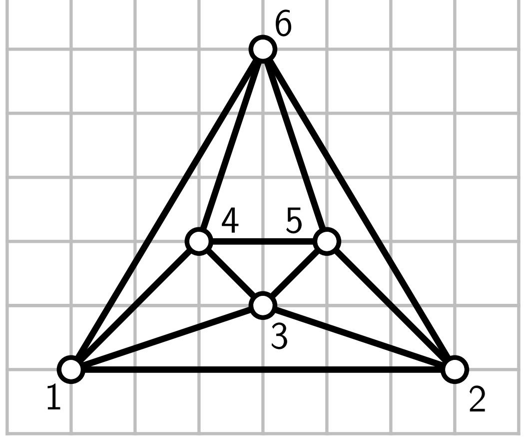


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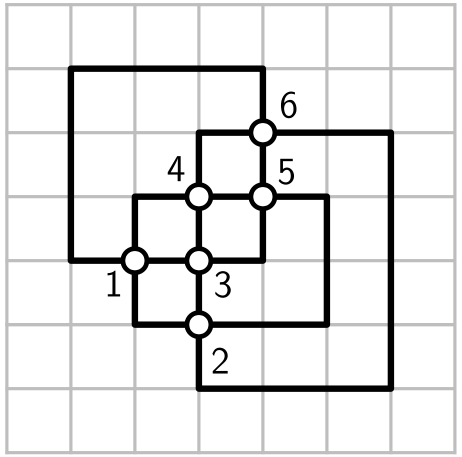


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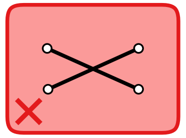

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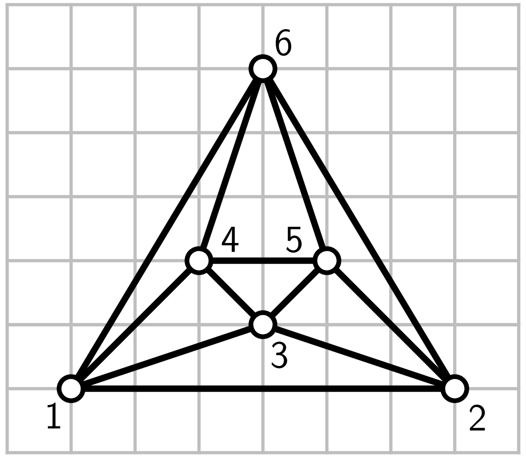


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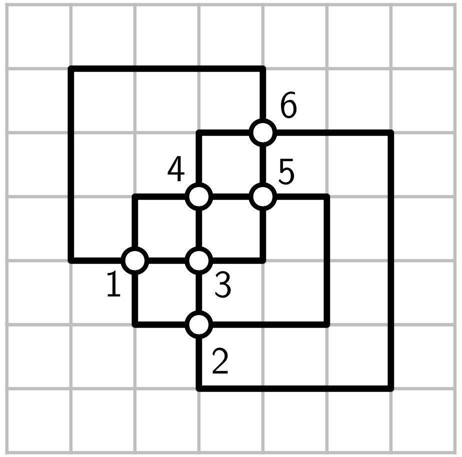


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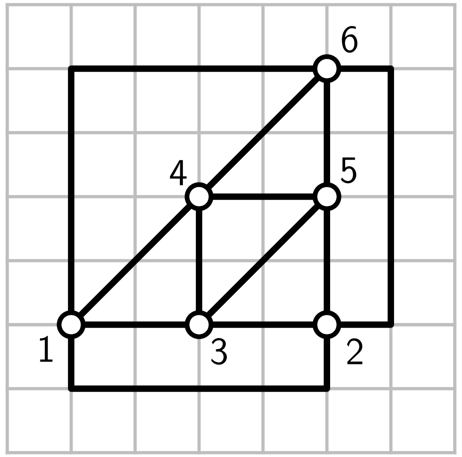


grid drawing with bends & 3 slopes

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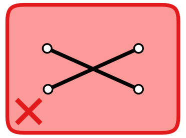

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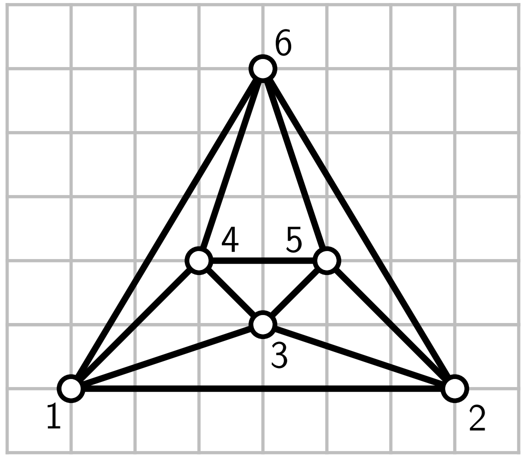


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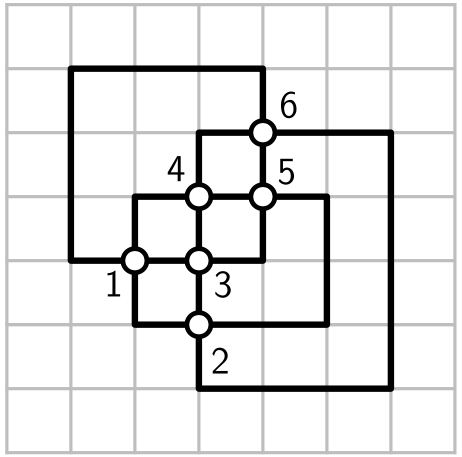


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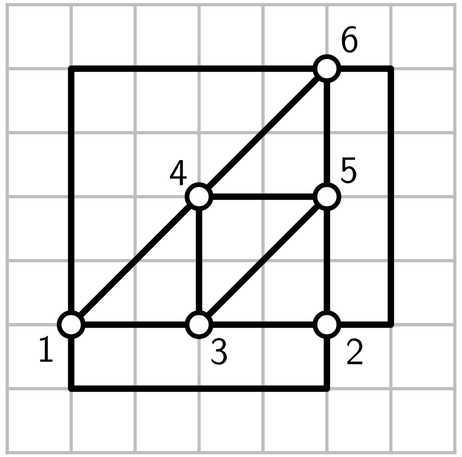


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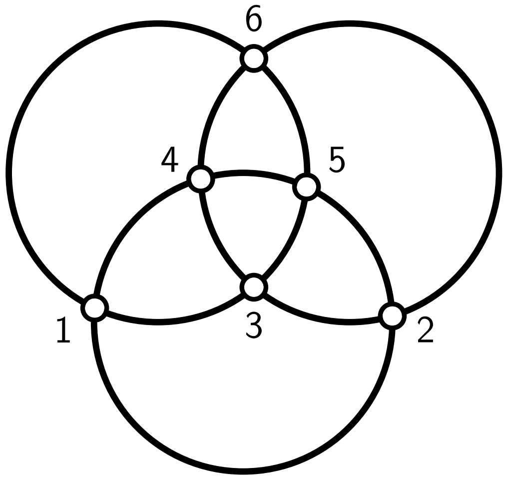


circular-arc drawing

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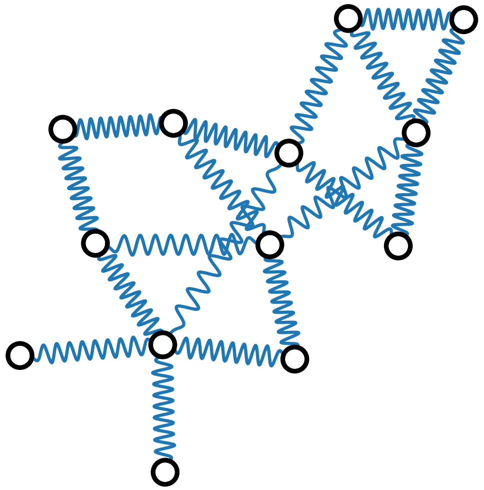

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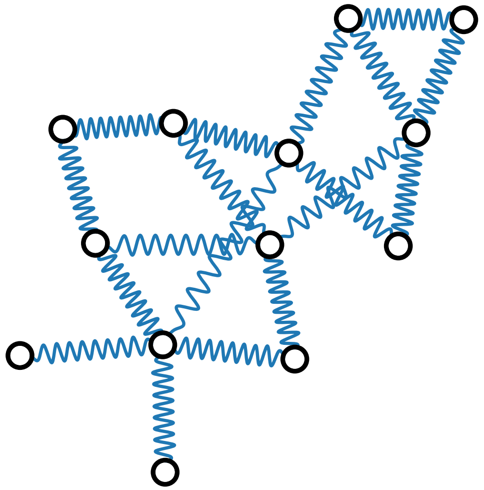

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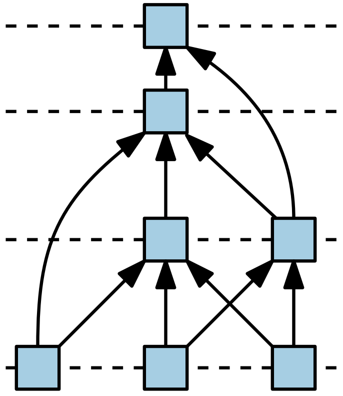

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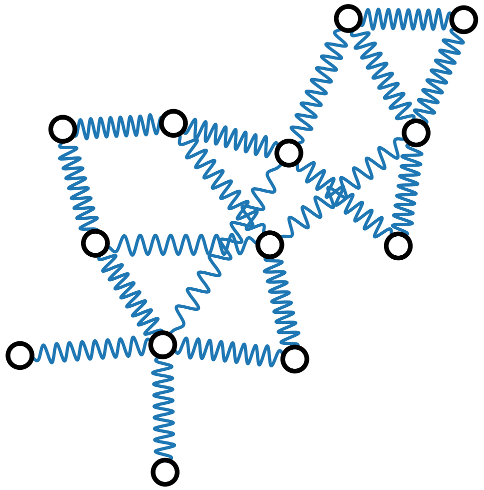

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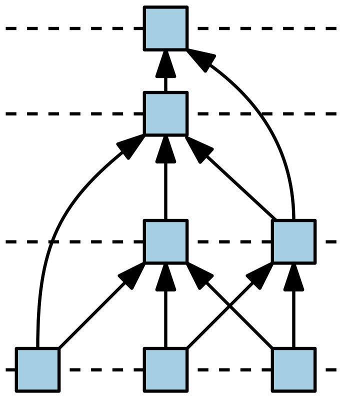

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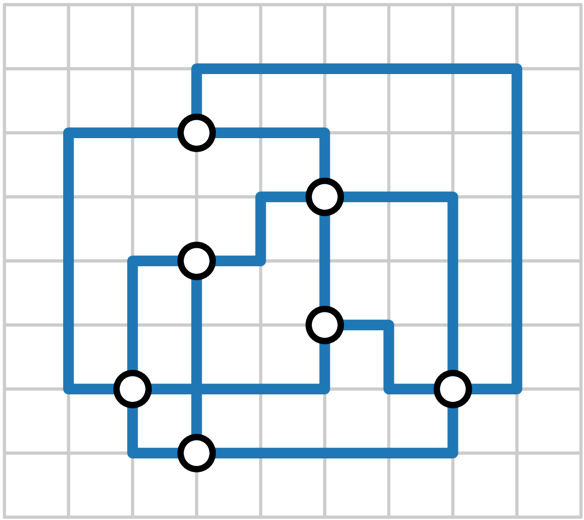

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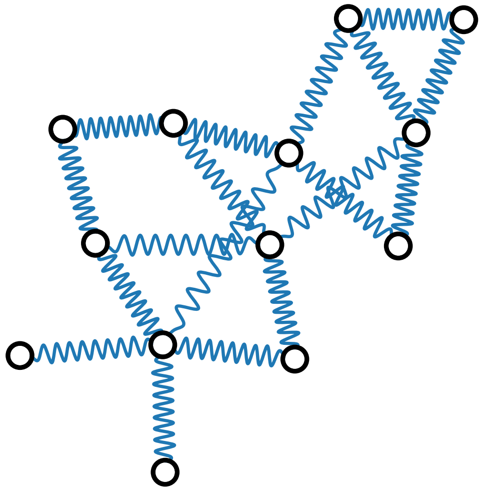

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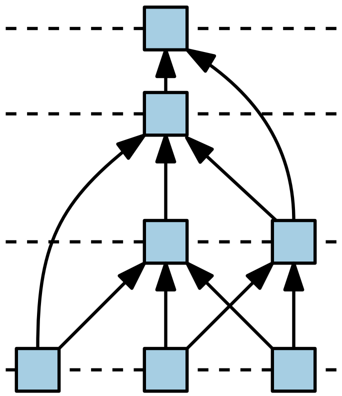

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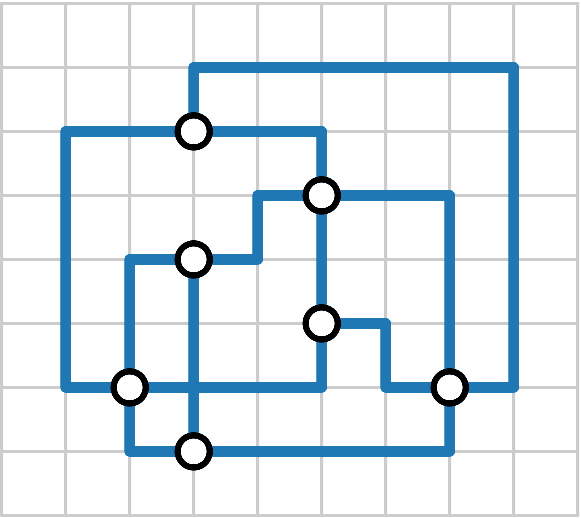

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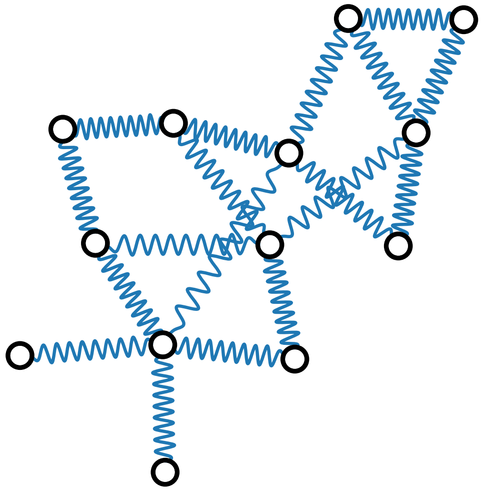

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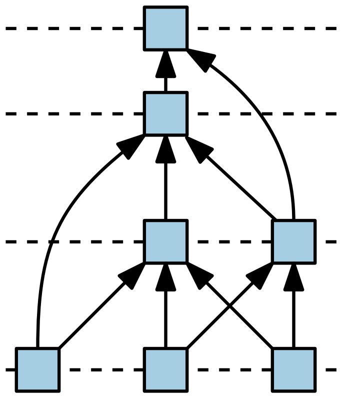

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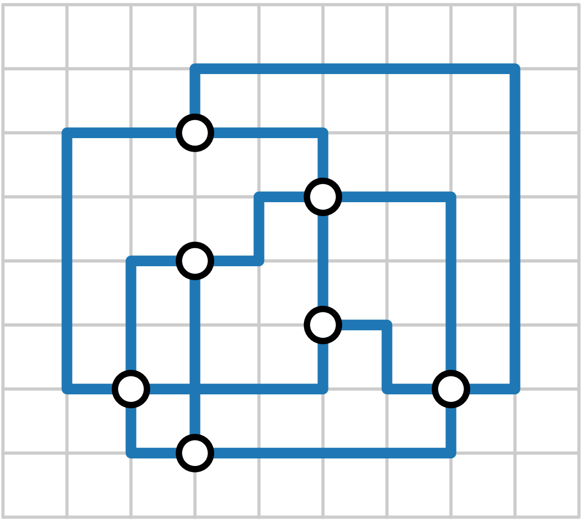

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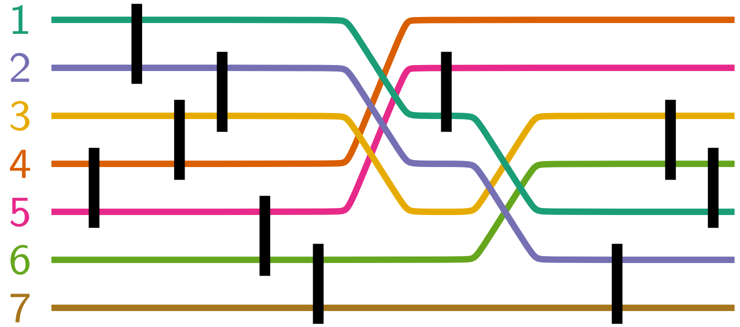


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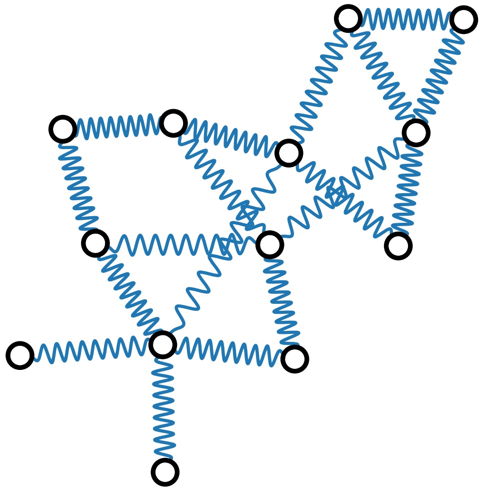

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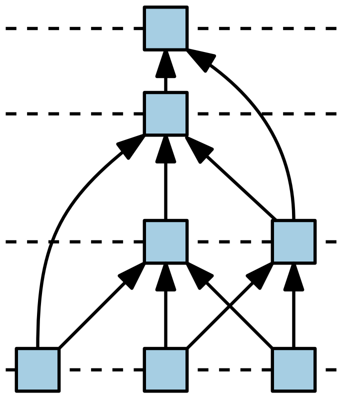

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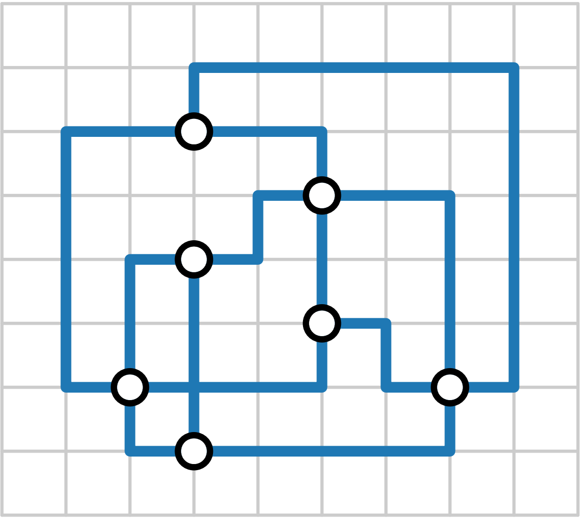

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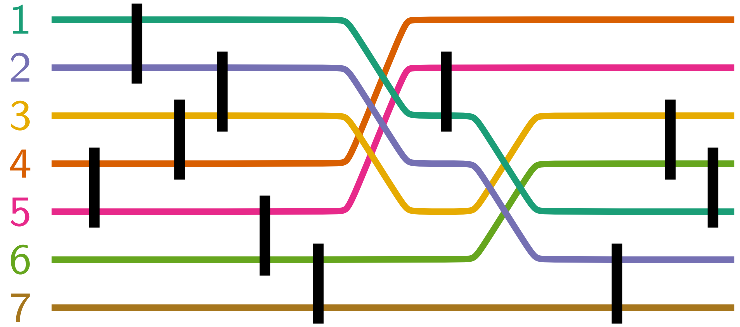


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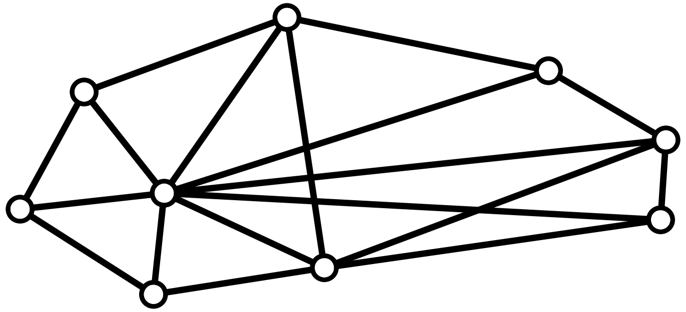

Which crossings feel worse?

# Eye-Tracking Experiment

[Eades, Hong & Huang 2008]

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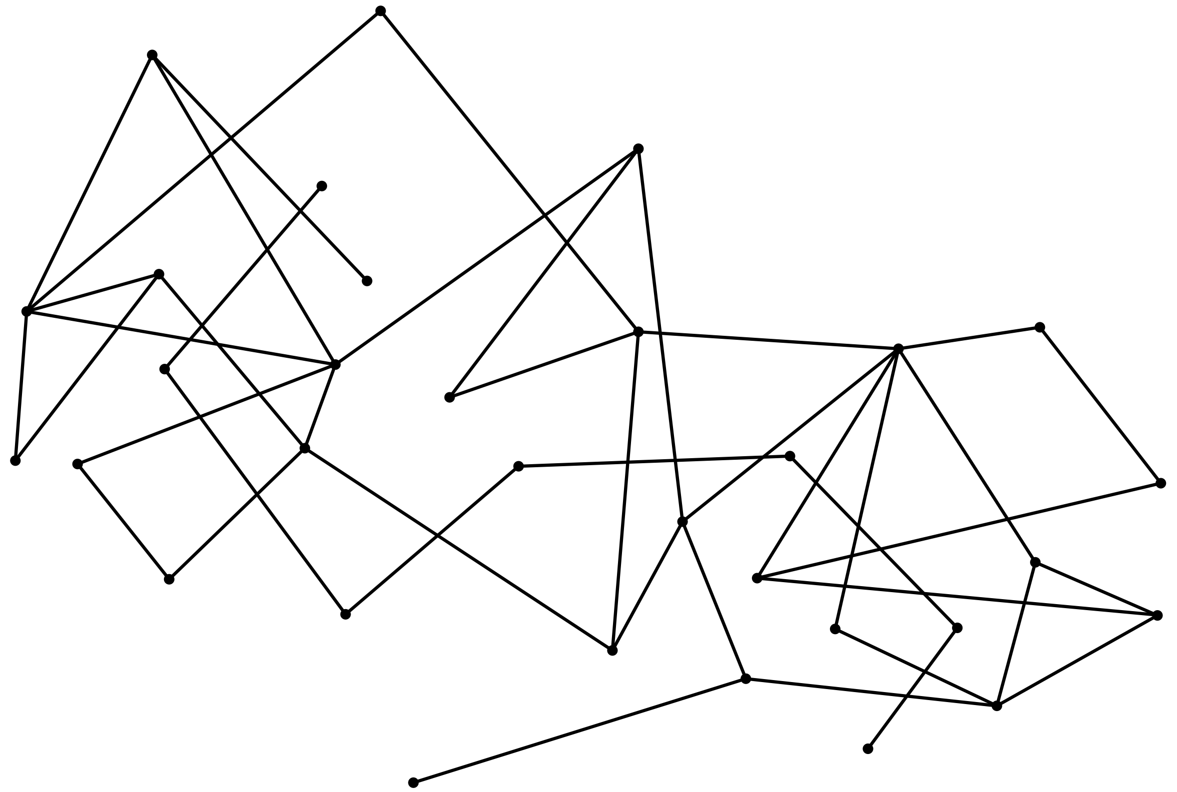

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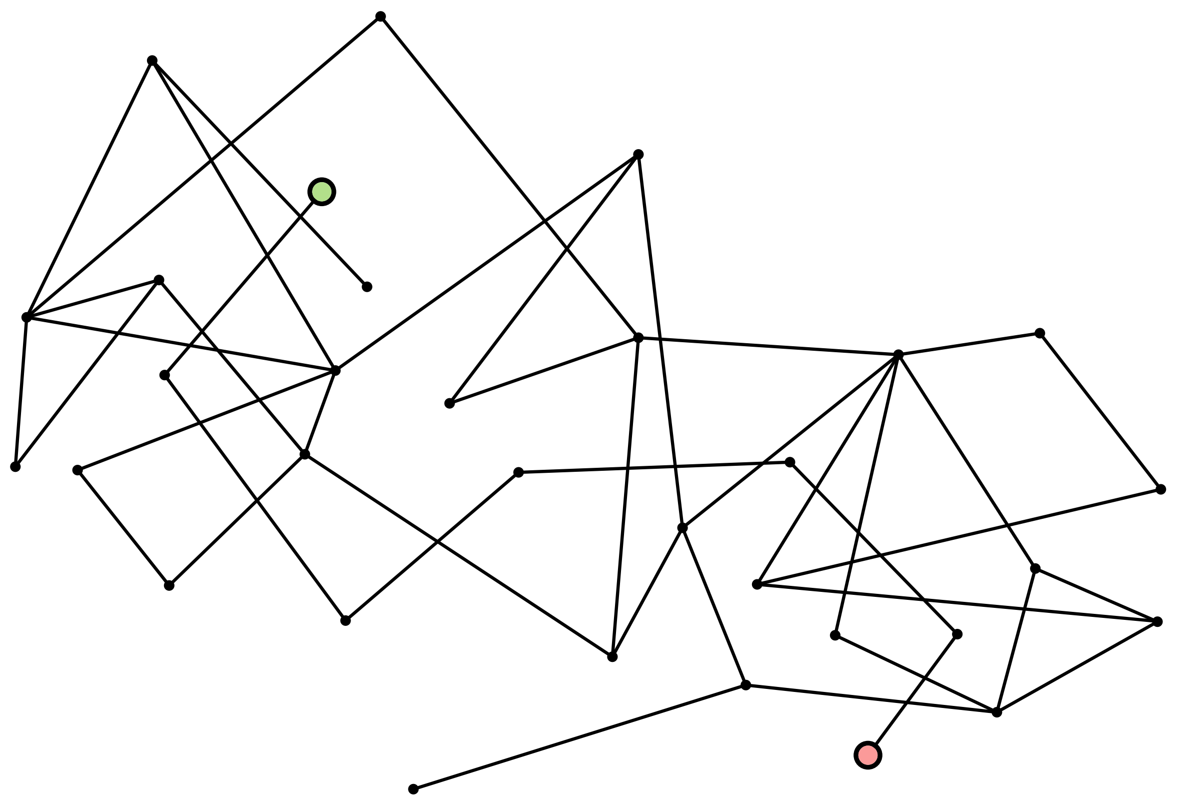

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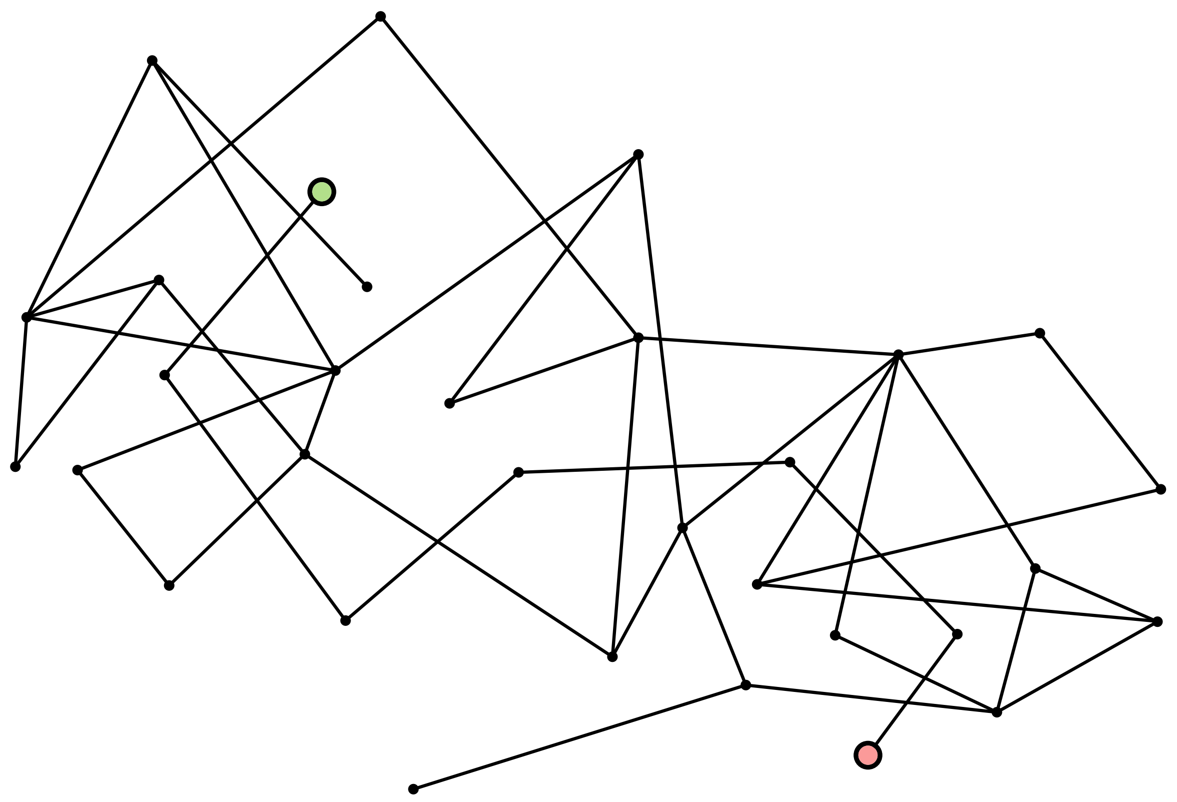

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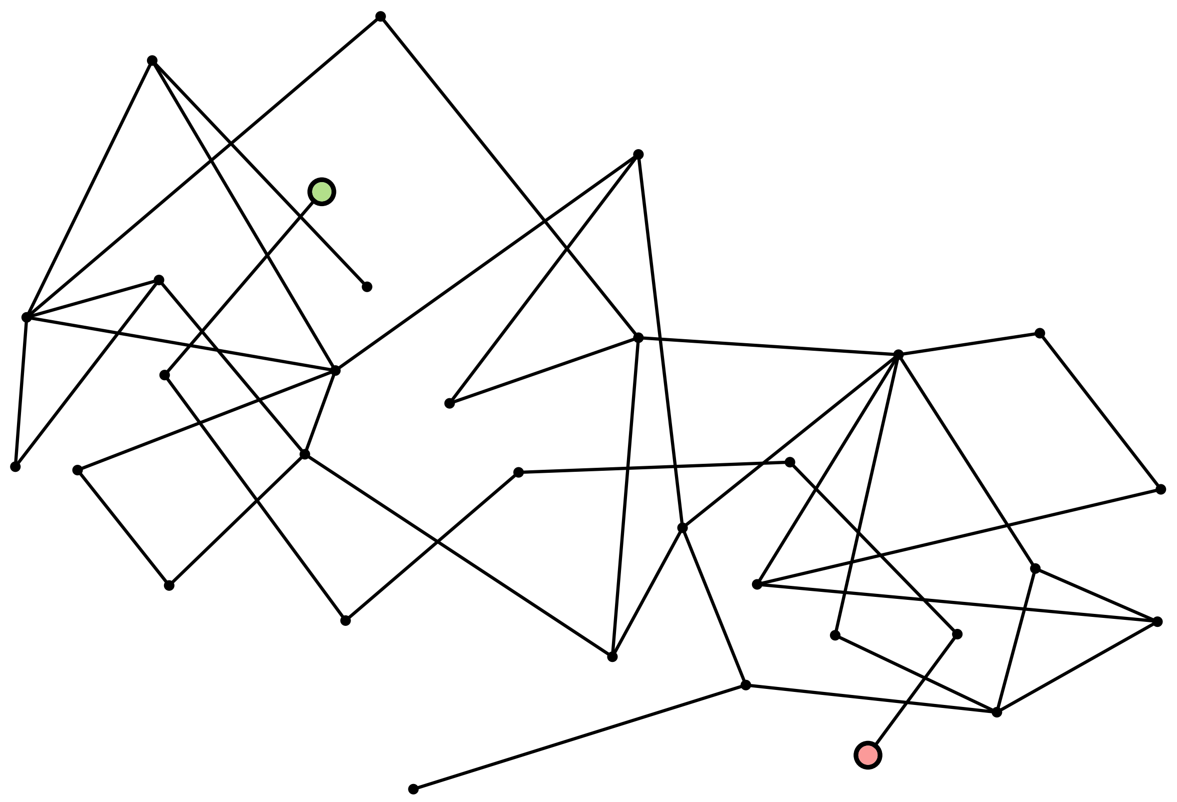

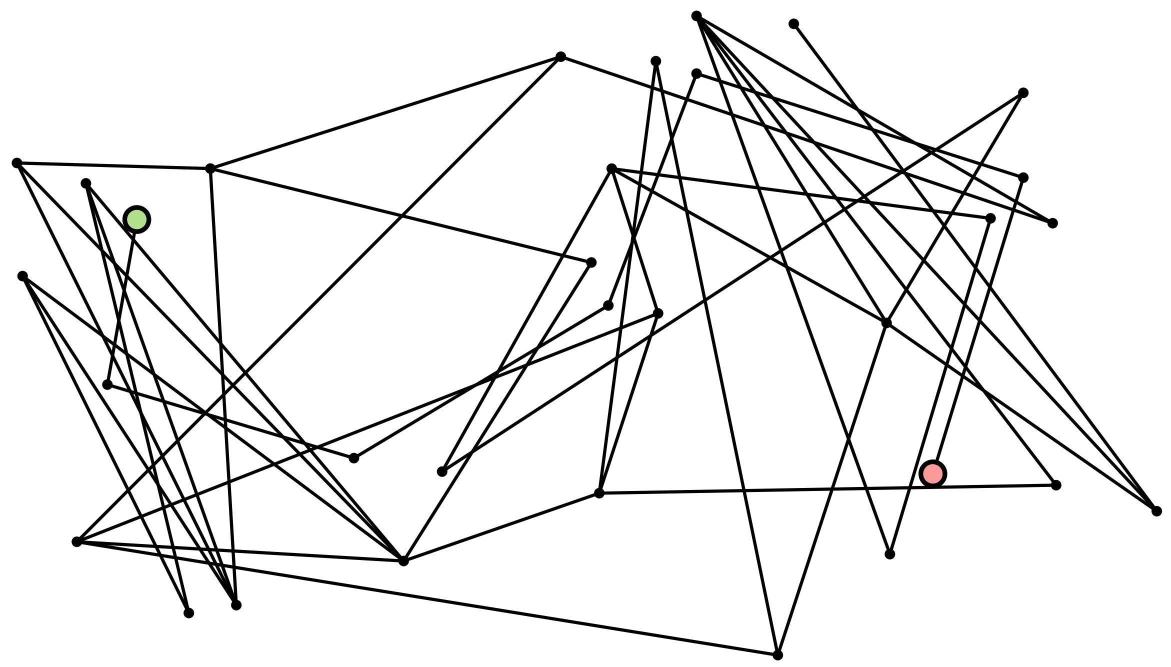

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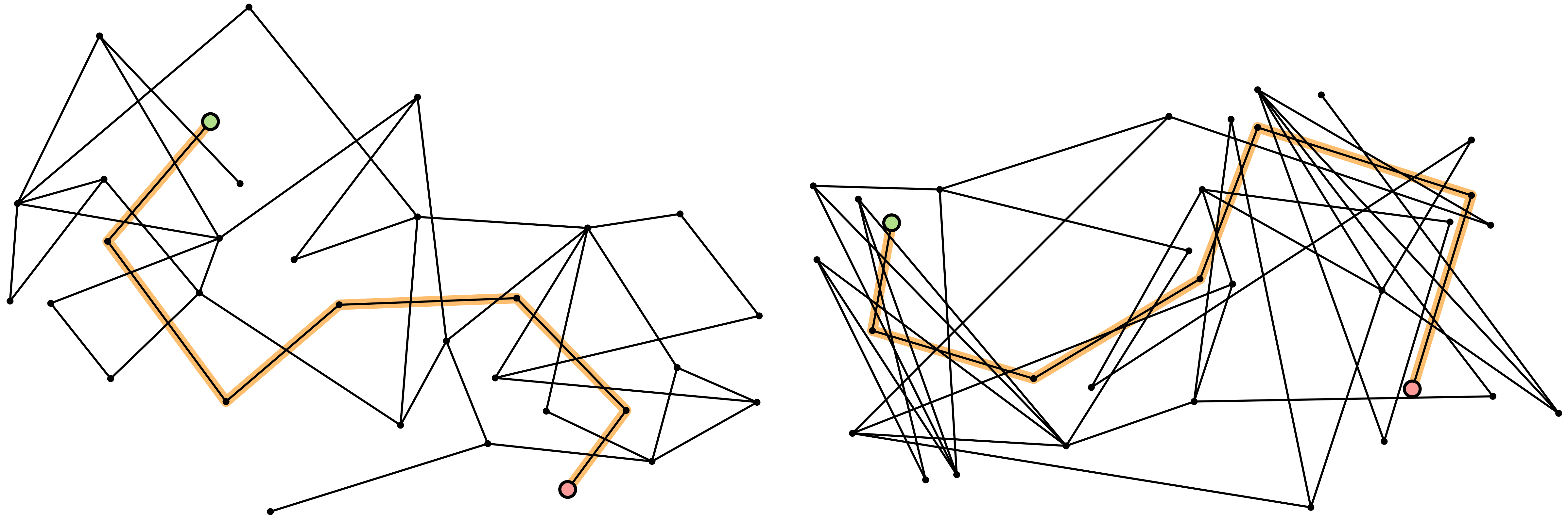

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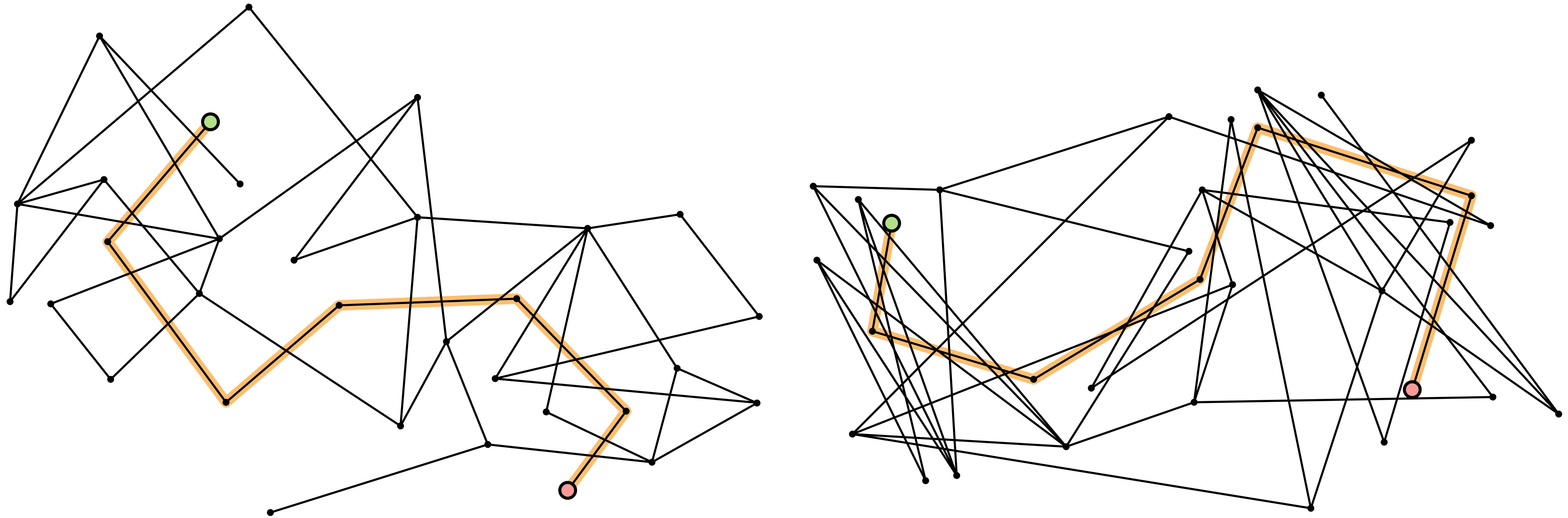

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**Input:** A graph drawing and designated path.

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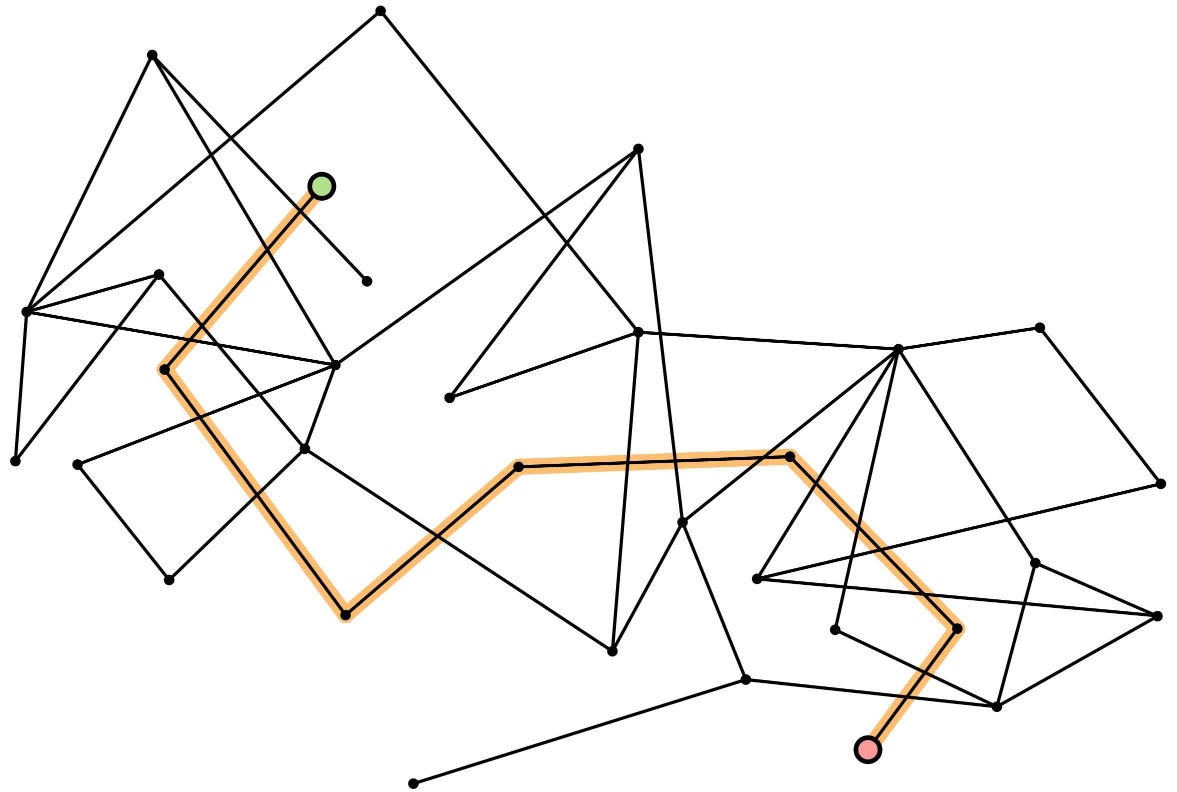
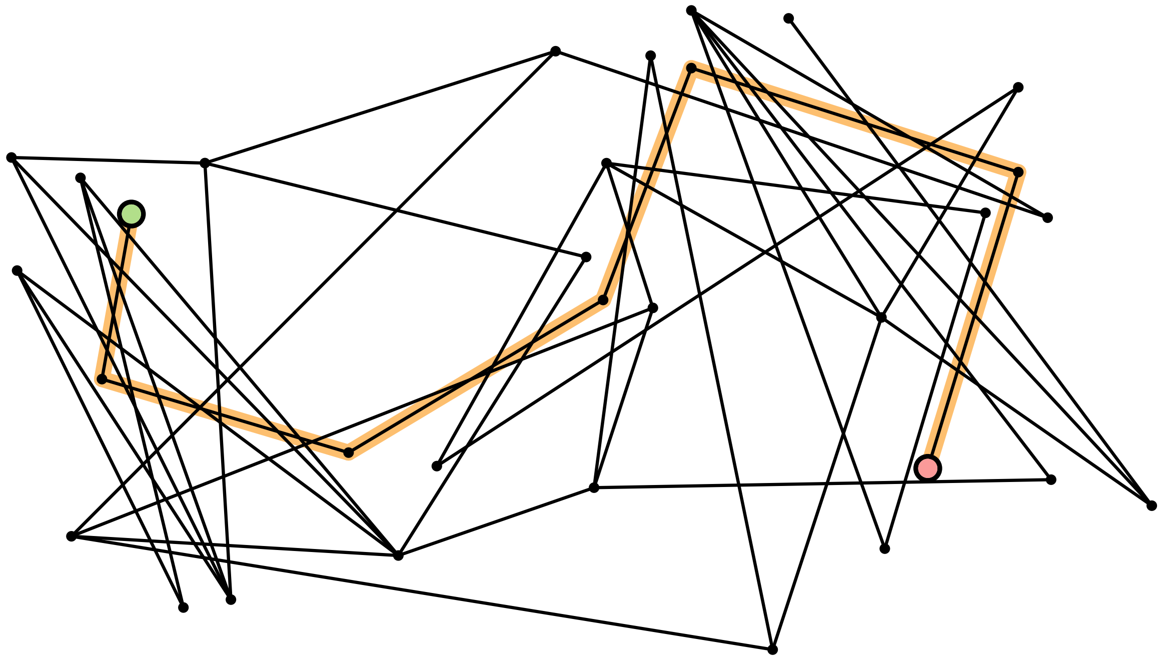
**Results:** no crossings eye movements smooth and fast





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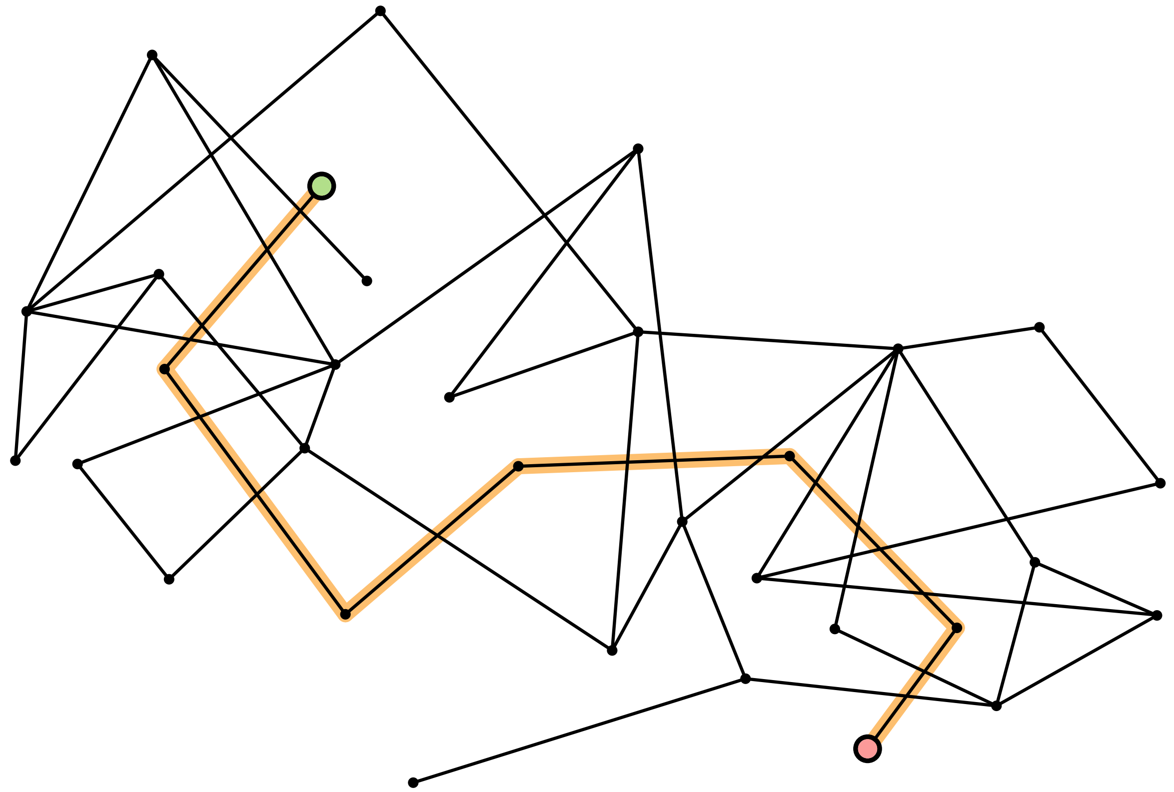
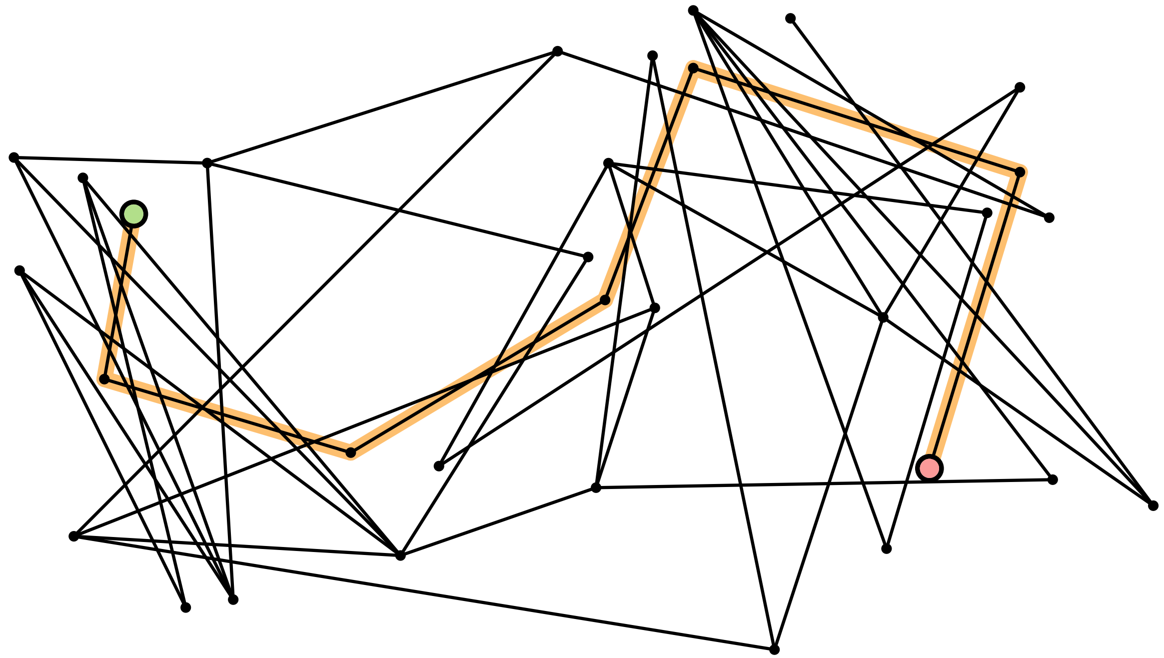
**Results:** no crossings — eye movements smooth and fast

large crossing angles — eye movements smooth but slightly slower





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**Results:**

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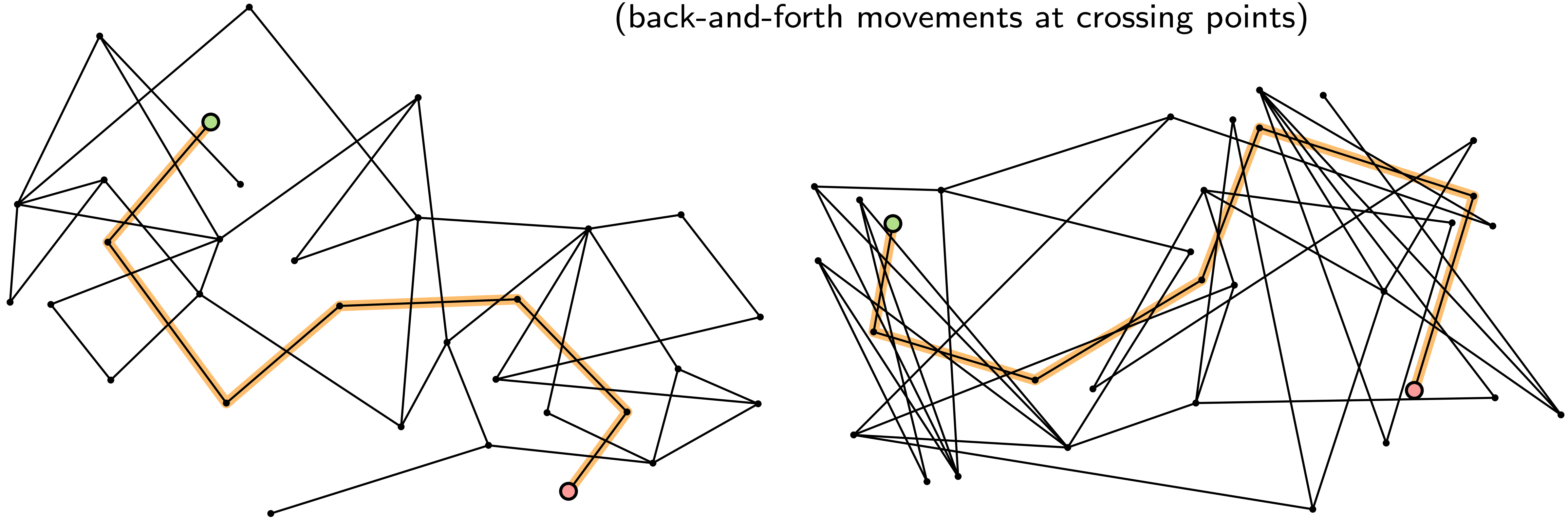
| no crossings | eye movements smooth and fast |
| large crossing angles | eye movements smooth but slightly slower |
| small crossing angles | eye movements no longer smooth and very slow (back-and-forth movements at crossing points) |



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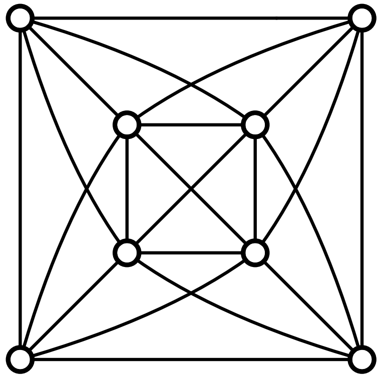

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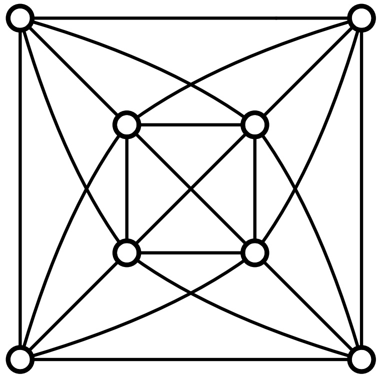

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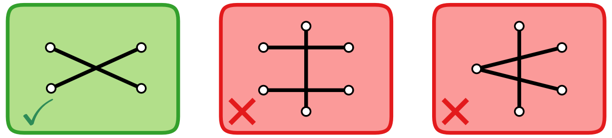

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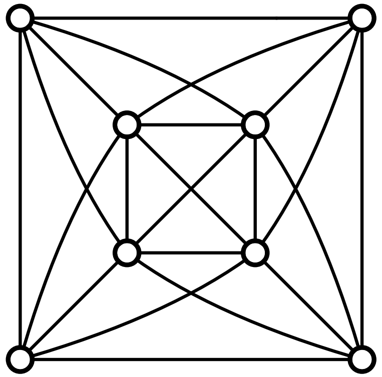

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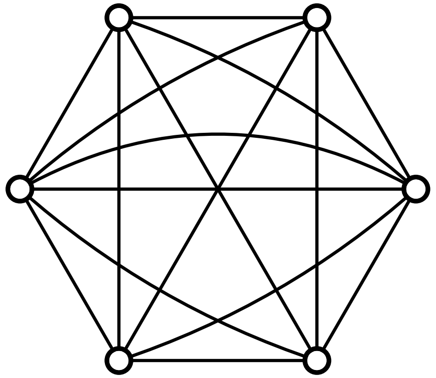

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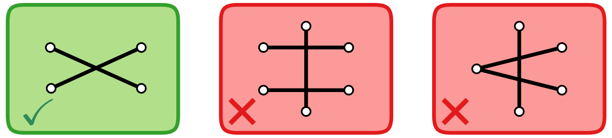

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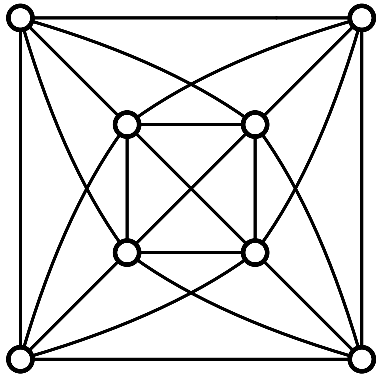

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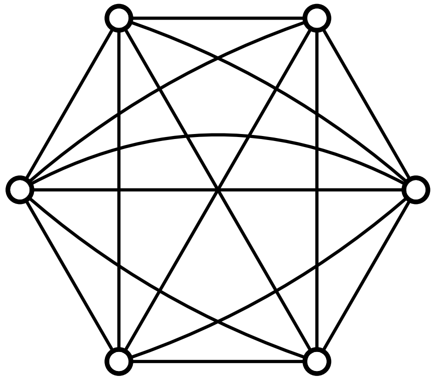

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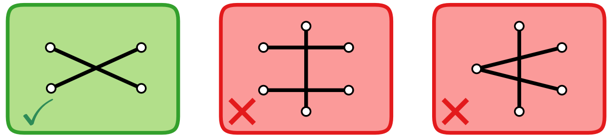

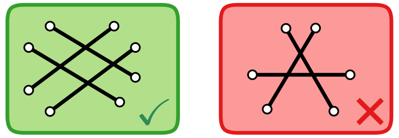

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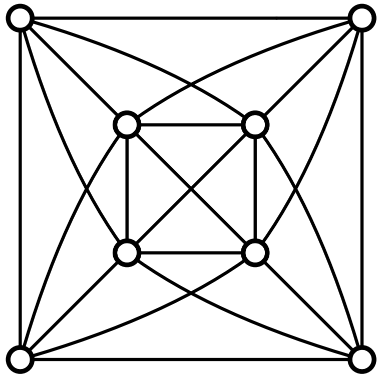
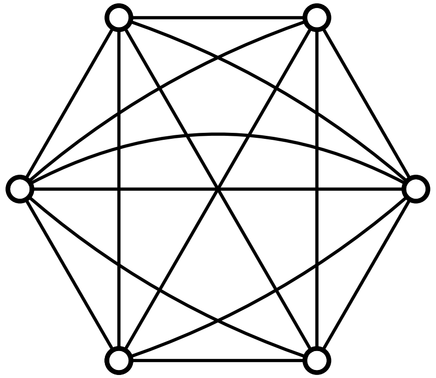
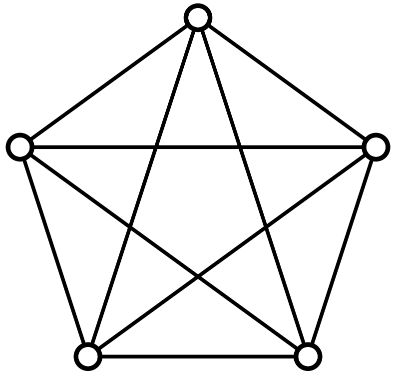


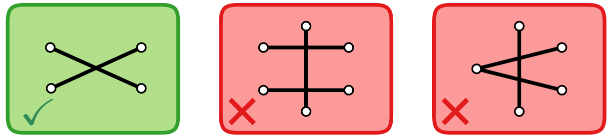
$k$-planar ($k = 1$)

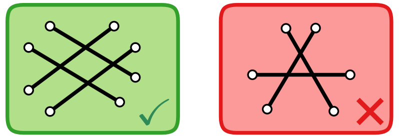


$k$-quasi-planar ($k = 3$)



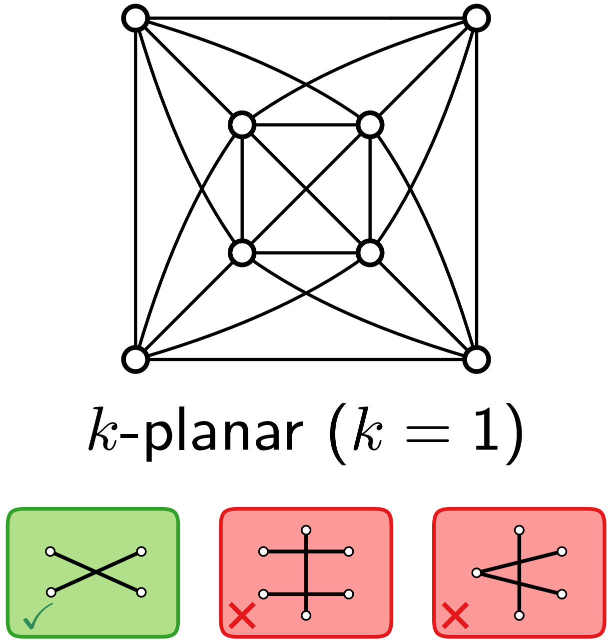
fan-planar



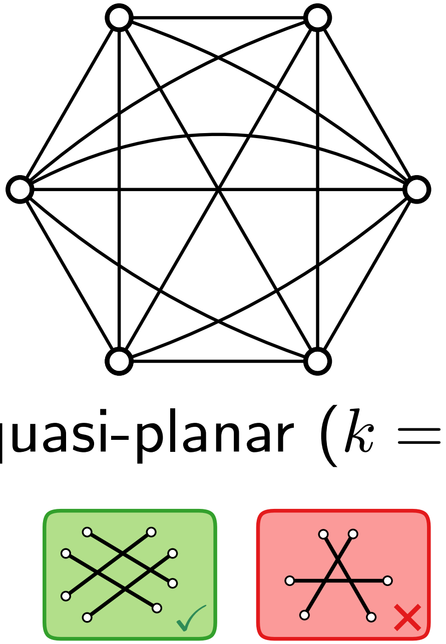


# Some Beyond-Planar Graph Classes

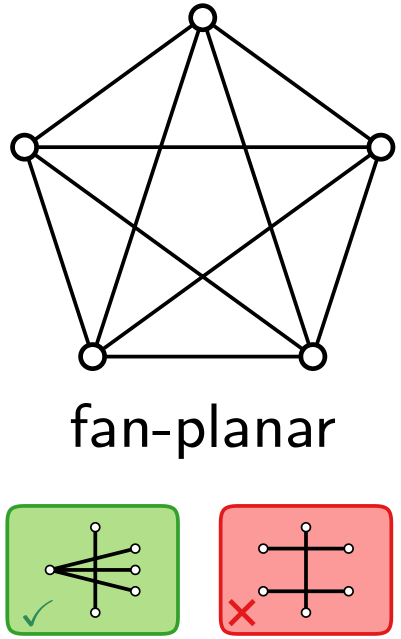
We define **aesthetics** for edge crossings and avoid/minimize “bad” crossing configurations.



$k$-planar ($k = 1$)



$k$-quasi-planar ($k = 3$)



fan-planar

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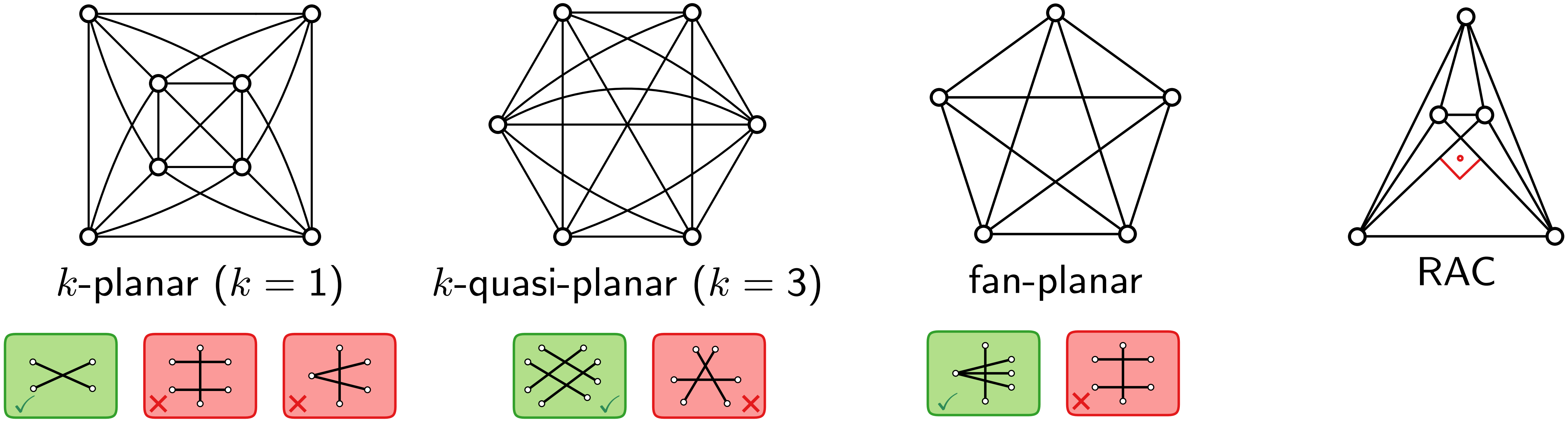
We define **aesthetics** for edge crossings and avoid/minimize "bad" crossing configurations.



# Some Beyond-Planar Graph Classes

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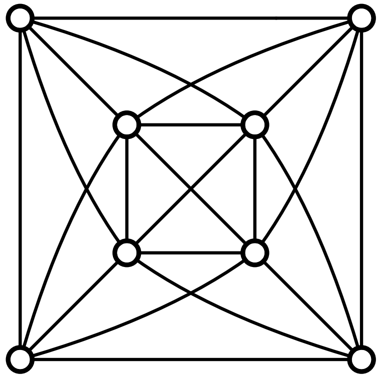
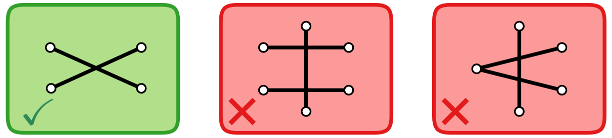
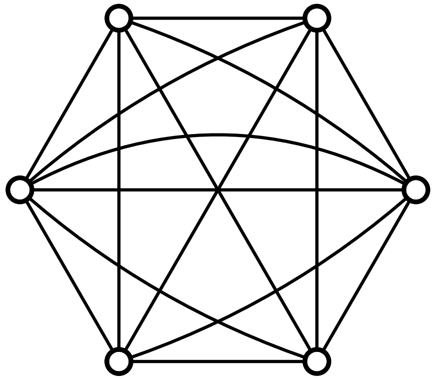
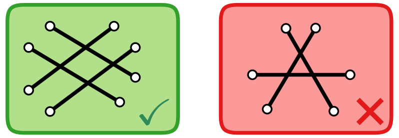


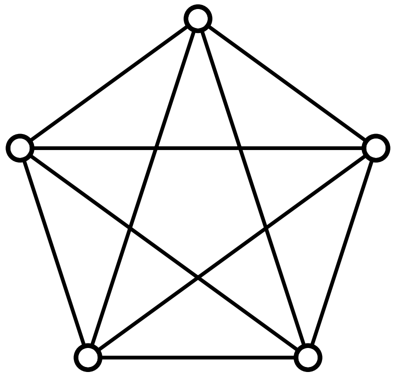
$k$-planar ($k = 1$)



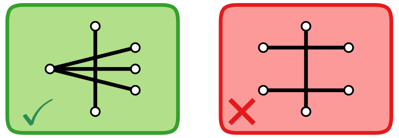


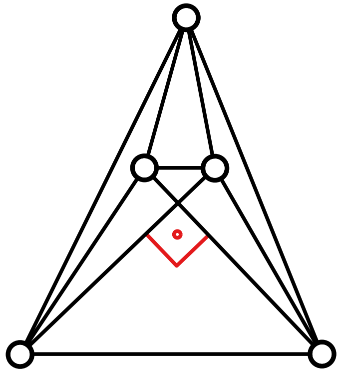
$k$-quasi-planar ($k = 3$)



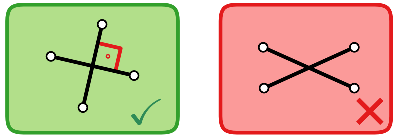


fan-planar



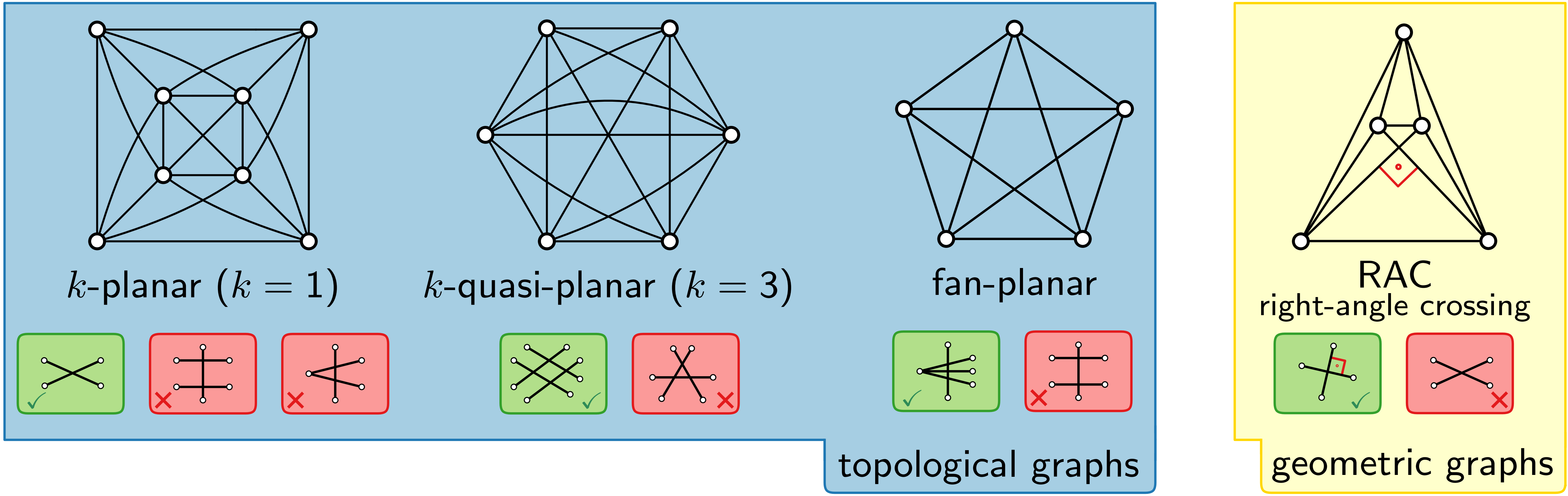


RAC
right-angle crossing



# Some Beyond-Planar Graph Classes

We define **aesthetics** for edge crossings and avoid/minimize "bad" crossing configurations.



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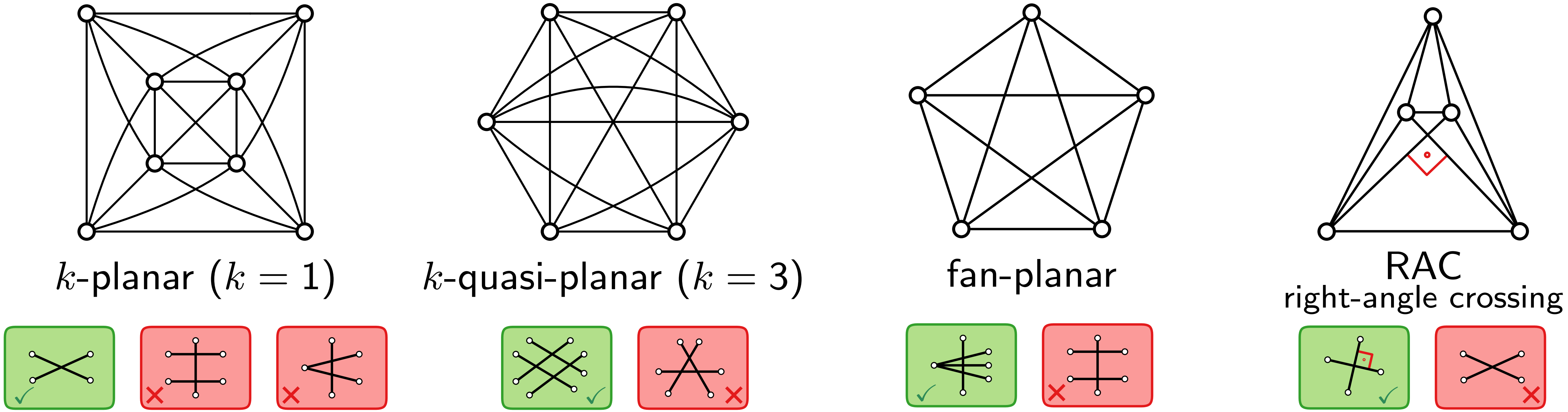
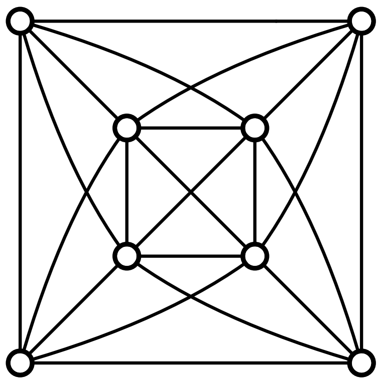


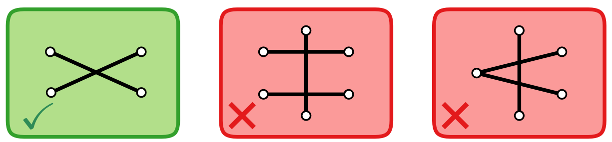

There are many more beyond planar graph classes...

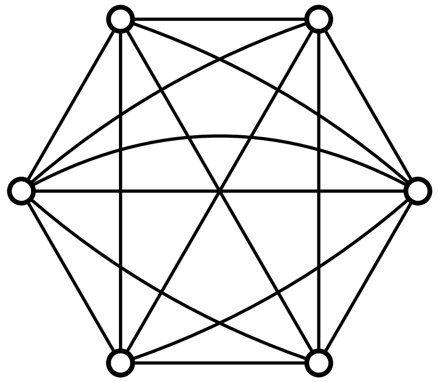
# Some Beyond-Planar Graph Classes

We define **aesthetics** for edge crossings and avoid/minimize "bad" crossing configurations.

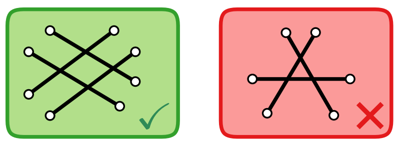


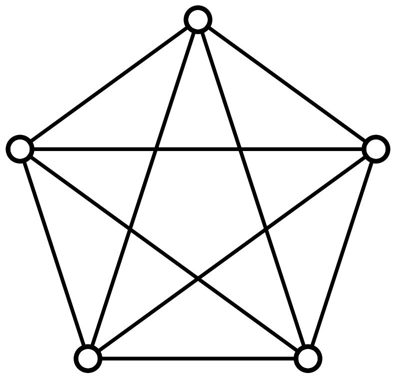
$k$-planar ($k = 1$)



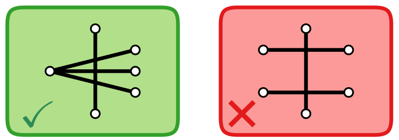


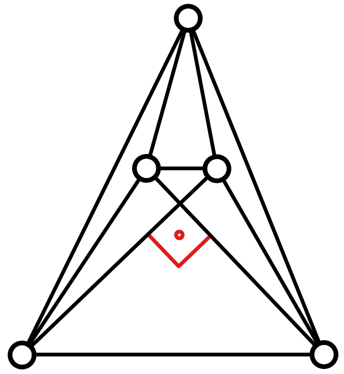
$k$-quasi-planar ($k = 3$)



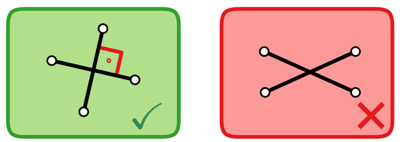


fan-planar

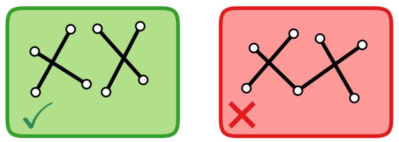




RAC
right-angle crossing



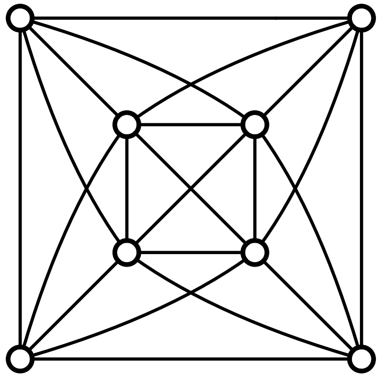
There are many more beyond planar graph classes. . .



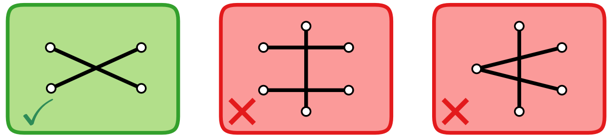
IC (independent crossing)

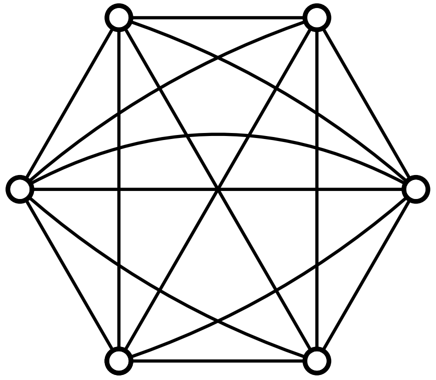
# Some Beyond-Planar Graph Classes

We define **aesthetics** for edge crossings and avoid/minimize "bad" crossing configurations.

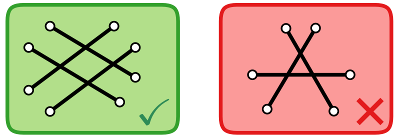


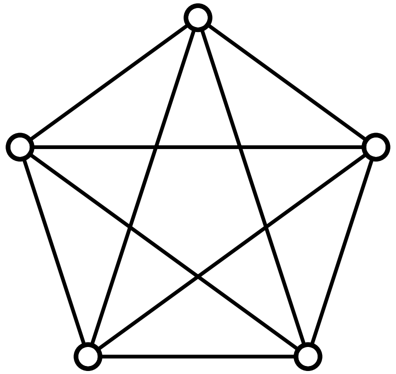
$k$-planar ($k = 1$)



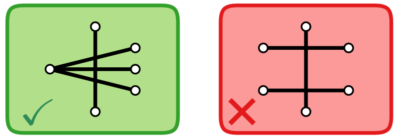


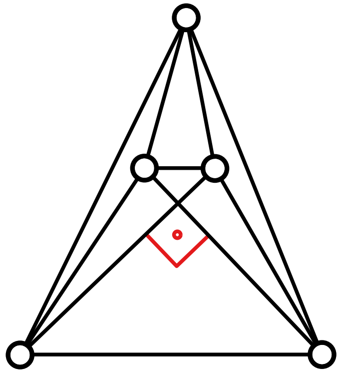
$k$-quasi-planar ($k = 3$)



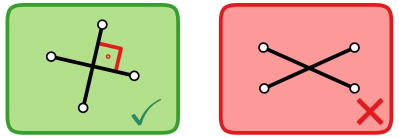


fan-planar

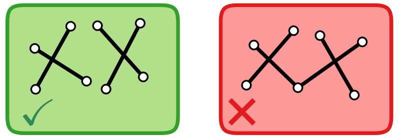




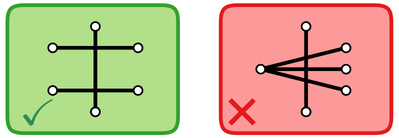
RAC
right-angle crossing



There are many more beyond planar graph classes. . .



IC (independent crossing)



fan-crossing-free

# Some Beyond-Planar Graph Classes

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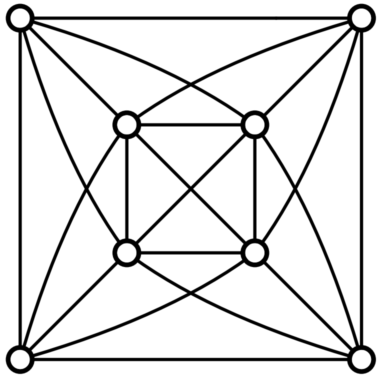
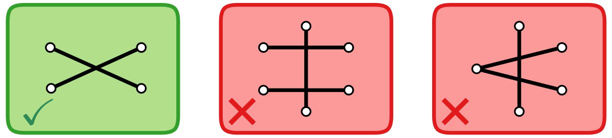
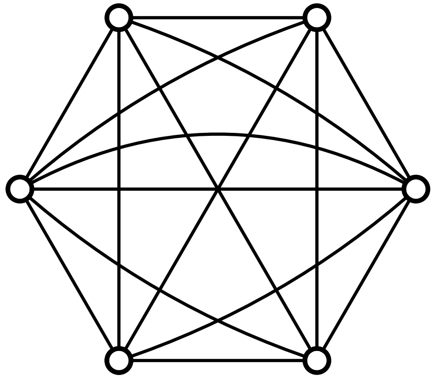
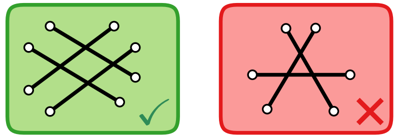


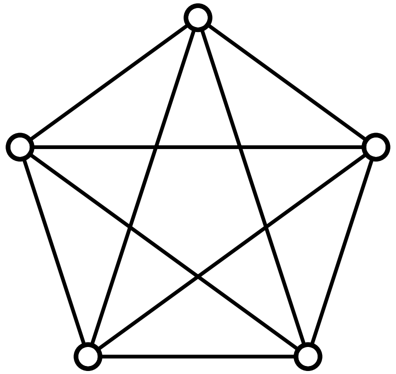
$k$-planar ($k = 1$)



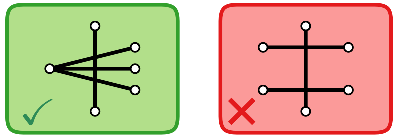


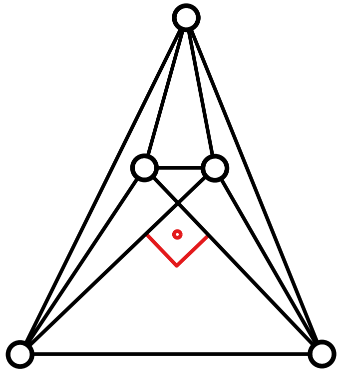
$k$-quasi-planar ($k = 3$)



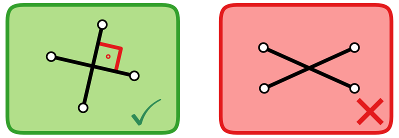


fan-planar

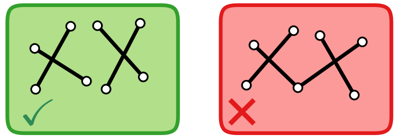




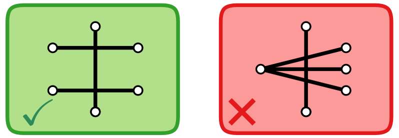
RAC
right-angle crossing



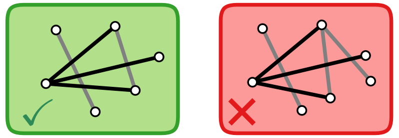
There are many more beyond planar graph classes...



IC (independent crossing)



fan-crossing-free



skewness-$k$ ($k = 2$)

# Some Beyond-Planar Graph Classes

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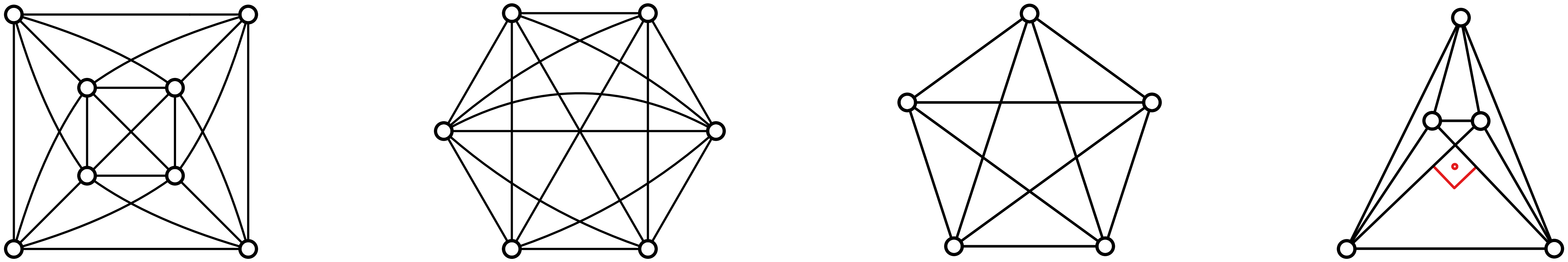
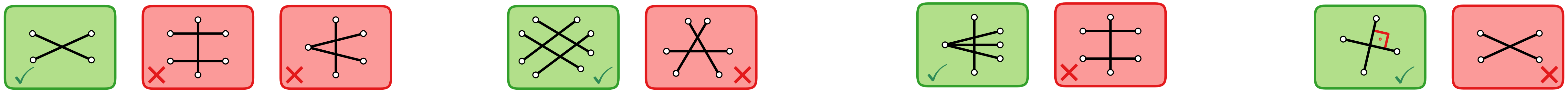


$k$-planar ($k = 1$) $k$-quasi-planar ($k = 3$) fan-planar RAC right-angle crossing



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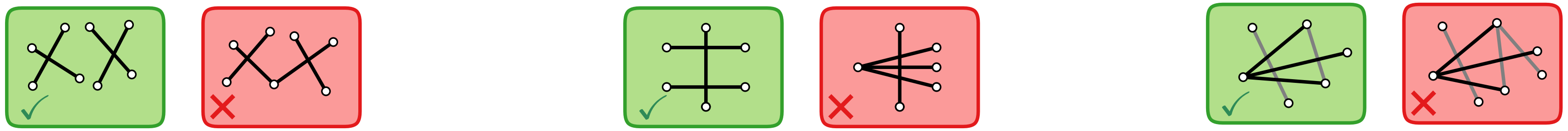
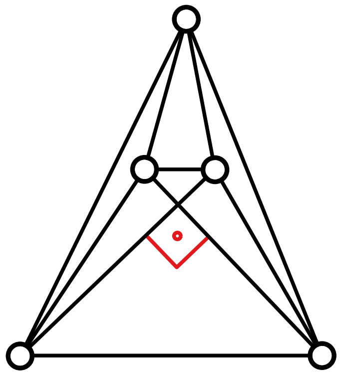


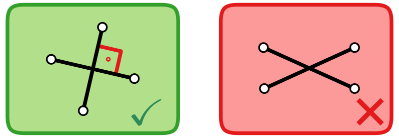
IC (independent crossing) fan-crossing-free skewness-$k$ ($k = 2$)

combinations, ...

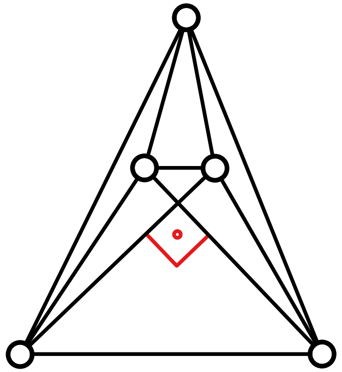
# Drawing Styles for Crossings



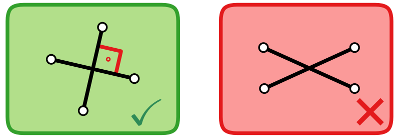
RAC
right-angle crossing

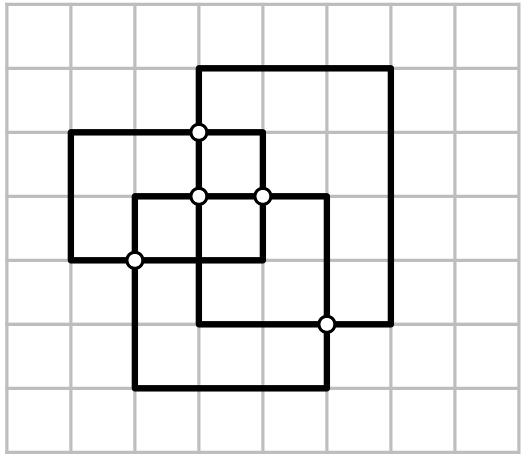


# Drawing Styles for Crossings

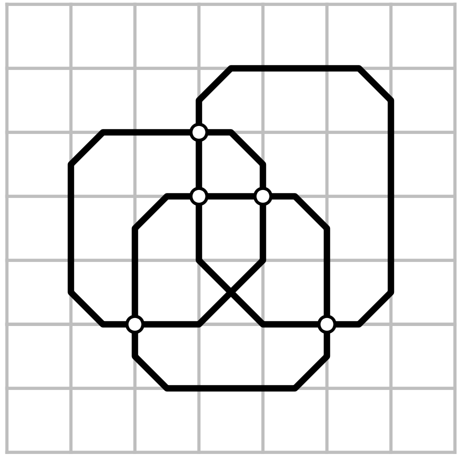


RAC
right-angle crossing



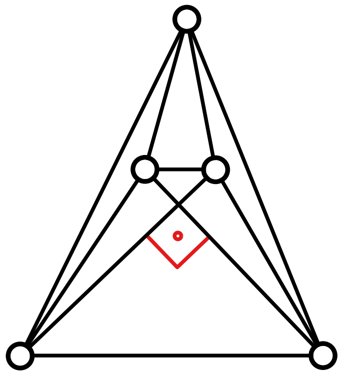


orthogonal

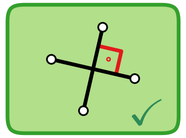


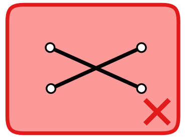
slanted orthogonal

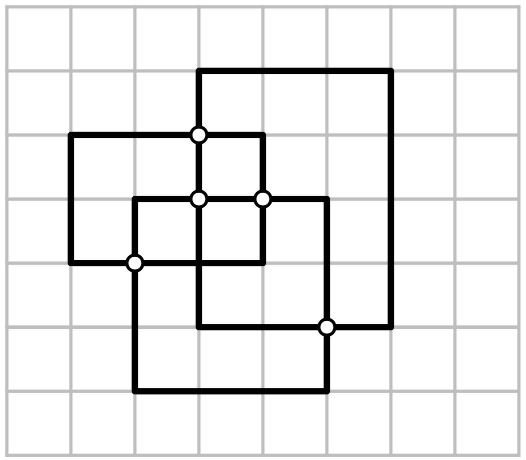
# Drawing Styles for Crossings



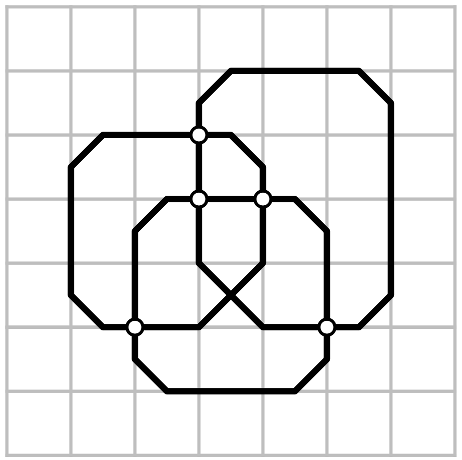
RAC
right-angle crossing



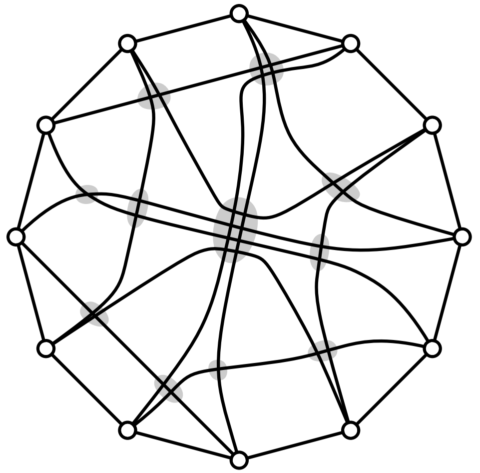




orthogonal



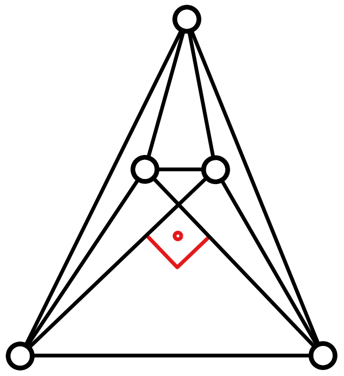
slanted orthogonal



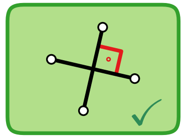
block/bundle crossings

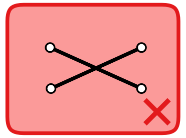
circular layout: 28 invididual
vs. 12 bundle crossings

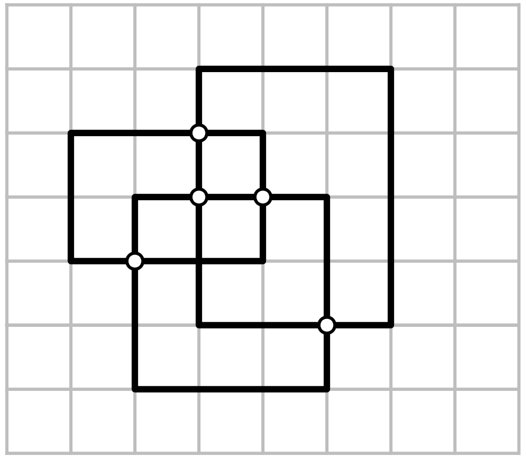
# Drawing Styles for Crossings



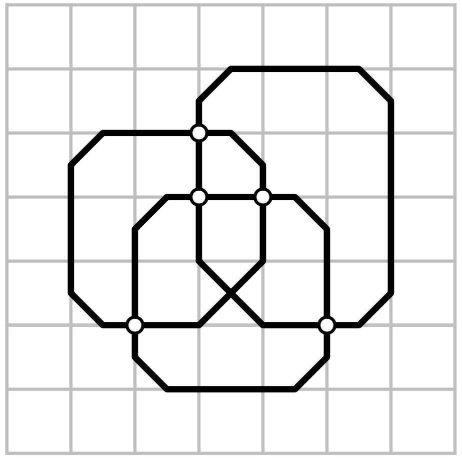
RAC
right-angle crossing



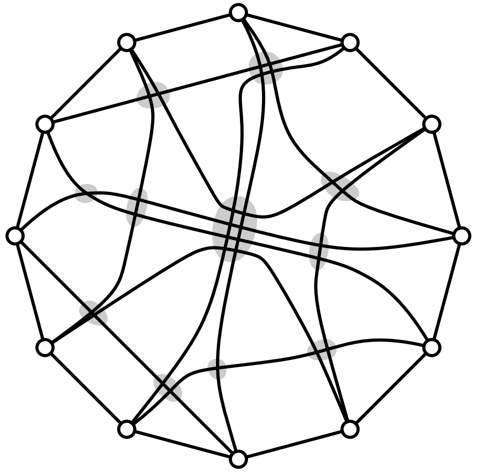




orthogonal

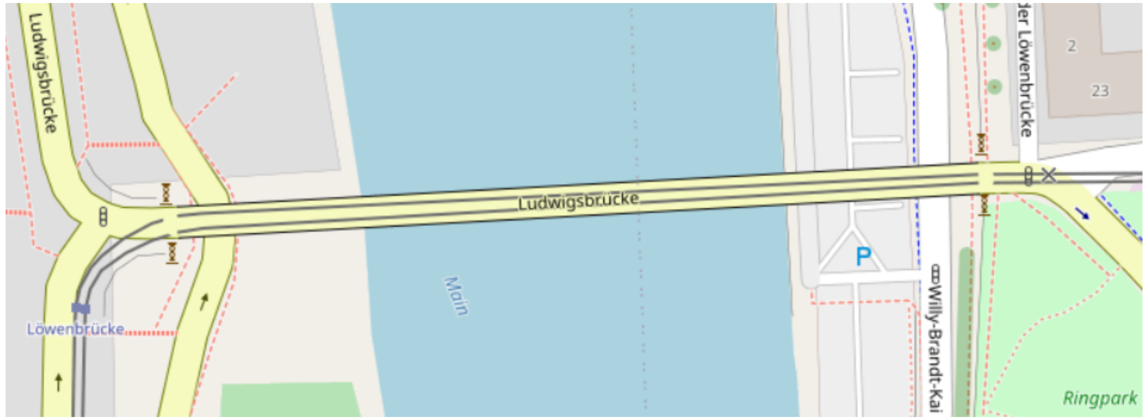


slanted orthogonal

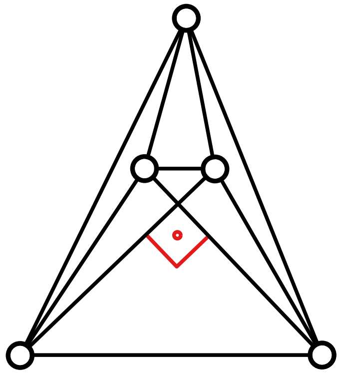


block/bundle crossings

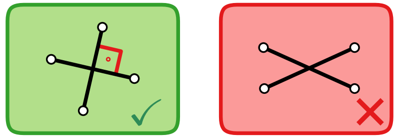
circular layout: 28 invididual
vs. 12 bundle crossings

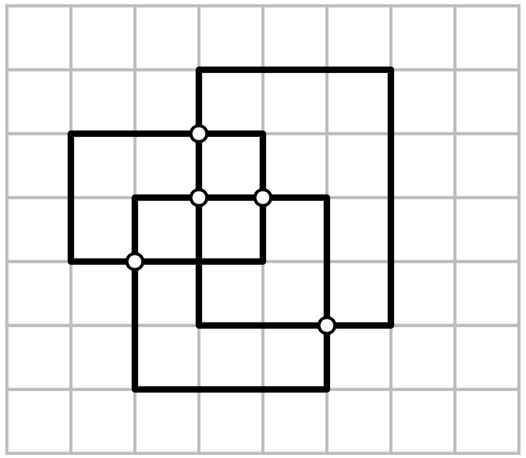


# Drawing Styles for Crossings

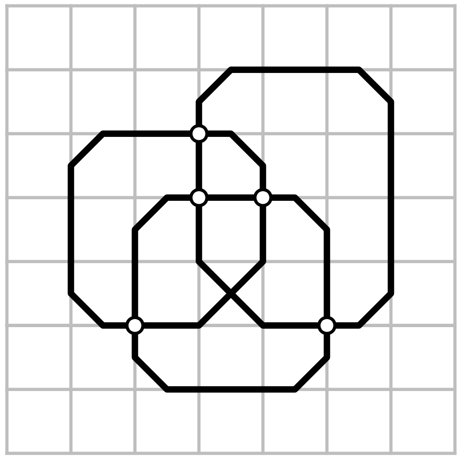


RAC
right-angle crossing

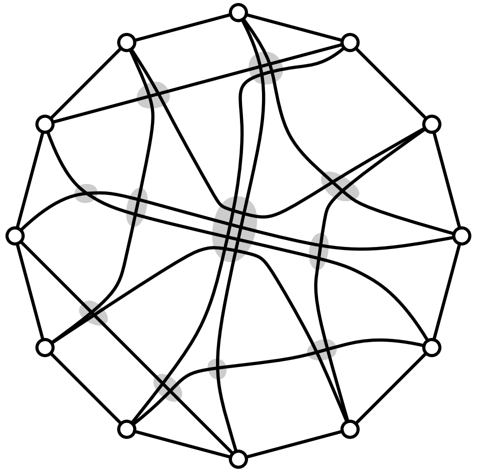




orthogonal



slanted orthogonal



block/bundle crossings

circular layout: 28 invididual
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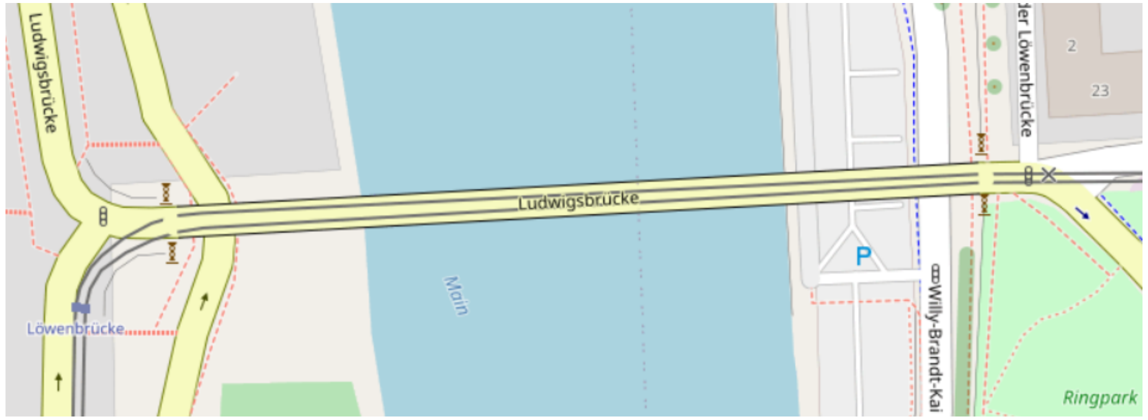
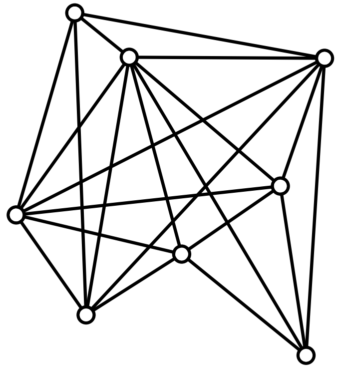
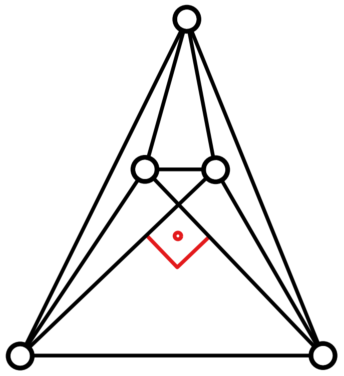
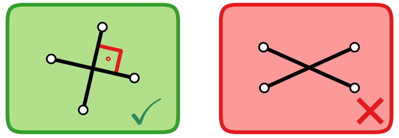


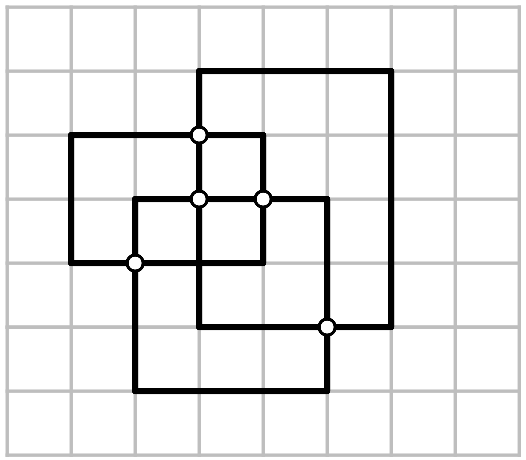



# Drawing Styles for Crossings

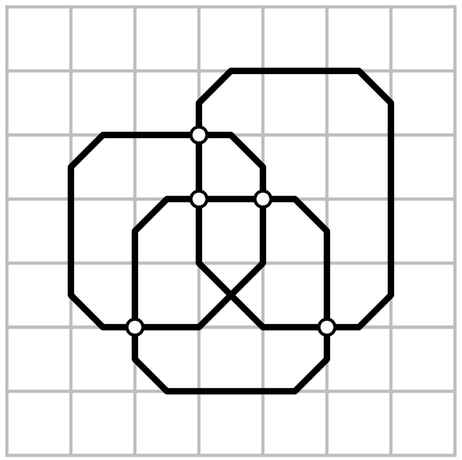


RAC
right-angle crossing

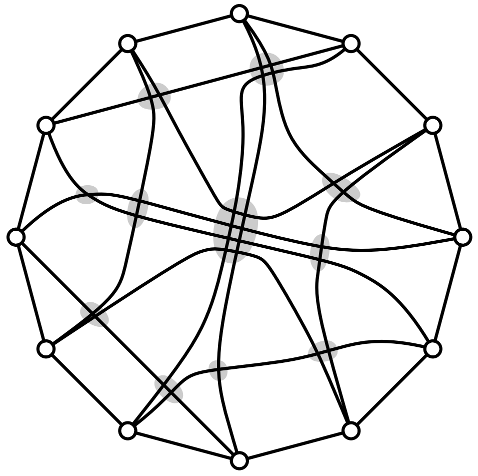




orthogonal



slanted orthogonal



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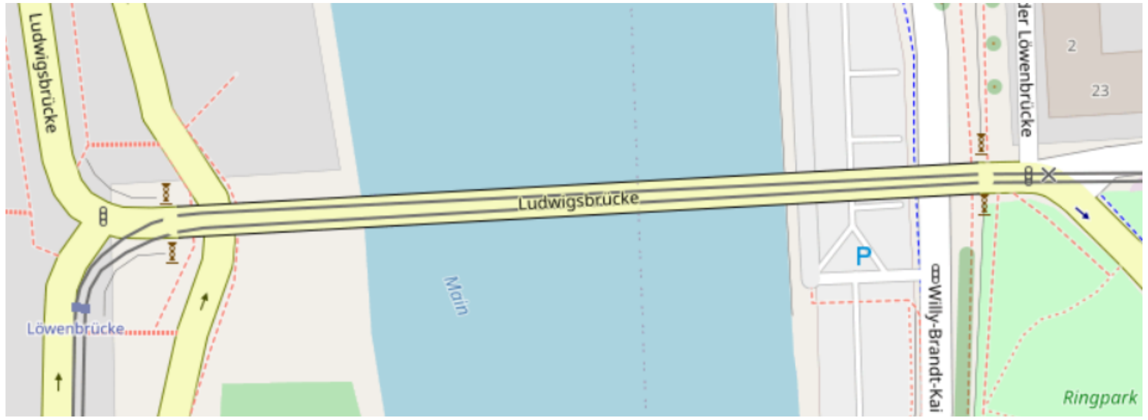
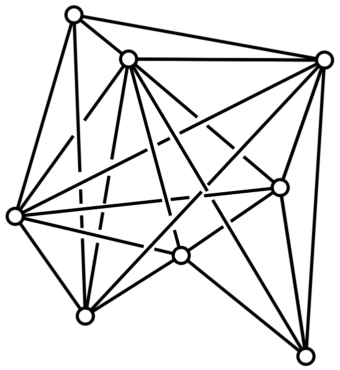
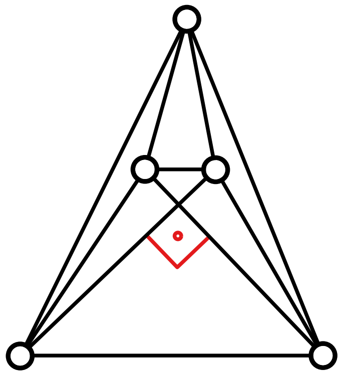


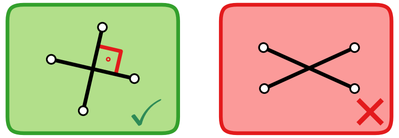



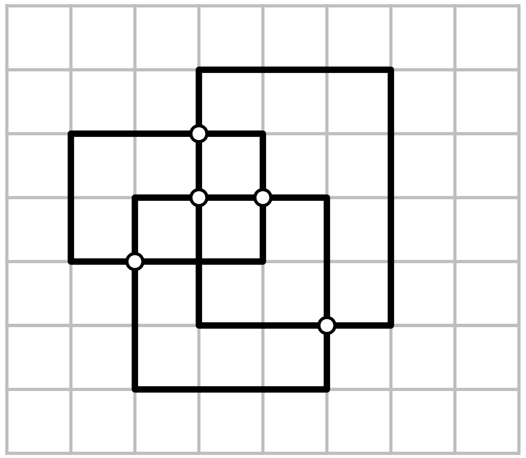
cased crossings

# Drawing Styles for Crossings

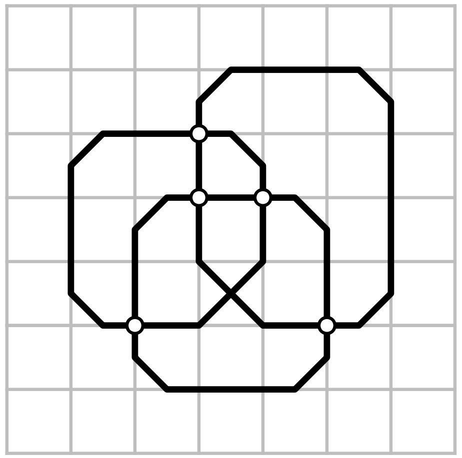


RAC
right-angle crossing

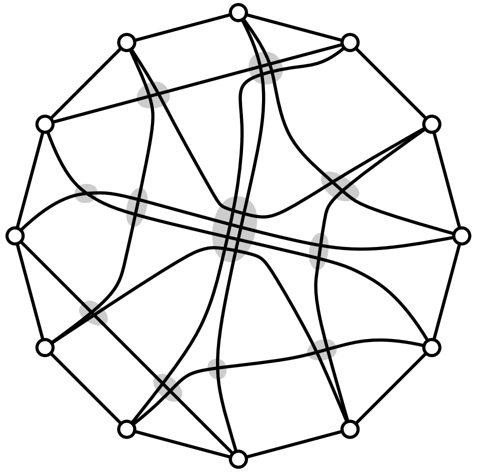




orthogonal



slanted orthogonal



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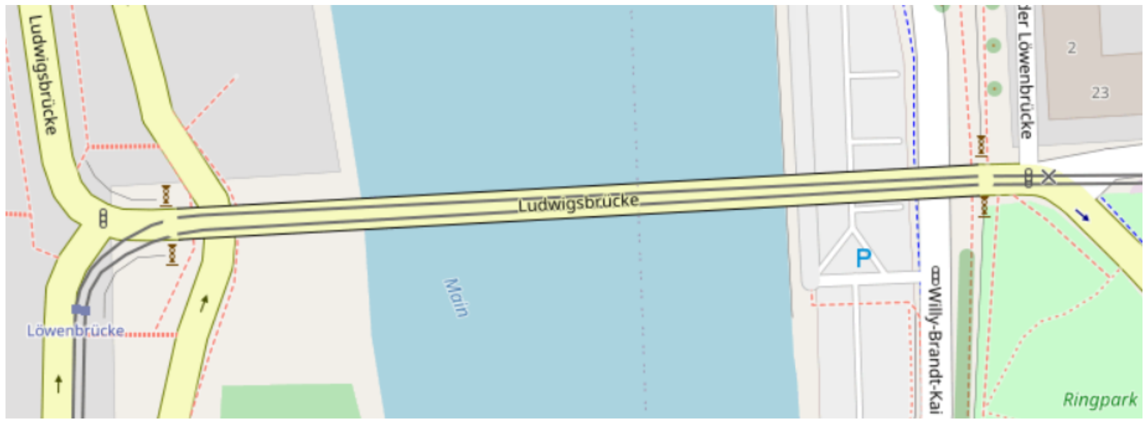
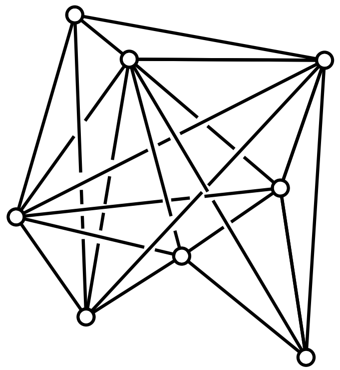
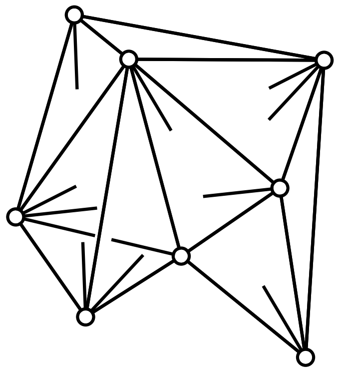


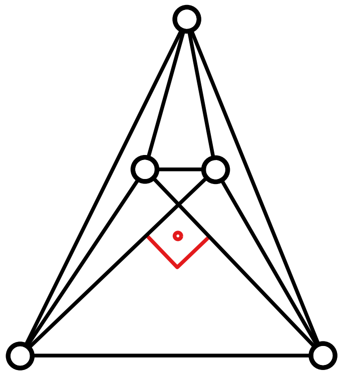



cased crossings

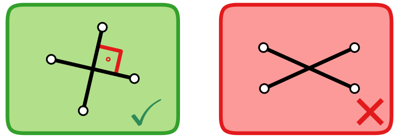


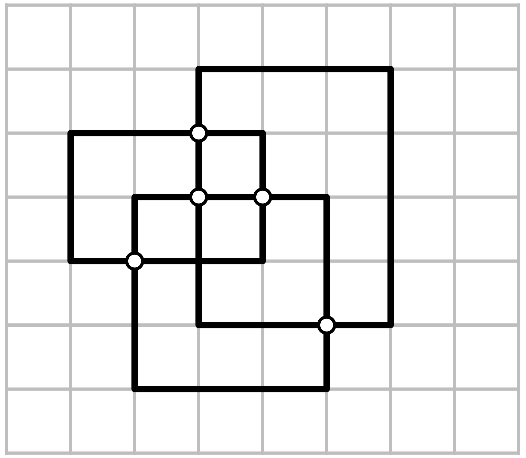
sym. partial
edge drawing

# Drawing Styles for Crossings

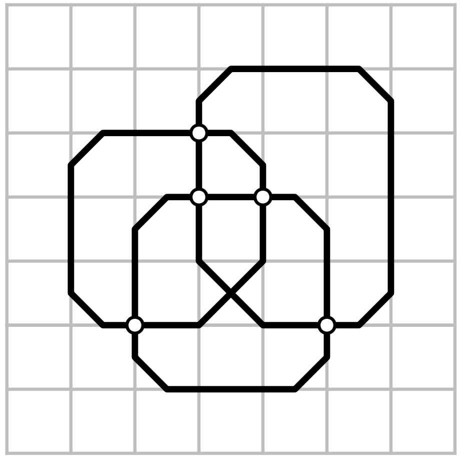


RAC
right-angle crossing

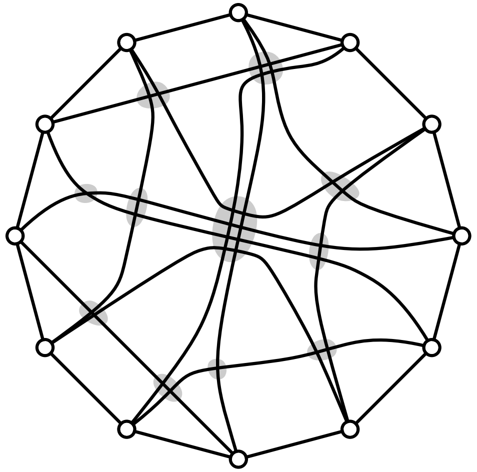




orthogonal



slanted orthogonal



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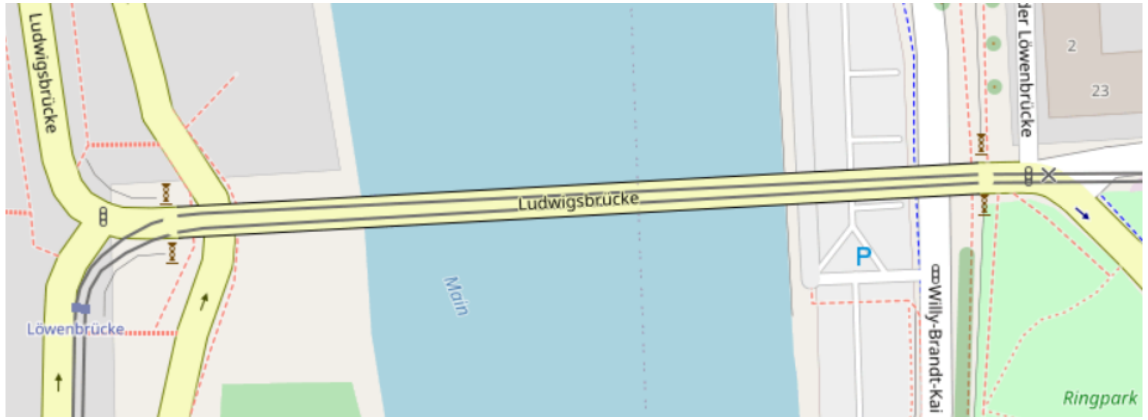
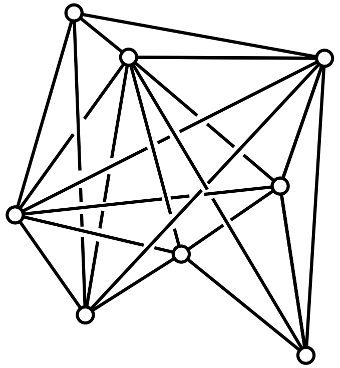
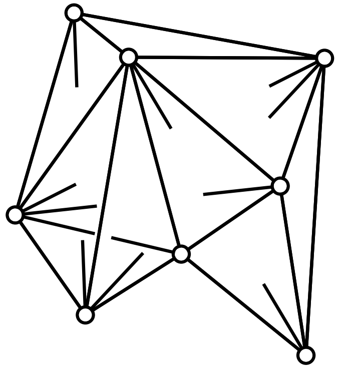
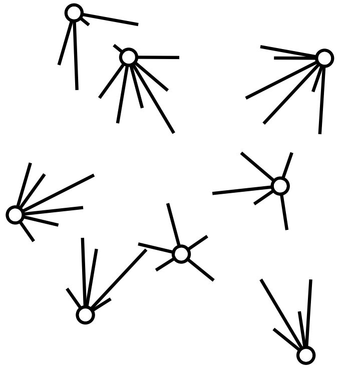


cased crossings

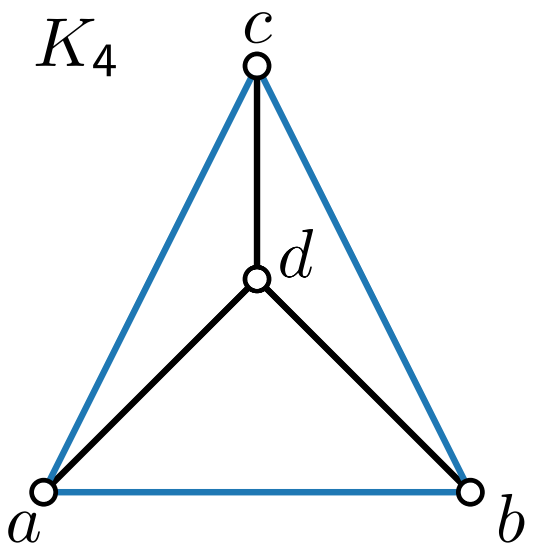


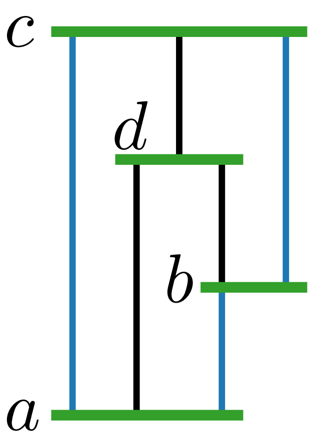
sym. partial
edge drawing



1/4-SHPED

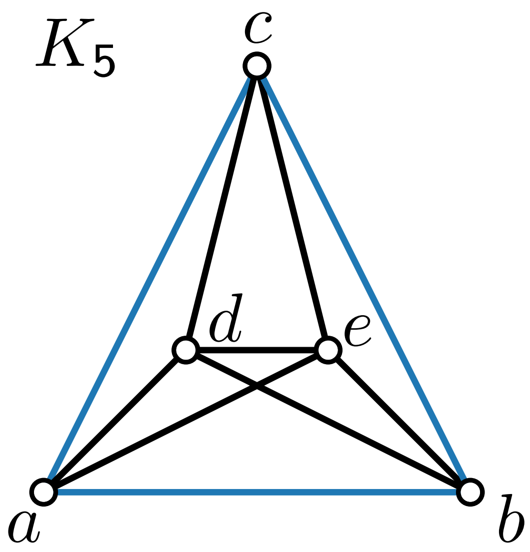
# Geometric Representations

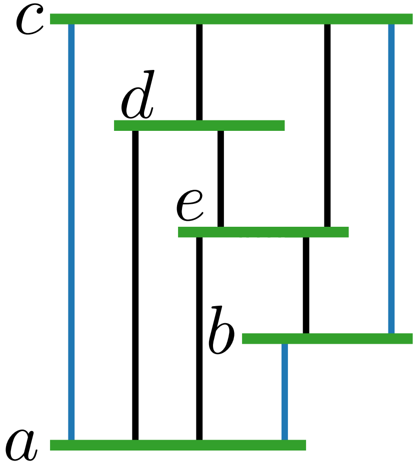



bar visibility
representation

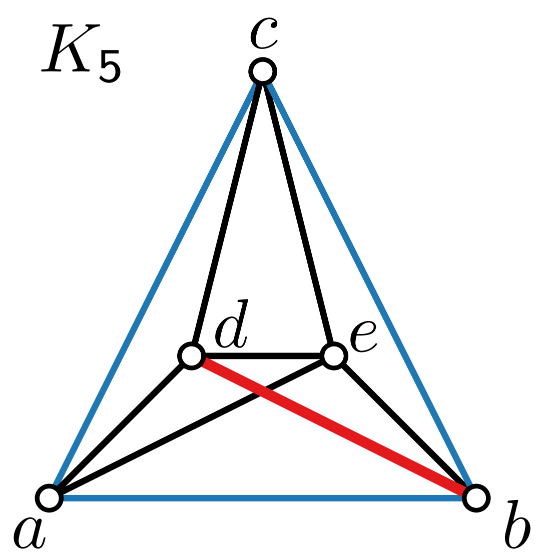
# Geometric Representations

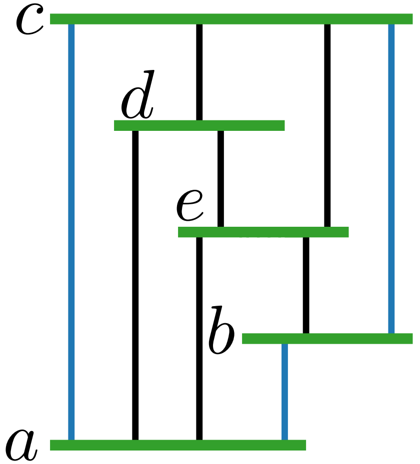



bar 1-visibility representation (B1VR)

# Geometric Representations

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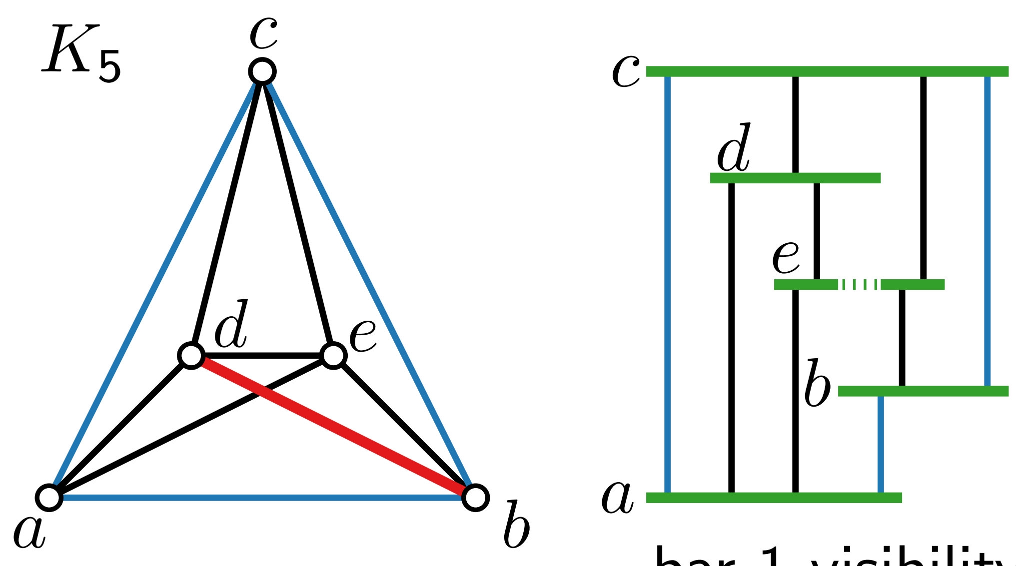


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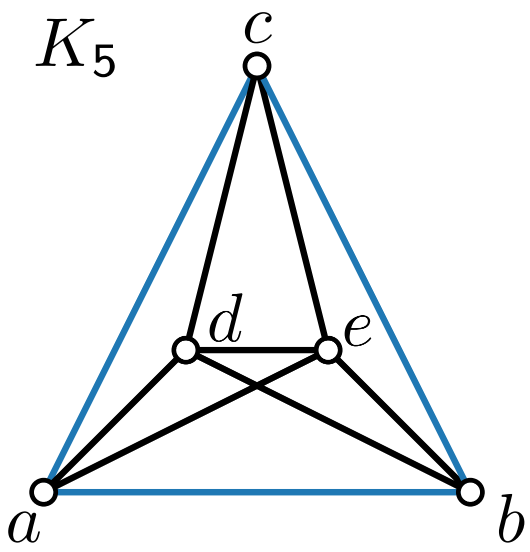
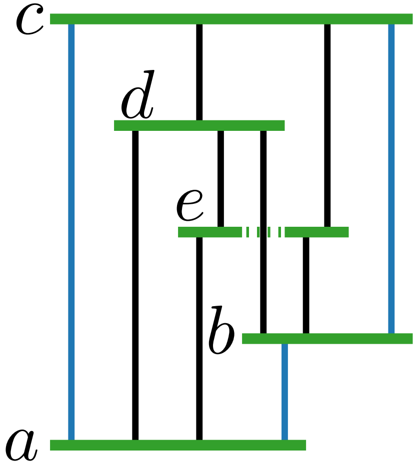
# Geometric Representations

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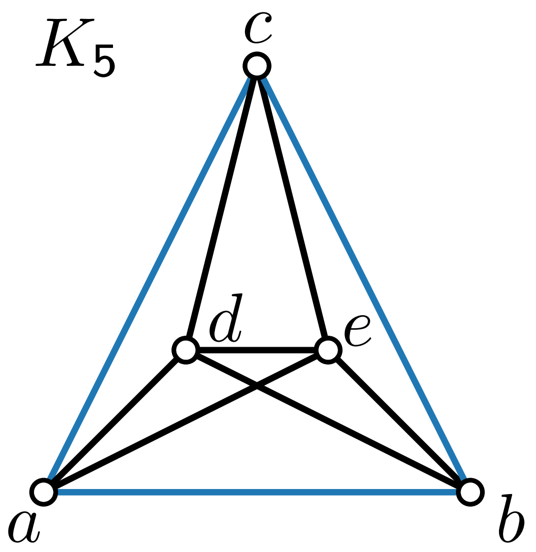
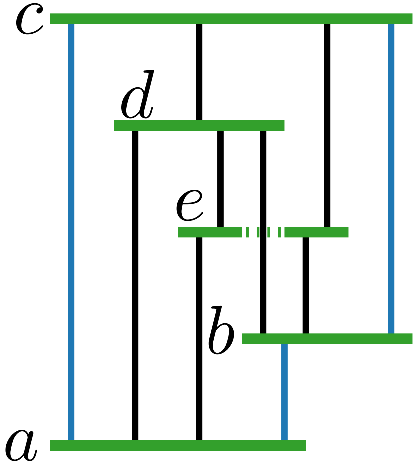
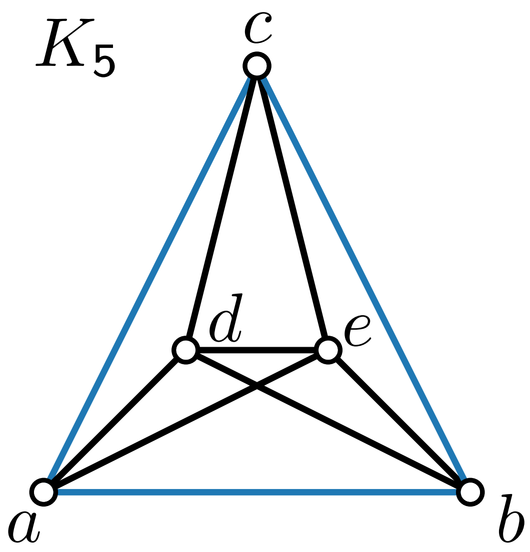


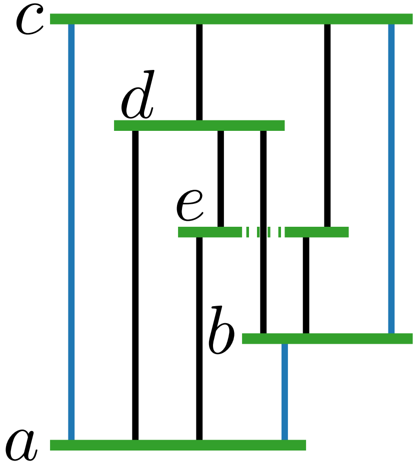




bar 1-visibility
representation (B1VR)

- Every 1-planar graph admits a B1VR.
  [Brandenburg 2014; Evans et al. 2014;
  Angelini et al. 2018]

# Geometric Representations




bar 1-visibility
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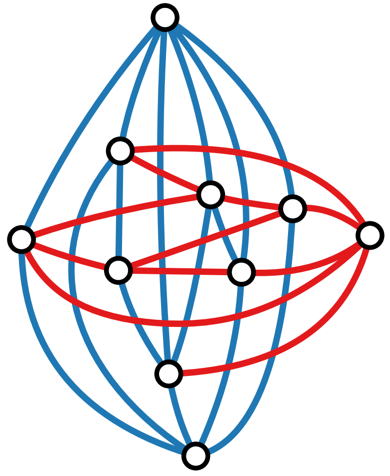

thickness
two graph

- Every 1-planar graph admits a B1VR.
  [Brandenburg 2014; Evans et al. 2014;
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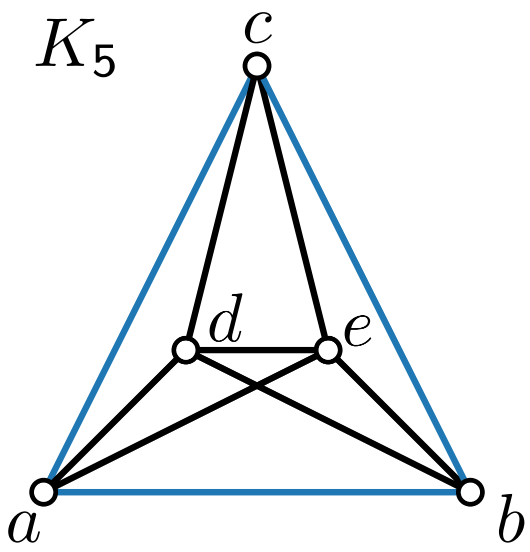
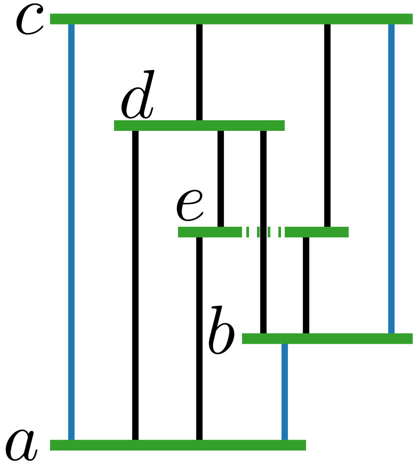
# Geometric Representations

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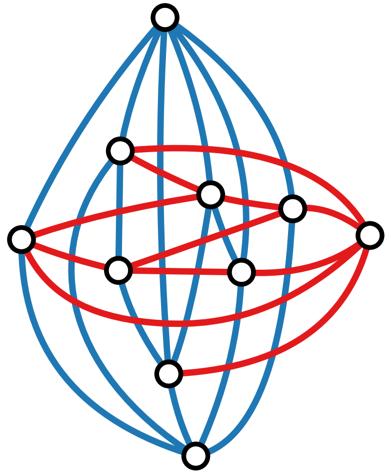
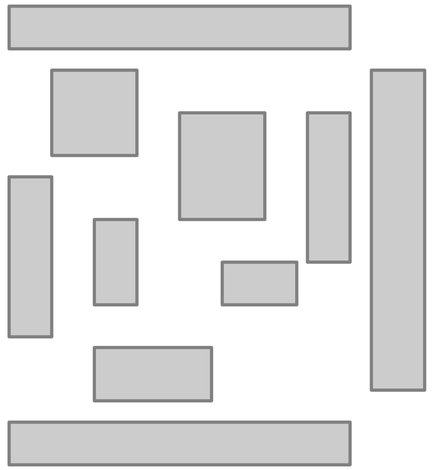


thickness two graph



rectangle visibility representation

- Every 1-planar graph admits a B1VR.
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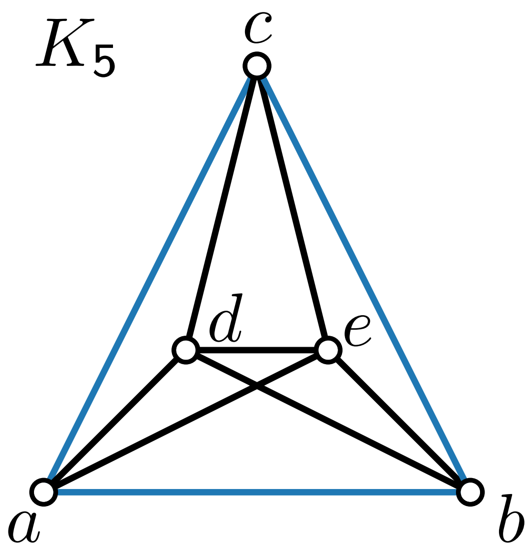
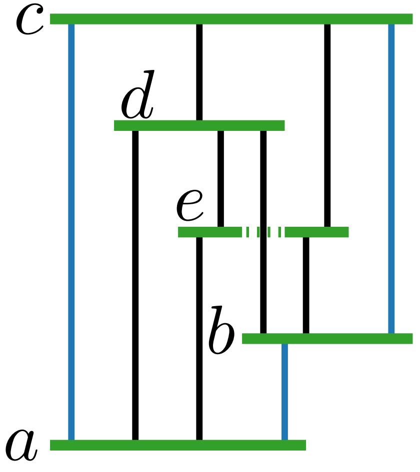
# Geometric Representations

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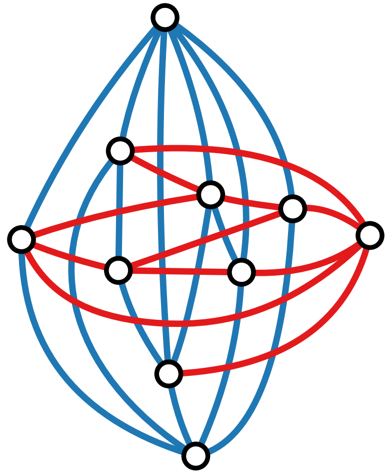

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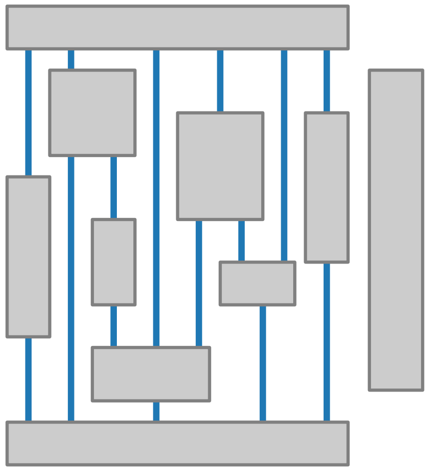

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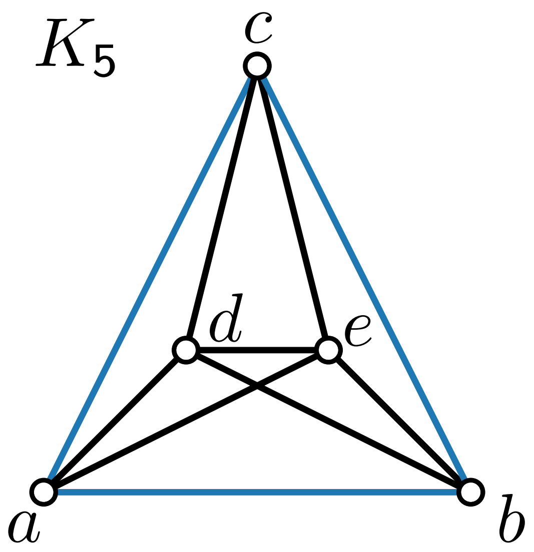
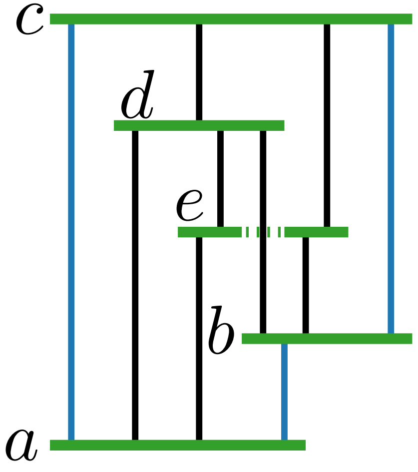
# Geometric Representations

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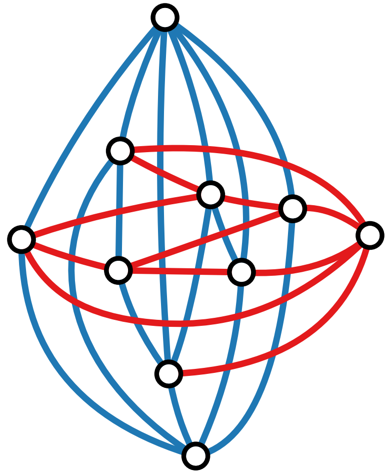

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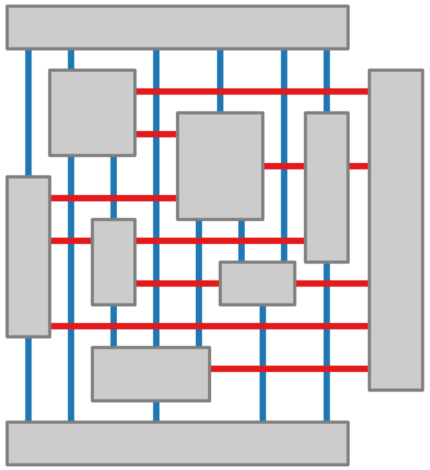

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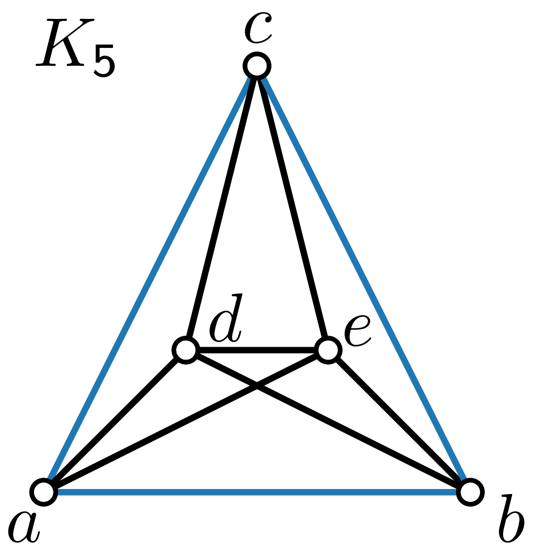
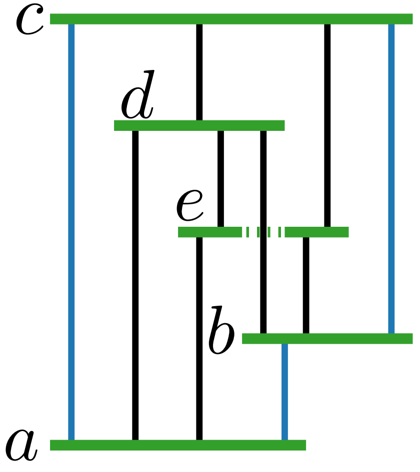
# Geometric Representations

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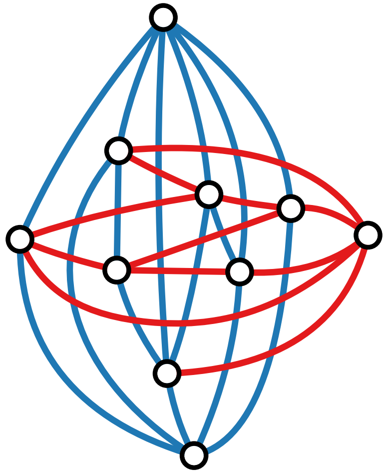

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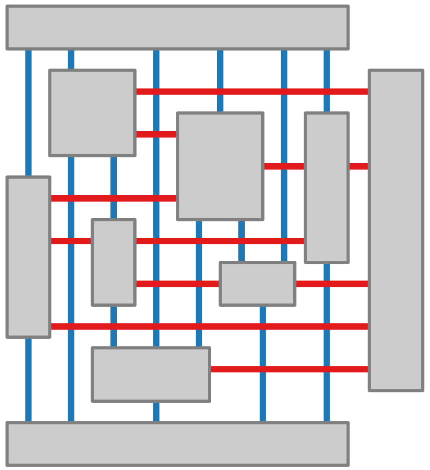
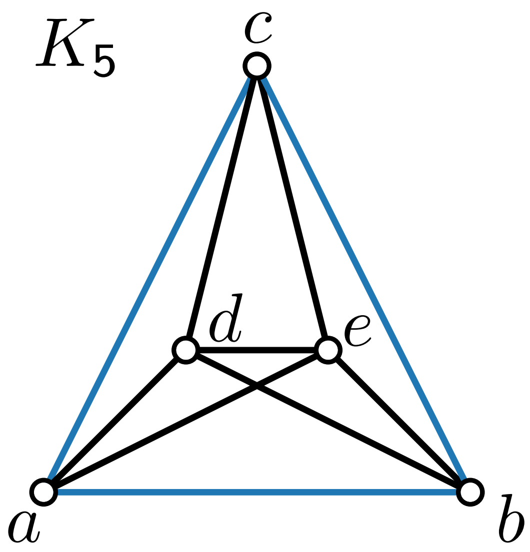
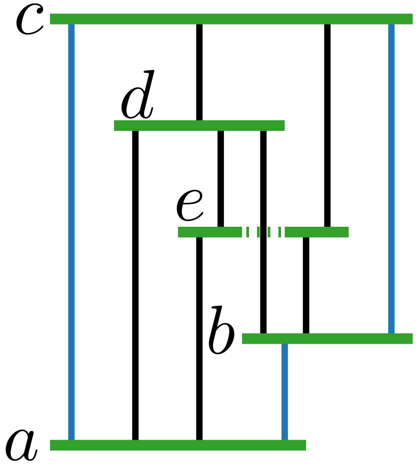


rectangle visibility representation

- Every 1-planar graph admits a B1VR. [Brandenburg 2014; Evans et al. 2014; Angelini et al. 2018]
- $G$ has at most $6n - 20$ edges [Bose et al. 1997]

# Geometric Representations

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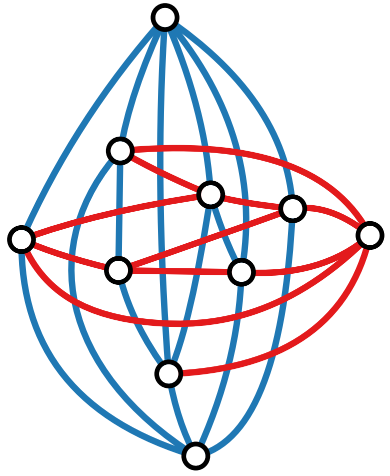

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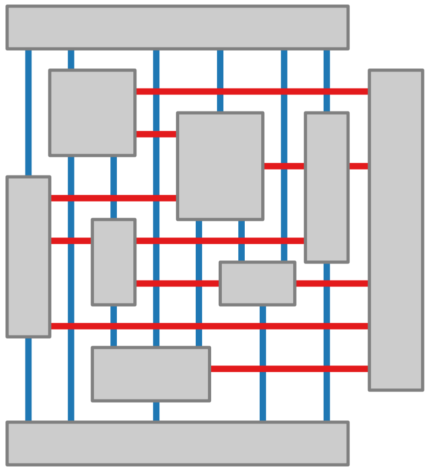

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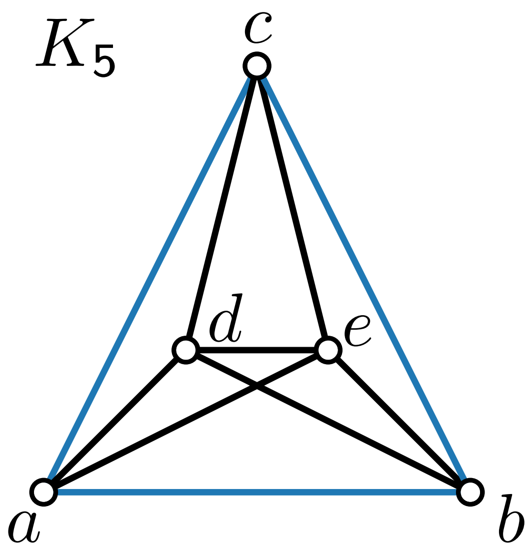
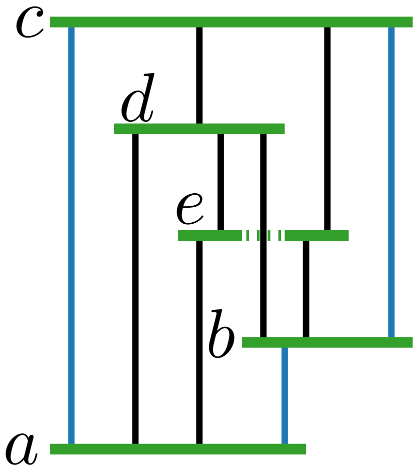
- Every 1-planar graph admits a B1VR. [Brandenburg 2014; Evans et al. 2014; Angelini et al. 2018]
- $G$ has at most $6n - 20$ edges [Bose et al. 1997]
- Recognition is NP-complete [Shermer 1996]

# Geometric Representations




bar 1-visibility
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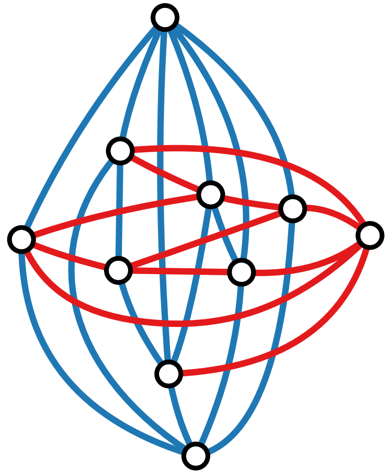
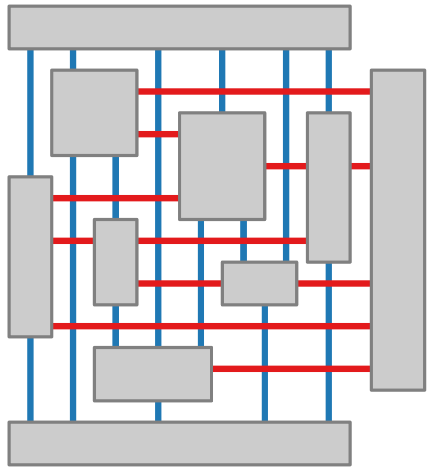


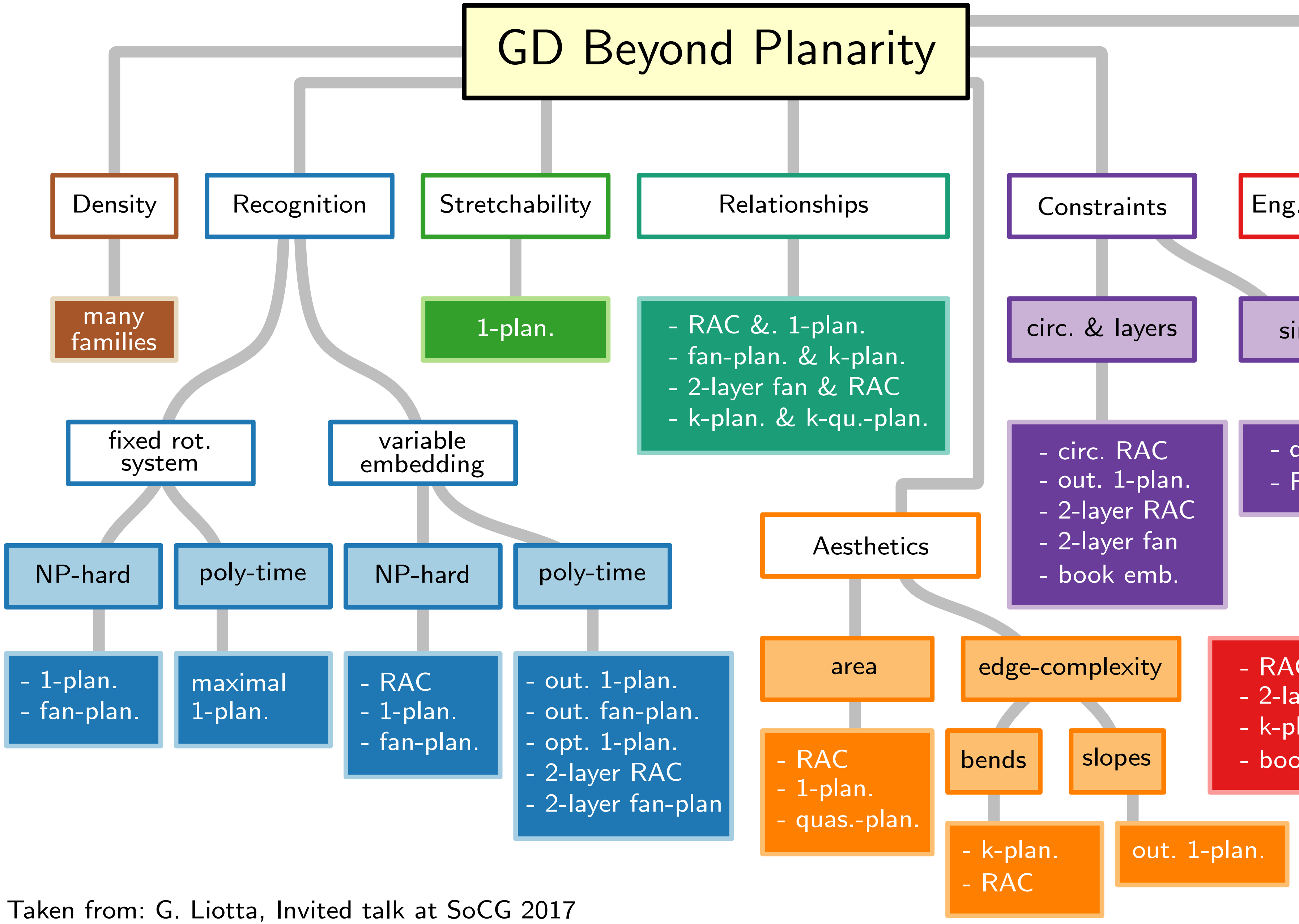
thickness
two graph



rectangle visibility
representation

- Every 1-planar graph admits a B1VR.
  [Brandenburg 2014; Evans et al. 2014; Angelini et al. 2018]
- $G$ has at most $6n - 20$ edges [Bose et al. 1997]
- Recognition is NP-complete [Shermer 1996]
- Recognition becomes polynomial if embedding is fixed [Biedl et al. 2018]

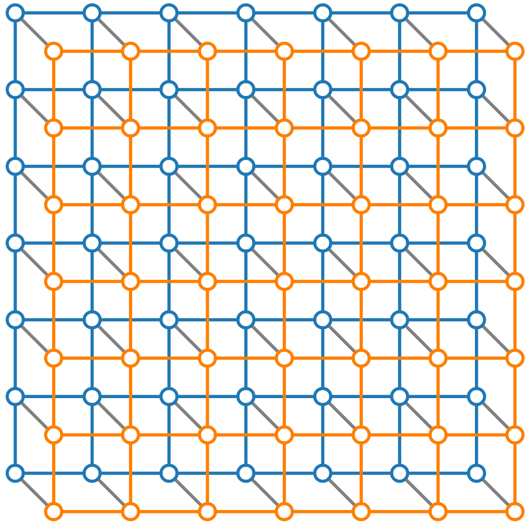
# GD Beyond Planarity: a Taxonomy

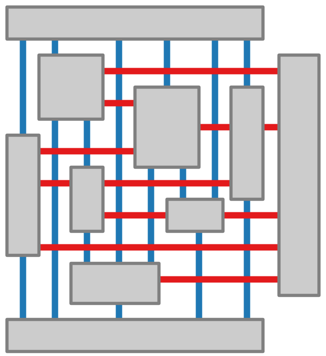



Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

# Visualization of Graphs

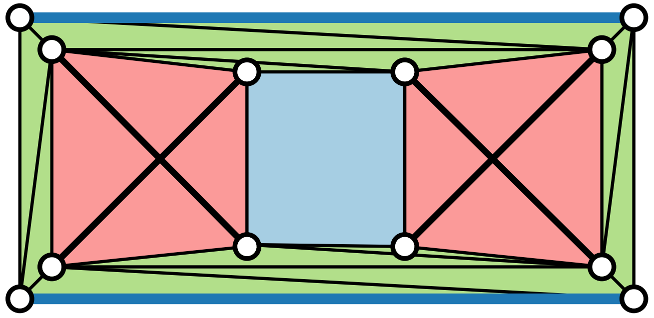


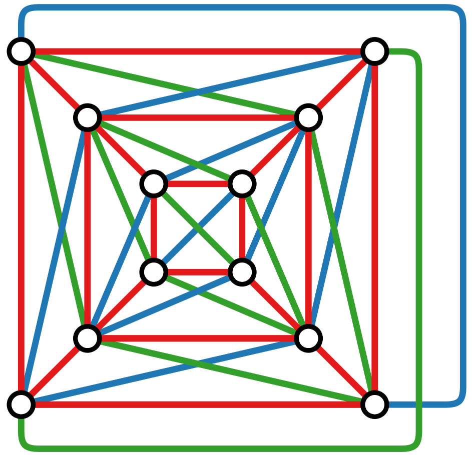


## Lecture 11:
## Beyond Planarity
### Drawing Graphs with Crossings

## Part II:
## Density & Relationships

Jonathan Klawitter





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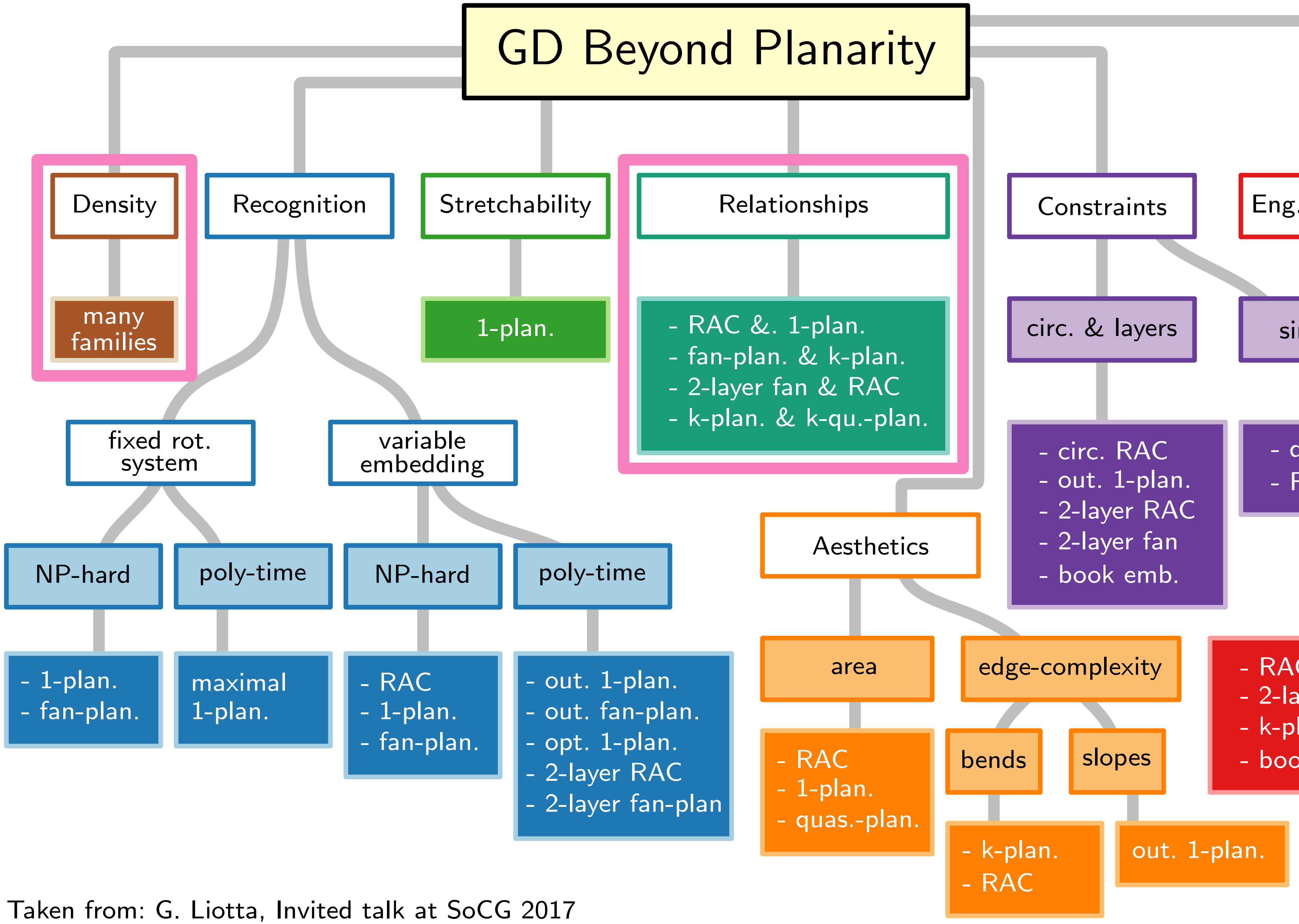


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# Density of 1-Planar Graphs

**Theorem.** [Ringel 1965, Pach & Tóth 1997]
A 1-planar graph with $n$ vertices has at most $4n - 8$ edges, which is a tight bound.

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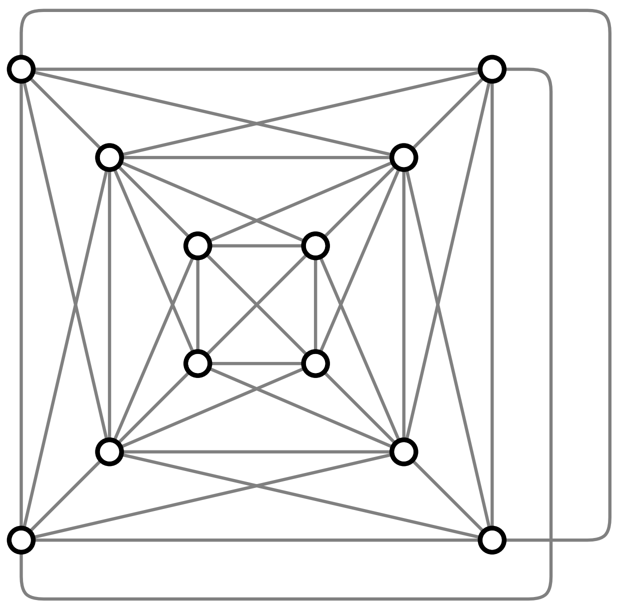

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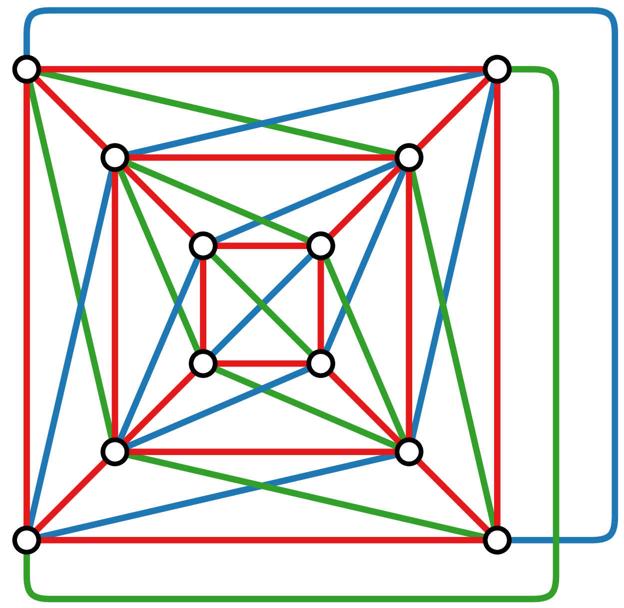

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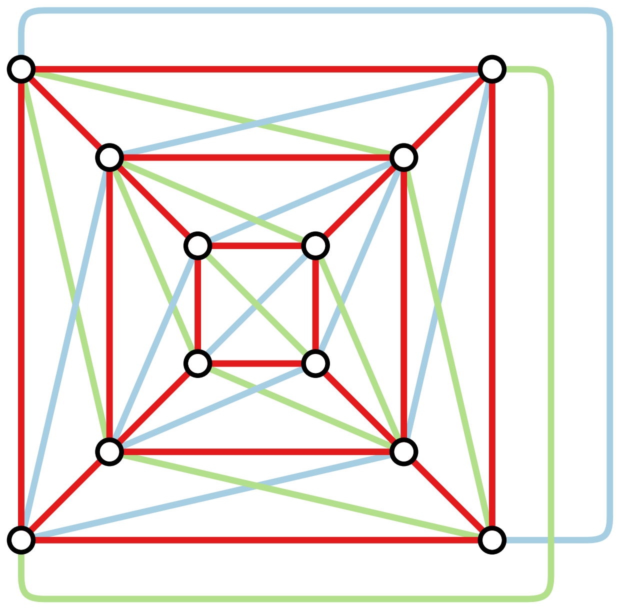

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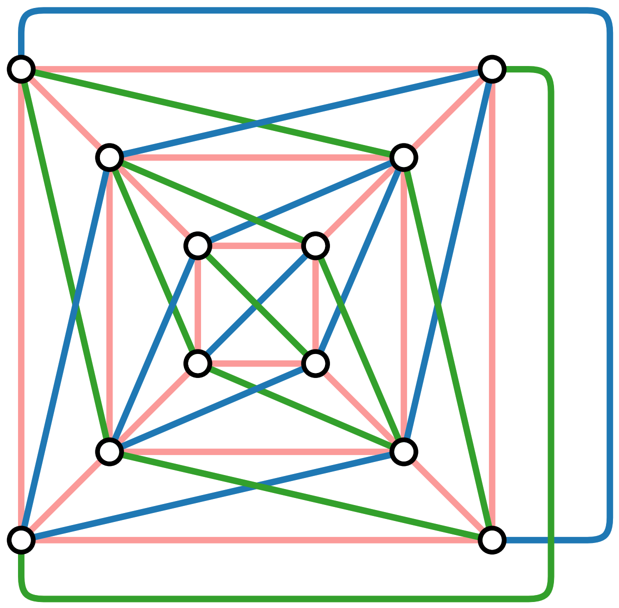

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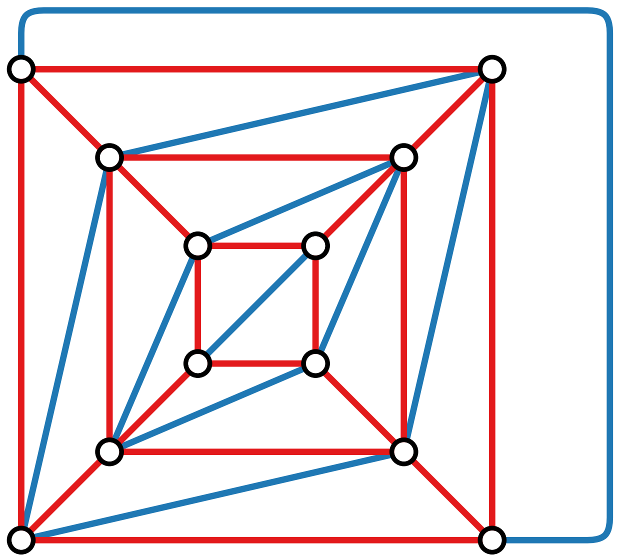

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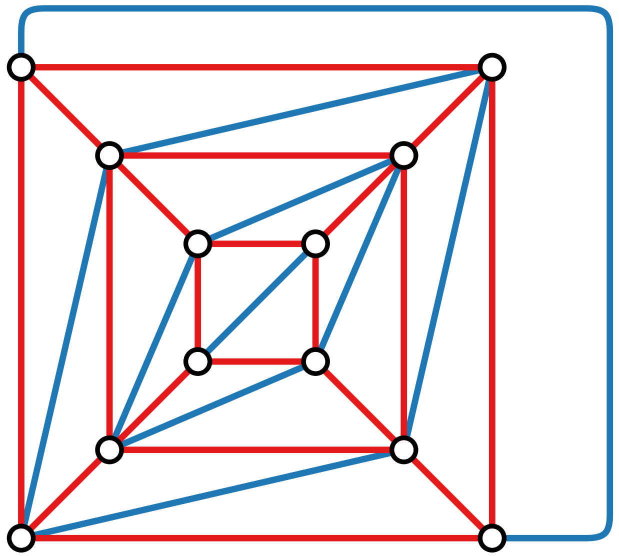

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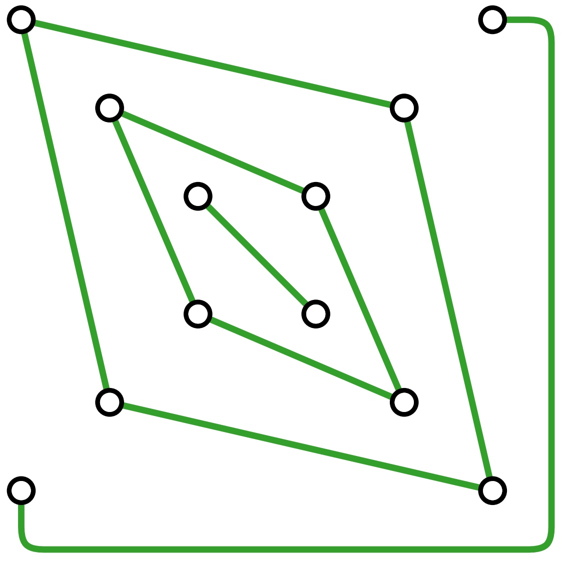

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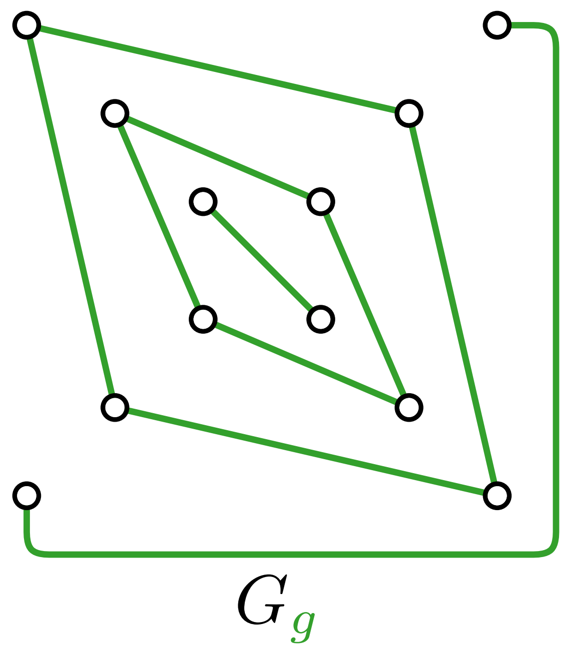

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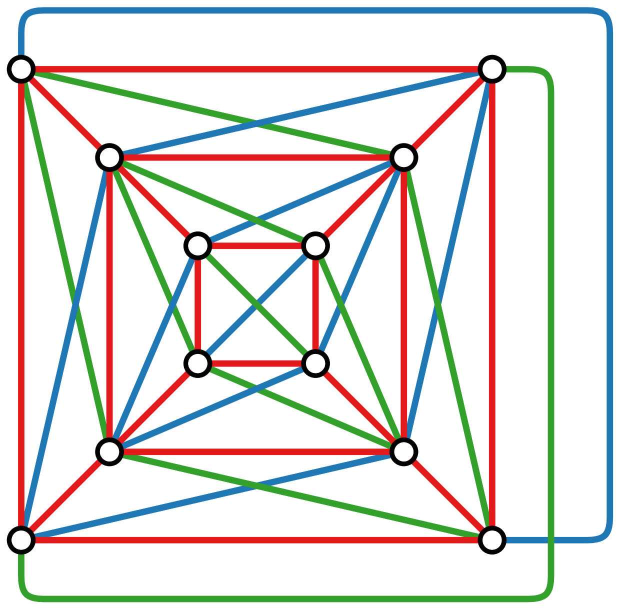

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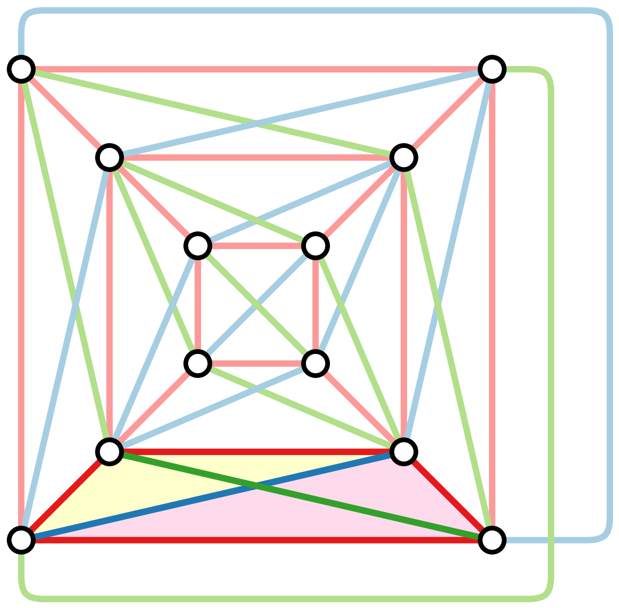

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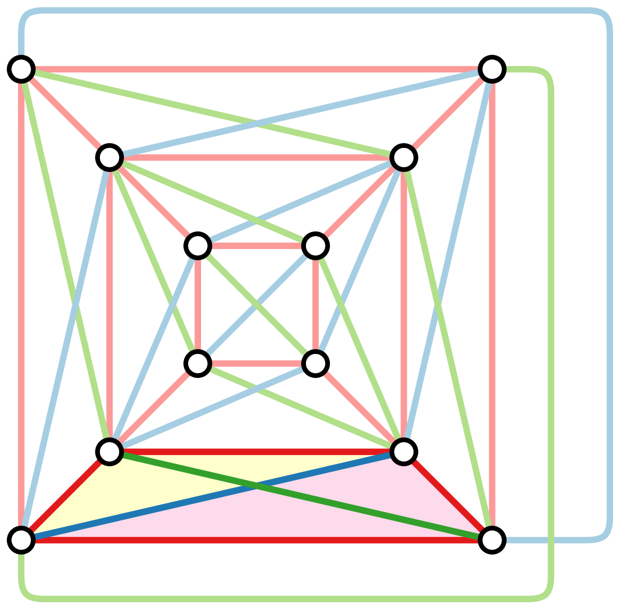

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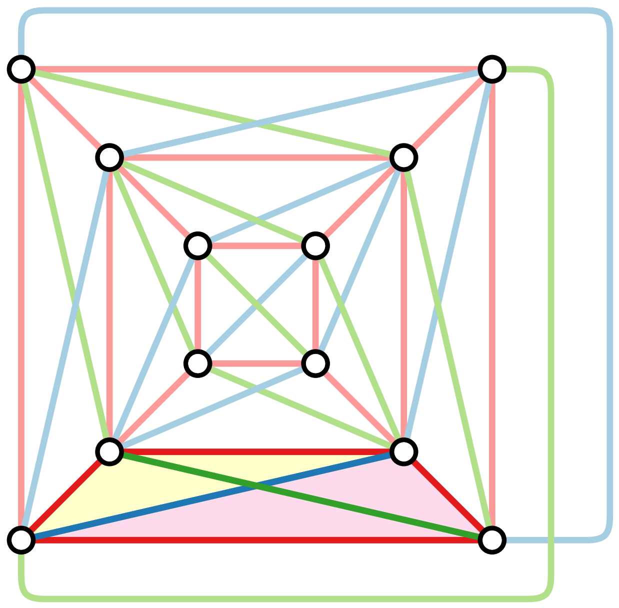

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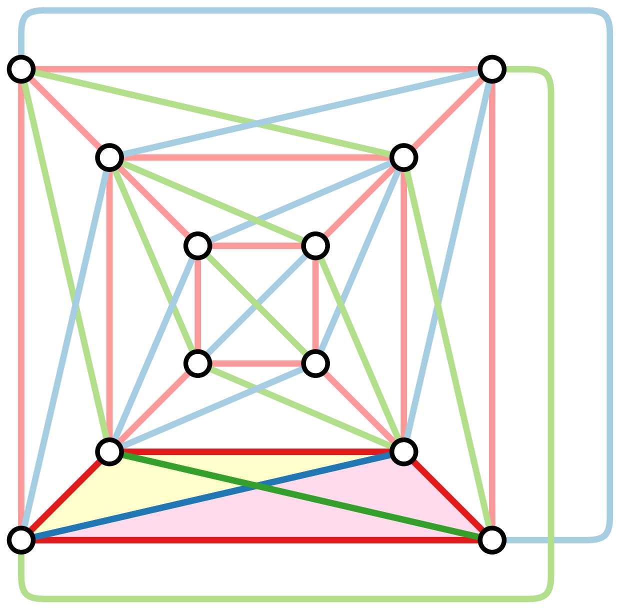

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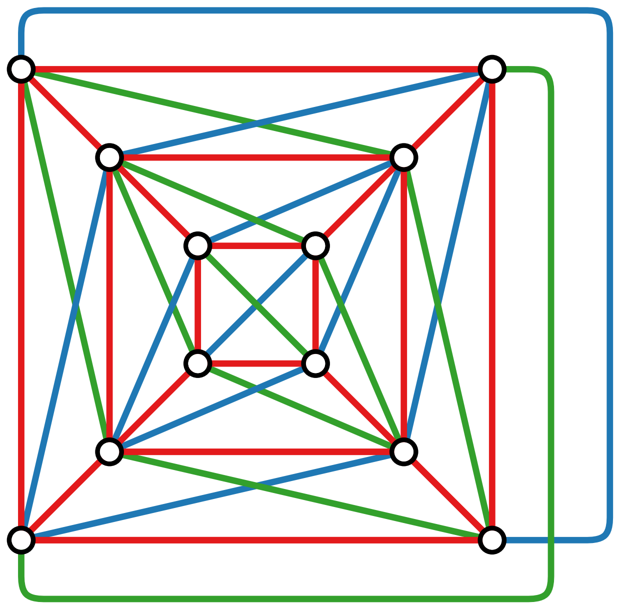

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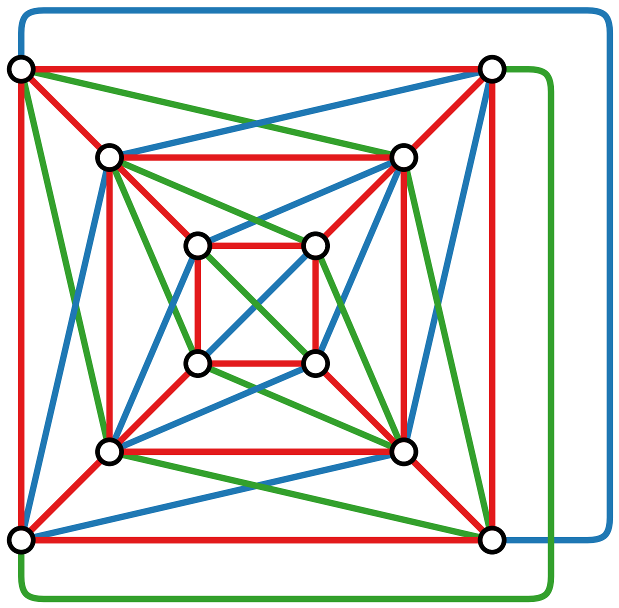

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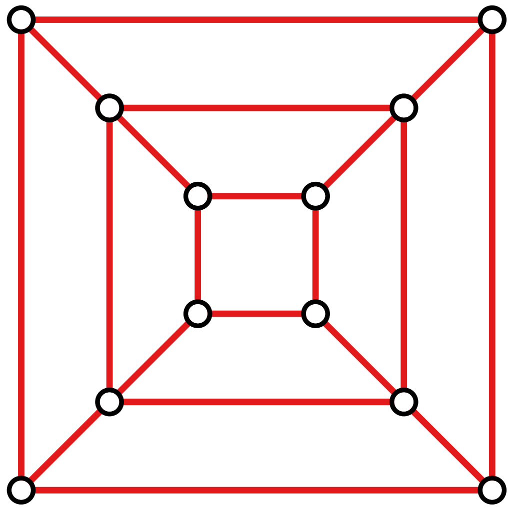

Planar structure:
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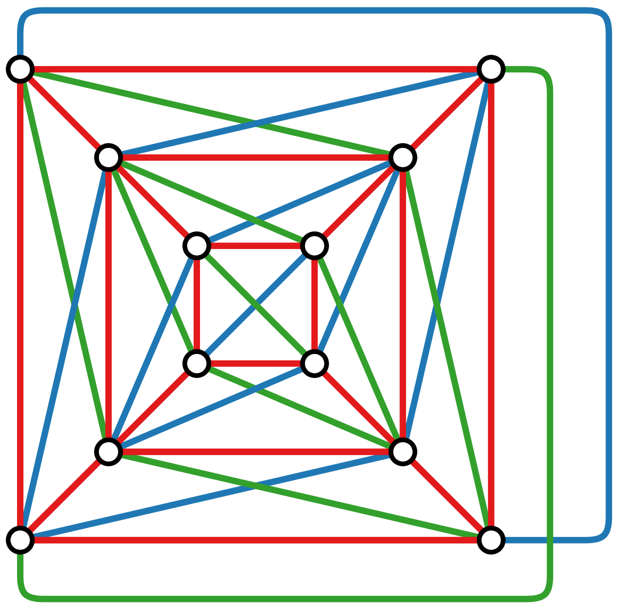

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Edges per face: 2 edges

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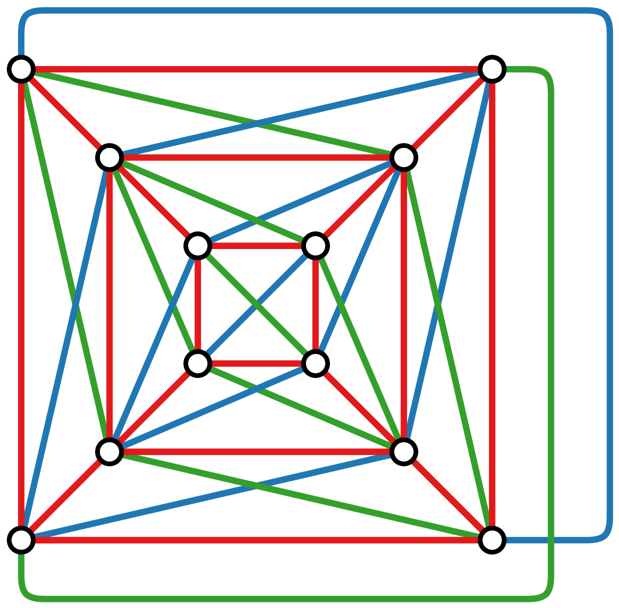

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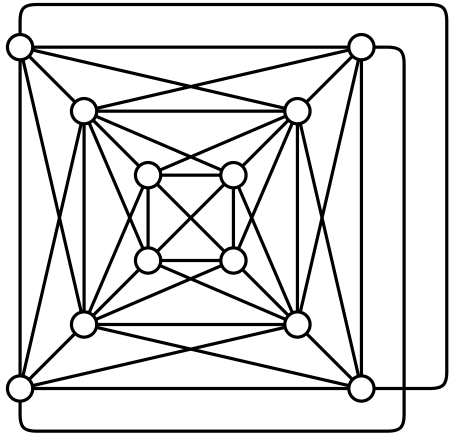
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**Theorem.** [Ringel 1965, Pach & Tóth 1997]
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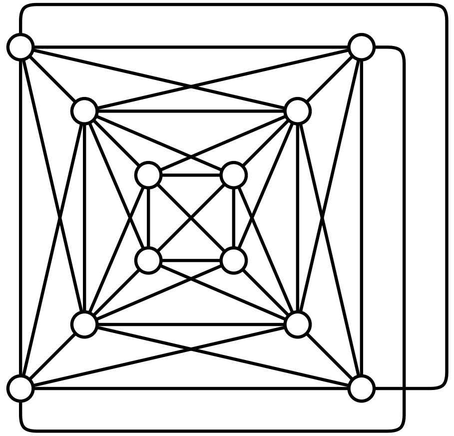
A 1-planar graph with $n$ vertices is called **optimal** if it has exactly $4n - 8$ edges.



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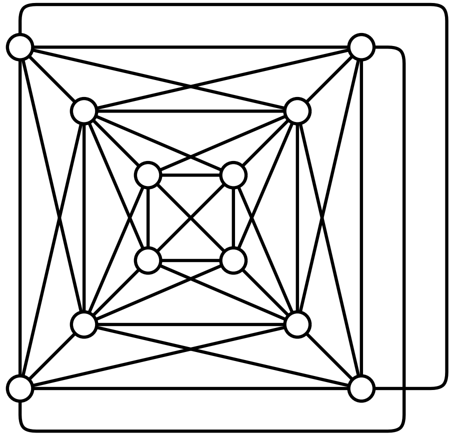

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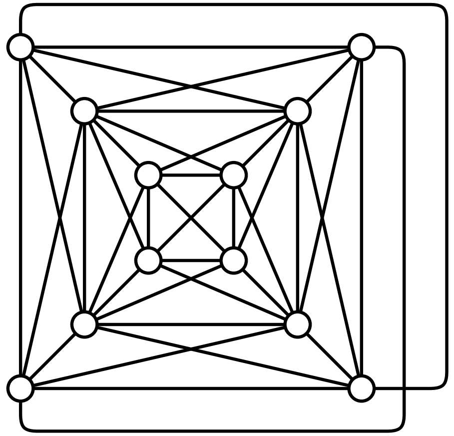

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There are **maximal** 1-planar graphs with $n$ vertices and $\underbrace{45/17n}_{\approx 2.65n} - O(1)$ edges.

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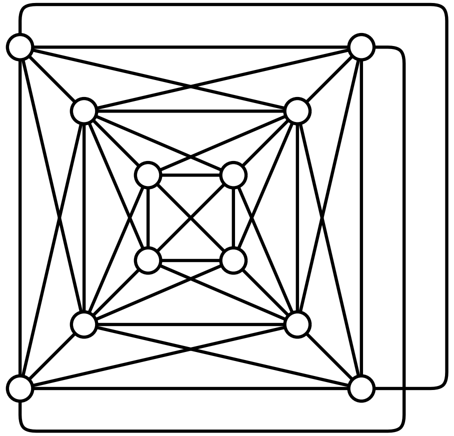


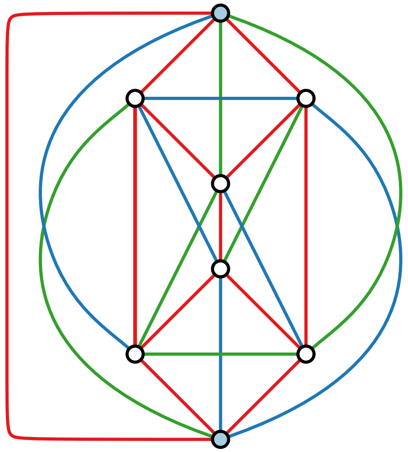

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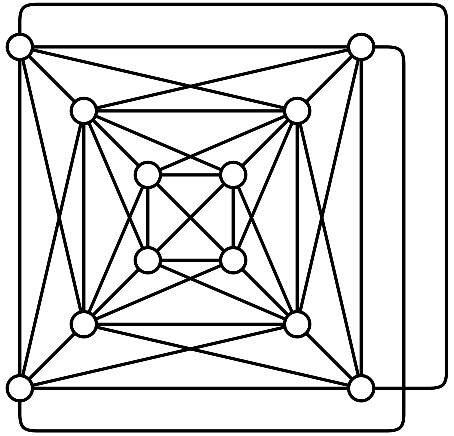

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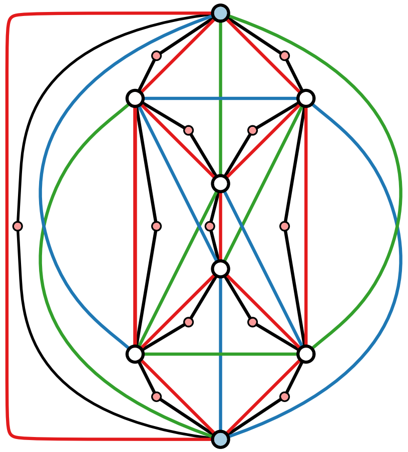

$n = 20, m = 48$

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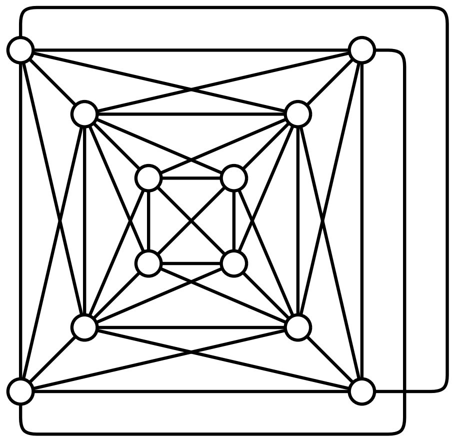


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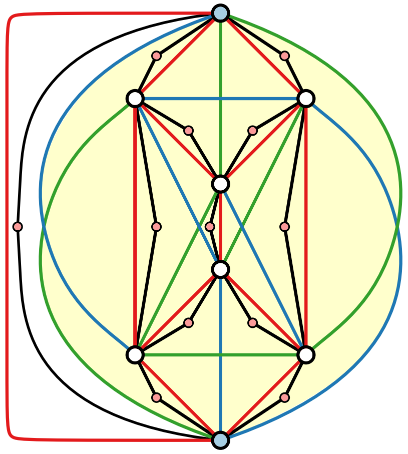


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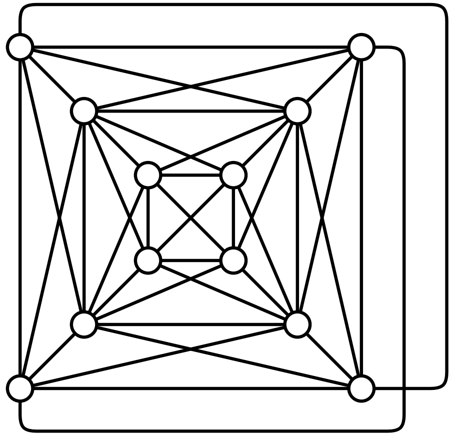


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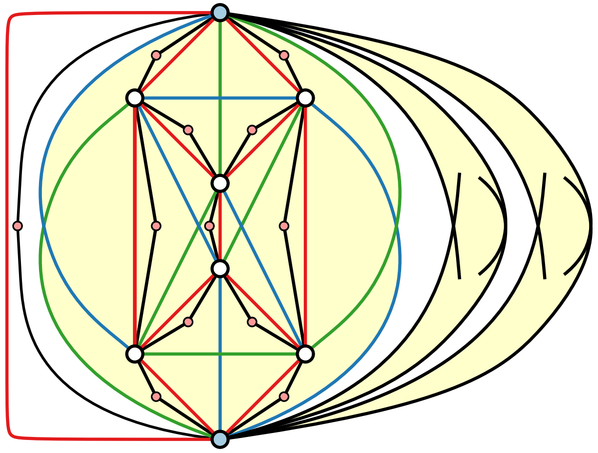
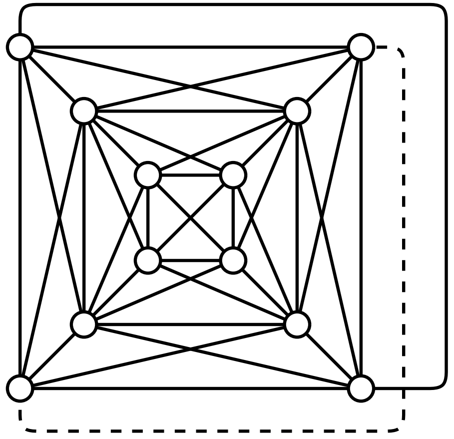
A 1-planar graph is called **maximal** if adding any edge would result in a non-1-planar graph.

**Theorem.** [Brandenburg et al. 2013]
There are **maximal** 1-planar graphs with $n$ vertices and $\underbrace{45/17n}_{\approx 2.65n} - O(1)$ edges.



# Density of 1-Planar Graphs

**Theorem.** [Ringel 1965, Pach & Tóth 1997]
A 1-planar graph with $n$ vertices has at most $4n - 8$ edges, which is a tight bound.

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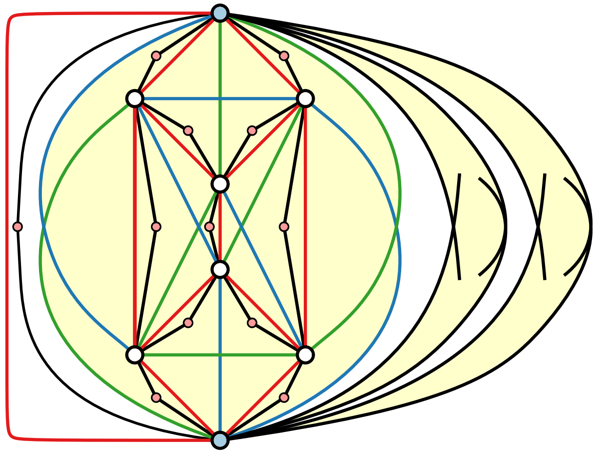

**Theorem.** [Brandenburg et al. 2013]
There are **maximal** 1-planar graphs with $n$ vertices and $\underbrace{45/17n}_{\approx 2.65n} - O(1)$ edges.

**Theorem.** [Didimo 2013]
A 1-planar graph with $n$ vertices that admits a **straight-line drawing** has at most $4n - 9$ edges.

# Density of $k$-Planar Graphs

**Theorem.**
A $k$-planar graph with $n$ vertices has at most:

| $k$ | number of edges |
| --- | --- |

# Density of $k$-Planar Graphs

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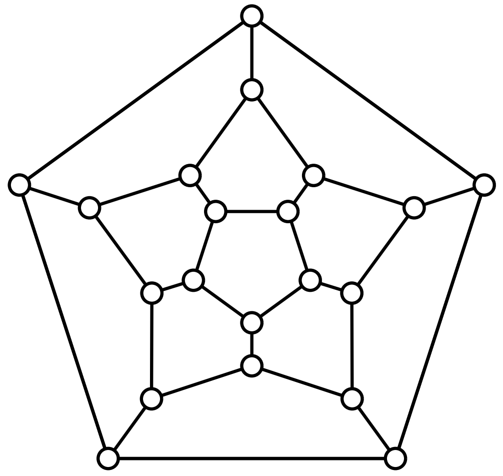

optimal 2-planar

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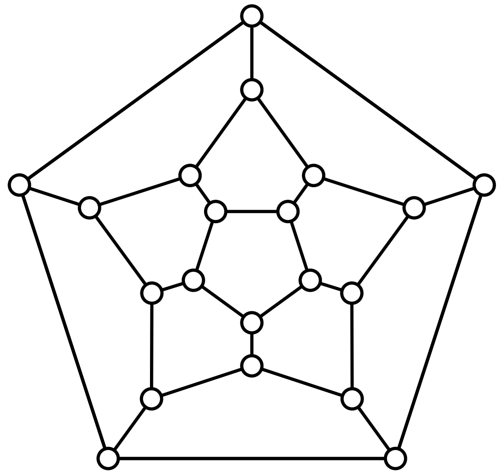

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Planar structure:

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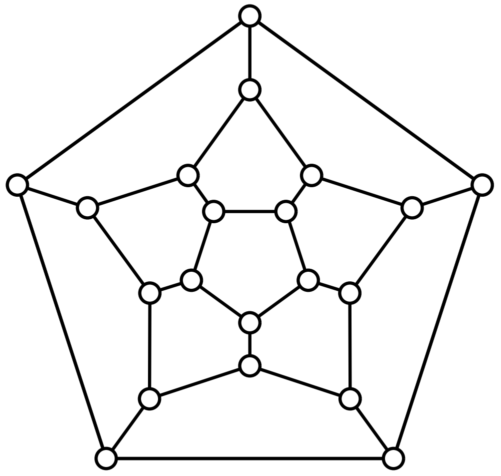

optimal 2-planar

Planar structure:

Edges per face:

Total:

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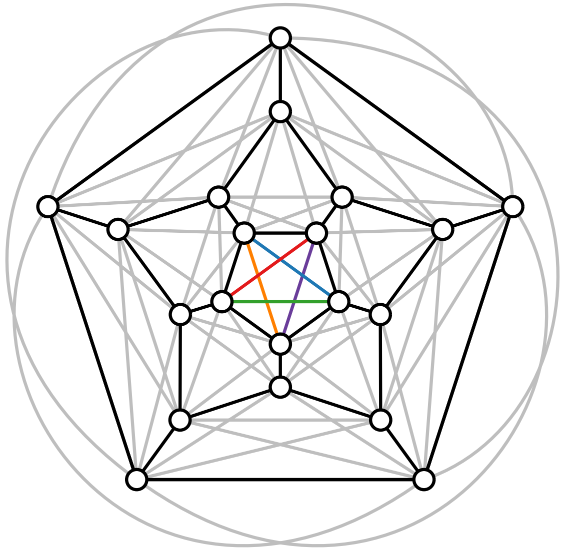

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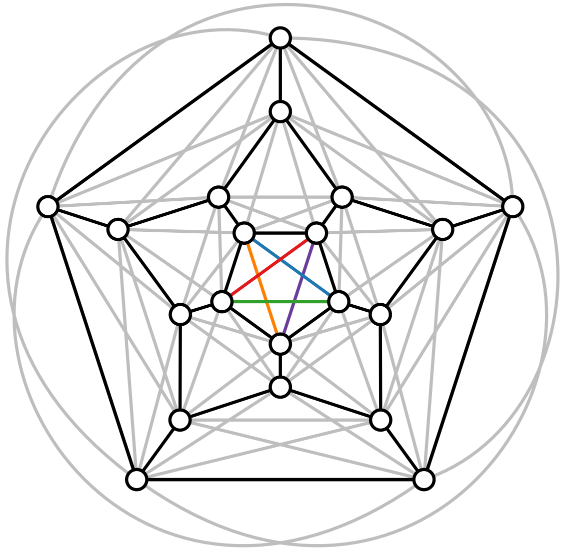

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Planar structure:

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Total:

$$n - m + f = 2$$
$$m = c \cdot f \ ?$$

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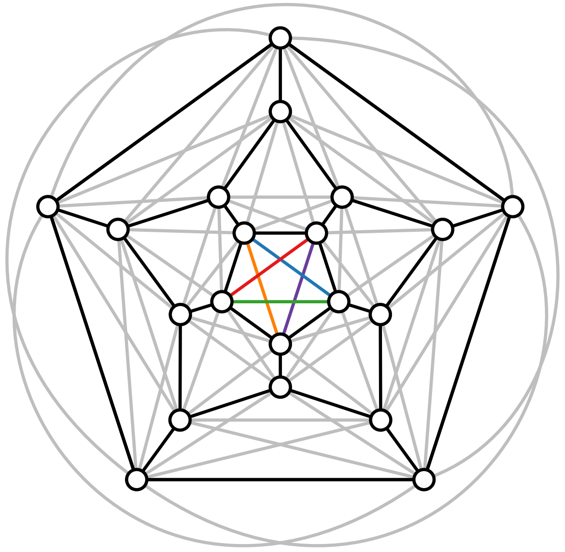

optimal 2-planar

Planar structure:
$\frac{5}{3}(n-2)$ edges
$\frac{2}{3}(n-2)$ faces

Edges per face:

Total:

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$m = c \cdot f$ ?
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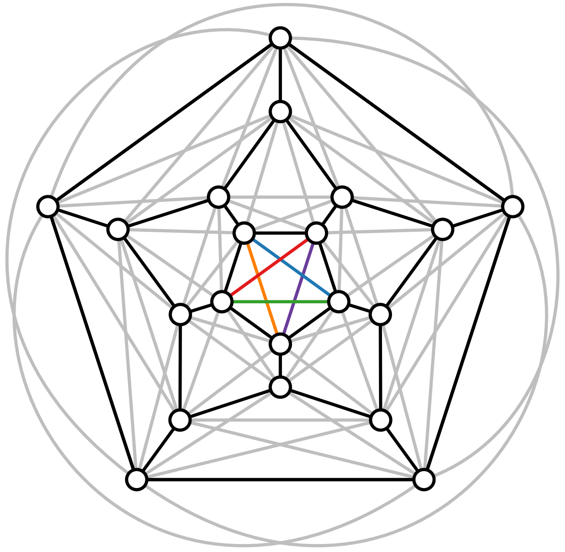

optimal 2-planar

$n - m + f = 2$

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Planar structure:

$\frac{5}{3}(n-2)$ edges

$\frac{2}{3}(n-2)$ faces

Edges per face: 5 edges

Total:

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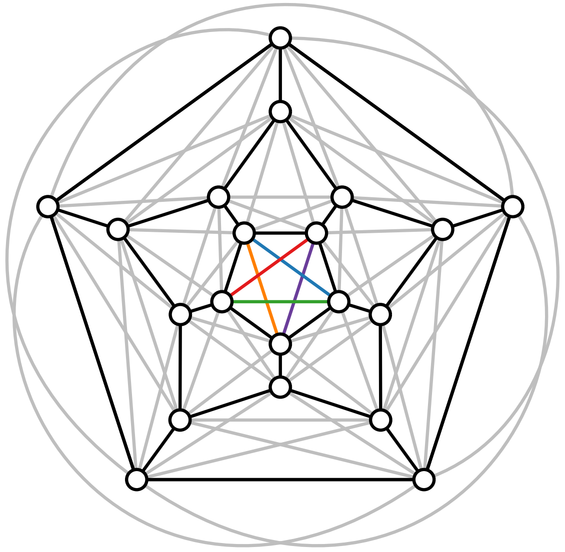

optimal 2-planar

Planar structure:
$\frac{5}{3}(n-2)$ edges
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Edges per face: 5 edges

Total: $5(n-2)$ edges

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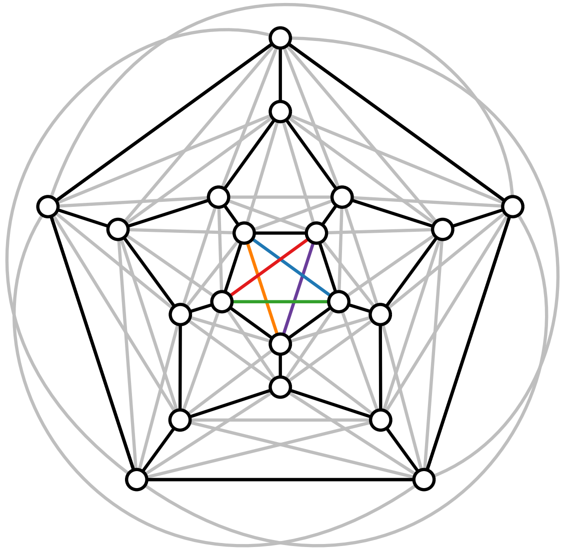

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Edges per face: 5 edges

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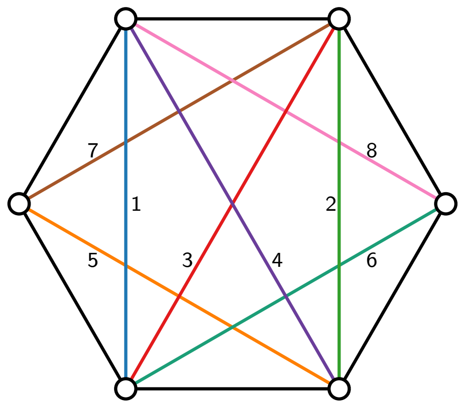


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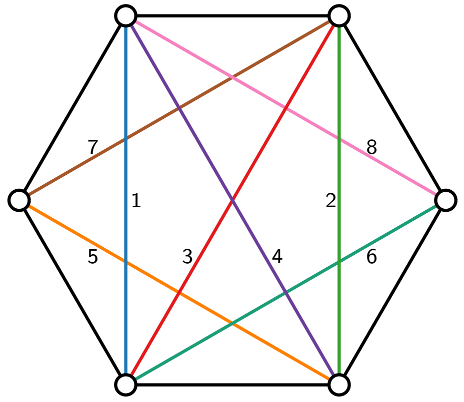

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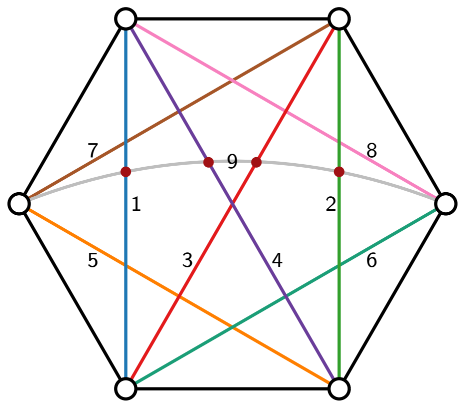


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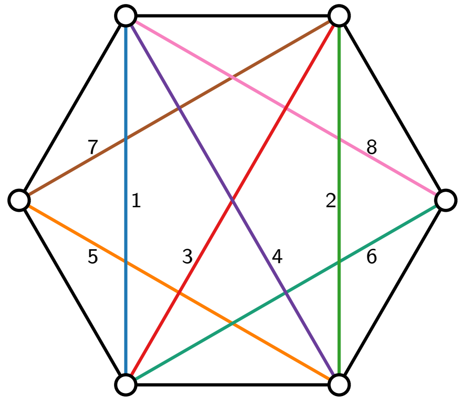

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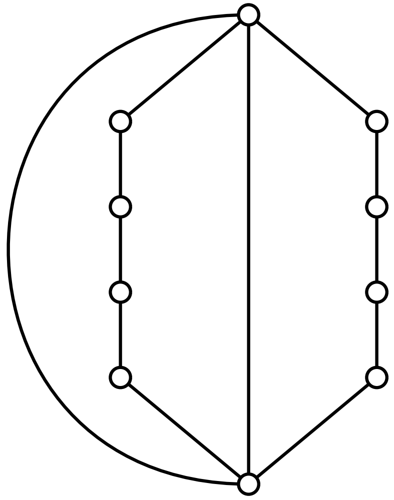

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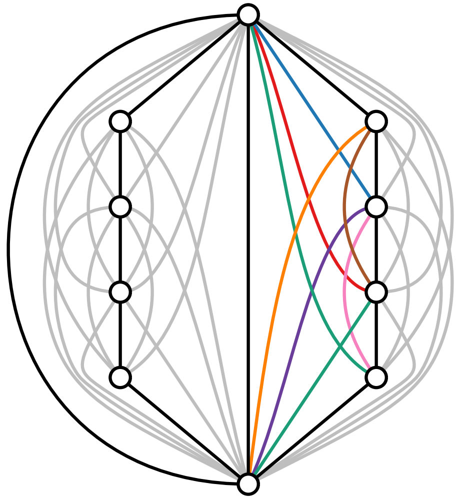

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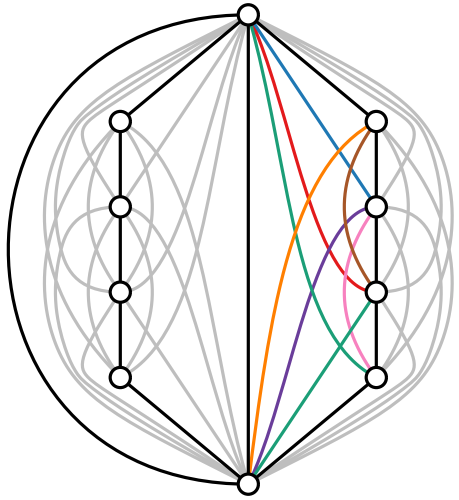

optimal 3-planar

Planar structure:
$\frac{3}{2}(n-2)$ edges
$\frac{1}{2}(n-2)$ faces

Edges per face: 8 edges

Total: $5.5(n-2)$ edges

# Density of $k$-Planar Graphs

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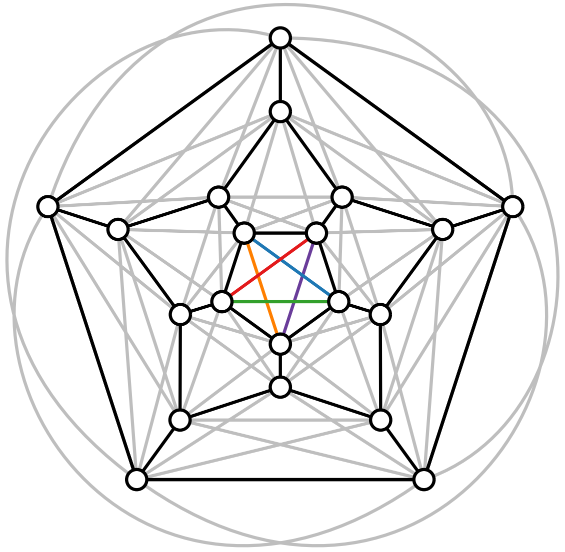
| 4 | $6(n-2)$ | [Ackerman 2015] |



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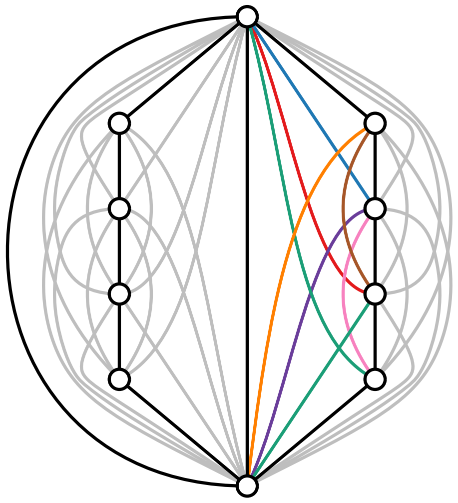

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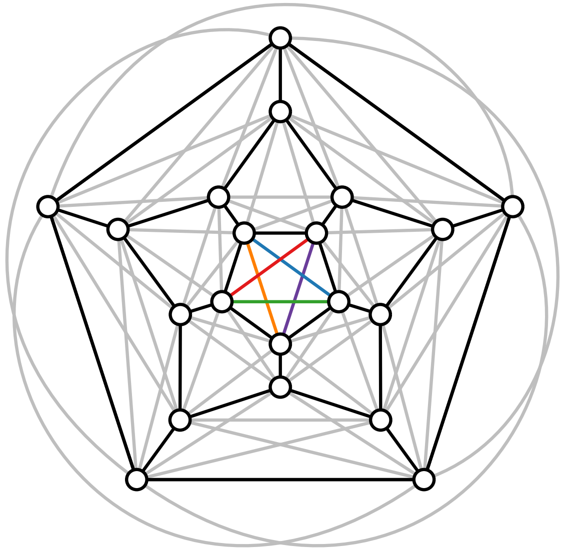

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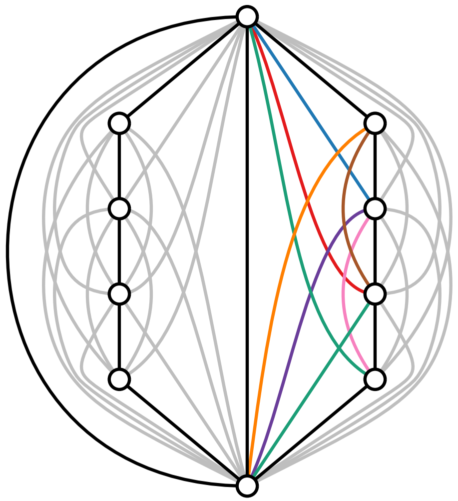

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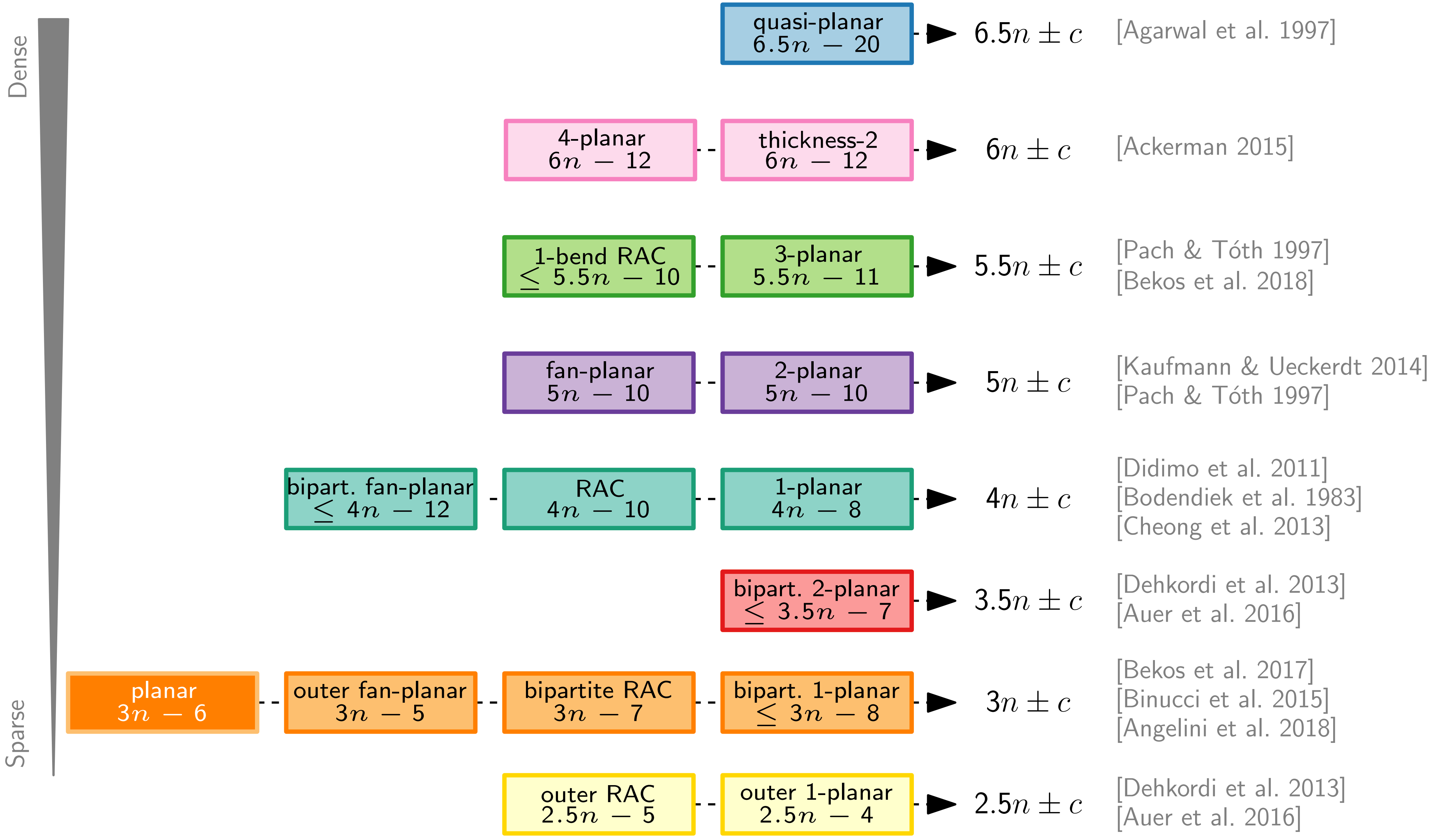
# GD Beyond Planarity: a Hierarchy



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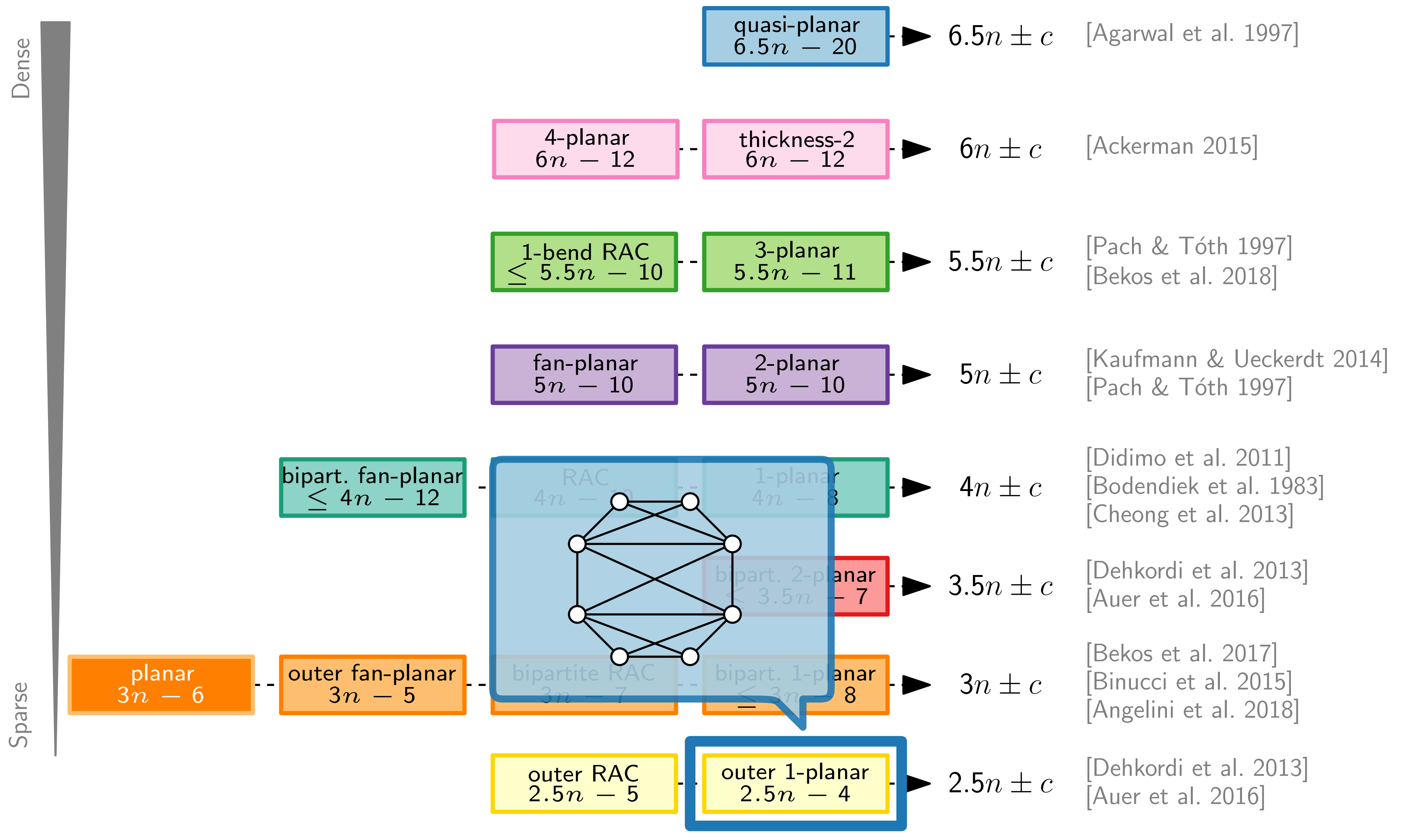

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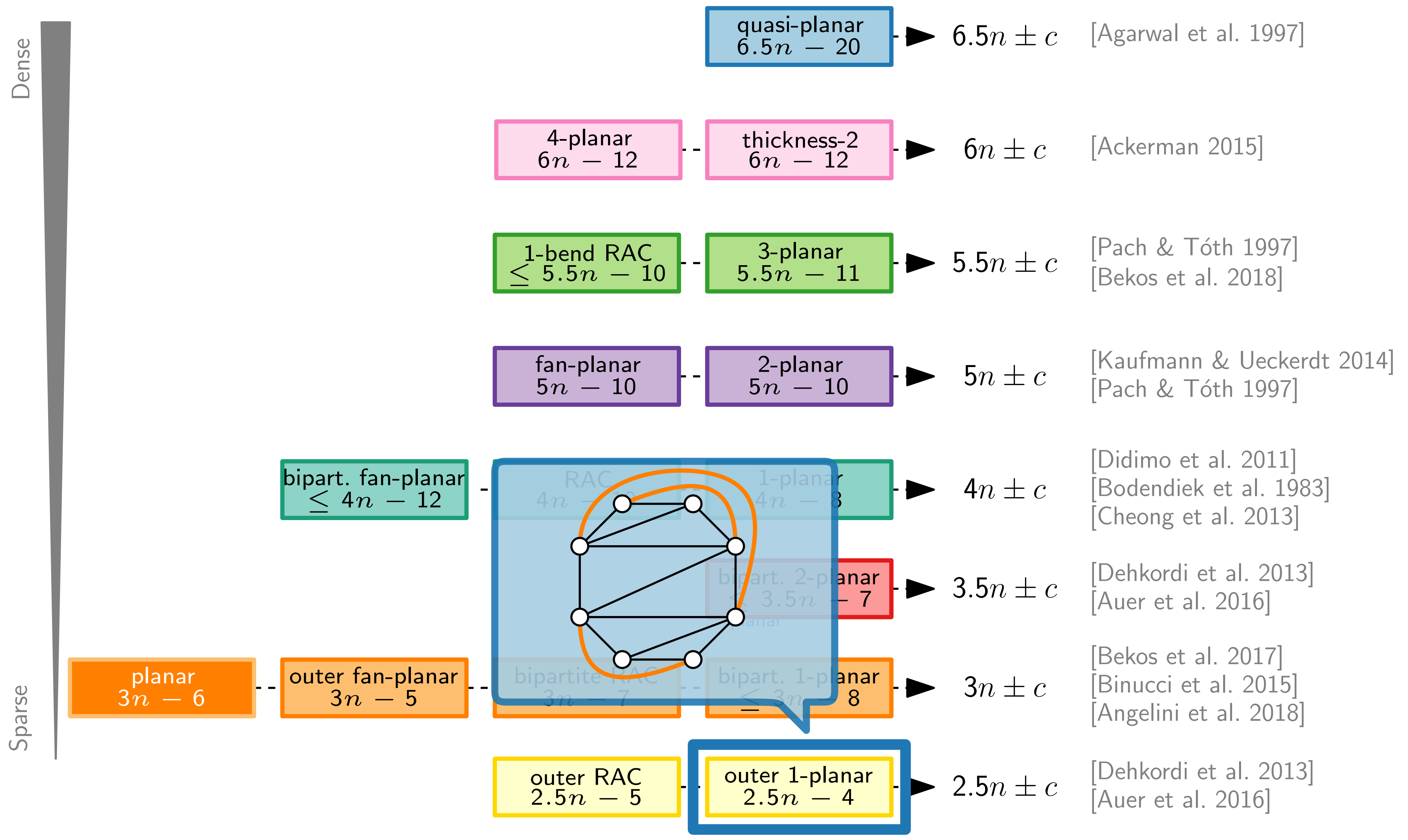
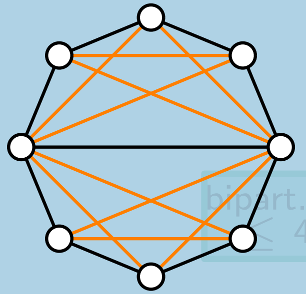


# GD Beyond Planarity: a Hierarchy

Dense

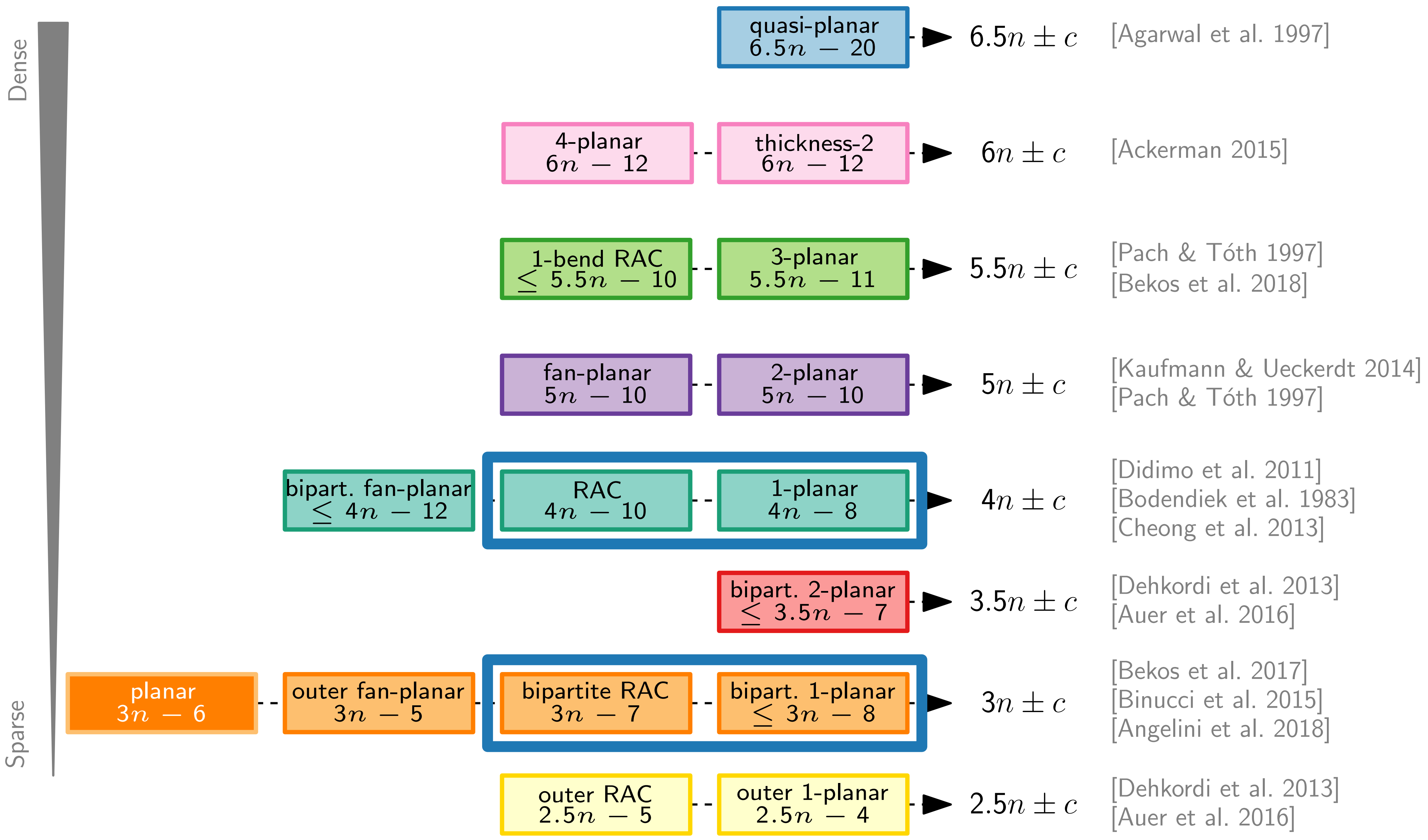
Sparse

| | | | | bound | references |
|---|---|---|---|---|---|
| | | | quasi-planar $6.5n - 20$ | $6.5n \pm c$ | [Agarwal et al. 1997] |
| | | 4-planar $6n - 12$ | thickness-2 $6n - 12$ | $6n \pm c$ | [Ackerman 2015] |
| | | 1-bend RAC $\leq 5.5n - 10$ | 3-planar $5.5n - 11$ | $5.5n \pm c$ | [Pach & Tóth 1997] [Bekos et al. 2018] |
| | | fan-planar $5n - 10$ | 2-planar $5n - 10$ | $5n \pm c$ | [Kaufmann & Ueckerdt 2014] [Pach & Tóth 1997] |
| | bipart. fan-planar $\leq 4n - 12$ | RAC $4n - 10$ | 1-planar $4n - 8$ | $4n \pm c$ | [Didimo et al. 2011] [Bodendiek et al. 1983] [Cheong et al. 2013] |
| | | | bipart. 2-planar $\leq 3.5n - 7$ | $3.5n \pm c$ | [Dehkordi et al. 2013] [Auer et al. 2016] |
| planar $3n - 6$ | outer fan-planar $3n - 5$ | bipartite RAC $3n - 7$ | bipart. 1-planar $\leq 3n - 8$ | $3n \pm c$ | [Bekos et al. 2017] [Binucci et al. 2015] [Angelini et al. 2018] |
| | | outer RAC $2.5n - 5$ | outer 1-planar $2.5n - 4$ | $2.5n \pm c$ | [Dehkordi et al. 2013] [Auer et al. 2016] |



For triconnected graphs, the bound is $3n - 6$.

# GD Beyond Planarity: a Hierarchy



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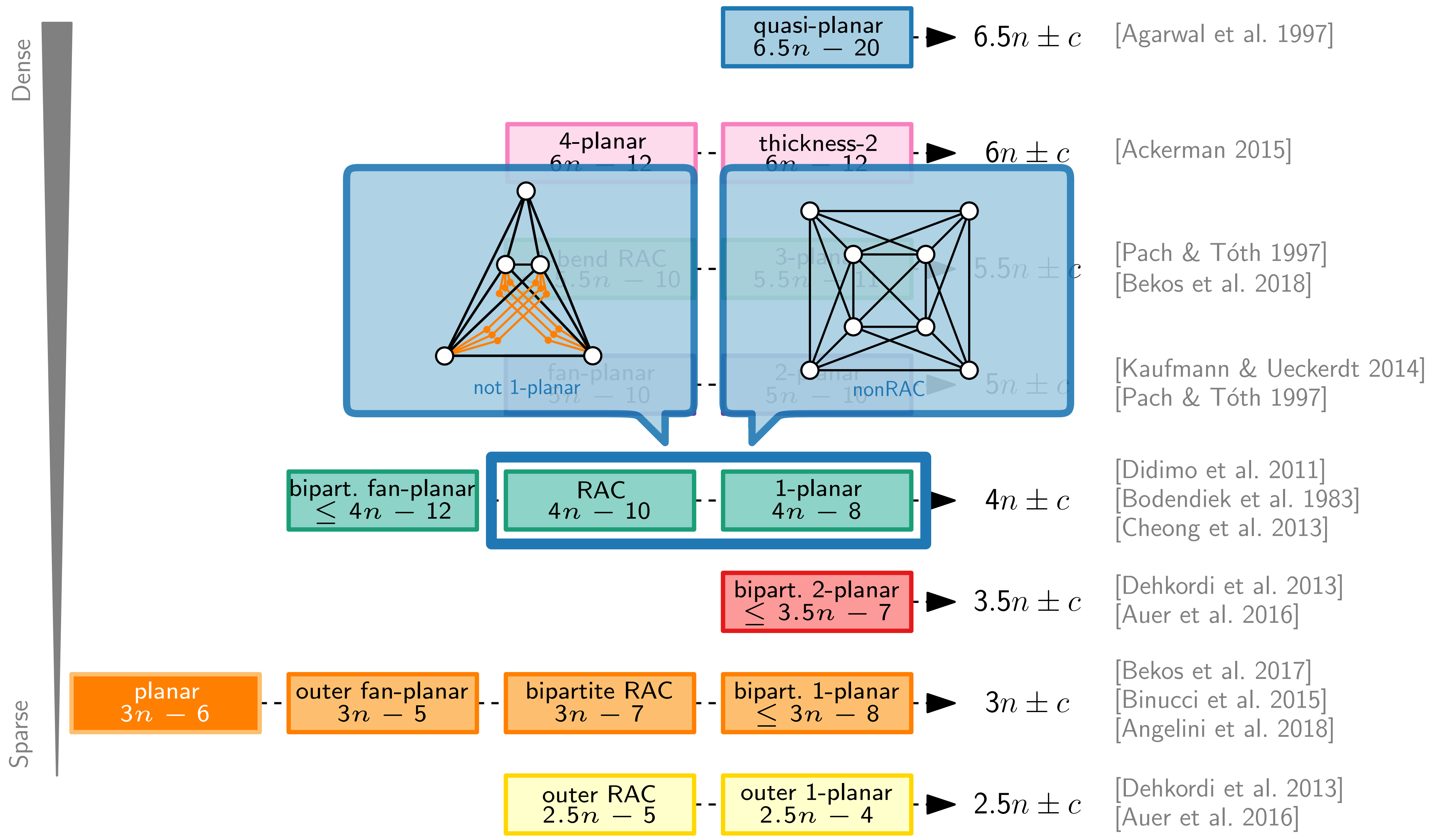

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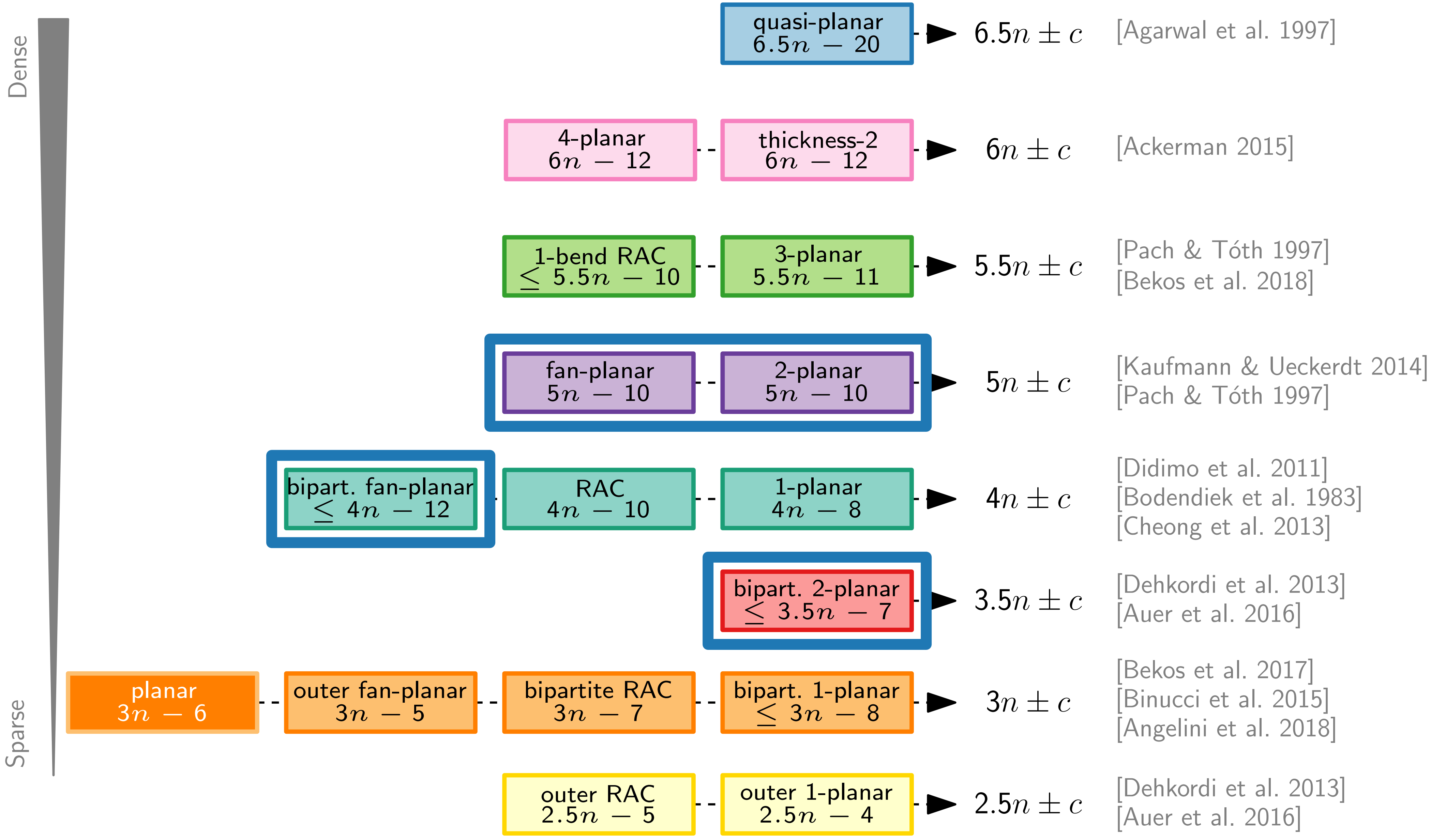

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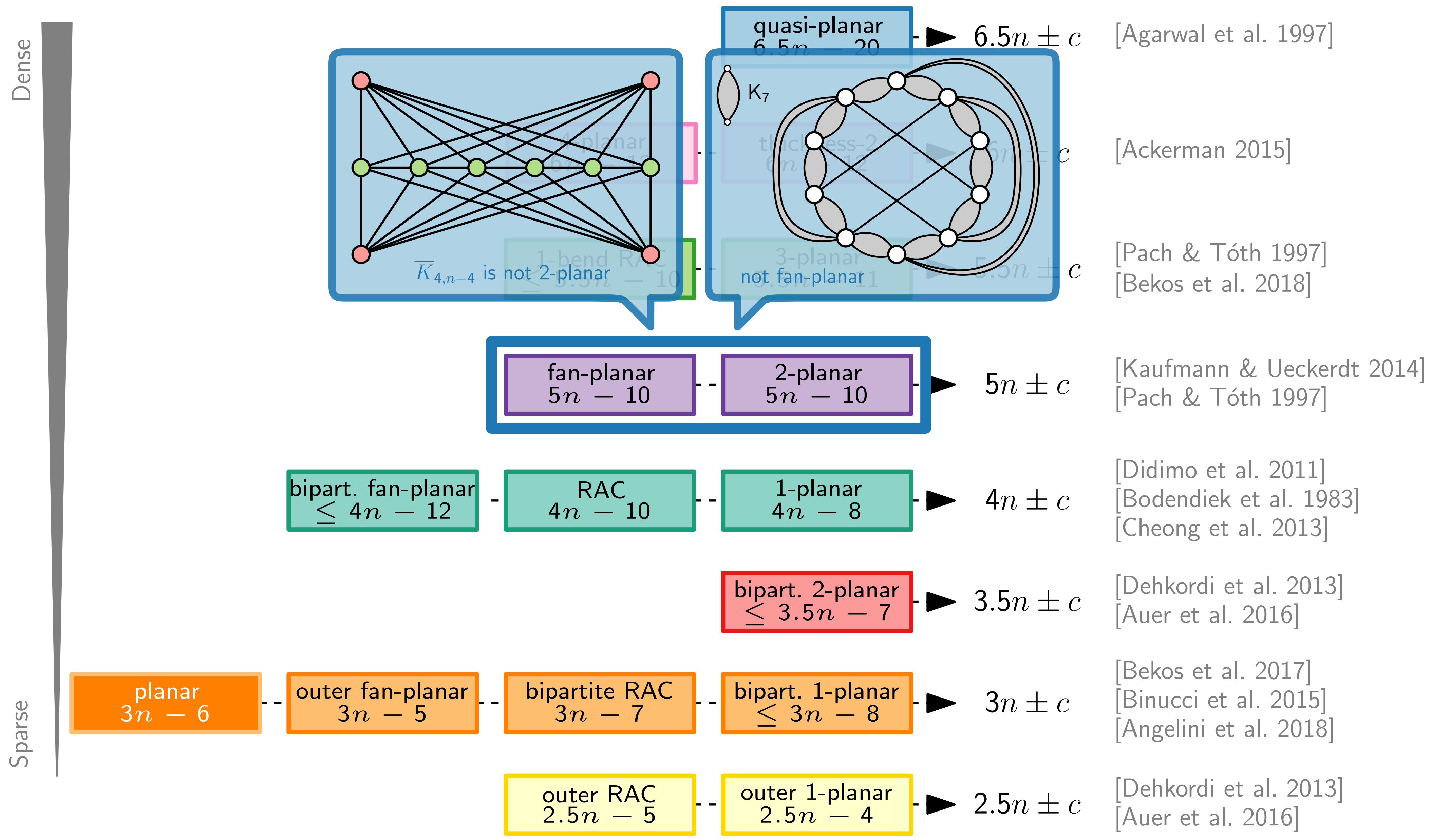

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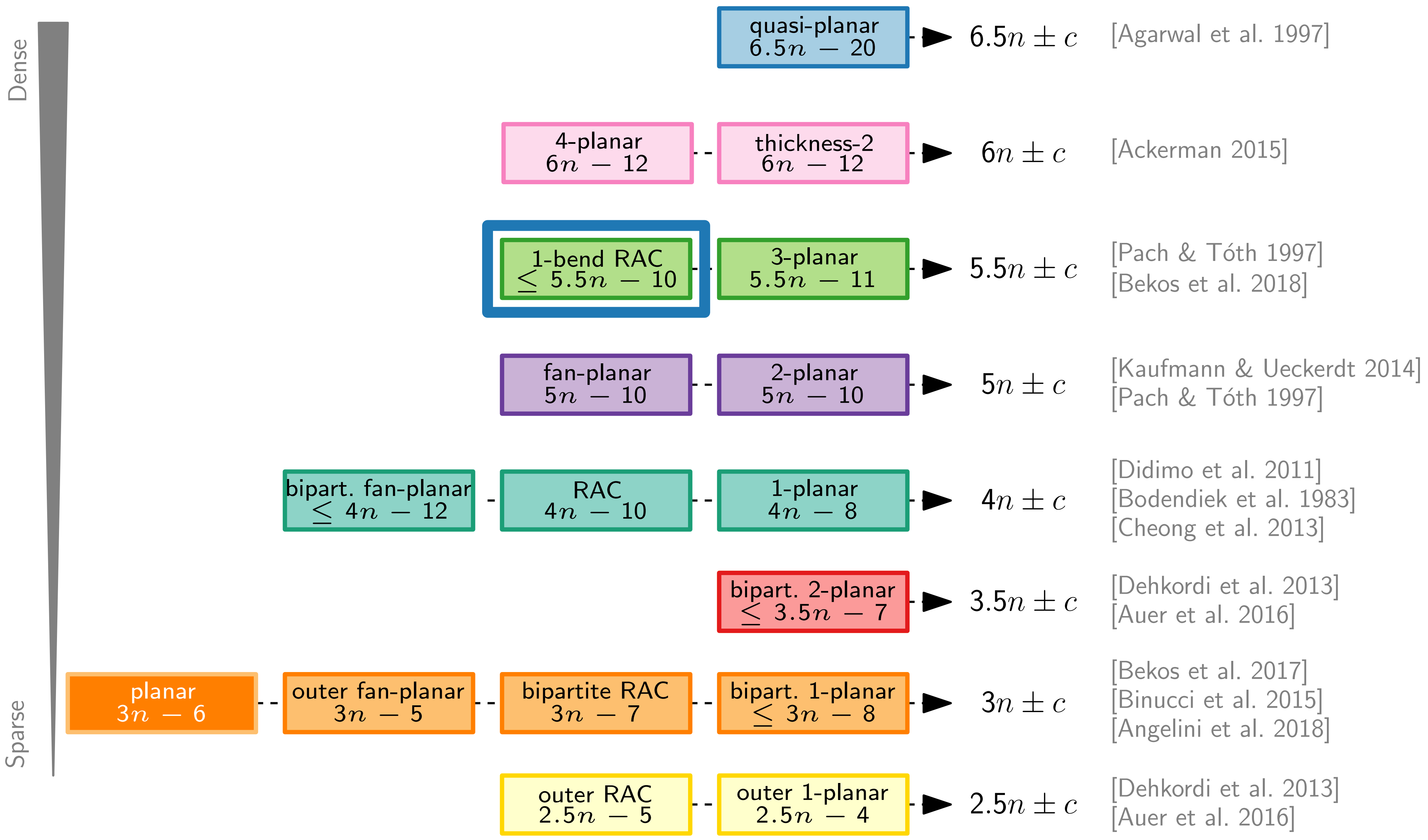

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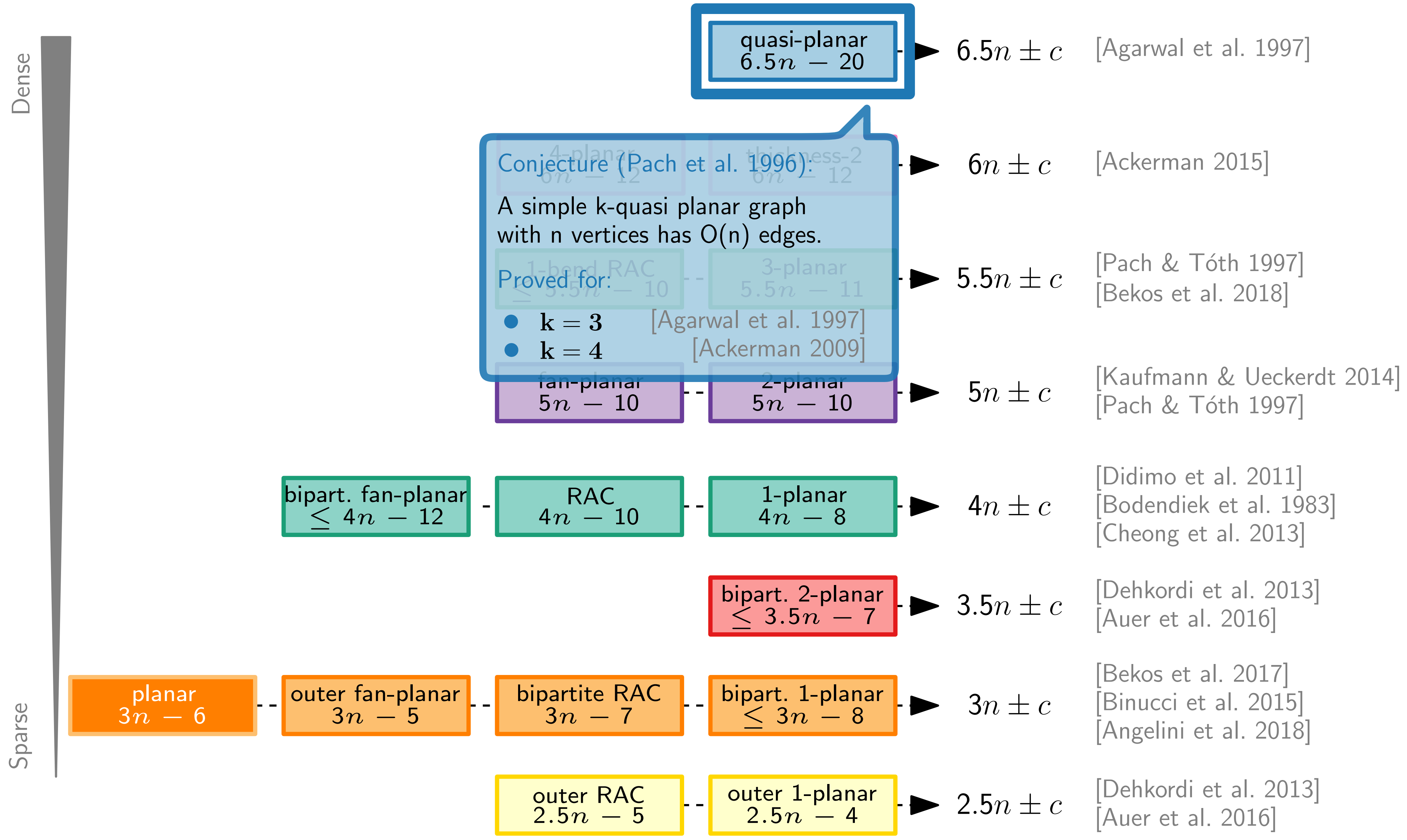


Conjecture (Pach et al. 1996):

A simple k-quasi planar graph with n vertices has O(n) edges.

Proved for:

- $k = 3$ [Agarwal et al. 1997]
- $k = 4$ [Ackerman 2009]

# Crossing Numbers

The **$k$-planar crossing number** $\mathrm{cr}_{k\text{-pl}}(G)$ of a graph $G$ is the number of crossings required in any $k$-planar drawing of $G$.

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**Theorem.** [Chimani, Kindermann, Montecchiani & Valtr 2019]
For every $\ell \geq 7$, there is a 1-planar graph $G$ with $n = 11\ell + 2$ vertices such that $\text{cr}(G) = 2$ and $\text{cr}_{1\text{-pl}}(G) = n - 2$.

# Crossing Numbers

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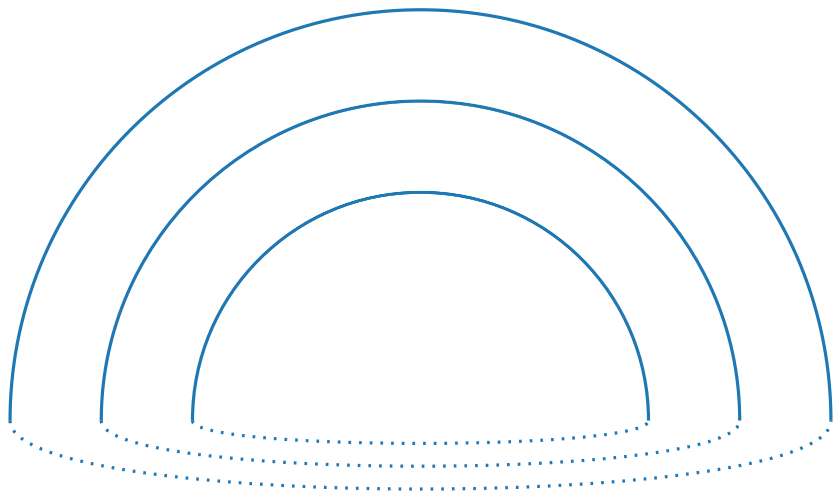

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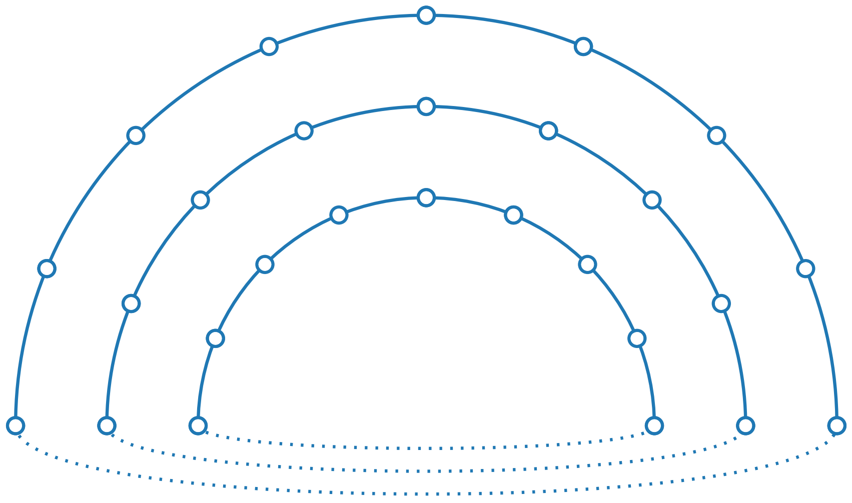

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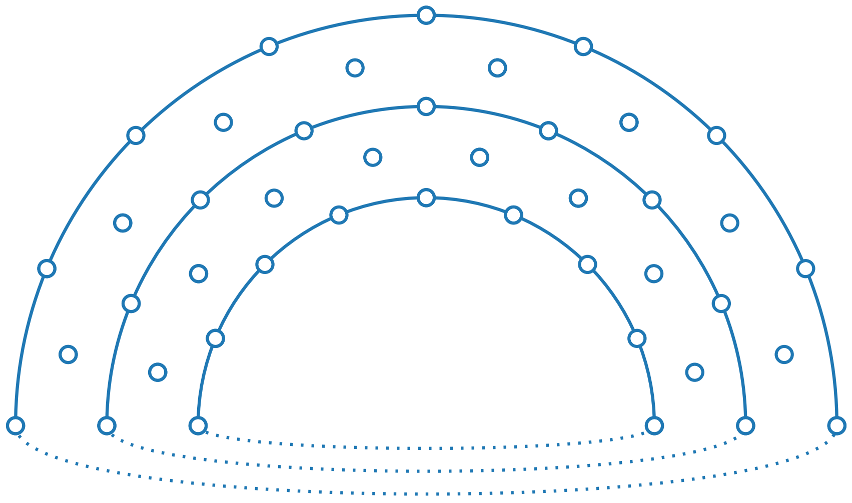

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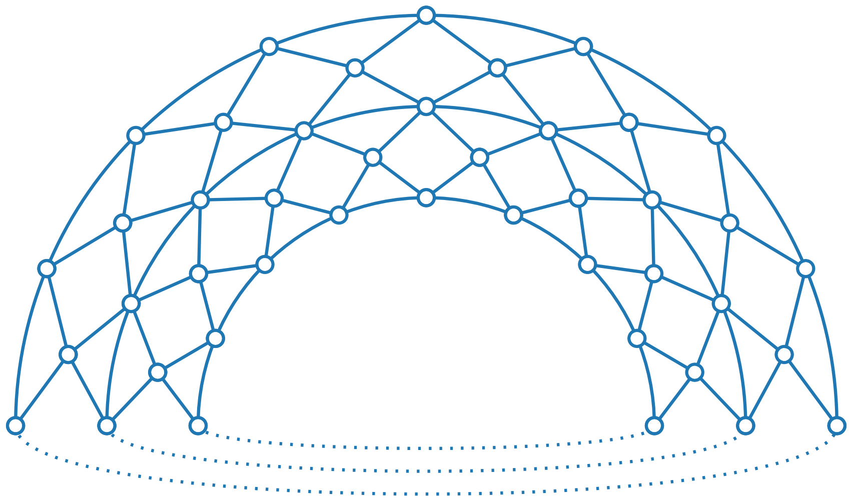

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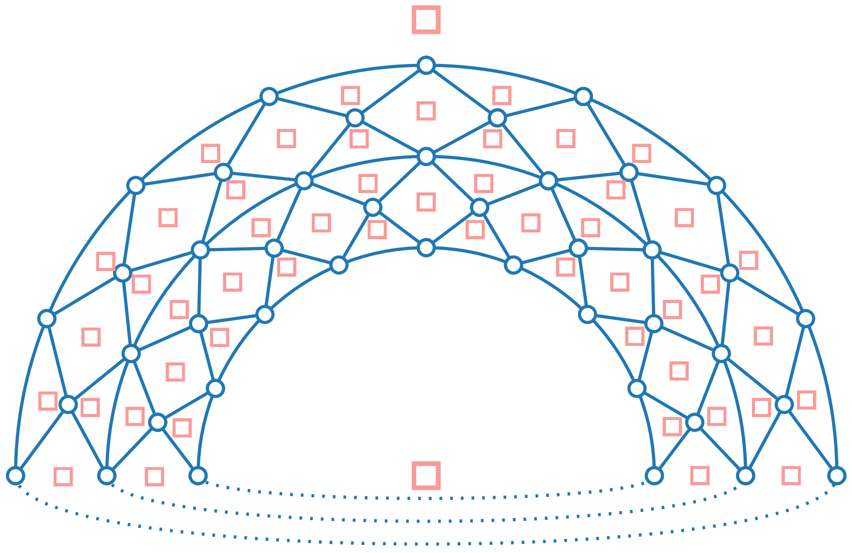

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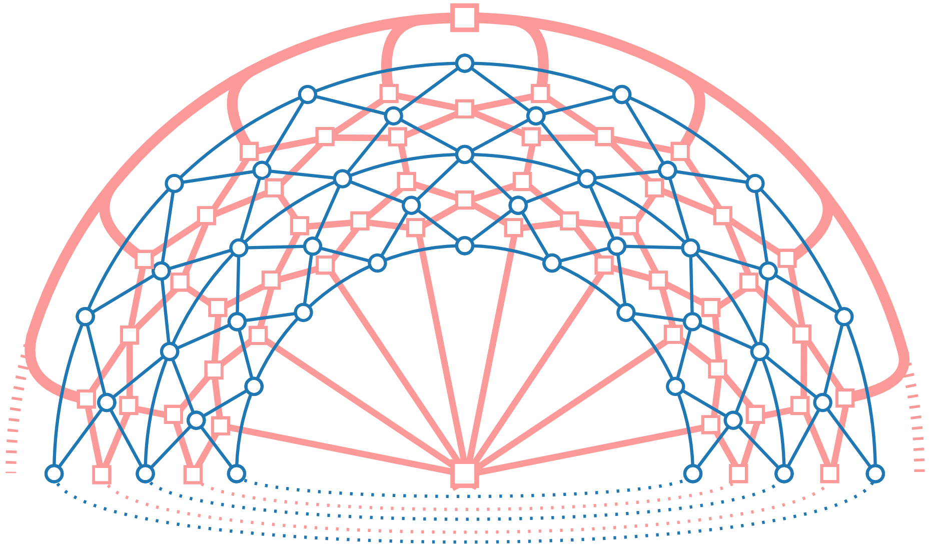

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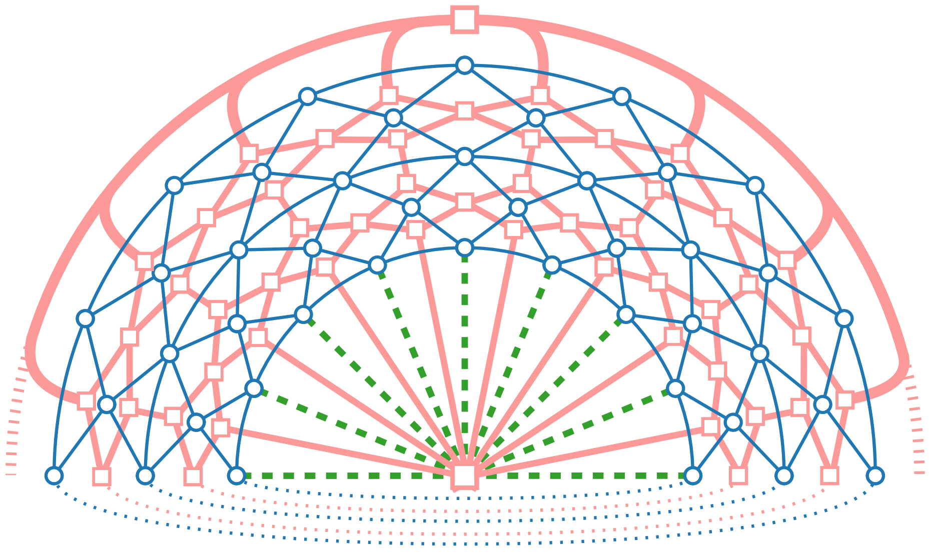

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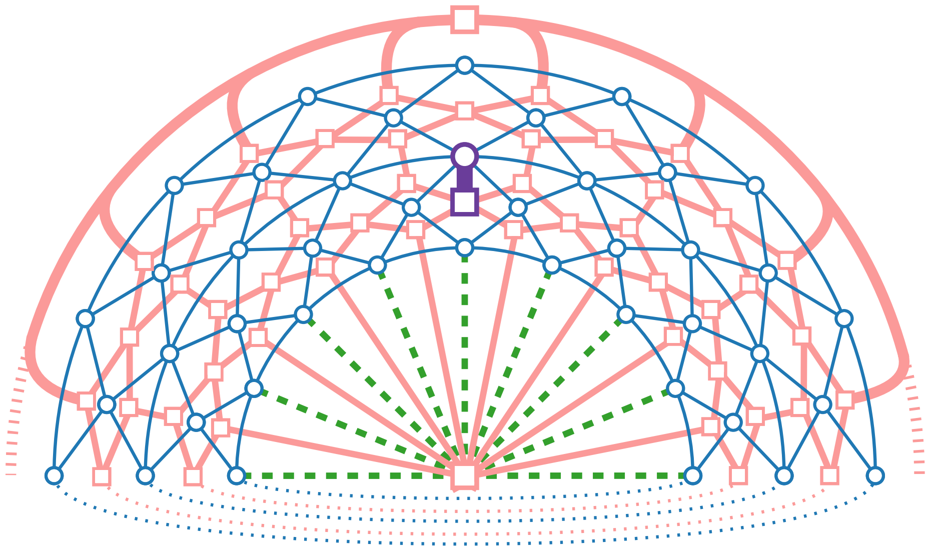

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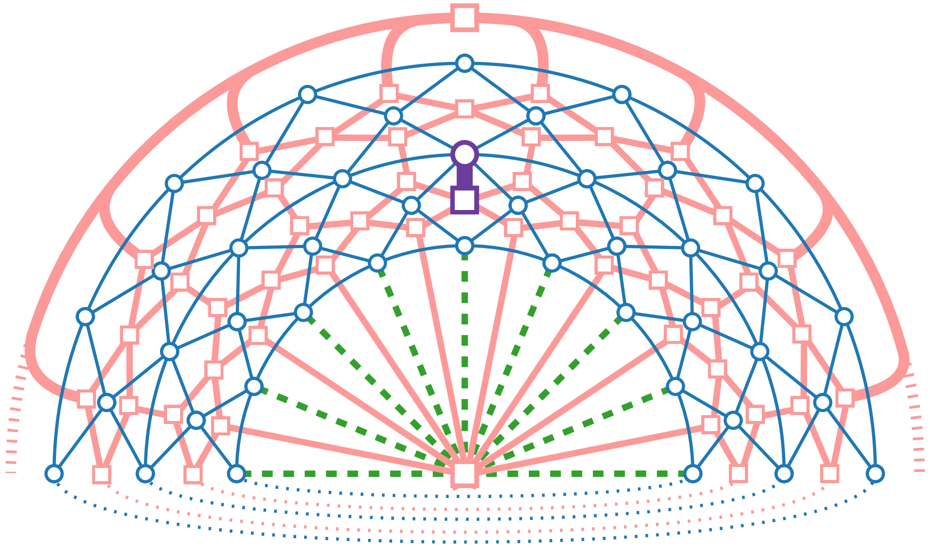

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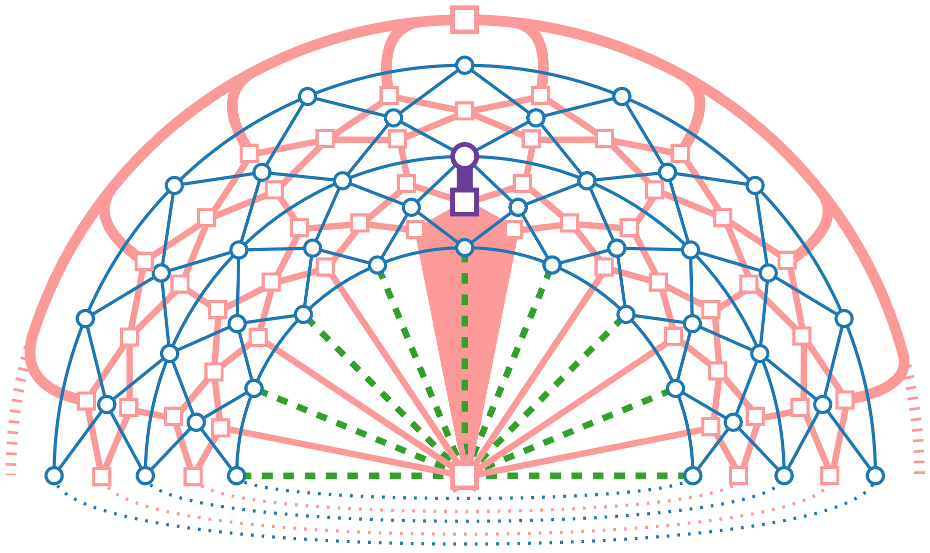

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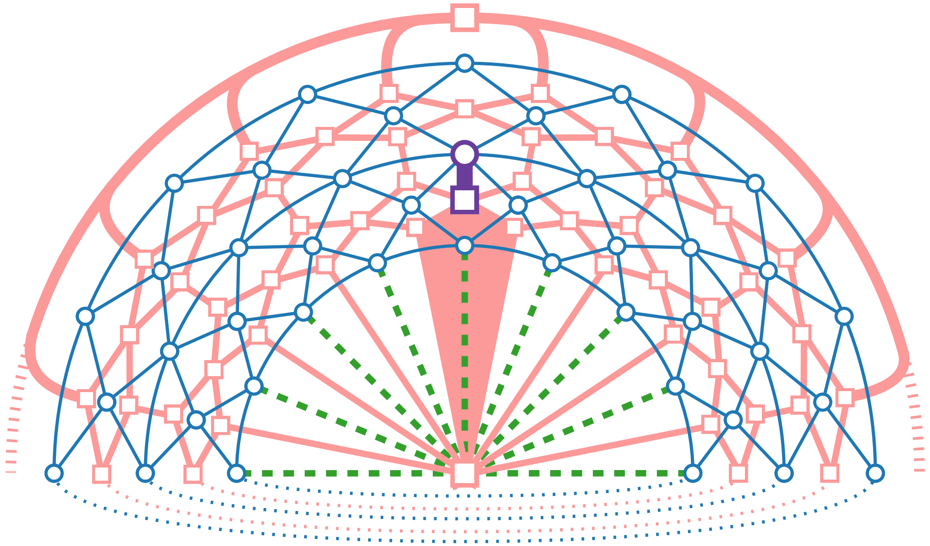

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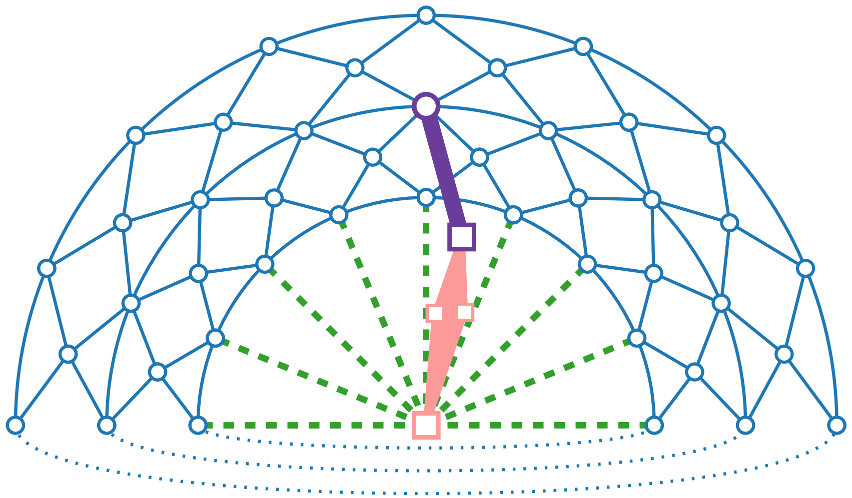

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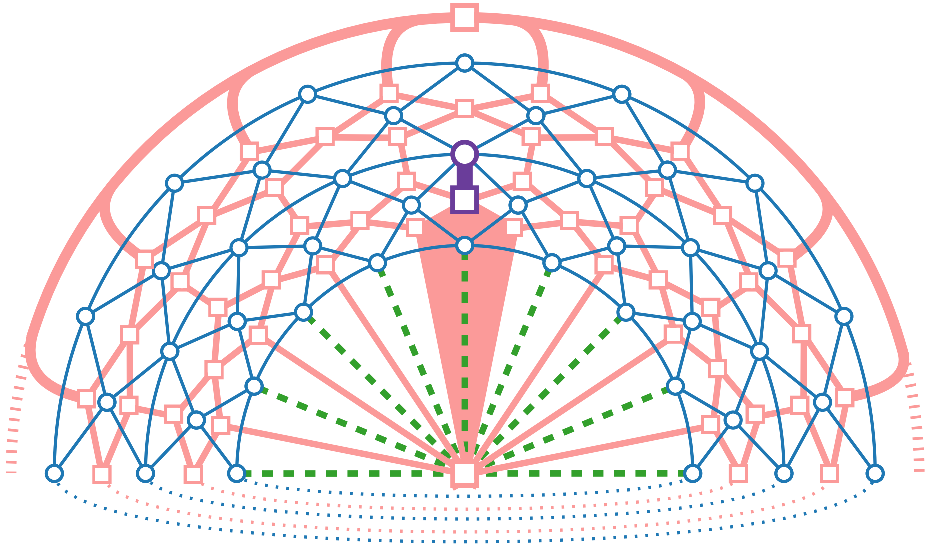

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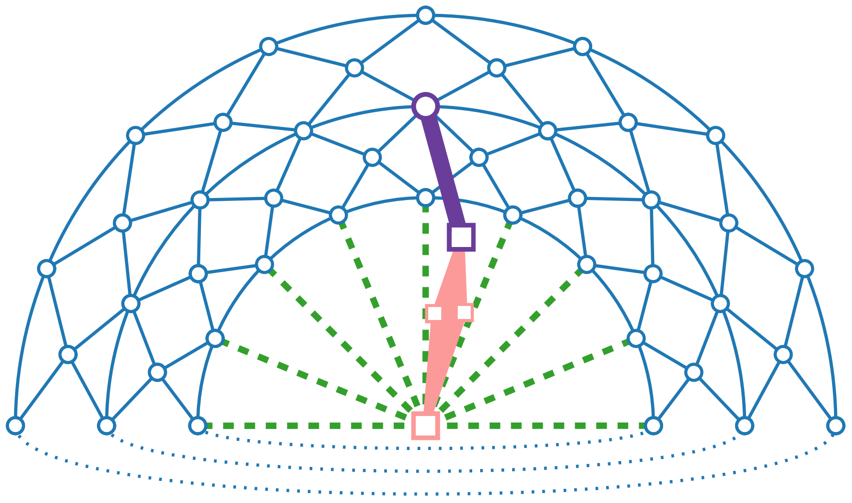

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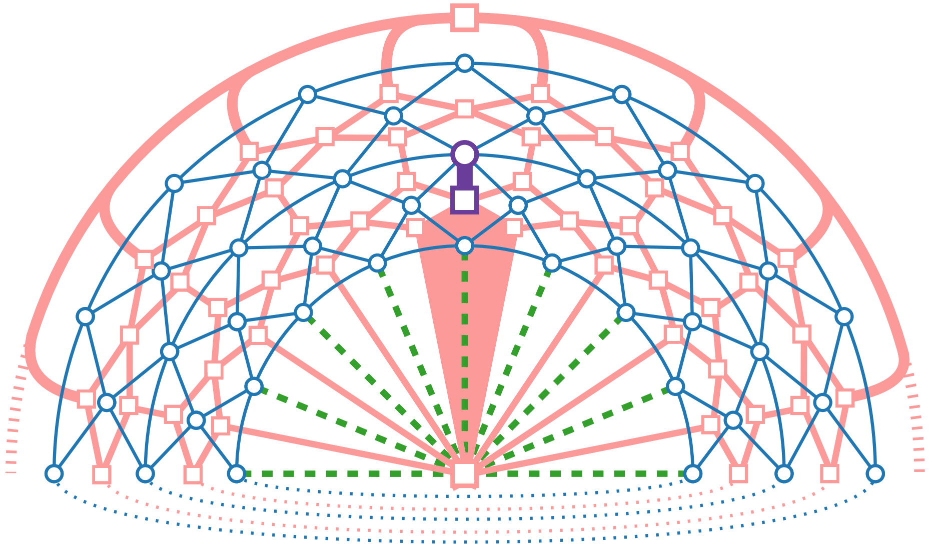
**Crossing ratio**
$\rho_{1\text{-pl}}(n) = (n-2)/2$



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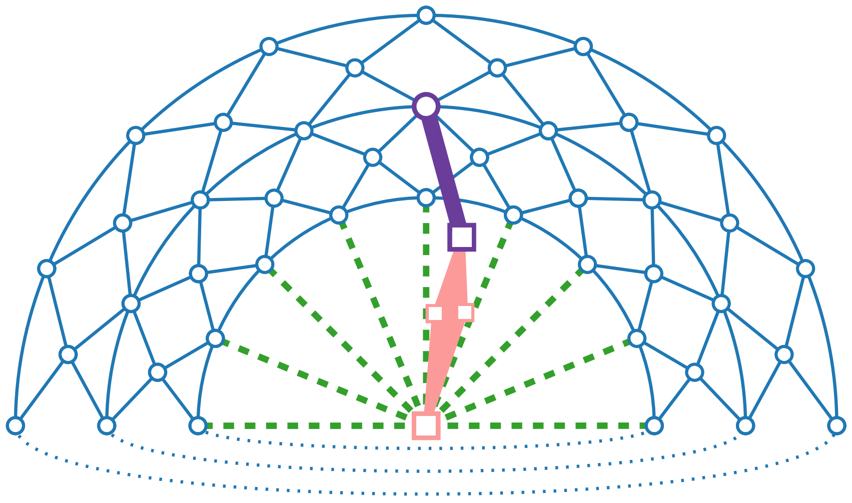

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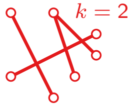
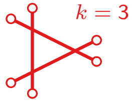
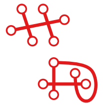
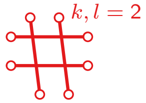
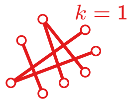
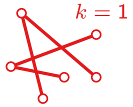
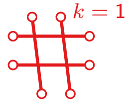
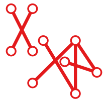
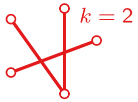
# Crossing Ratios

Table from "Crossing Numbers of Beyond-Planar Graphs Revisited" [van Beusekom, Parada & Speckmann 2021]

| Family | Forbidden Configurations | | Lower | Upper |
|---|---|---|---|---|
| $k$-planar | An edge crossed more than $k$ times |   | $\boldsymbol{\Omega(n/k)}$ | $\boldsymbol{O(k\sqrt{k}n)}$ |
| $k$-quasi-planar | $k$ pairwise crossing edges |   | $\Omega(n/k^3)$ | $f(k)n^2 \log^2 n$ |
| Fan-planar | Two independent edges crossing a third or two adjacent edges crossing another edge from different "side" |  | $\Omega(n)$ | $O(n^2)$ |
| $(k,l)$-grid-free | Set of $k$ edges such that each edge crosses each edge from a set of $l$ edges. |   | $\Omega\left(\frac{n}{kl(k+l)}\right)$ | $g(k,l)n^2$ |
| $k$-gap-planar | More than $k$ crossings mapped to an edge in an optimal mapping |   | $\boldsymbol{\Omega(n/k^3)}$ | $\boldsymbol{O(k\sqrt{k}n)}$ |
| Skewness-$k$ | Set of crossings not covered by at most $k$ edges |   | $\boldsymbol{\Omega(n/k)}$ | $\boldsymbol{O(kn + k^2)}$ |
| $k$-apex | Set of crossings not covered by at most $k$ vertices |   | $\Omega(n/k)$ | $O(k^2n^2 + k^4)$ |
| Planarly connected | Two crossing edges that do not have two of their endpoint connected by a crossing-free edge |  | $\Omega(n^2)$ | $\boldsymbol{O(n^2)}$ |
| $k$-fan-crossing-free | An edge that crosses $k$ adjacent edges |   | $\boldsymbol{\Omega(n^2/k^3)}$ | $\boldsymbol{O(k^2n^2)}$ |
| Straight-line RAC | Two edges crossing at an angle $< \frac{\pi}{2}$ |  | $\Omega(n^2)$ | $\boldsymbol{O(n^2)}$ |

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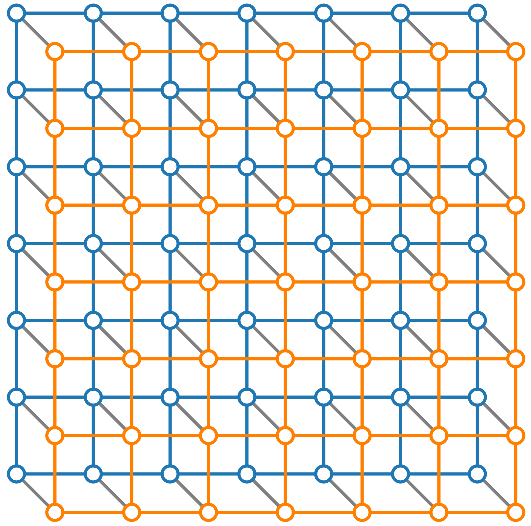
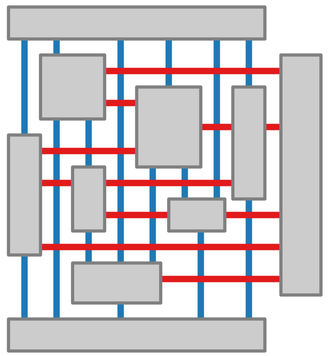
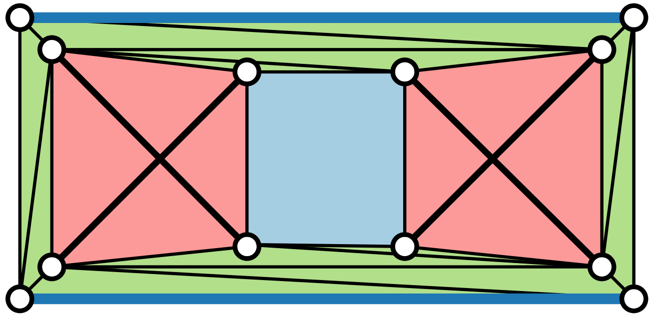
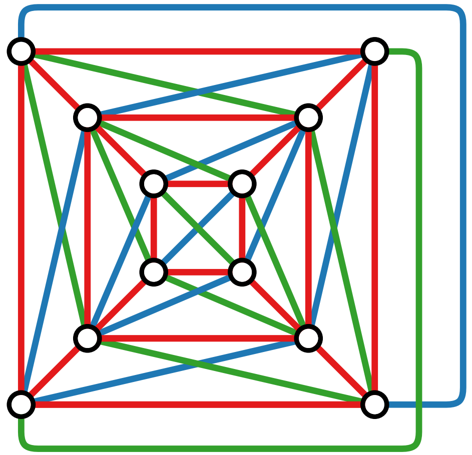
# Visualization of Graphs





## Lecture 11:
## Beyond Planarity
### Drawing Graphs with Crossings

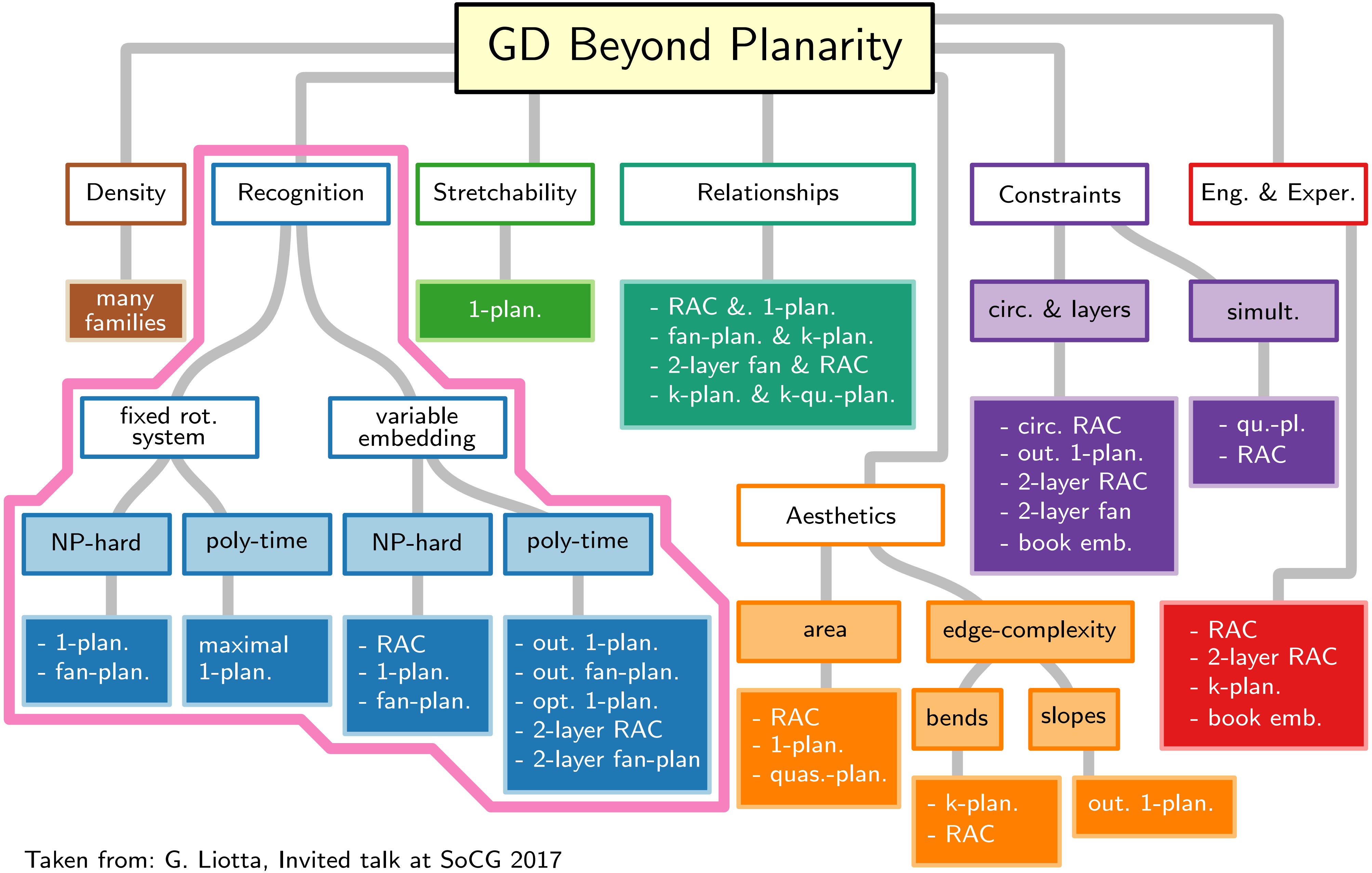




## Part III:
## Recognition

Jonathan Klawitter

# GD Beyond Planarity: a Taxonomy




Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

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**Theorem.** [Kuratowski 1930]
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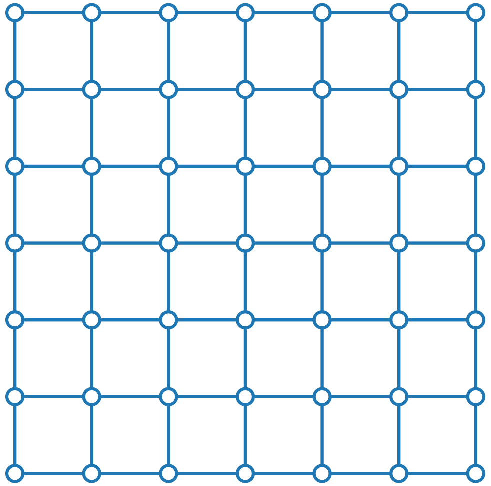

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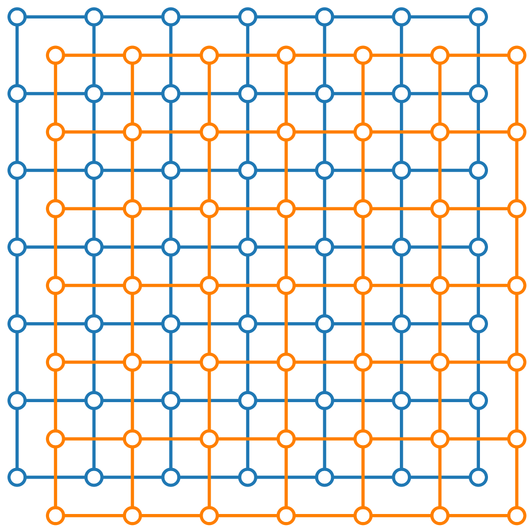

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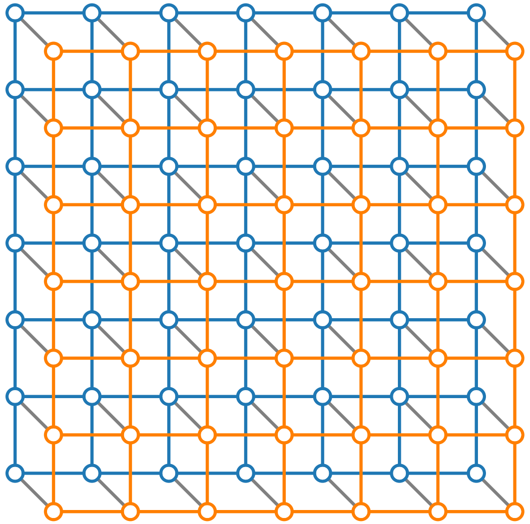

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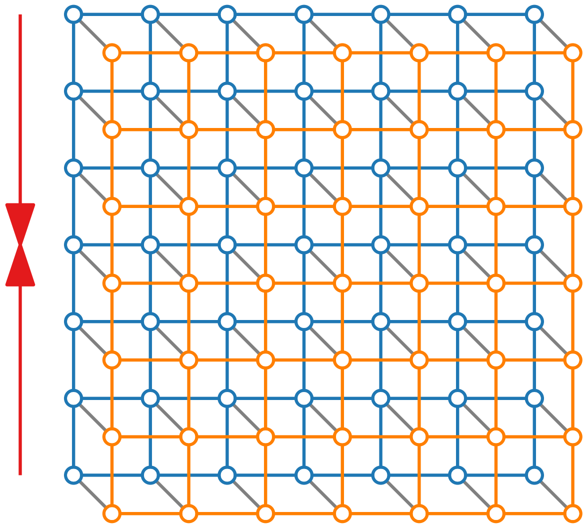

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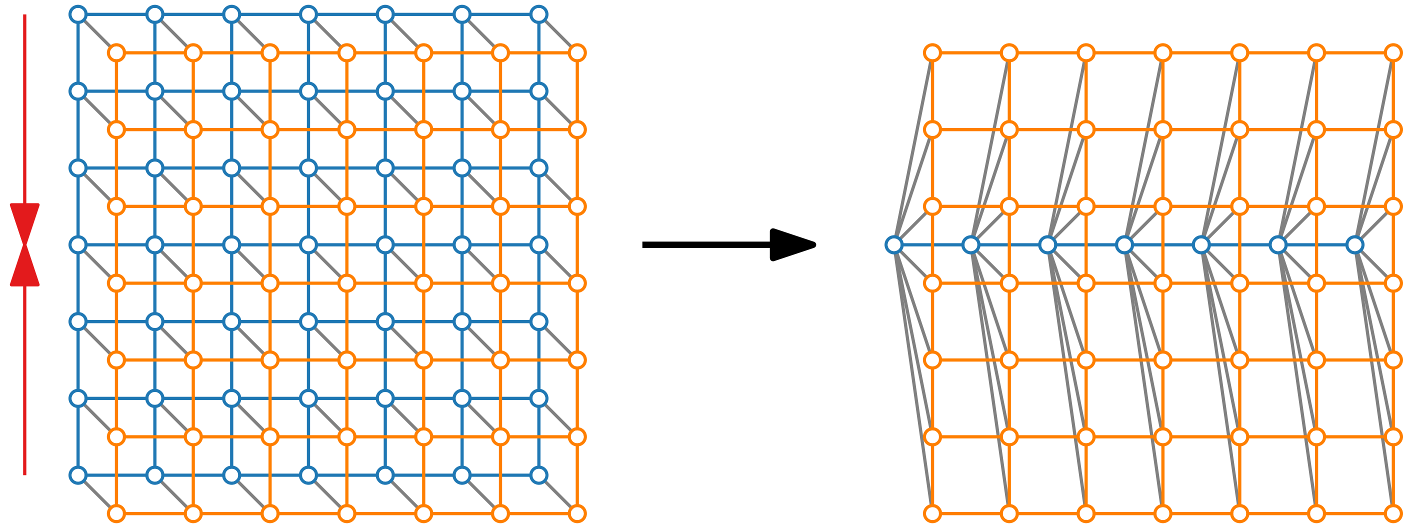

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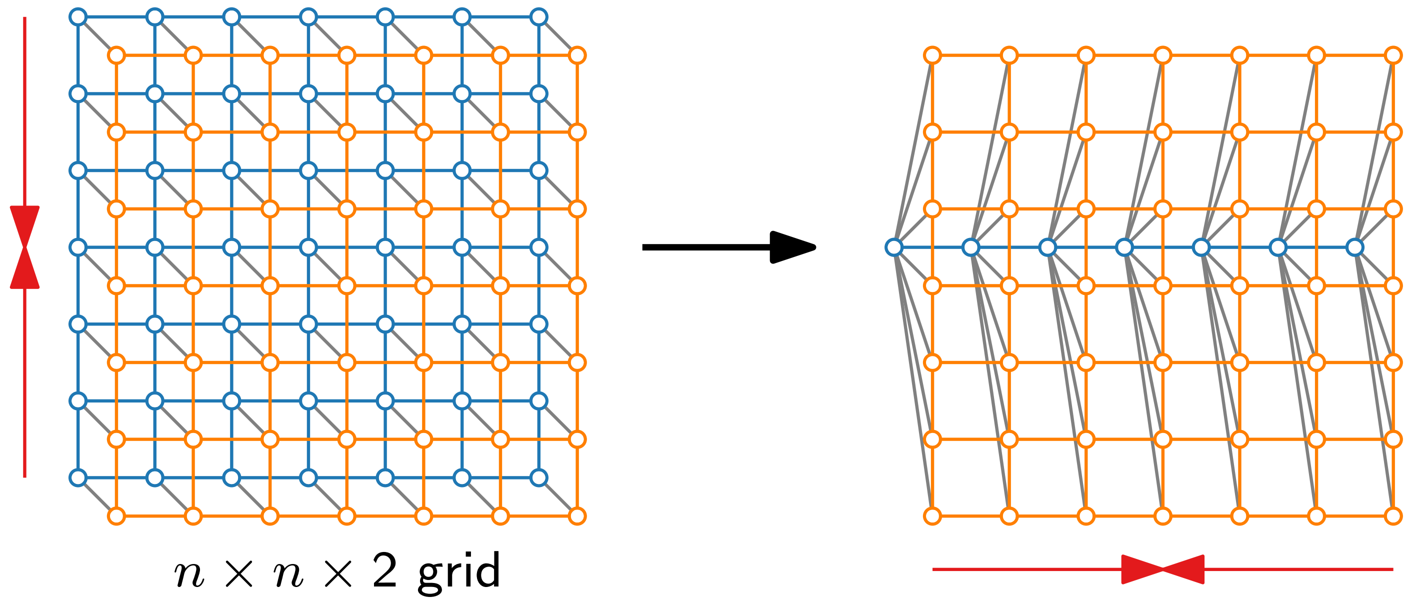

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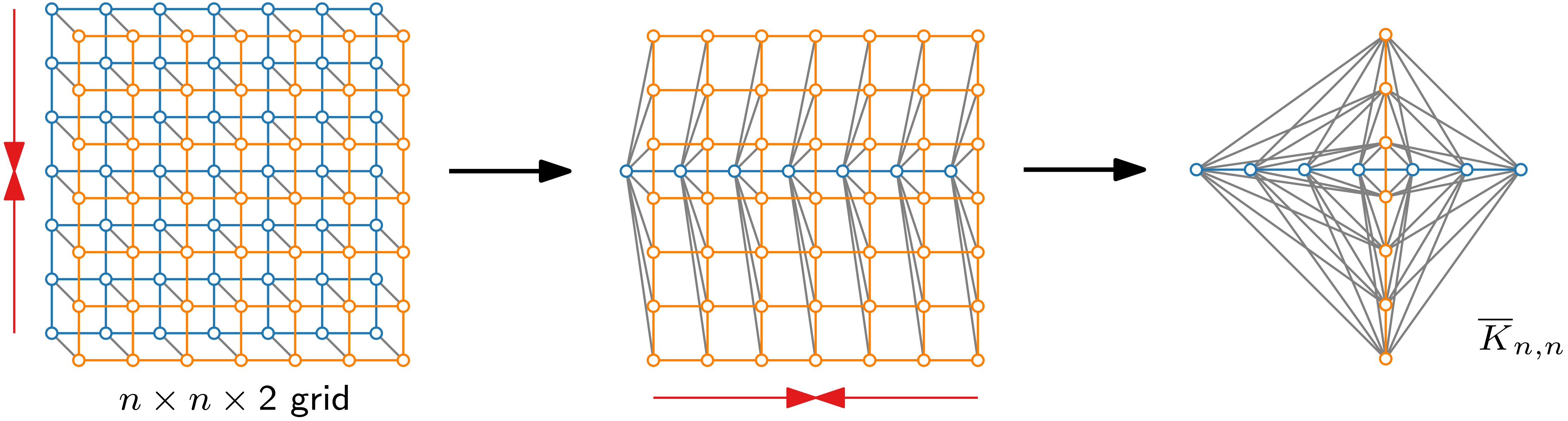

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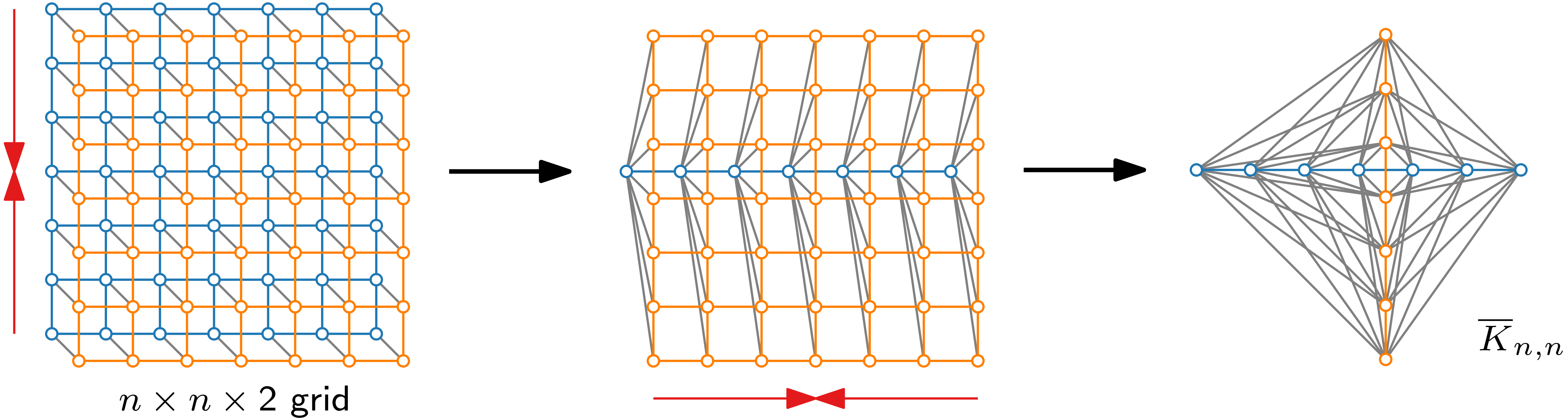


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$G$ planar $\Leftrightarrow$ neither $K_5$ nor $K_{3,3}$ minor of $G$

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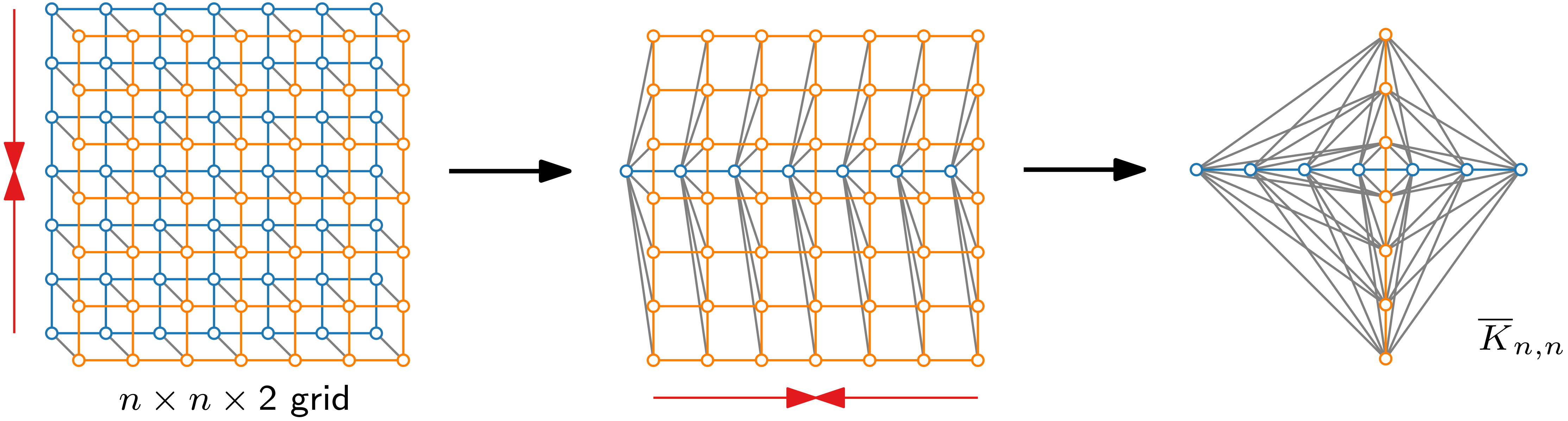


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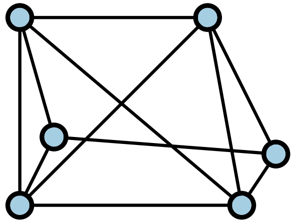

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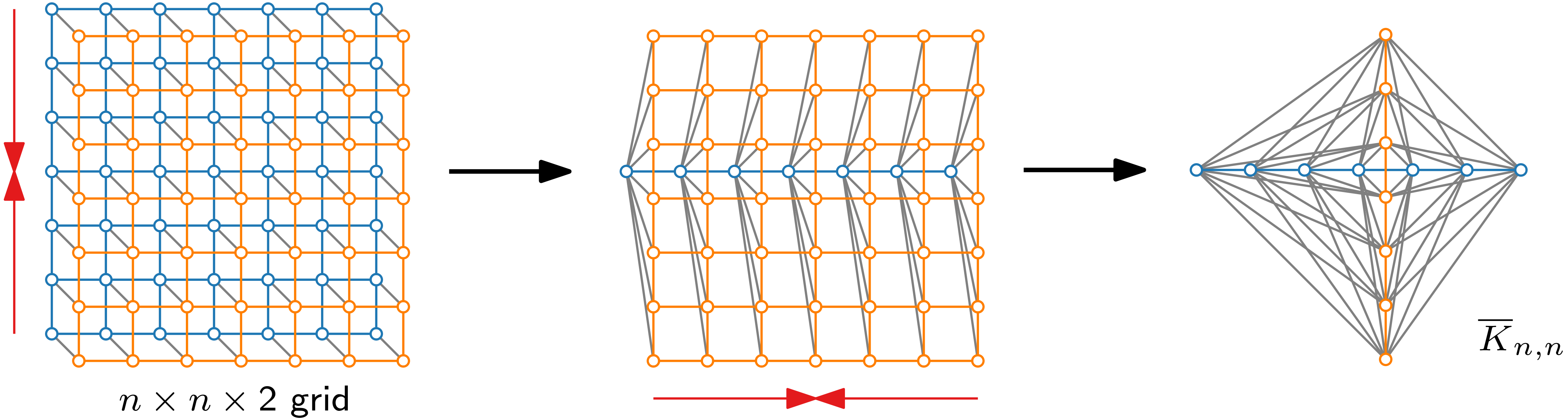


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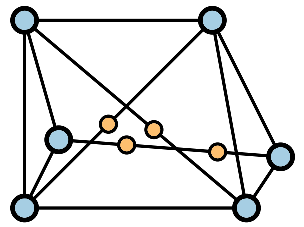

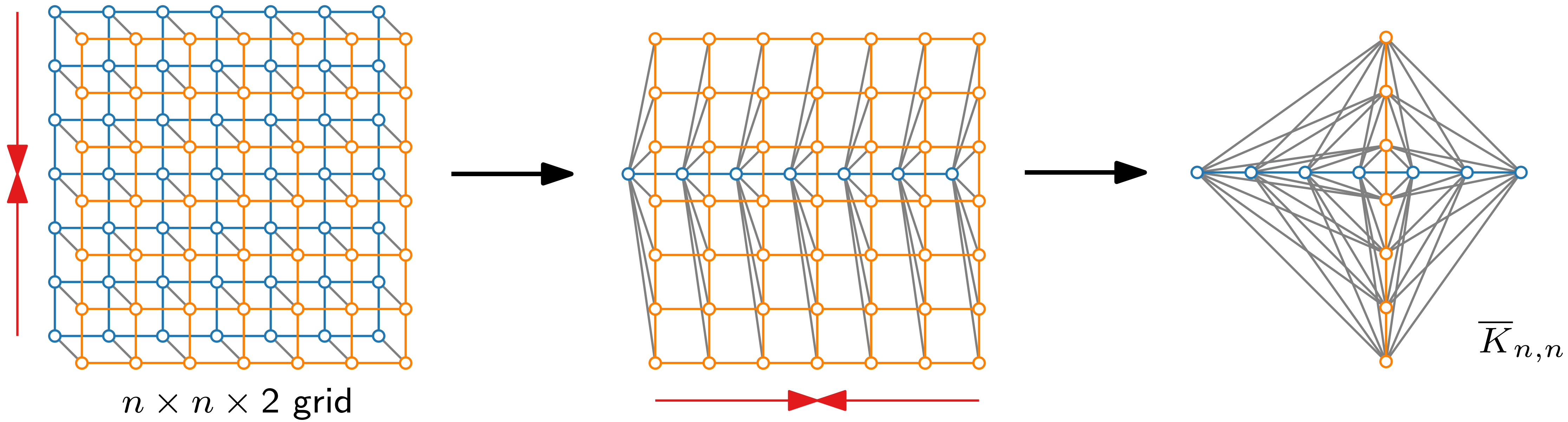


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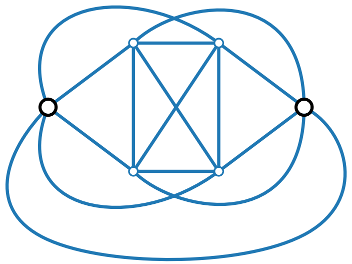

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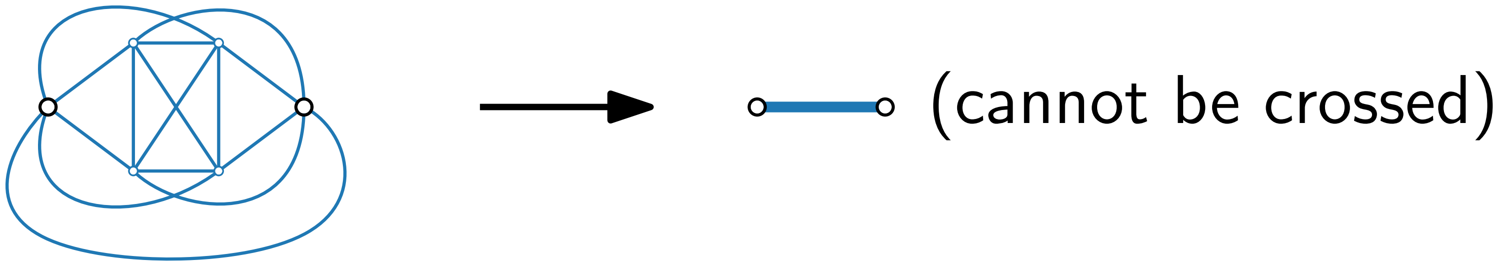


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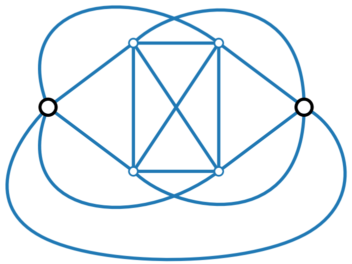

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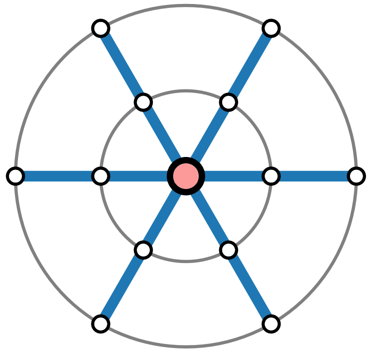

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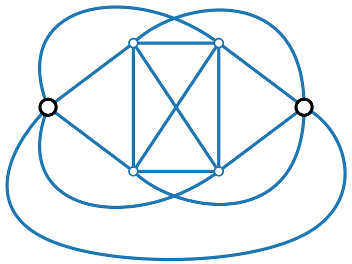

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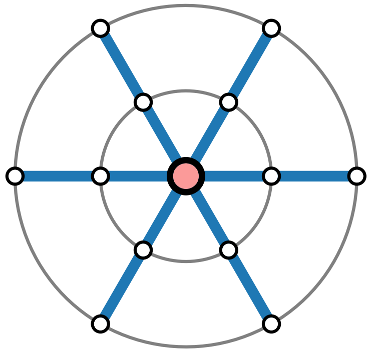

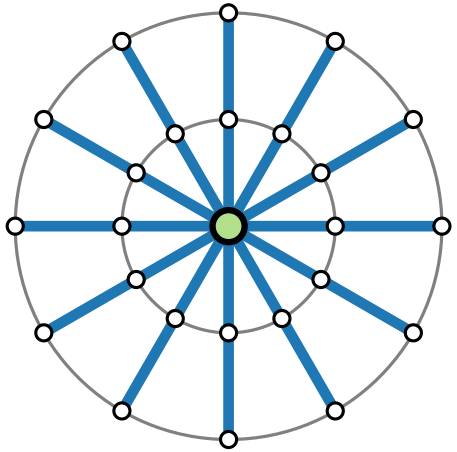

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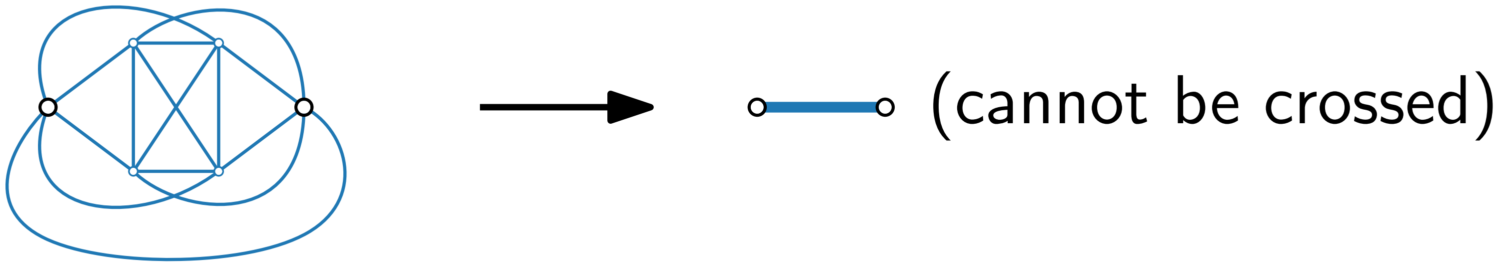


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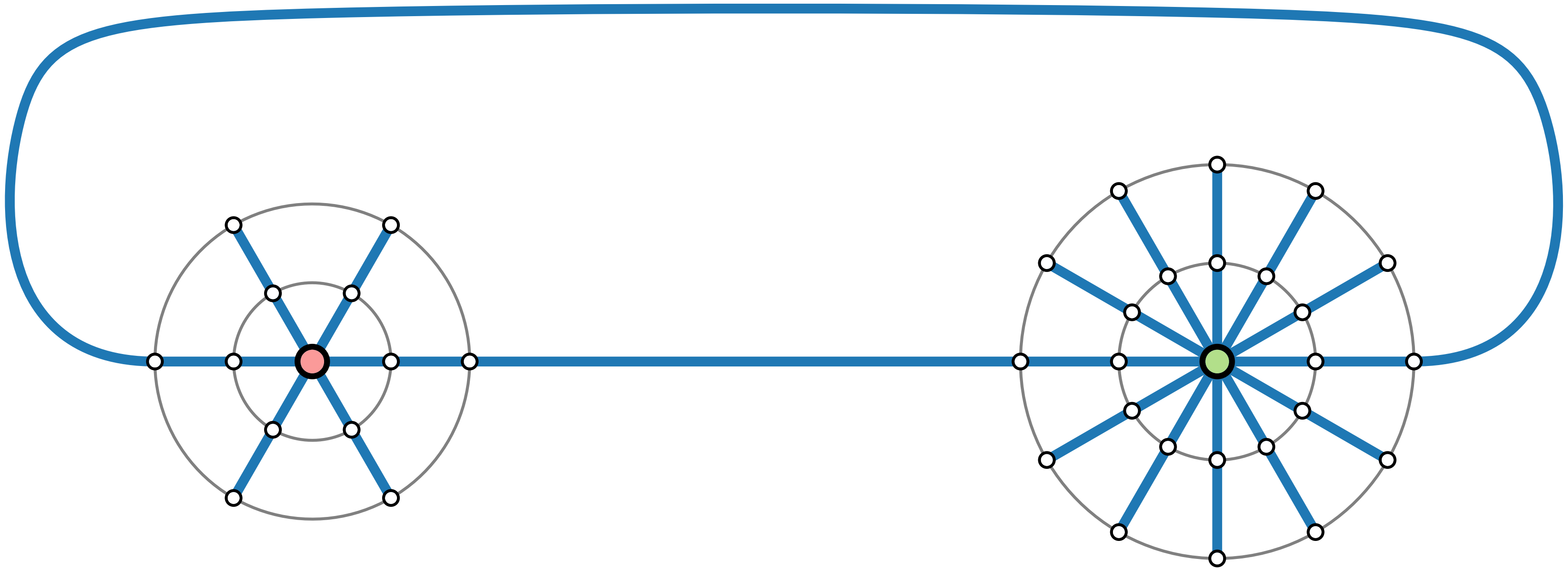

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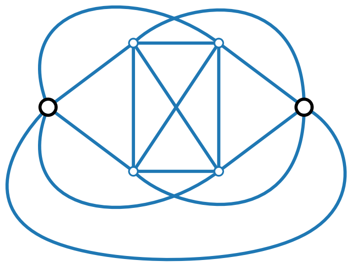

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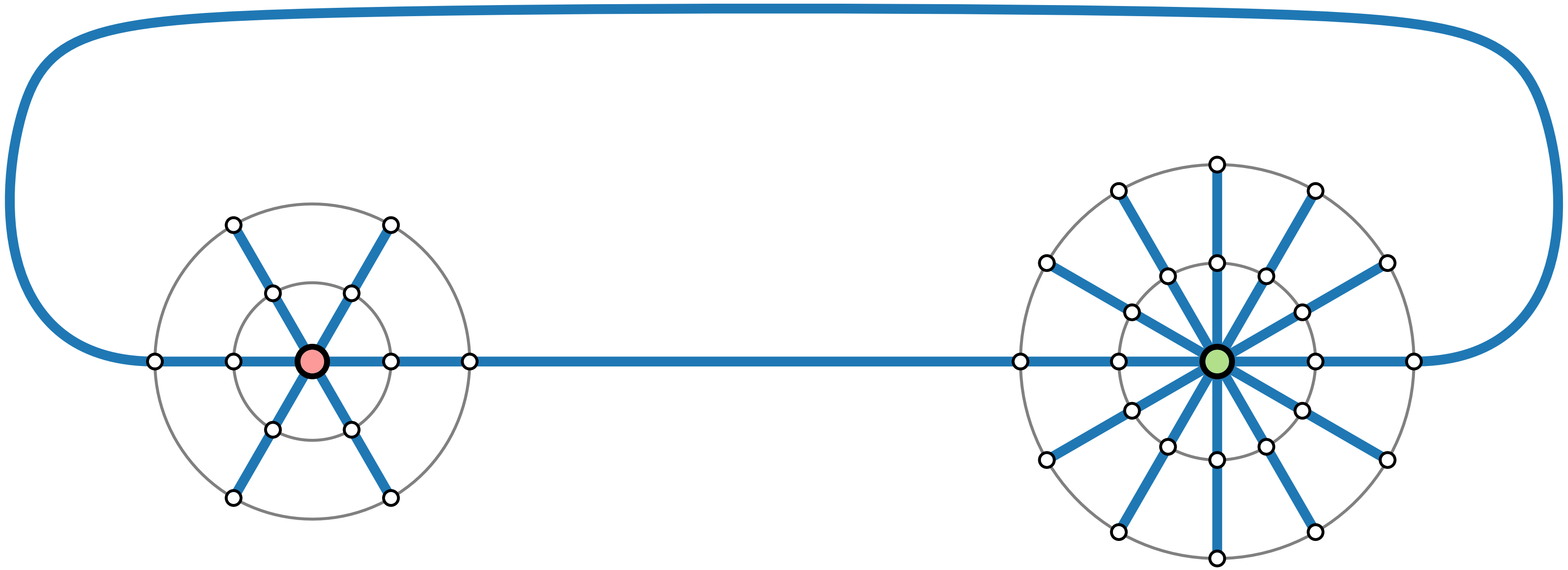

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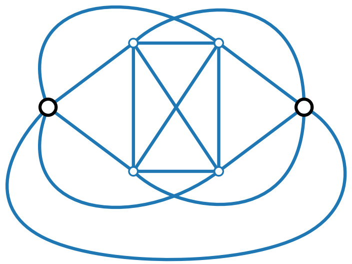

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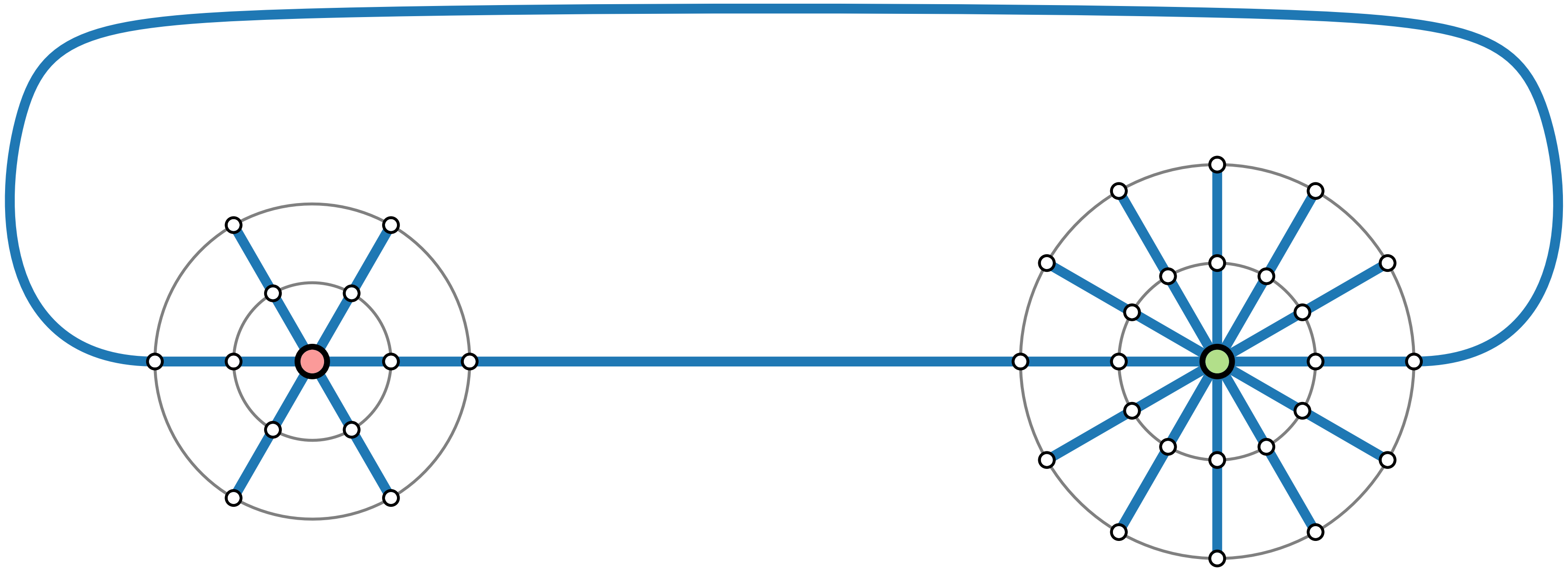

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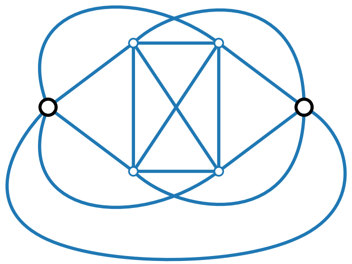

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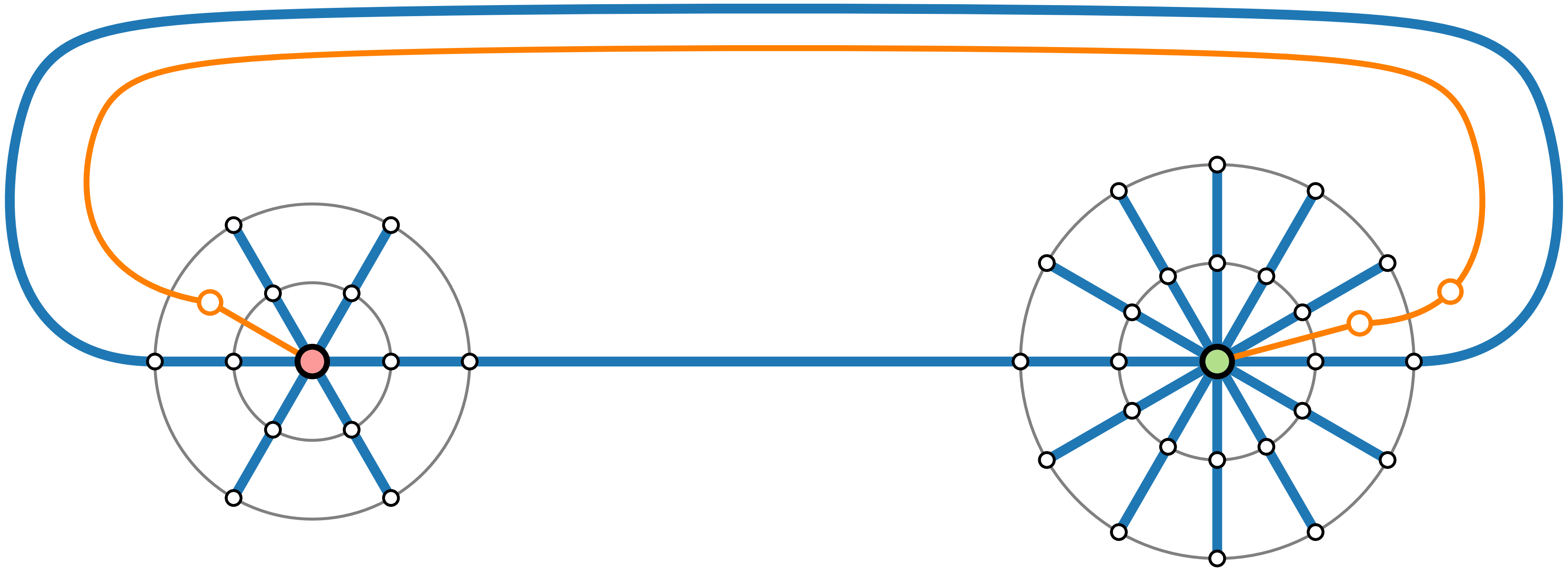

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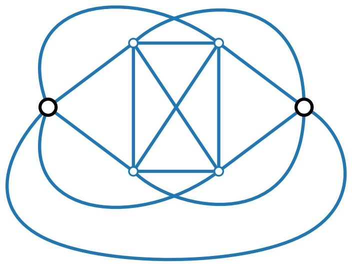

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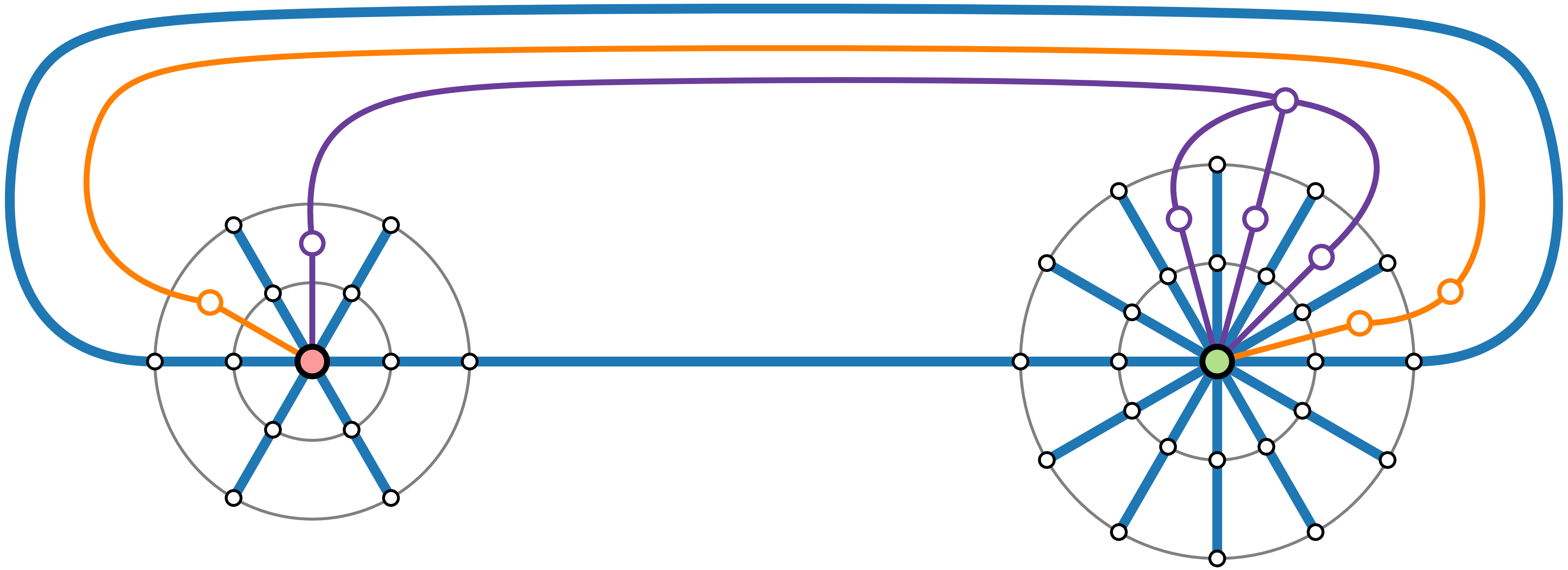

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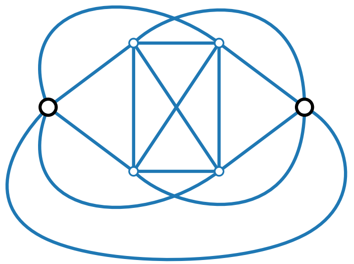

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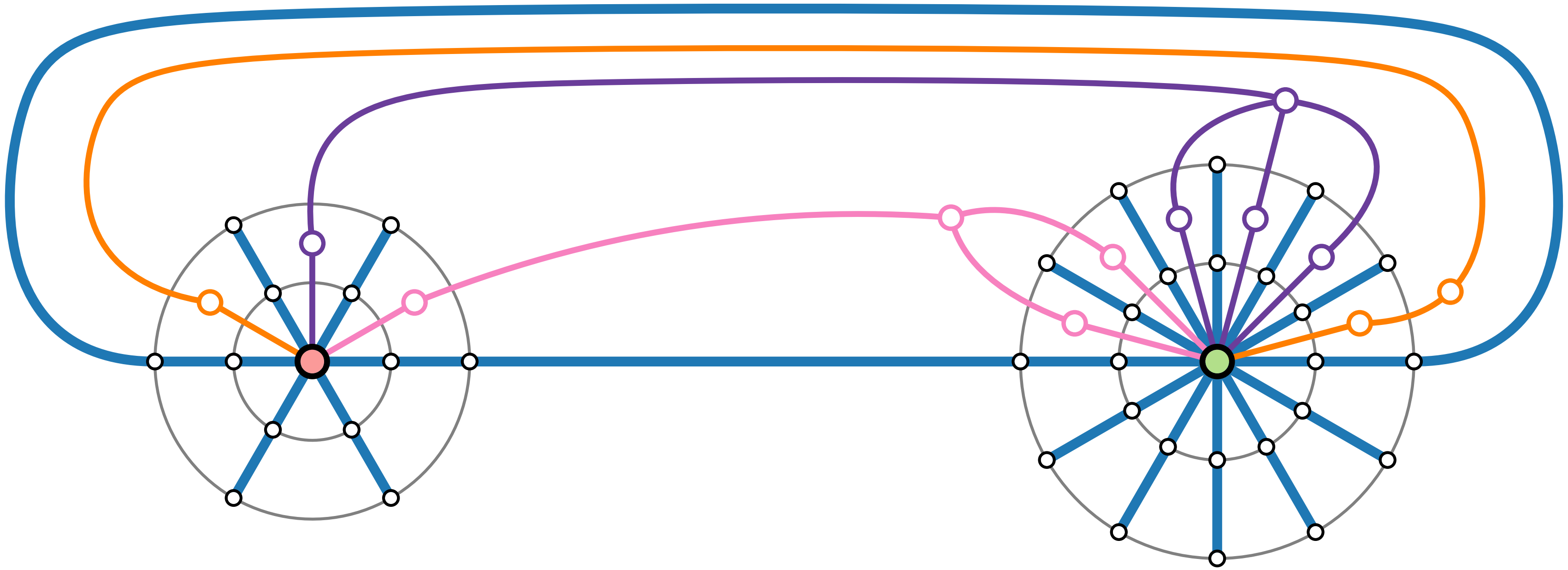

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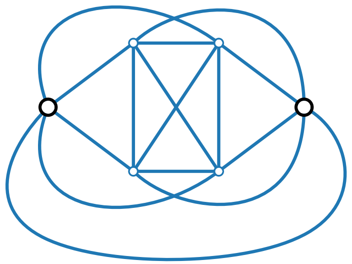

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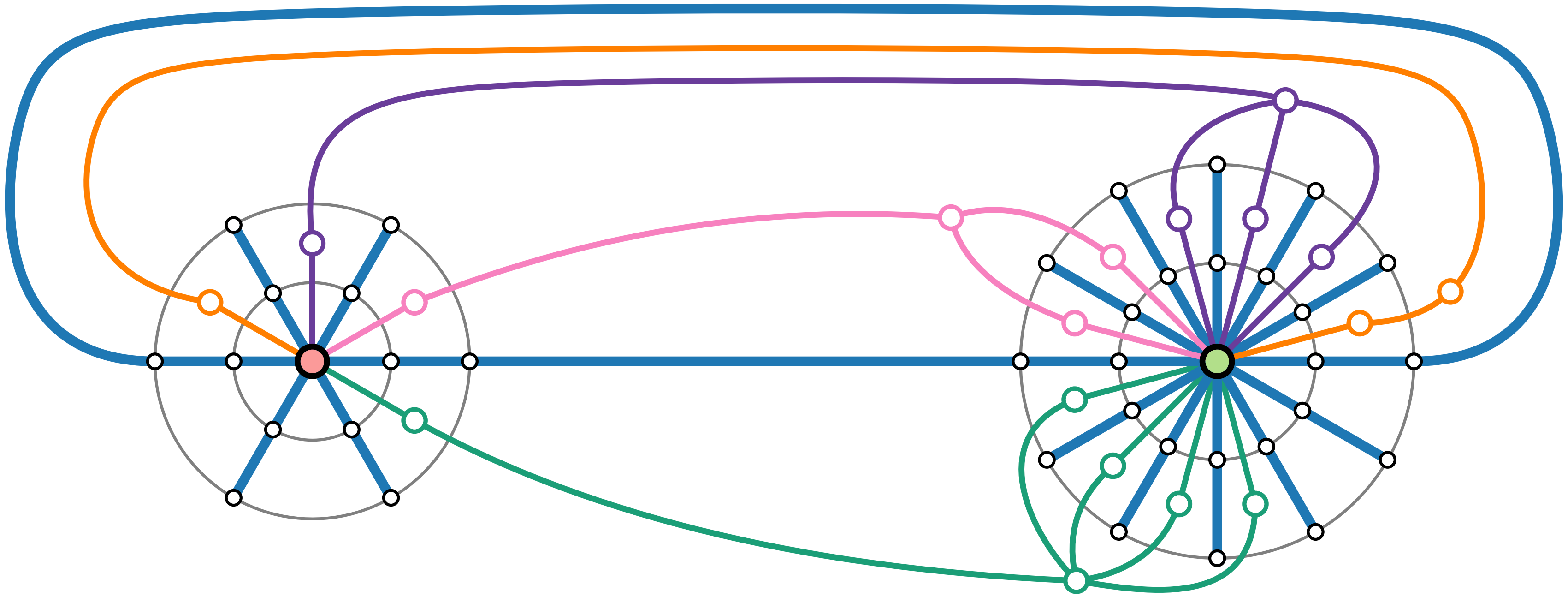

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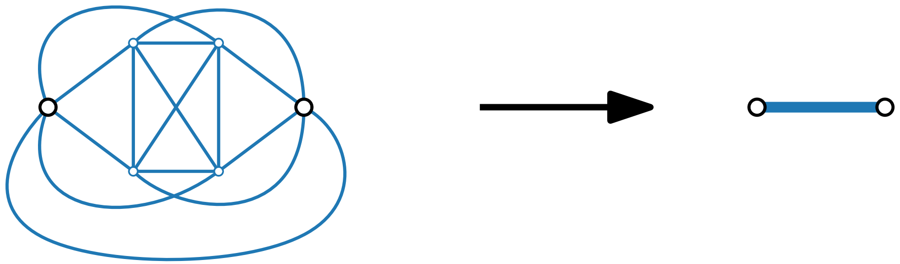

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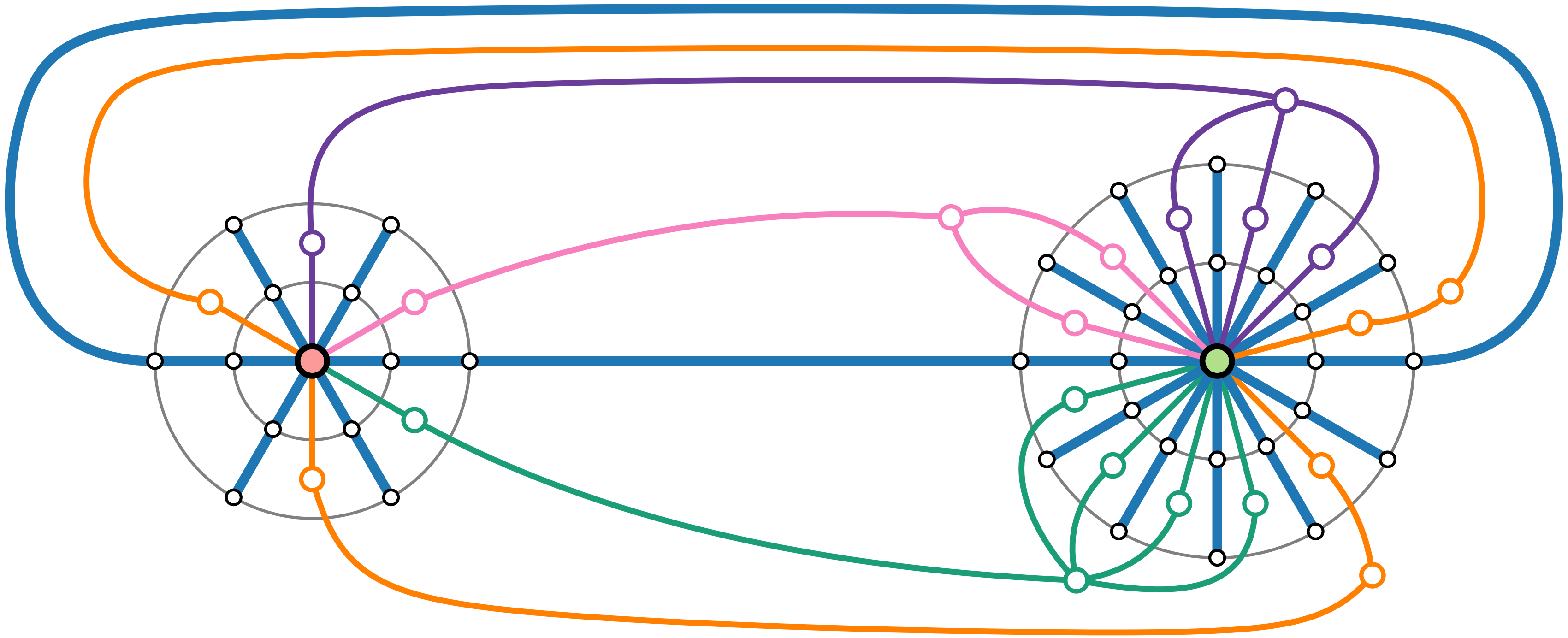

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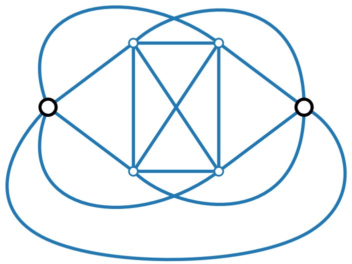

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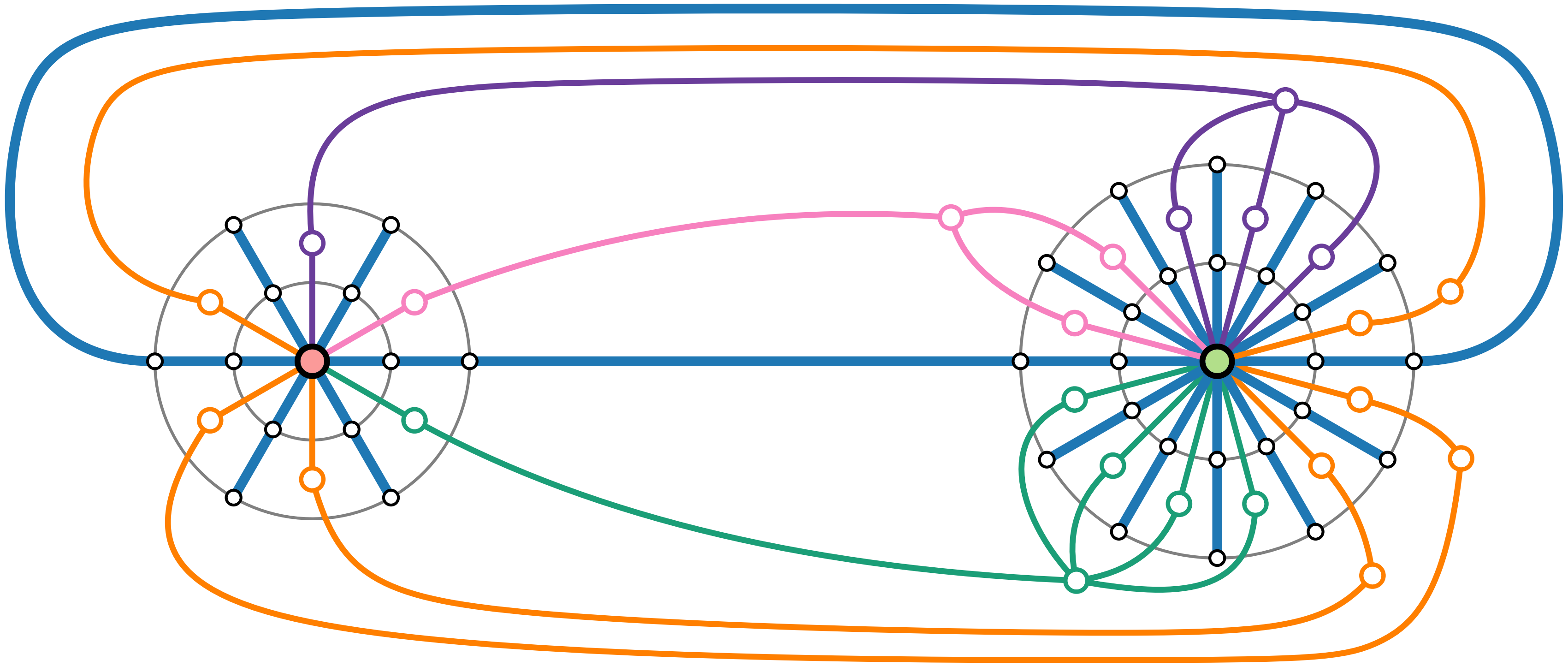

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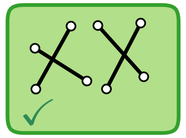

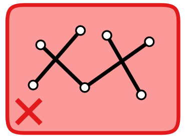

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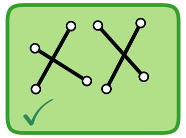

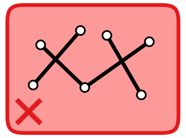

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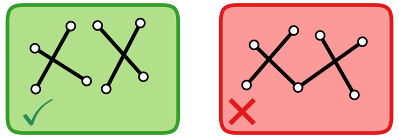

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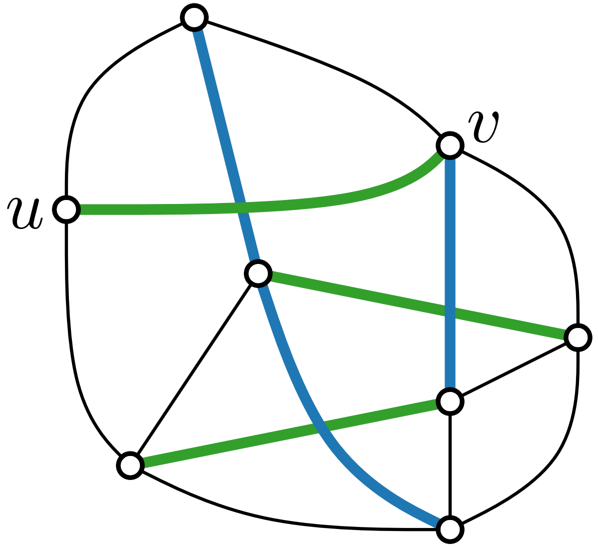

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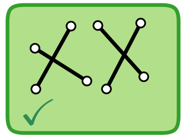

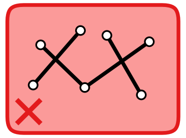

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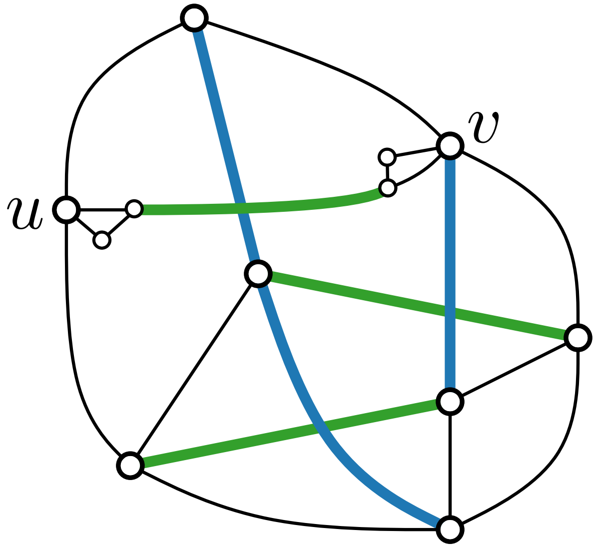

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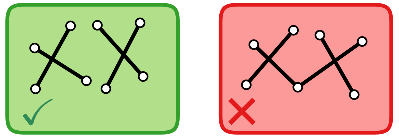

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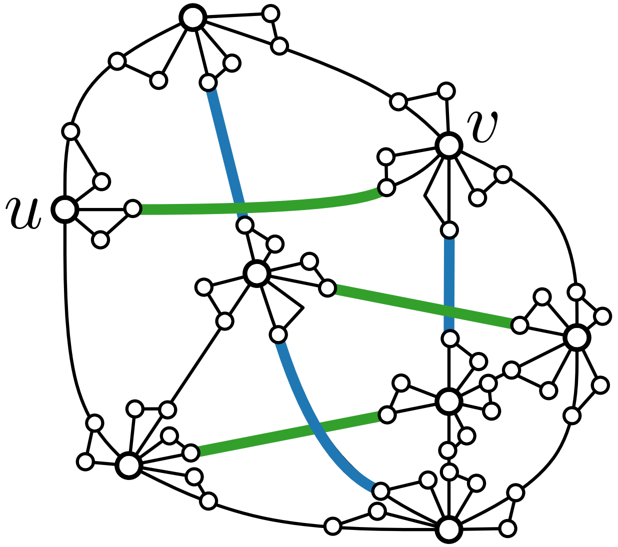

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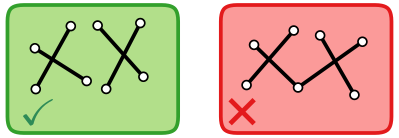

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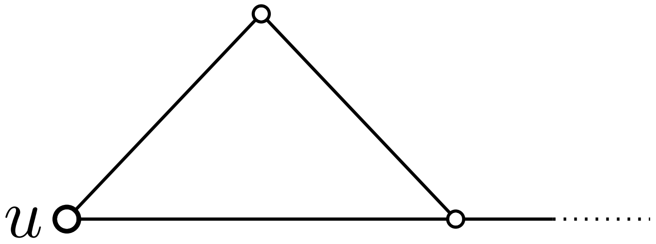


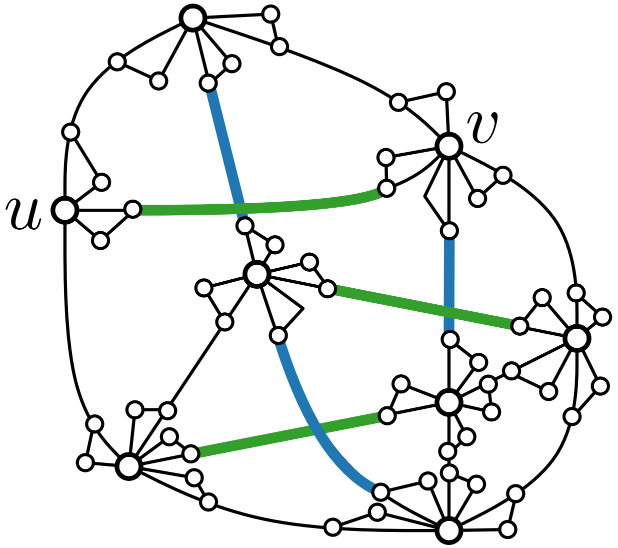

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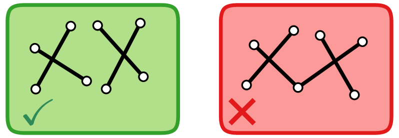

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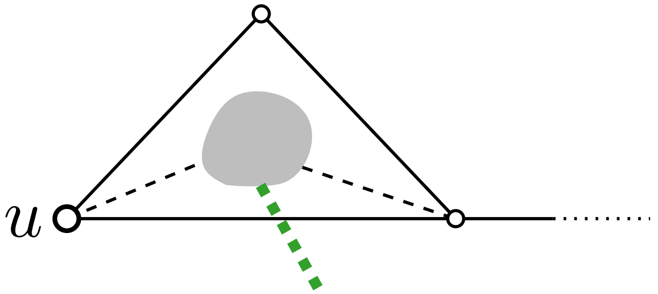


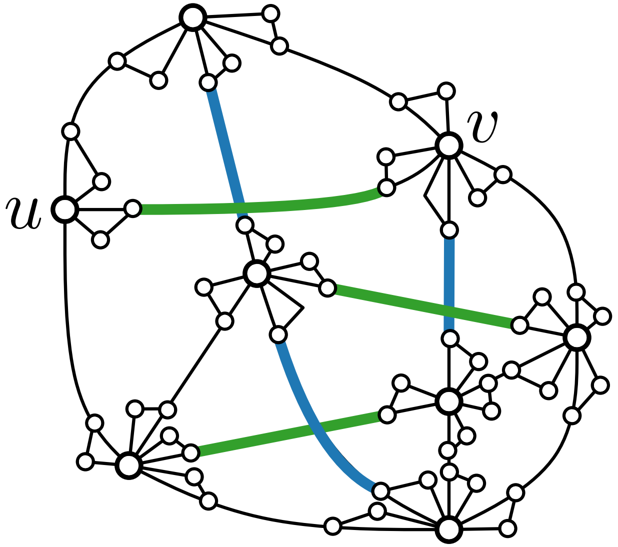

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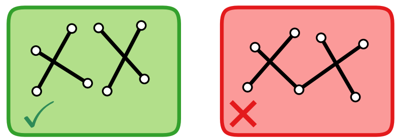

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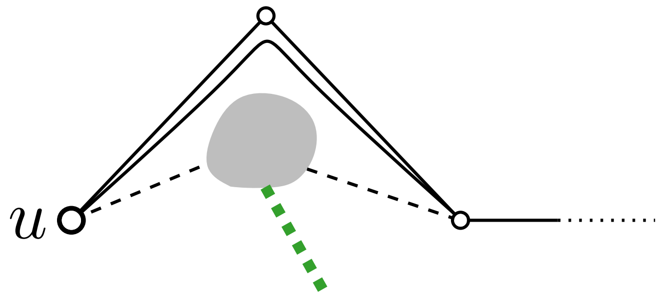


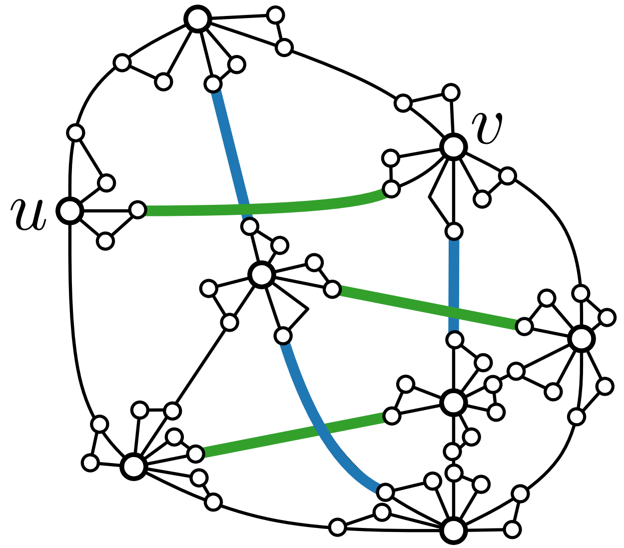

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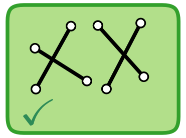
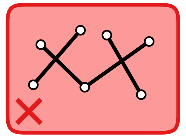
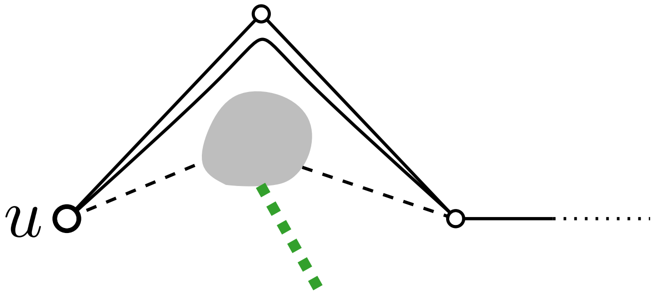
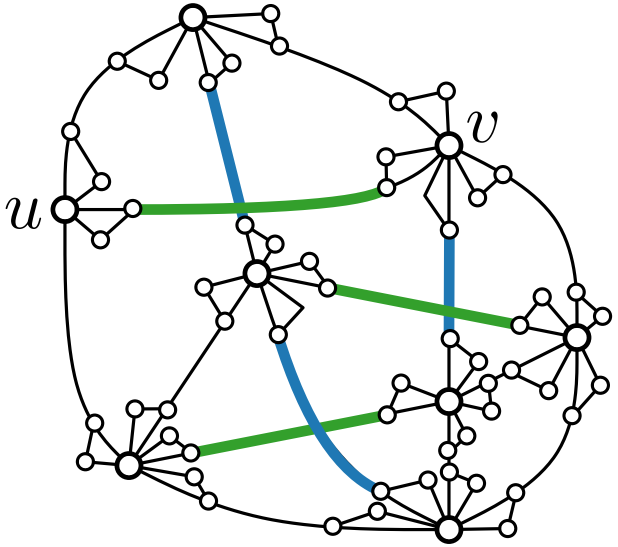




**Theorem.** [Brandenburg, Didimo, Evans, Kindermann, Liotta & Montecchiani 2015]
Testing IC-planarity is NP-complete.

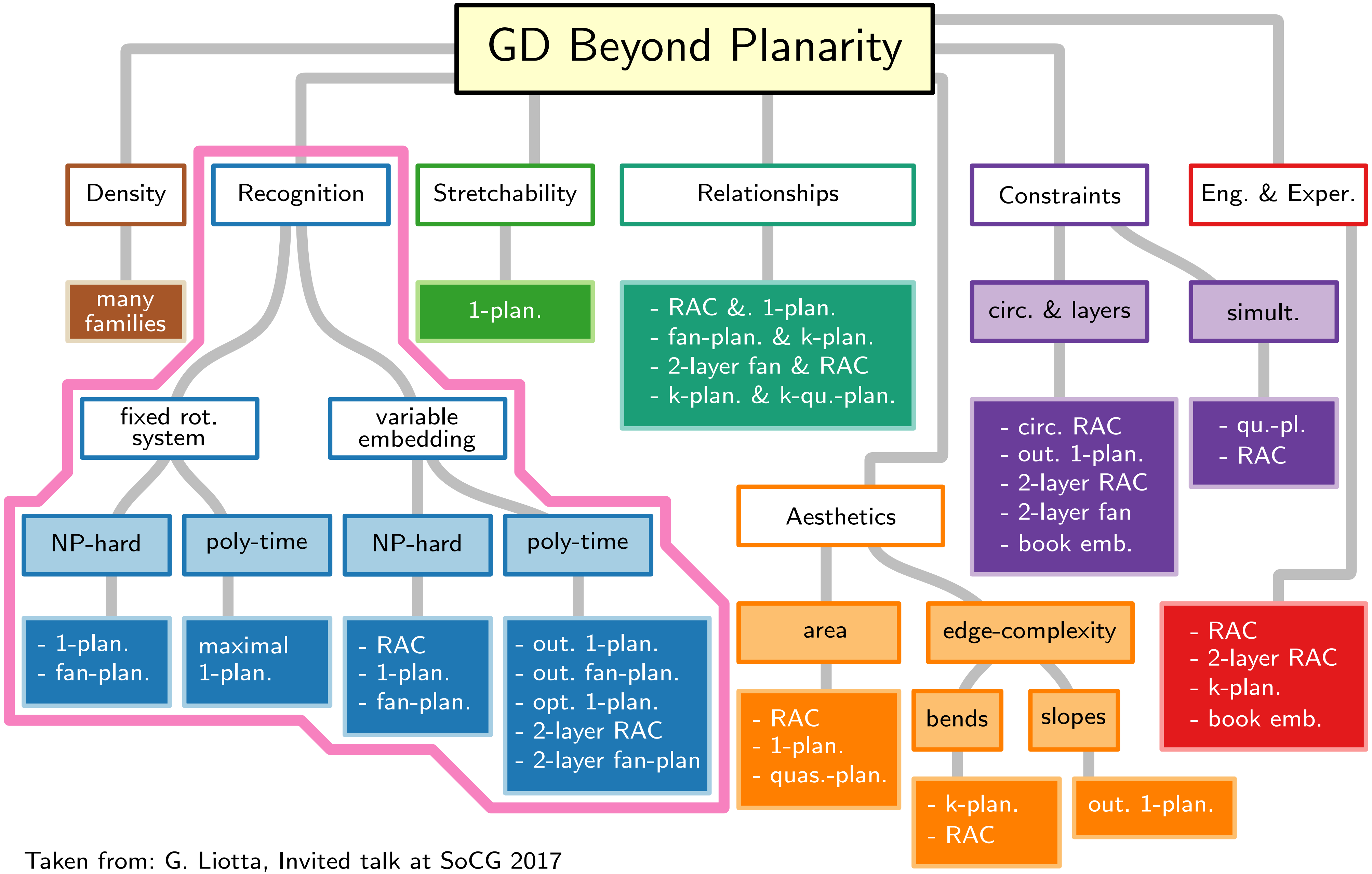
**Proof.**
Reduction from 1-planarity testing.



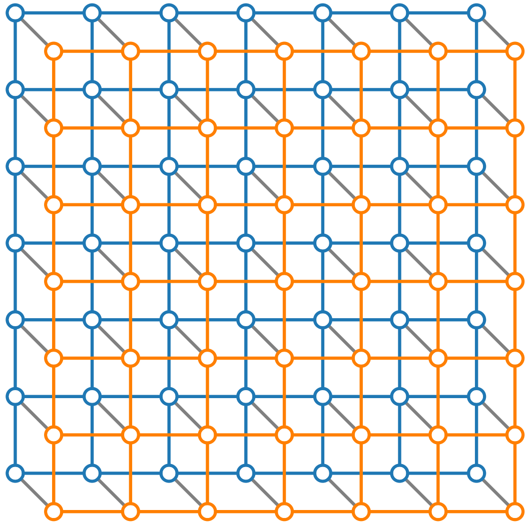




**Theorem.** [Brandenburg, Didimo, Evans, Kindermann, Liotta & Montecchiani 2015]
Testing IC-planarity is NP-complete,
even if the rotation system is given.

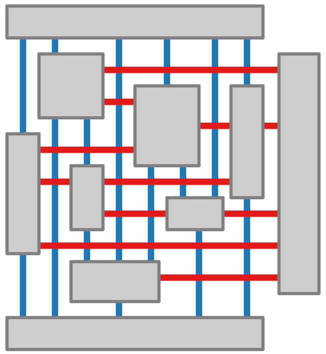
# GD Beyond Planarity: a Taxonomy



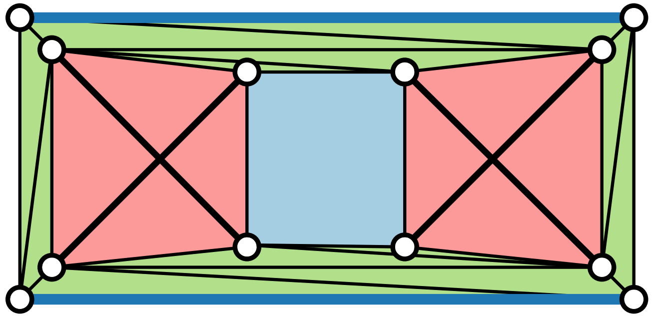

Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

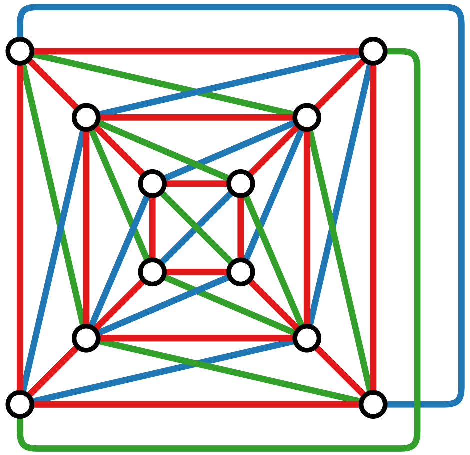


# Visualization of Graphs



## Lecture 11:
## Beyond Planarity
### Drawing Graphs with Crossings

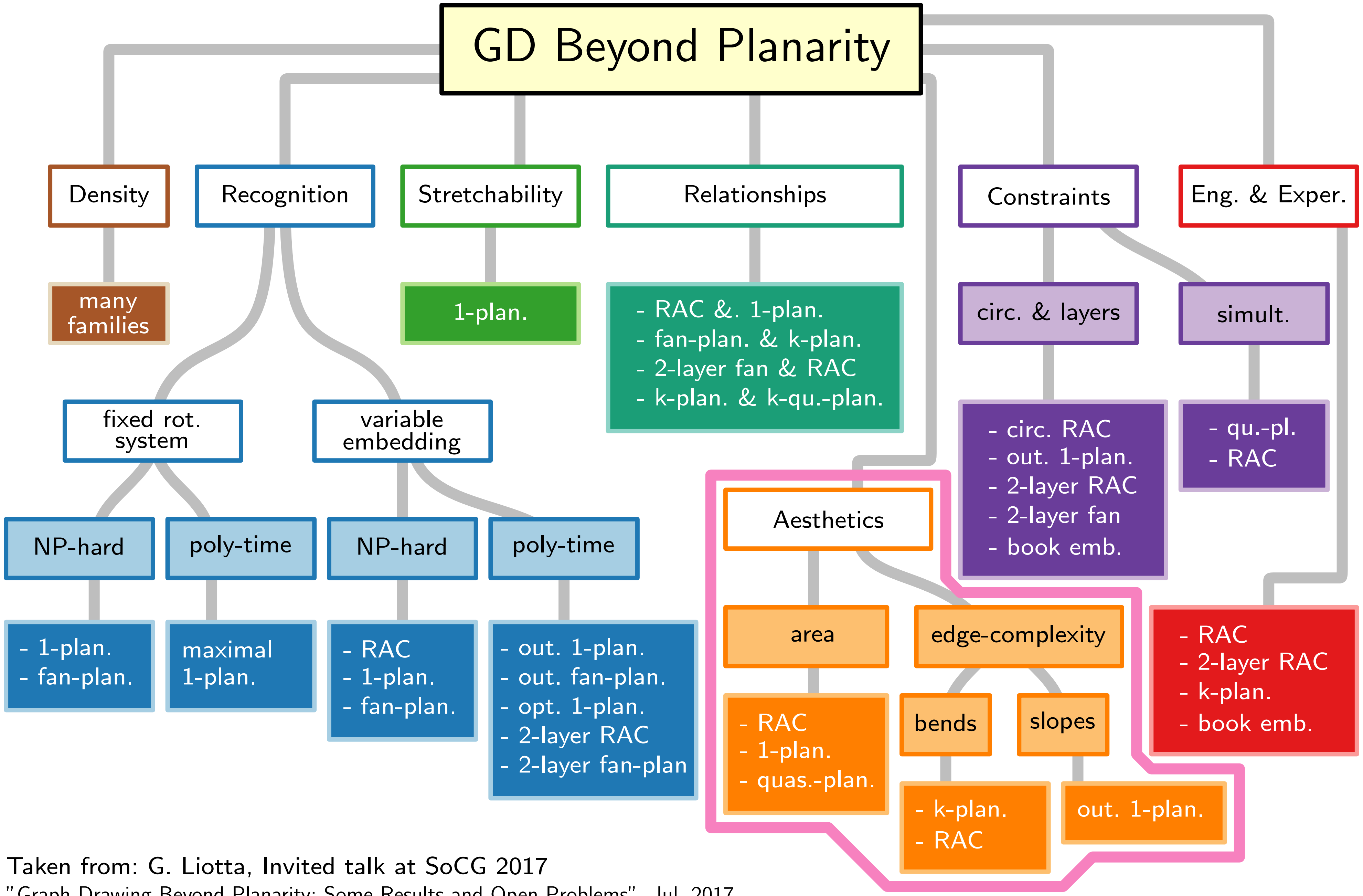




## Part IV:
## RAC Drawings

Jonathan Klawitter

# GD Beyond Planarity: a Taxonomy

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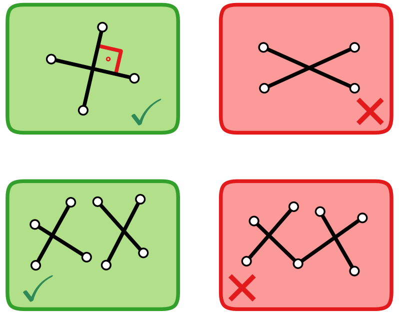
# Area of Straight-Line RAC Drawings



**Theorem.** [Brandenburg, Didimo, Evans, Kindermann, Liotta & Montecchiani 2015]
IC-planar straight-line RAC drawings may require exponential area.

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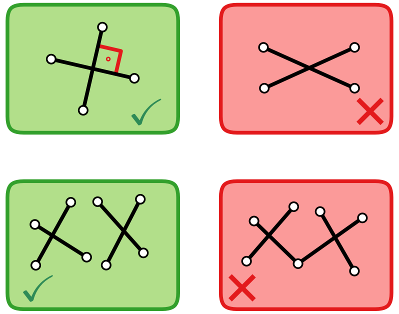

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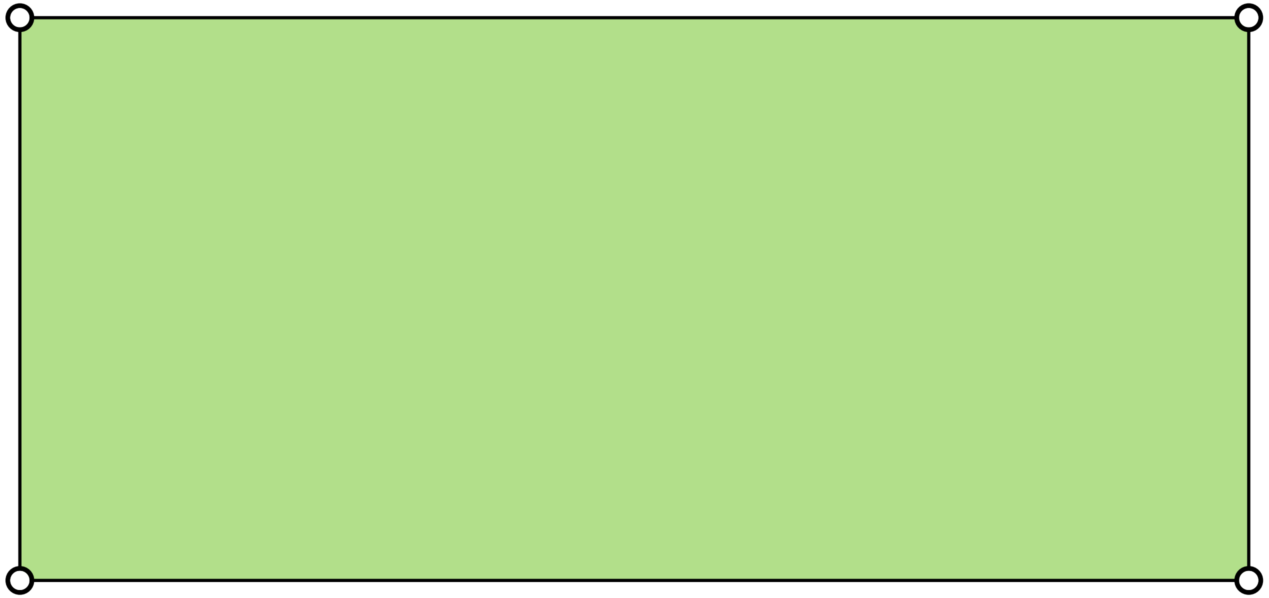

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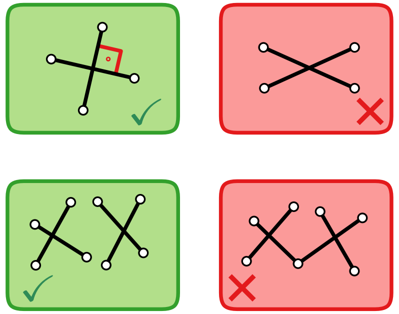

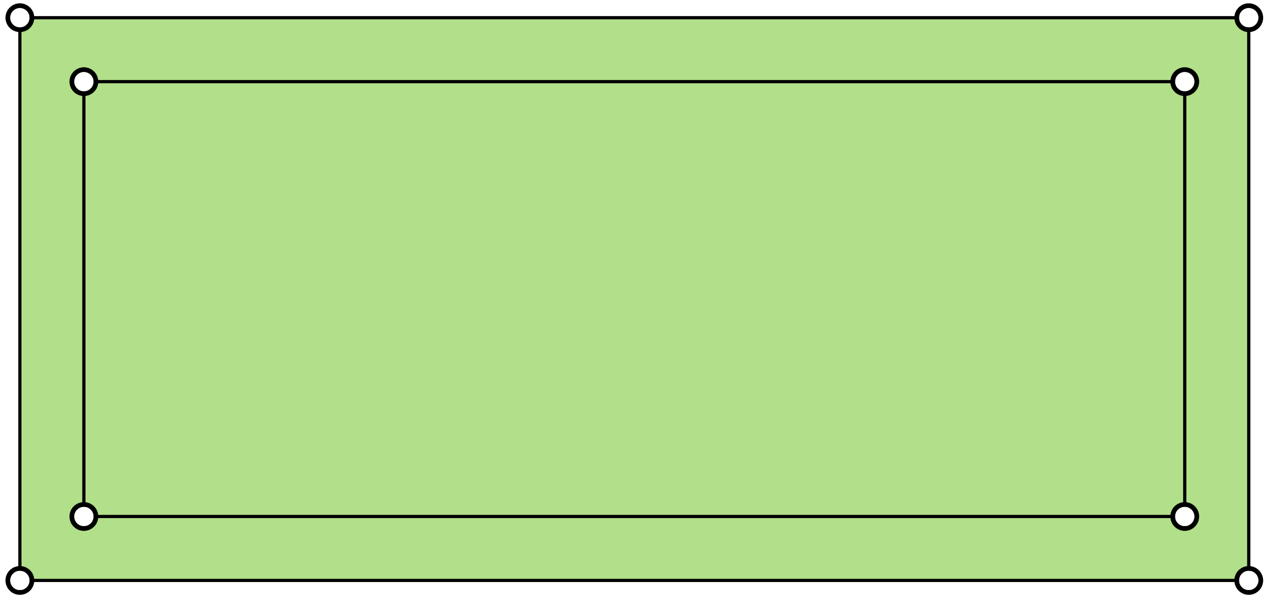

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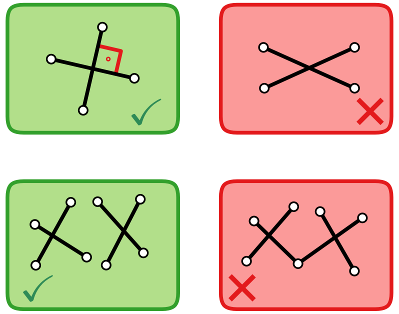

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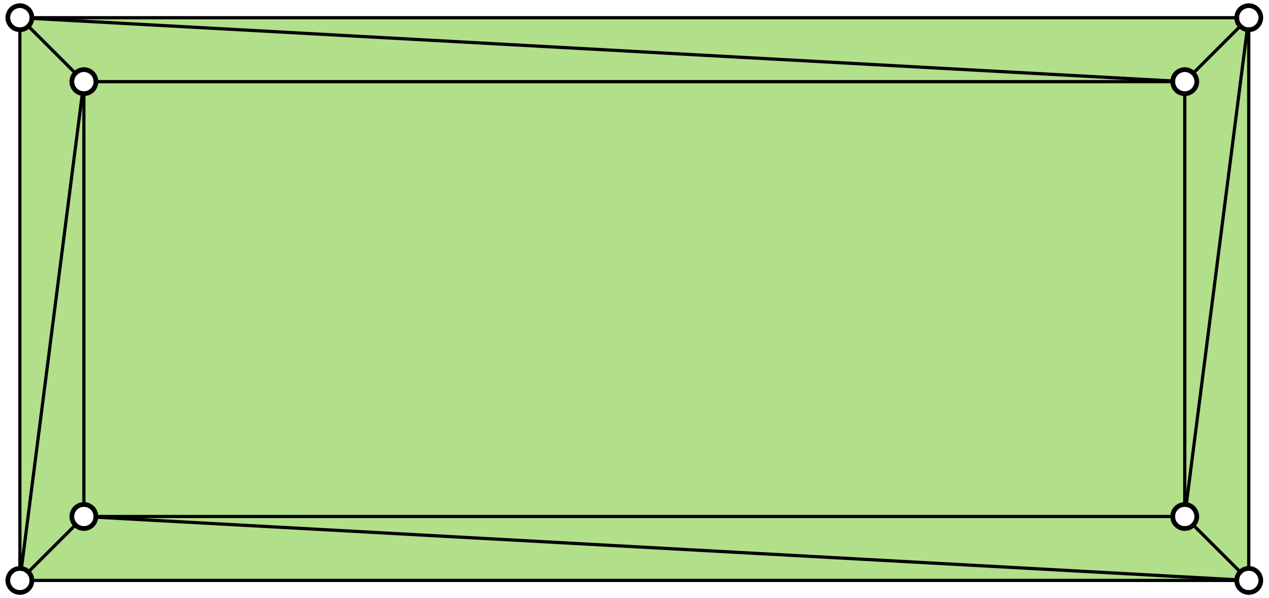

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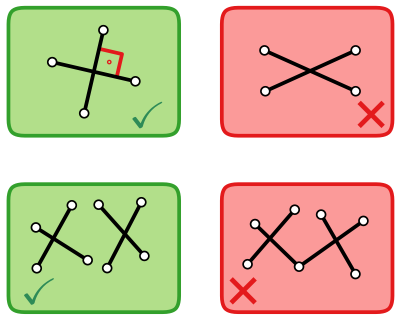

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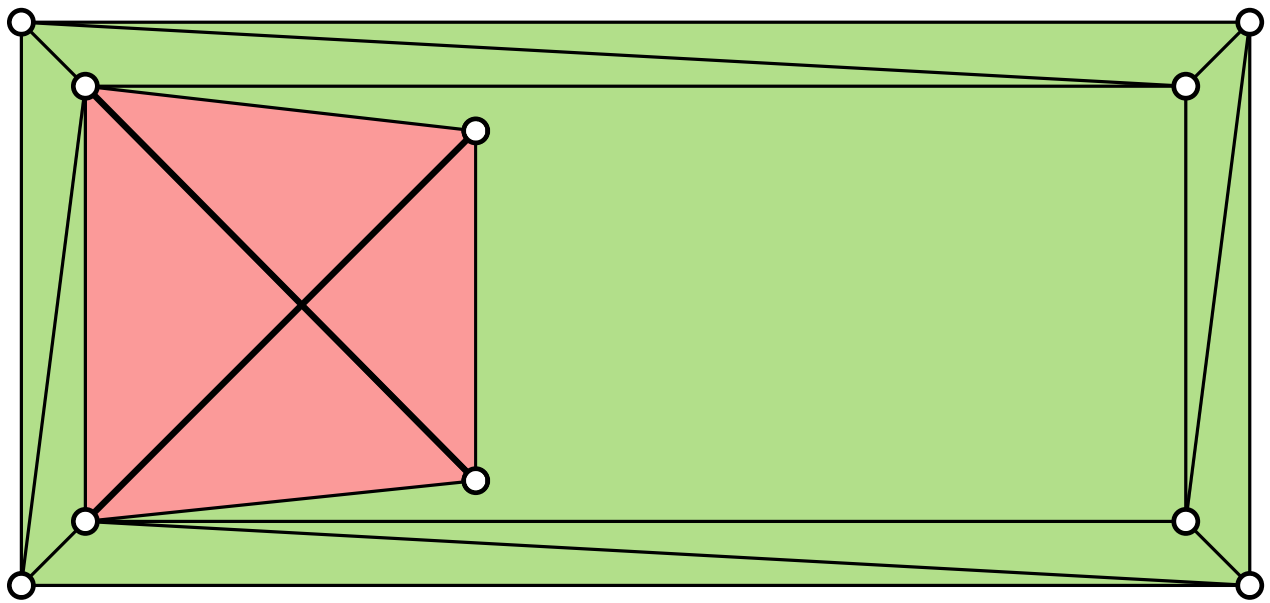

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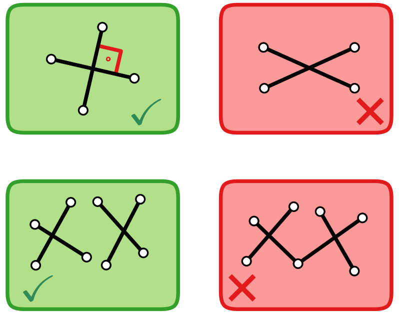

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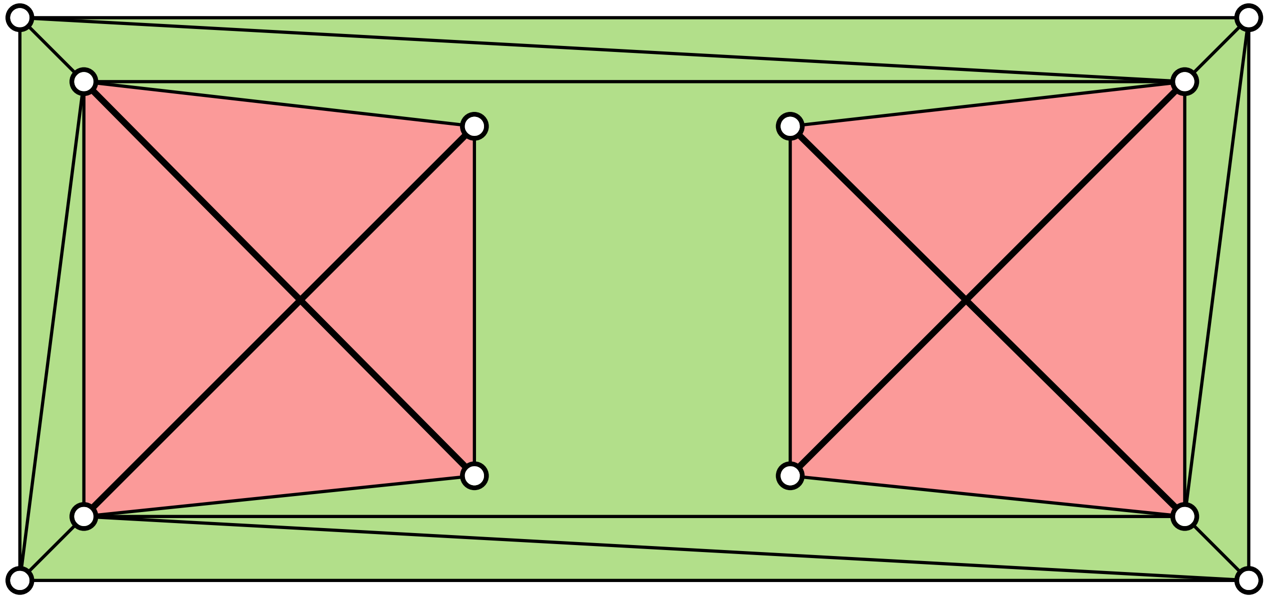

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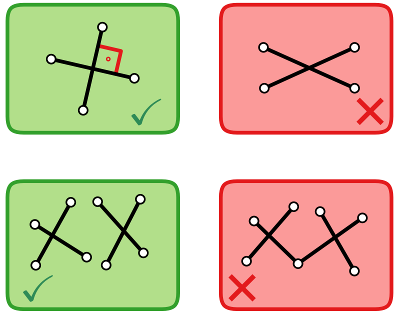

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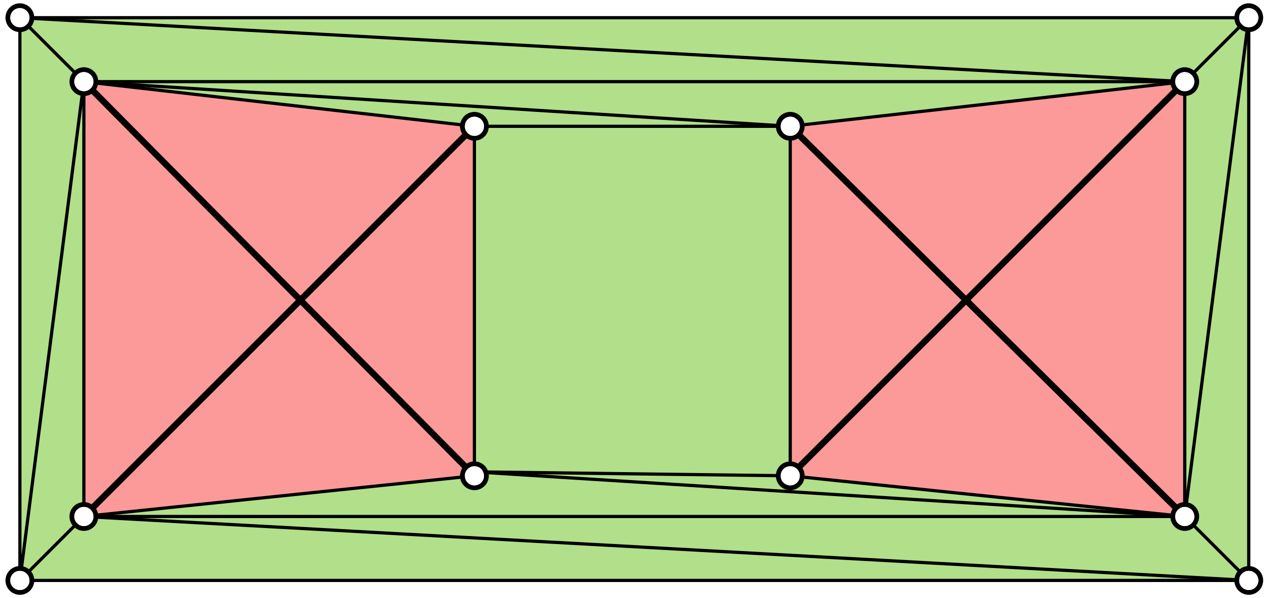

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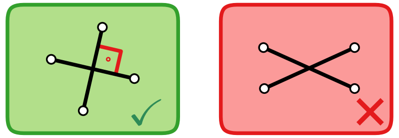

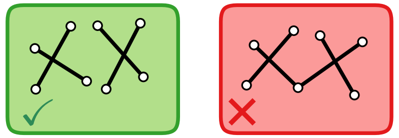

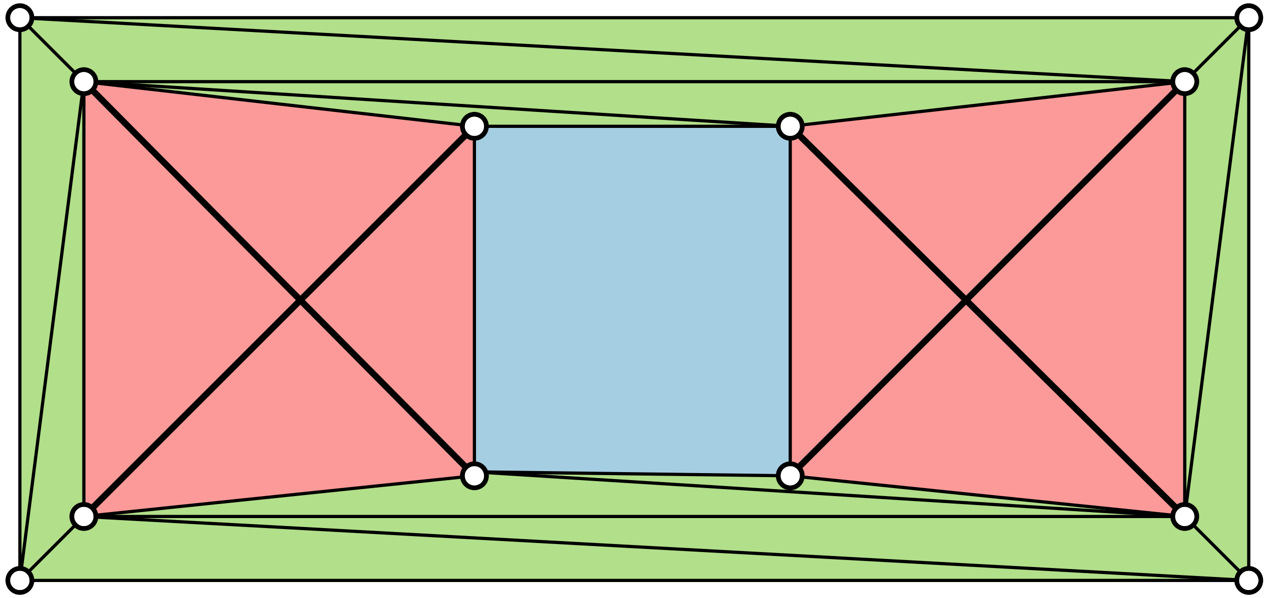

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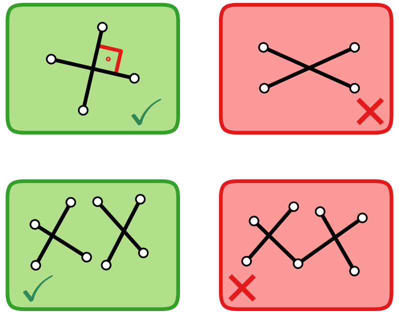

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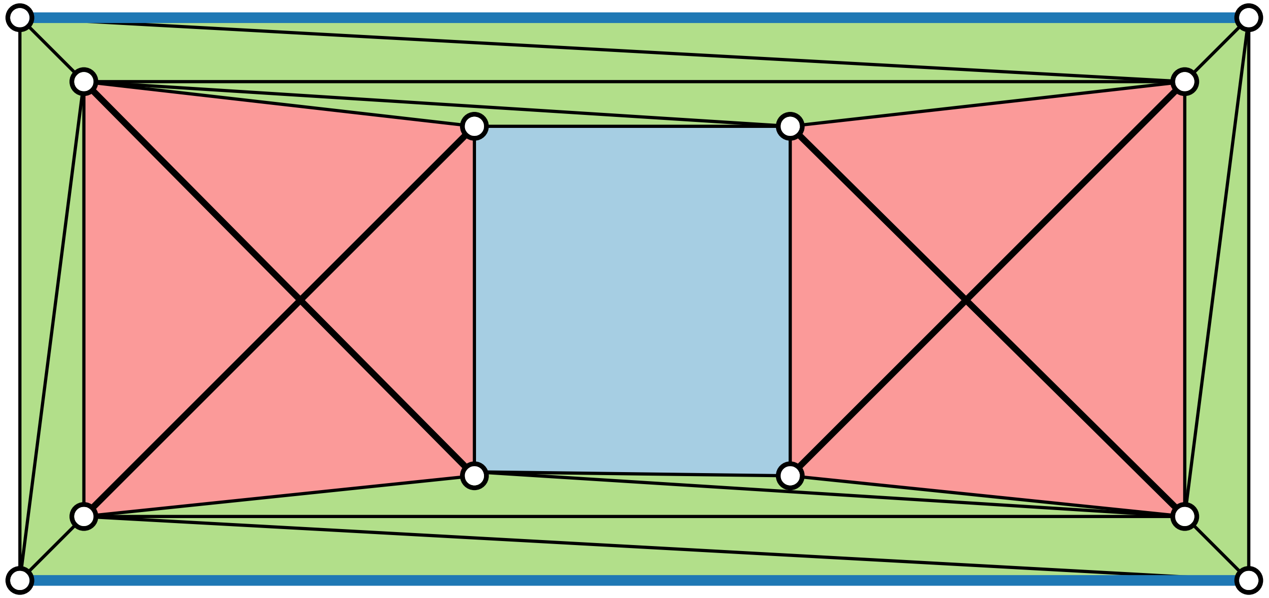

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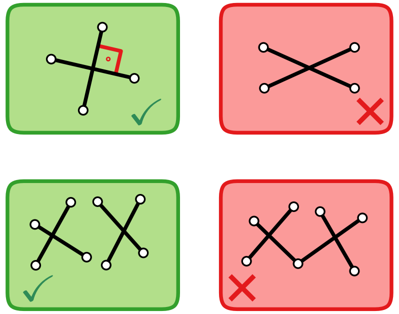

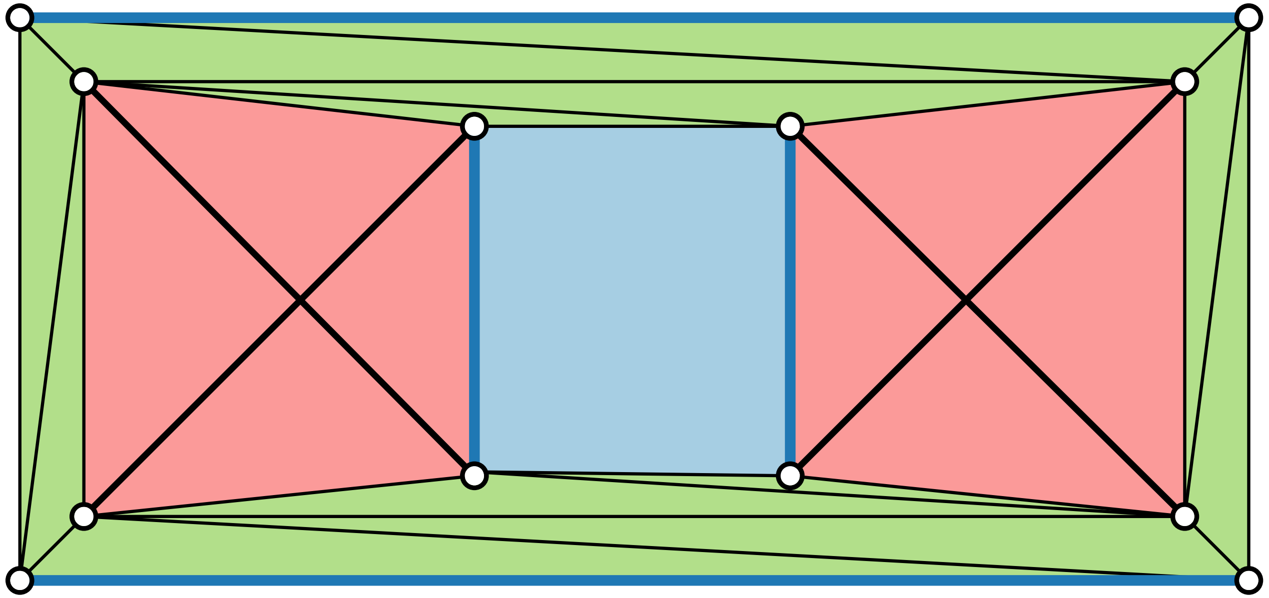

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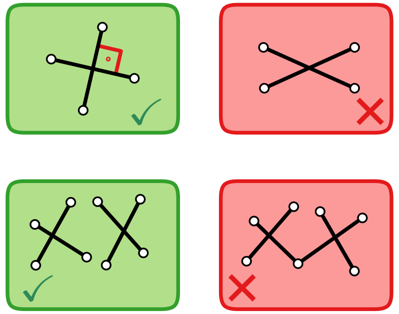

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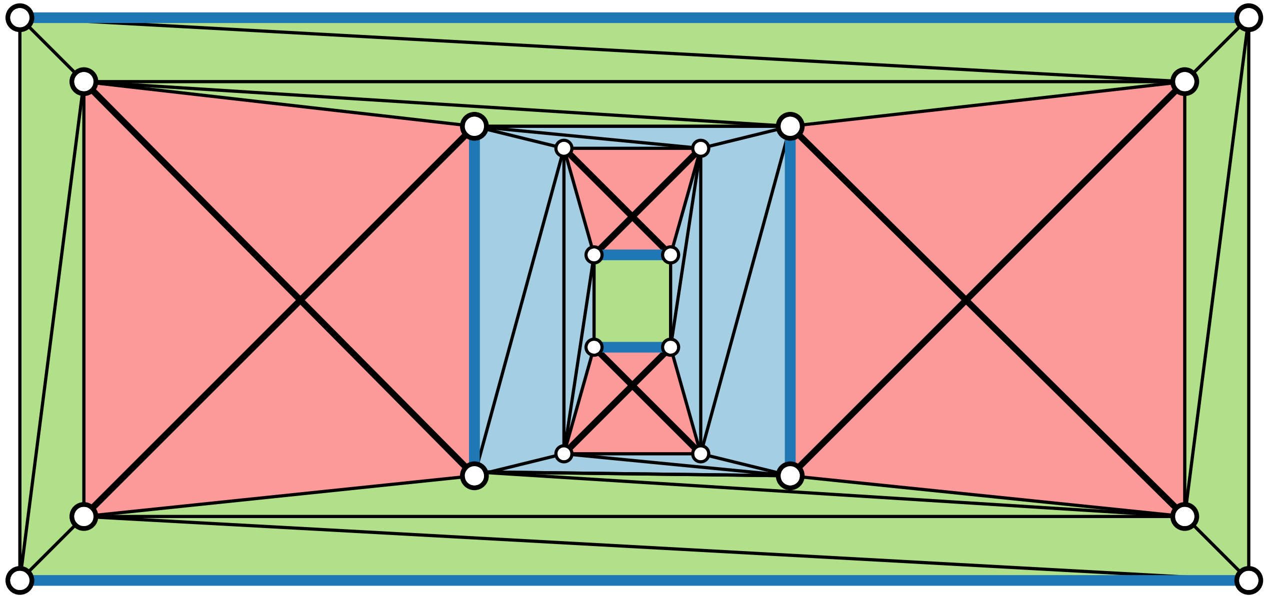

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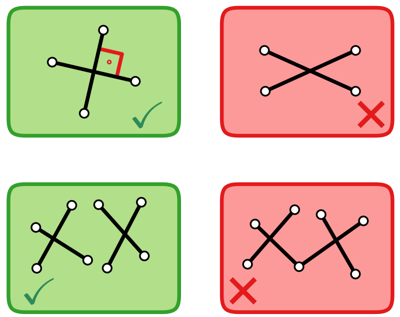

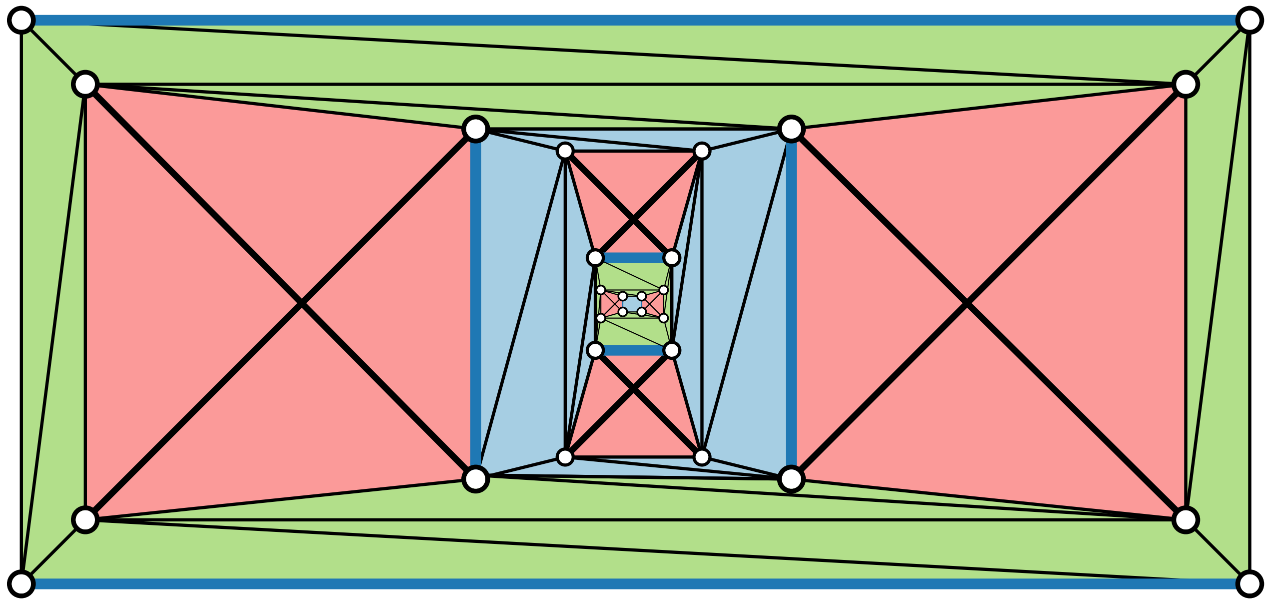

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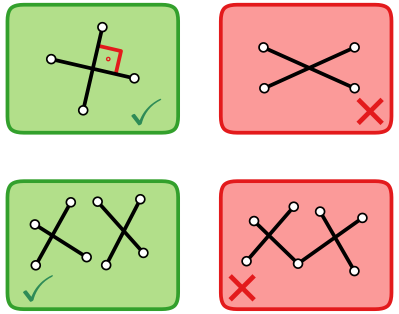

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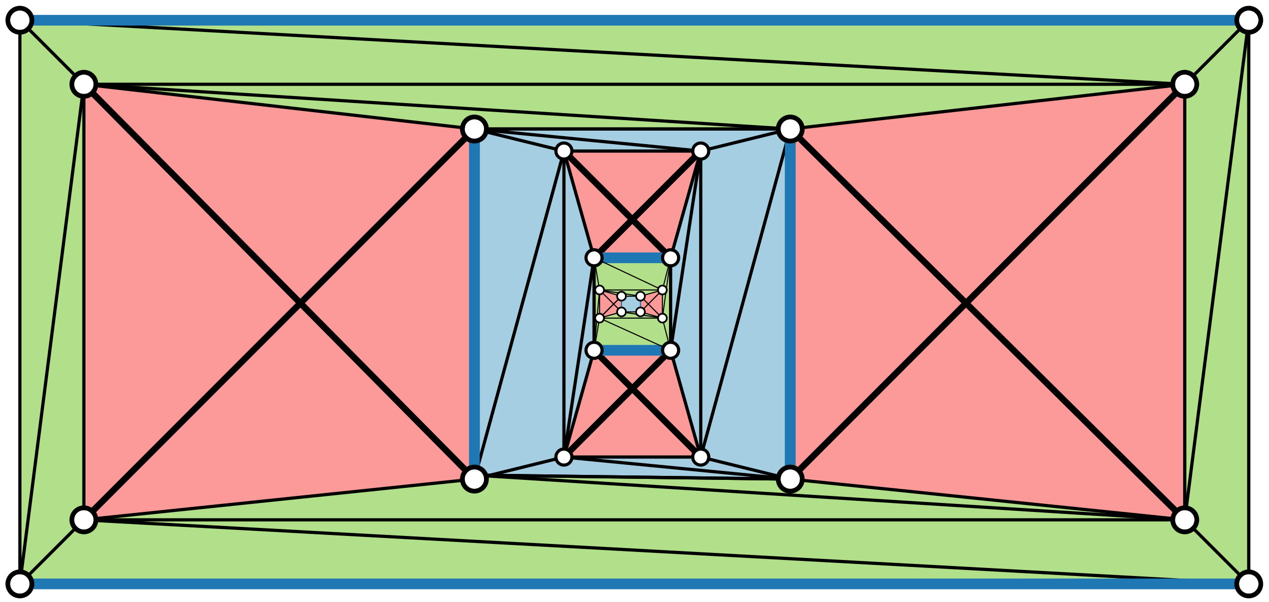
All IC-planar graphs have an IC-planar straight-line RAC drawing, and it can be found in polynomial time.



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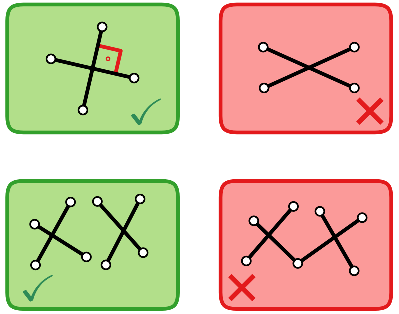

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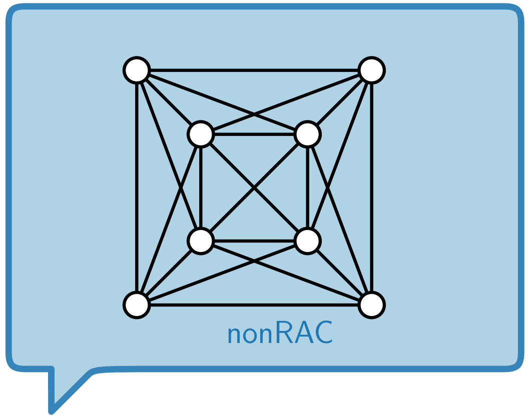
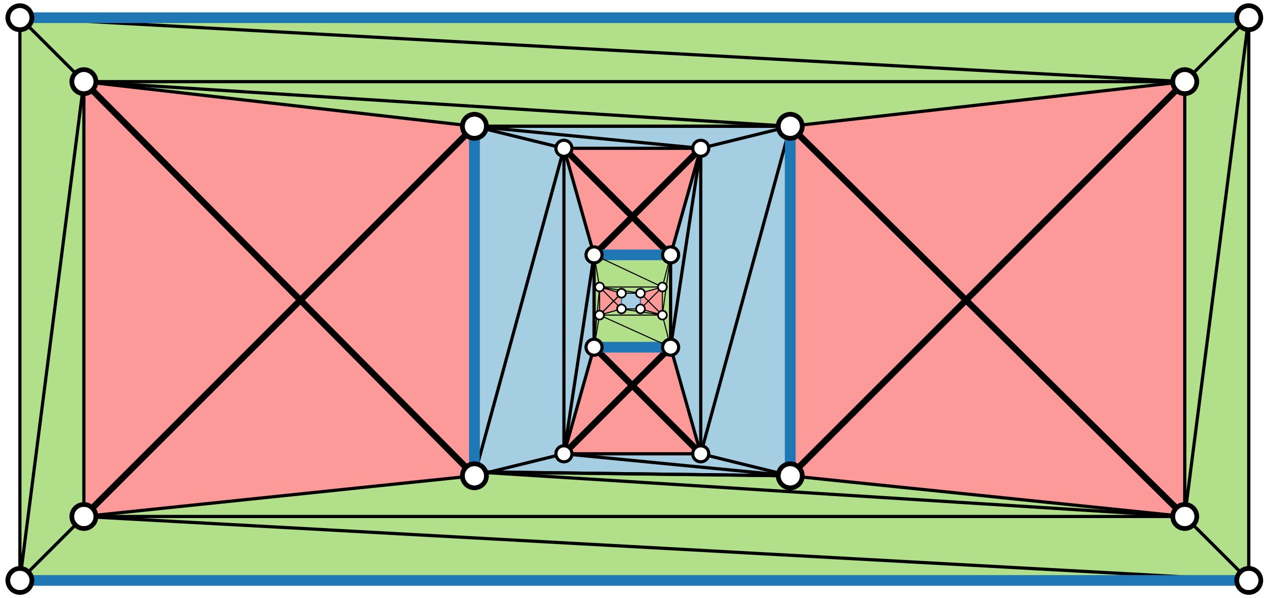
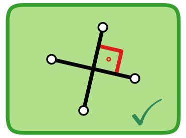
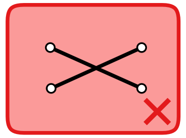
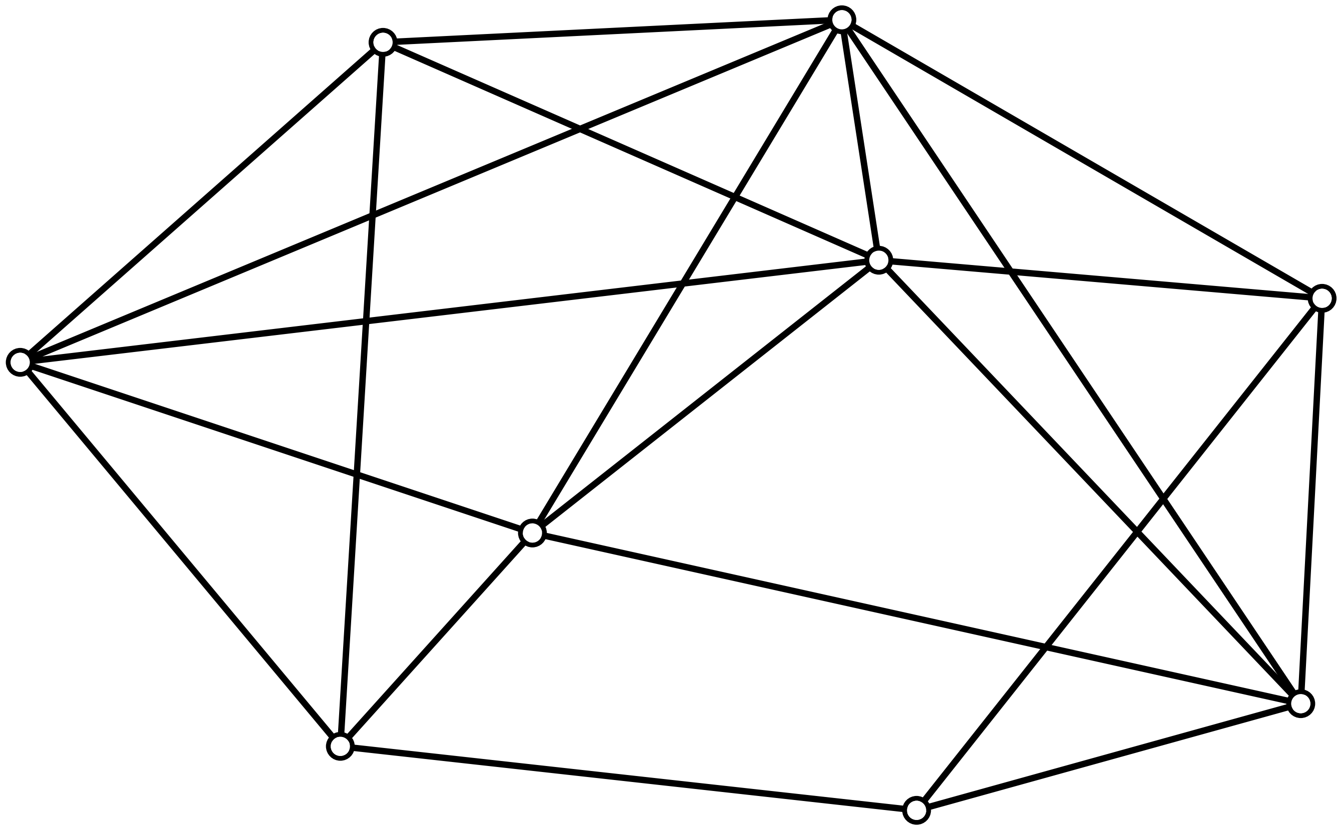


# RAC Drawings







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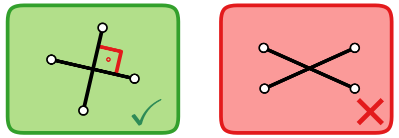

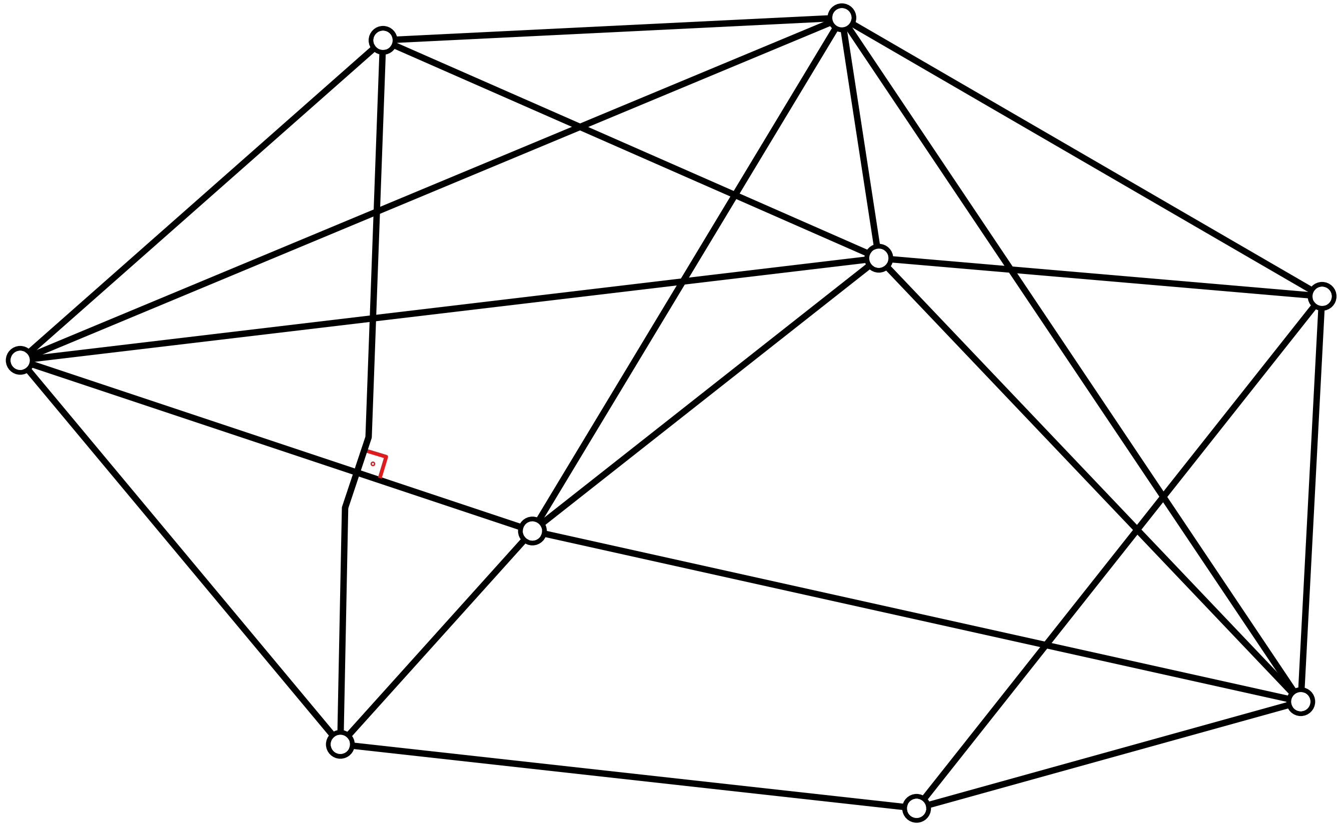

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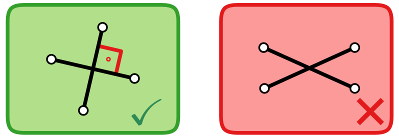

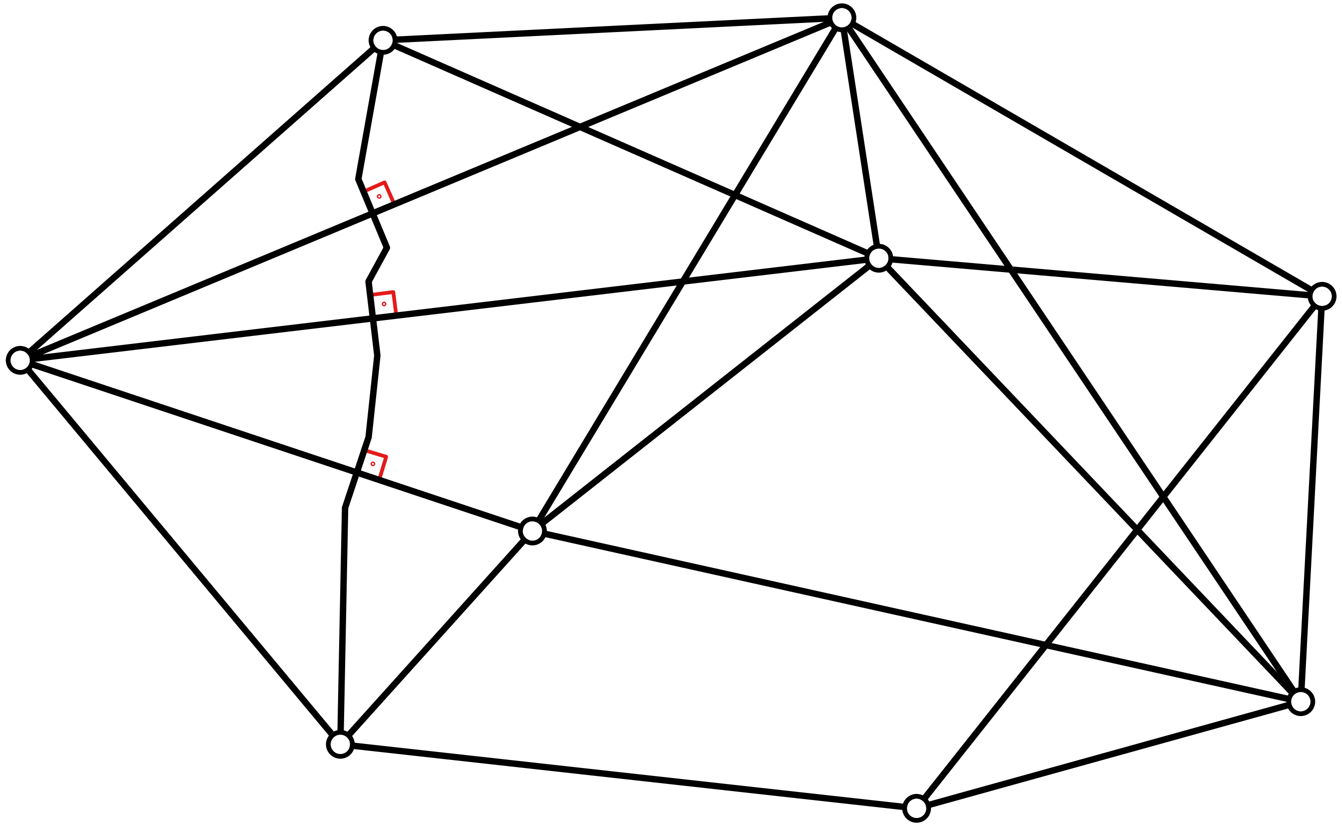

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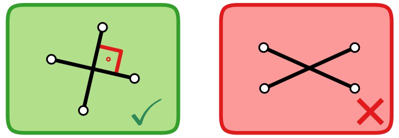

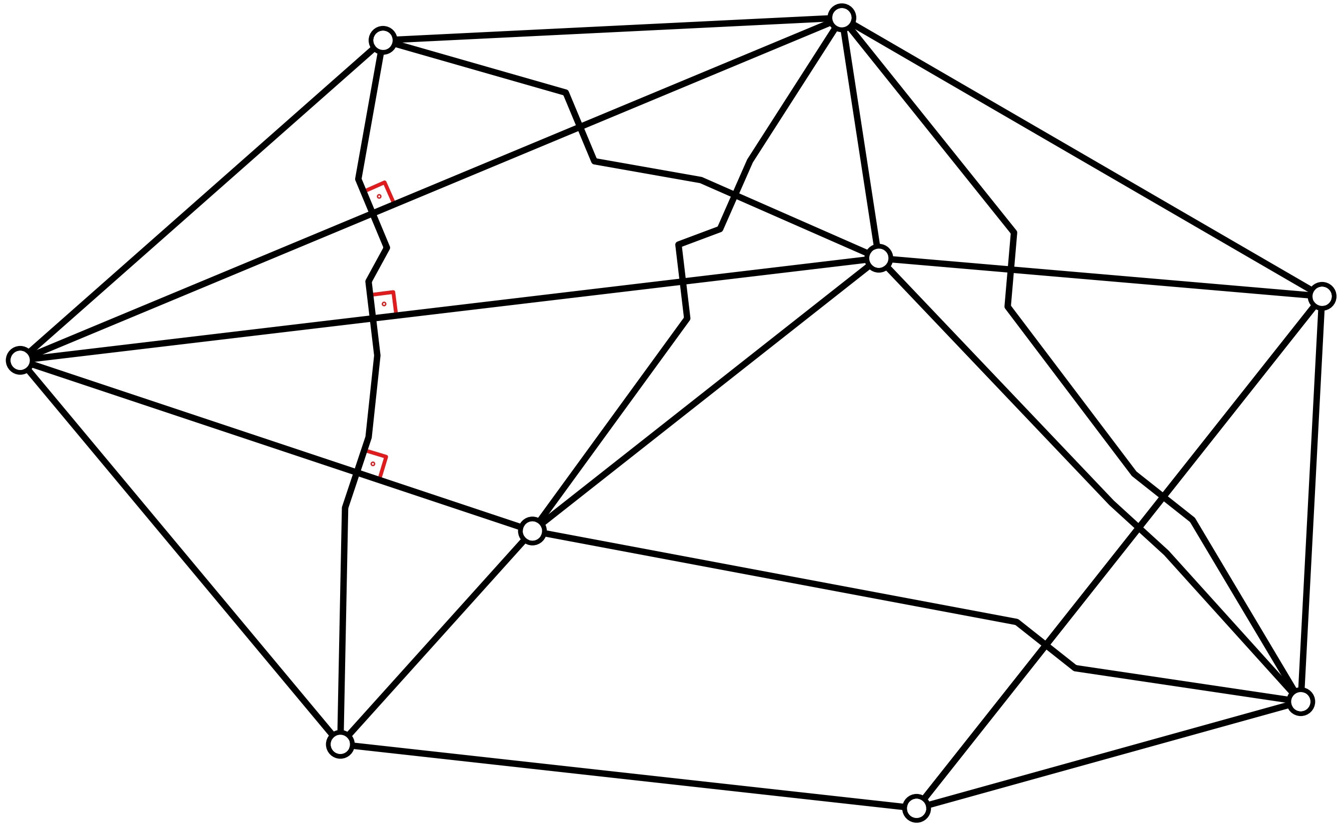

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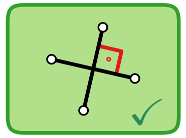
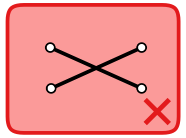
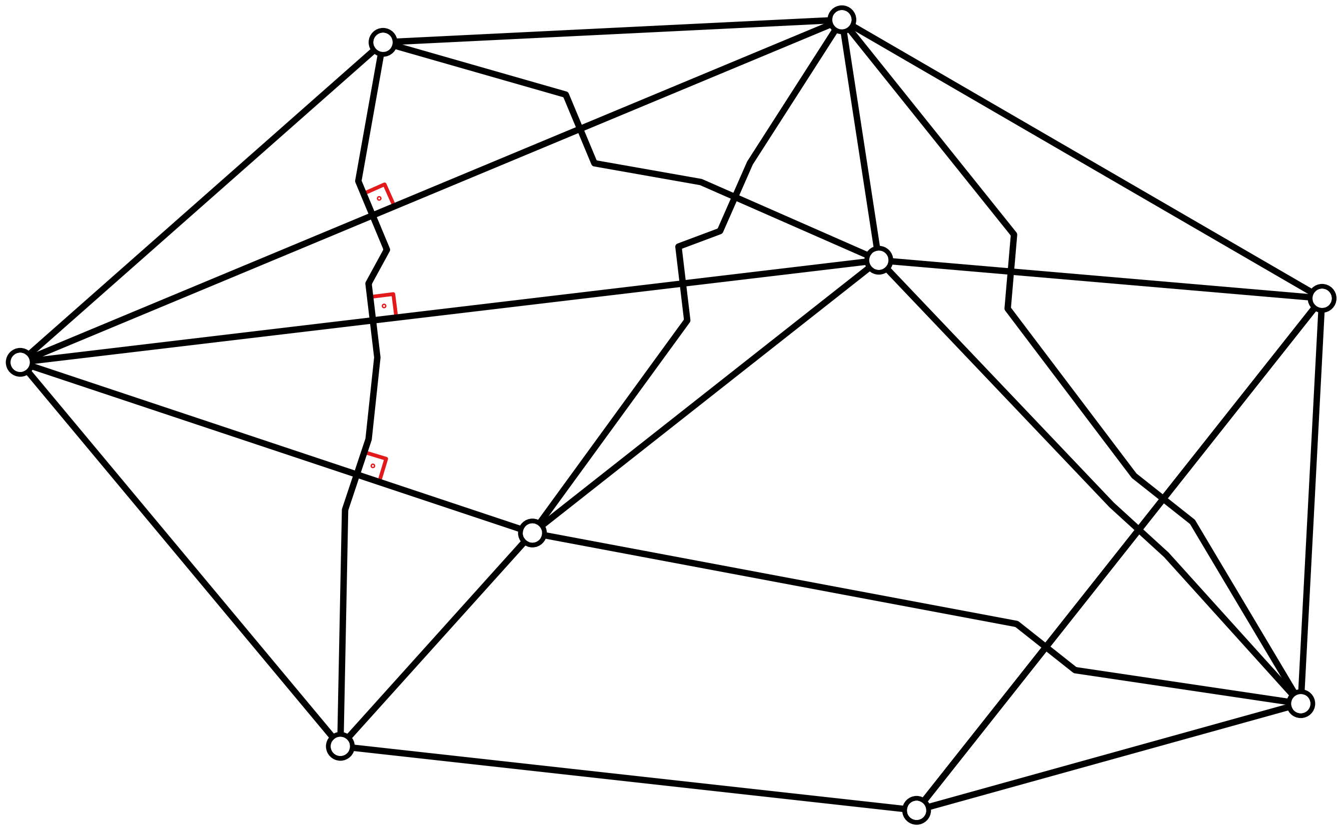
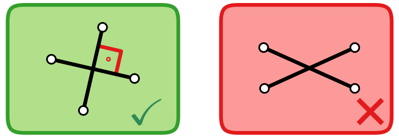


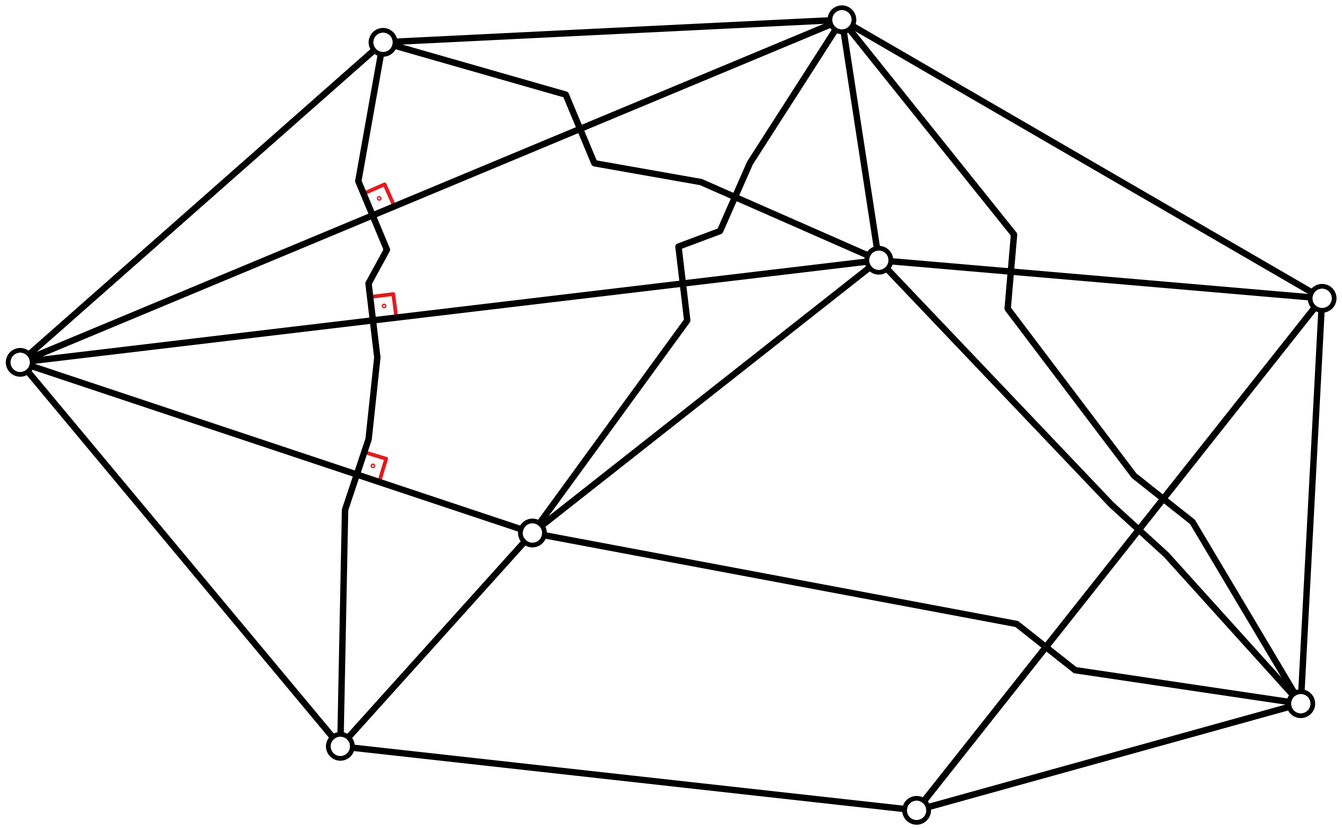




Every graph admits a RAC drawing . . .

# RAC Drawings With Enough Bends





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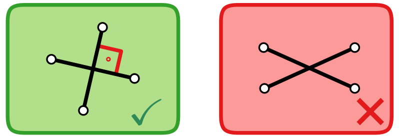

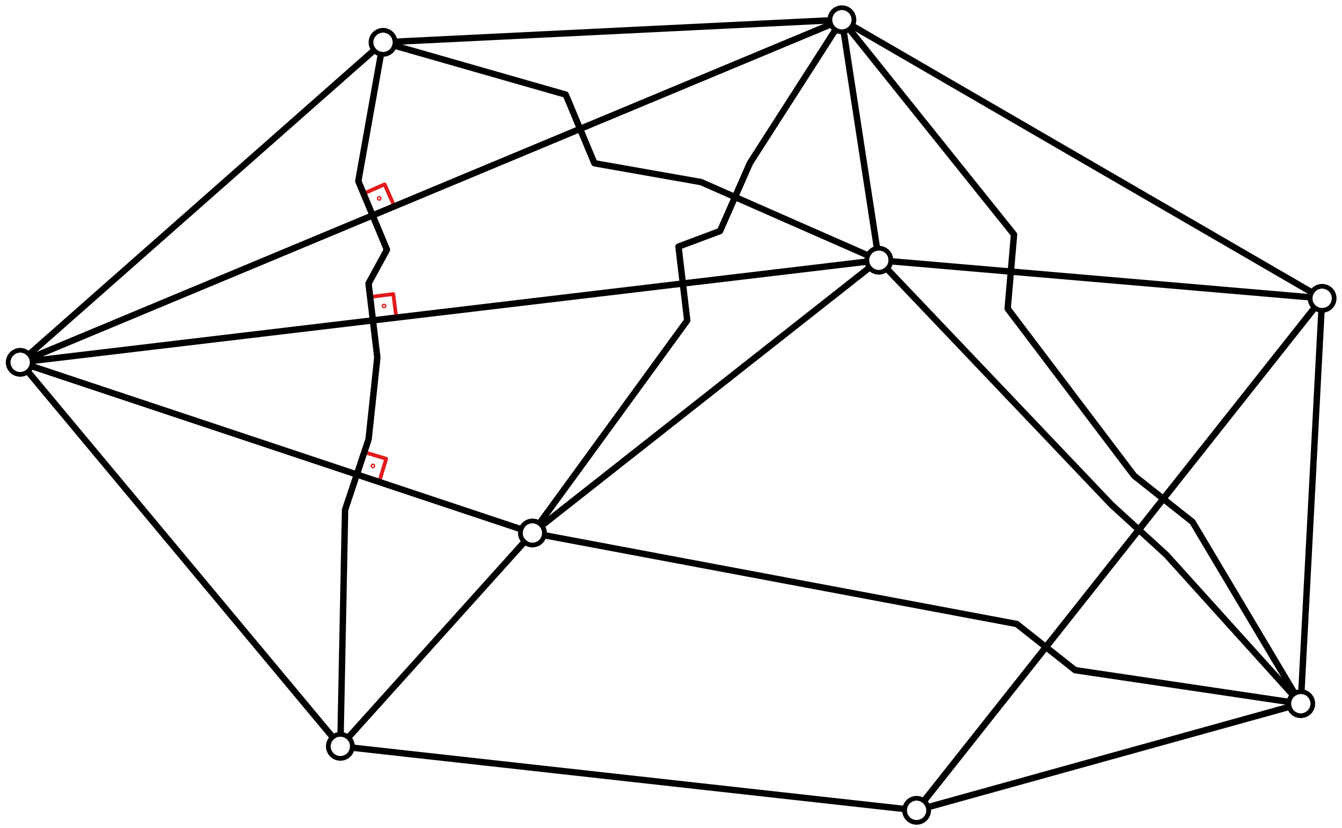

Every graph admits a RAC drawing . . .

. . . if we use enough bends.

How many do we need at most in total or per edge?

# 3-Bend RAC Drawings

**Theorem.** [Didimo, Eades & Liotta 2017]
Every graph admits a 3-bend RAC drawing, that is, a RAC drawing where every edge has at most 3 bends.

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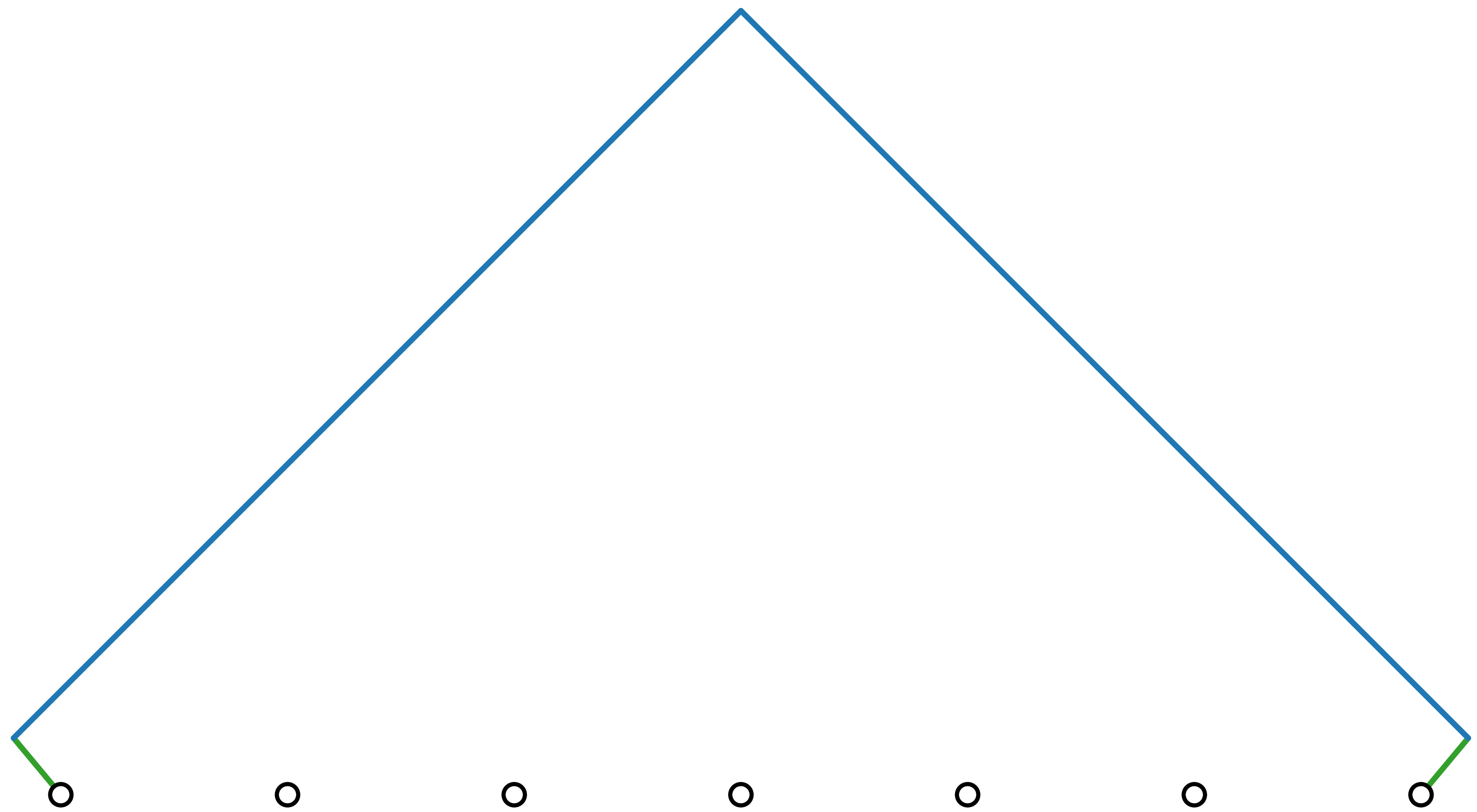

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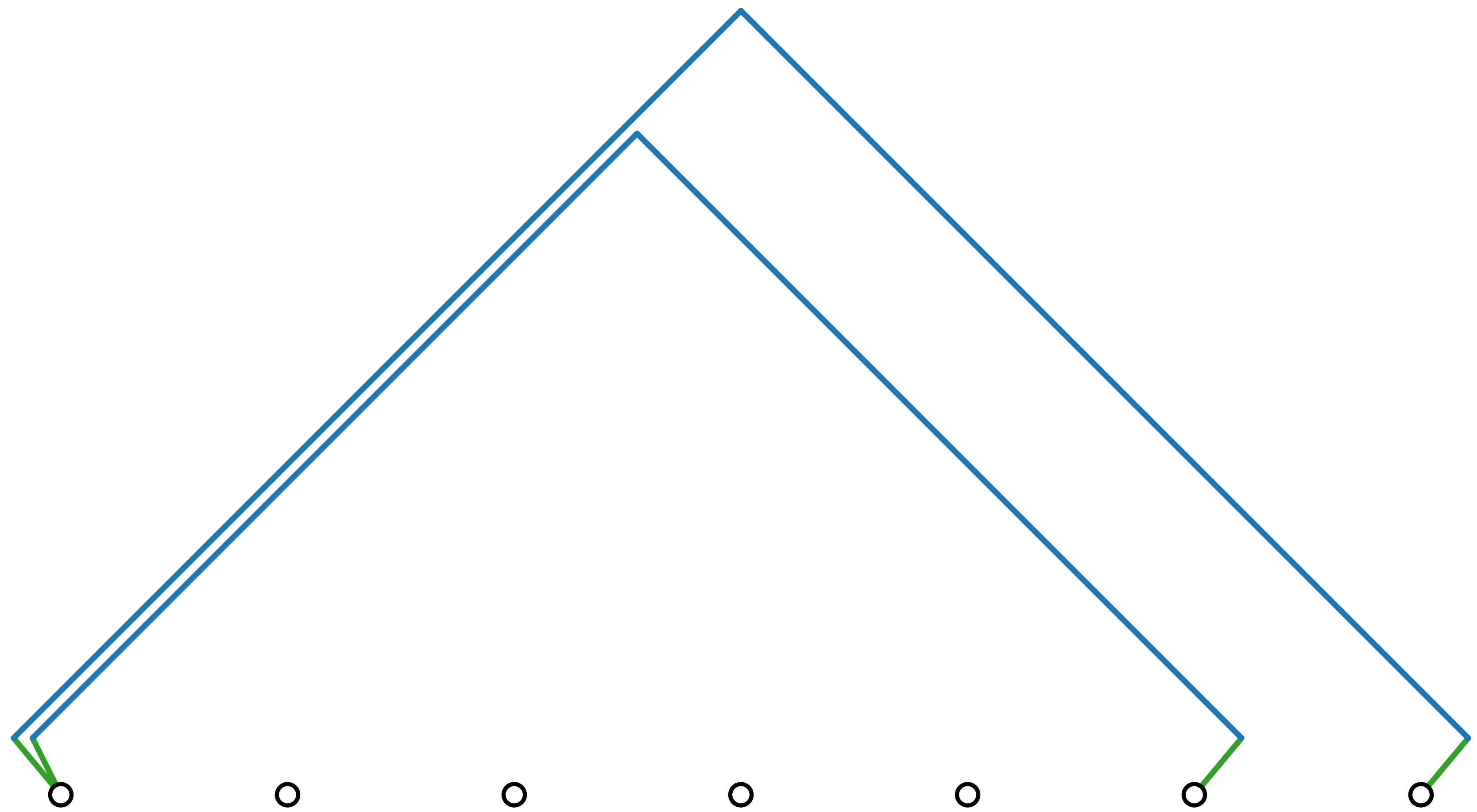

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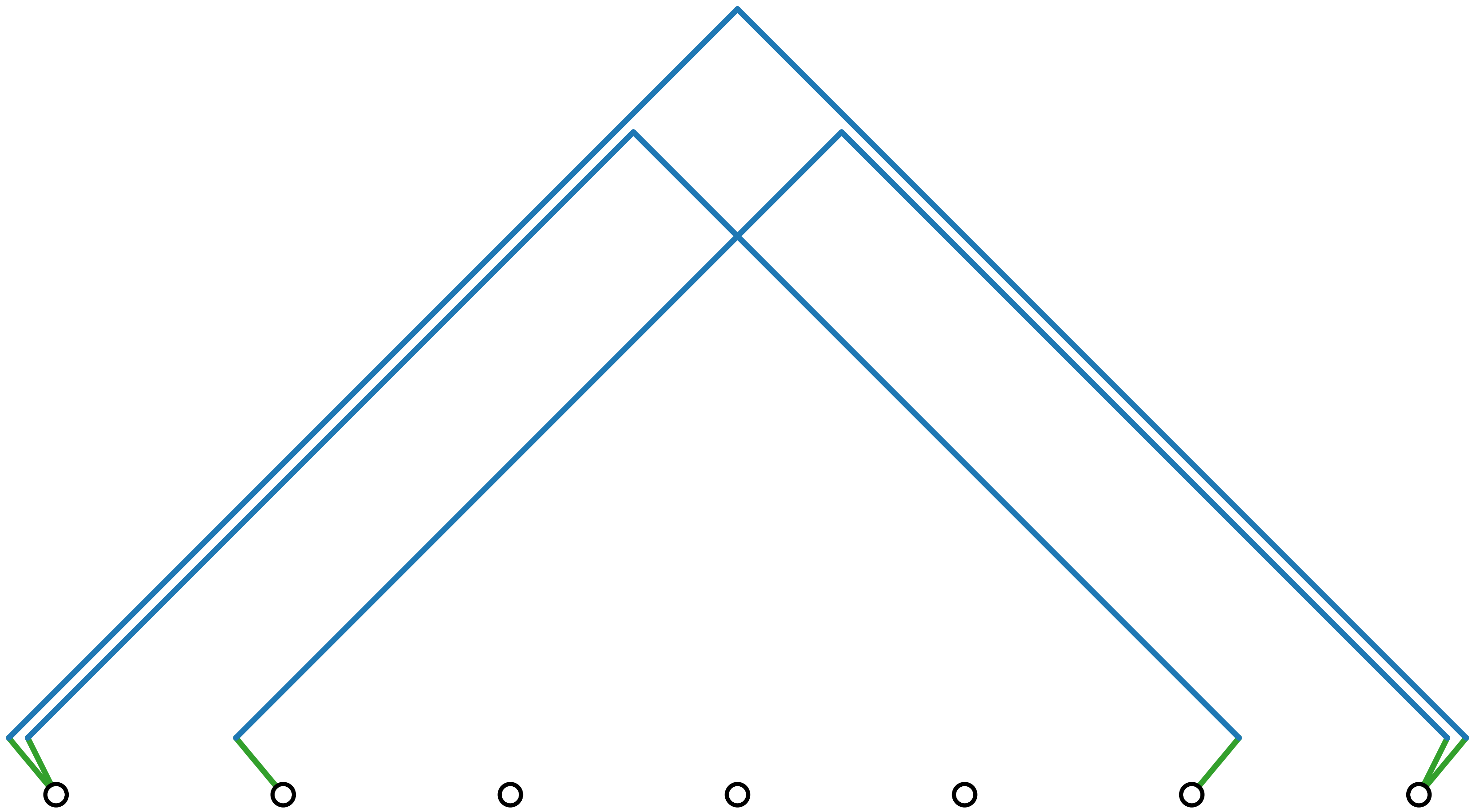

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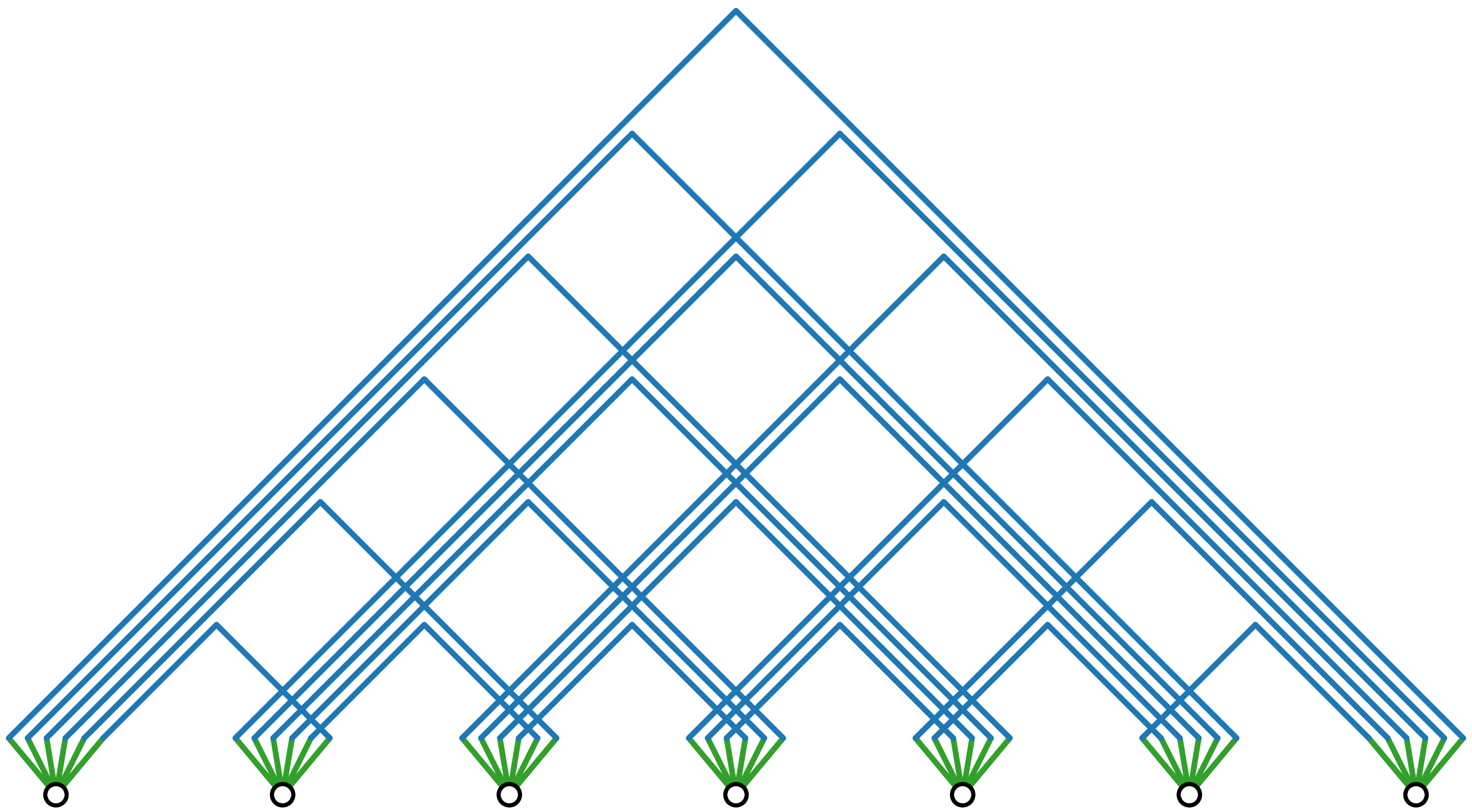
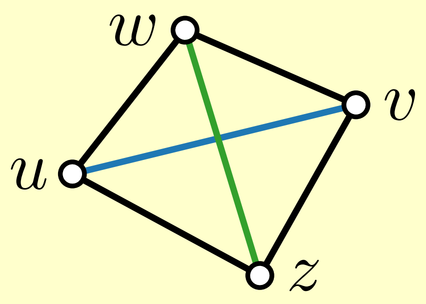


# Kite Triangulations

This is a **kite**:



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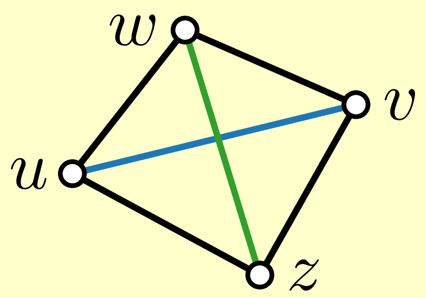


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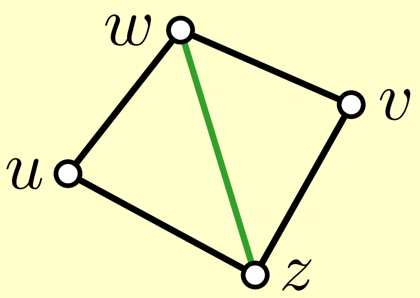

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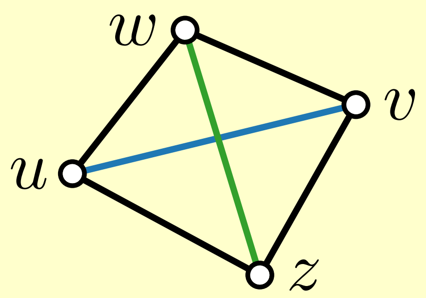


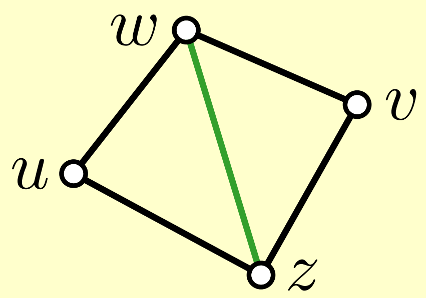


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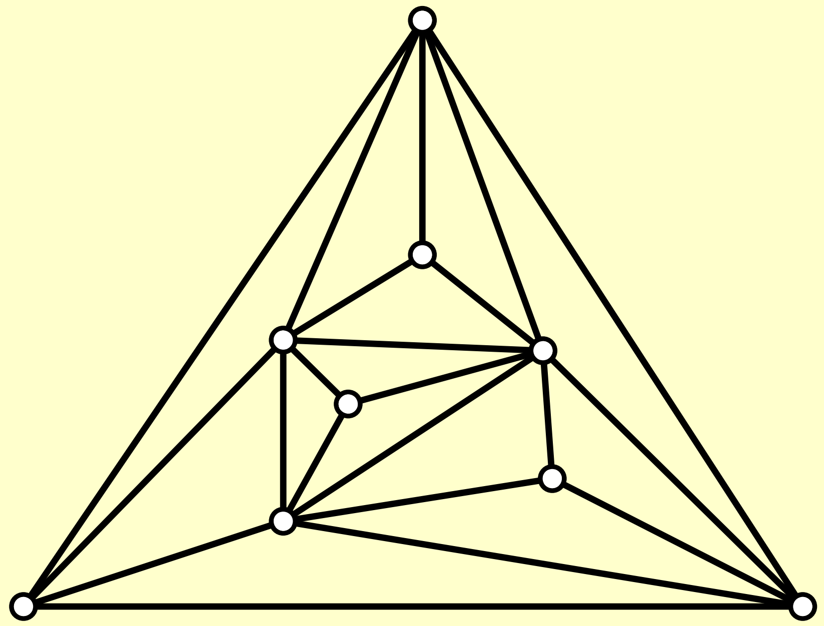

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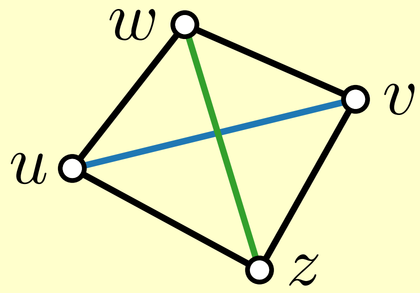


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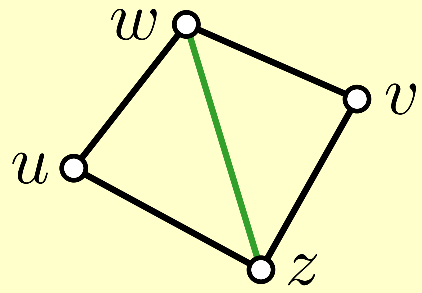


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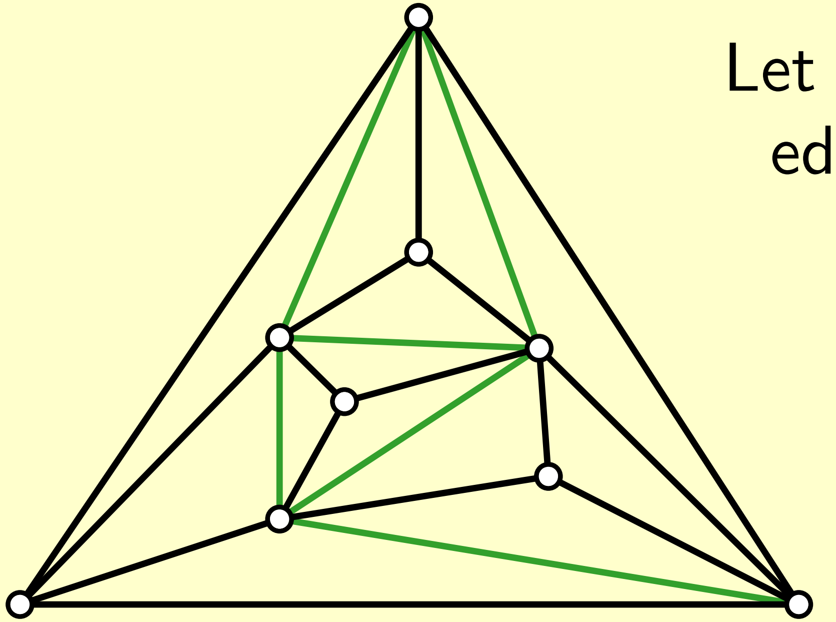

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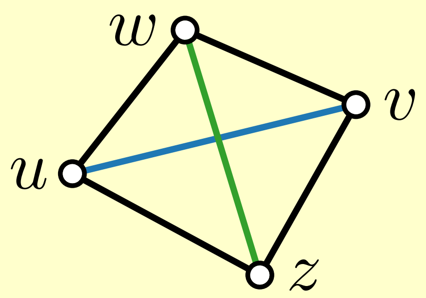


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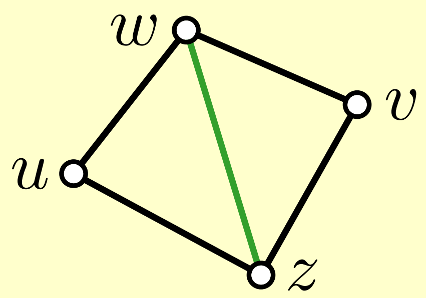


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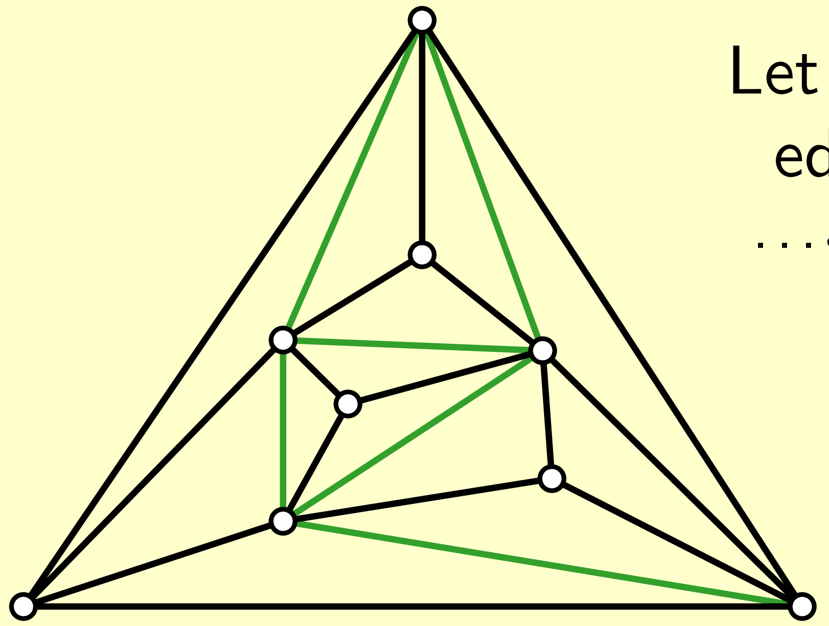

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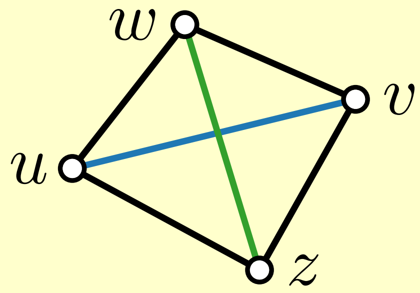


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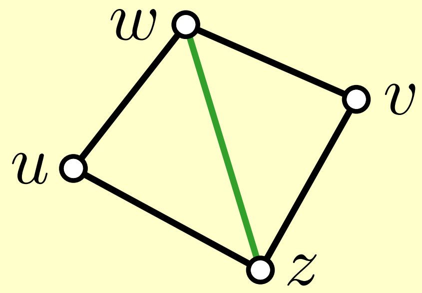


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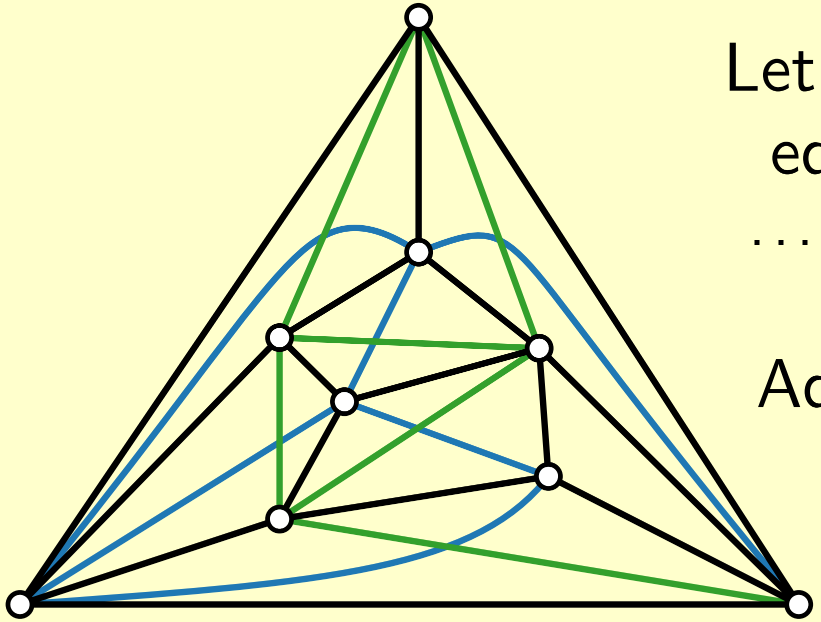

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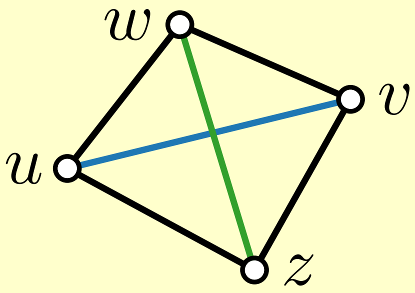


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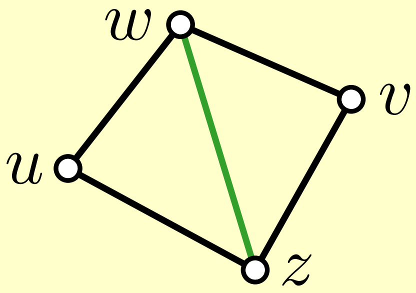


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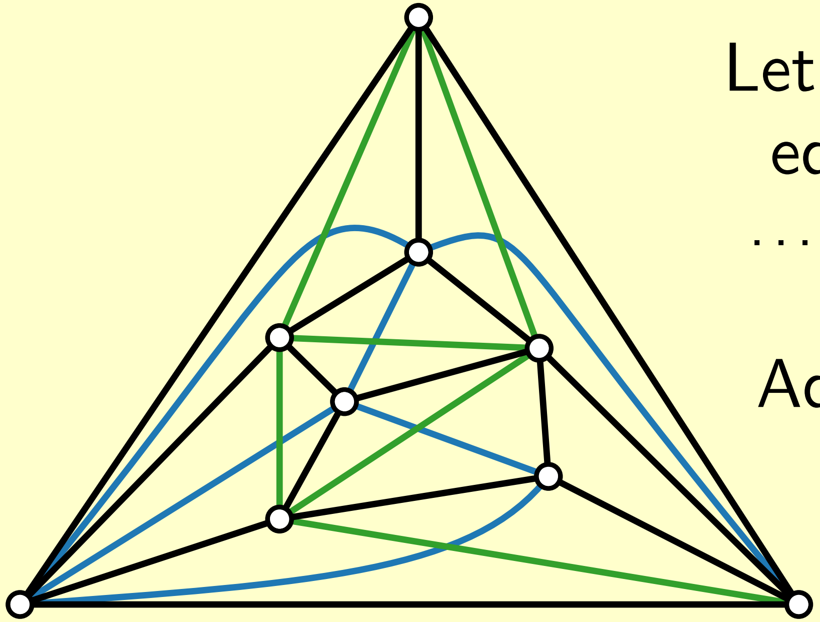

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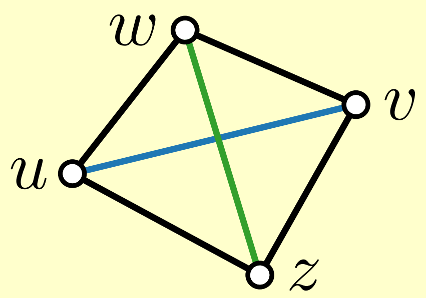


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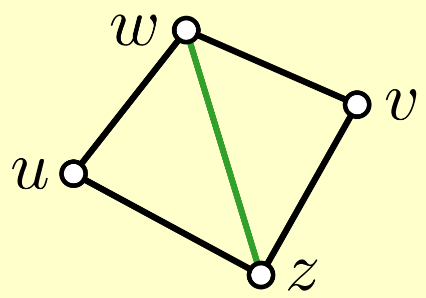


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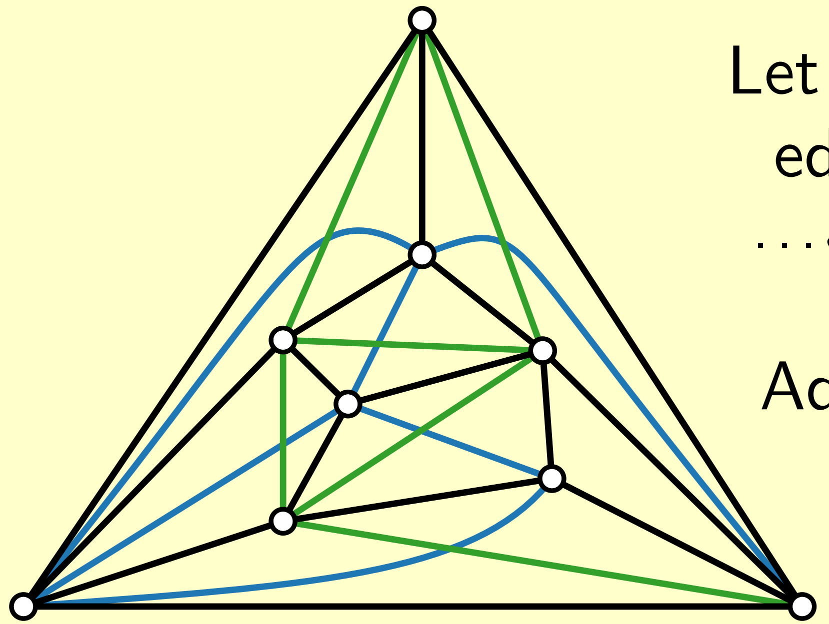

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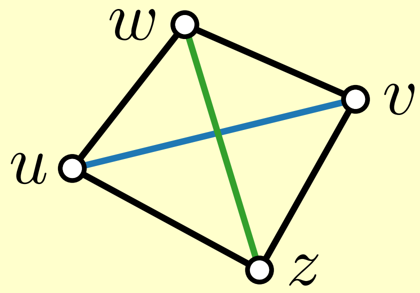


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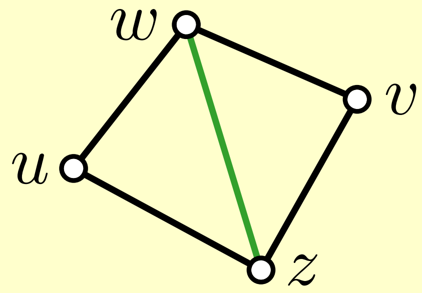


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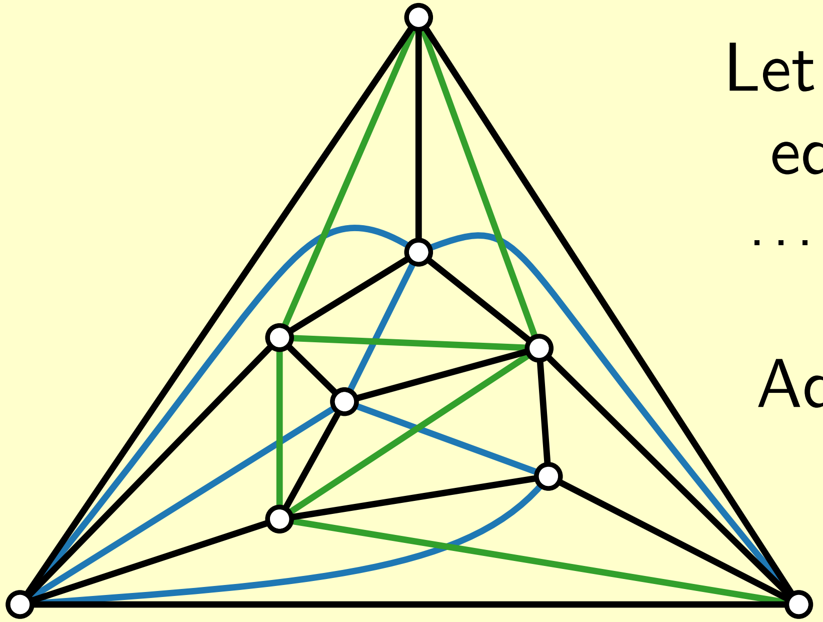

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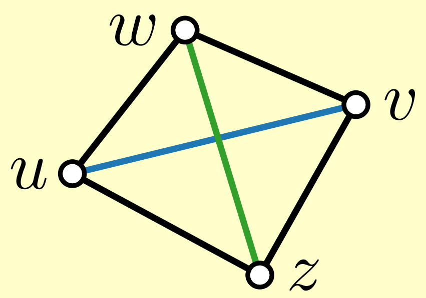


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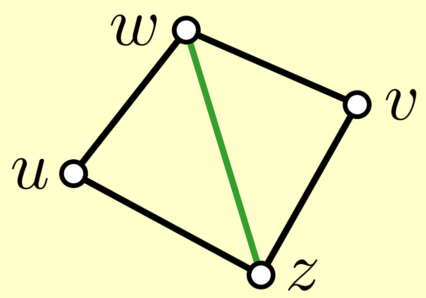


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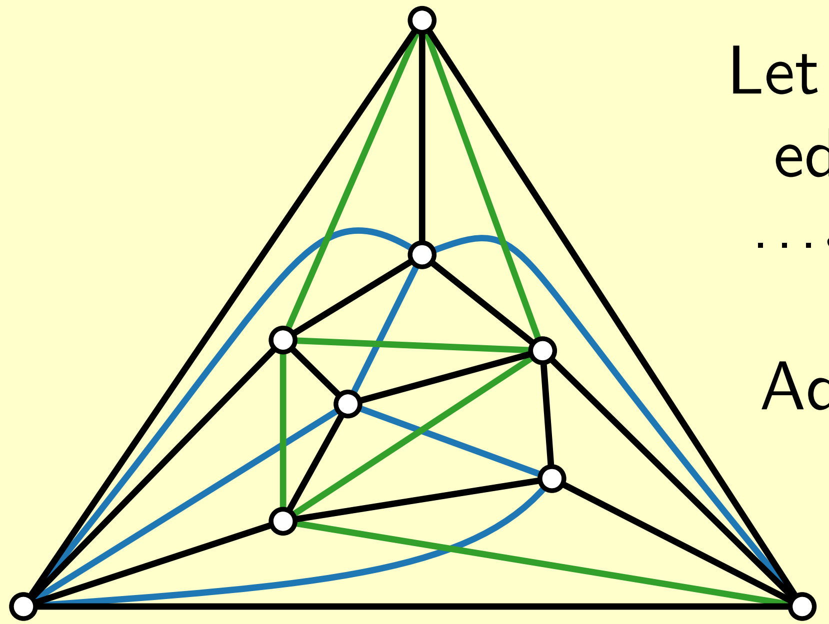

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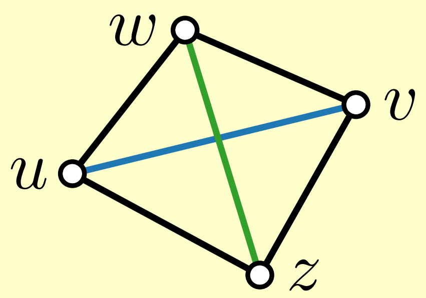


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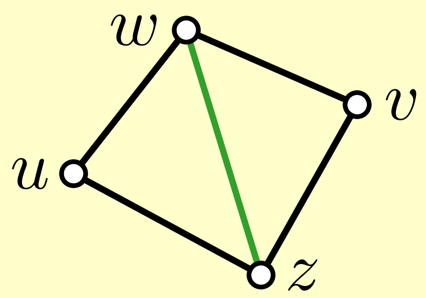


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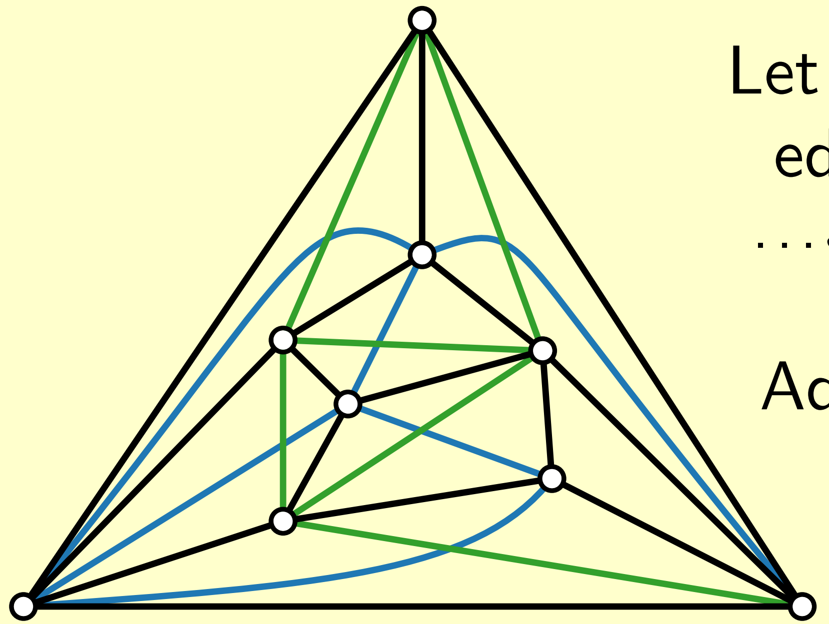

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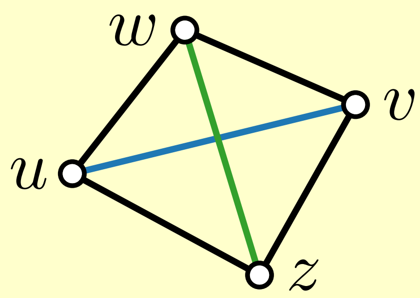


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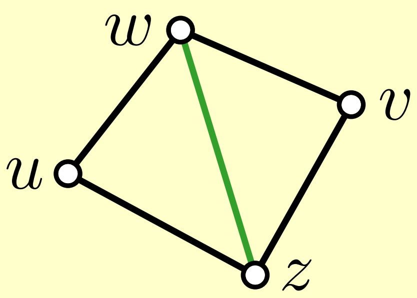


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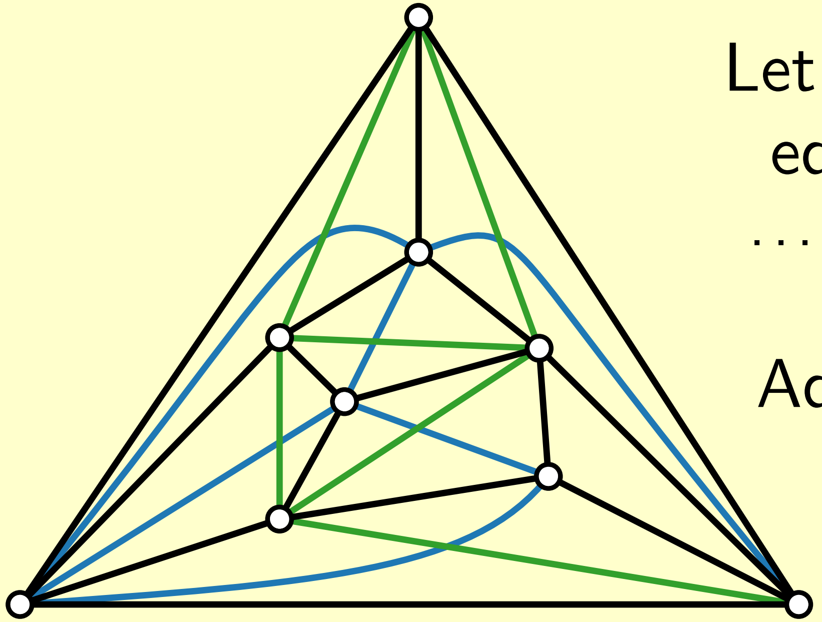

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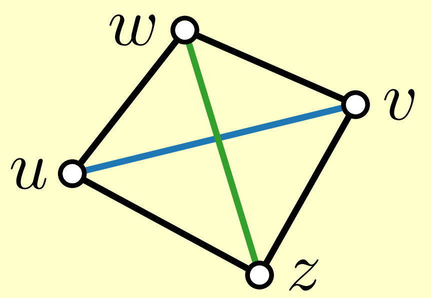


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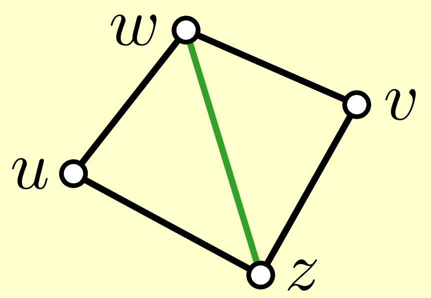


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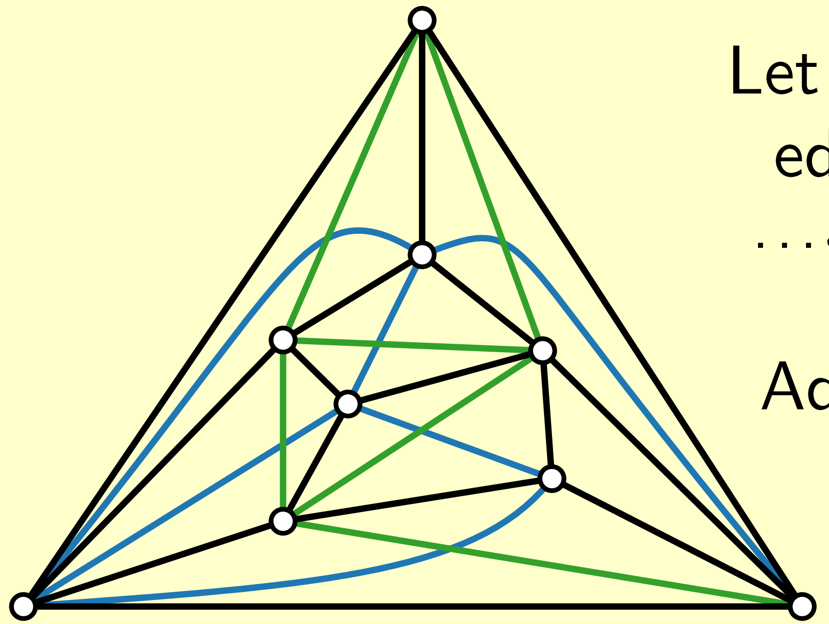

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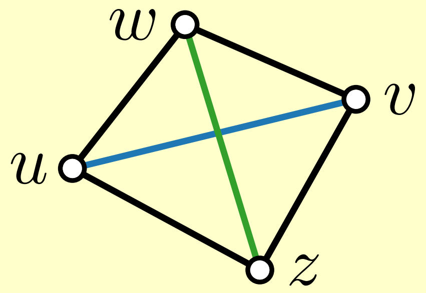


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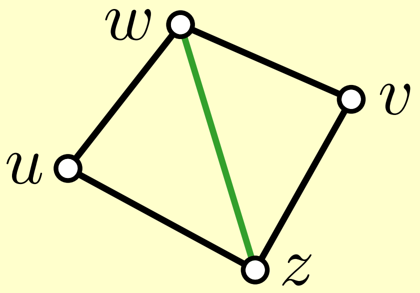


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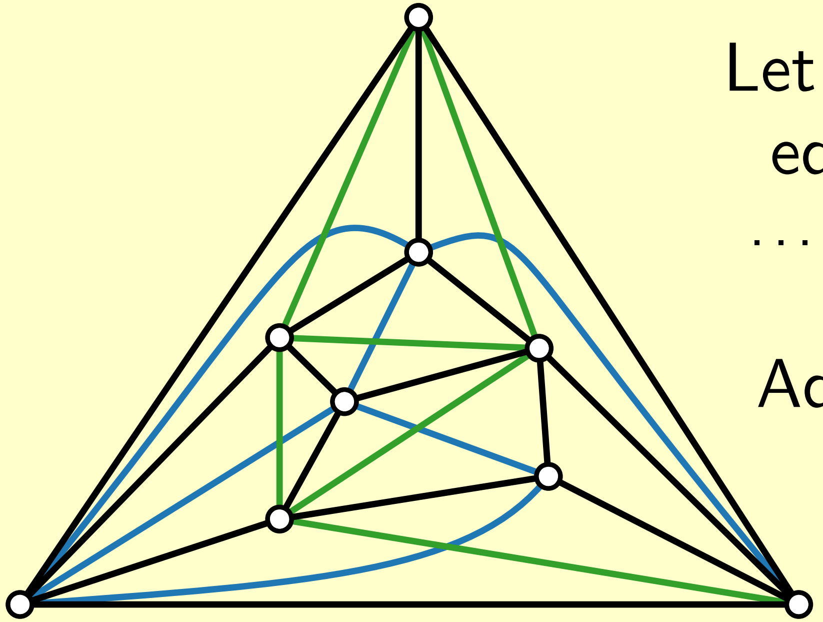

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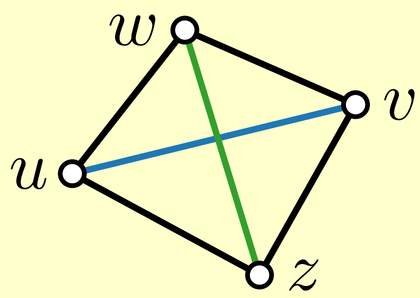


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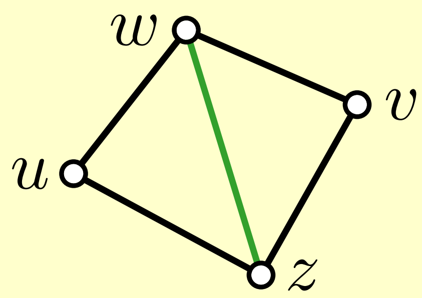


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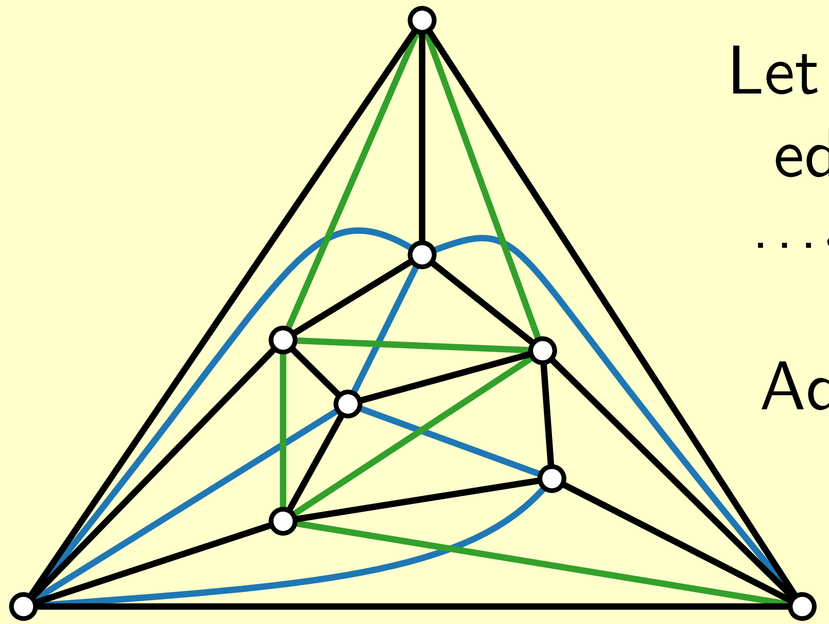

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Fill faces as follows:

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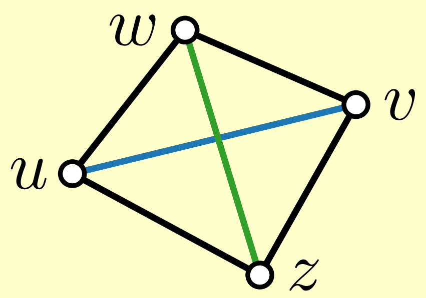


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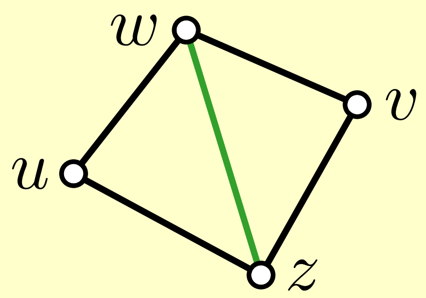


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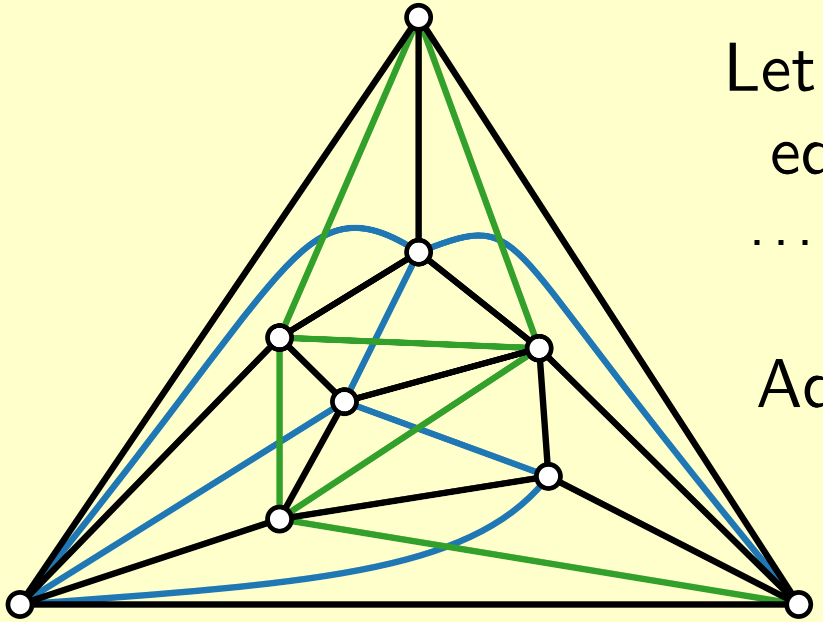

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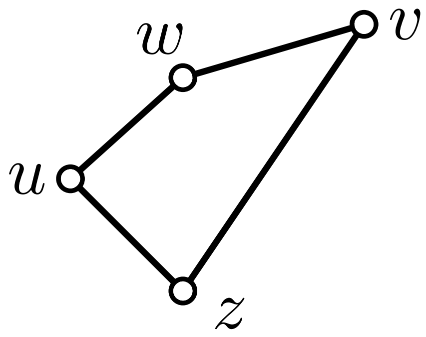


strictly convex face

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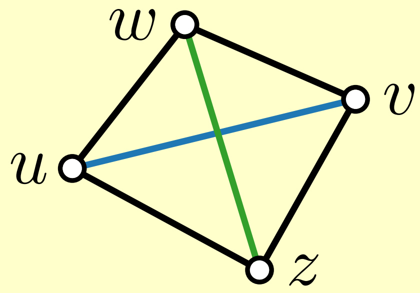


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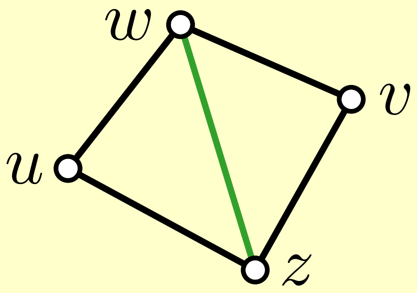


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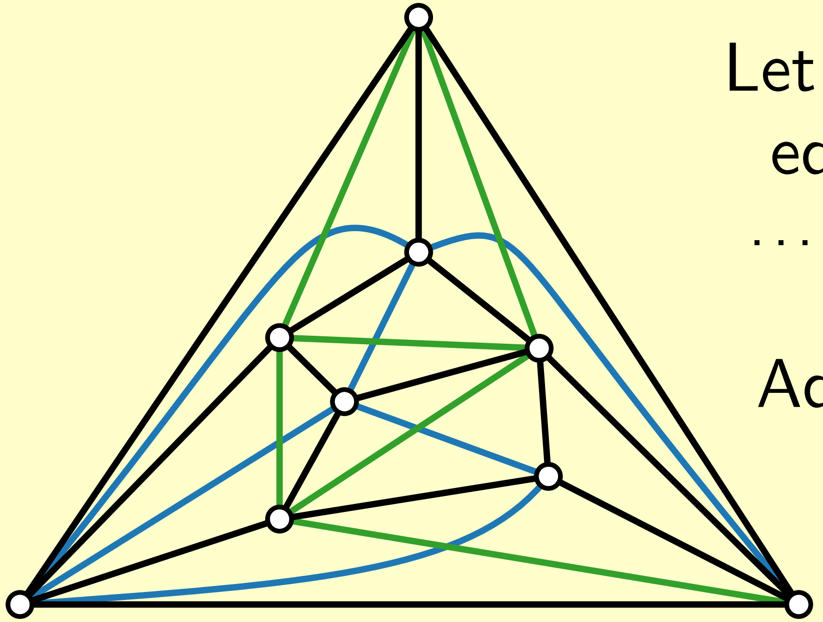

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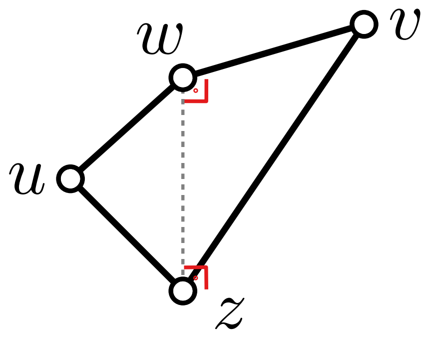


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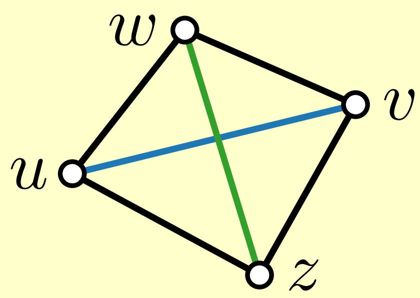


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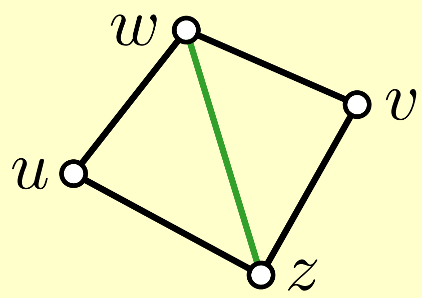


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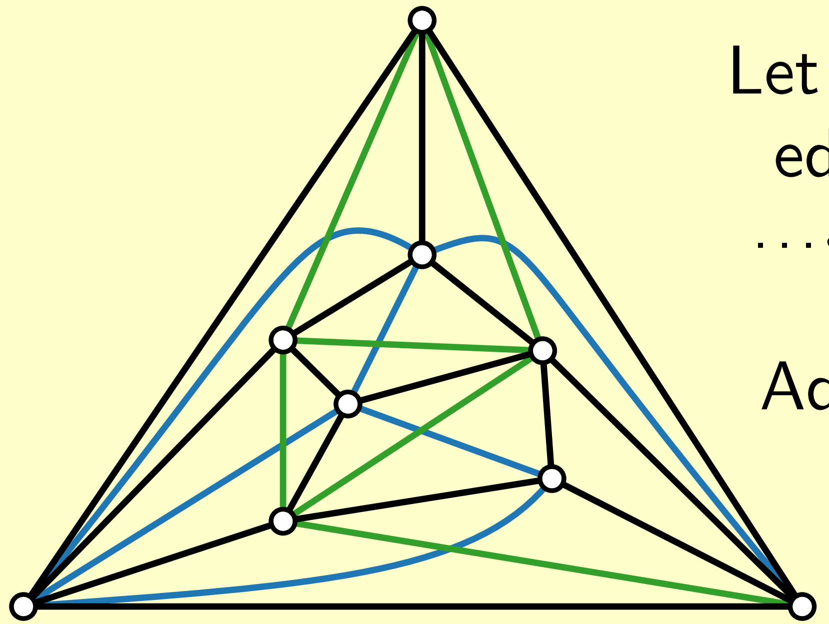

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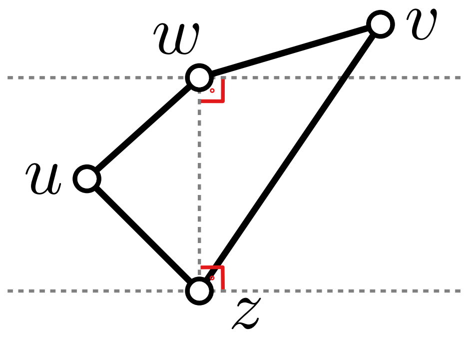


strictly convex face

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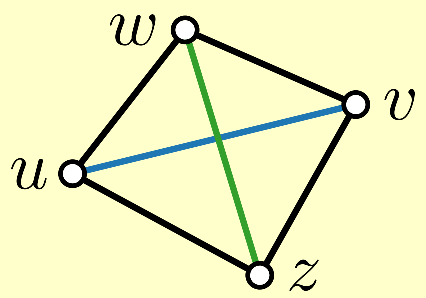


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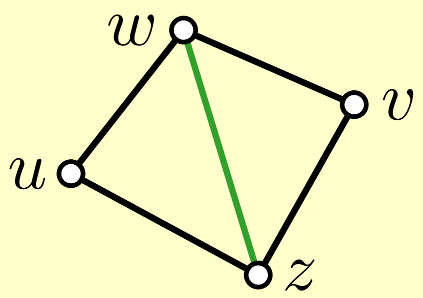


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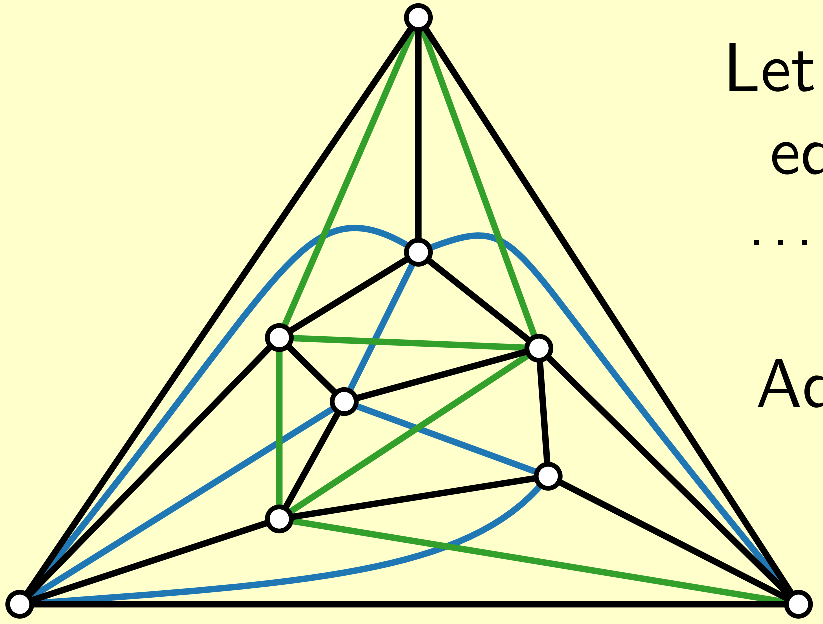

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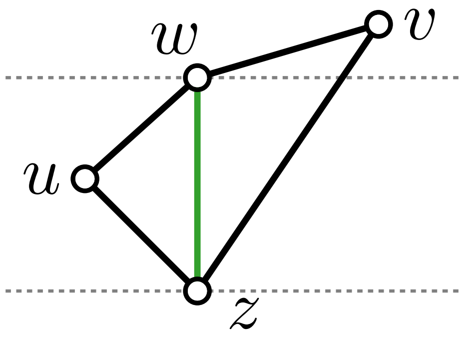


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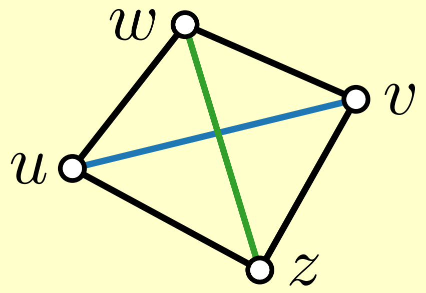


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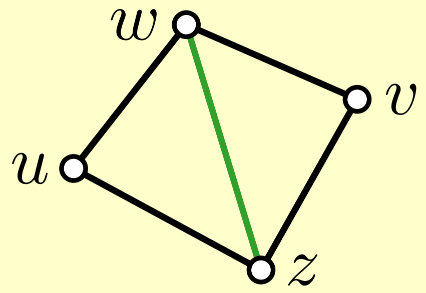


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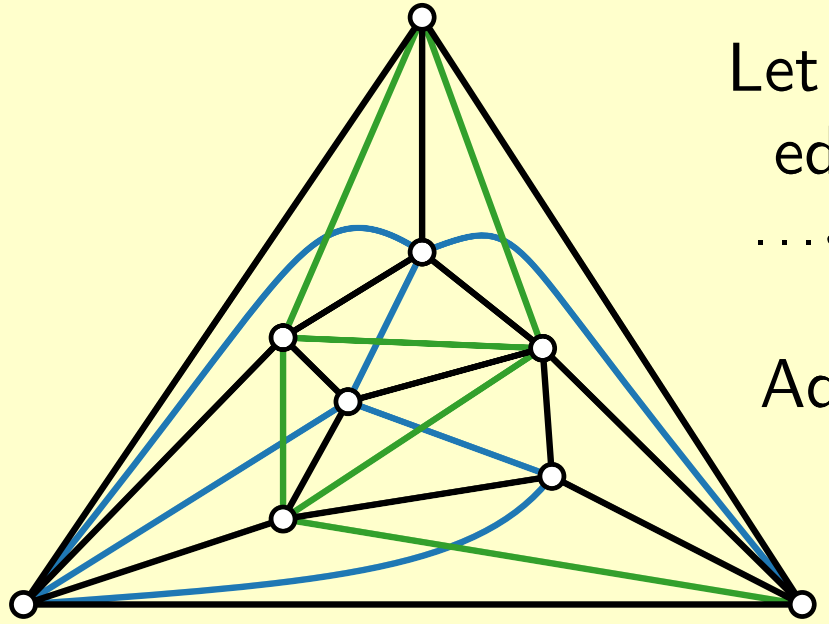

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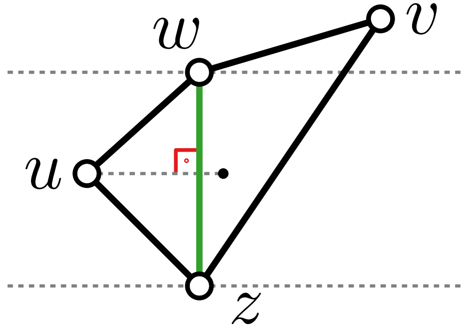


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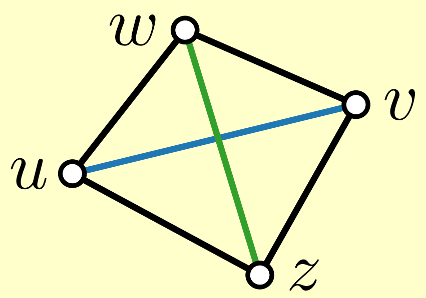


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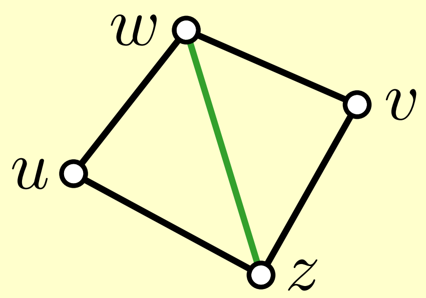


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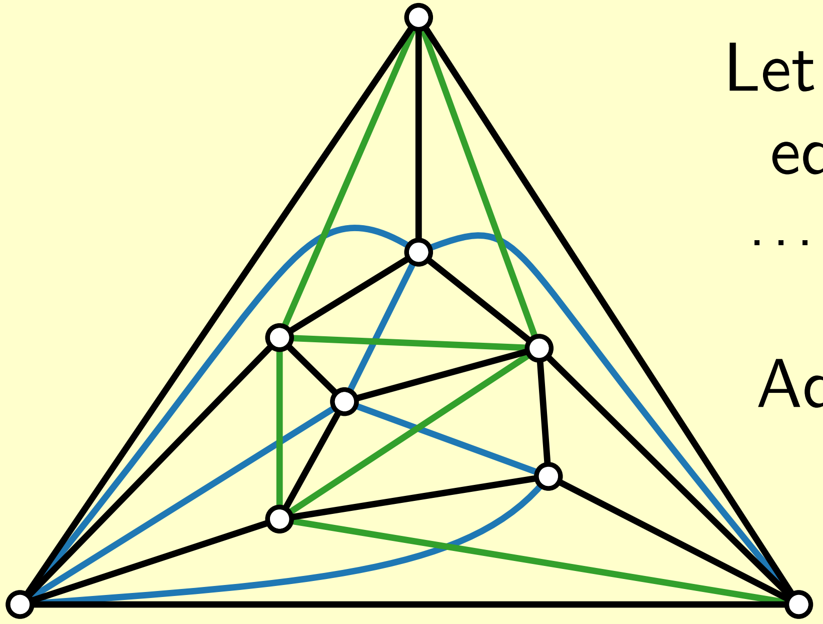

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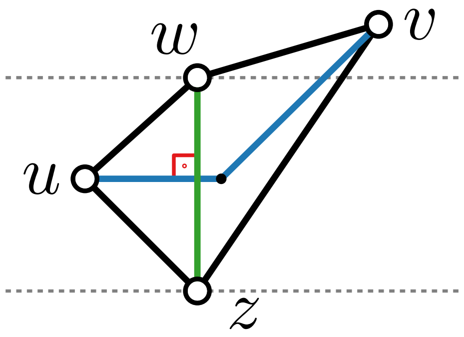


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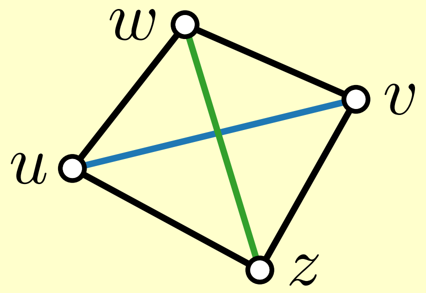


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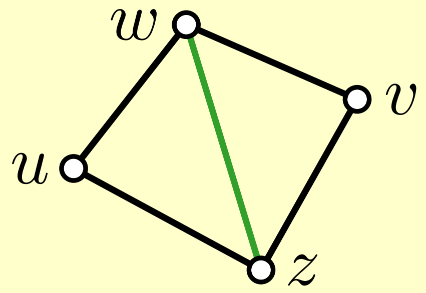


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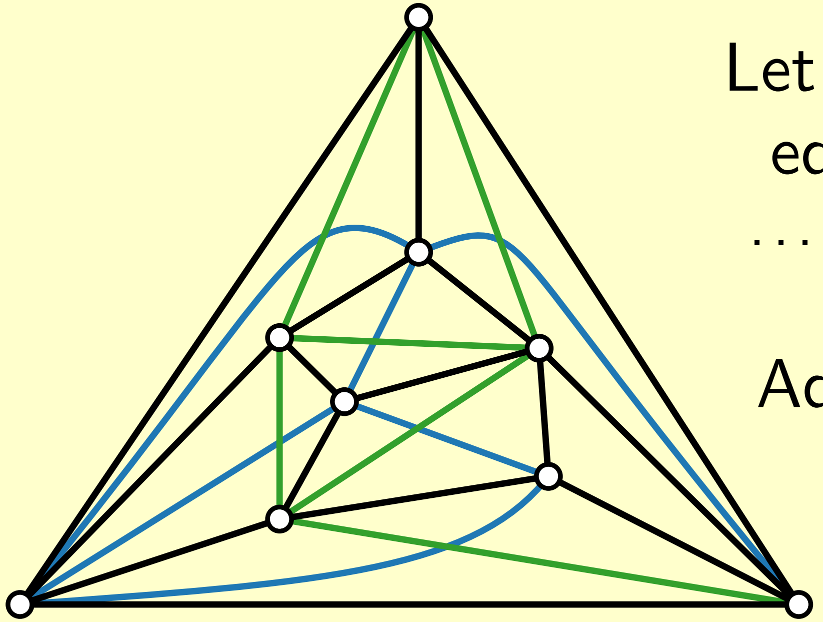

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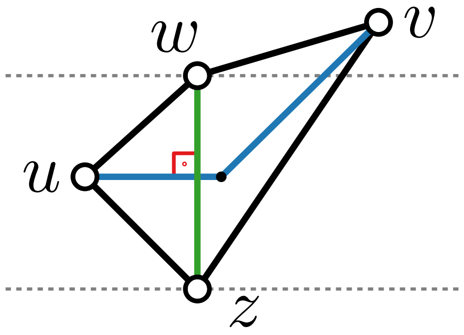


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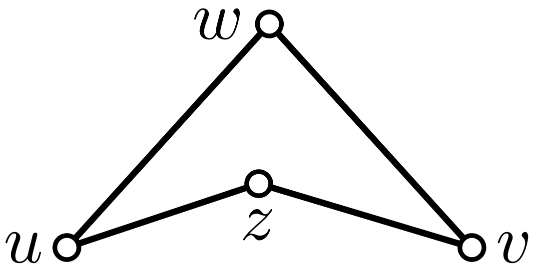


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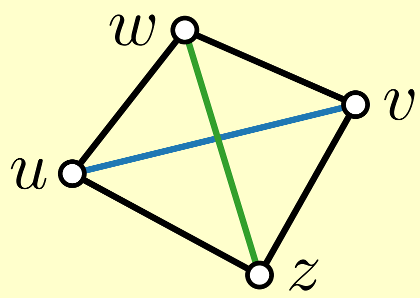


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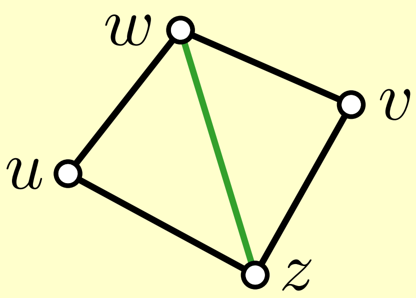


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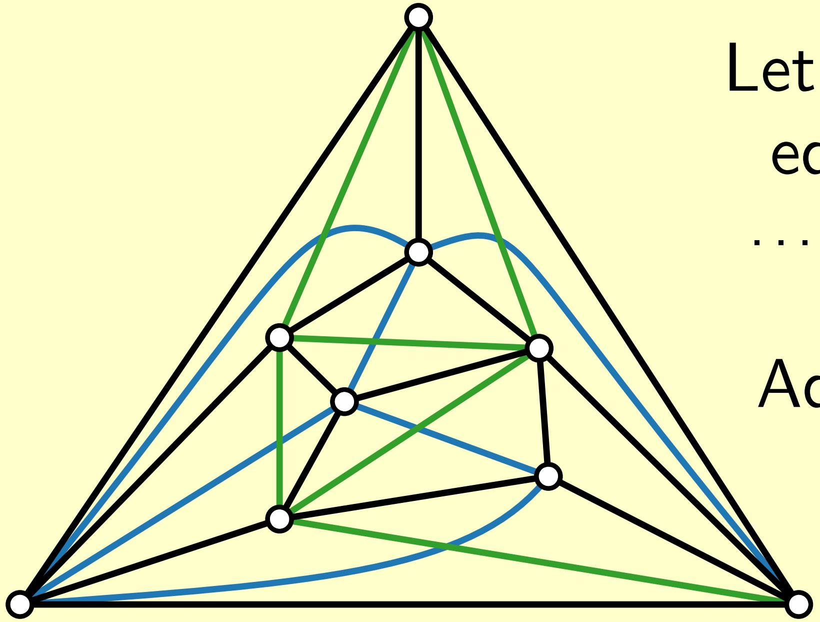

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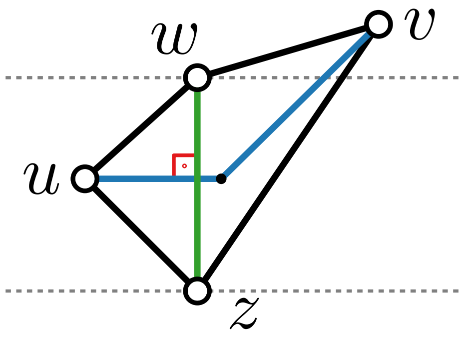


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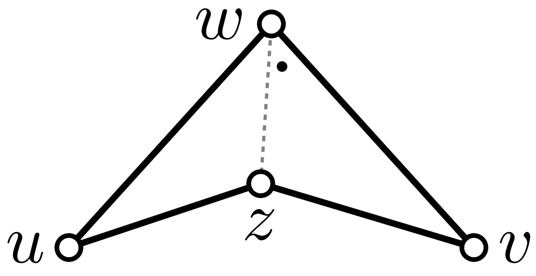


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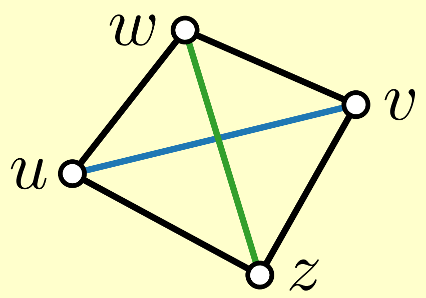


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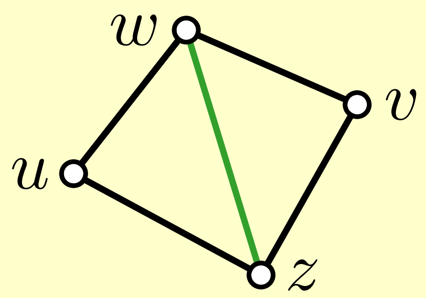


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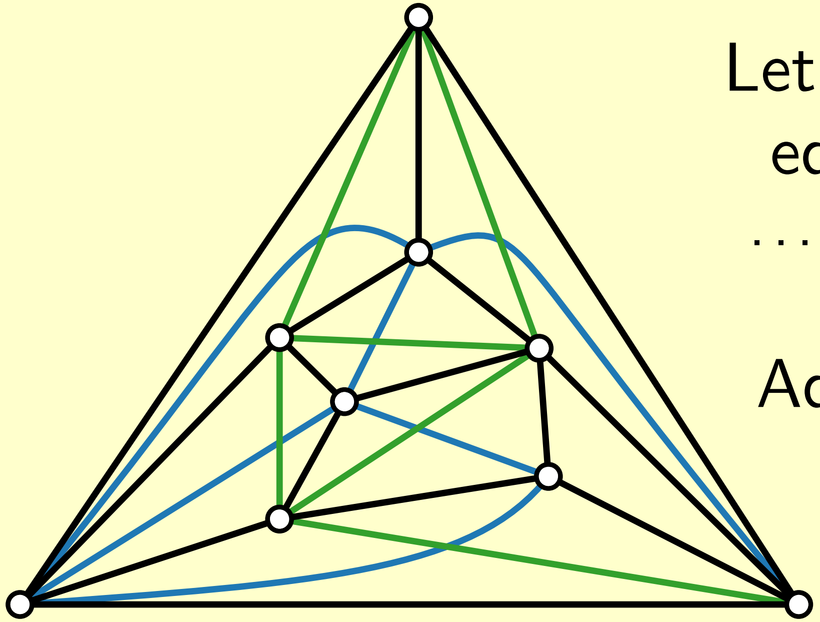

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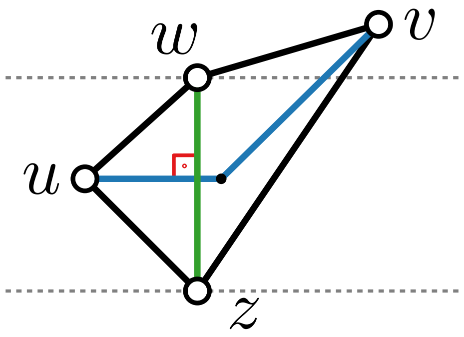


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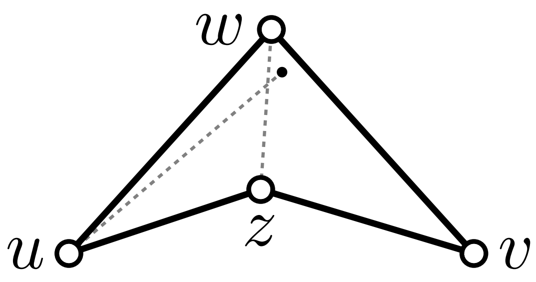


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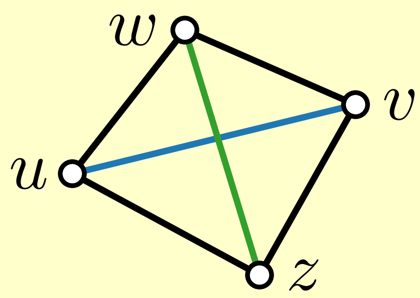


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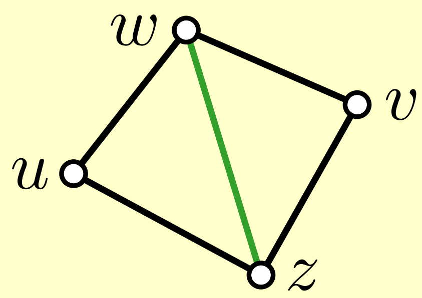


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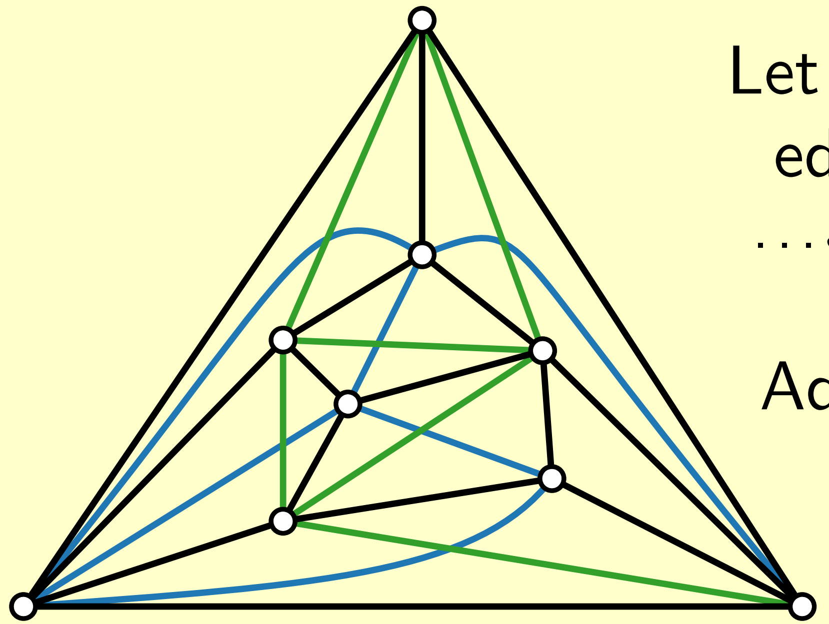

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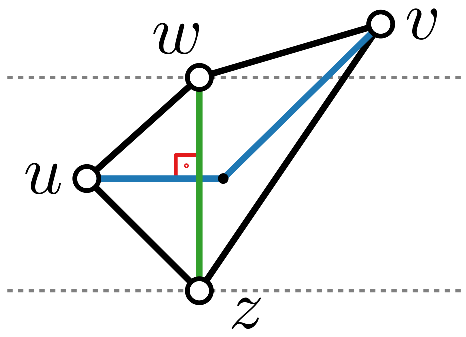


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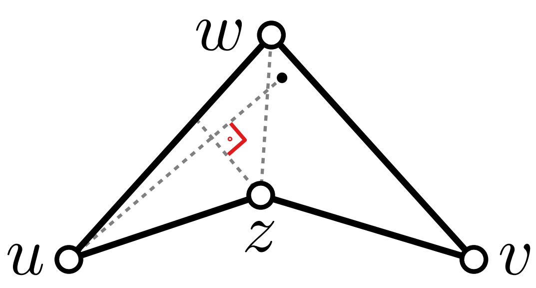


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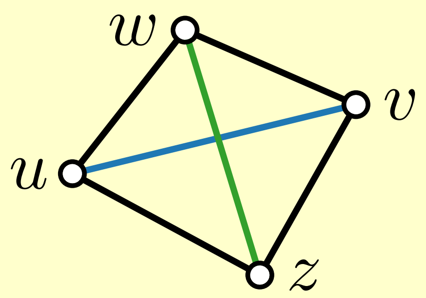


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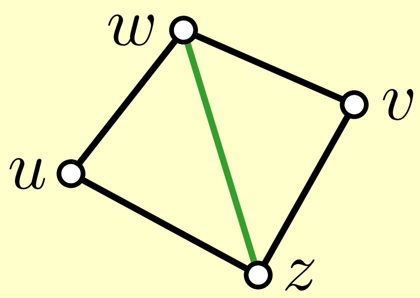


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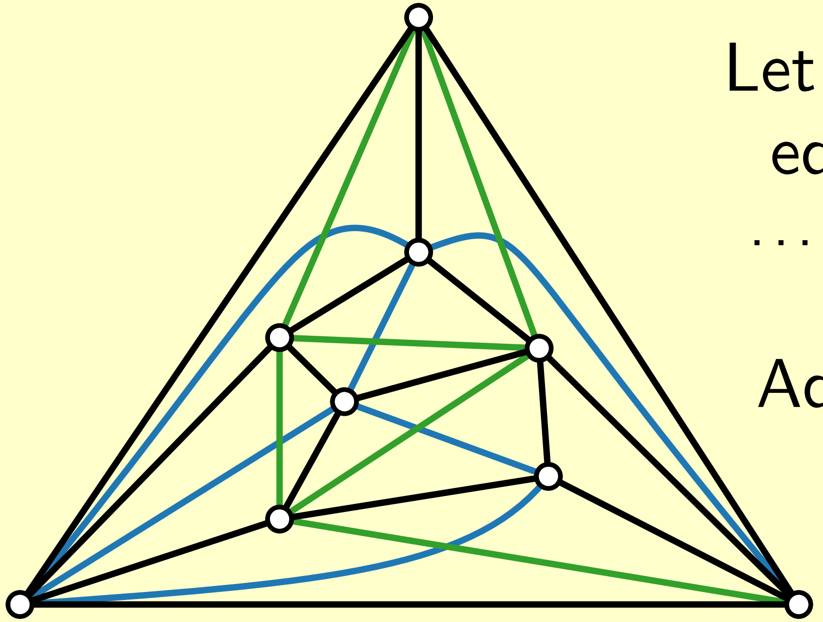

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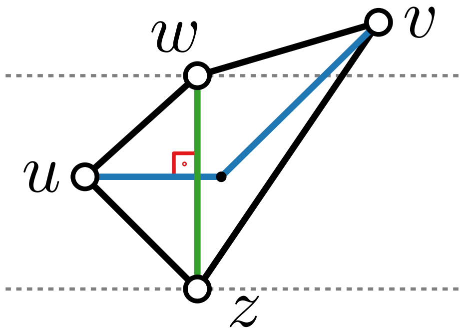


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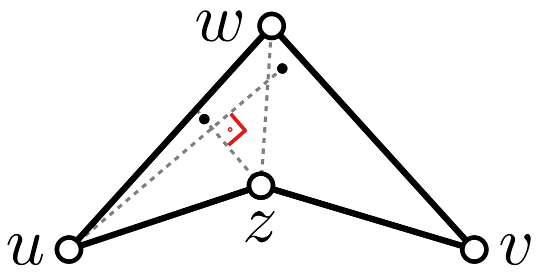


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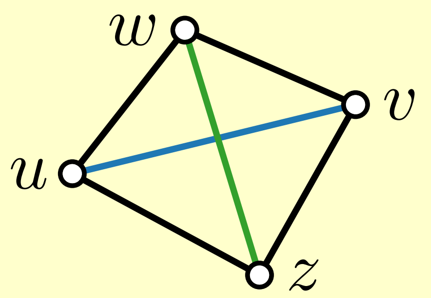


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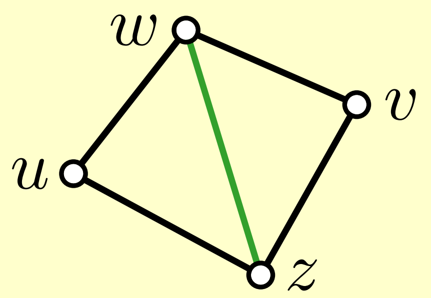


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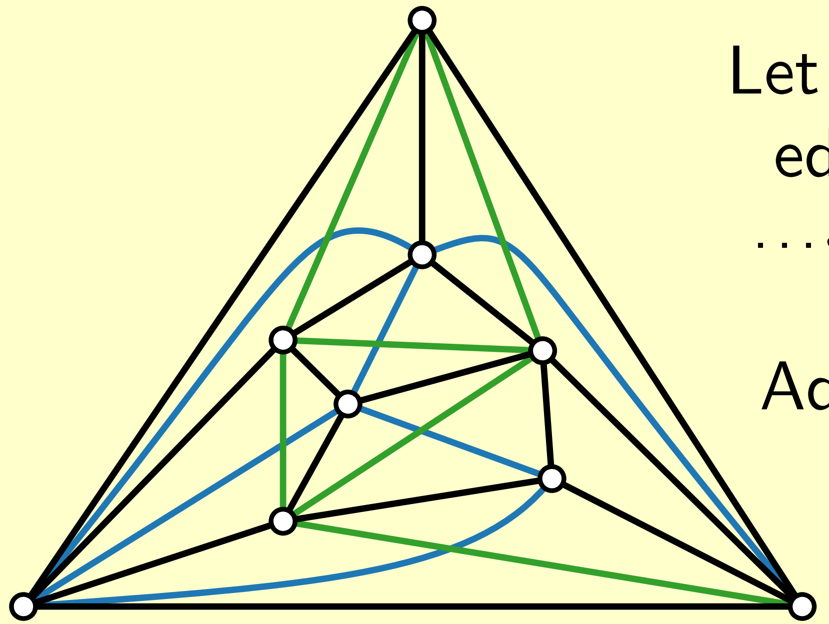

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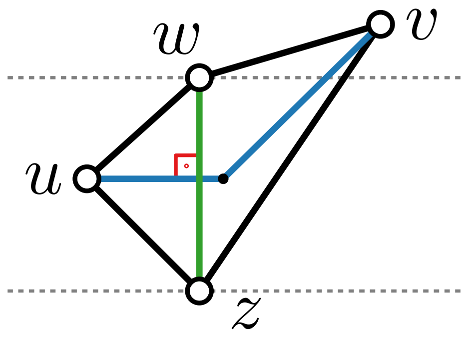


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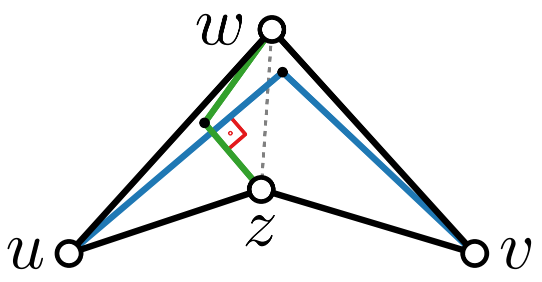


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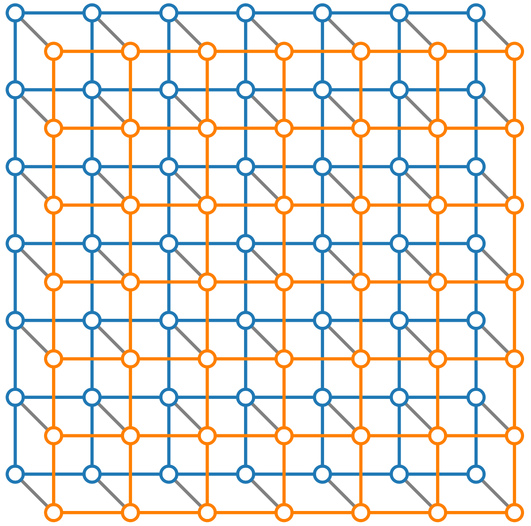
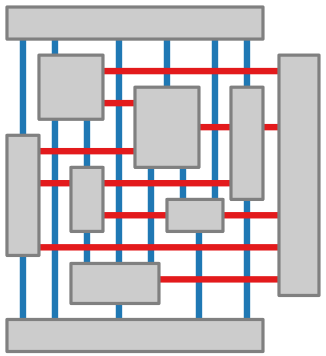
UNIVERSITÄT
WÜRZBURG

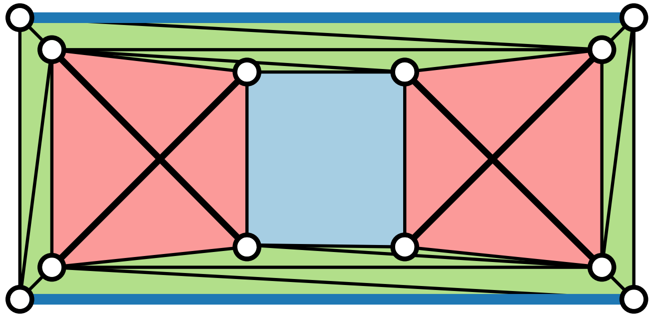
# Visualization of Graphs

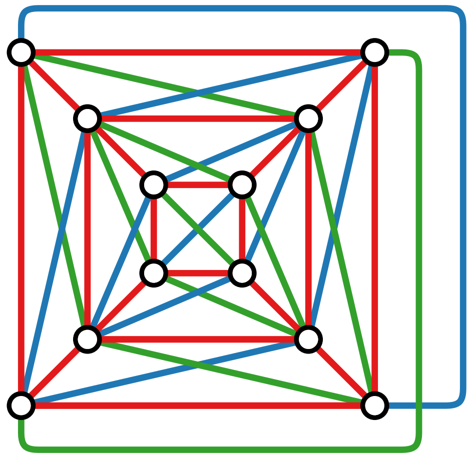




## Lecture 11:
## Beyond Planarity
### Drawing Graphs with Crossings

## Part V:
## 1-Planar 1-Bend RAC Drawings





Jonathan Klawitter

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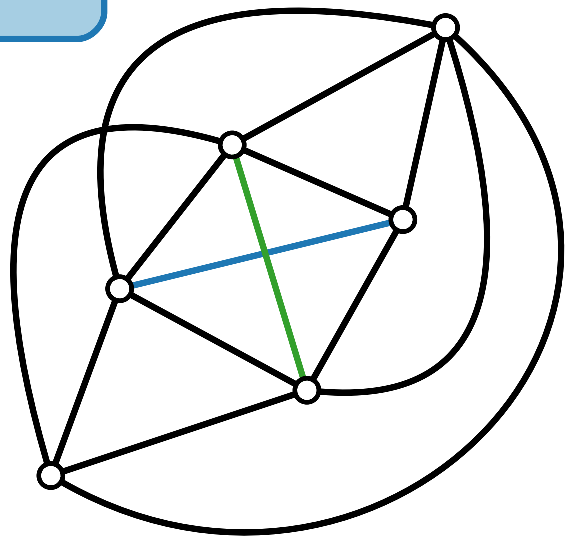
**Observation.**
In a triangulated 1-plane graph (not necessarily simple), each pair of crossing edges of $G$ forms an (empty) kite,



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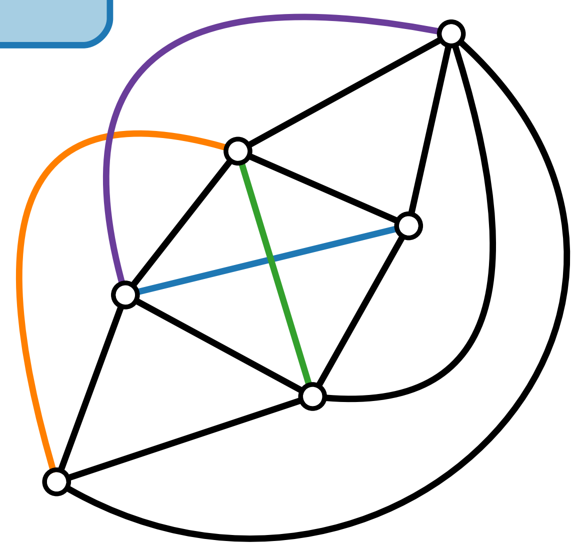
Also, if a 1-planar embedding of $G$ is given as part of the input, Γ can be computed in $\mathcal{O}(n)$ time.

**Observation.**
In a triangulated 1-plane graph (not necessarily simple), each pair of crossing edges of $G$ forms an (empty) kite, except for at most one pair if their crossing point is on the outer face of $G$.



# 1-Planar 1-Bend RAC Drawings

**Theorem.** [Bekos, Didimo, Liotta, Mehrabi & Montecchiani 2017]
Every 1-planar graph $G$ on $n$ vertices admits a 1-planar 1-bend RAC drawing $\Gamma$.
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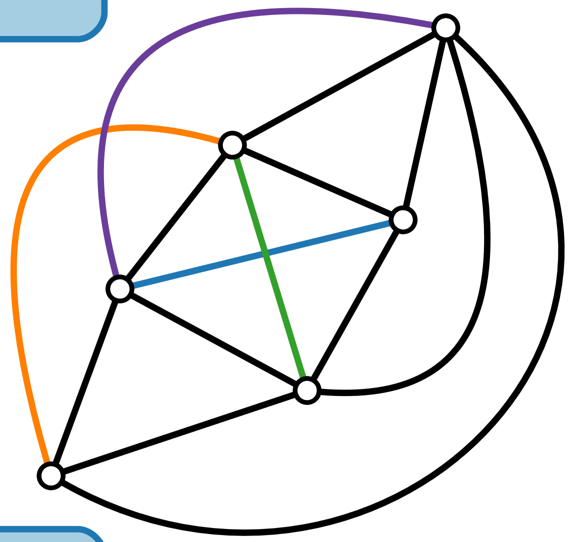
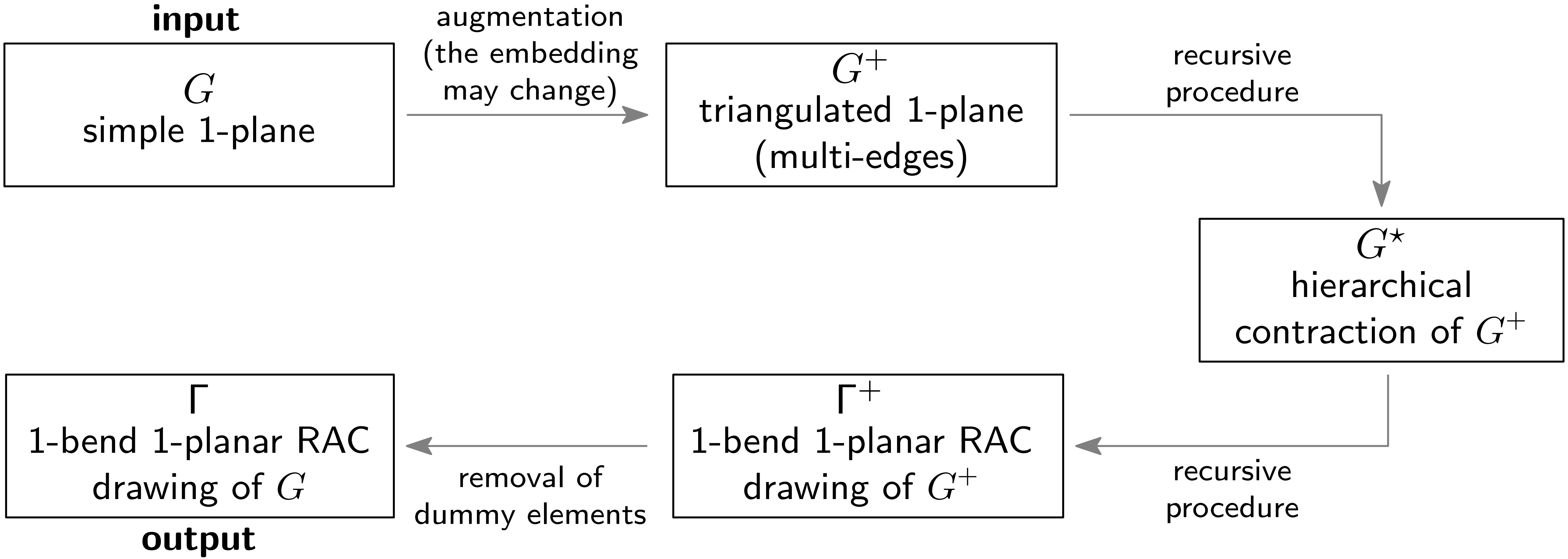
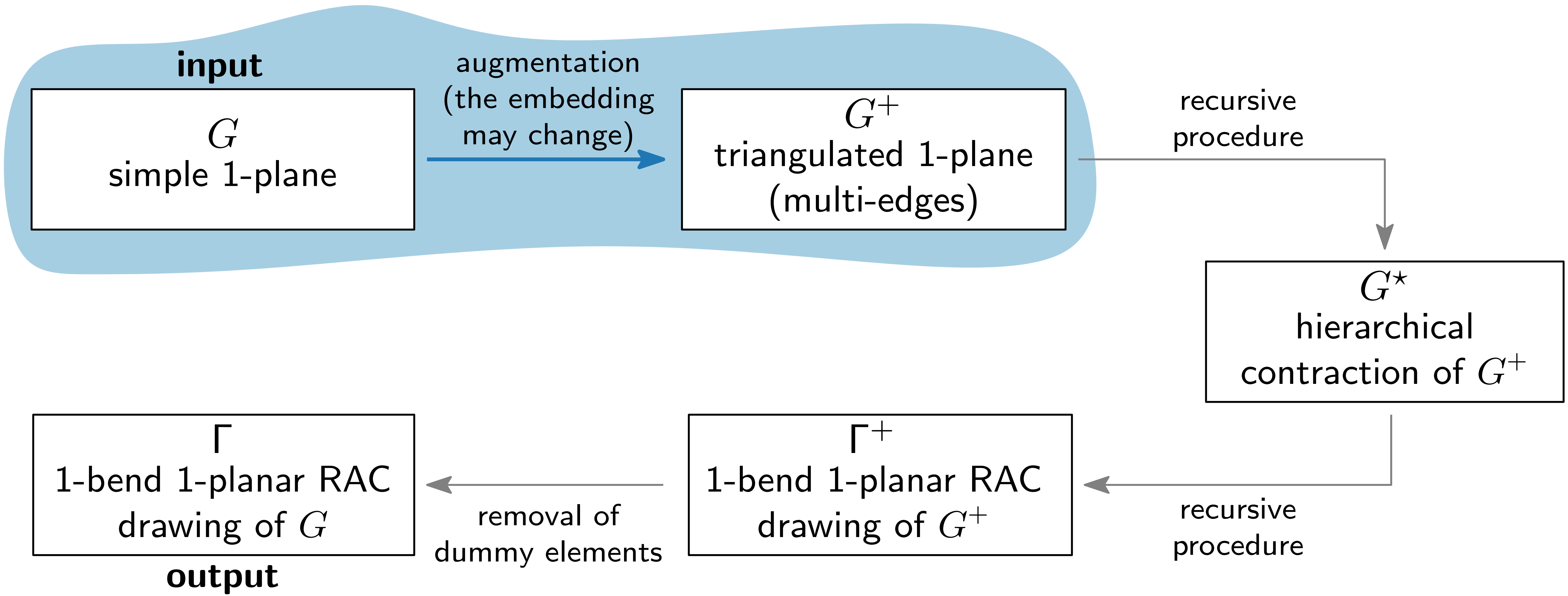


**Theorem.** [Chiba, Yamanouchi & Nishizeki 1984]
For every planar graph $G$ and convex polygon $P$, a strictly convex planar straight-line drawing of $G$ where the outer face coincides with $P$ can be computed in $O(n)$ time.

# Algorithm Outline

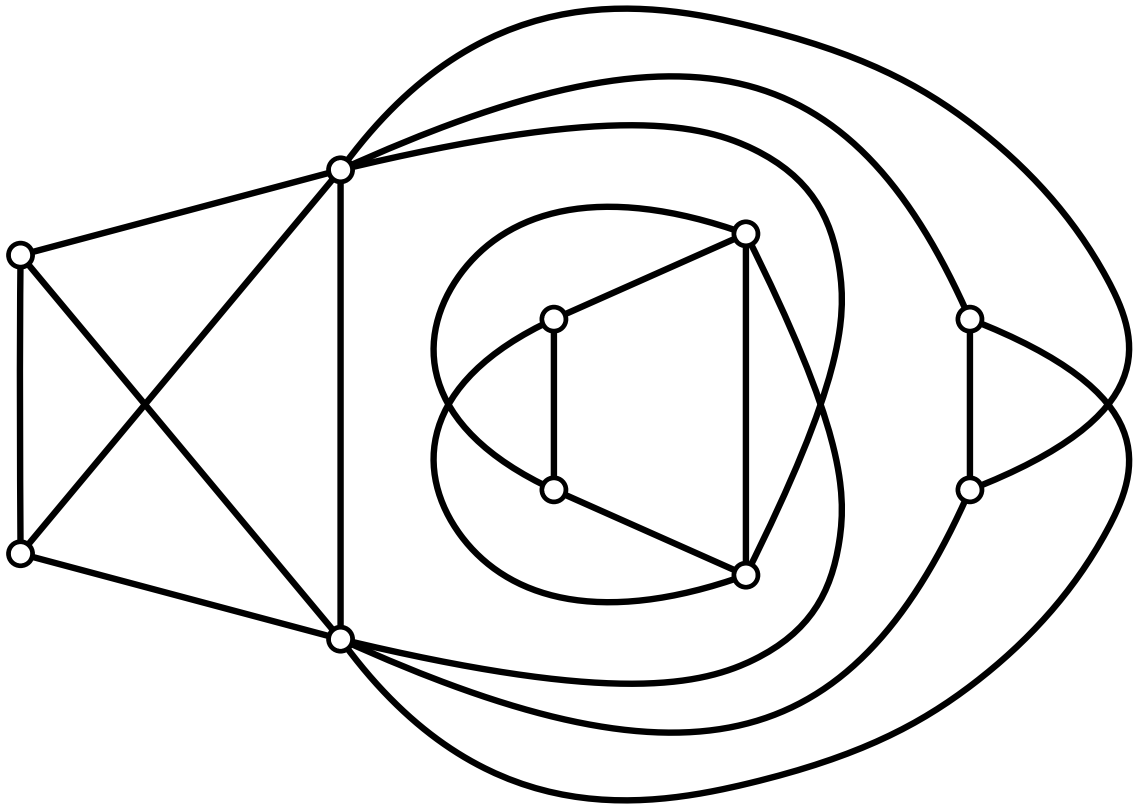


# Algorithm Outline



# Algorithm Step 1: Augmentation

$G$
simple 1-plane



# Algorithm Step 1: Augmentation

1. For each pair of crossing edges add an enclosing 4-cycle.

$G$
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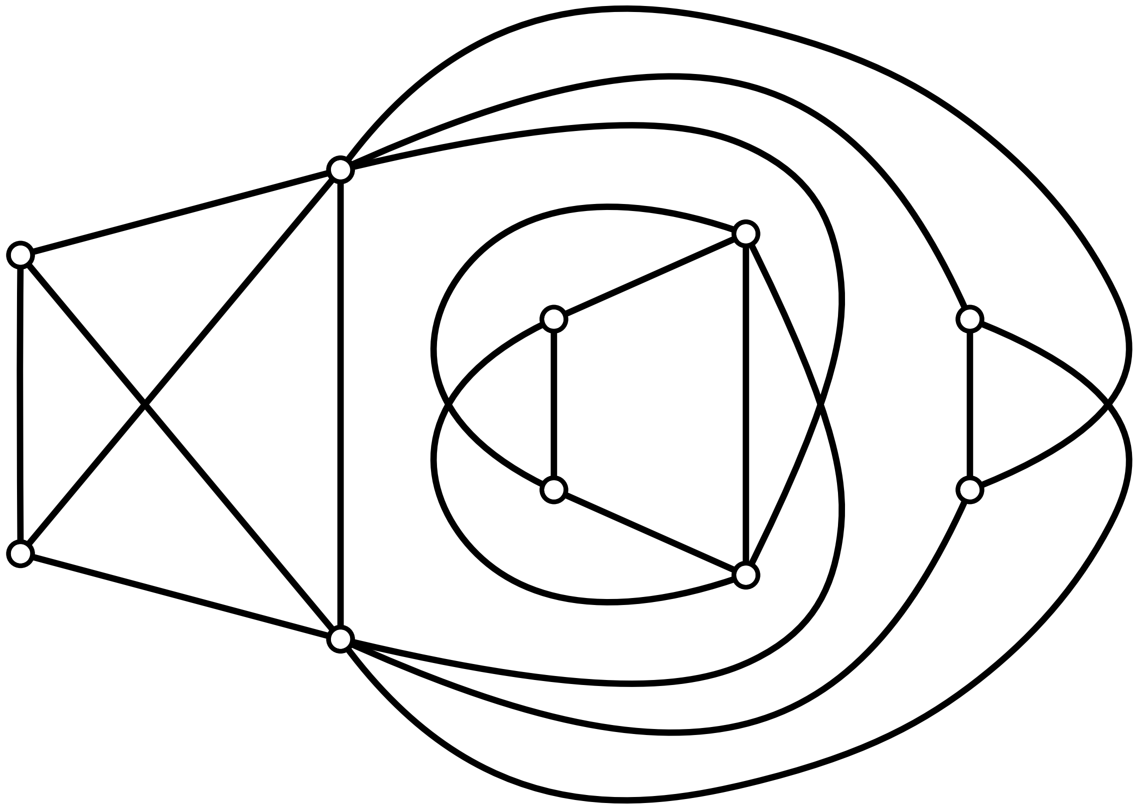

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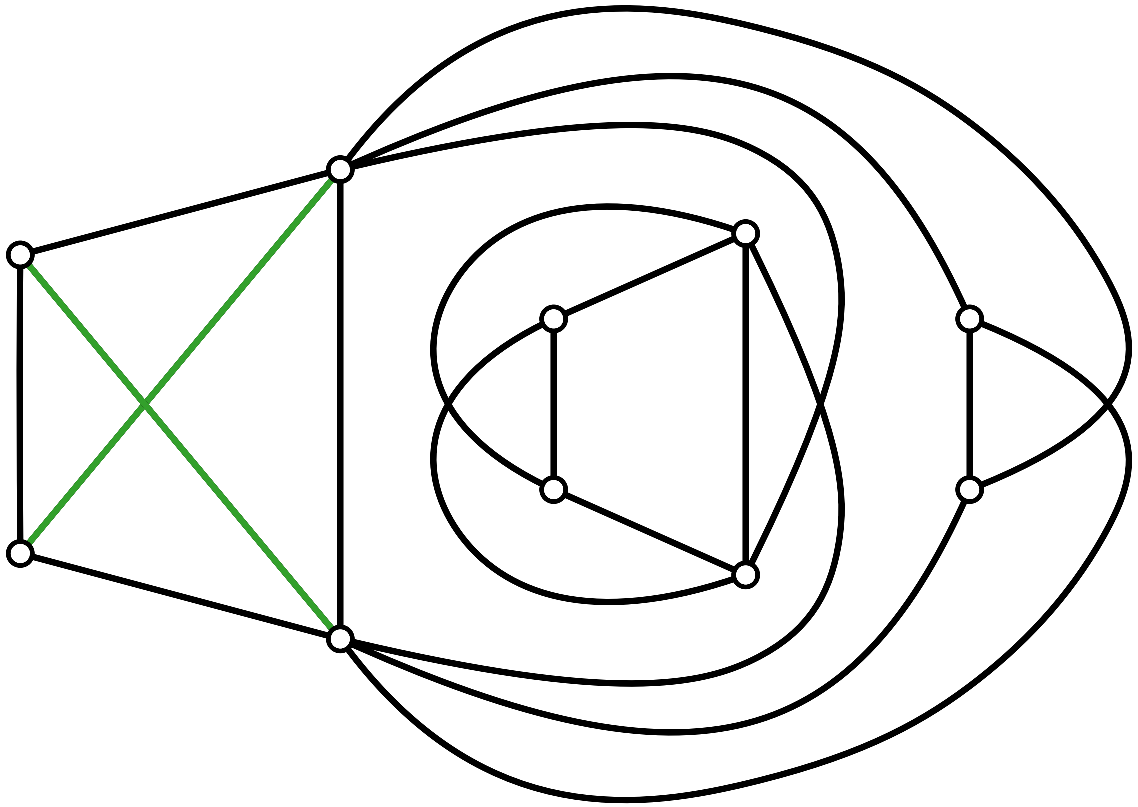

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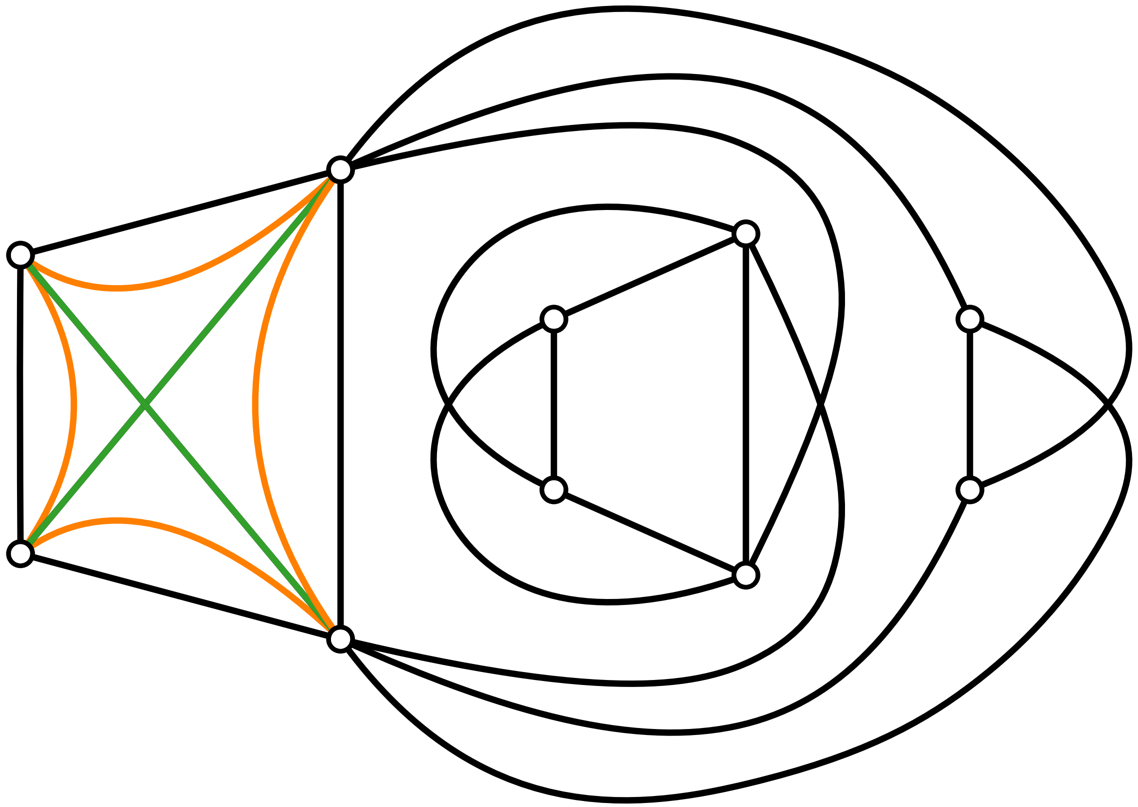

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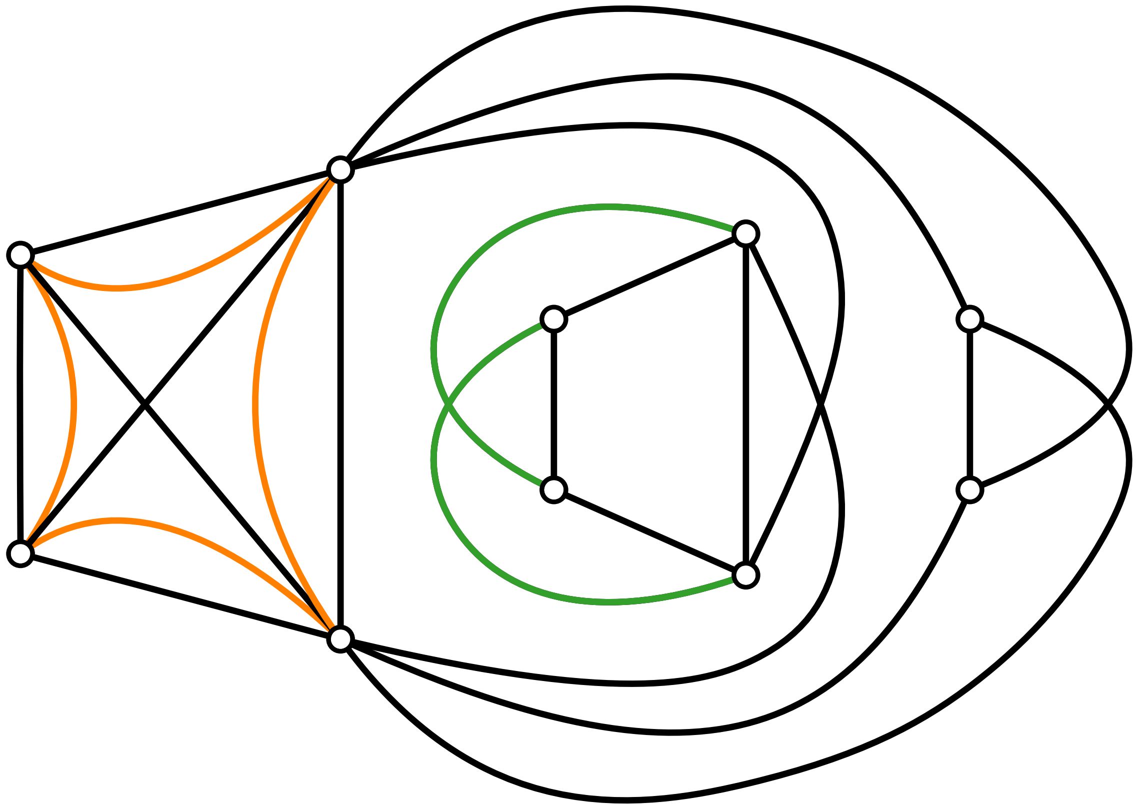

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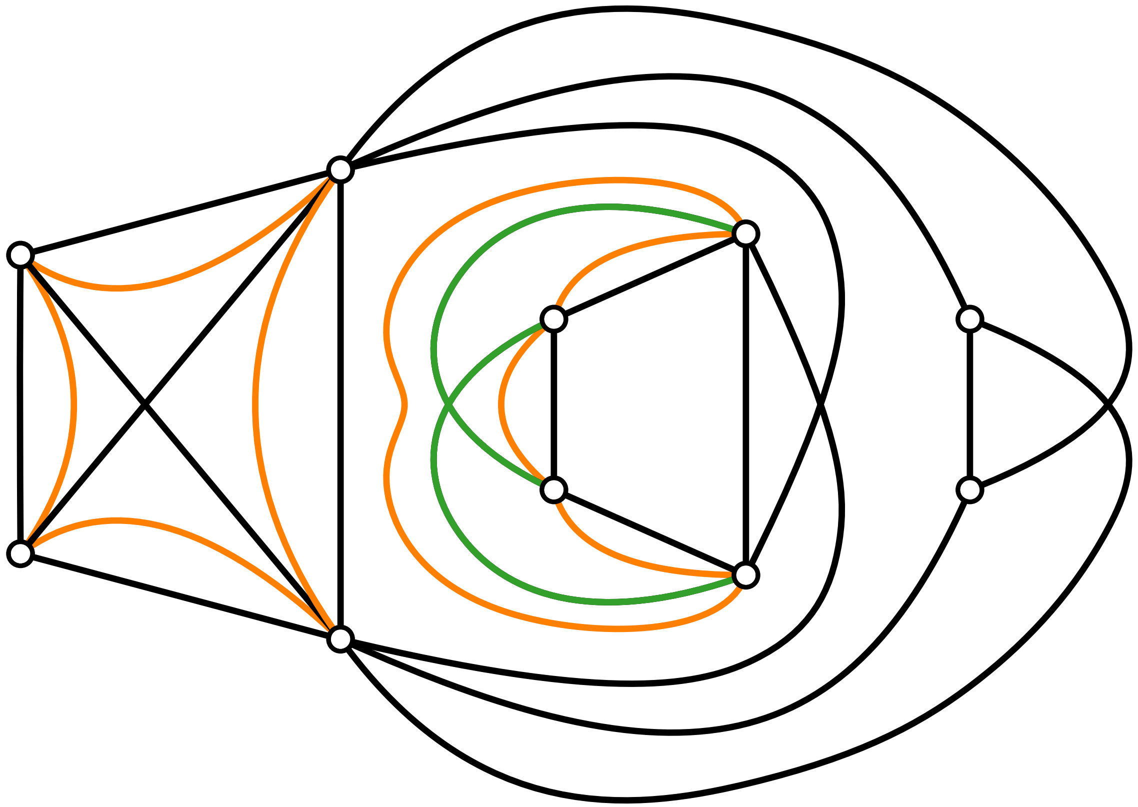

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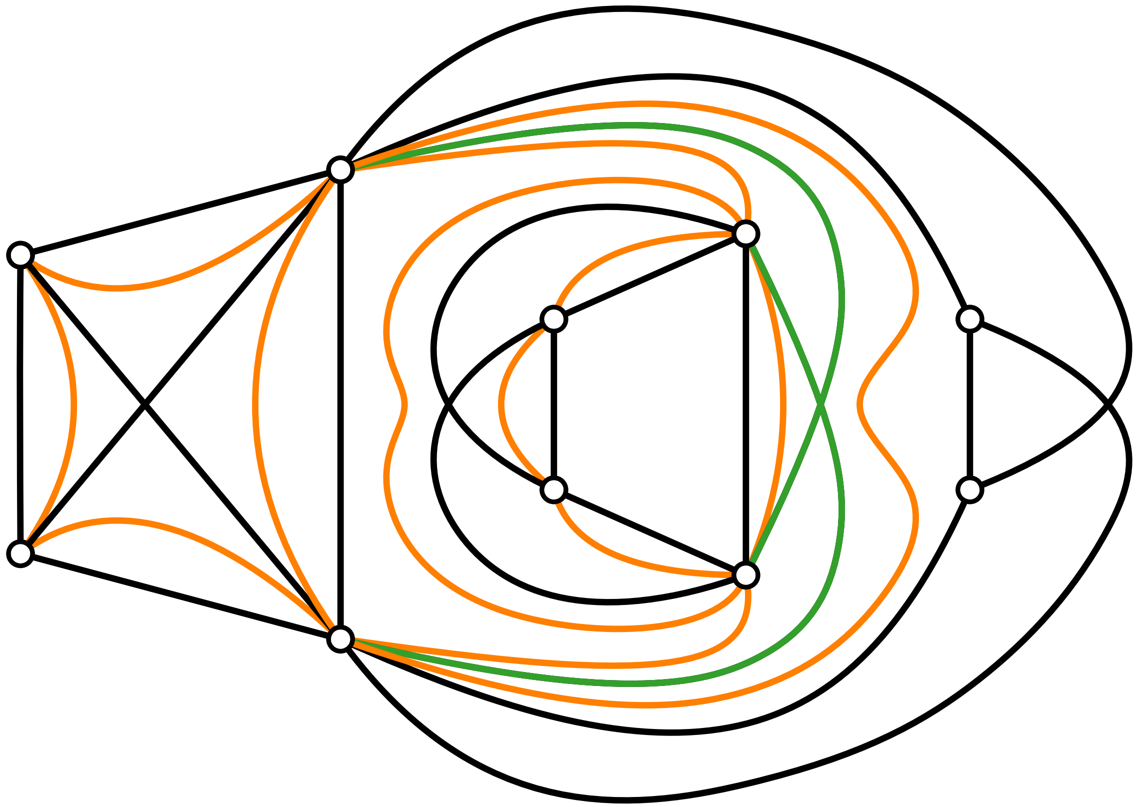

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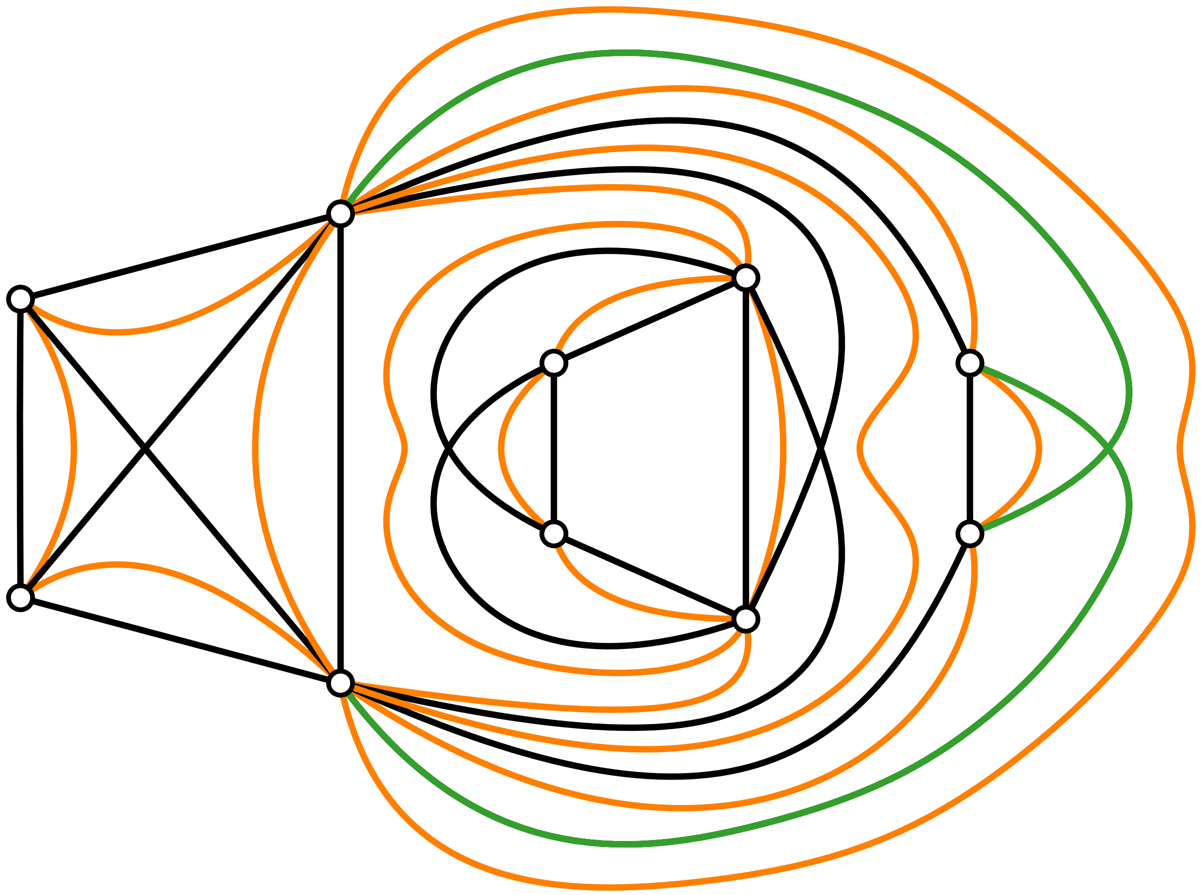
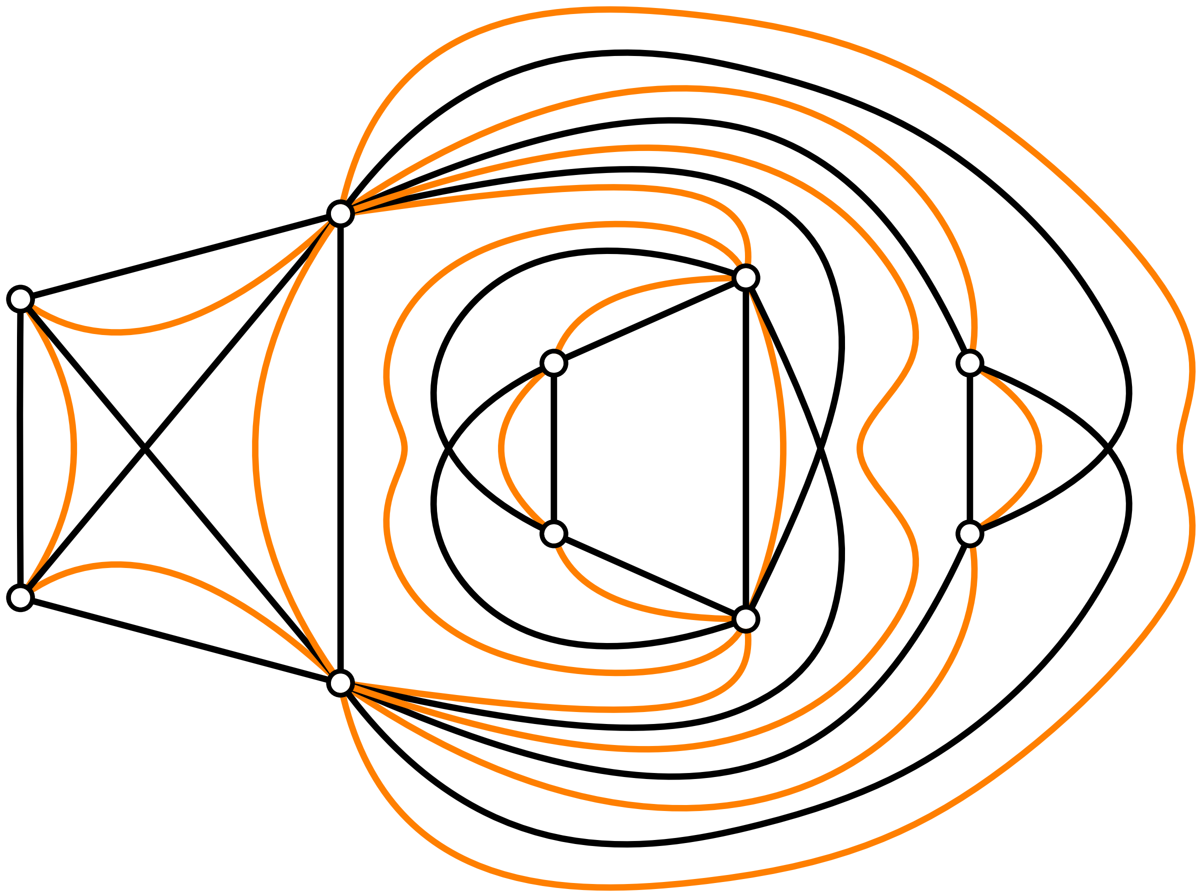


# Algorithm Step 1: Augmentation

1. For each pair of crossing edges add an enclosing 4-cycle.

2. Remove those multiple edges that belong to $G$.



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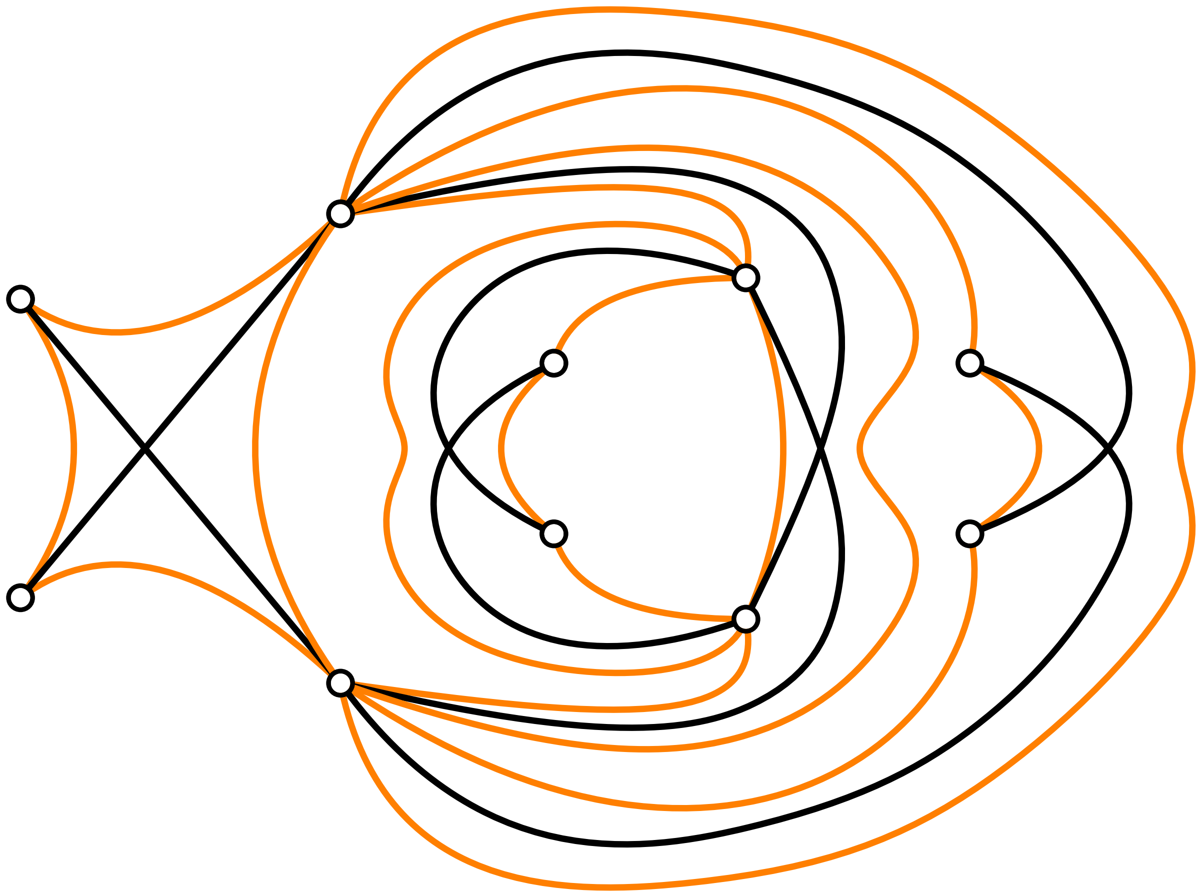
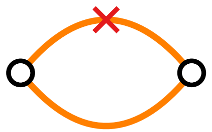


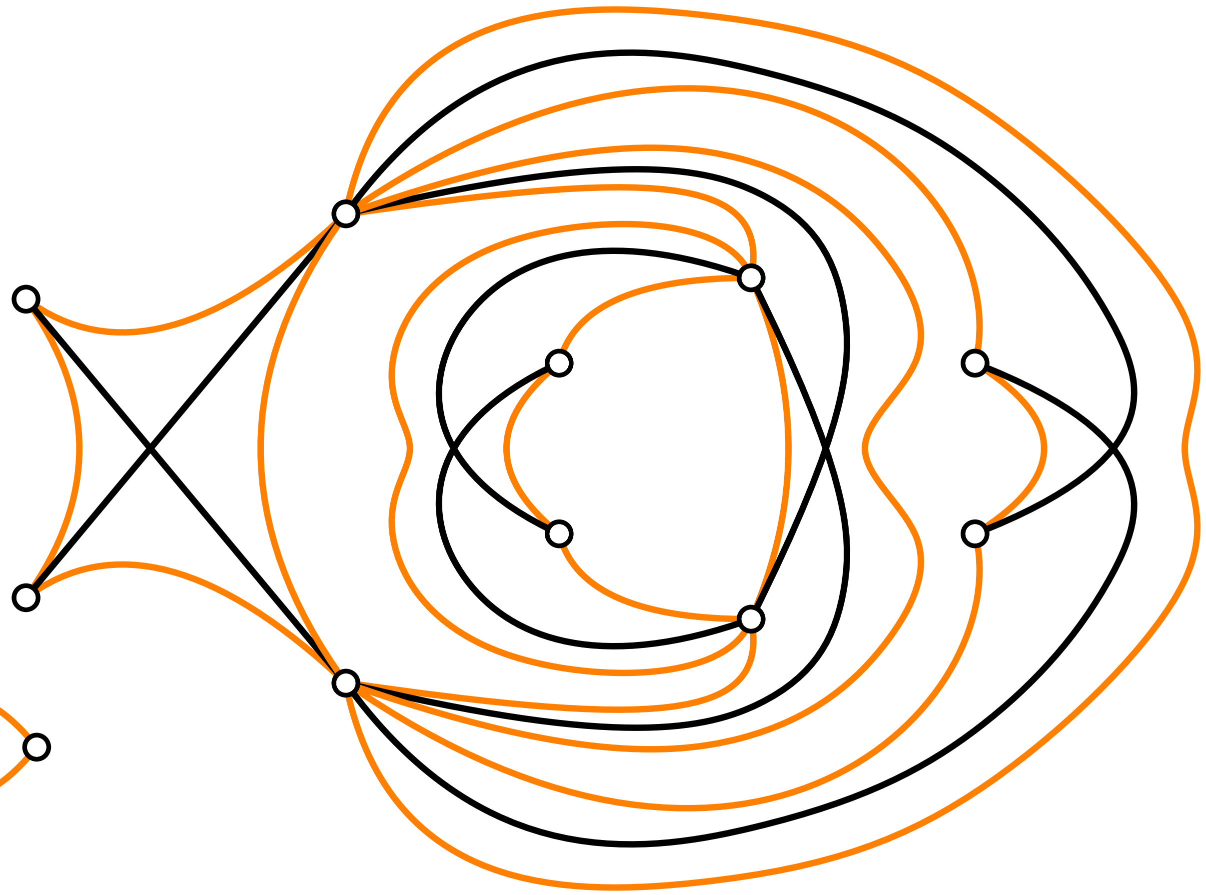
# Algorithm Step 1: Augmentation

1. For each pair of crossing edges add an enclosing 4-cycle.

2. Remove those multiple edges that belong to $G$.

3. Remove one (multiple) edge from each face of degree two (if any).





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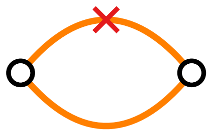
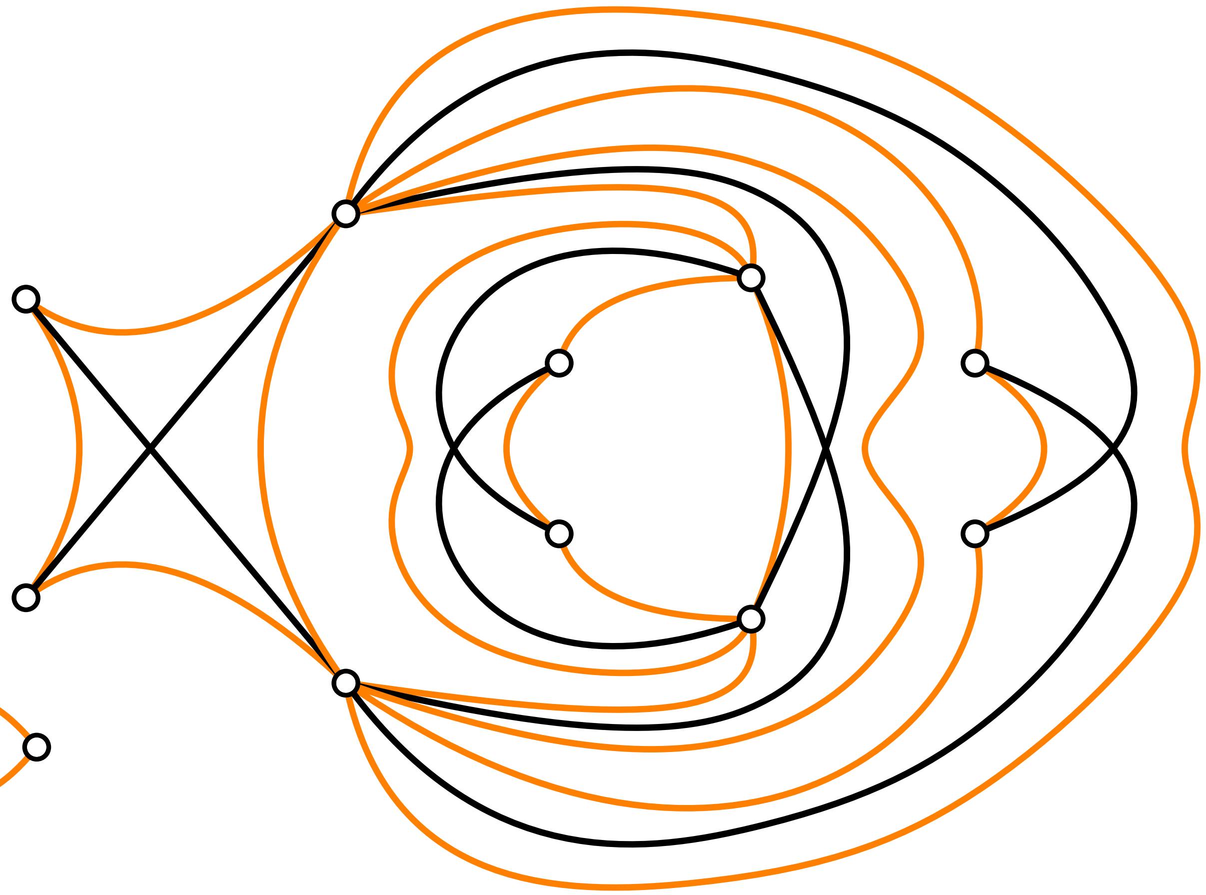


4. Triangulate faces of degree $> 3$ by inserting a star inside them.



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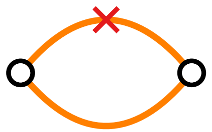

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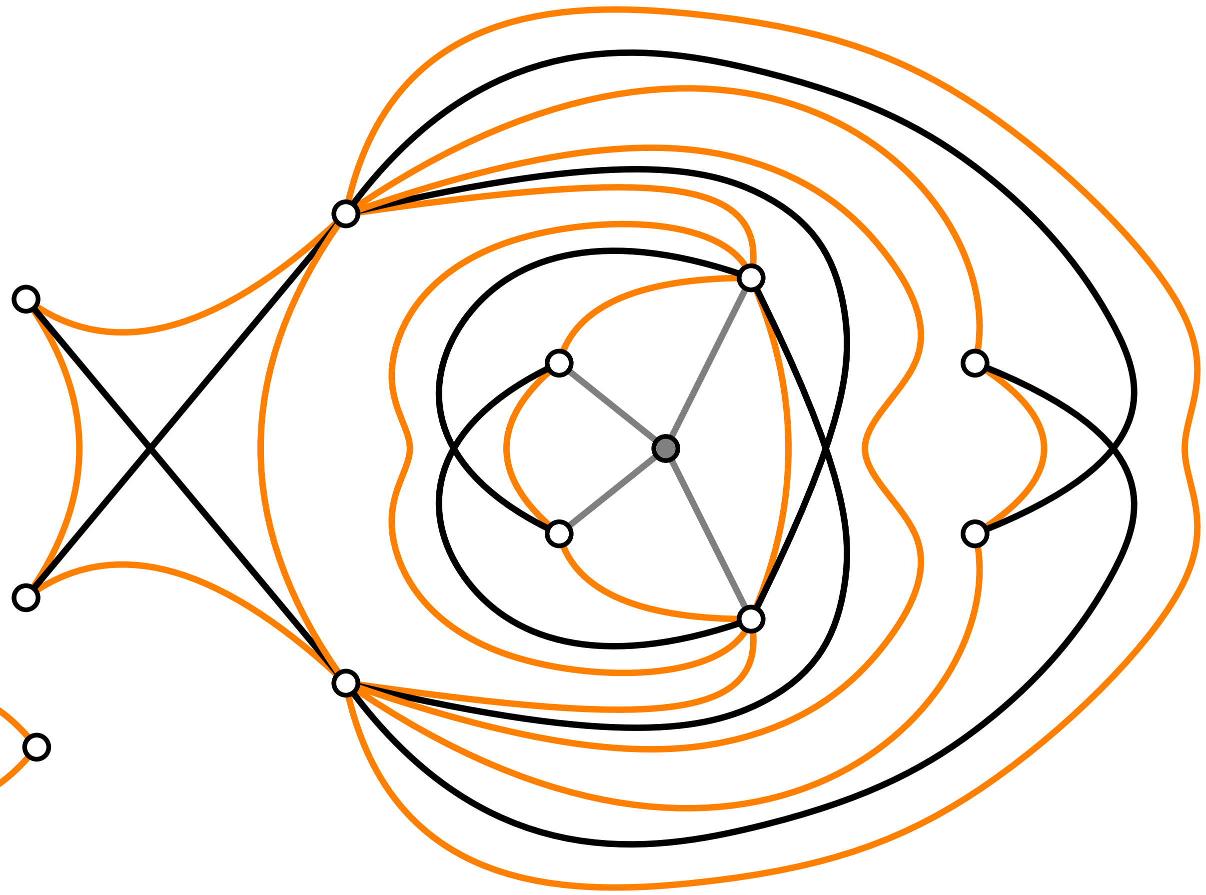

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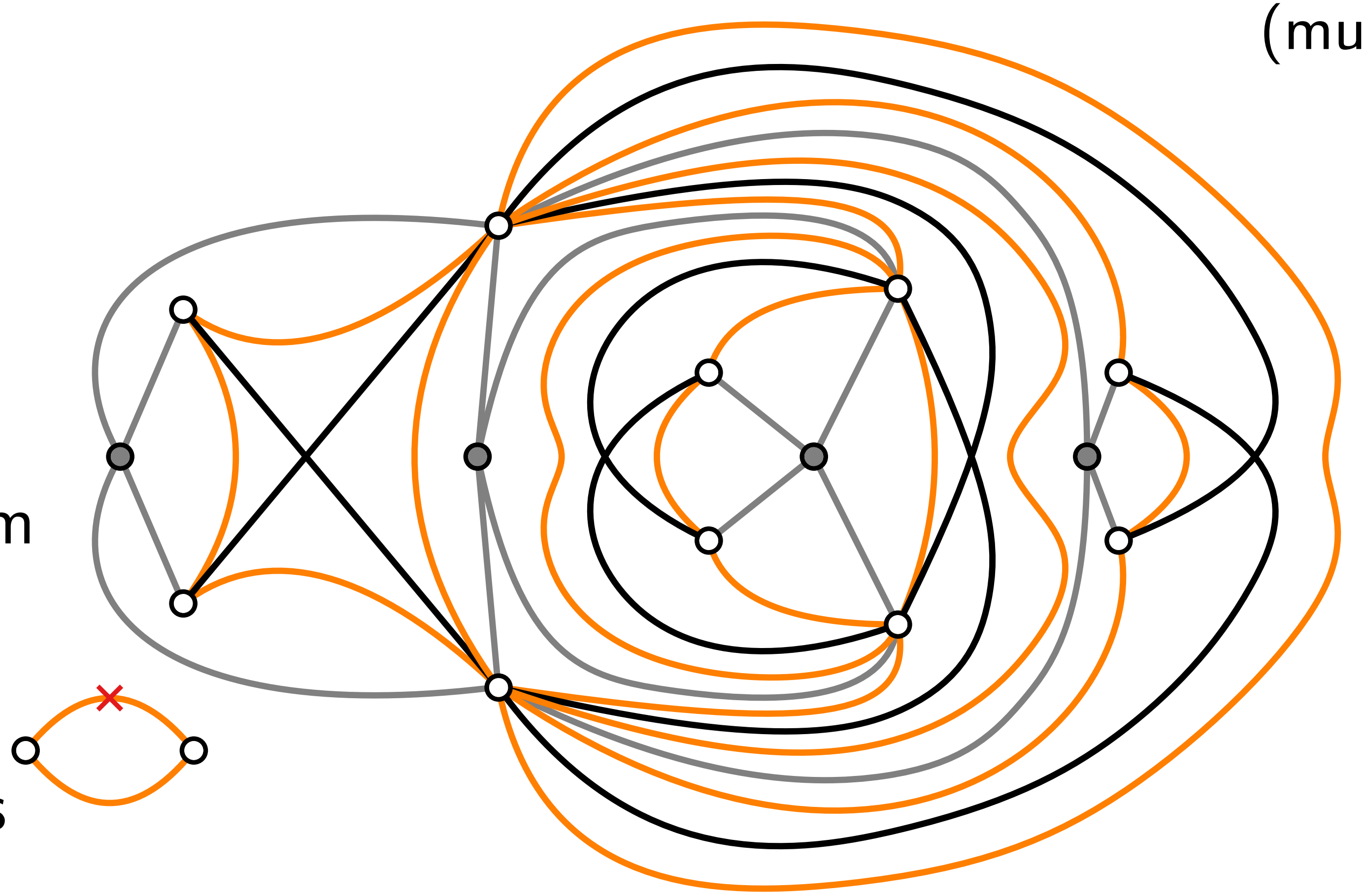
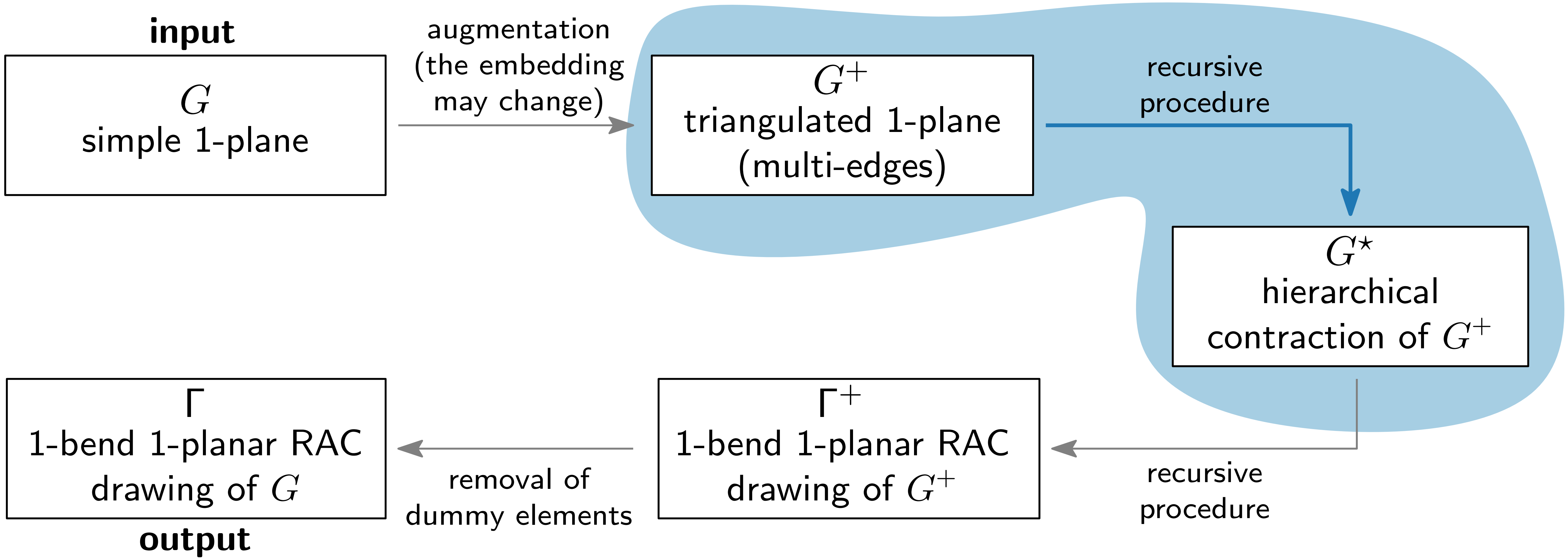


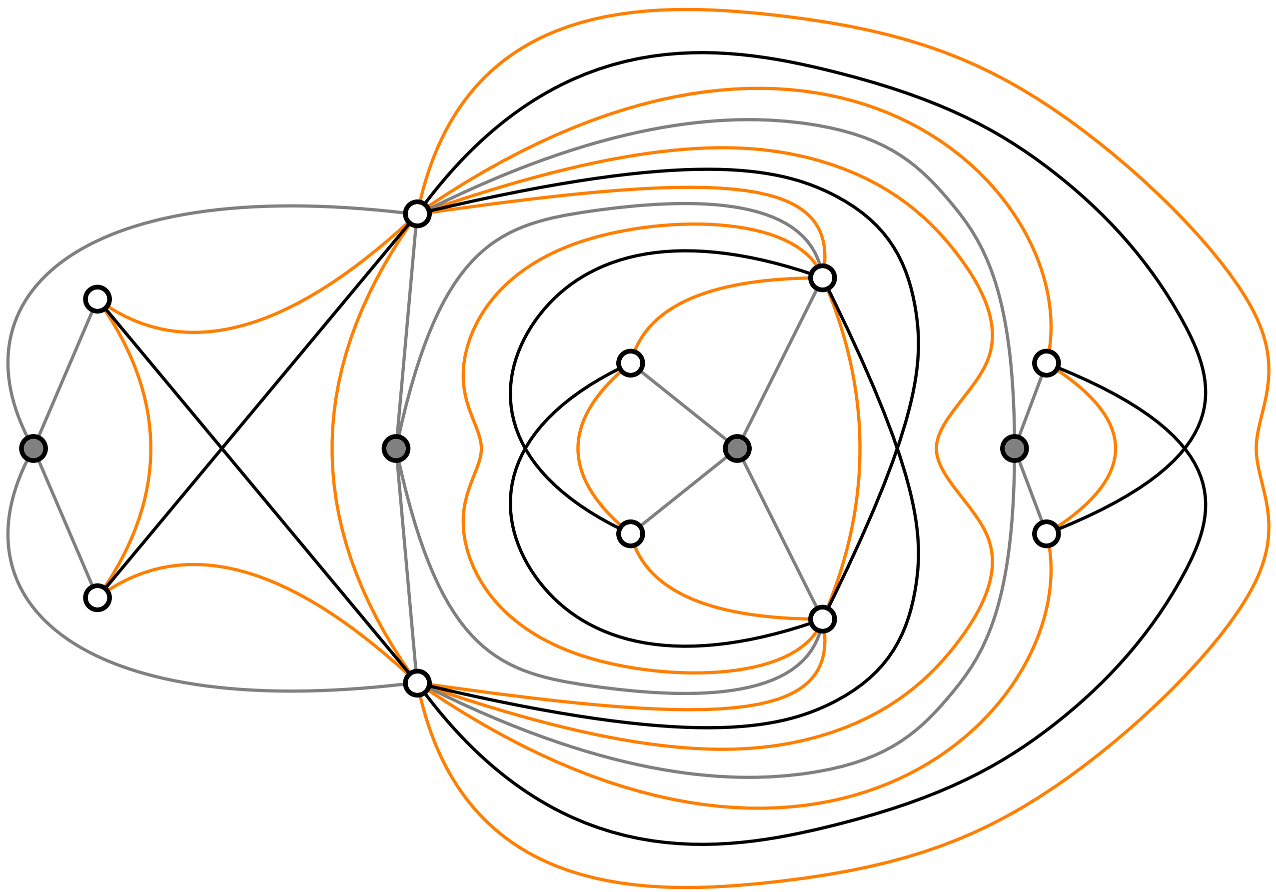
$G^+$
triangulated 1-plane
(multi-edges)

# Algorithm Outline



# Algoritm Step 2: Hierarchical Contractions

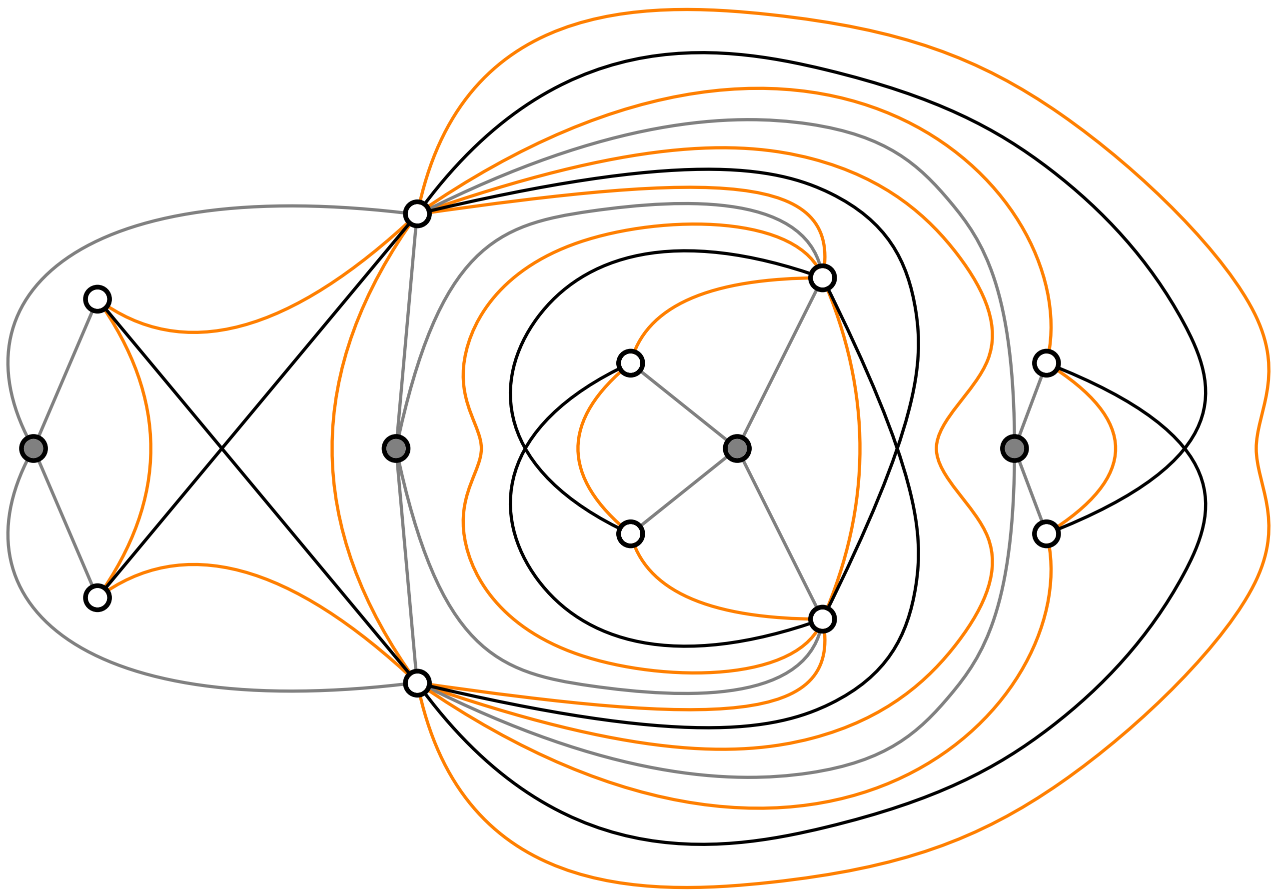
$G^+$
triangulated 1-plane
(multi-edges)



# Algoritm Step 2: Hierarchical Contractions

$G^+$
triangulated 1-plane
(multi-edges)

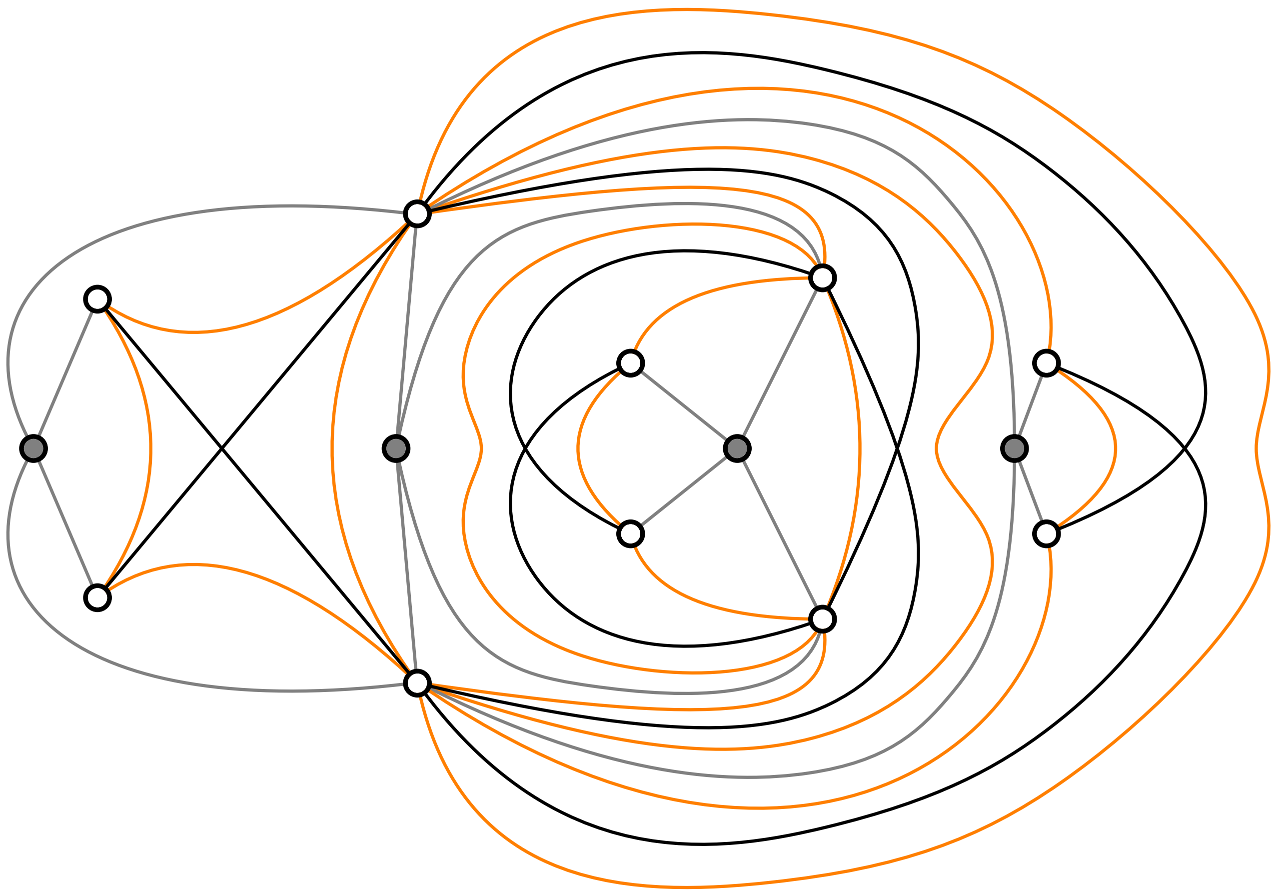
- triangular faces



# Algoritm Step 2: Hierarchical Contractions

$G^+$
triangulated 1-plane
(multi-edges)

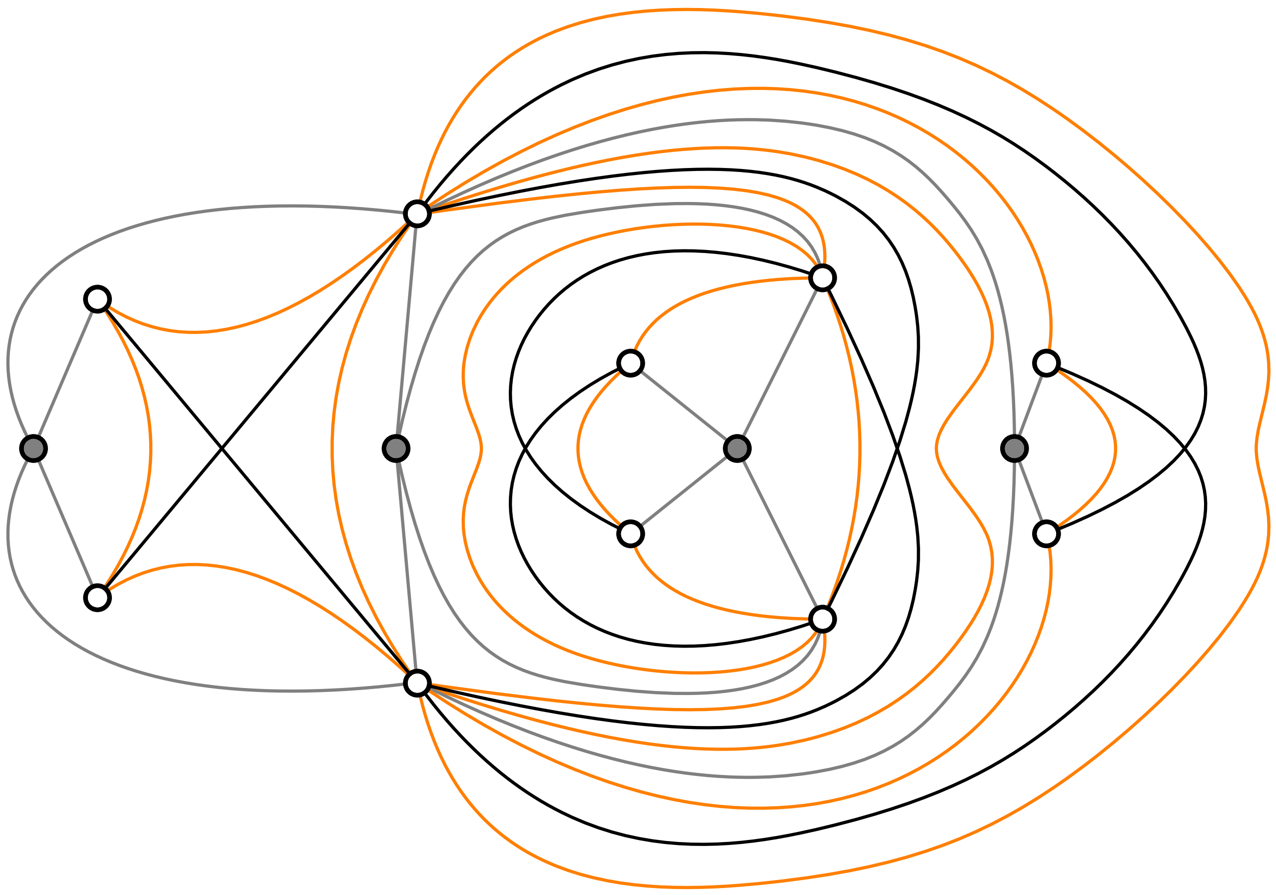
- triangular faces
- multiple edges never crossed



# Algoritm Step 2: Hierarchical Contractions

$G^+$
triangulated 1-plane
(multi-edges)

- triangular faces
- multiple edges never crossed
- only empty kites



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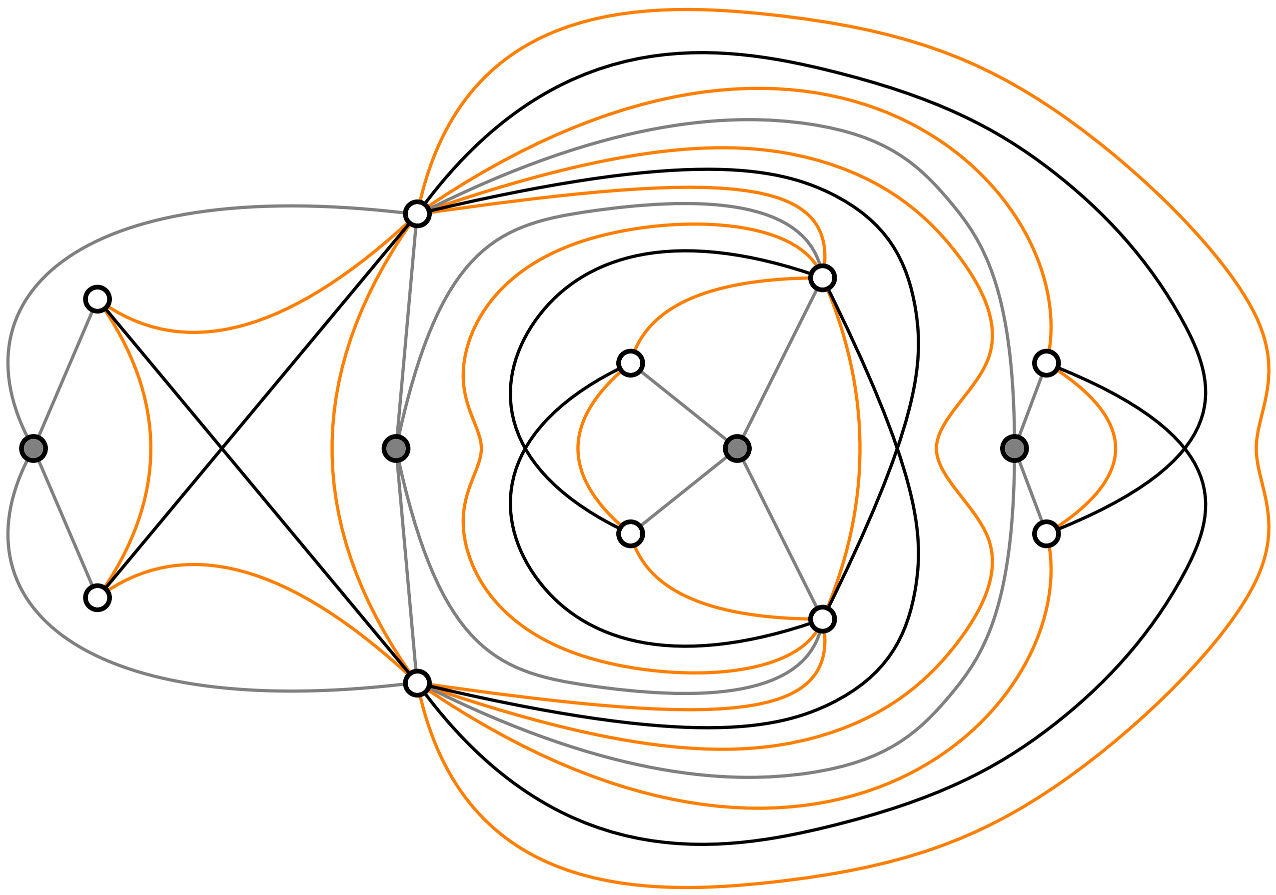
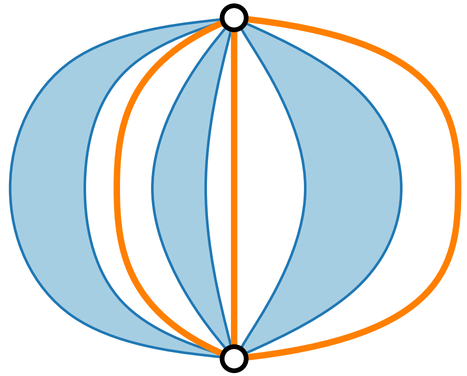


⇒



structure of each separation pair

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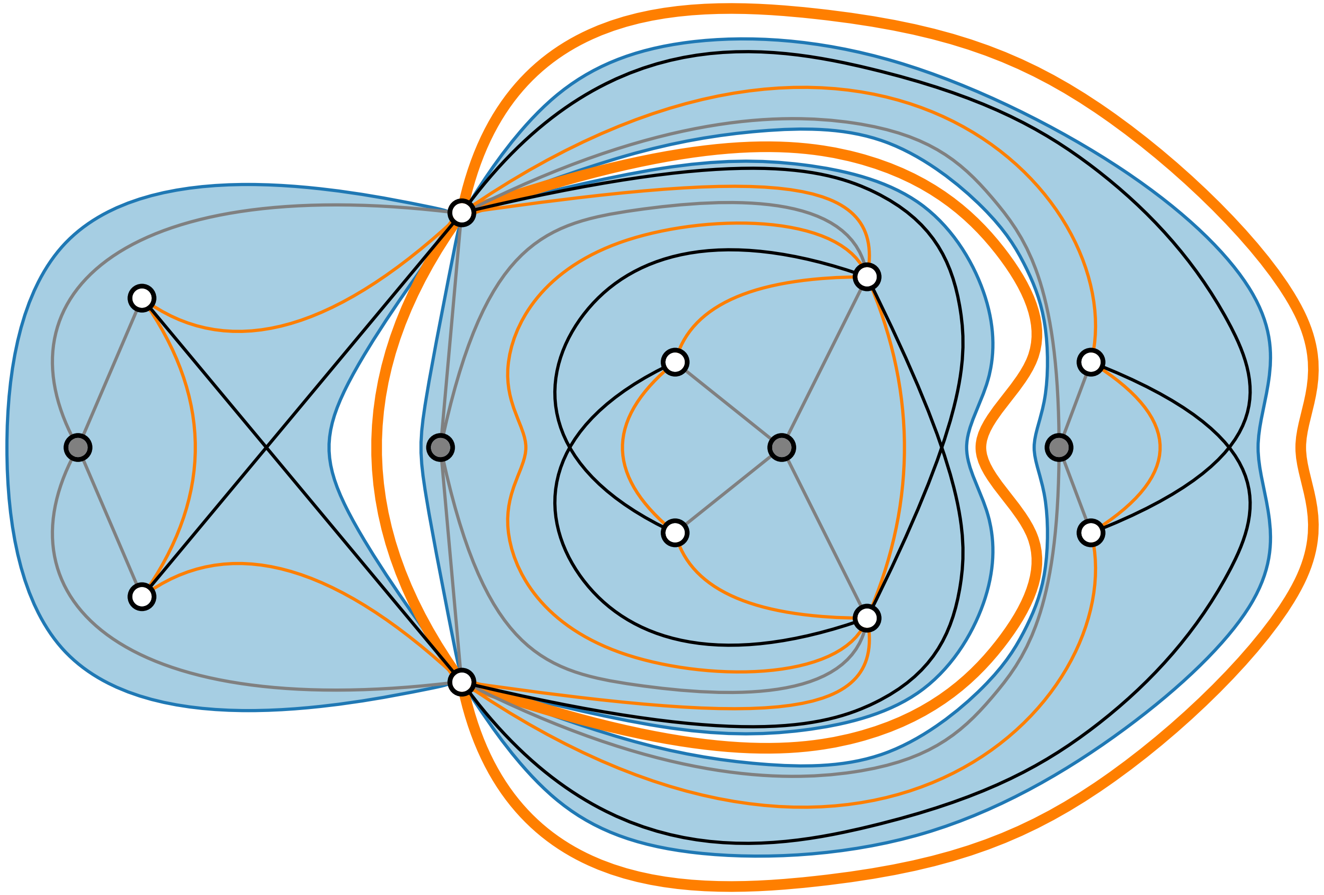
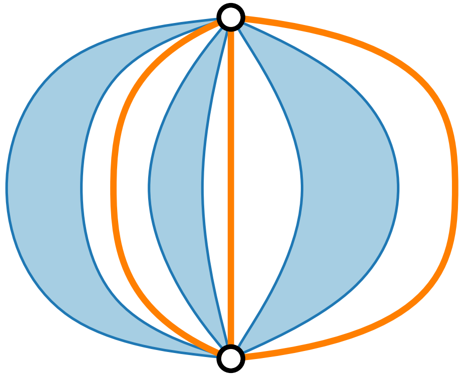


$\Rightarrow$



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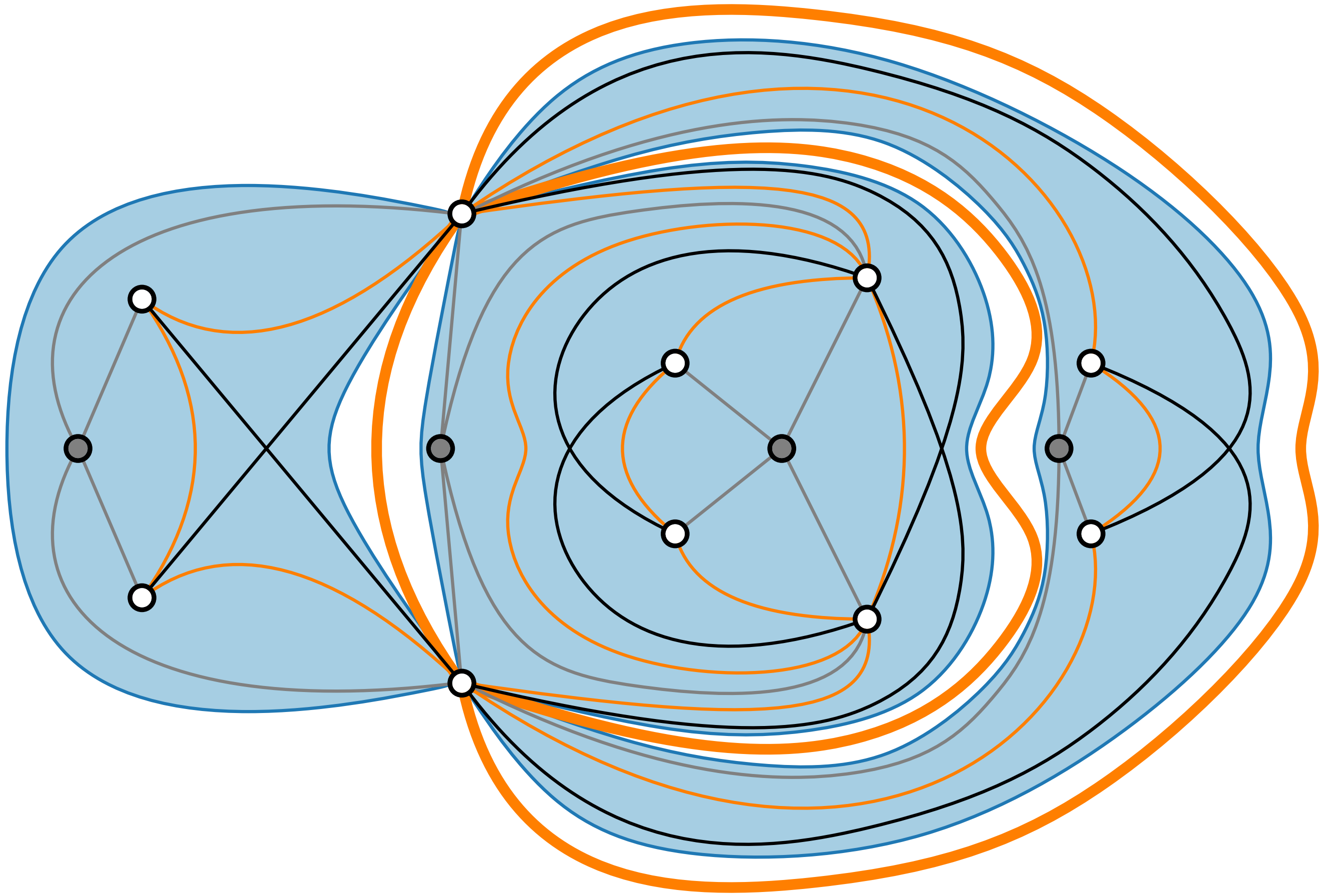

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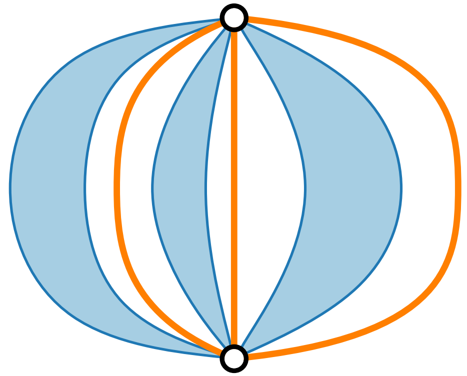

structure of each separation pair

Contract all inner components of each separation pair into a thick edge.

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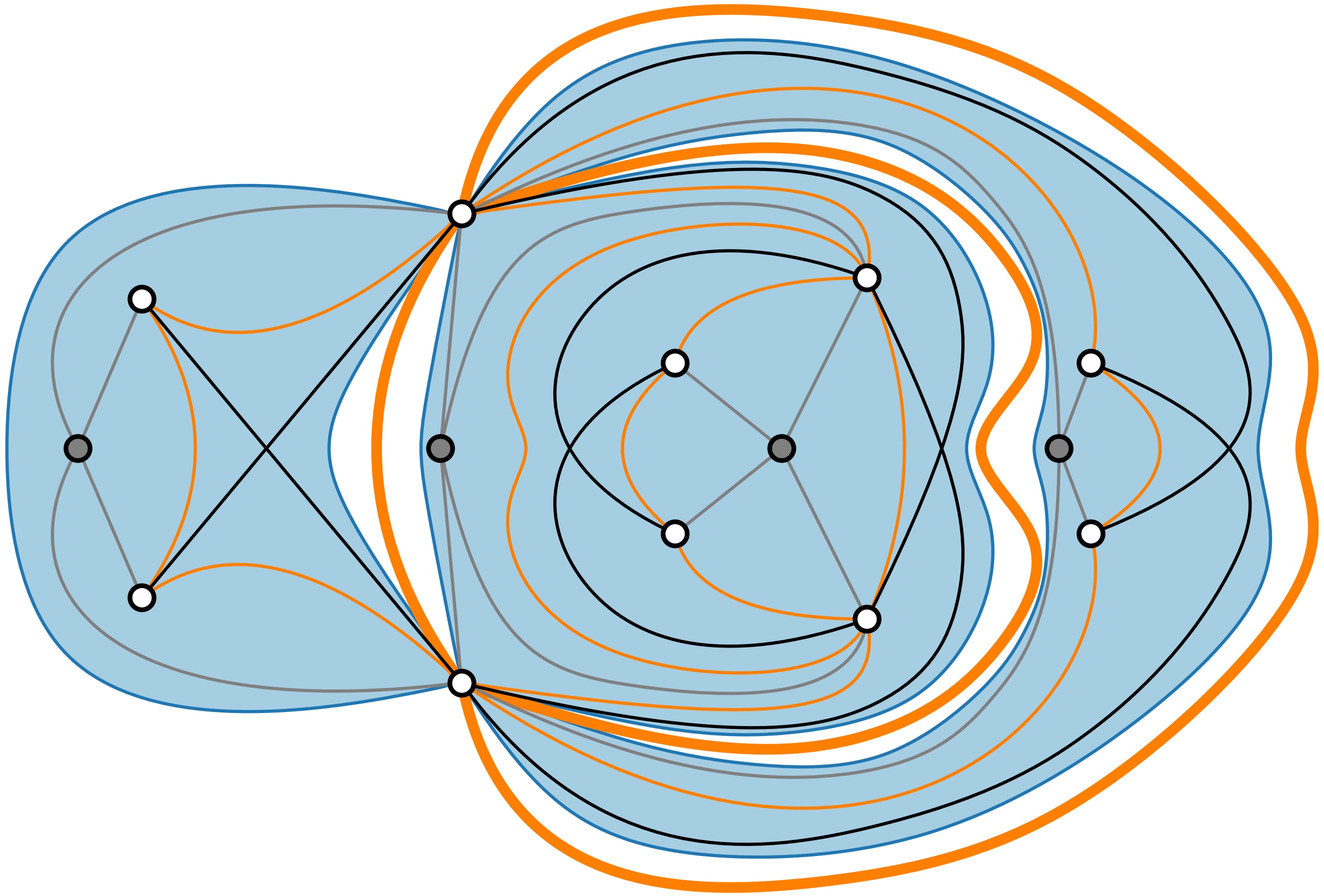

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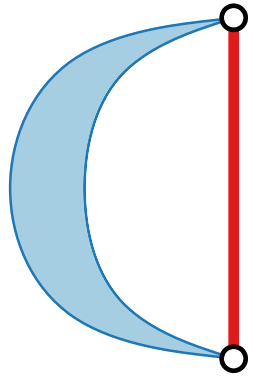

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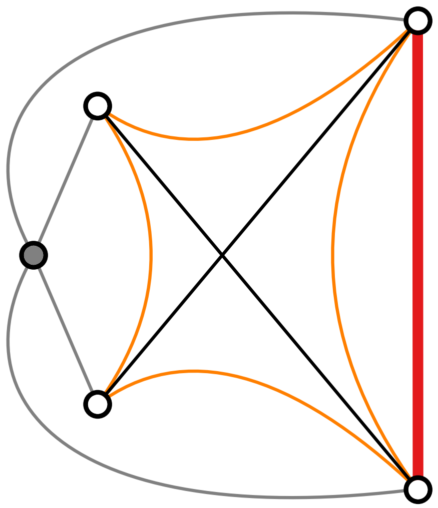

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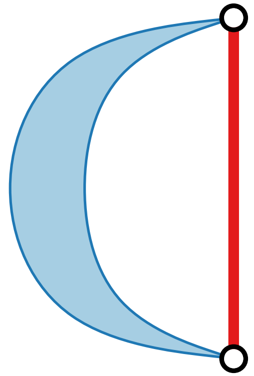

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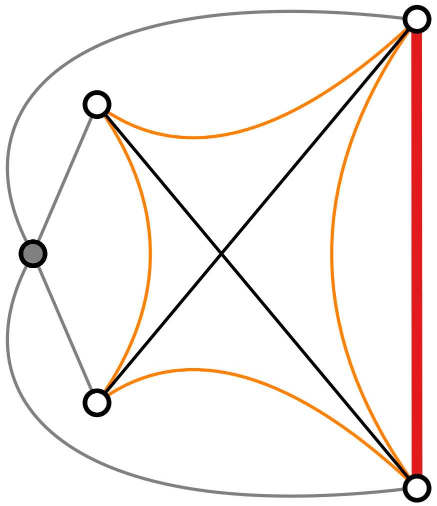

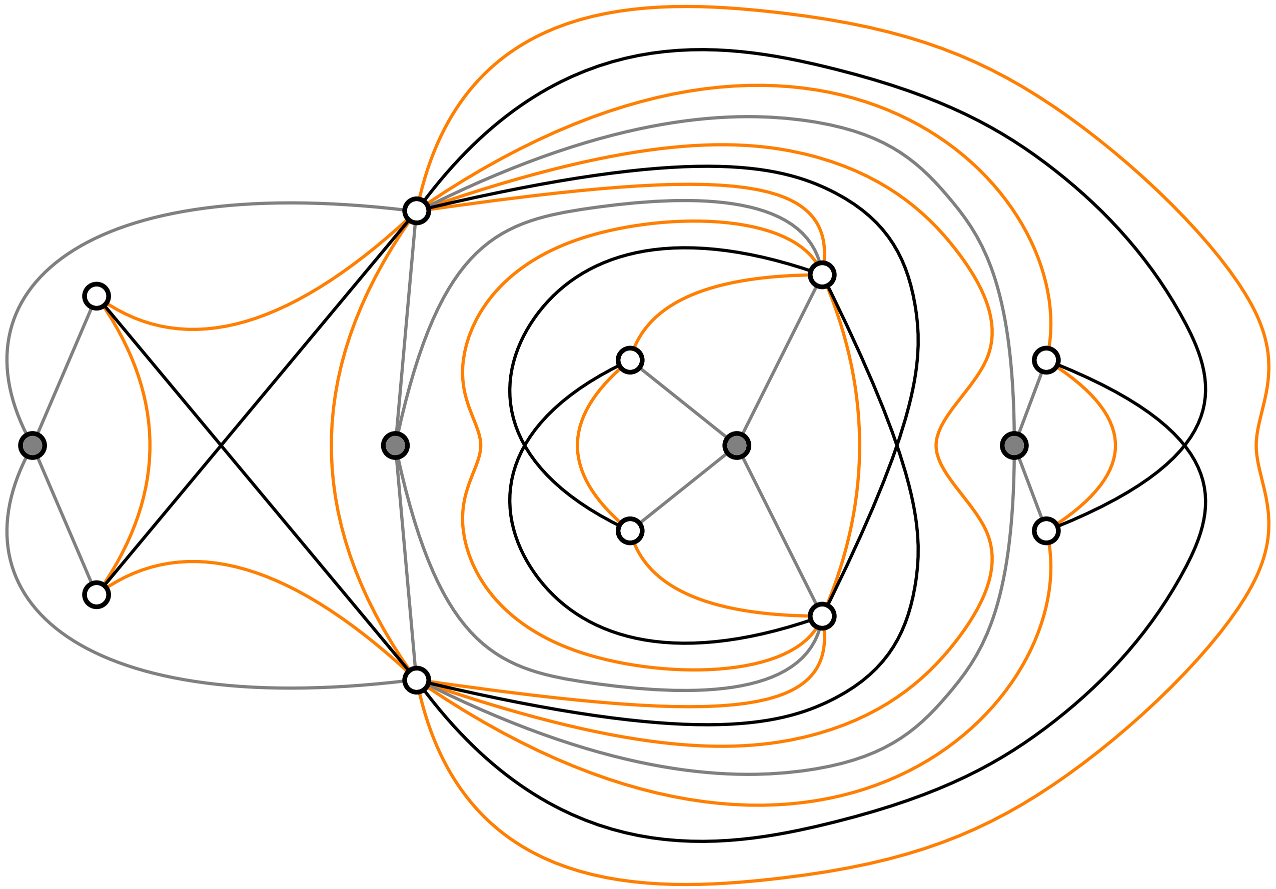

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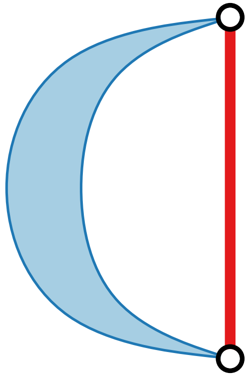

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# Algoritm Step 2: Hierarchical Contractions

$G^+$
triangulated 1-plane
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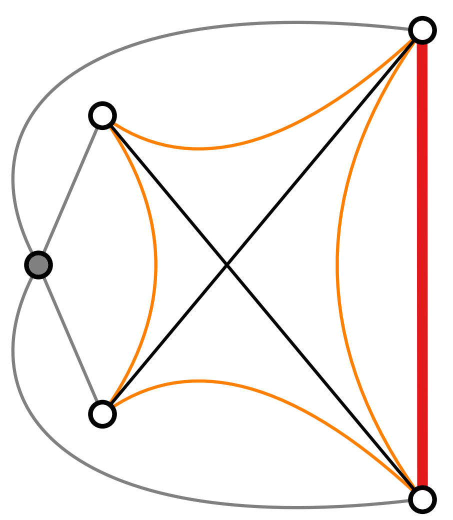

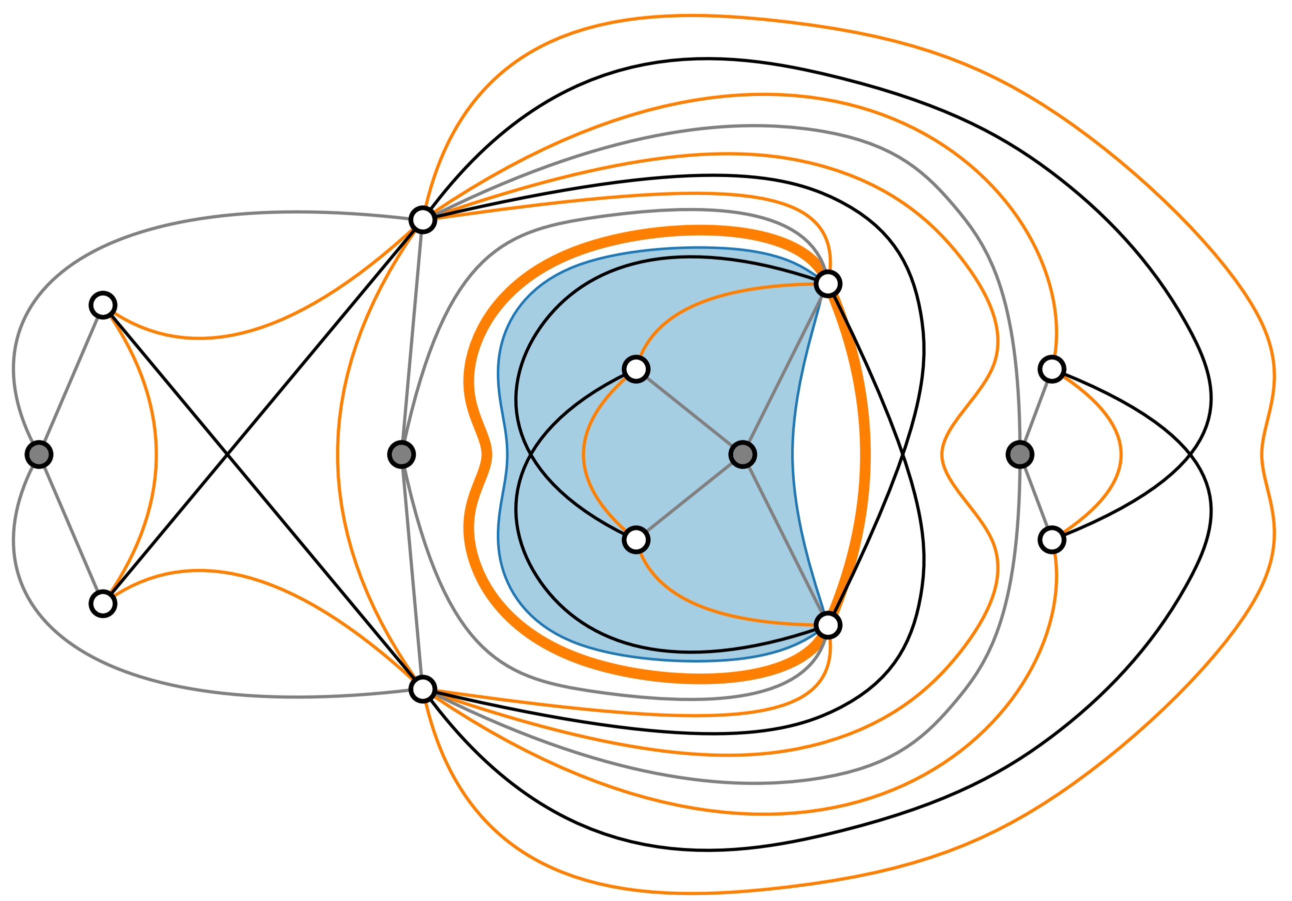

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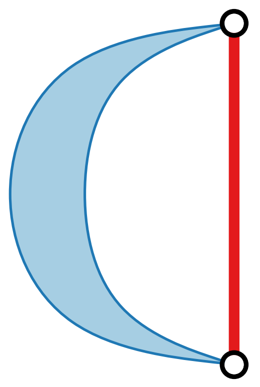

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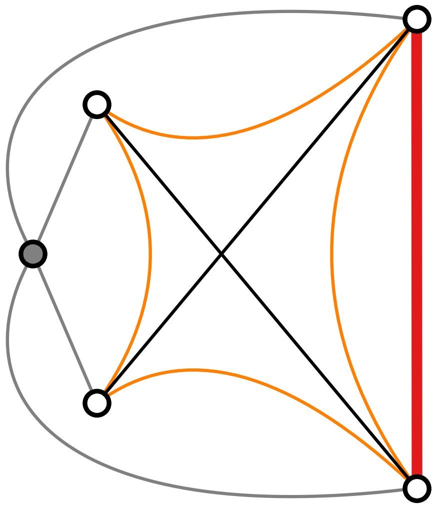

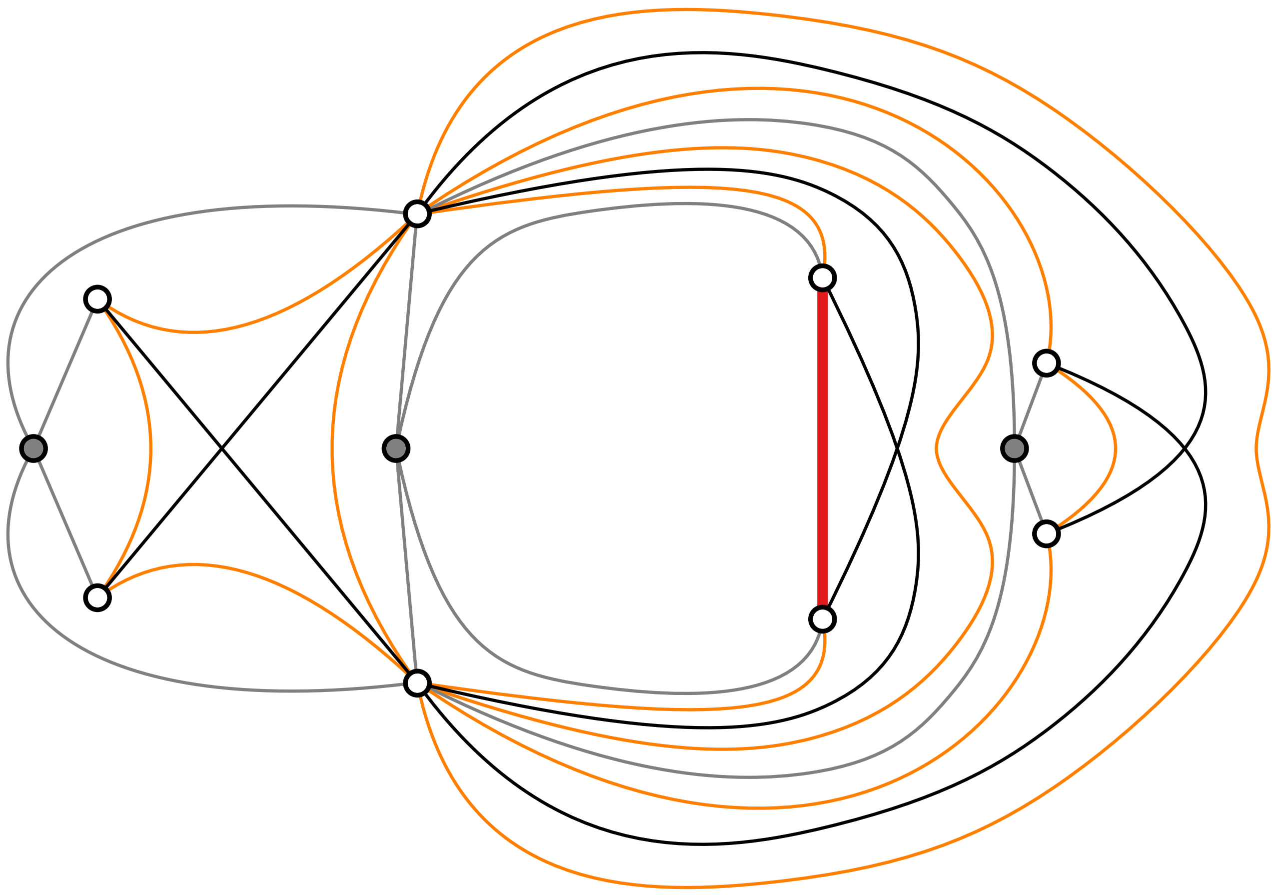

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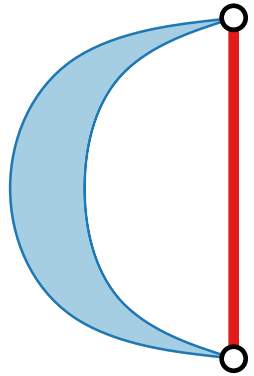

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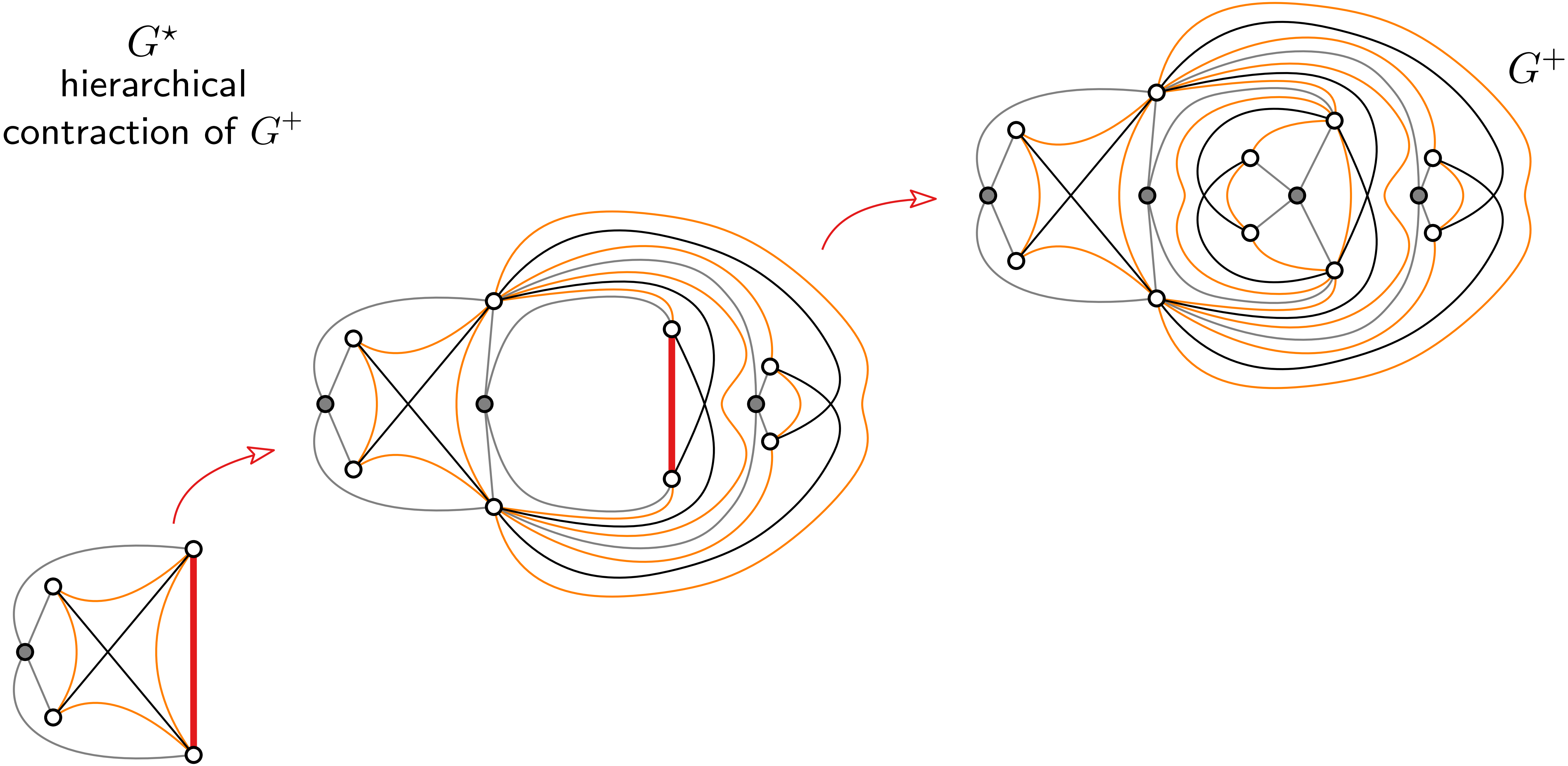
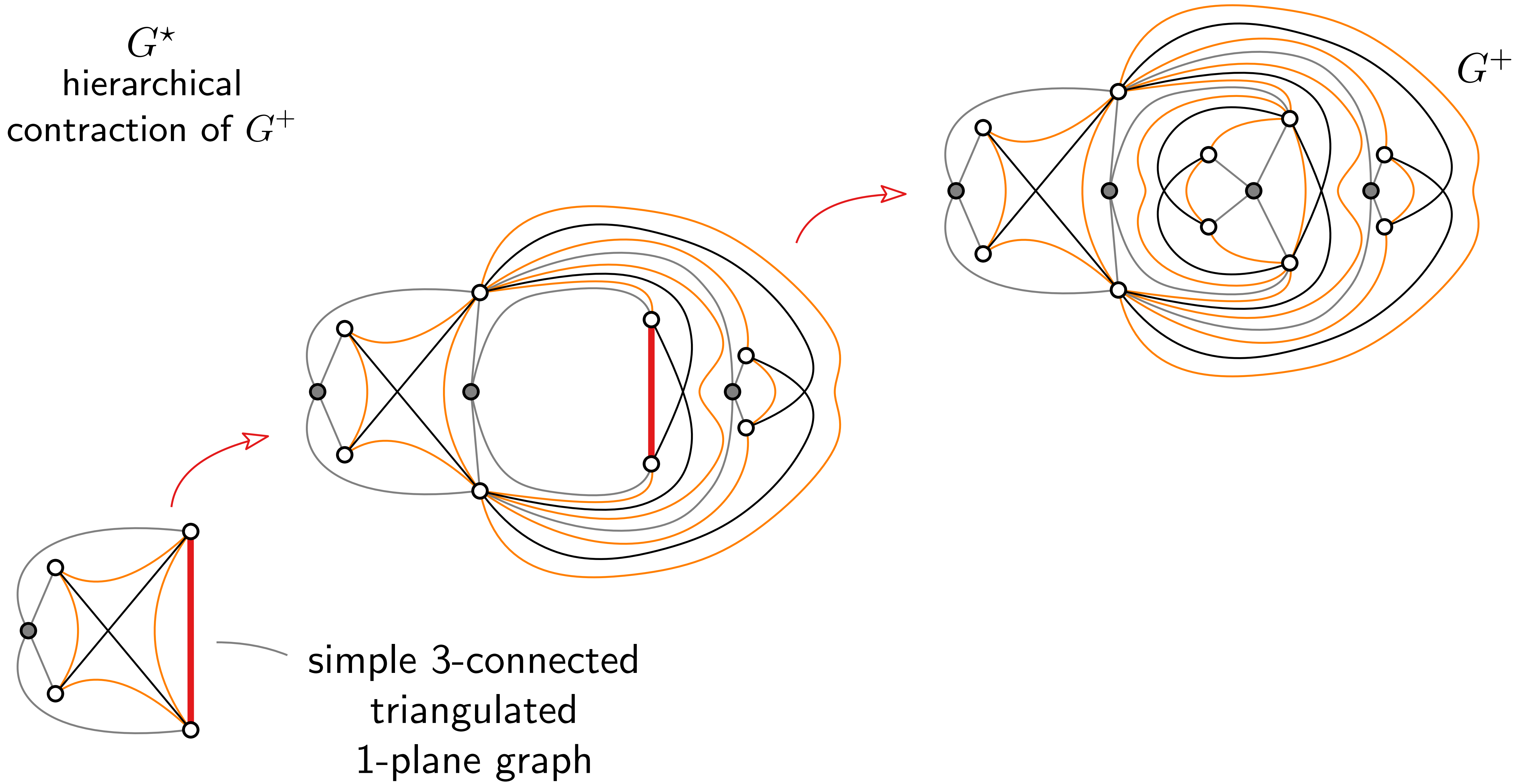
# Algoritm Step 2: Hierarchical Contractions

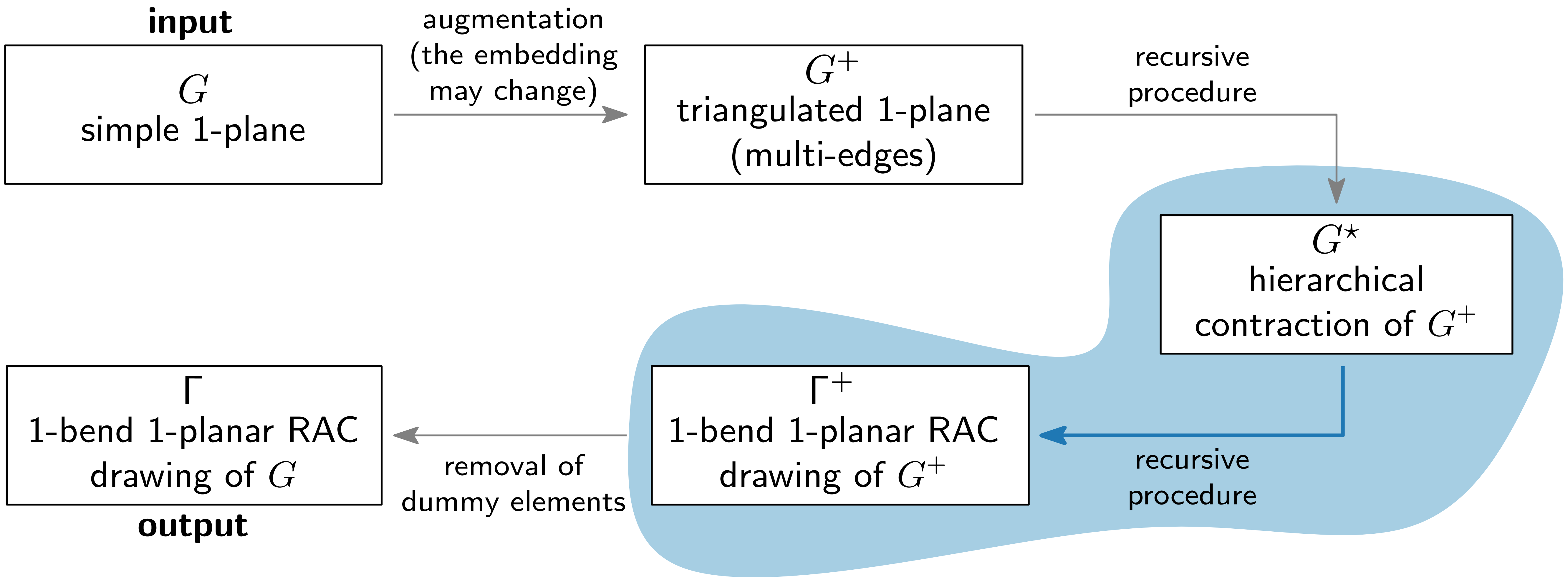
$G^\star$
hierarchical
contraction of $G^+$



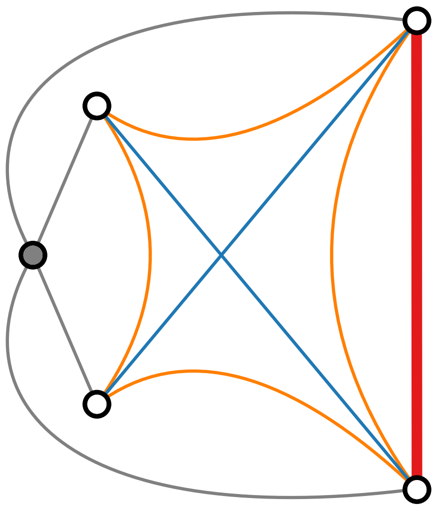
# Algoritm Step 2: Hierarchical Contractions



# Algorithm Outline



# Algorithm Step 3: Drawing Procedure



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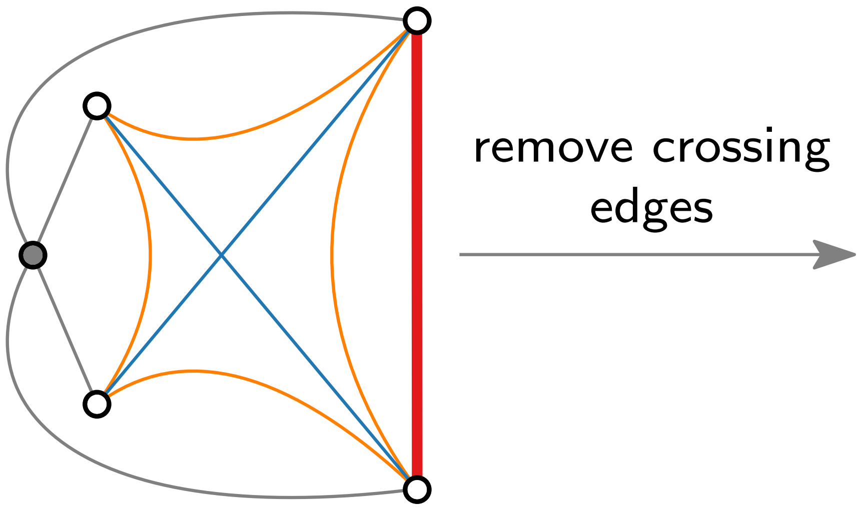

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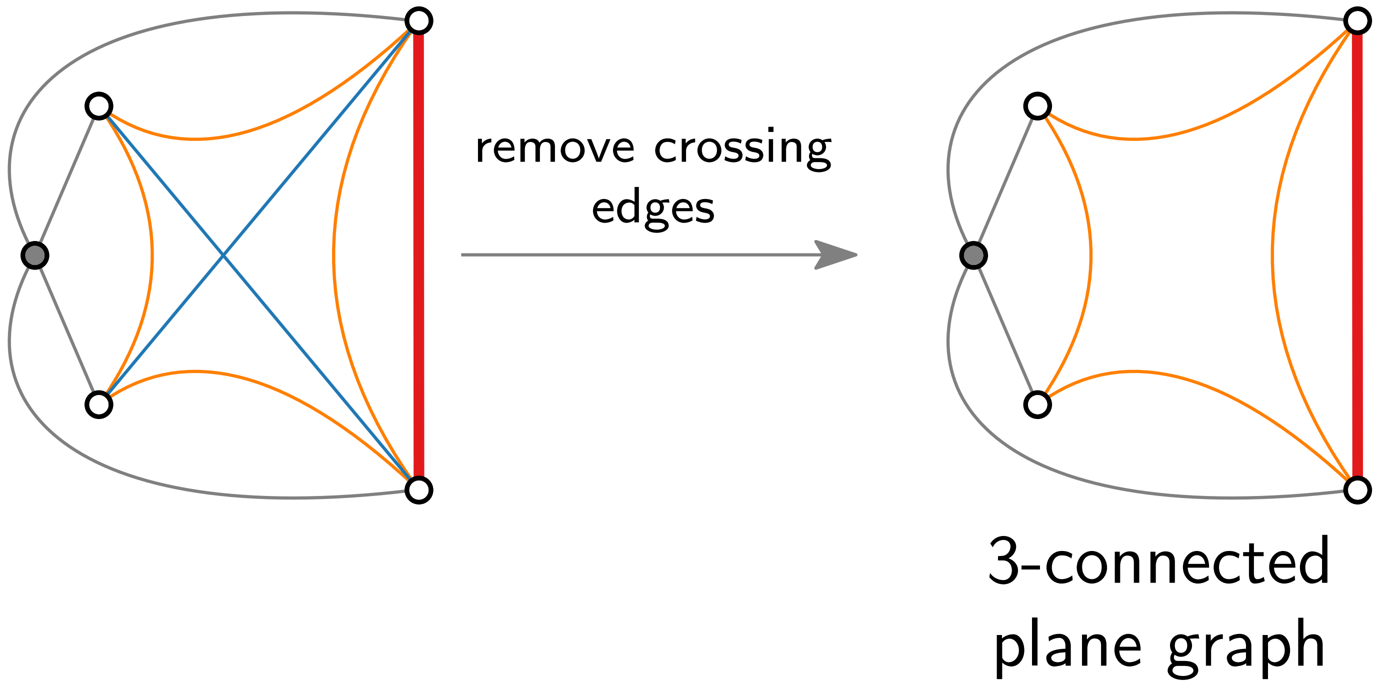

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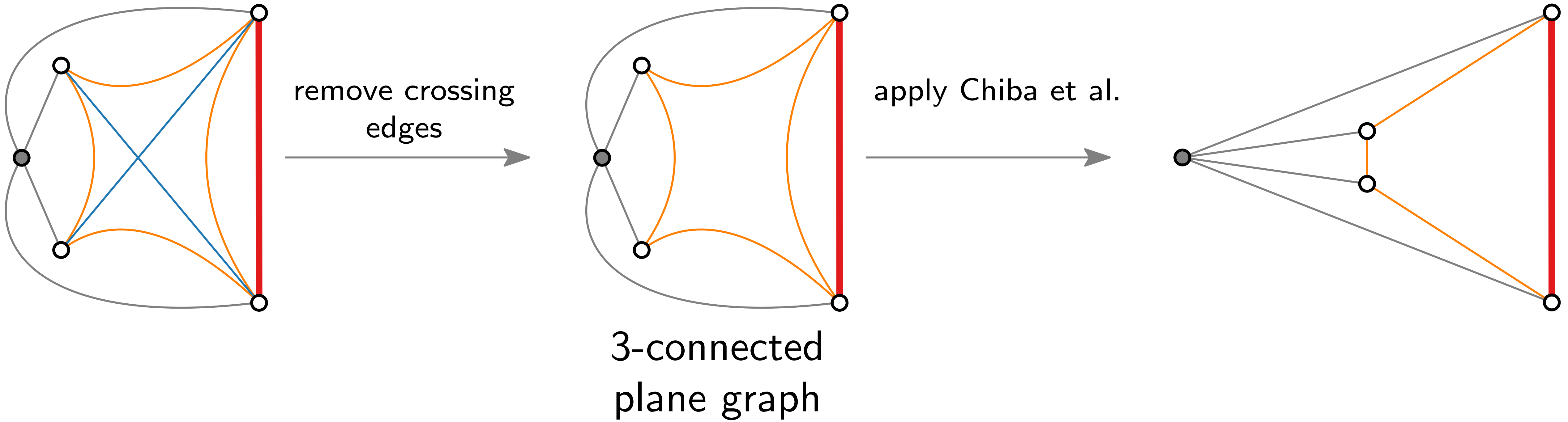

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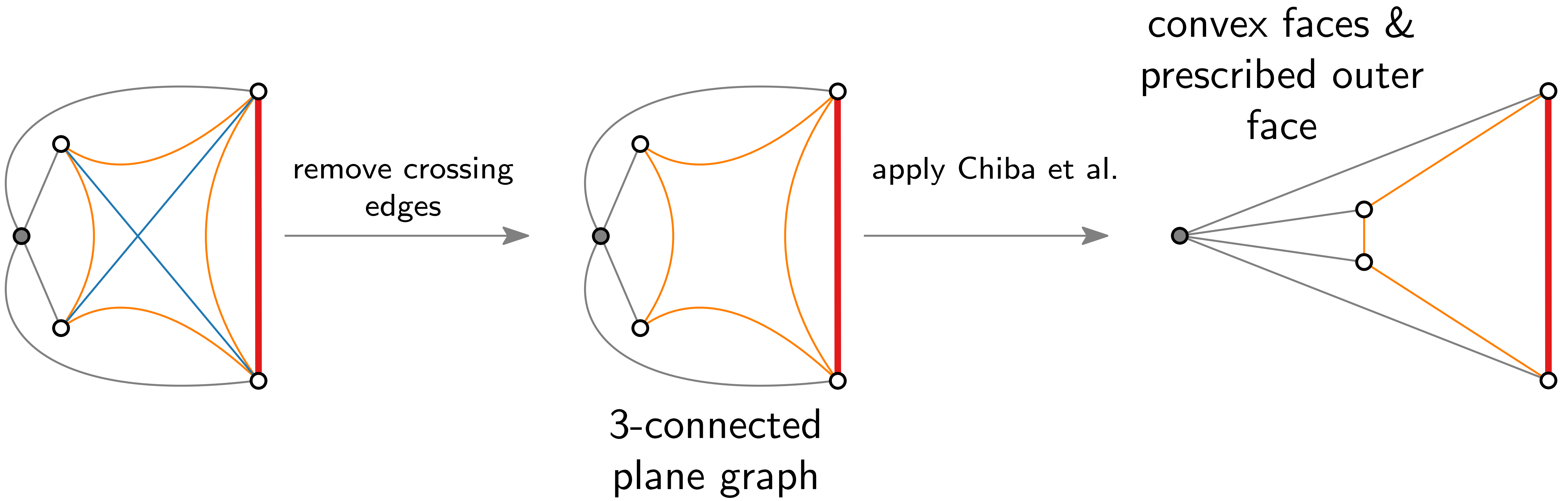

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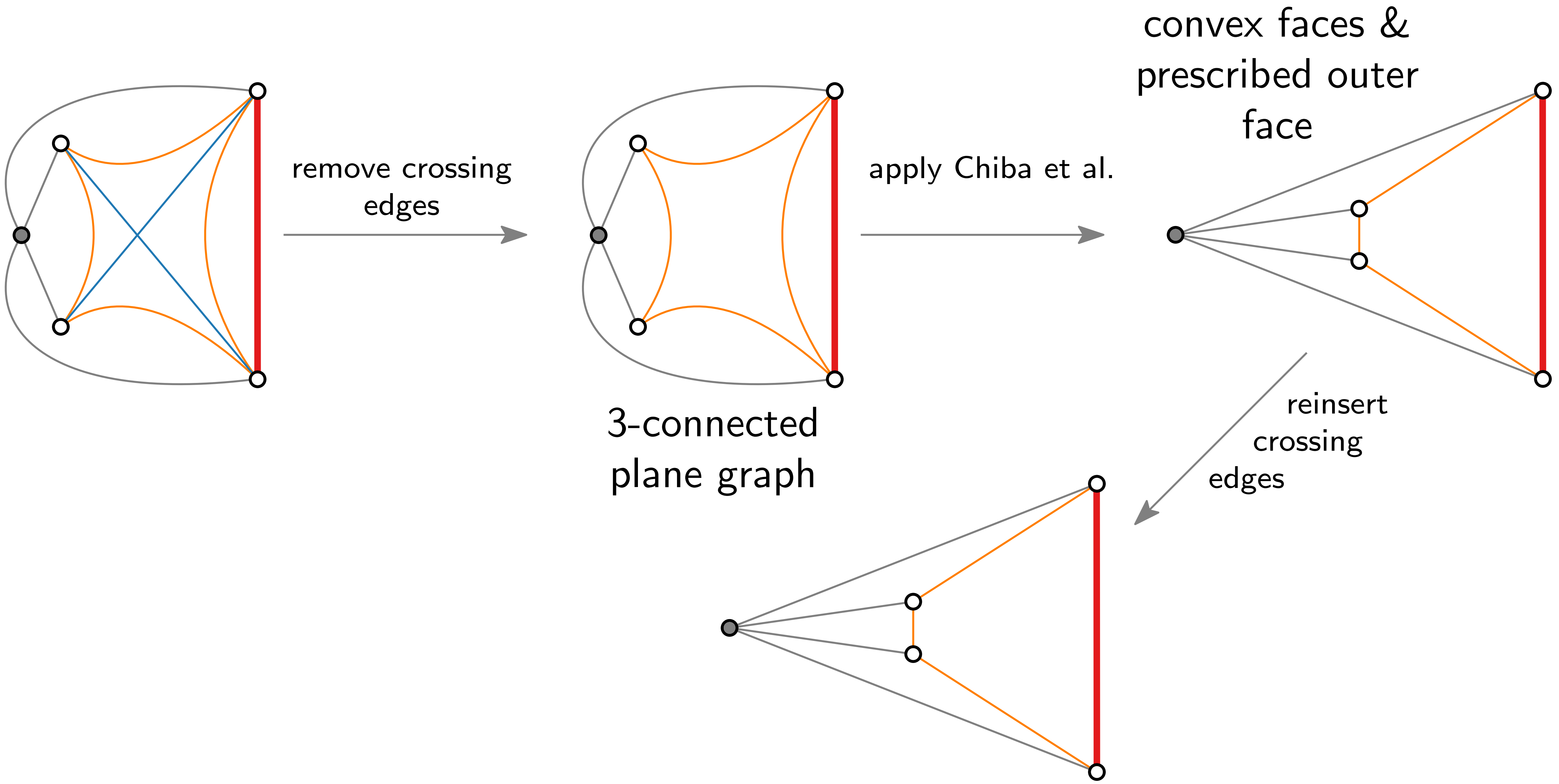

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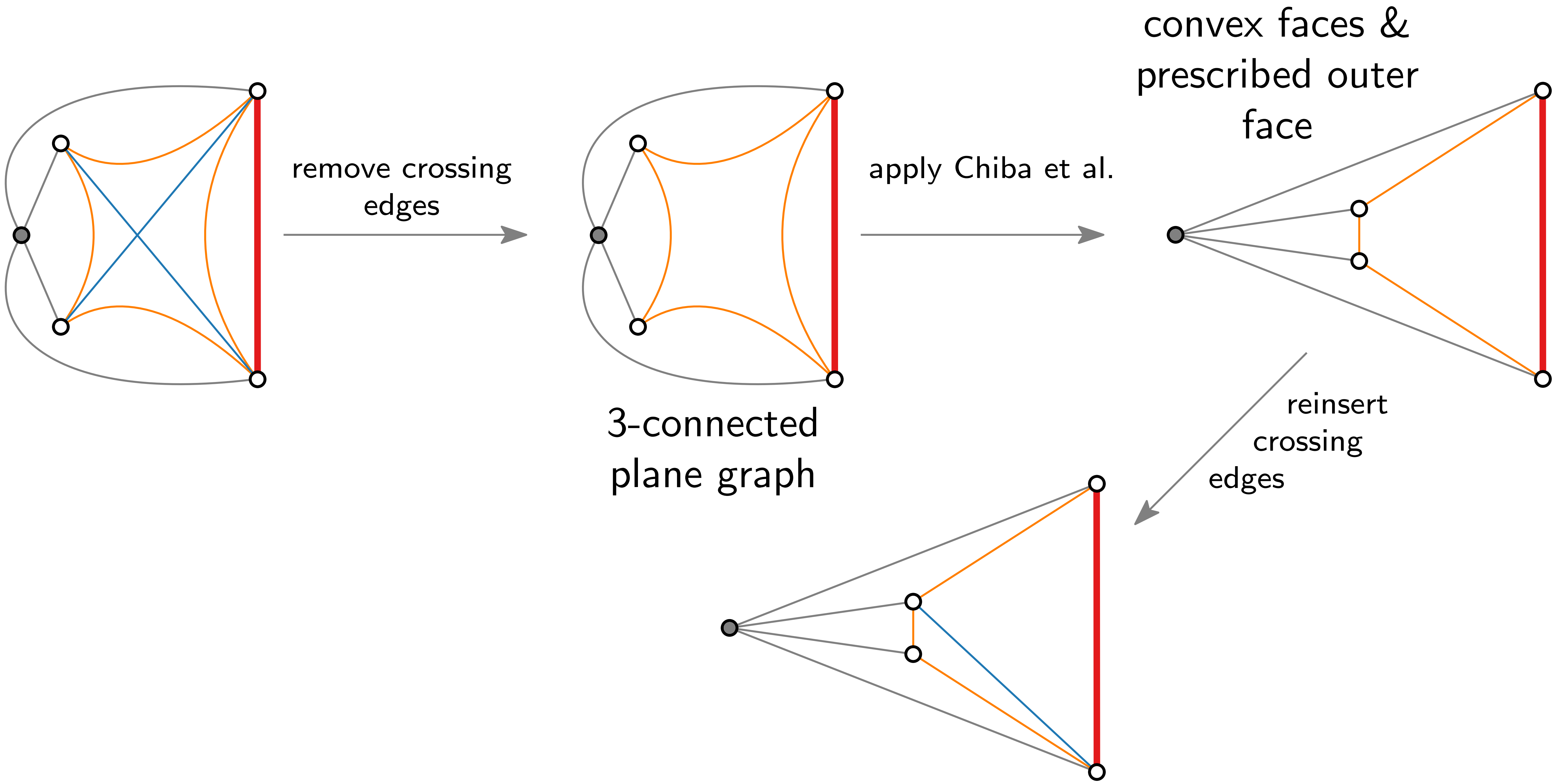

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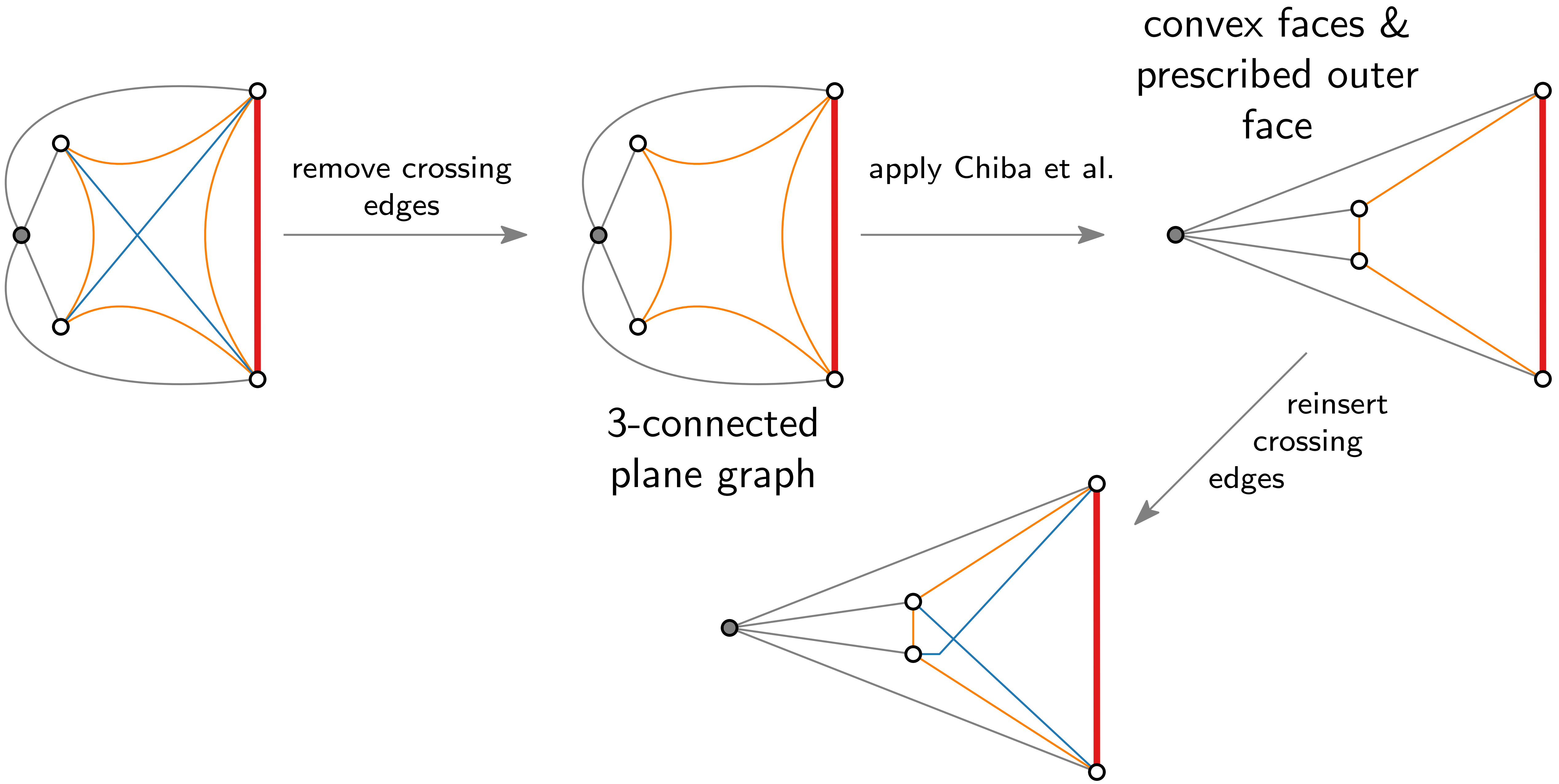

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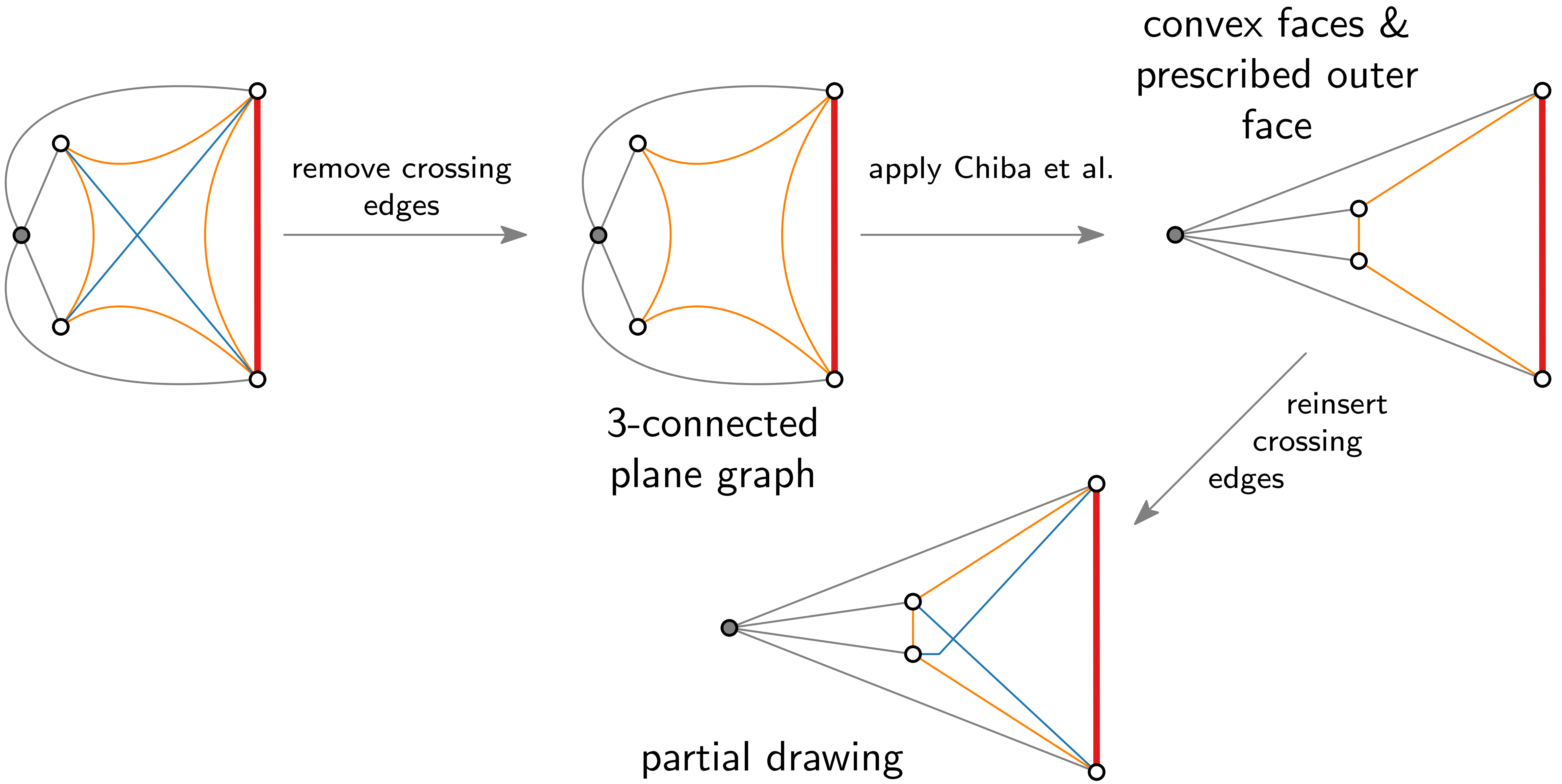

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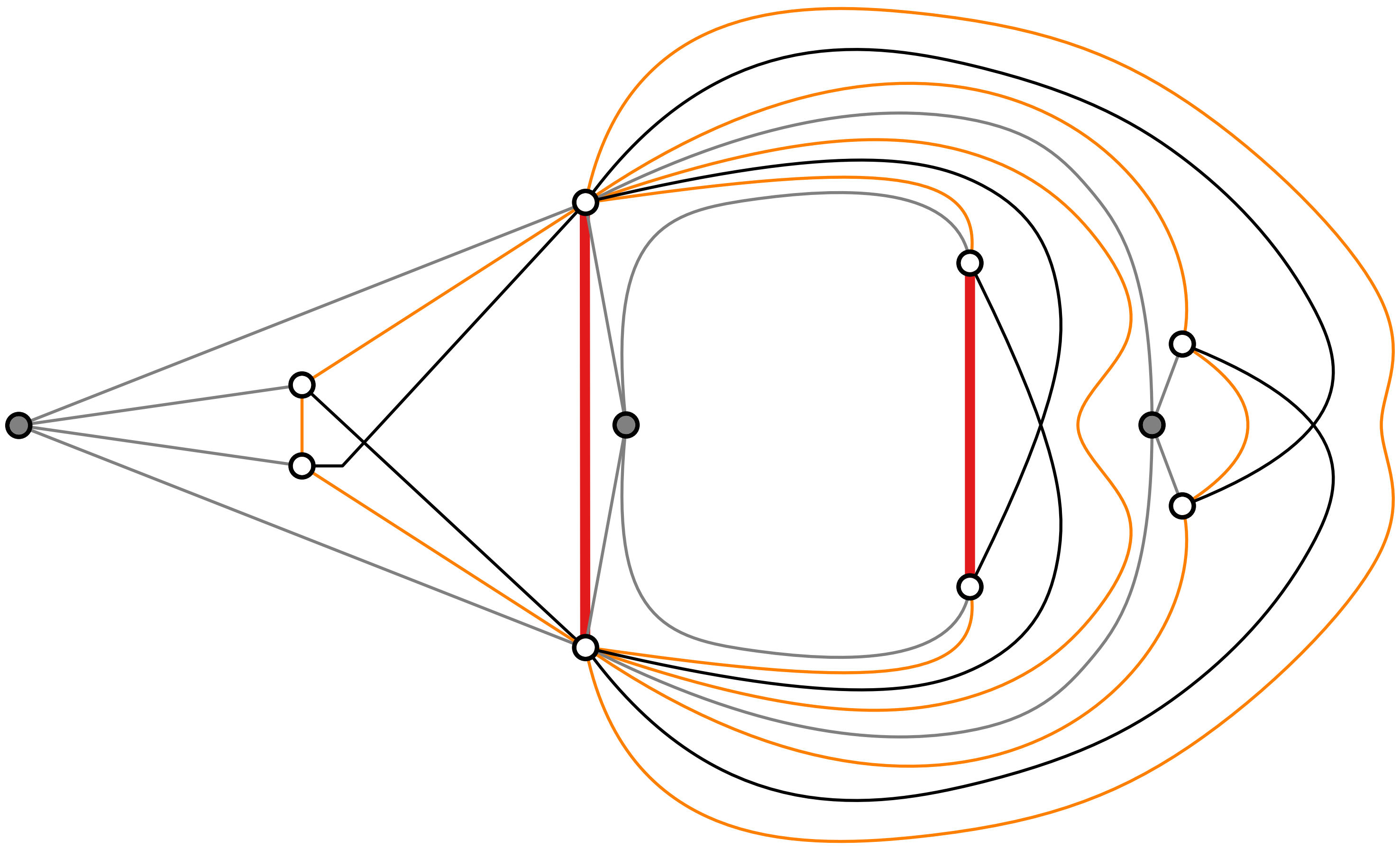

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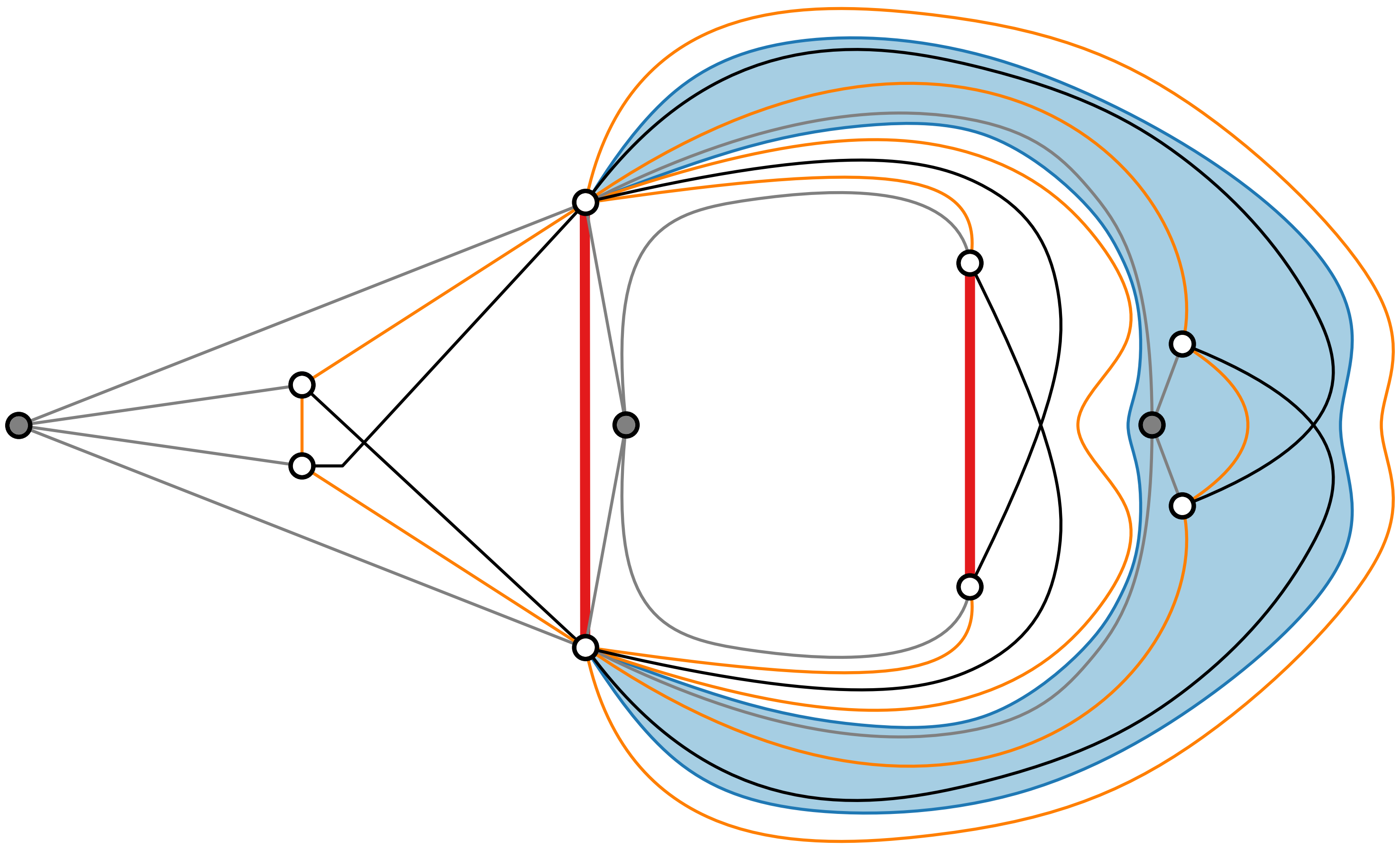
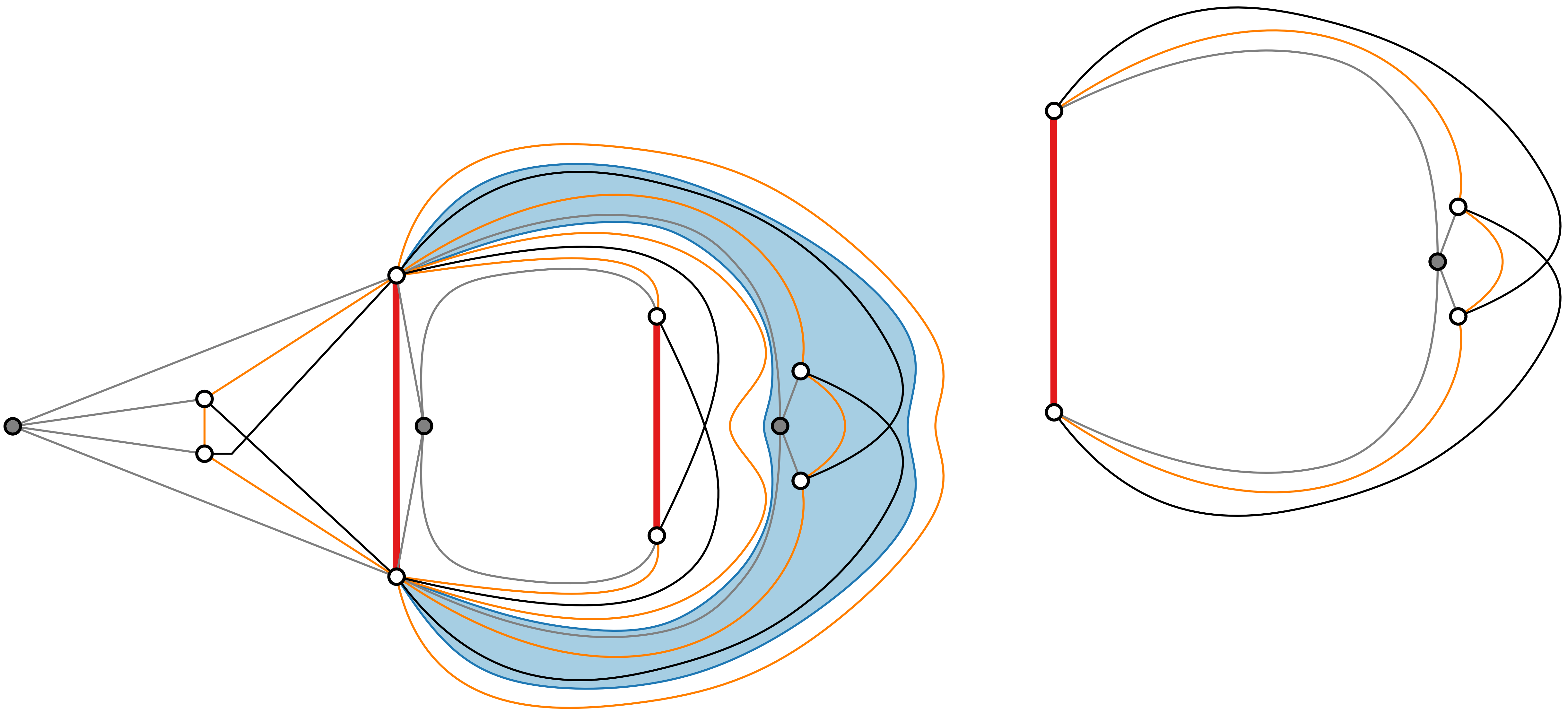
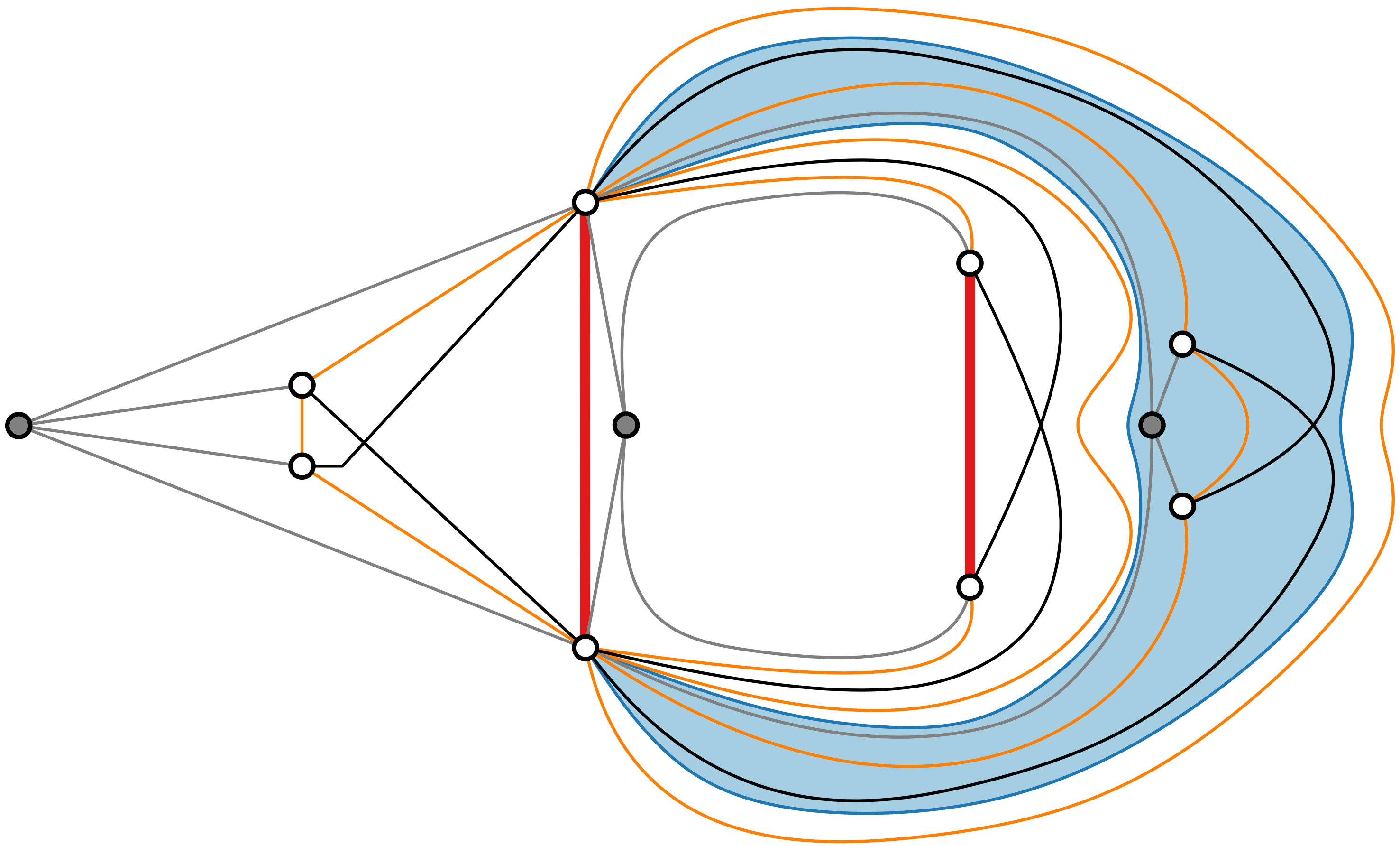
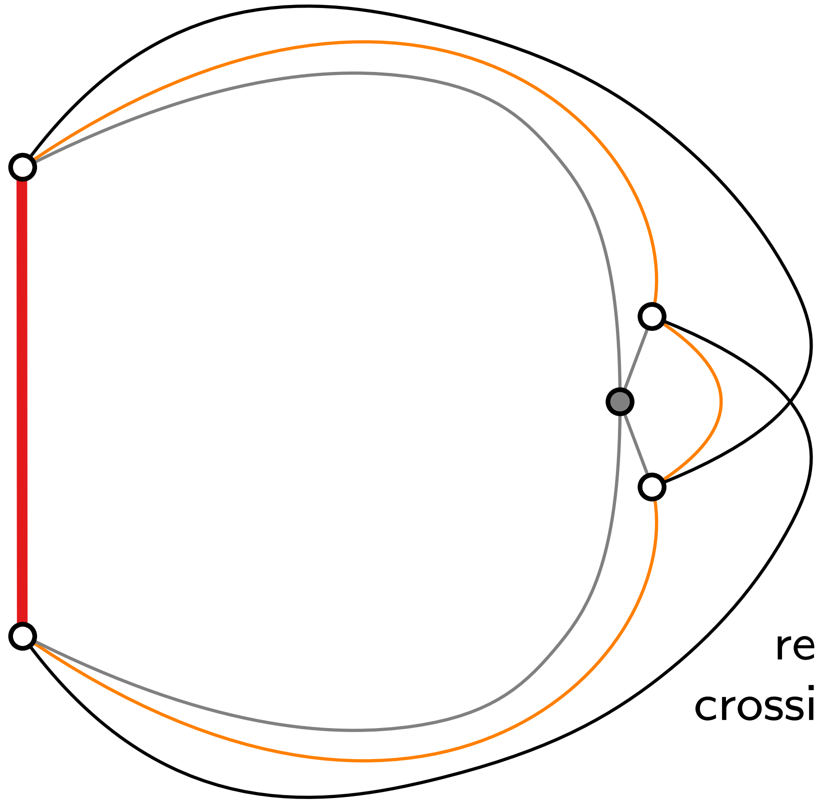


# Algorithm Step 3: Drawing Procedure

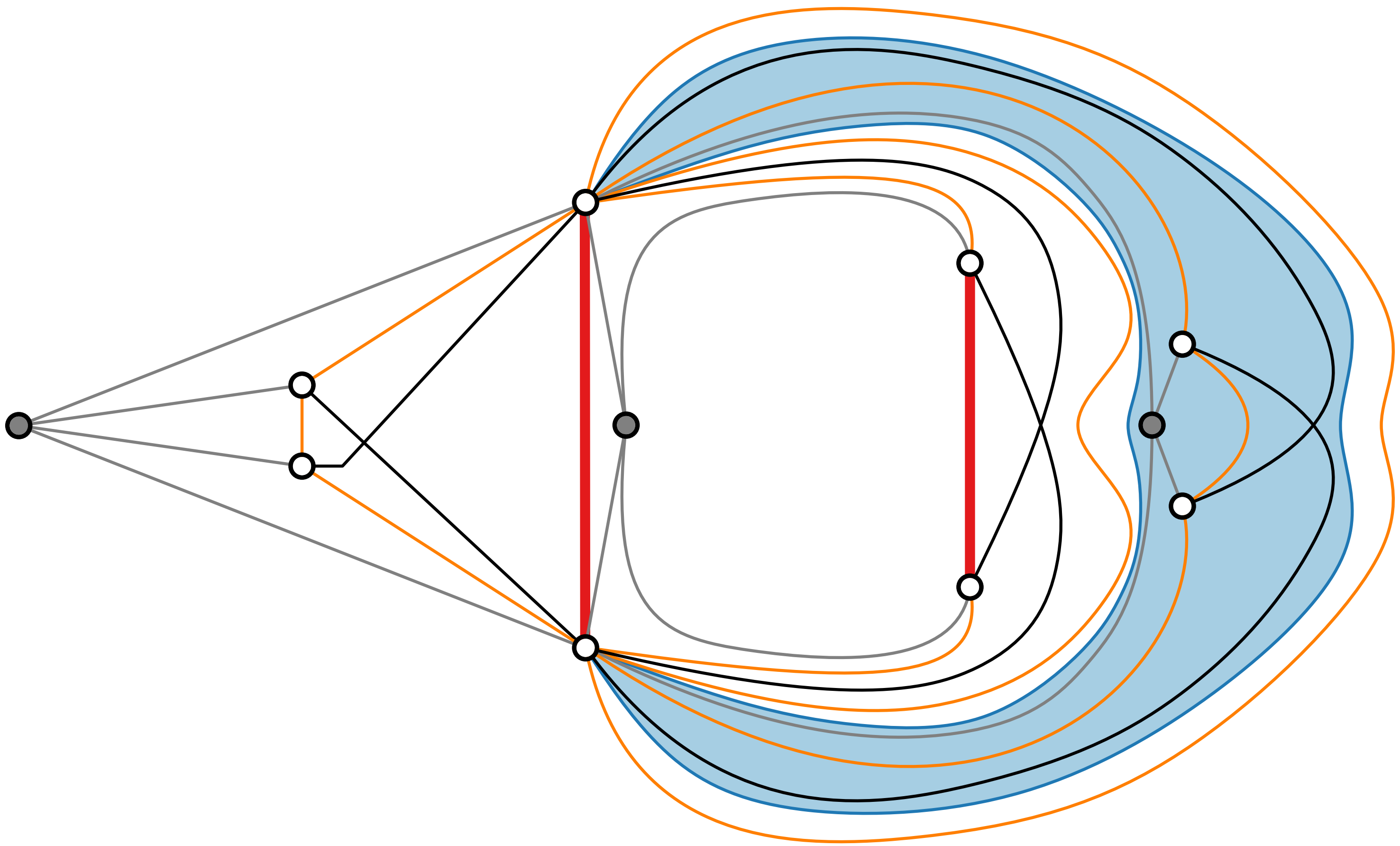


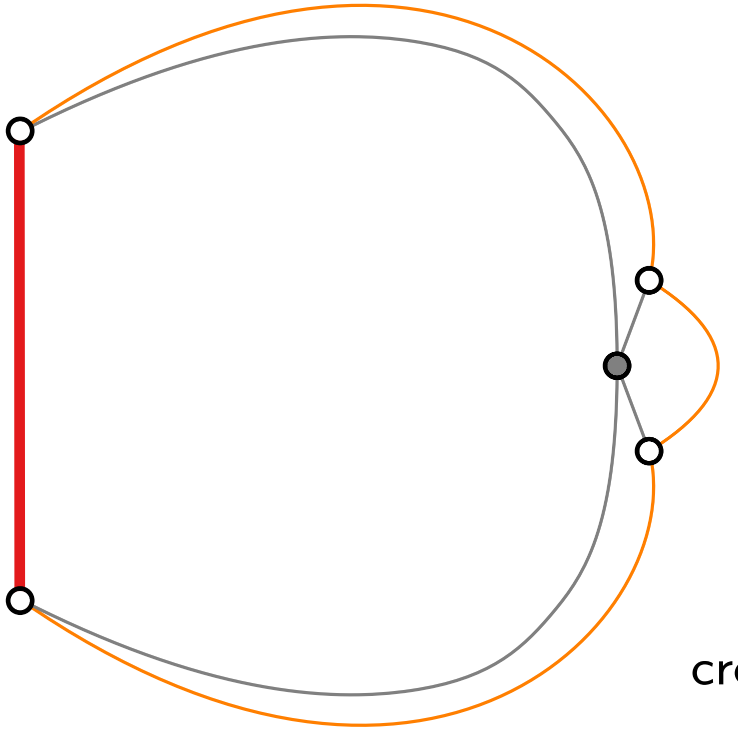
# Algorithm Step 3: Drawing Procedure





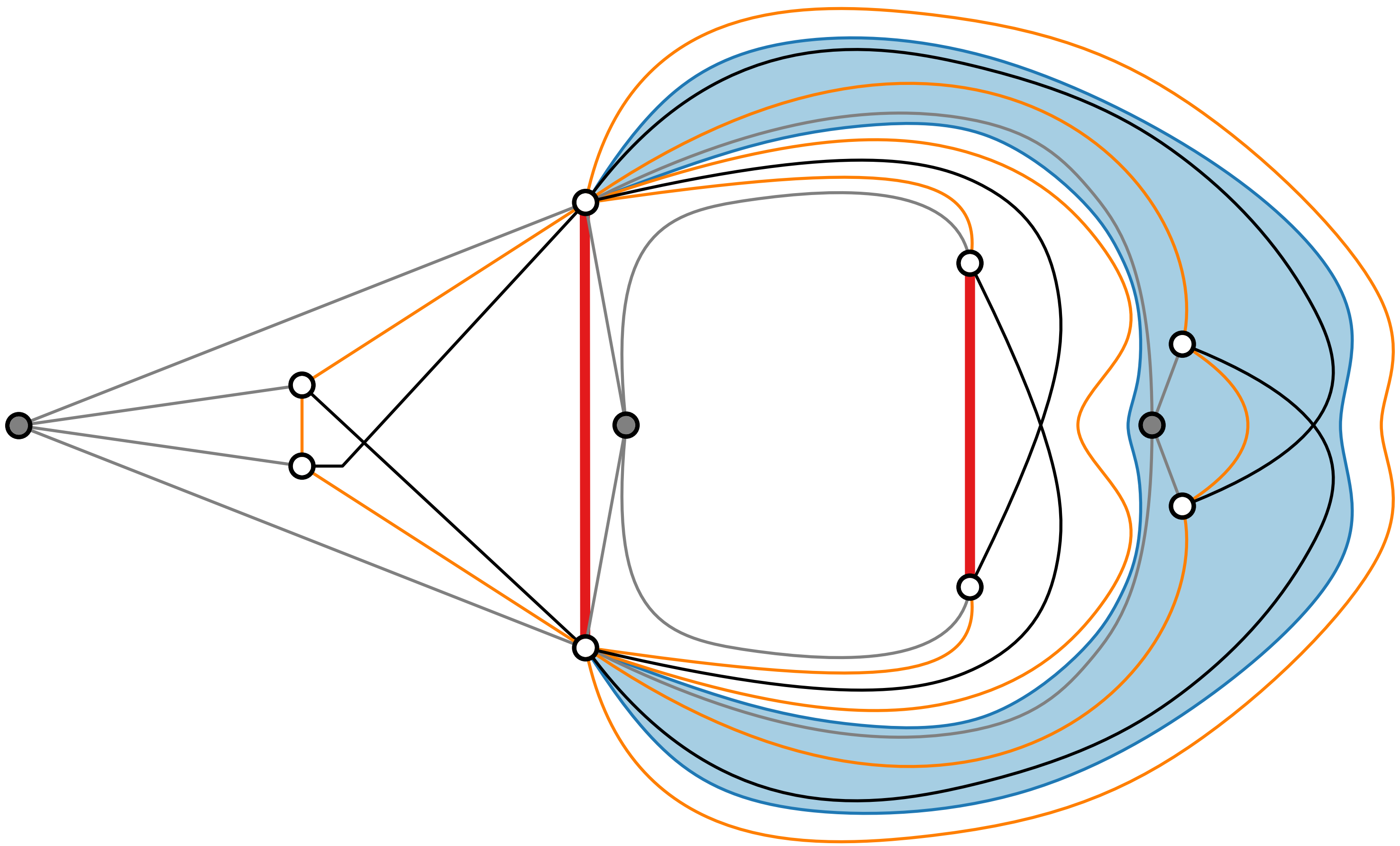
# Algorithm Step 3: Drawing Procedure

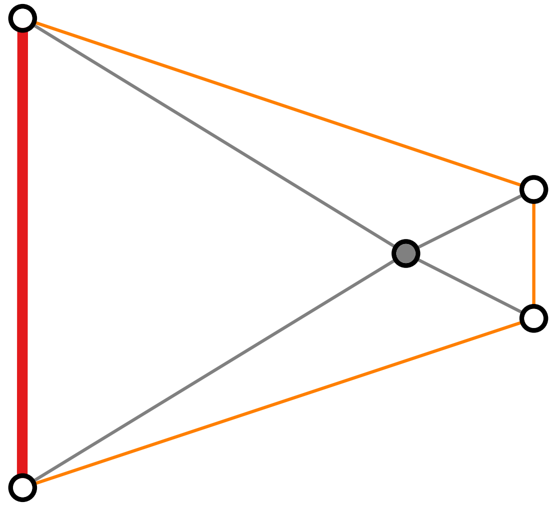




remove
crossing edges

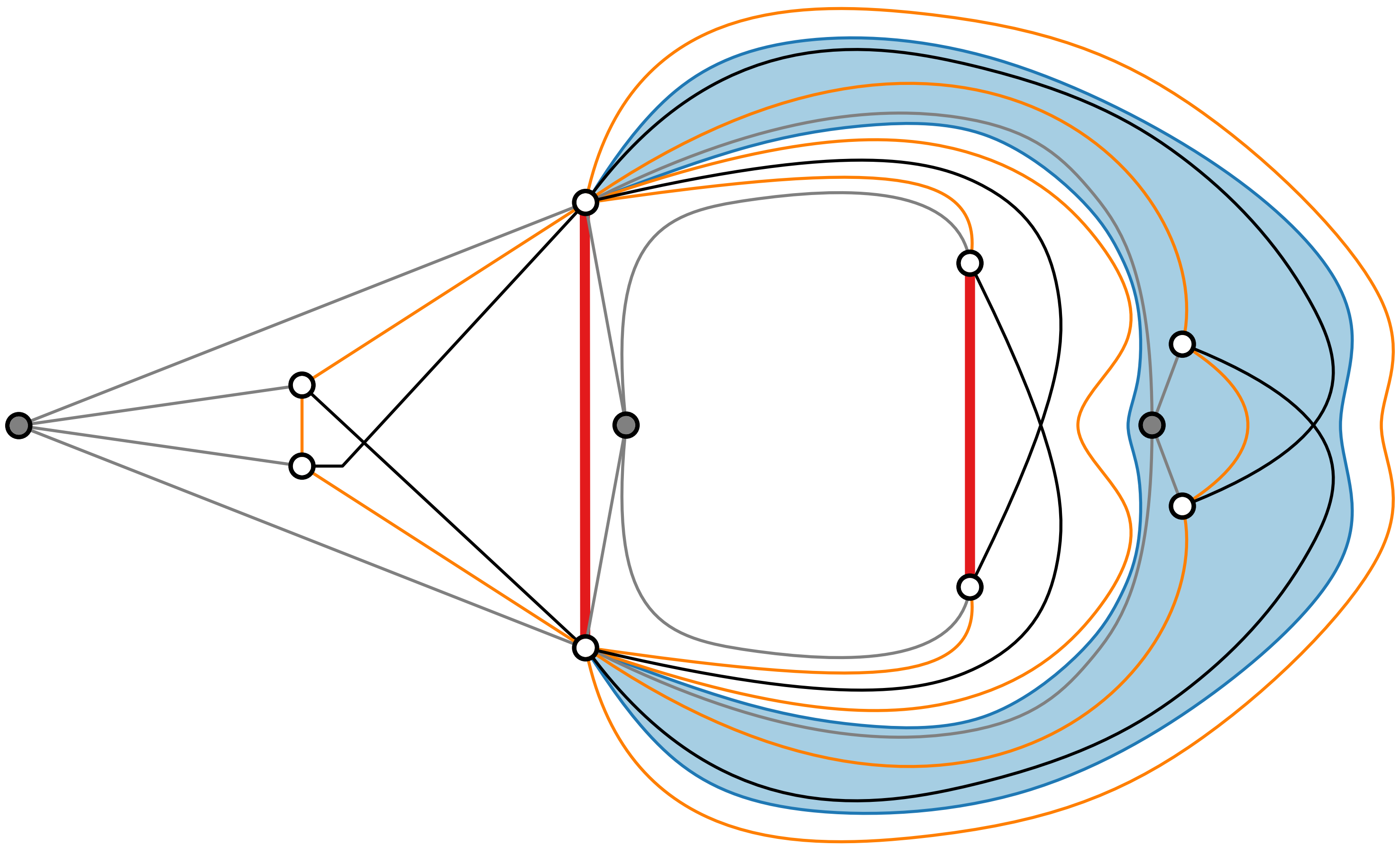
# Algorithm Step 3: Drawing Procedure

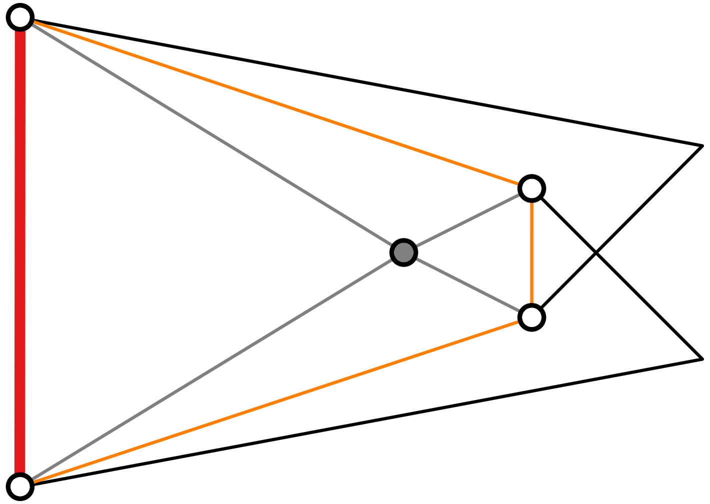




apply Chiba et al.

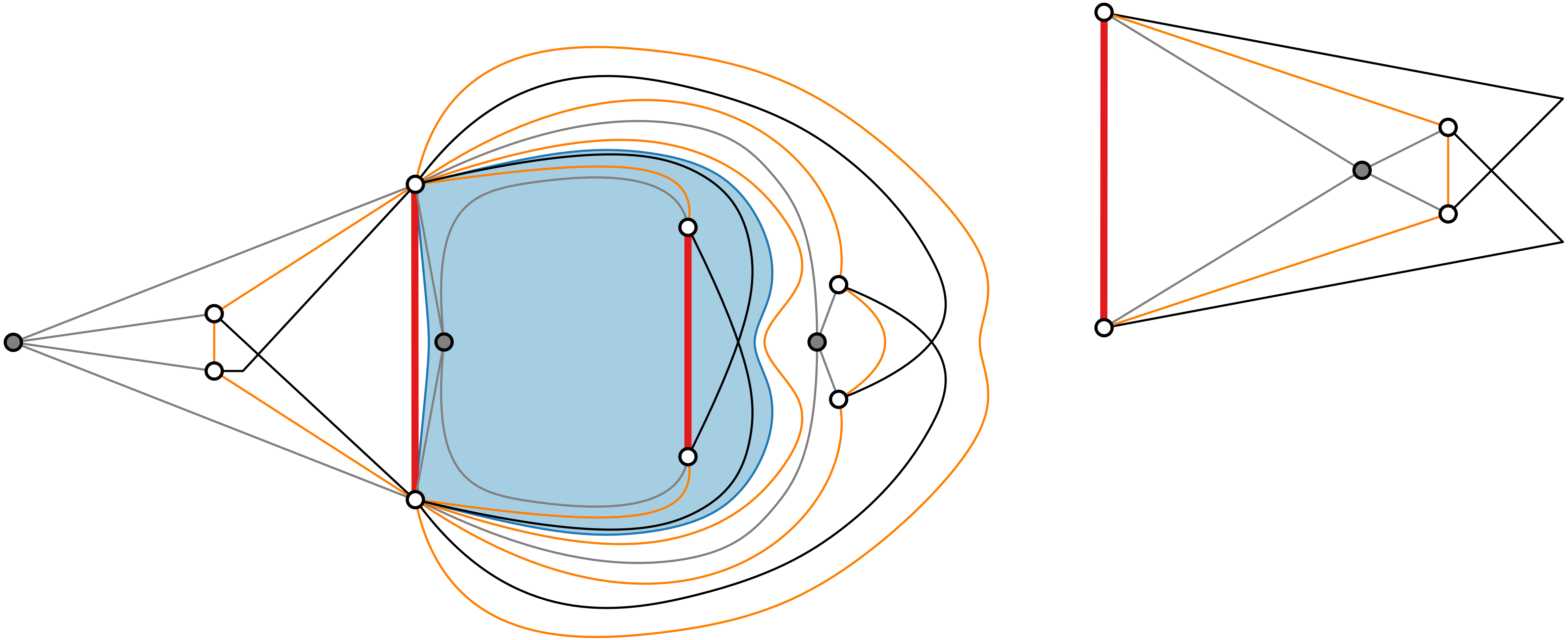
# Algorithm Step 3: Drawing Procedure



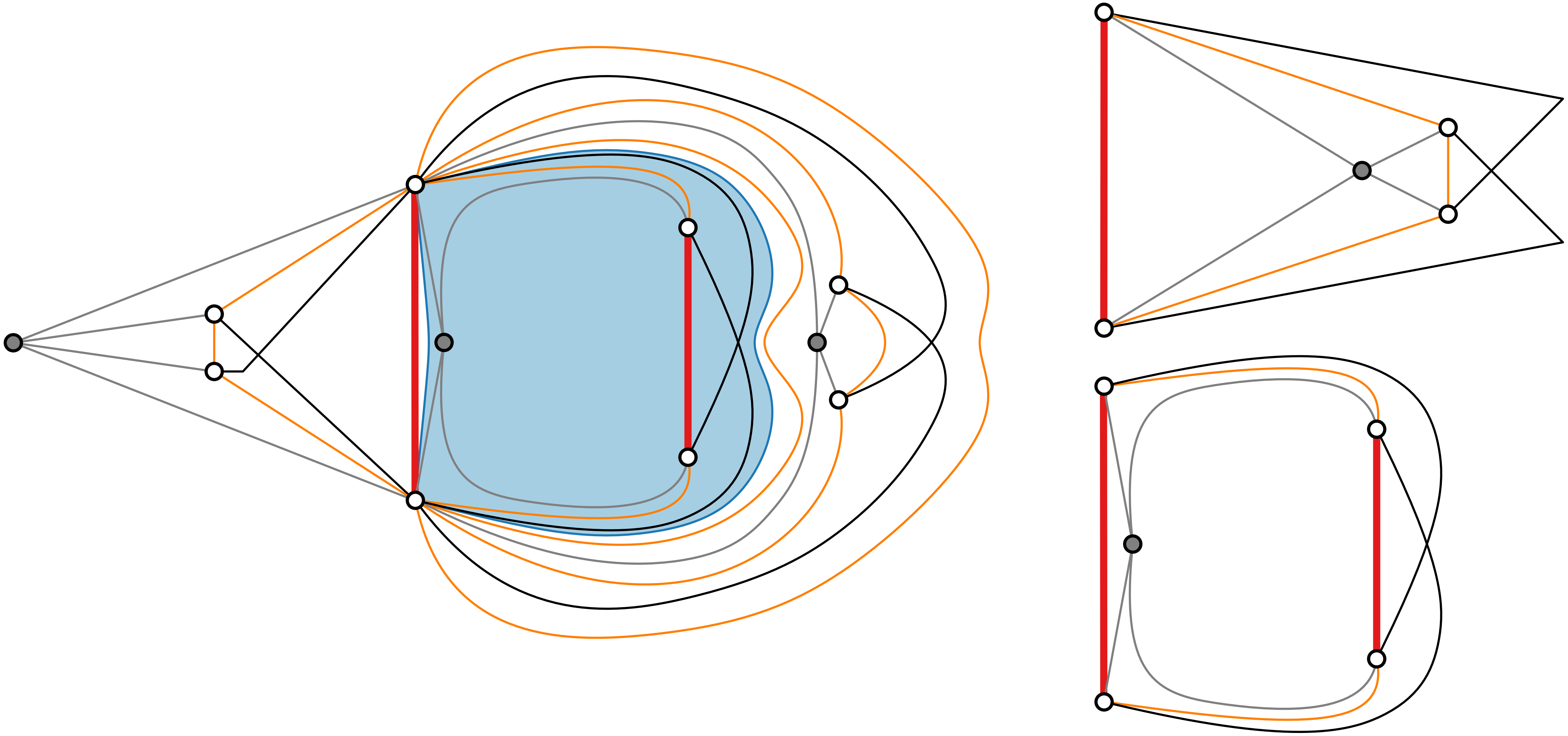


reinsert
crossing edges

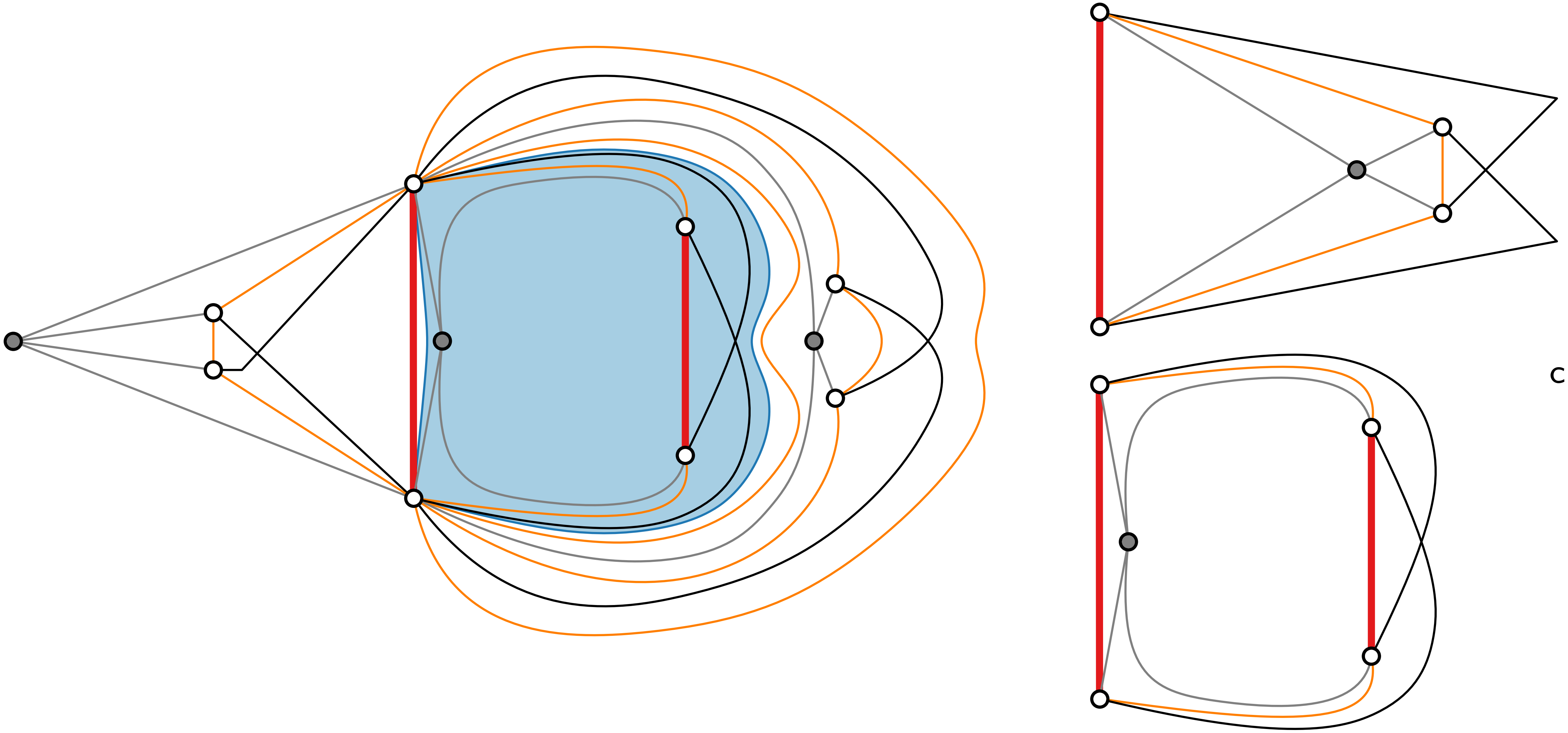
# Algorithm Step 3: Drawing Procedure



# Algorithm Step 3: Drawing Procedure

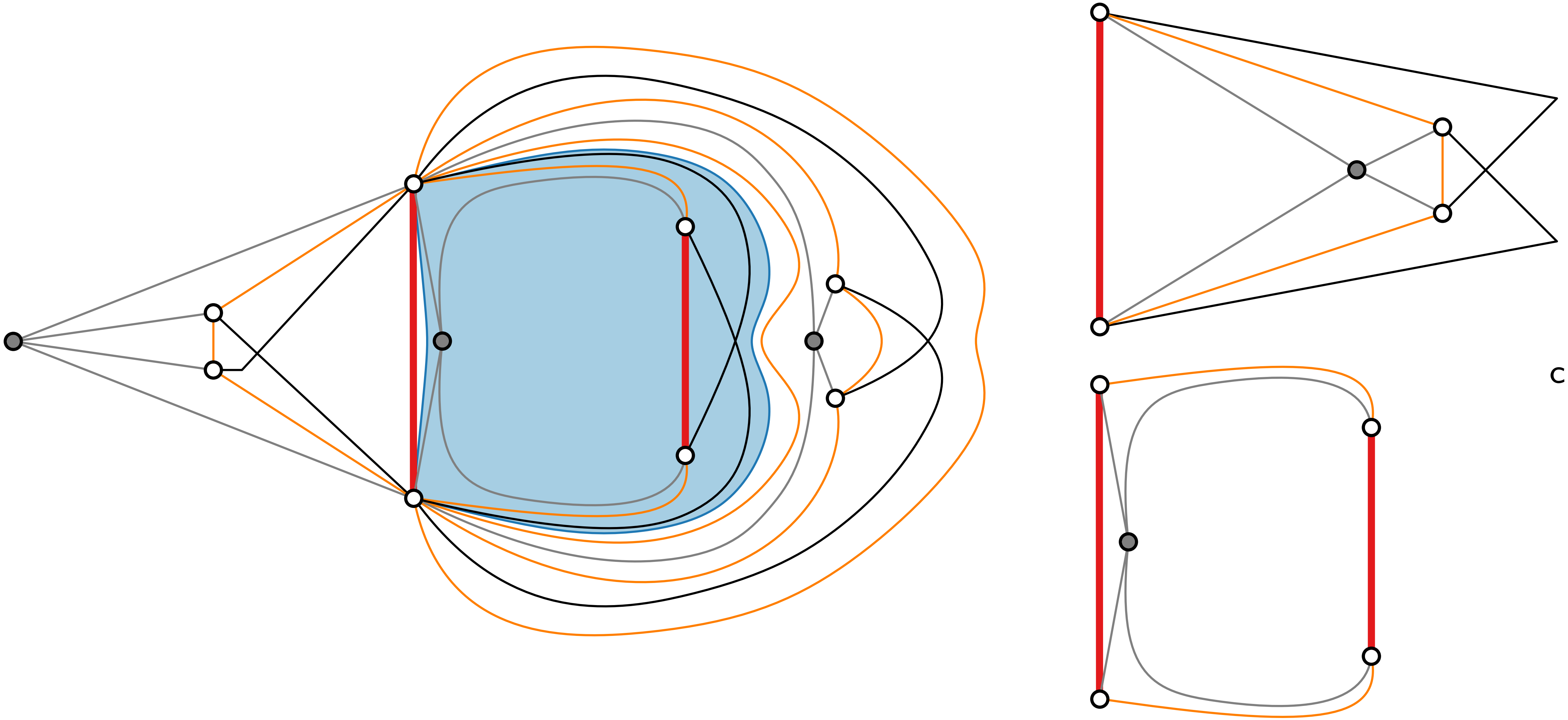


# Algorithm Step 3: Drawing Procedure



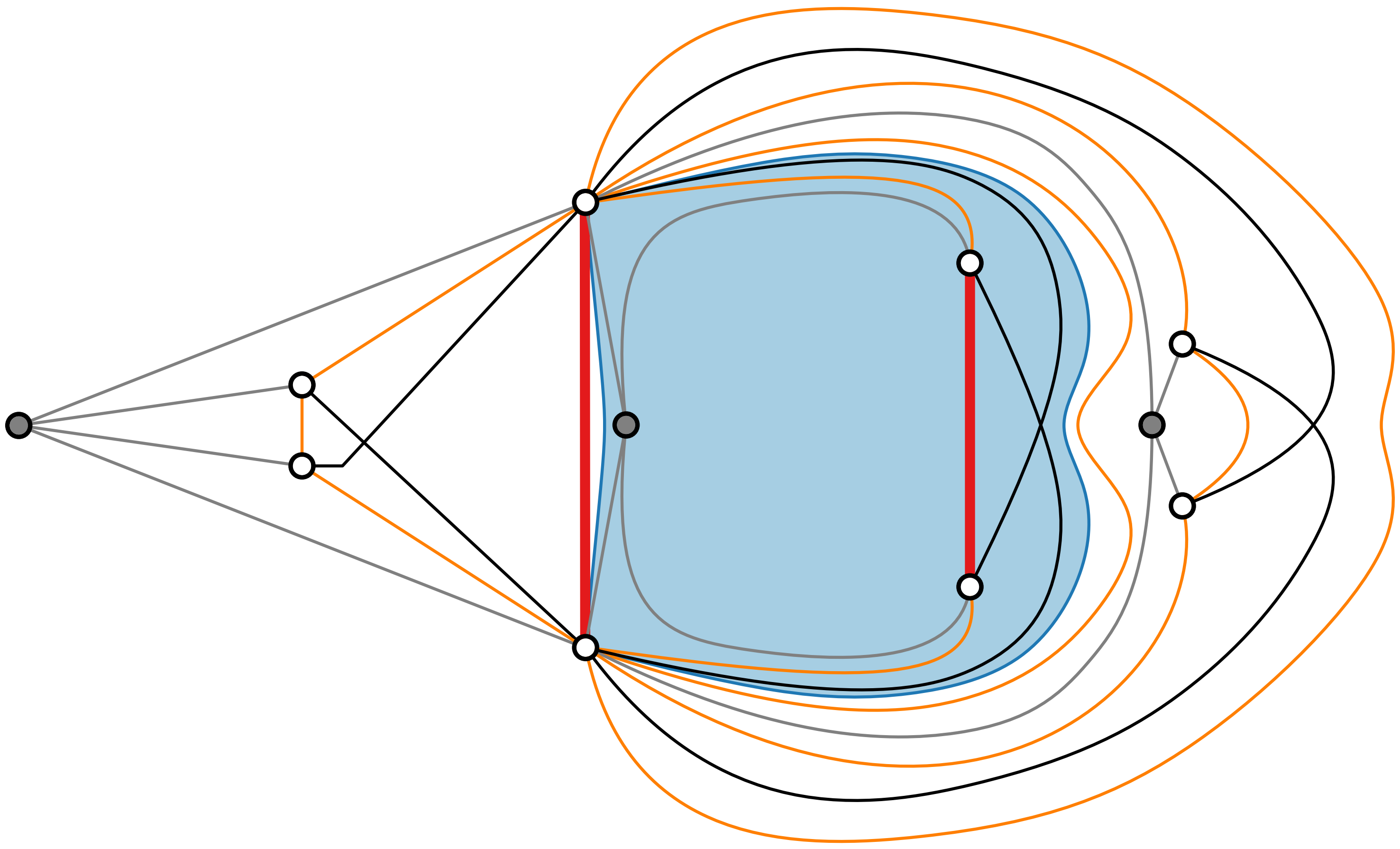
remove
crossing edges

# Algorithm Step 3: Drawing Procedure

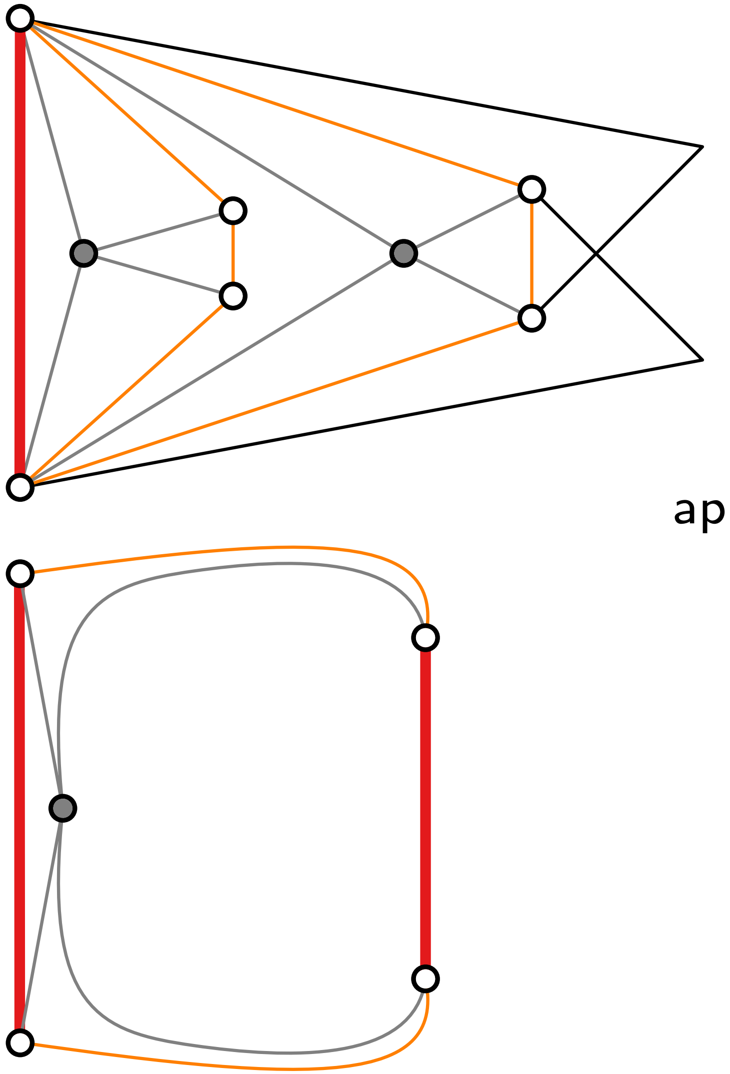


remove
crossing edges

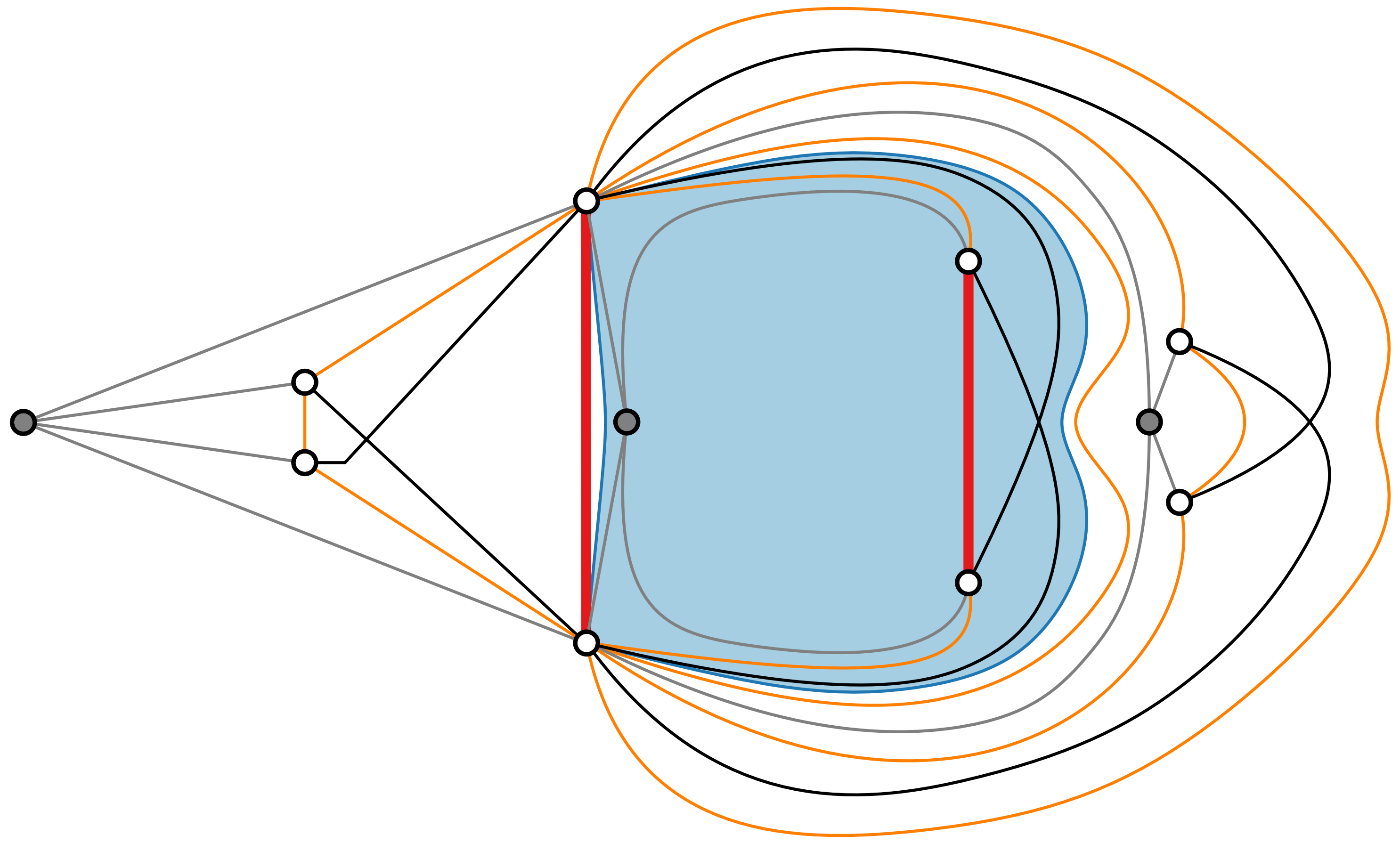
# Algorithm Step 3: Drawing Procedure

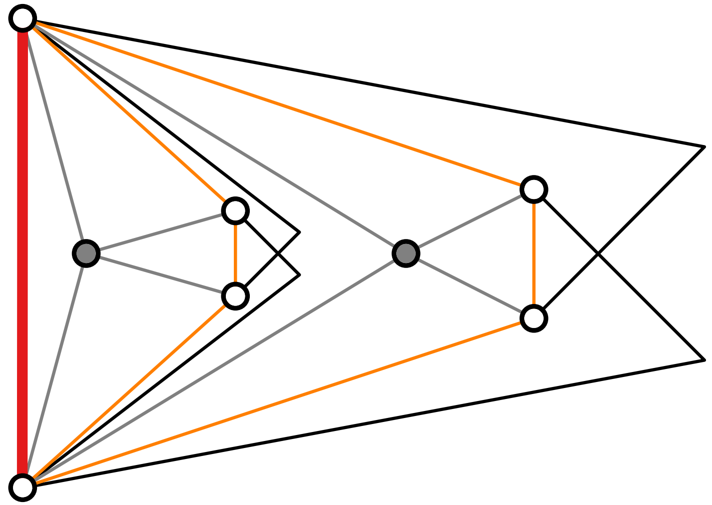


apply Chiba et al.

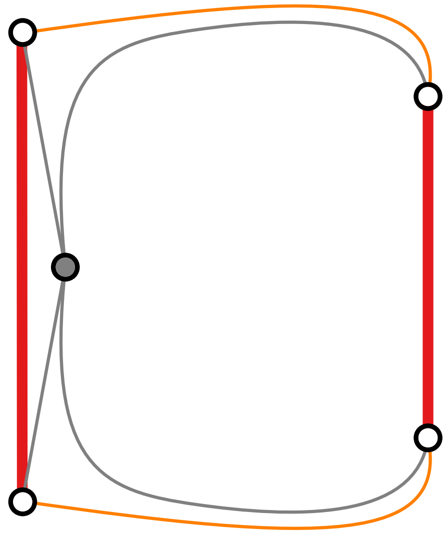


# Algorithm Step 3: Drawing Procedure

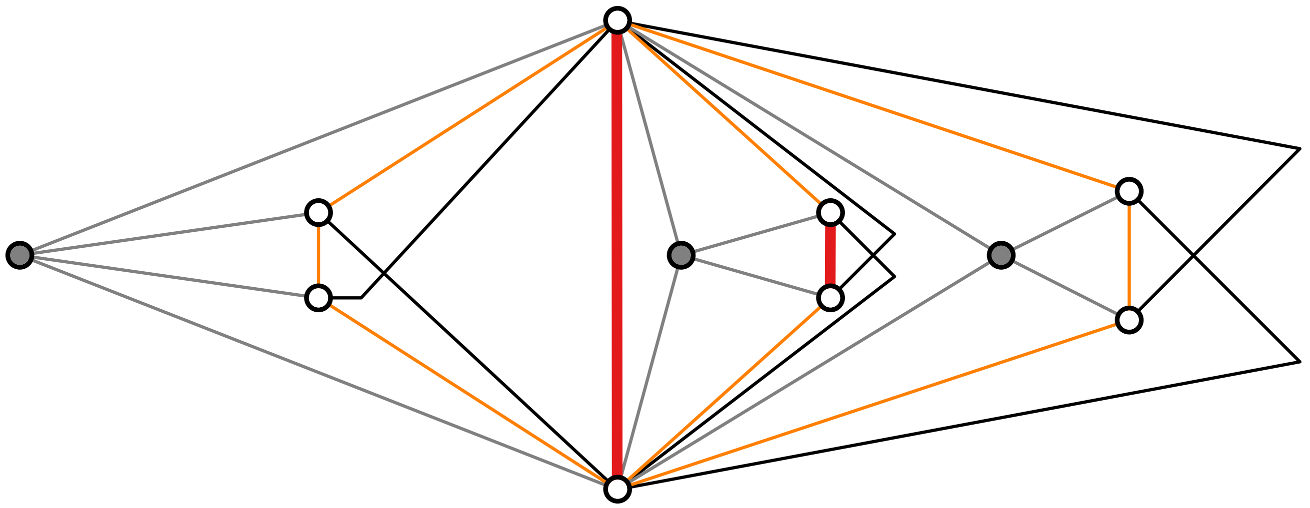




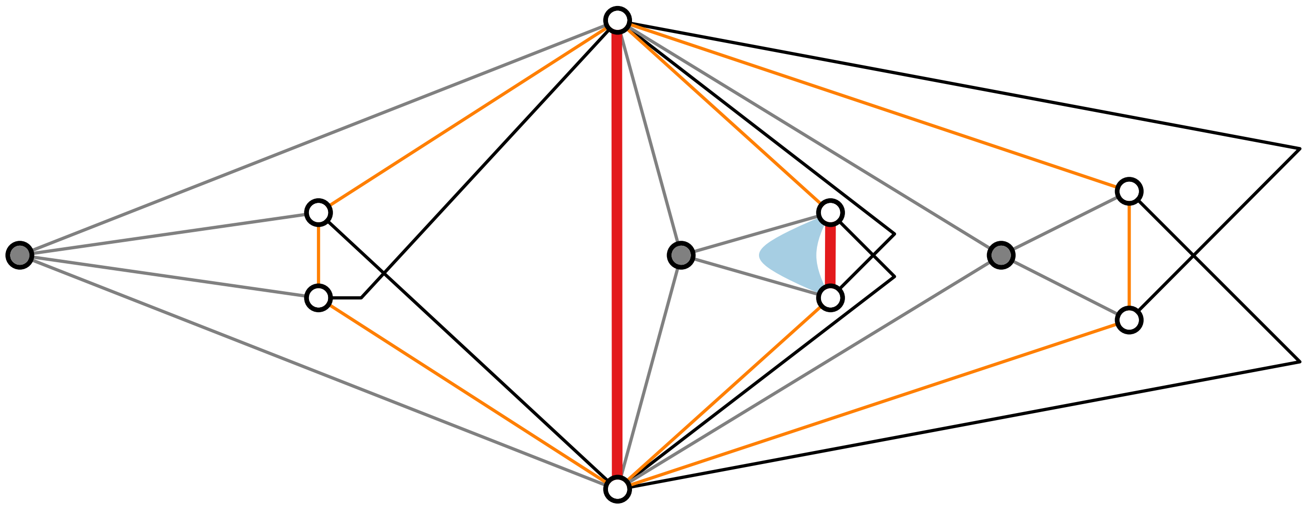
reinsert
crossing edges

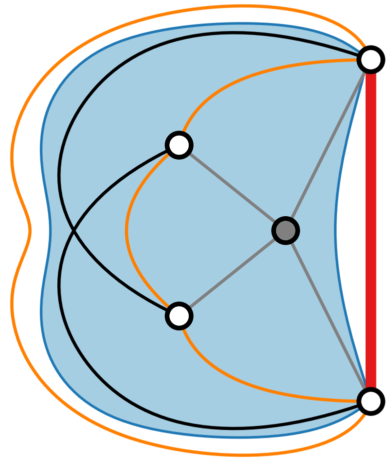


# Algorithm Step 3: Drawing Procedure

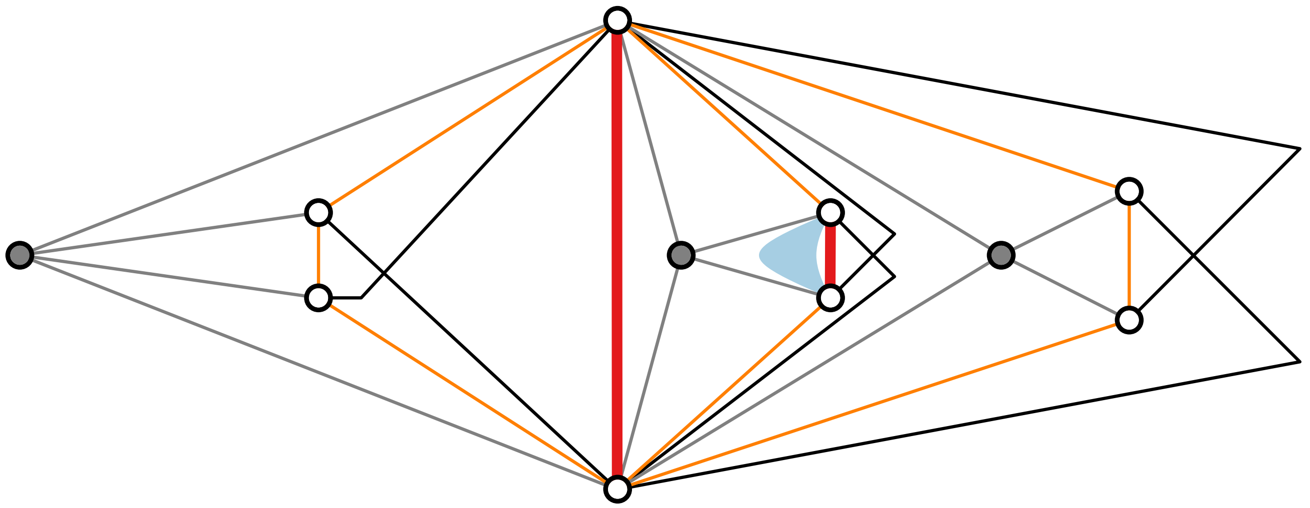


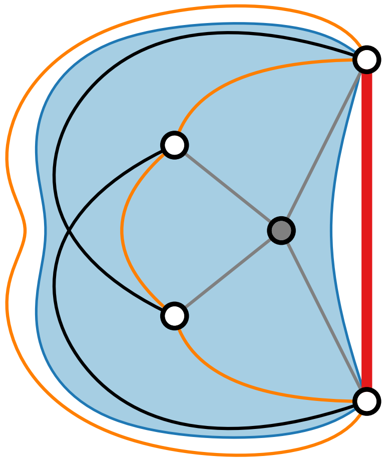
# Algorithm Step 3: Drawing Procedure

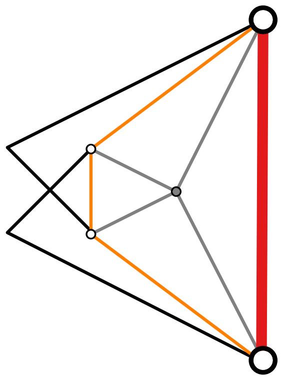




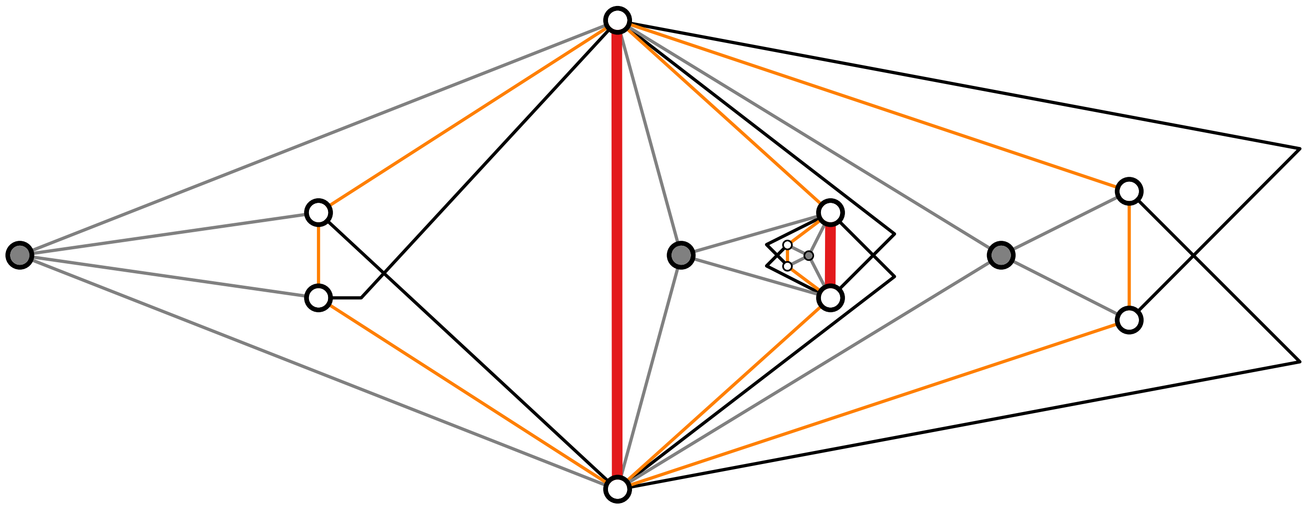
# Algorithm Step 3: Drawing Procedure

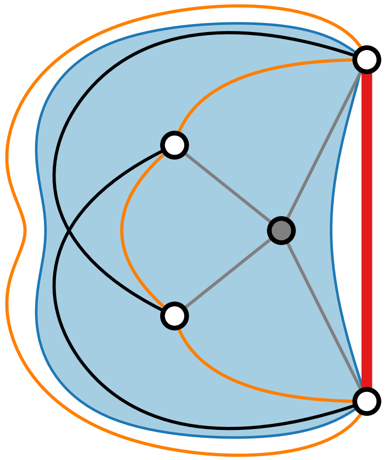


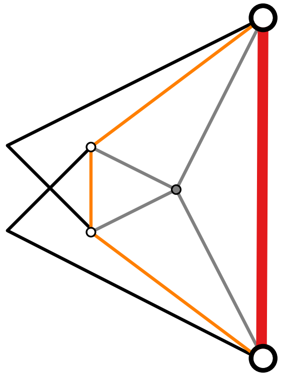




# Algorithm Step 3: Drawing Procedure

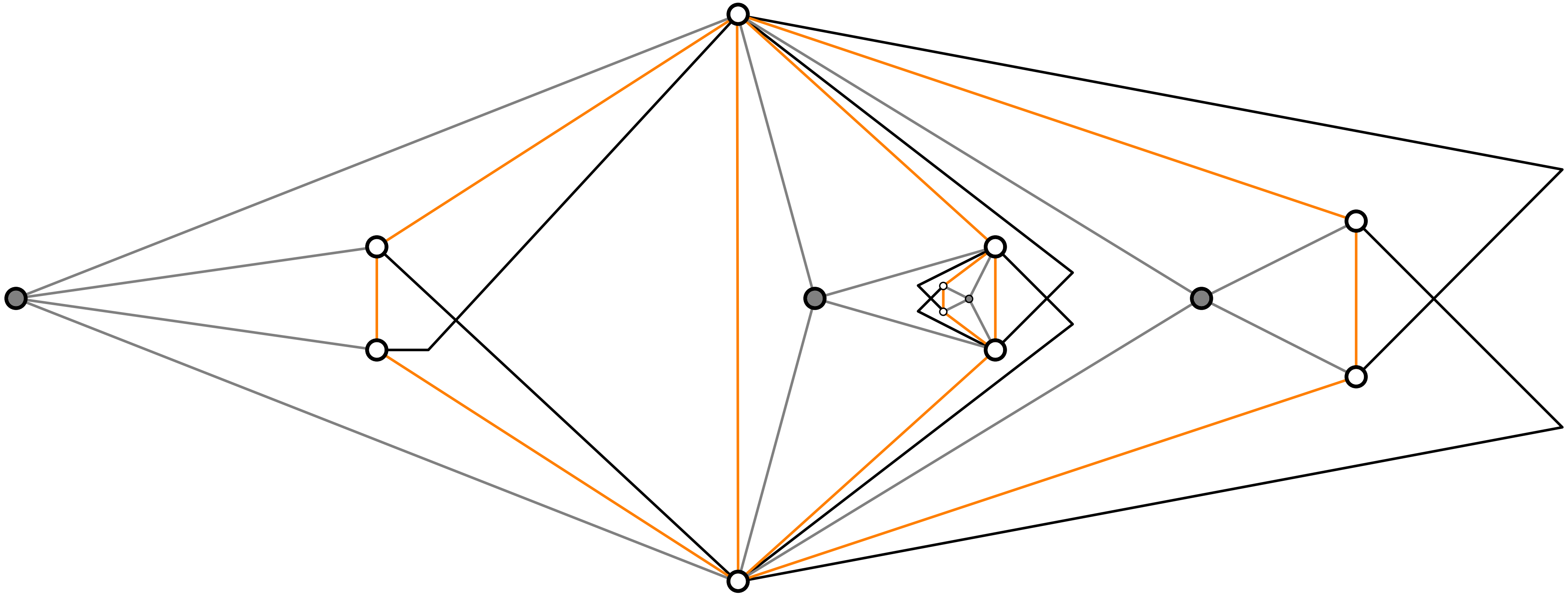




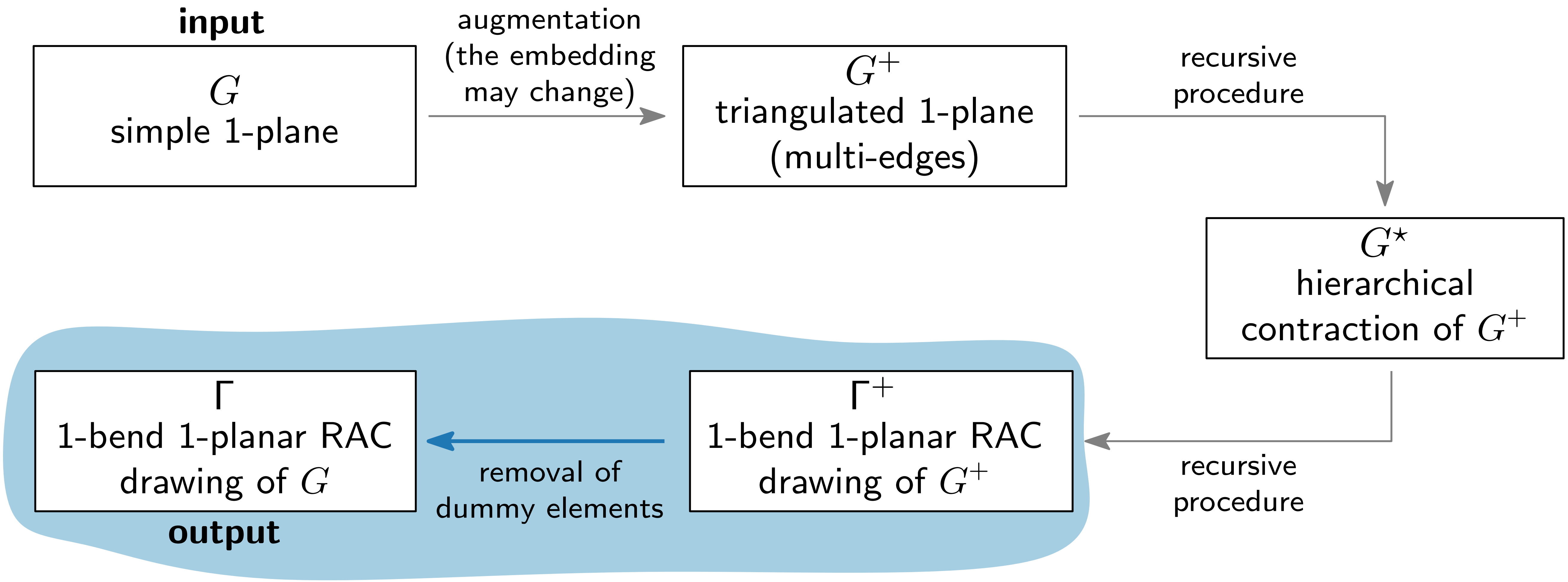


# Algorithm Step 3: Drawing Procedure

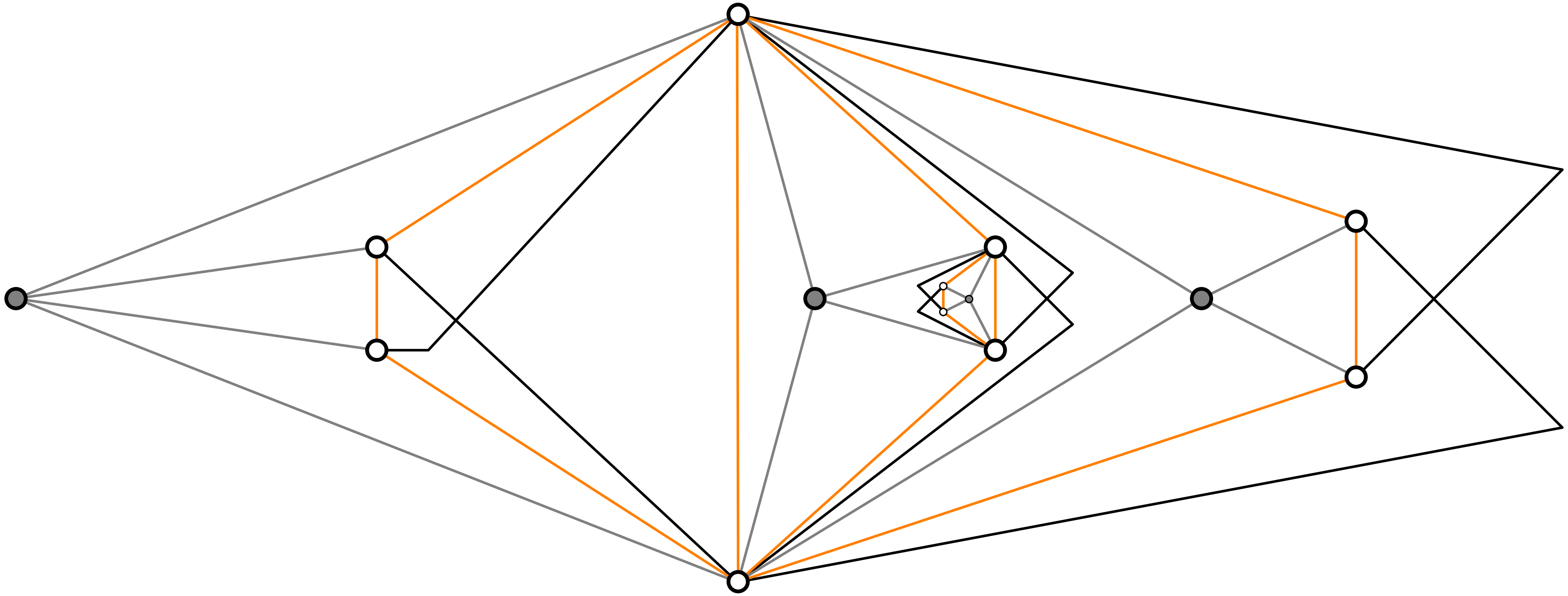
$\Gamma^+$
1-bend 1-planar RAC
drawing of $G^+$



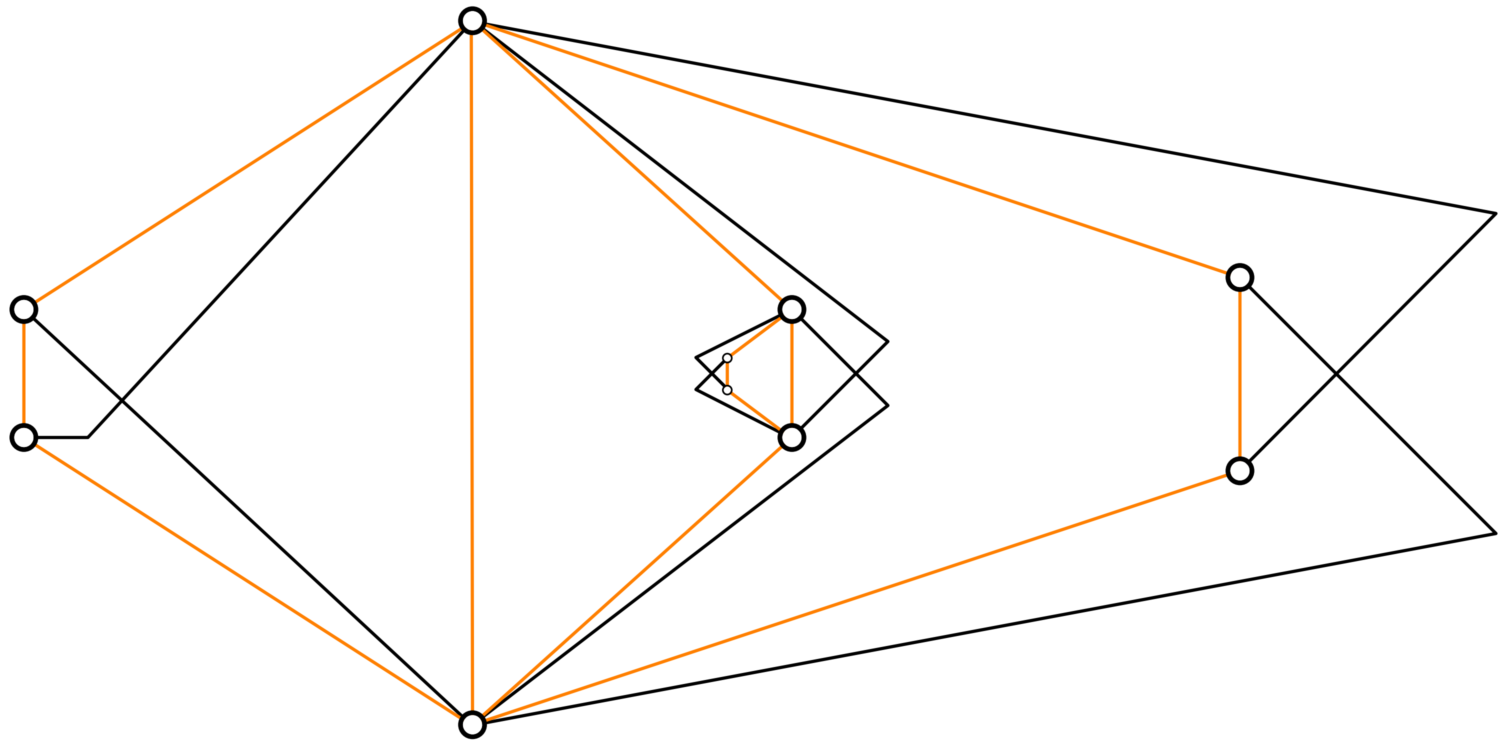
# Algorithm Outline



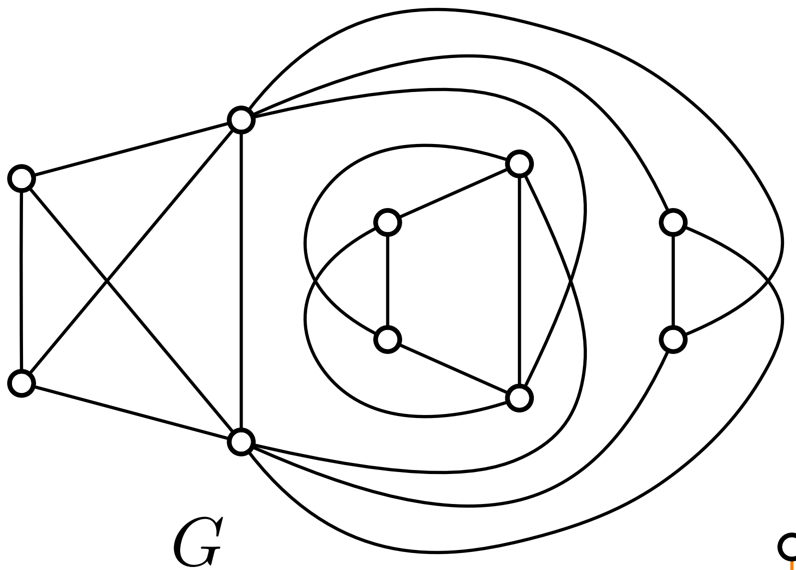
# Algorithm Step 4: Removal of Dummy Vertices

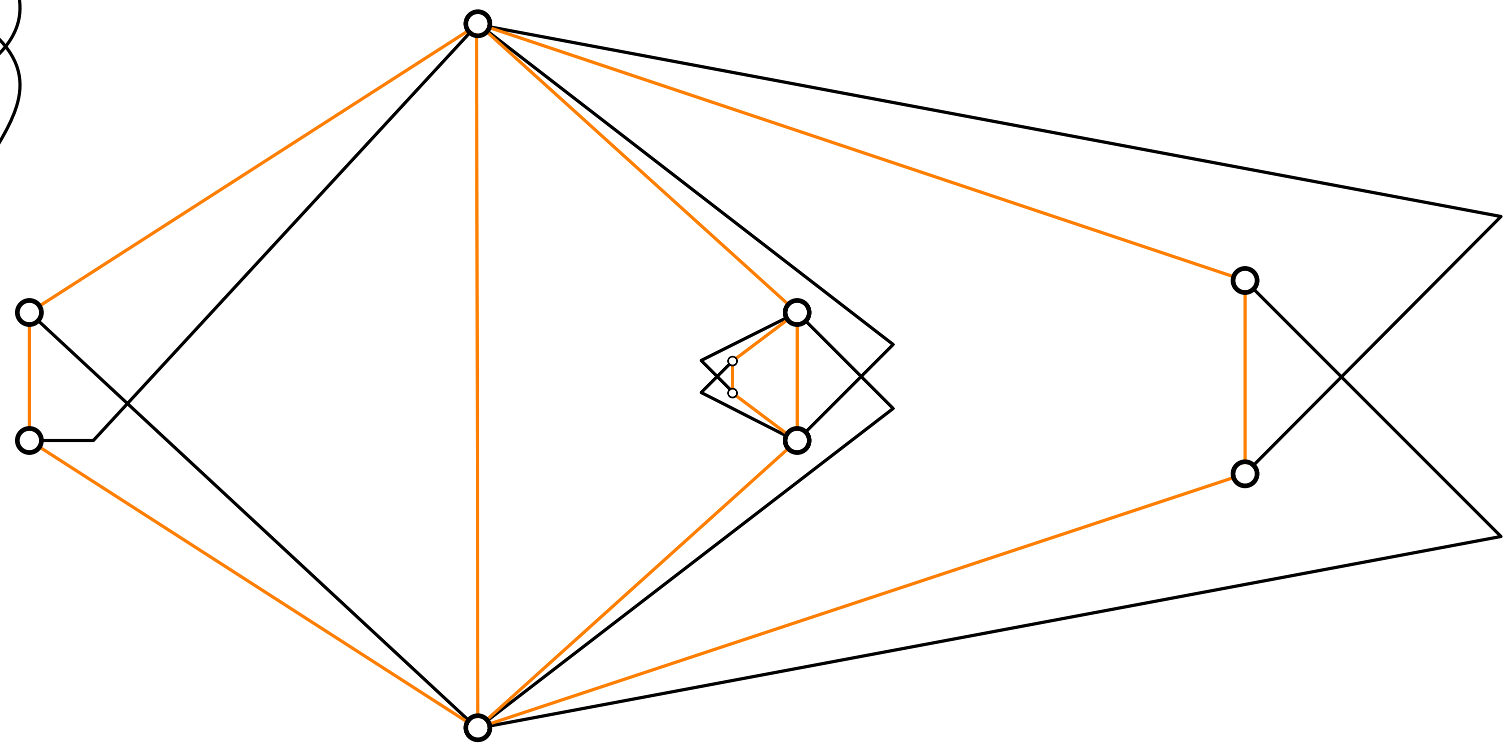


# Algorithm Step 4: Removal of Dummy Vertices

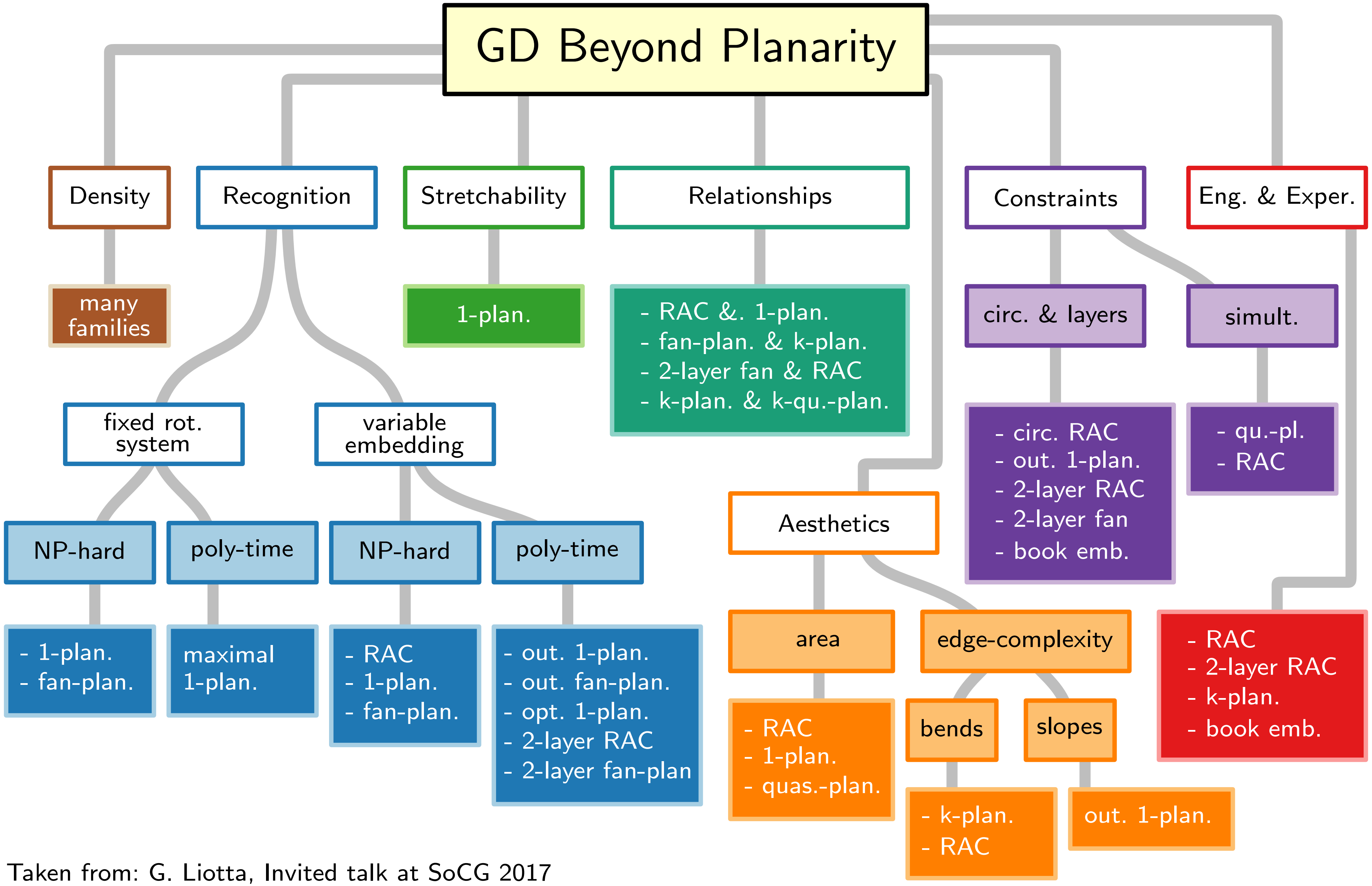


# Algorithm Step 4: Removal of Dummy Vertices



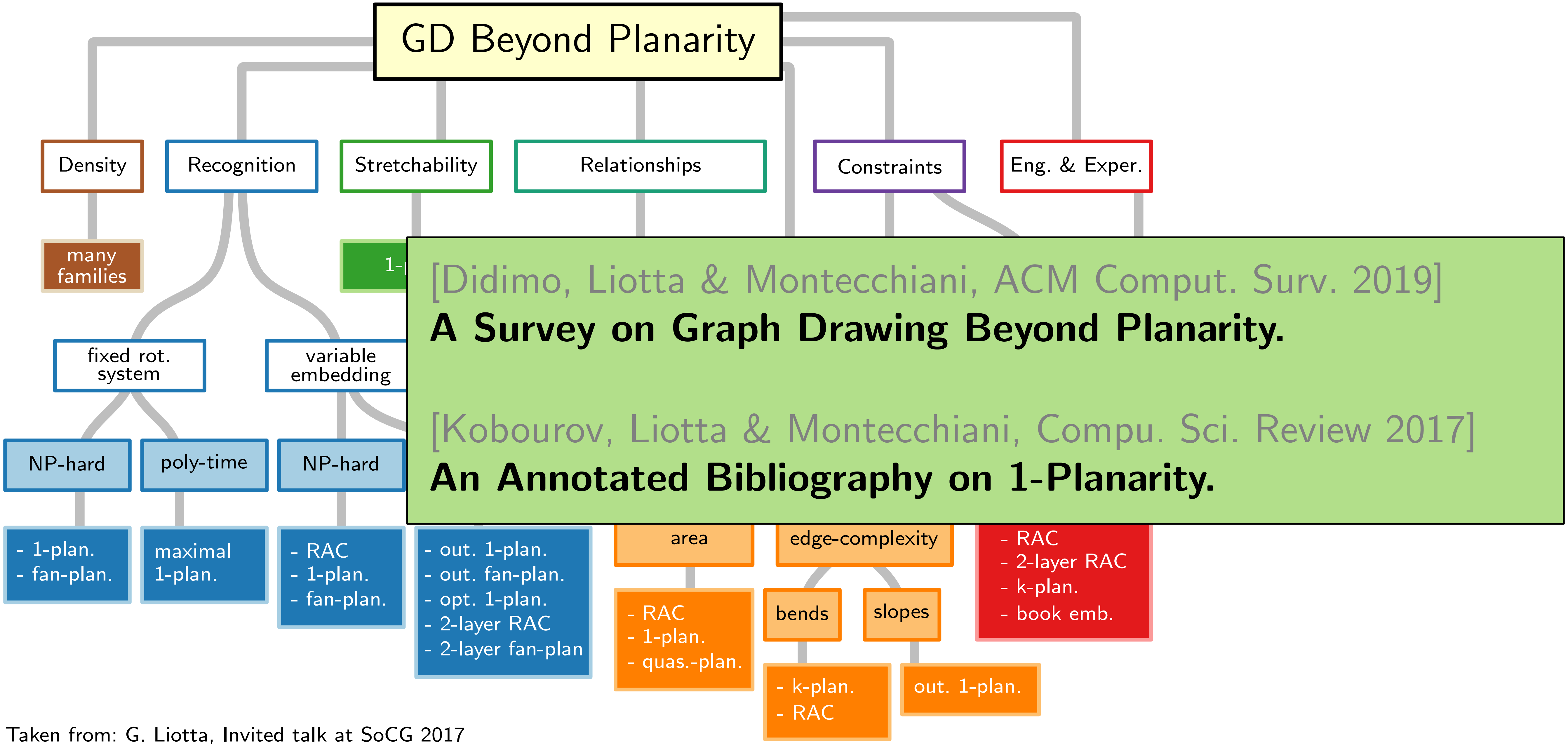



# GD Beyond Planarity: a Taxonomy




Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

# GD Beyond Planarity: a Taxonomy




[Didimo, Liotta & Montecchiani, ACM Comput. Surv. 2019]
**A Survey on Graph Drawing Beyond Planarity.**

[Kobourov, Liotta & Montecchiani, Compu. Sci. Review 2017]
**An Annotated Bibliography on 1-Planarity.**

Taken from: G. Liotta, Invited talk at SoCG 2017
"Graph Drawing Beyond Planarity: Some Results and Open Problems", Jul. 2017

# Literature

Books and surveys:

- [Didimo, Liotta & Montecchiani 2019] A Survey on Graph Drawing Beyond Planarity
- [Kobourov, Liotta & Montecchiani '17] An Annotated Bibliography on 1-Planarity
- [Eds. Hong and Tokuyama '20] Beyond Planar Graphs

Some references for proofs:

- [Eades, Huang, Hong '08] Effects of Crossing Angles
- [Brandenburg et al. '13] On the density of maximal 1-planar graphs
- [Chimani, Kindermann, Montecchani, Valter '19] Crossing Numbers of Beyond-Planar Graphs
- [Grigoriev and Bodlaender '07] Algorithms for graphs embeddable with few crossings per edge
- [Angilini et al. '11] On the Perspectives Opened by Right Angle Crossing Drawings
- [Didimo, Eades, Liotta '17] Drawing graphs with right angle crossings
- [Bekos et al. '17] On RAC drawings of 1-planar graphs