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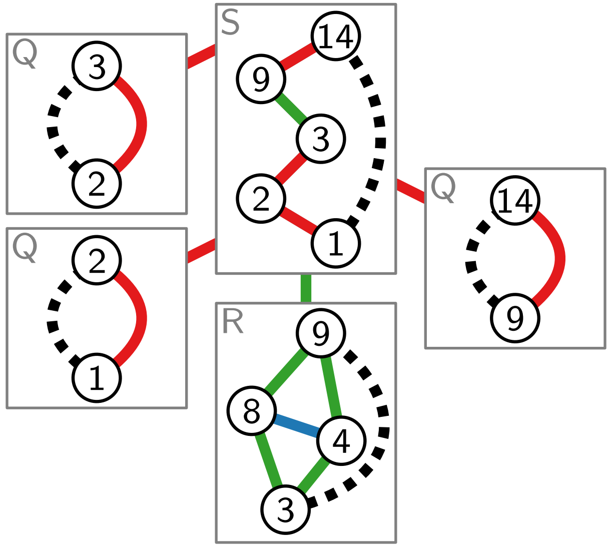
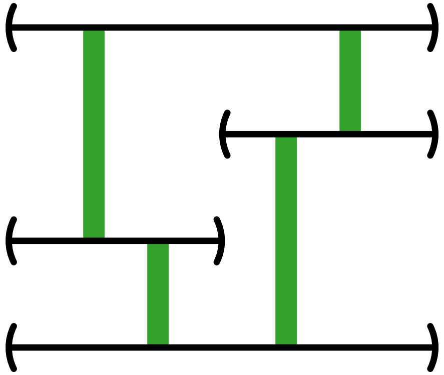
Visualization of Graphs

Lecture 9:
Partial Visibility Representation Extension



Part I:
Problem Definition



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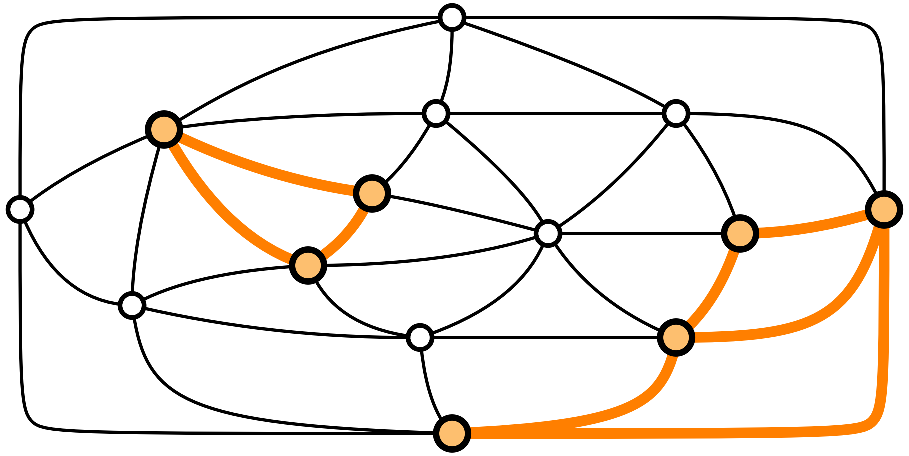
Partial Representation Extension Problem

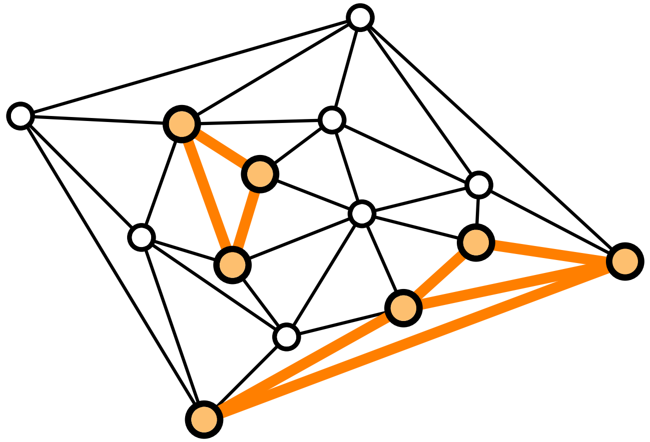
Let $G = (V, E)$ be a graph.

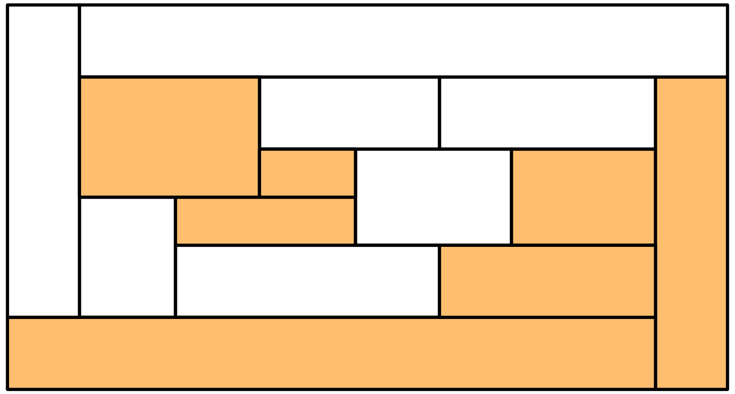
Let $V' \subseteq V$ and $H = G[V']$

Let Γ_H be representation of H

Find representation Γ_G that *extends* Γ_H

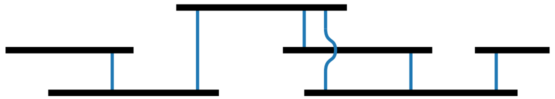




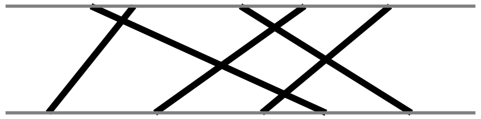


Polytime for:

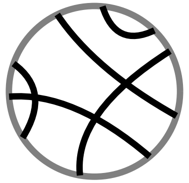
- (unit) interval graphs



- permutation graphs

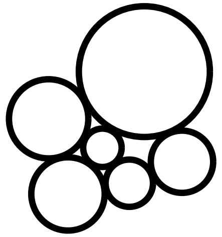


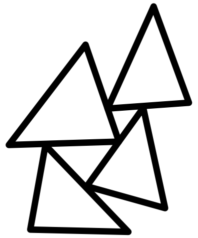
- circle graphs

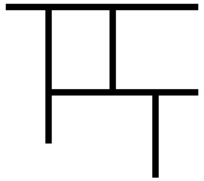


NP-hard for:

- planar straight-line drawings
- contacts of
  - disks
  - triangles
  - orthogonal segments

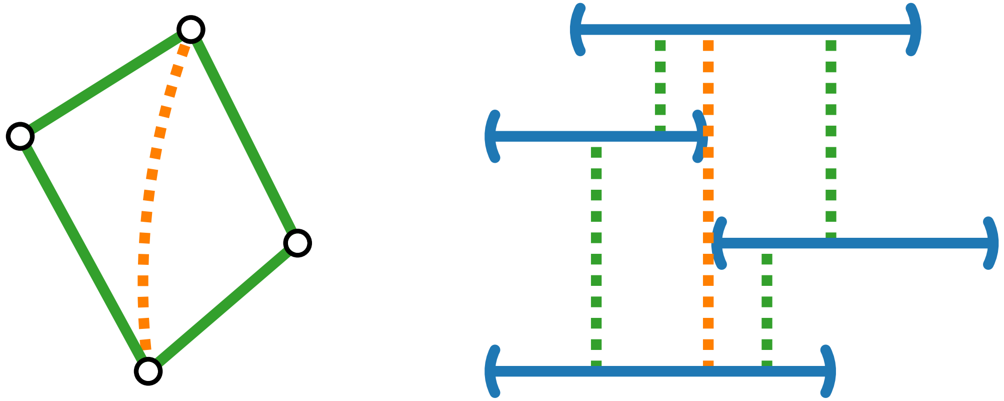






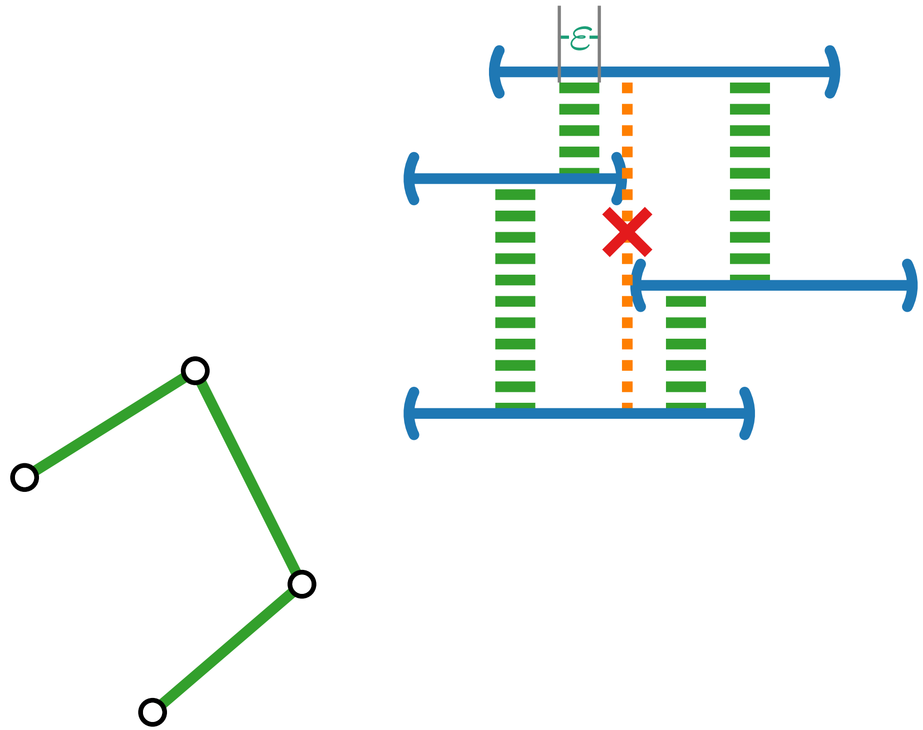
Bar Visibility Representation

- Vertices correspond to horizontal open line segments called **bars**
- **Edges** correspond to vertical unobstructed vertical sightlines
- What about unobstructed **0-width** vertical sightlines? Do all visibilities induce edges?

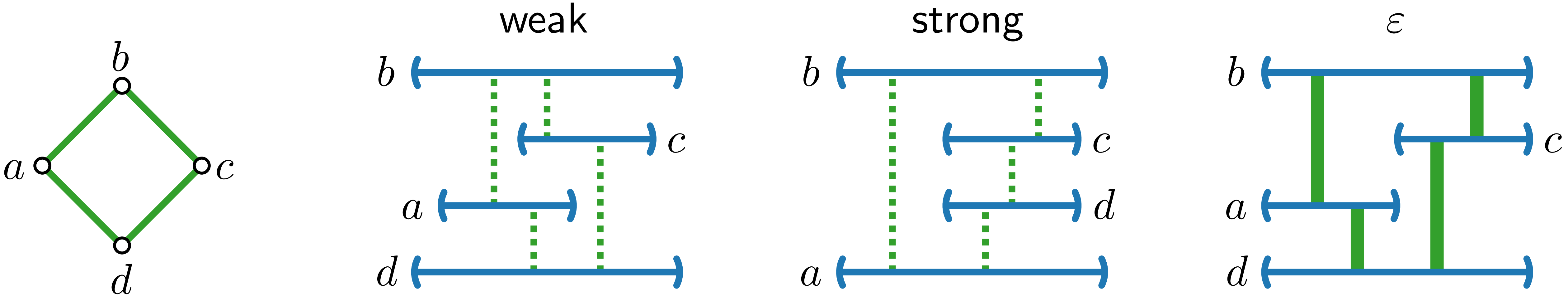


Models.

- **Strong:** Edge $uv \Leftrightarrow$ unobstructed **0-width** vertical sightlines
- ε**:** Edge $uv \Leftrightarrow \varepsilon$ wide vertical sightlines for $\varepsilon > 0$
- **Weak:** Edge $uv \Rightarrow$ unobstructed vertical sightlines exists, i. e., any subset of *visible* pairs



Problems

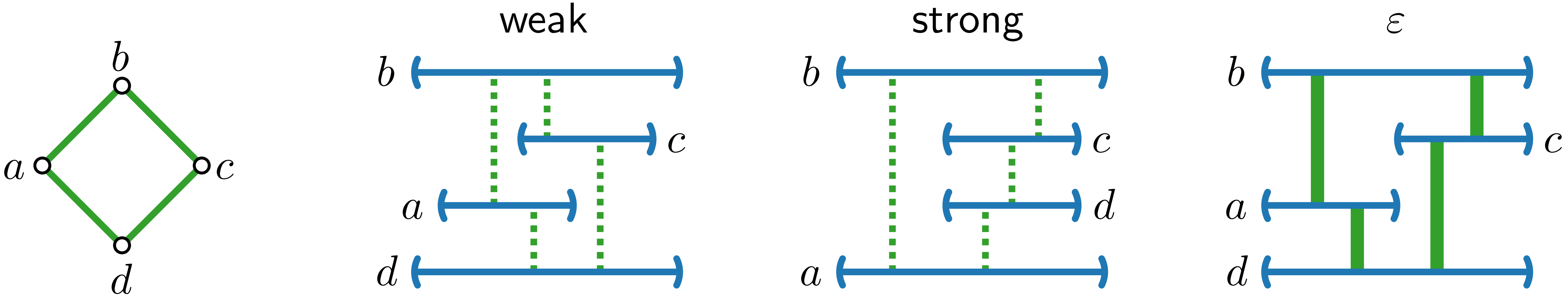


Recognition Problem.
Given a graph G, **decide** if there exists a weak/strong/ε bar visibility representation ψ of G.

Construction Problem.
Given a graph G, **construct** a weak/strong/ε bar visibility representation ψ of G when one exists.

Partial Representation Extension Problem.
Given a graph G and a **set of bars ψ' of $V' \subset V(G)$**, **decide** if there exists a weak/strong/ε bar visibility representation ψ of G **where $\psi|_{V'} = \psi'$** (and **construct** ψ when it exists).

Background



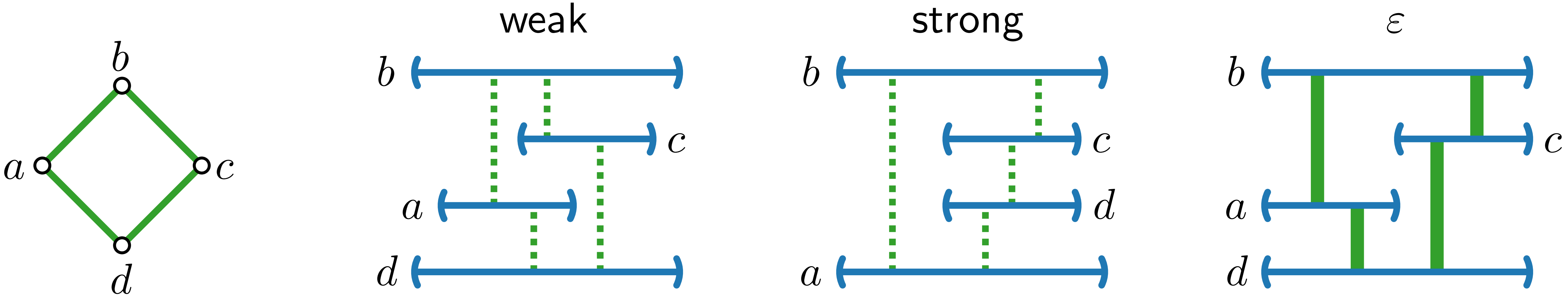
Weak Bar Visibility.

- All planar graphs. [Tamassia & Tollis '86; Wismath '85]
- Linear time recognition and construction [T&T '86]
- Representation Extension is NP-complete [Chaplick et al. '14]

Strong Bar Visibility.

- NP-complete to recognize [Andreae '92]

Background



ε-Bar Visibility.

- Planar graphs that can be embedded with all **cut vertices** on the outerface. [T&T '86, Wismath '85]
- Linear time recognition and construction [T&T '86]
- Representation Extension? This Lecture!

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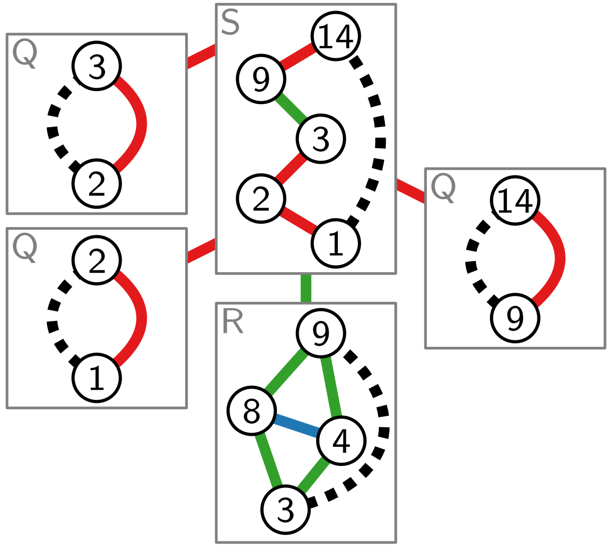
Visualization of Graphs

Lecture 9:
Partial Visibility Representation Extension



Part II:
Recognition & Construction

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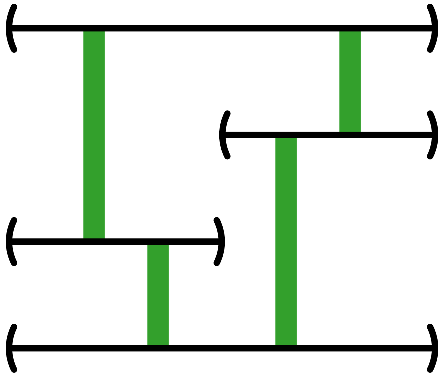
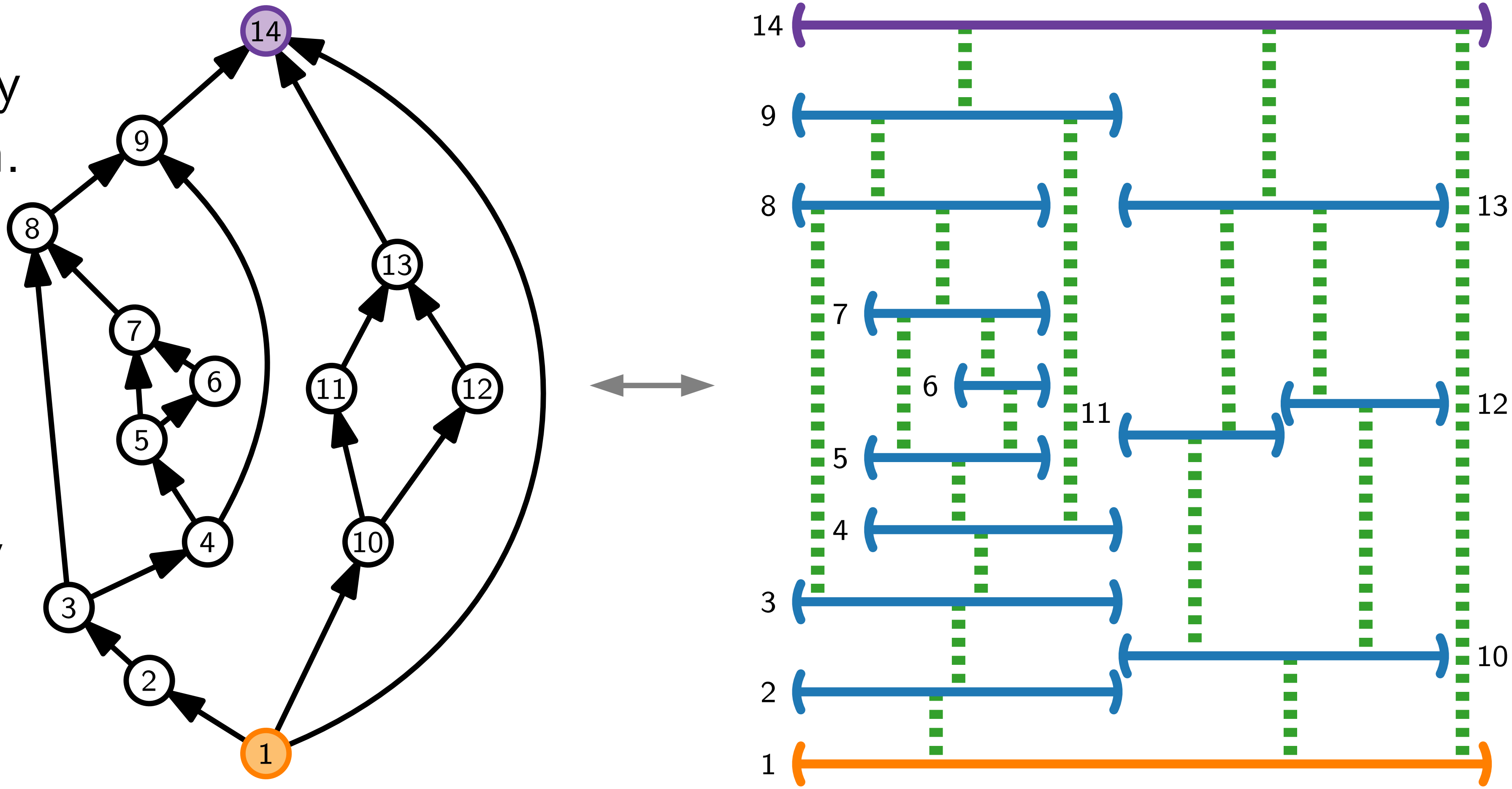


ε-bar Visibility and st-Graphs

Recall that an **st-graph** is a planar digraph G with exactly one source s and one sink t where s and t occur on the outer face of an embedding of G.

Observation.
st-orientations correspond to ε-bar visibility representations.

- ε-bar visibility testing is easily done via st-graph recognition.
- Strong bar visibility recognition. . . open!
- In a **rectangular** bar visibility representation $\psi(s)$ and $\psi(t)$ span an enclosing rectangle.



Results and Outline

Theorem 1. [Chaplick et al. '18]
Rectangular ε-Bar Visibility Representation Extension can be solved in $\mathcal{O}(n \log^2 n)$ time for st-graphs.

- Dynamic program via SPQR-trees
- Easier version: $\mathcal{O}(n^2)$

Theorem 2. [Chaplick et al. '18]
ε-Bar Visibility Representation Ext. is NP-complete.

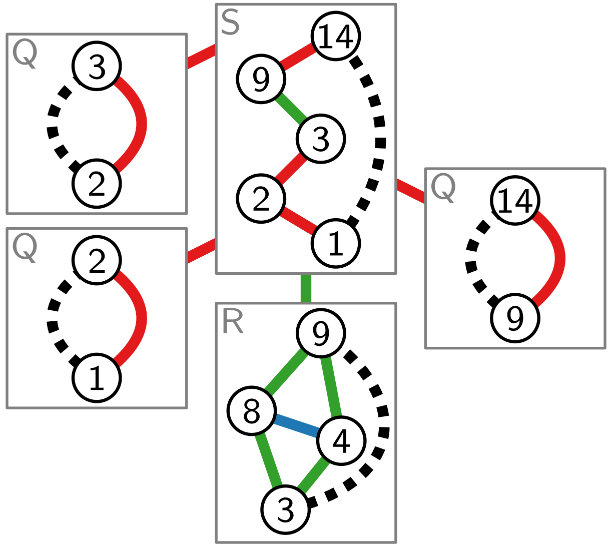
- Reduction from Planar Monotone 3-SAT

Theorem 3. [Chaplick et al. '18]
ε-Bar Visibility Representation Ext. is NP-complete for (series-parallel) st-graphs when restricted to the **integer grid** (or if any fixed $\varepsilon > 0$ is specified).

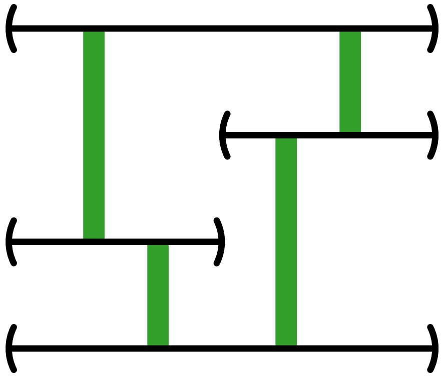
- Reduction from 3-Partition

Visualization of Graphs

Lecture 9:
Partial Visibility Representation Extension



Part III:
SPQR-Trees



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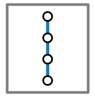
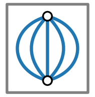
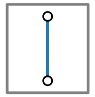
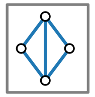
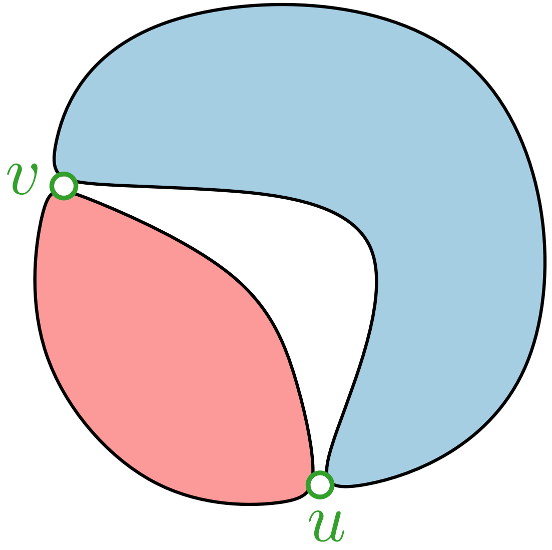
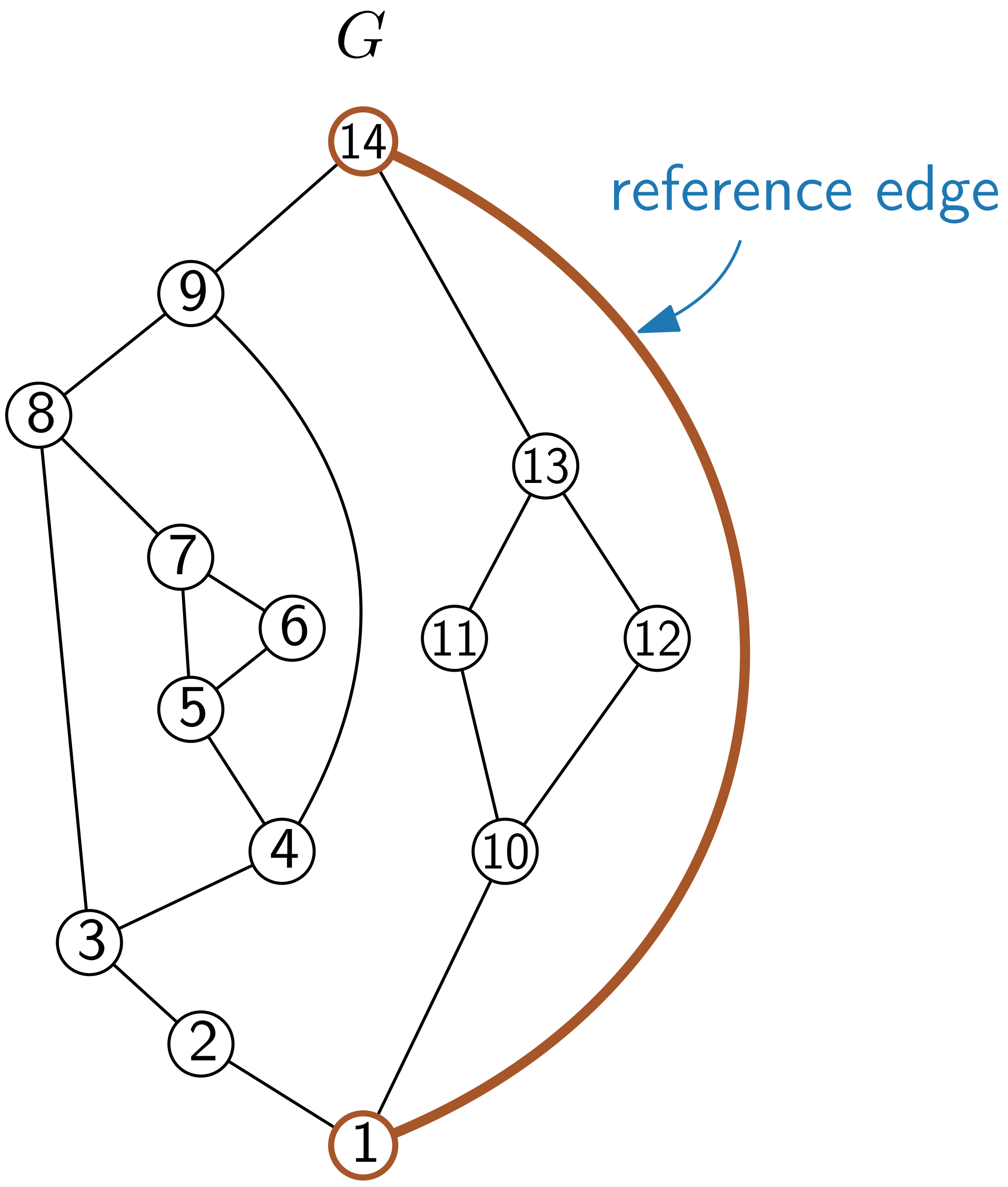
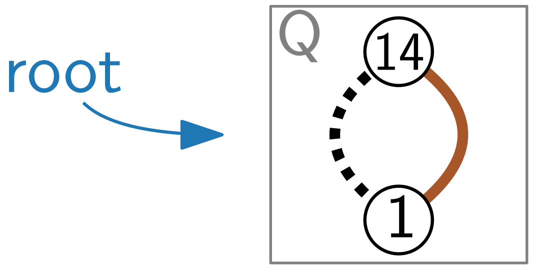
SPQR-Tree

- An **SPQR-tree** T is a decomposition of a planar graph G by **separation pairs**.
- The nodes of T are of four types:
  - **S** nodes represent a series composition 
  - **P** nodes represent a parallel composition 
  - **Q** nodes represent a single edge 
  - **R** nodes represent 3-connected (*rigid*) subgraphs 
- A decomposition tree of a series-parallel graph is an SPQR-tree without **R** nodes.
- T represents all planar embeddings of G.
- T can be computed in $\mathcal{O}(n)$ time. [Gutwenger, Mutzel '01]

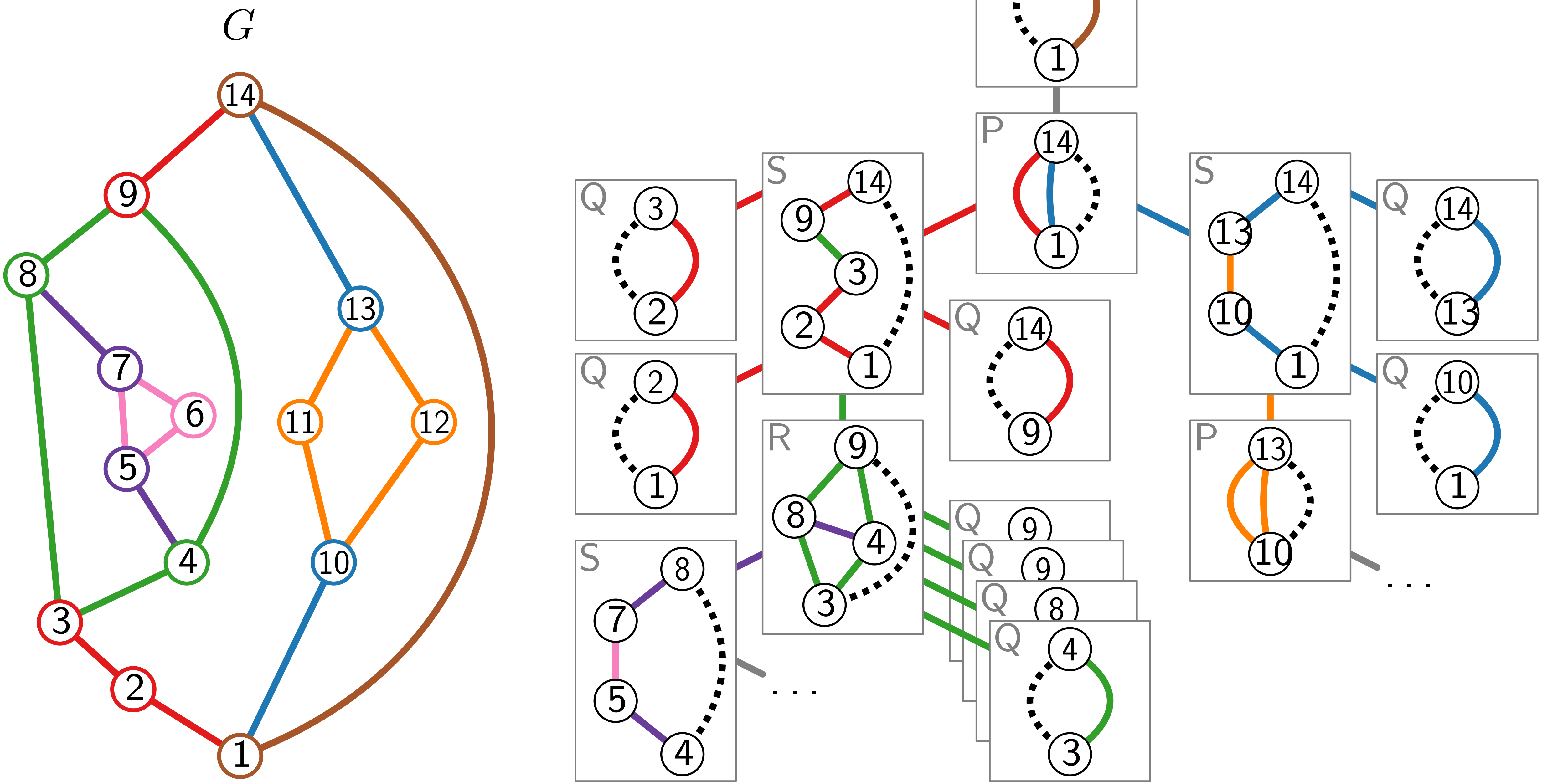


SPQR-Tree Example





SPQR-Tree Example



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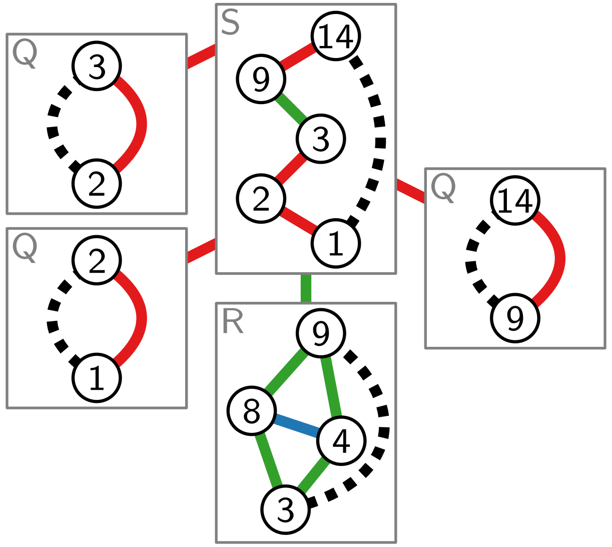
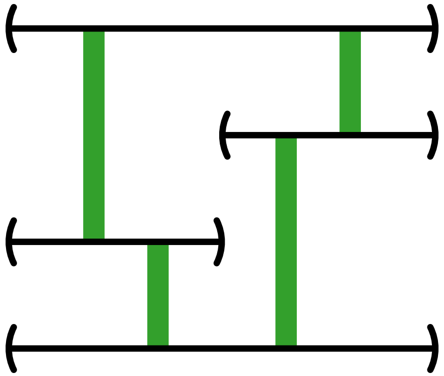


Part IV:
Rectangular Representation Extension



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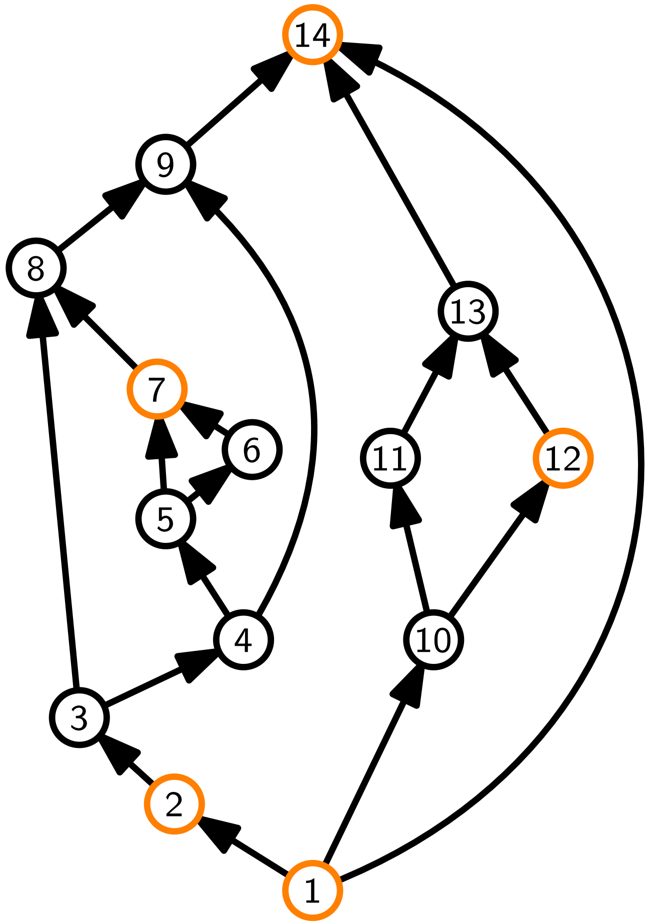
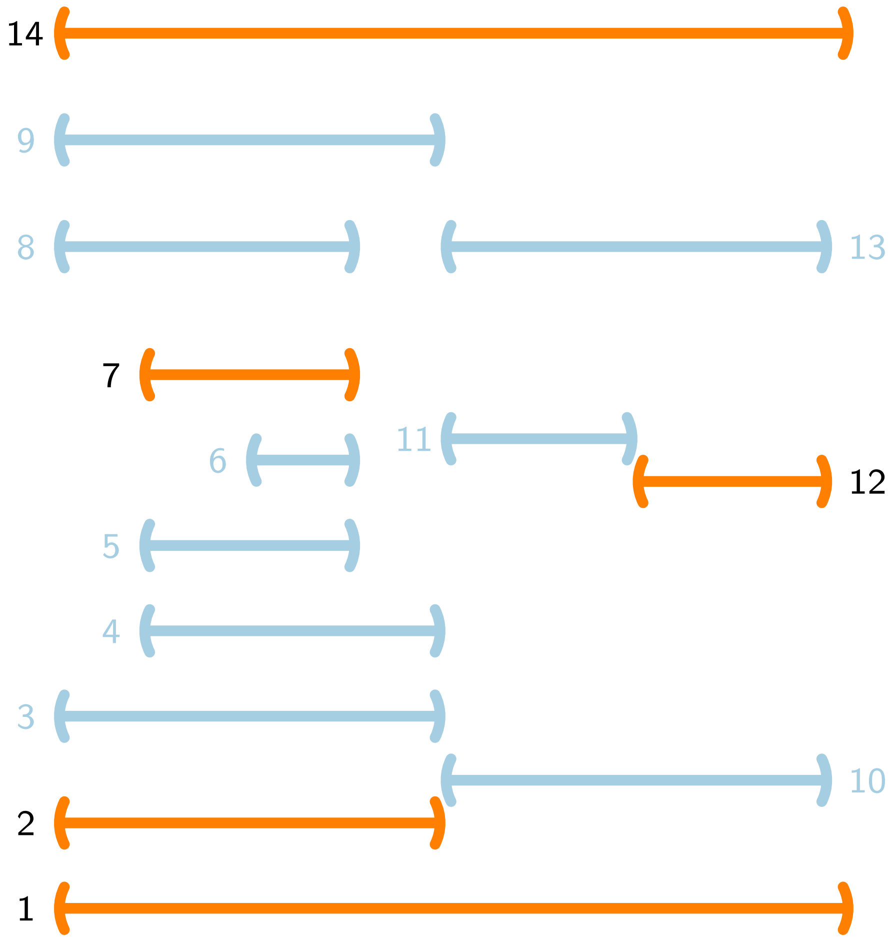
Representation Extension for st-Graphs

Theorem 1'.
Rectangular ε-Bar Visibility Representation Extension can be solved in $\mathcal{O}(n^2)$ time for st-graphs.





- Simplify with assumption on y-coordinates

Representation Extension for st-Graphs

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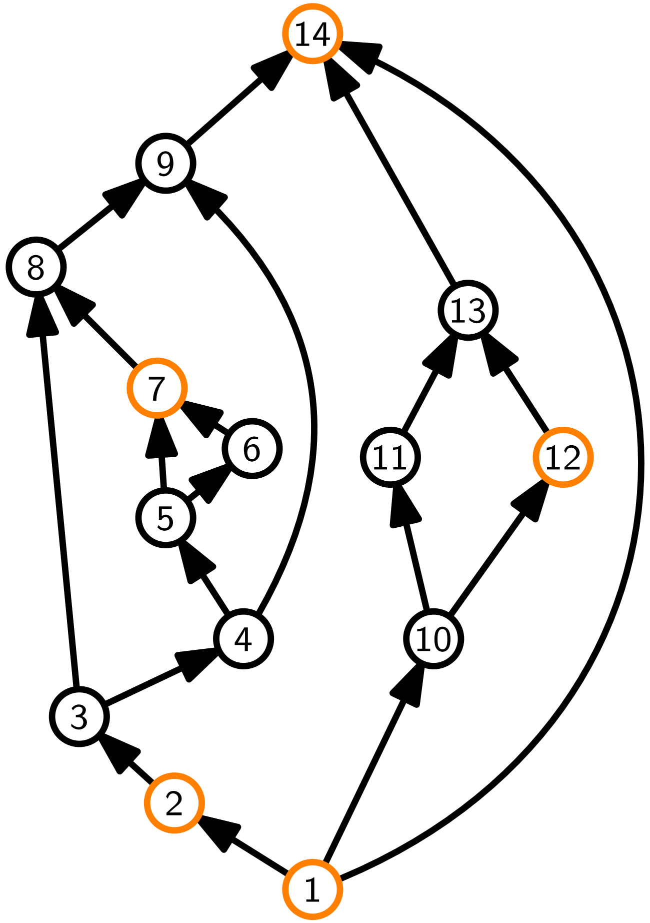
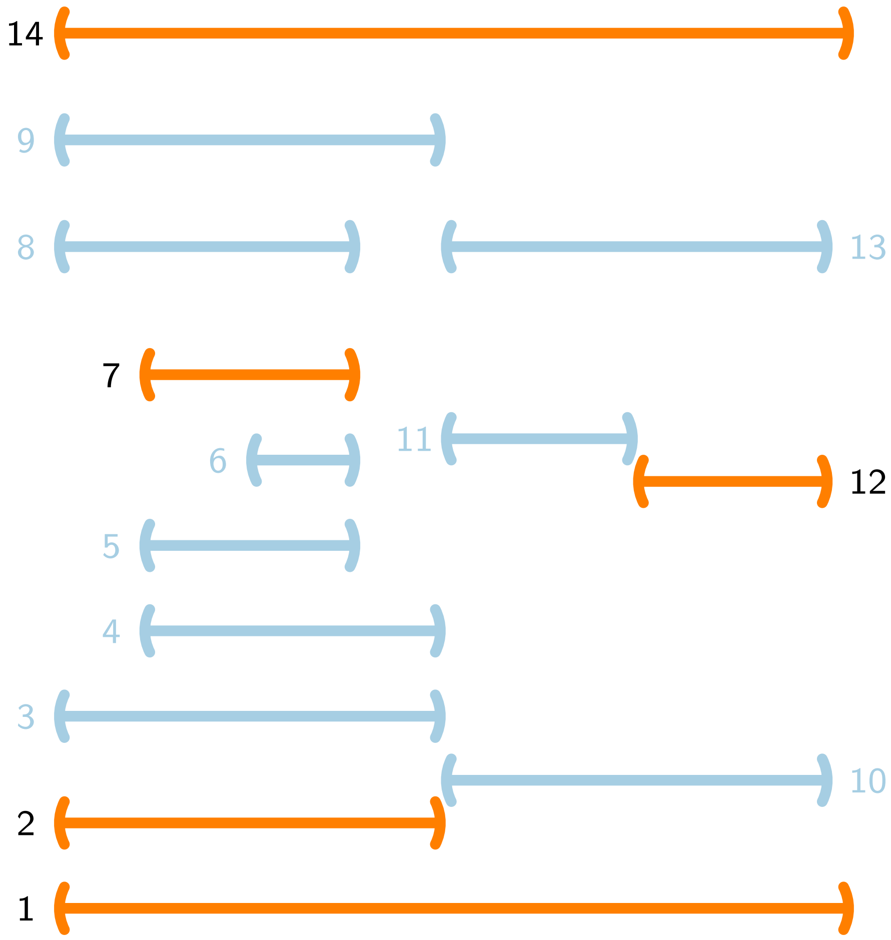




- Simplify with assumption on y-coordinates
- Look at connection to SPQR-trees – tiling
- Solve problems for **S**, **P** and **R** nodes
- Dynamic program via SPQR-tree

y-Coordinate Invariant

- Let $G = (V, E)$ be an st-graph, and ψ' be a representation of $V' \subseteq V$.
- Let $y : V \to \mathbb{R}$ such that
  - for each $v \in V'$, $y(v) =$ the y-coordinate of $\psi'(v)$.
  - for each edge (u, v), $y(u) < y(v)$.

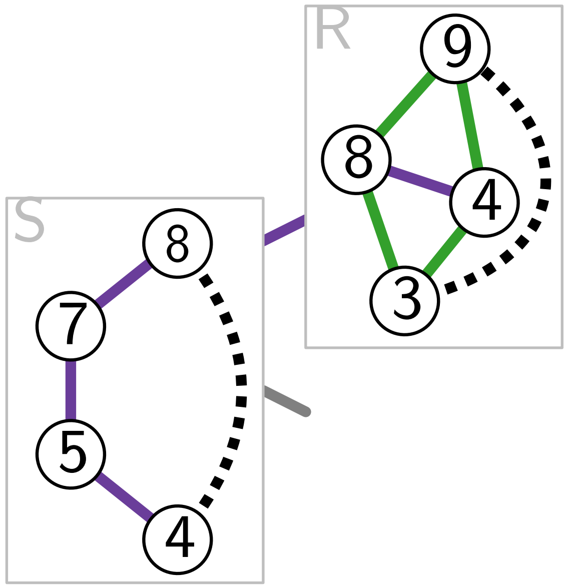
Lemma 1.
G has a representation extending ψ' if and only if G has a representation ψ extending ψ' where the y-coordinates of the bars are as in y.

We can now assume all y-coordinates are given!

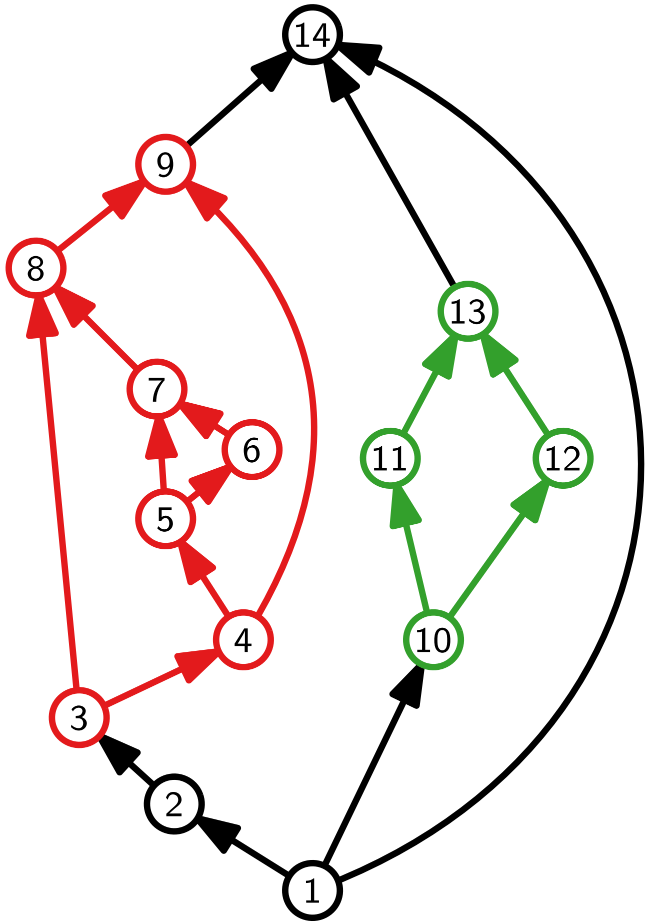
Proof idea. The relative positions of **adjacent** bars must match the order given by y.
So, we can adjust the y-coordinates of any solution to be as in y by sweeping from bottom-to-top.

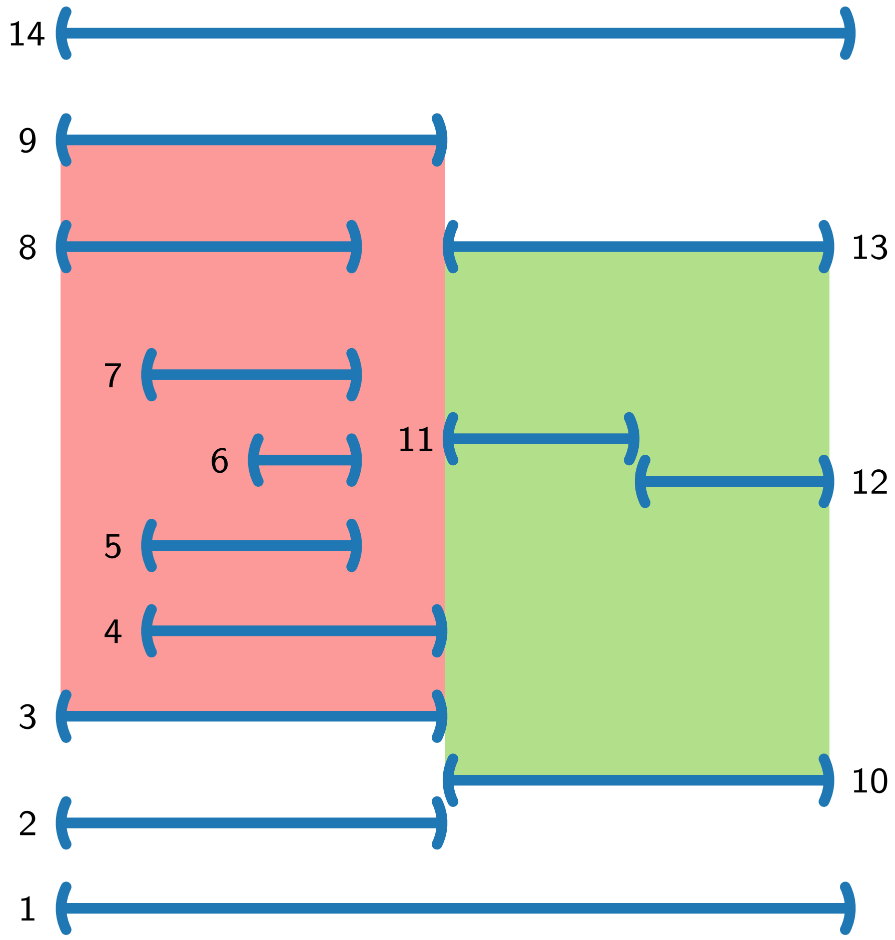
But why do SPQR-Trees help?

Lemma 2.
The SPQR-tree of an st-graph G induces a recursive **tiling** of any ε-bar visibility representation of G.



Solve tiles bottom-up





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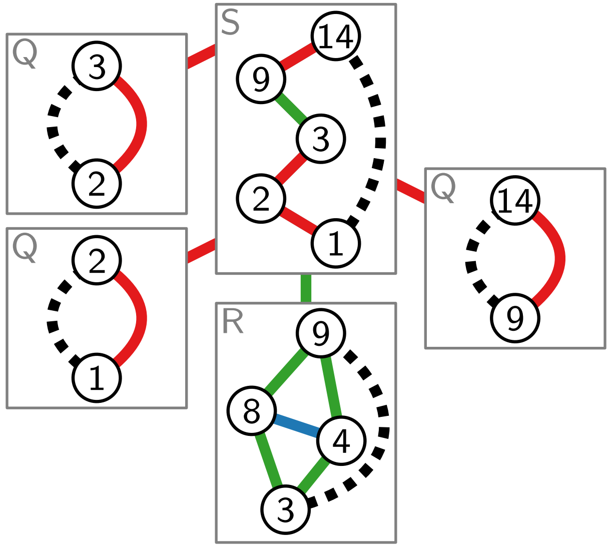

Part V:
Dynamic Program

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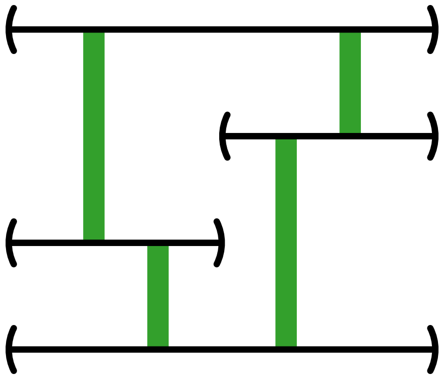
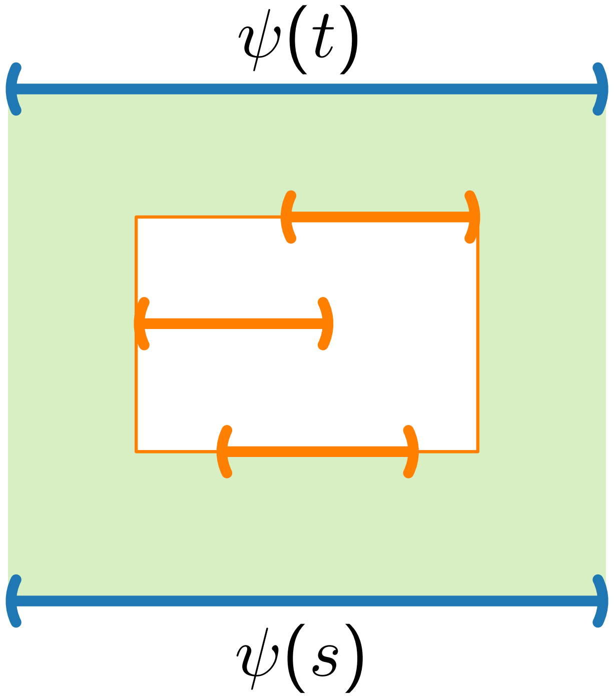


Tiles

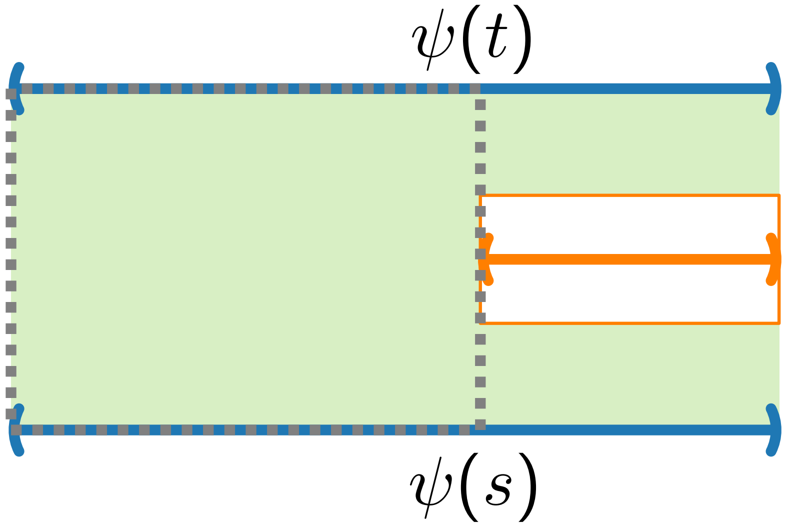
Convention. **Orange** bars are from the partial representation



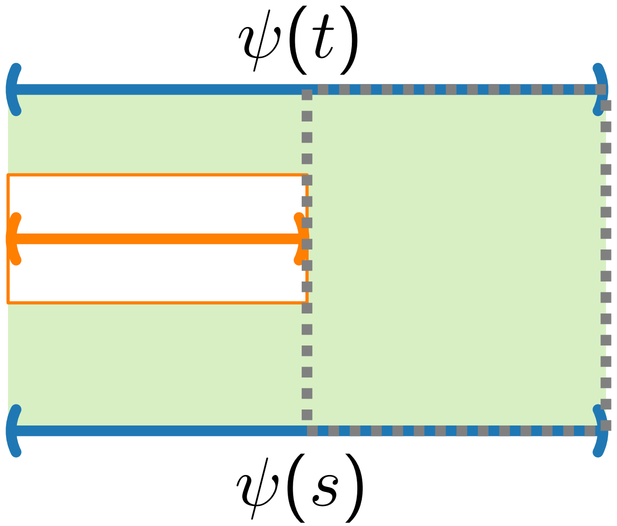
Observation.
The bounding box (tile) of any solution ψ **contains** the bounding box of the partial representation.

How many **different** tiles can we really have?

Types of Tiles



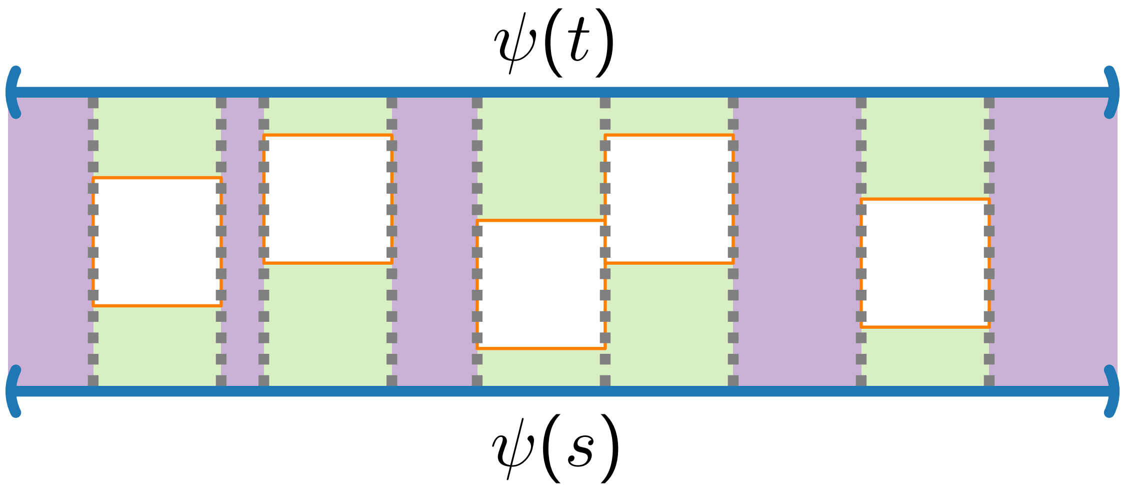
- Right **F**ixed – due to the orange bar
- Left **L**oose – due to the orange bar

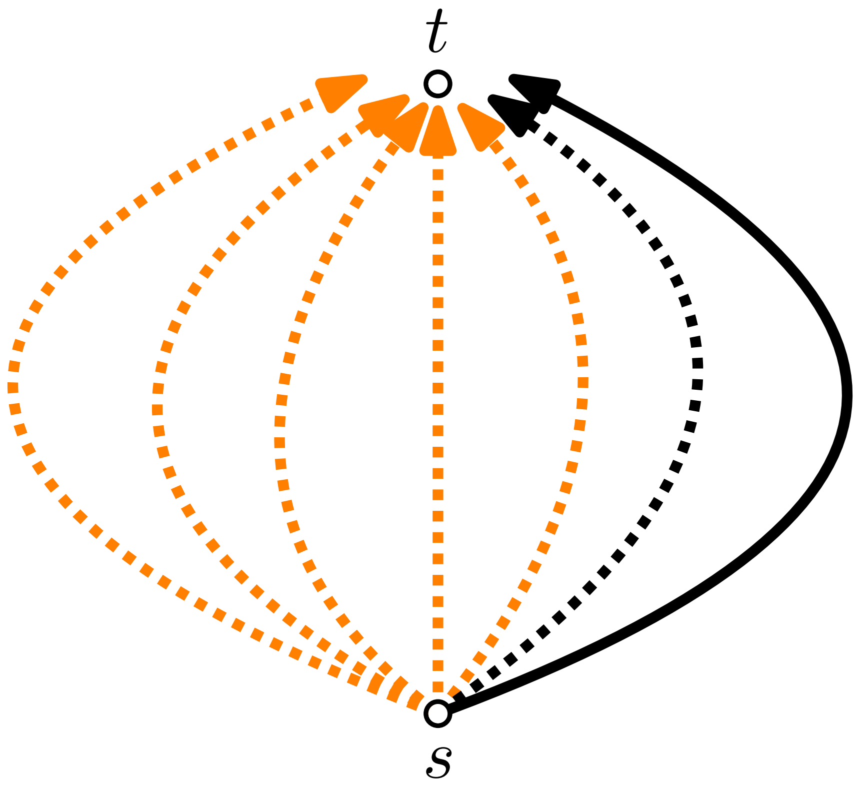


- Left **F**ixed – due to the orange bar
- Right **L**oose – due to the orange bar

Four different types: **FF, FL, LF, LL**

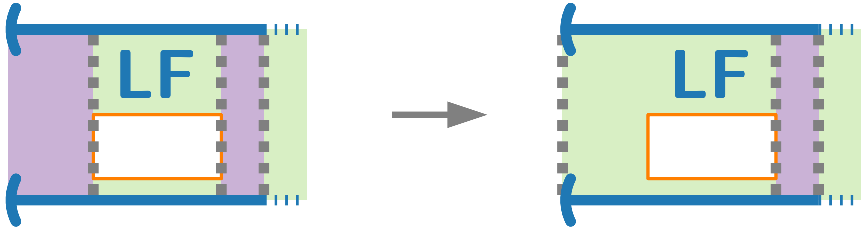
P Nodes





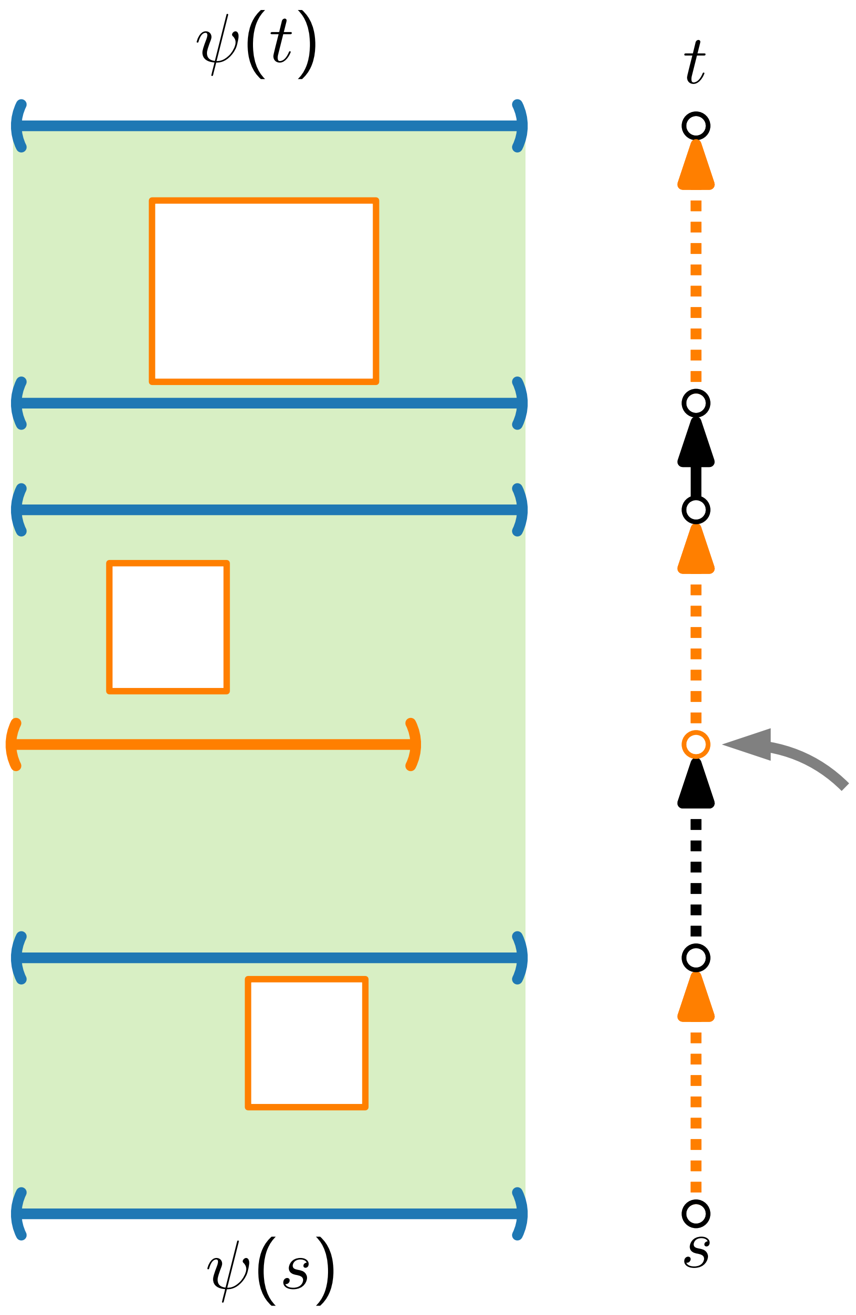
- Children of **P** node with prescribed bars occur in given left-to-right order
- But there might be some gaps. . .

Idea.
Greedily *fill* the gaps by preferring to "stretch" the children with prescribed bars.



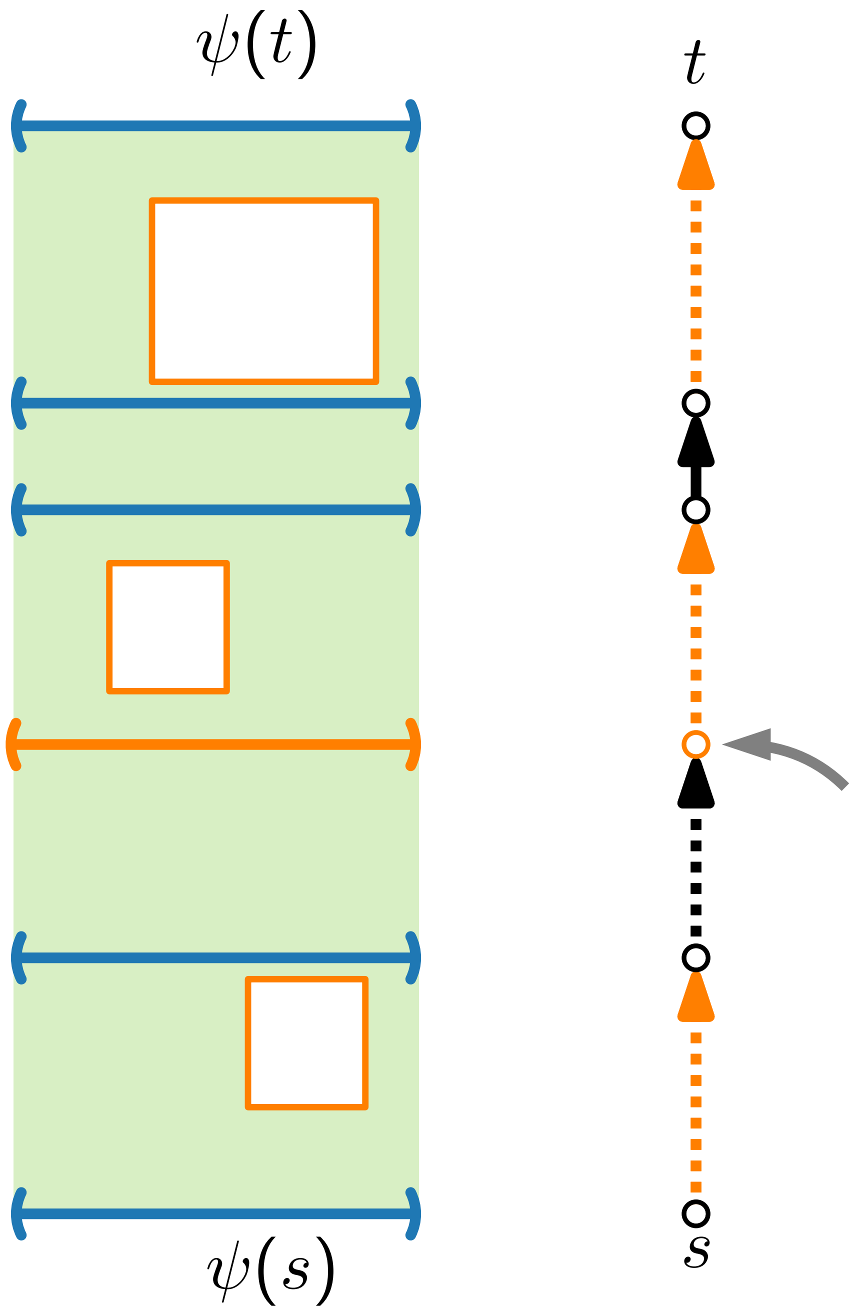
Outcome.
After processing, we must know the valid types for the corresponding subgraphs.

S Nodes



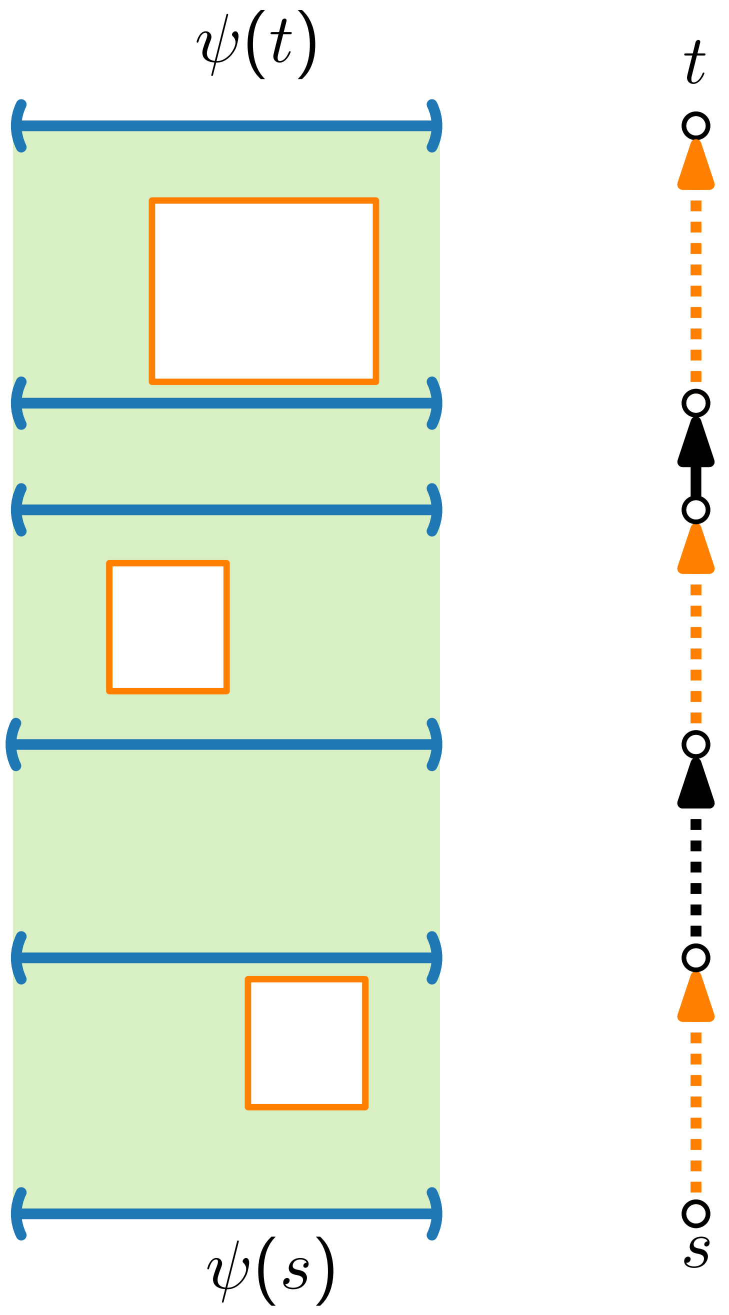
This fixed vertex means we can only make a Fixed-Fixed representation!

S Nodes



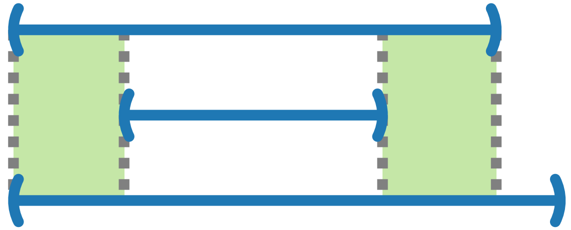
This fixed vertex means we can only make a Fixed-Fixed representation!

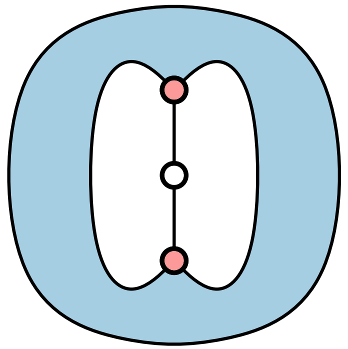
Here we have a chance to make all (**LL, FL, LF, FF**) types.



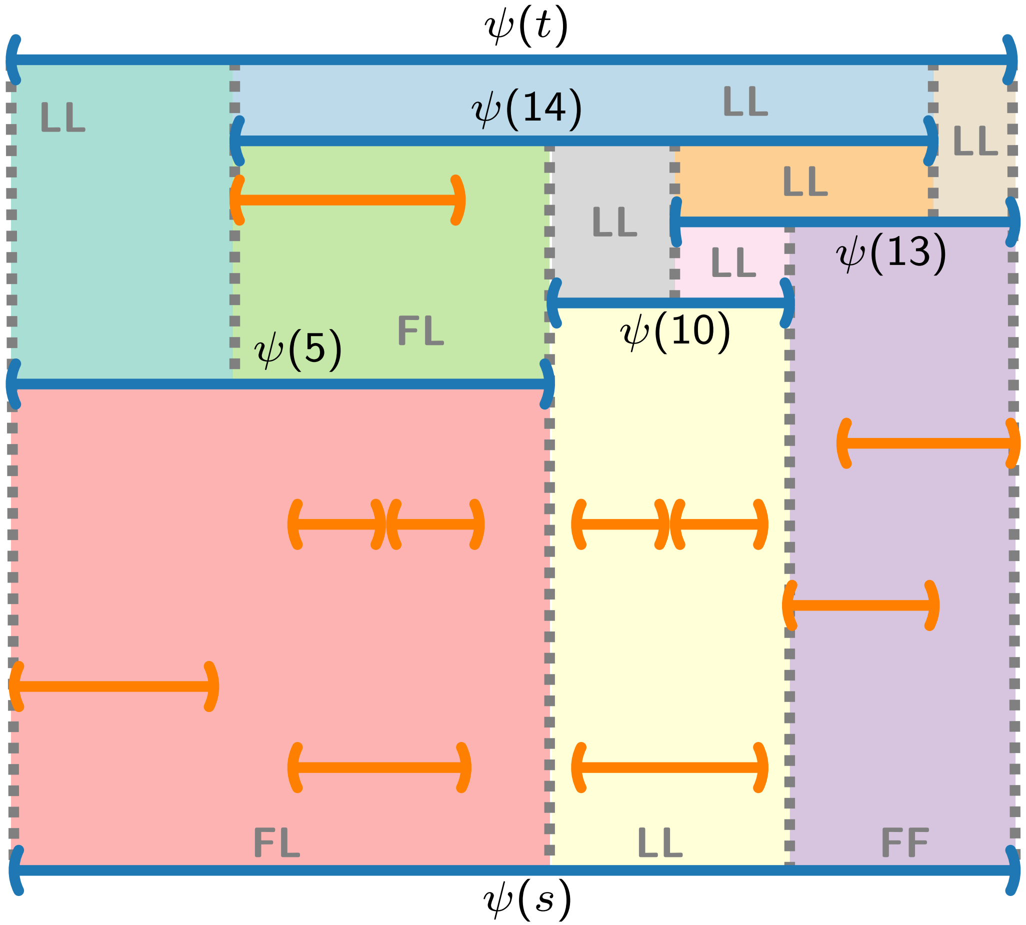
R Nodes with 2-SAT Formulation

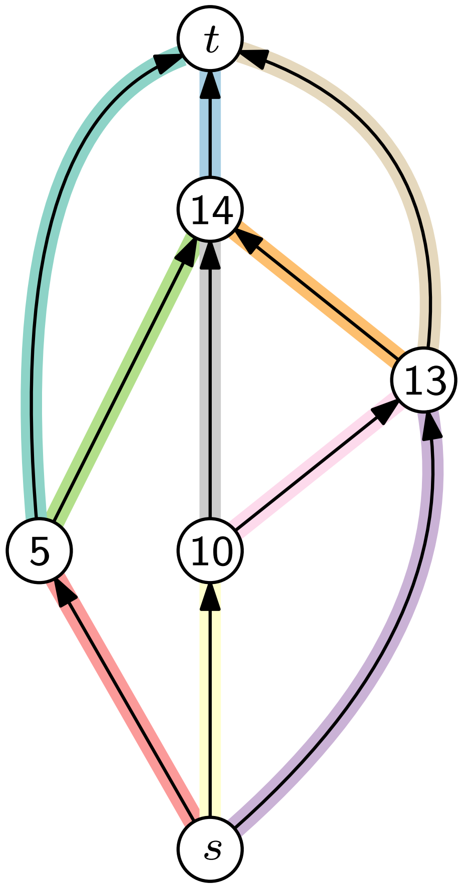
- for each child (edge) e:
  - find all types of $\{$**FF,FL,LF,LL**$\}$ that admit a drawing
  - 2 variables l_e, r_e encoding fixed/loose type of its tile
  - consistency clauses





cutting pair!





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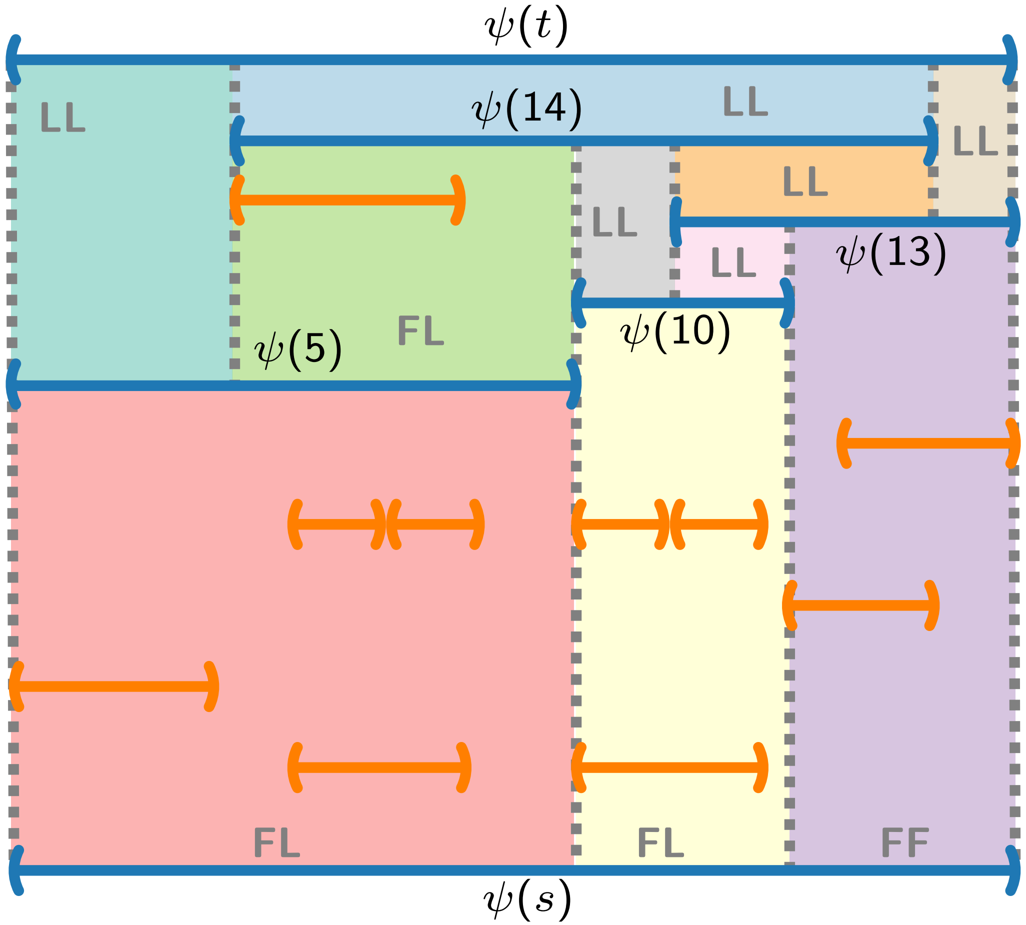

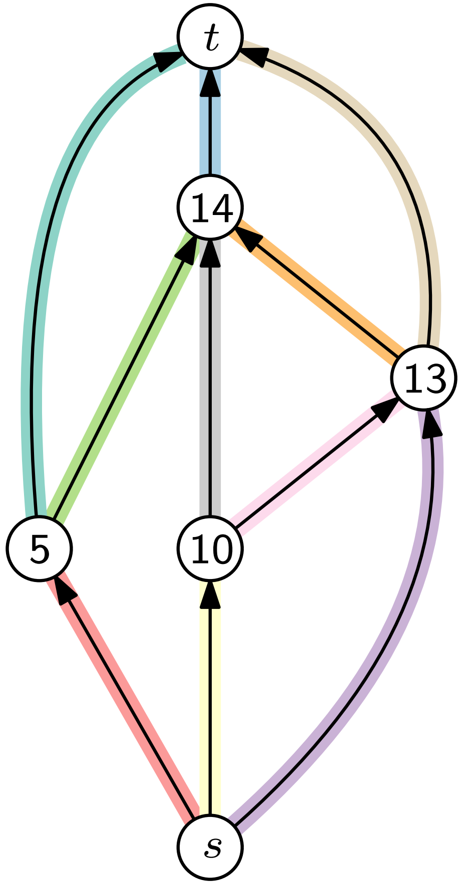

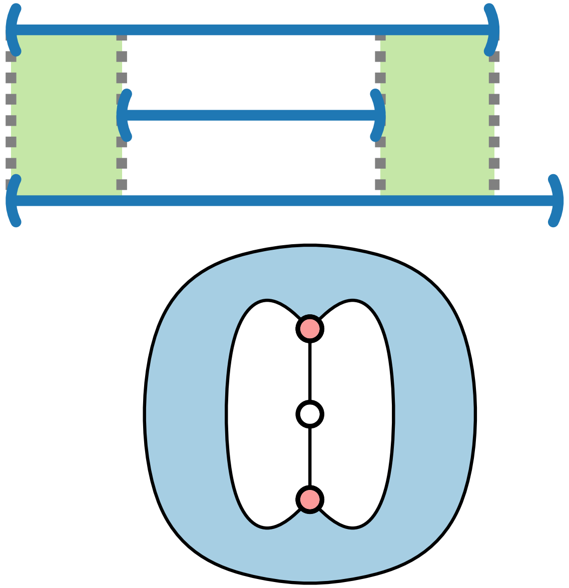

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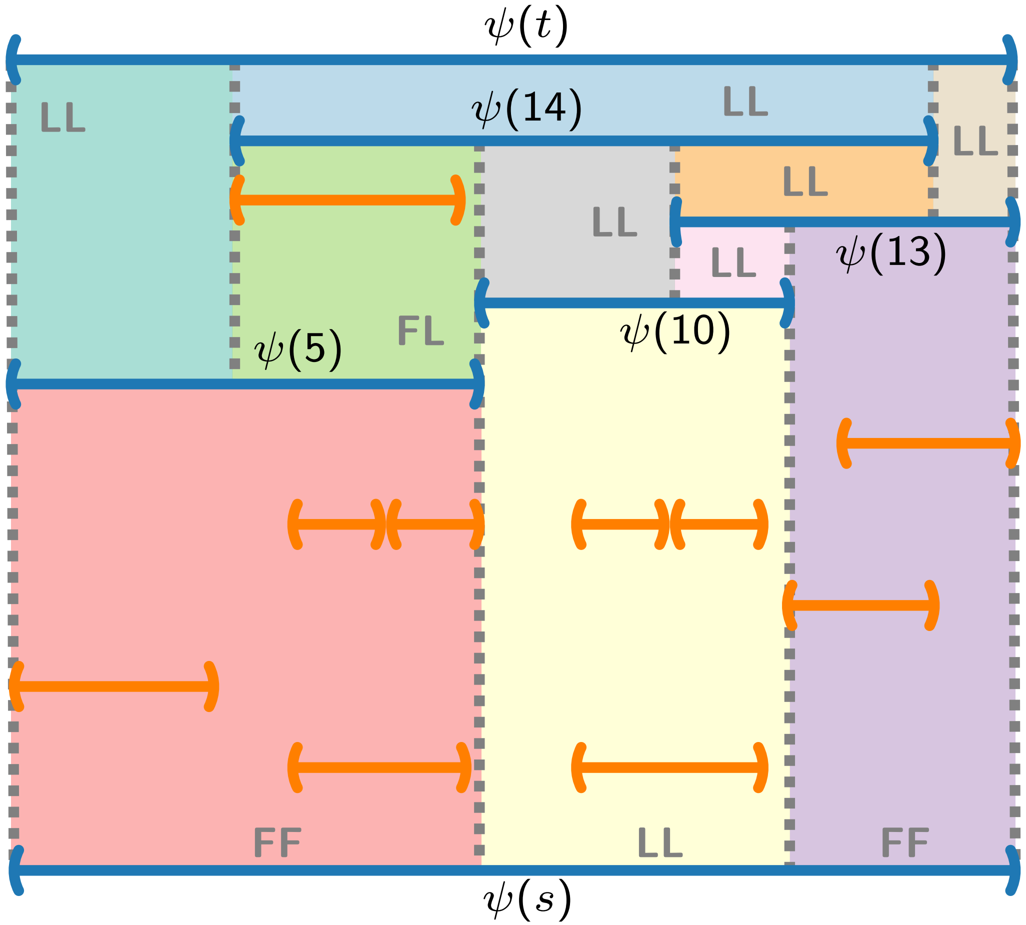
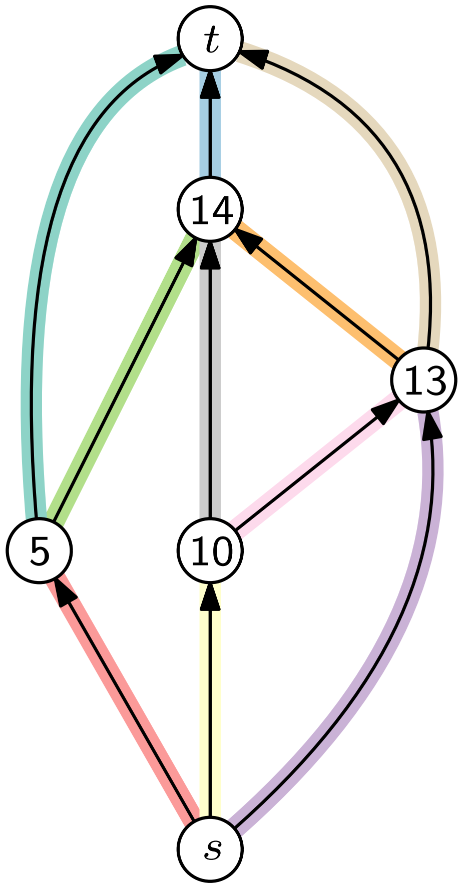
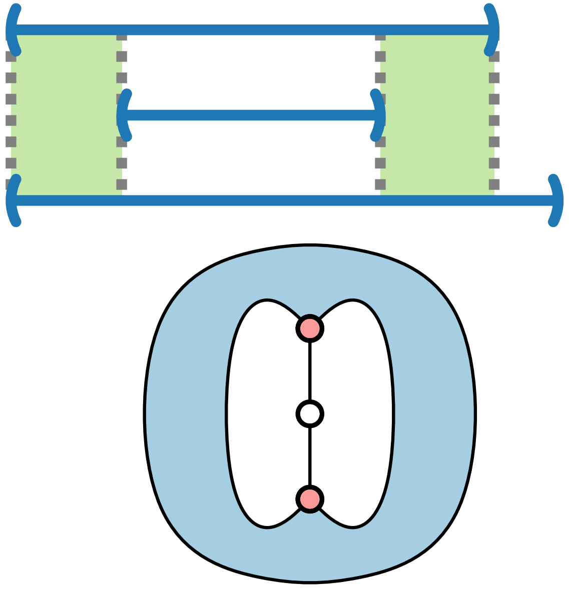
  - consistency clauses



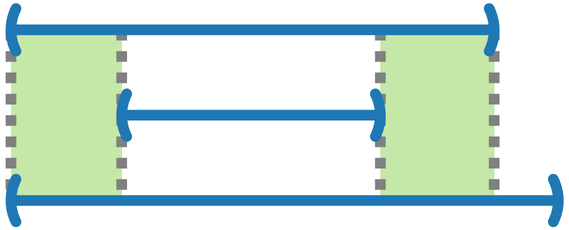


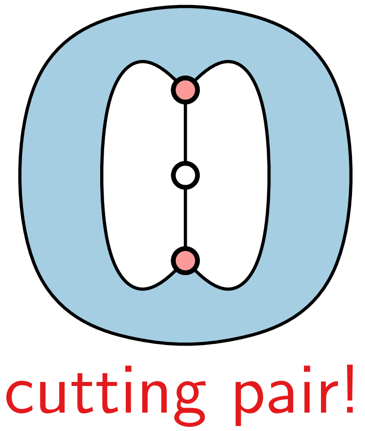


cutting pair!

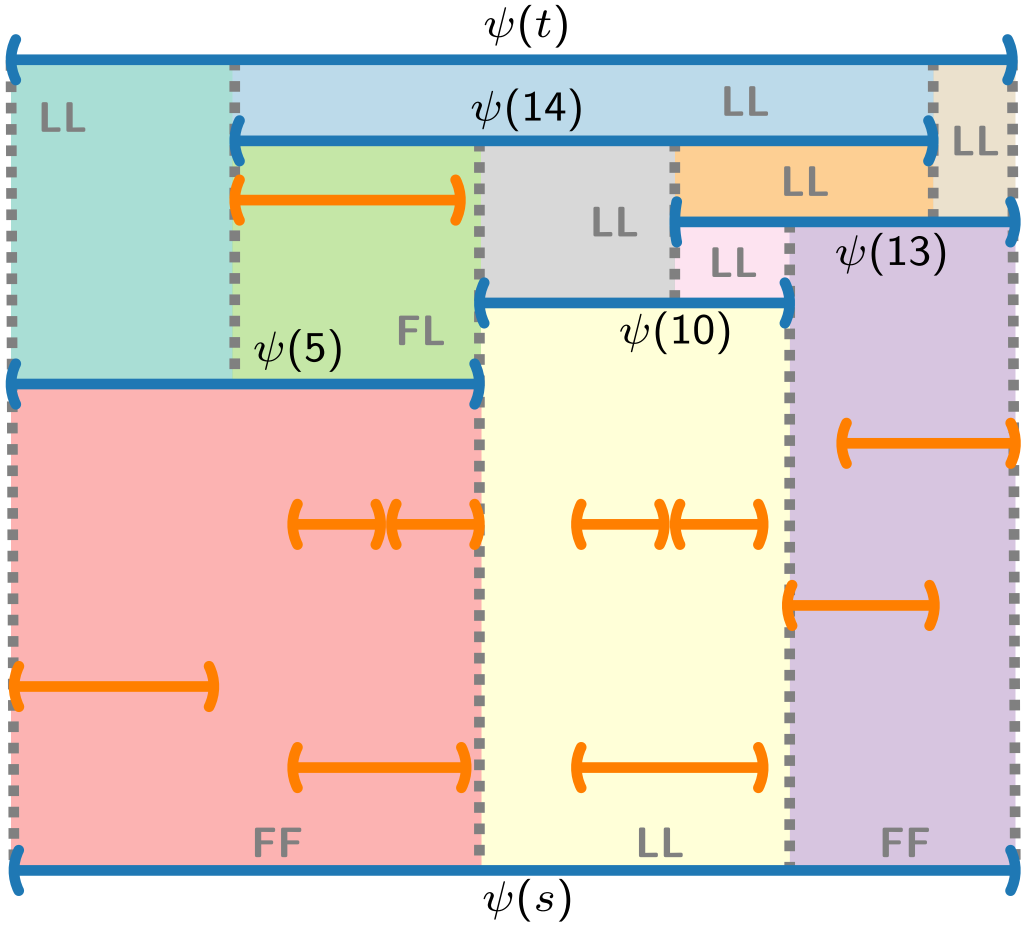
R Nodes with 2-SAT Formulation

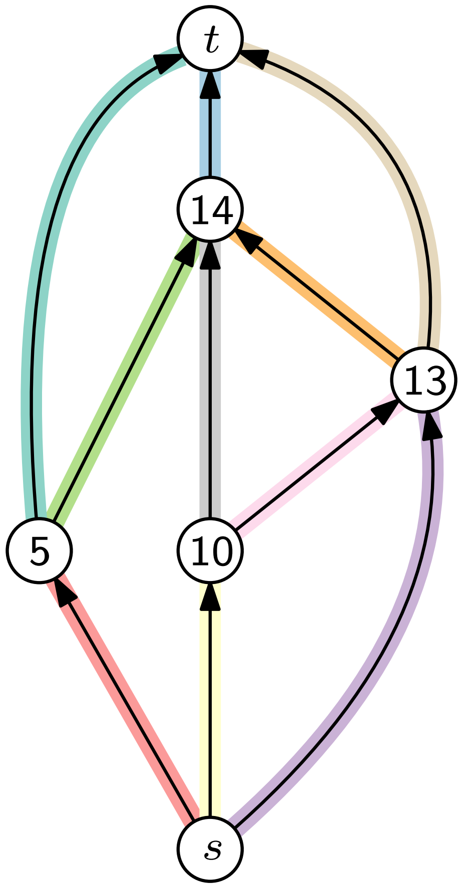
- for each child (edge) e:
  - find all types of $\{$**FF,FL,LF,LL**$\}$ that admit a drawing
  - 2 variables l_e, r_e encoding fixed/loose type of its tile
  - consistency clauses – $O(n^2)$ many, but can be reduced to $O(n \log^2 n)$





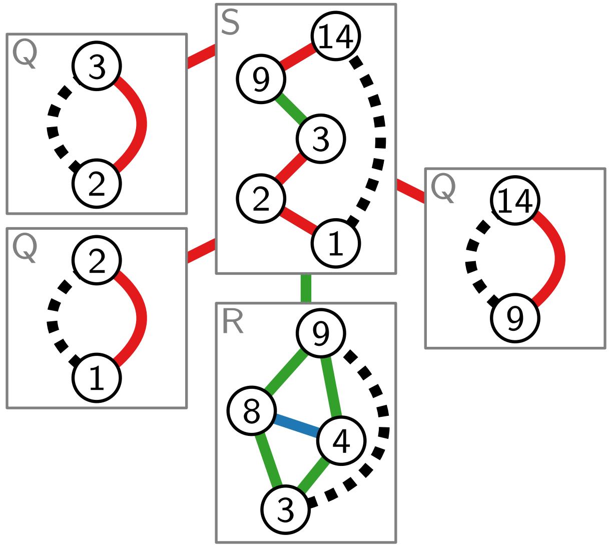
cutting pair!





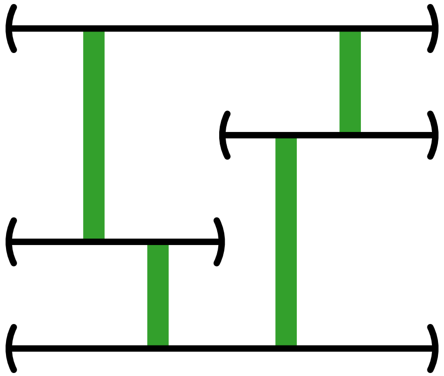
Visualization of Graphs

Lecture 9:
Partial Visibility Representation Extension



Part VI:
NP-Hardness
of General Case

Jonathan Klawitter



NP-Hardness of RepExt in General Case

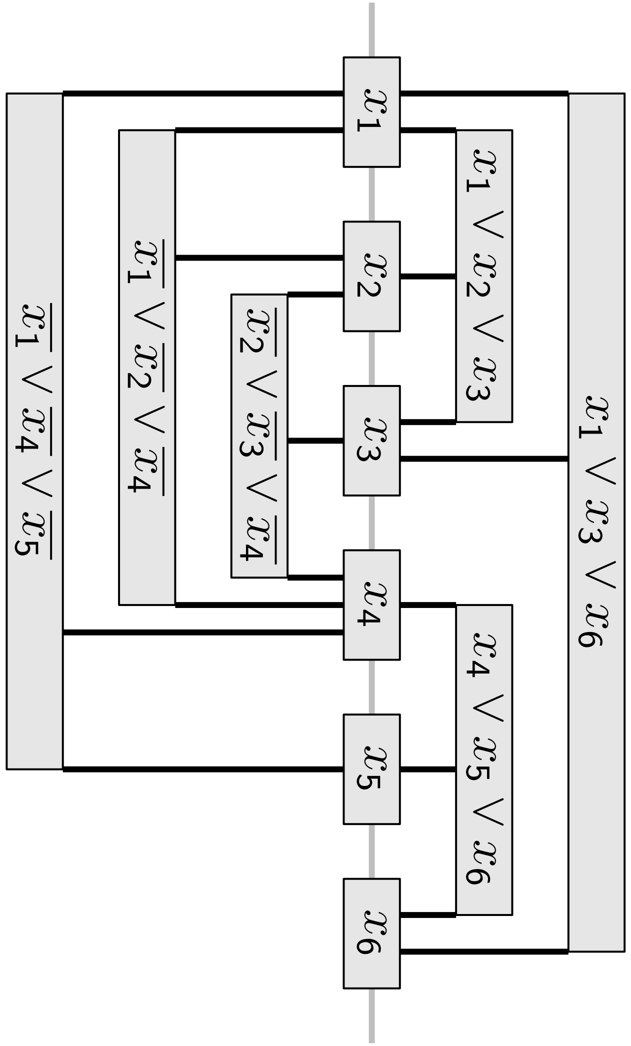
Theorem 2.
ε-Bar Visibility Representation Ext. is NP-complete.

- Reduction from Planar Monotone 3-SAT

NP-Hardness of RepExt in General Case

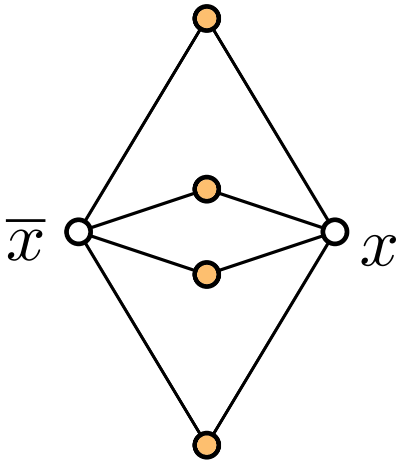
Theorem 2.
ε-Bar Visibility Representation Ext. is NP-complete.

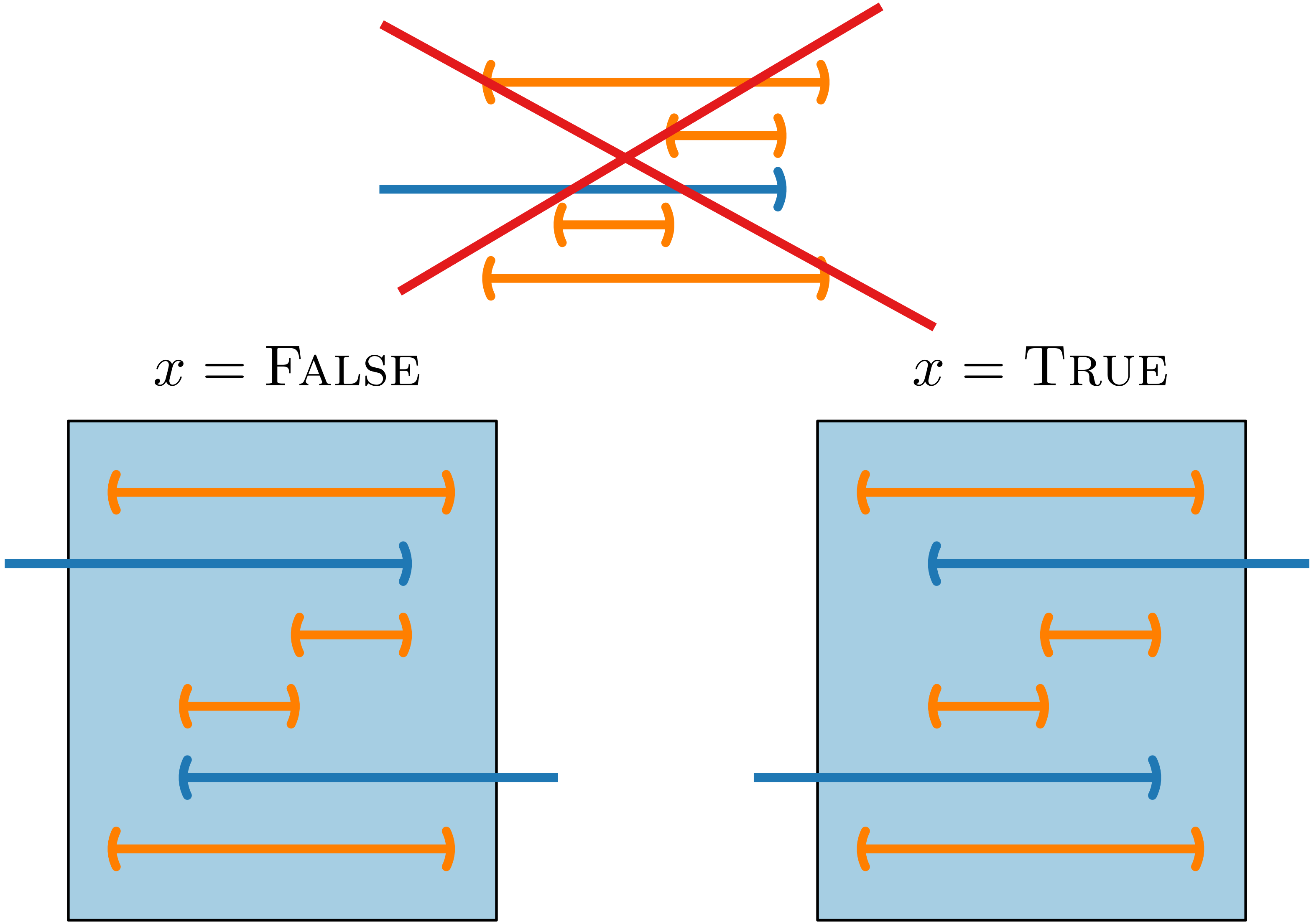
- Reduction from Planar Monotone 3-SAT


x_1
x_2
x_3
x_4
x_5
x_6
$\overline{x_1} \vee \overline{x_4} \vee \overline{x_5}$
$\overline{x_1} \vee \overline{x_2} \vee \overline{x_4}$
$\overline{x_2} \vee \overline{x_3} \vee \overline{x_4}$
$x_1 \vee x_2 \vee x_3$
$x_1 \vee x_3 \vee x_6$
$x_4 \vee x_5 \vee x_6$

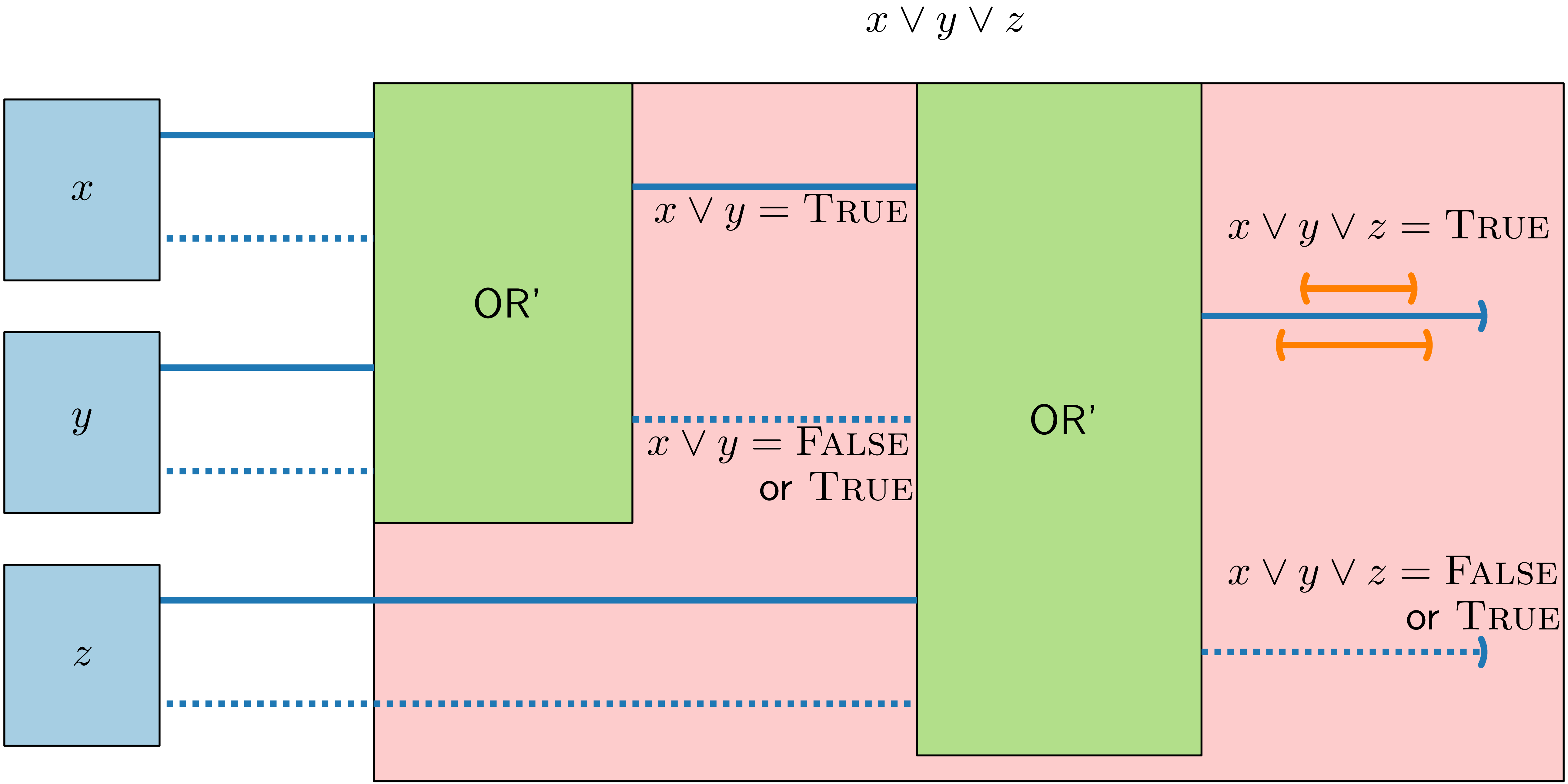
- NP-complete
  [Berg & Khosravi ’10]

Variable Gadget

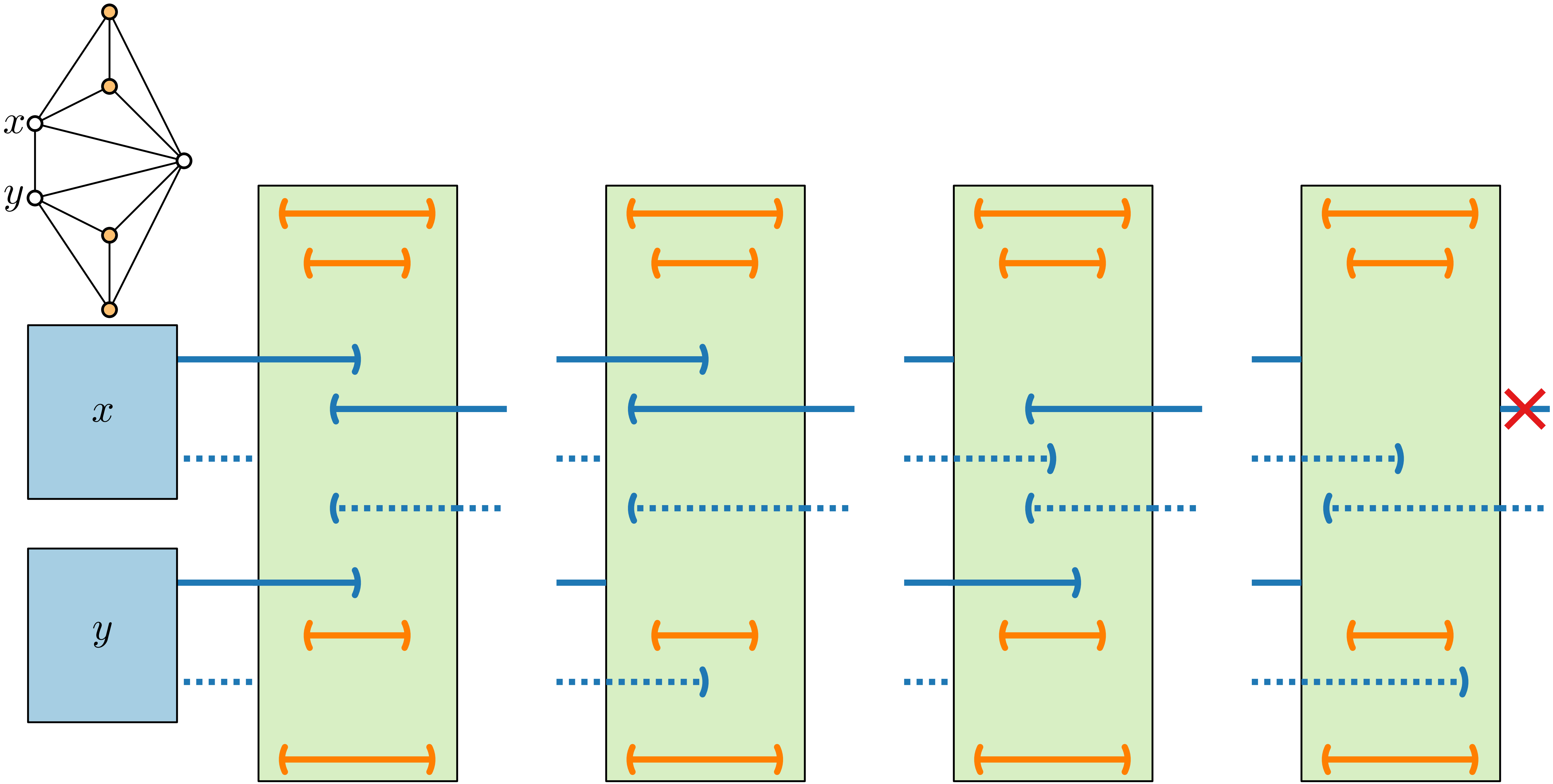




Clause Gadget



OR' Gadget



Discussion

- *rectangular* ε-Bar Visibility Representation Extension can be solved in $O(n \log^2 n)$ time for st-graphs.
- ε-Bar Visibility Representation Extension is NP-complete.
- ε-Bar Visibility Representation Extension is NP-complete for (series-parallel) st-graphs when restricted to the *Integer Grid* (or if any fixed $\varepsilon > 0$ is specified).

Open Problems:

- Can ~~*rectangular*~~ ε-Bar Visibility Representation Extension can be solved in polynomial time on st-graphs? DAGs?
- Can **Strong** Bar Visibility Recognition / Representation Extension can be solved in polynomial time on st-graphs?

Literature

Main source:

- [Chaplick, Guśpiel, Gutowski, Krawczyk, Liotta '18] The Partial Visibility Representation Extension Problem

Referenced papers:

- [Gutwenger, Mutzel '01] A Linear Time Implementation of SPQR-Trees
- [Wismath '85] Characterizing bar line-of-sight graphs
- [Tamassia, Tollis '86] Algorithms for visibility representations of planar graphs
- [Andreae '92] Some results on visibility graphs
- [Chaplick, Dorbec, Kratchovíl, Montassier, Stacho '14] Contact representations of planar graphs: Extending a partial representation is hard