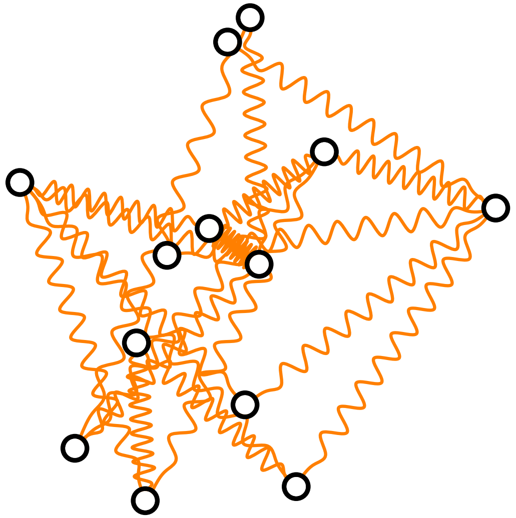
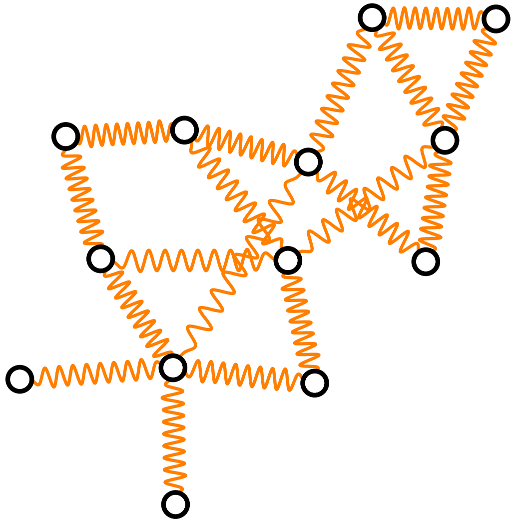
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Visualization of Graphs

Lecture 2:
Force-Directed Drawing Algorithms



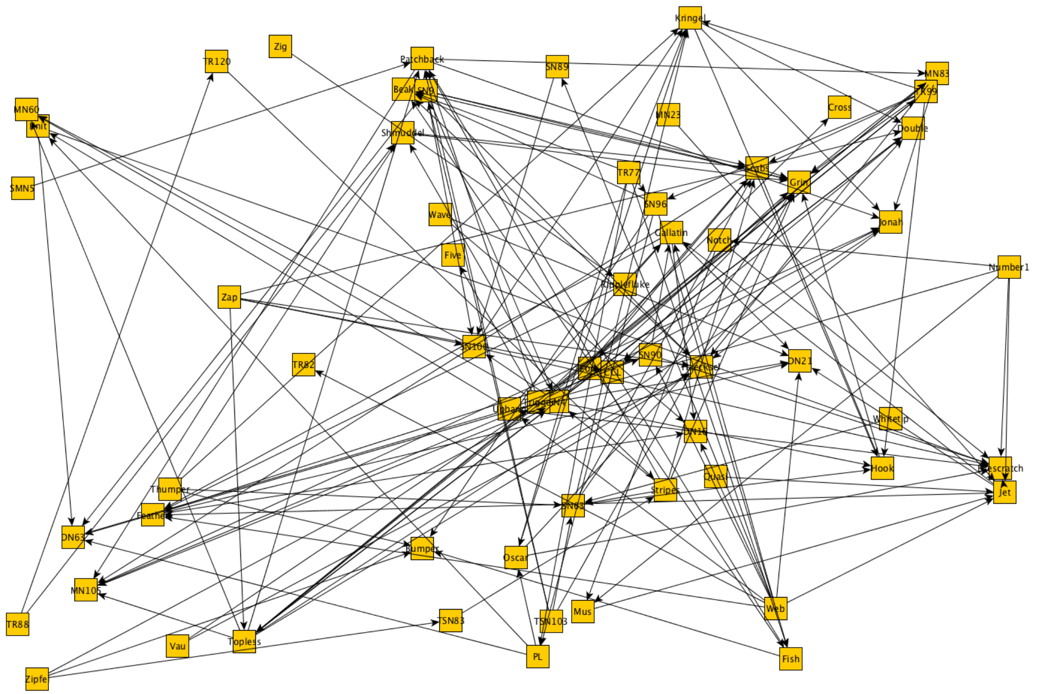
Part I:
Algorithm Framework



Jonathan Klawitter

General Layout Problem

Input: Graph $G = (V, E)$



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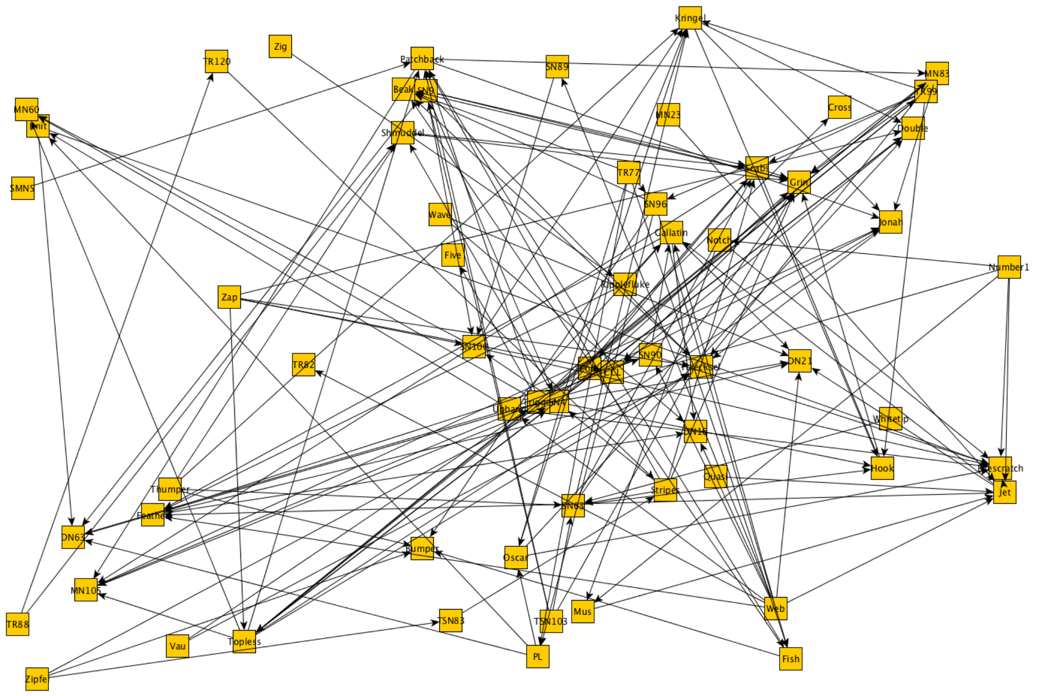
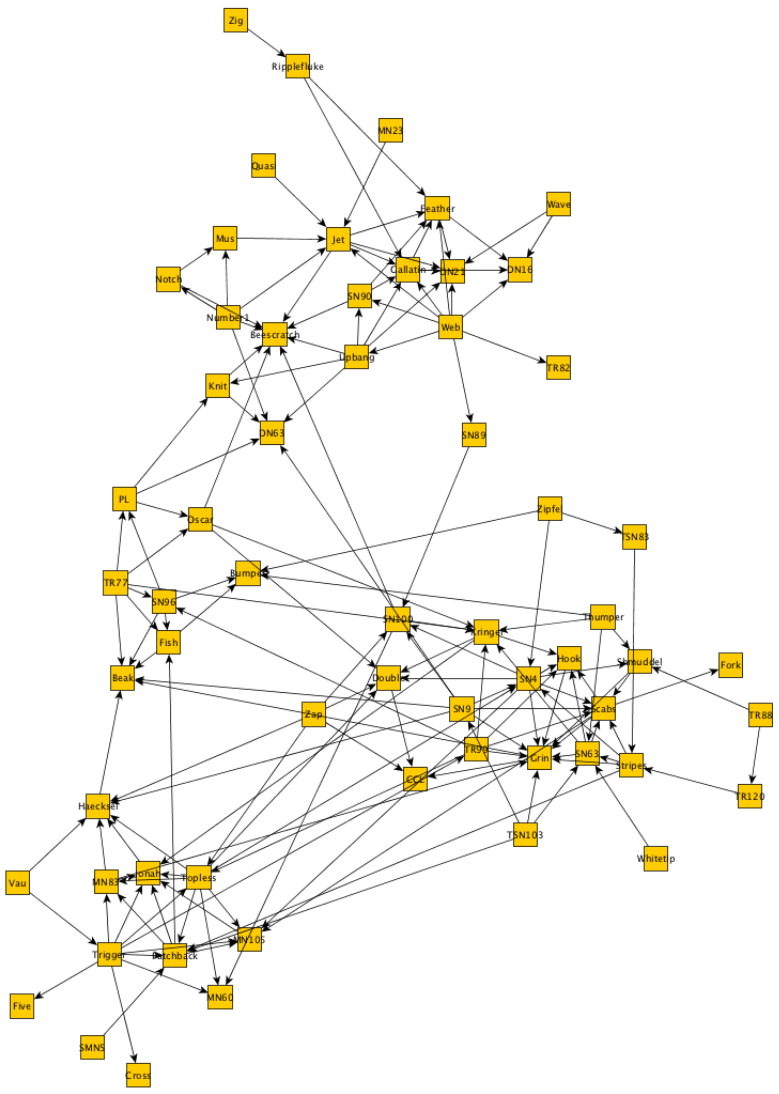
Input: Graph $G = (V, E)$

Output: Clear and readable straight-line drawing of G





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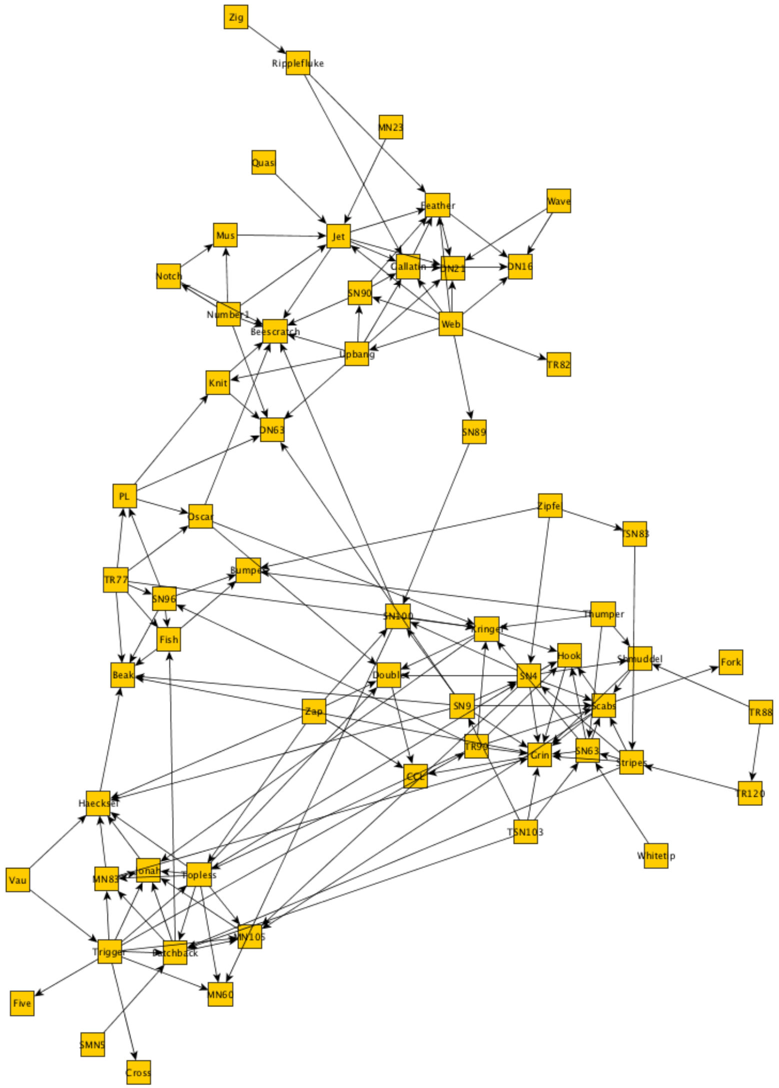
Drawing aesthetics:



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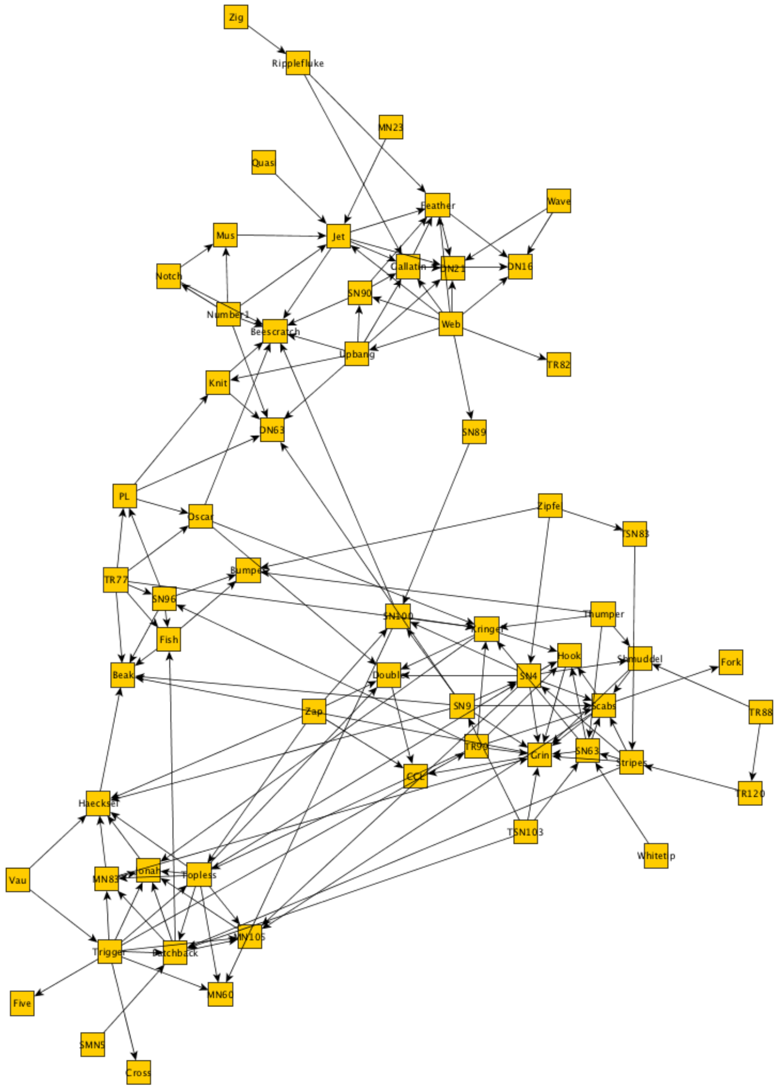
Drawing aesthetics:

- adjacent vertices are close



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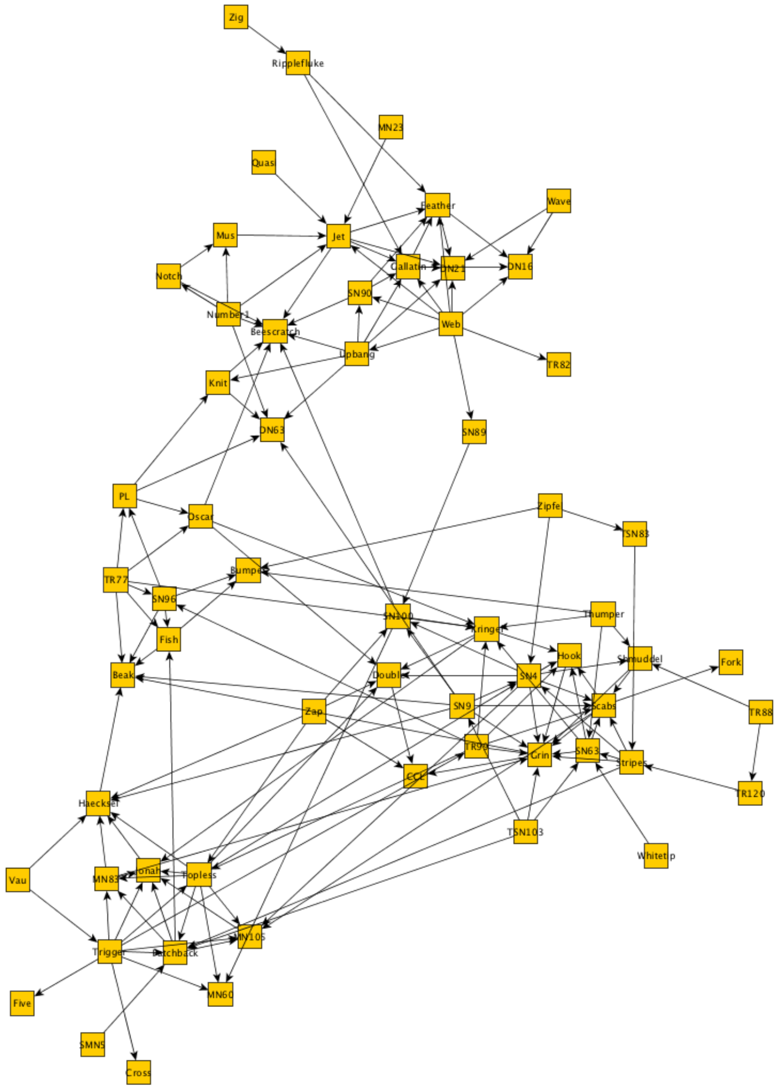

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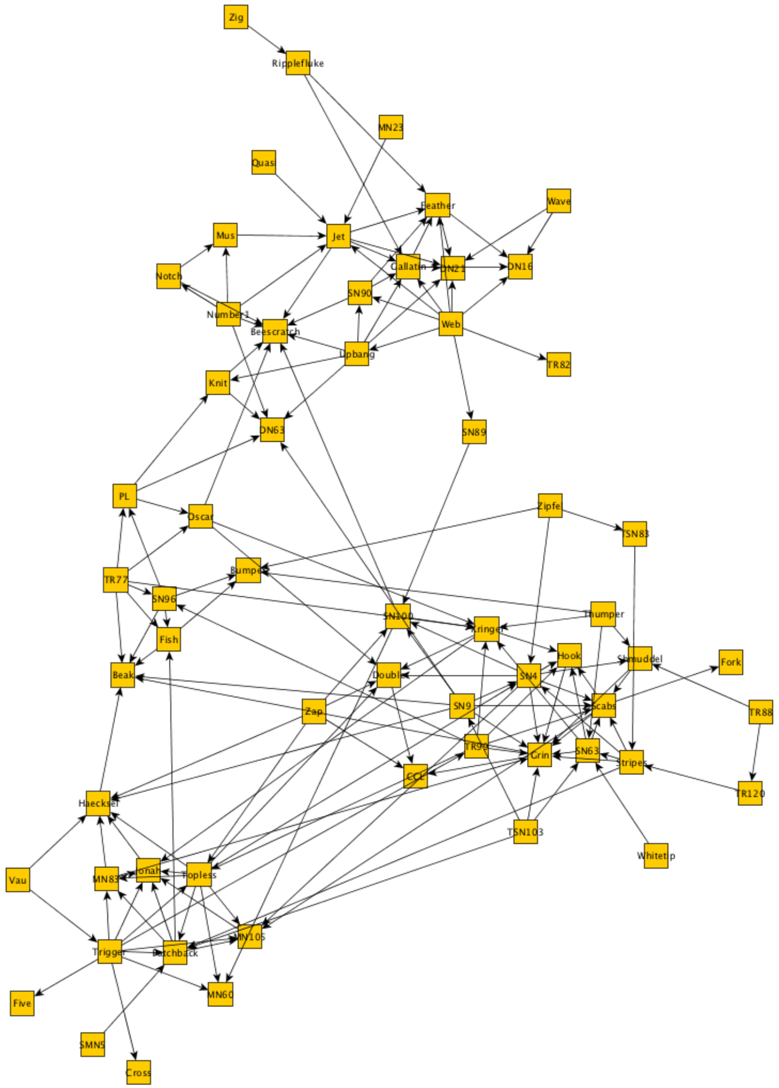
- adjacent vertices are close
- non-adjacent vertices are far apart
- edges short, straight-line, **similar length**



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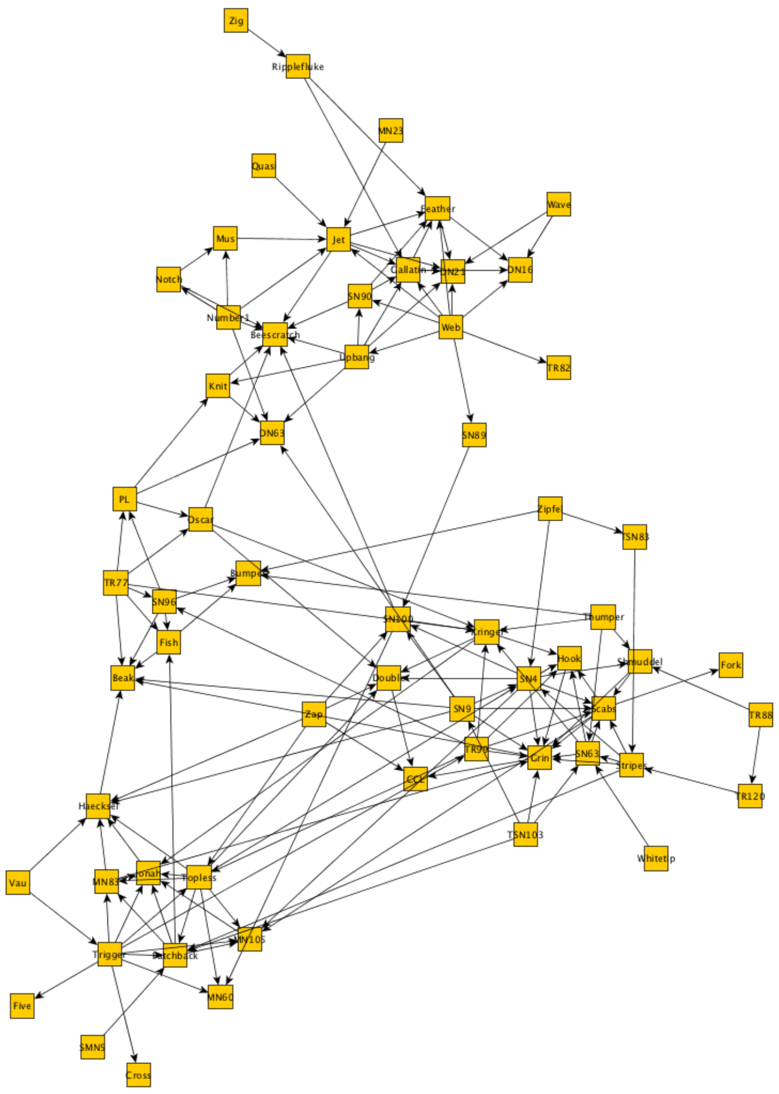

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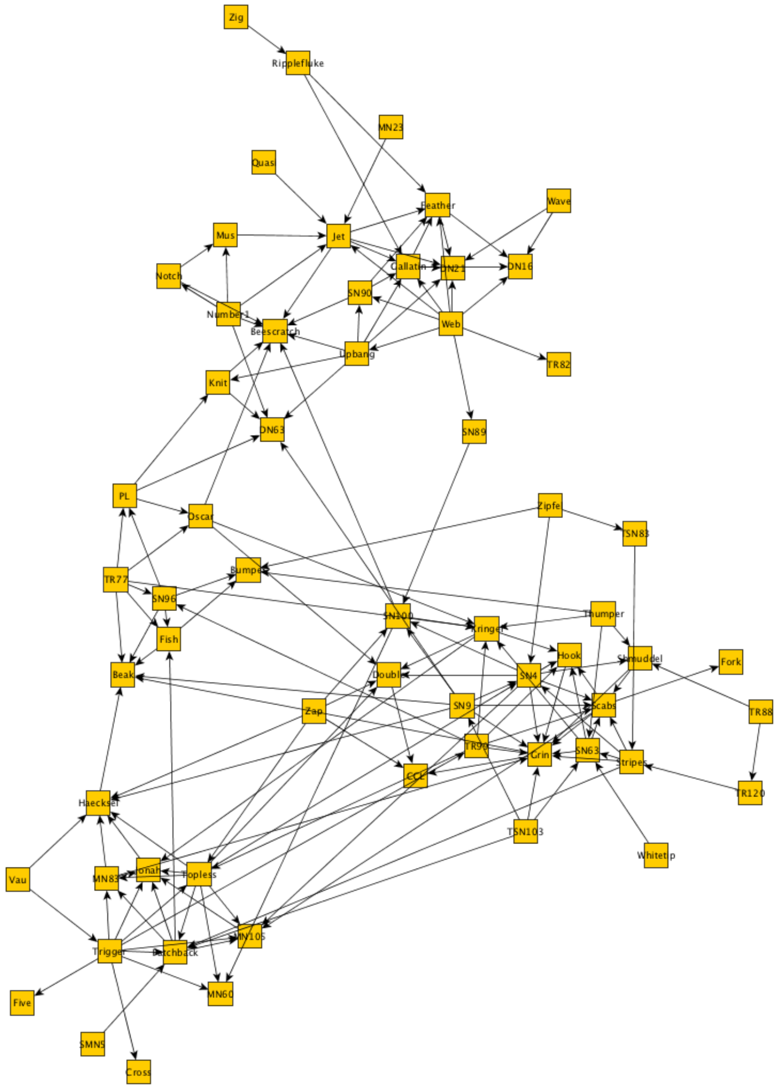

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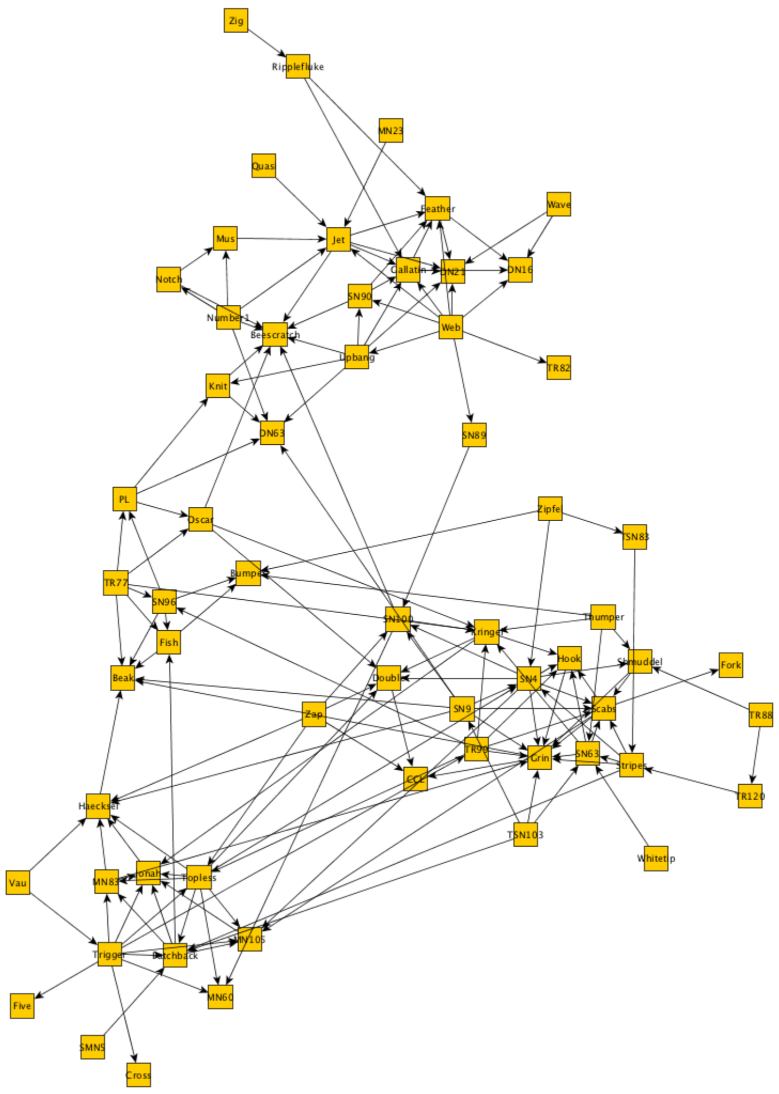

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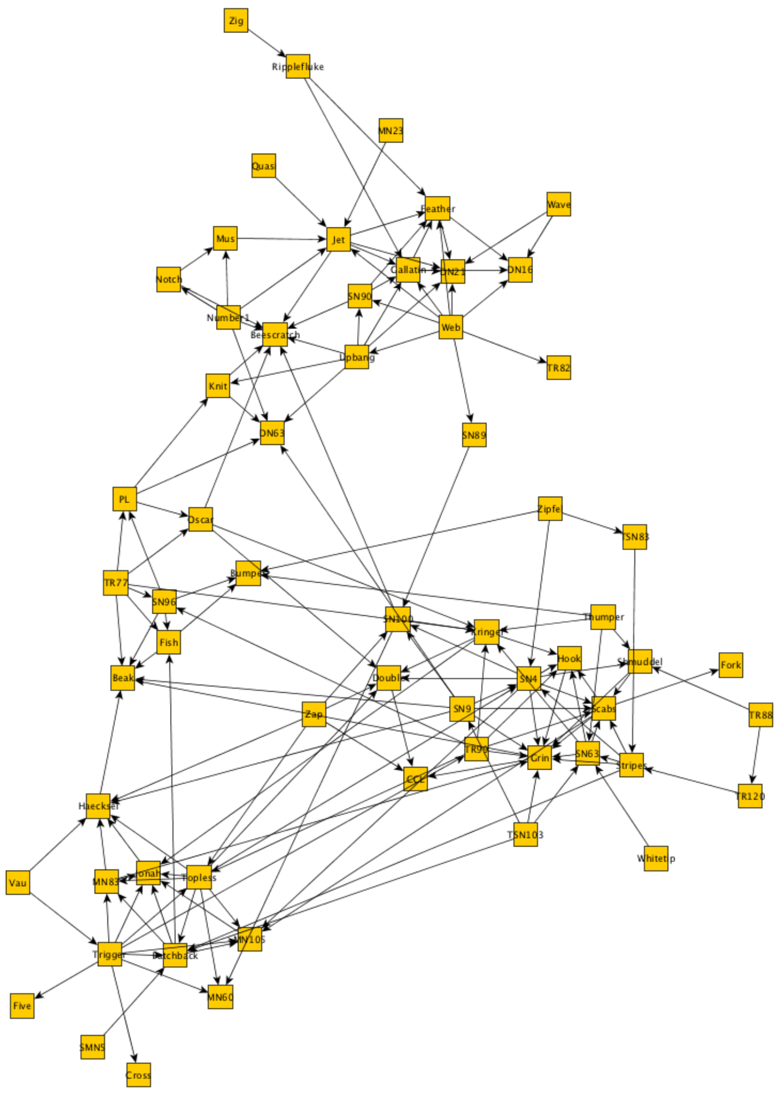
Optimization criteria partially contradict each other



Fixed Edge Lengths?

Input: Graph $G = (V, E)$, required edge length $\ell(e)$, $\forall e \in E$

Output: Drawing of G which realizes all the edge lengths

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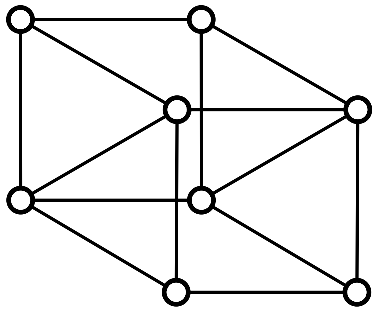

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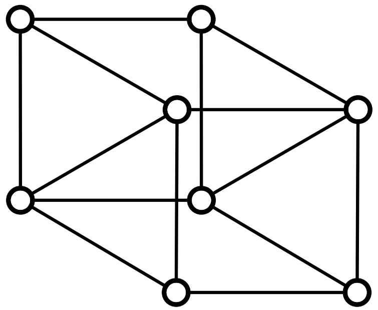

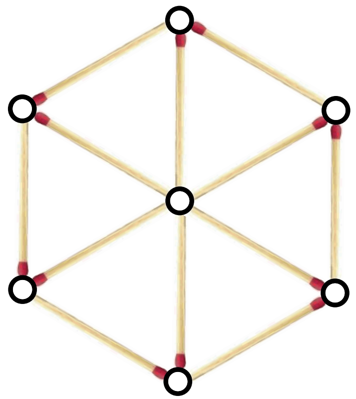

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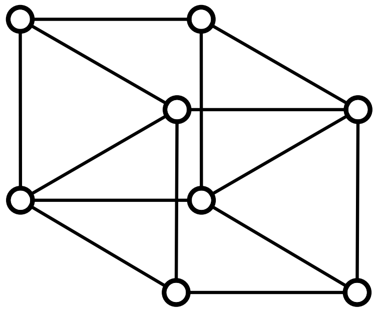

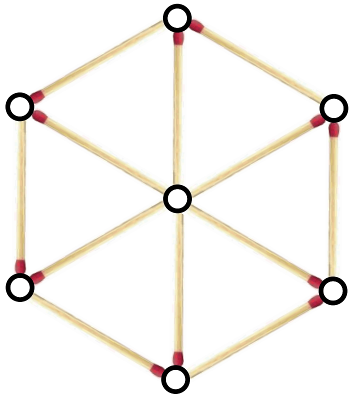

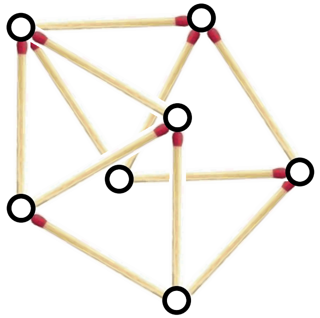

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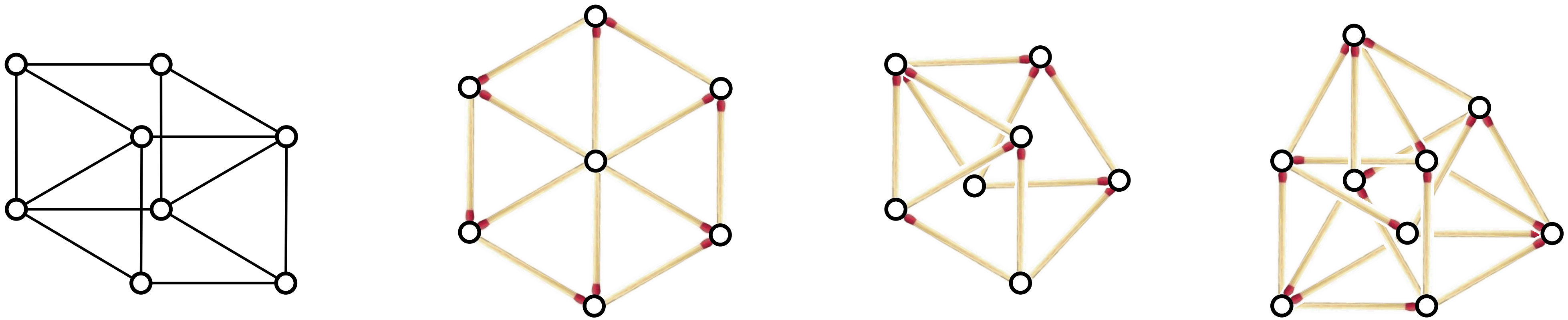

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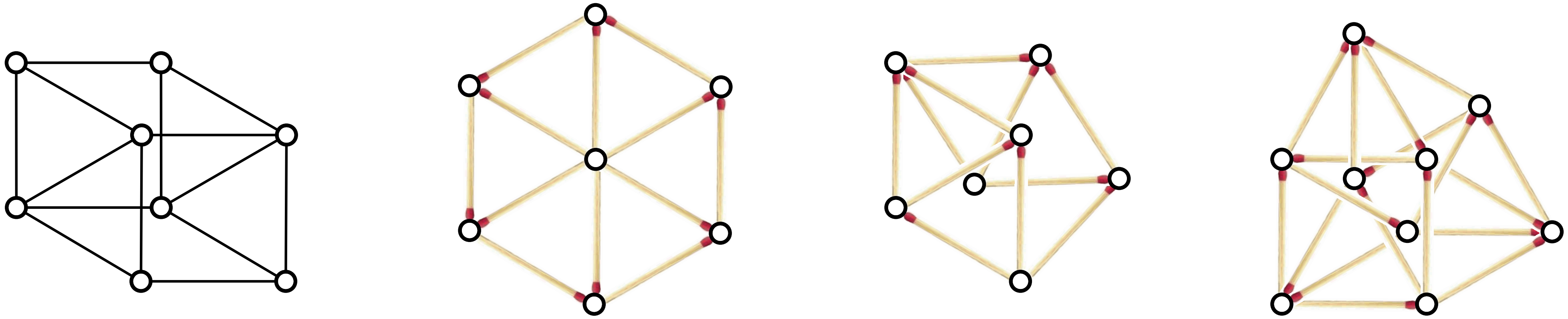

NP-hard for

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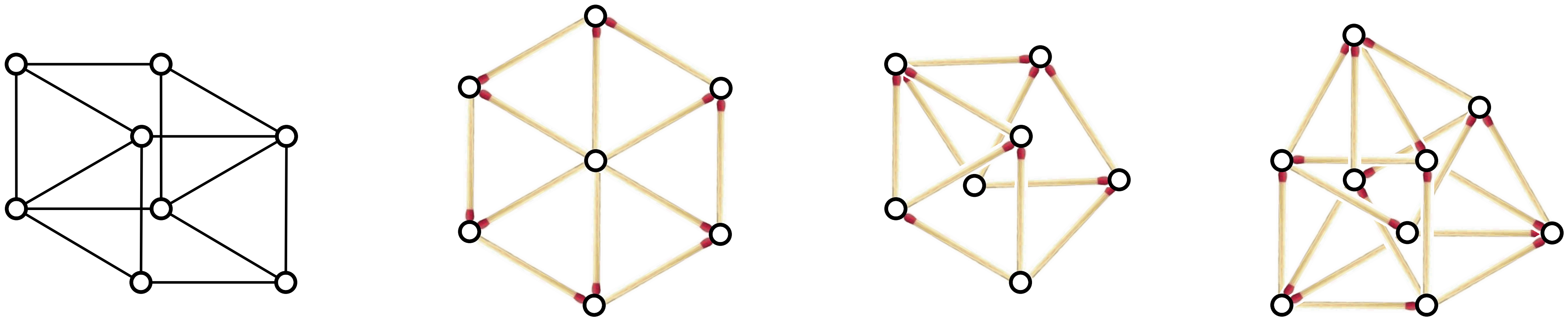

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- uniform edge lengths in any dimension [Johnson '82]

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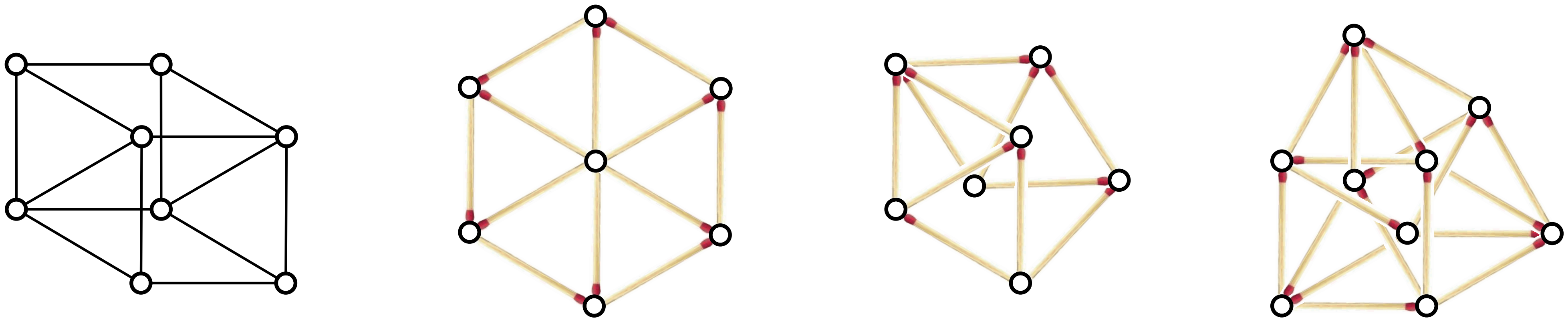

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- uniform edge lengths in planar drawings [Eades, Wormald '90]

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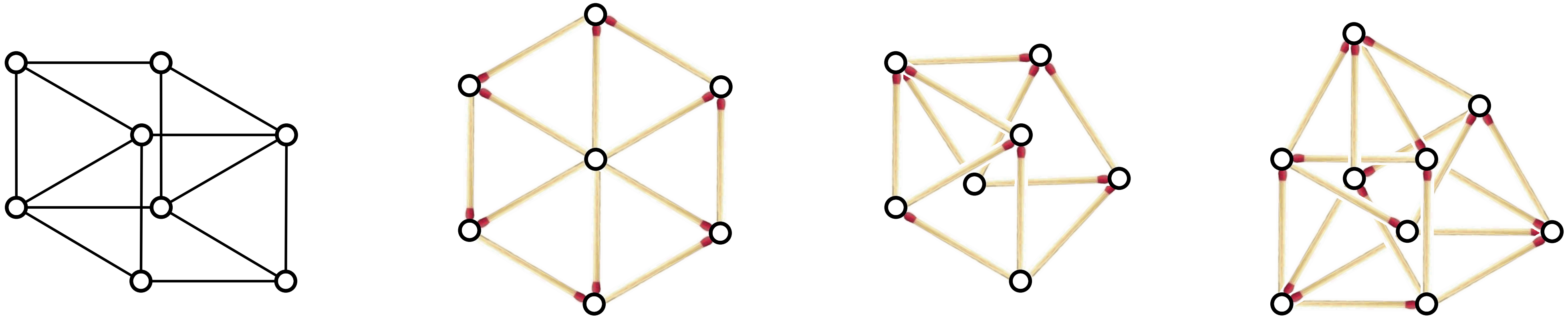

NP-hard for

- uniform edge lengths in any dimension [Johnson '82]
- uniform edge lengths in planar drawings [Eades, Wormald '90]
- edge lengths $\{1, 2\}$ [Saxe '80]

Physical Analogy

Idea. [Eades '84]

"To embed a graph we replace the vertices by steel rings and replace each edge with a **spring** to form a mechanical system . . .

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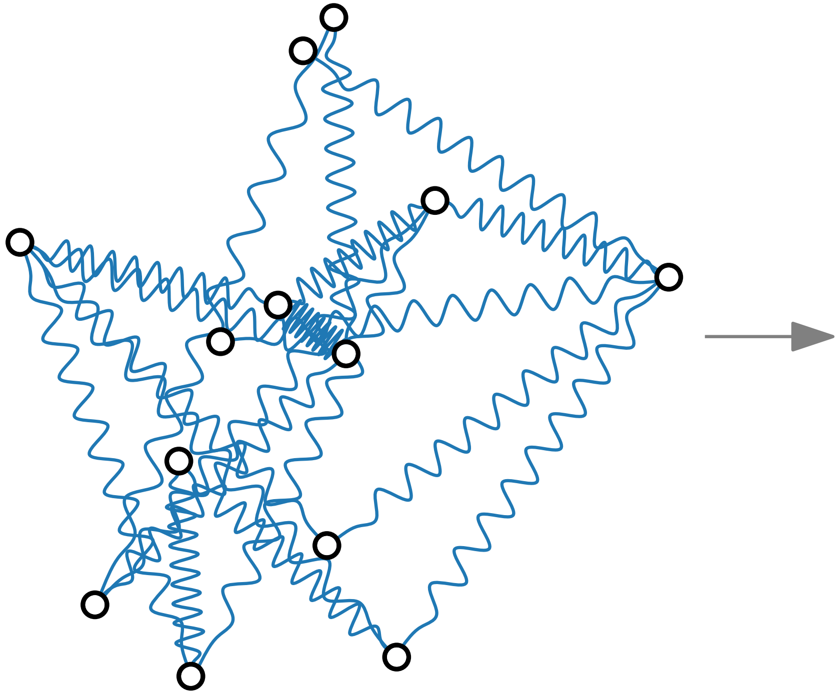

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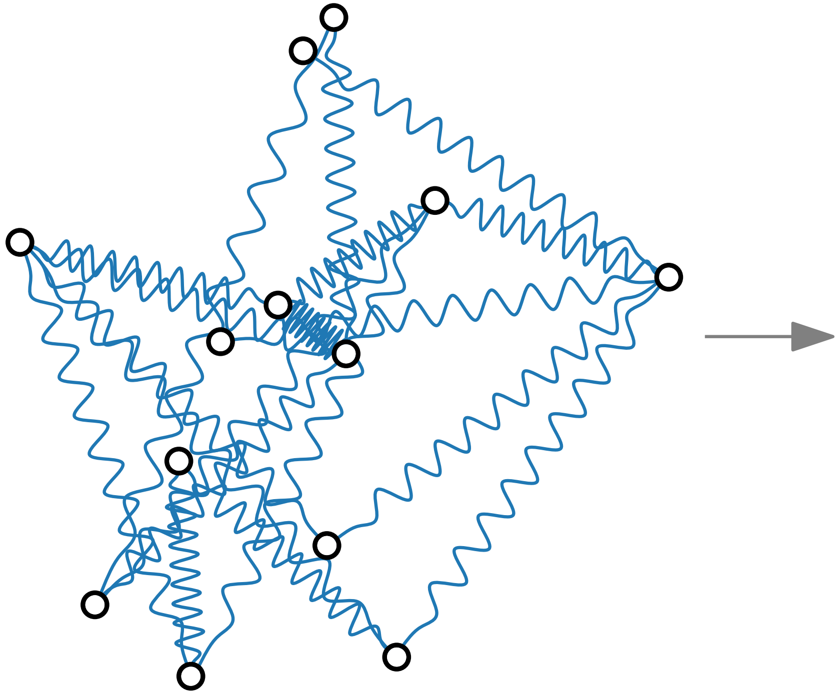

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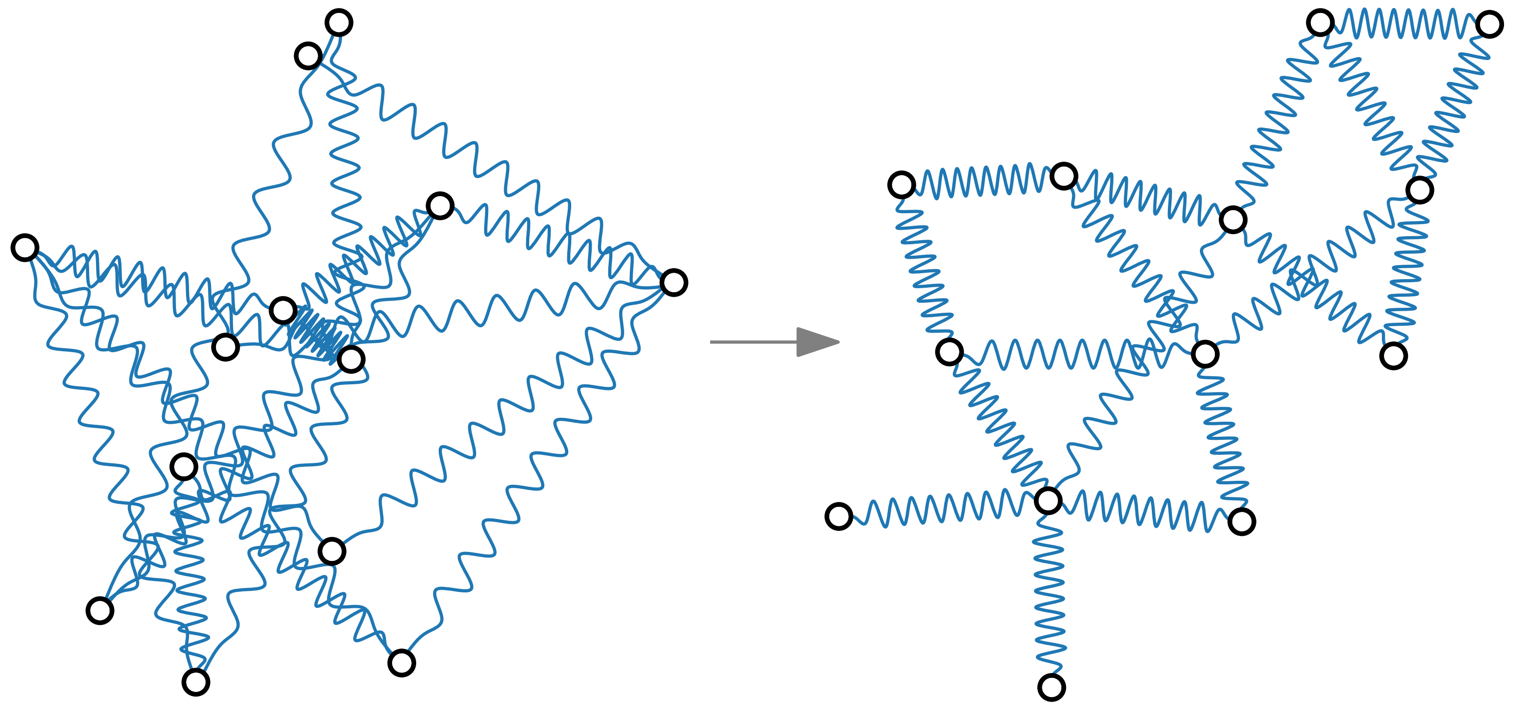

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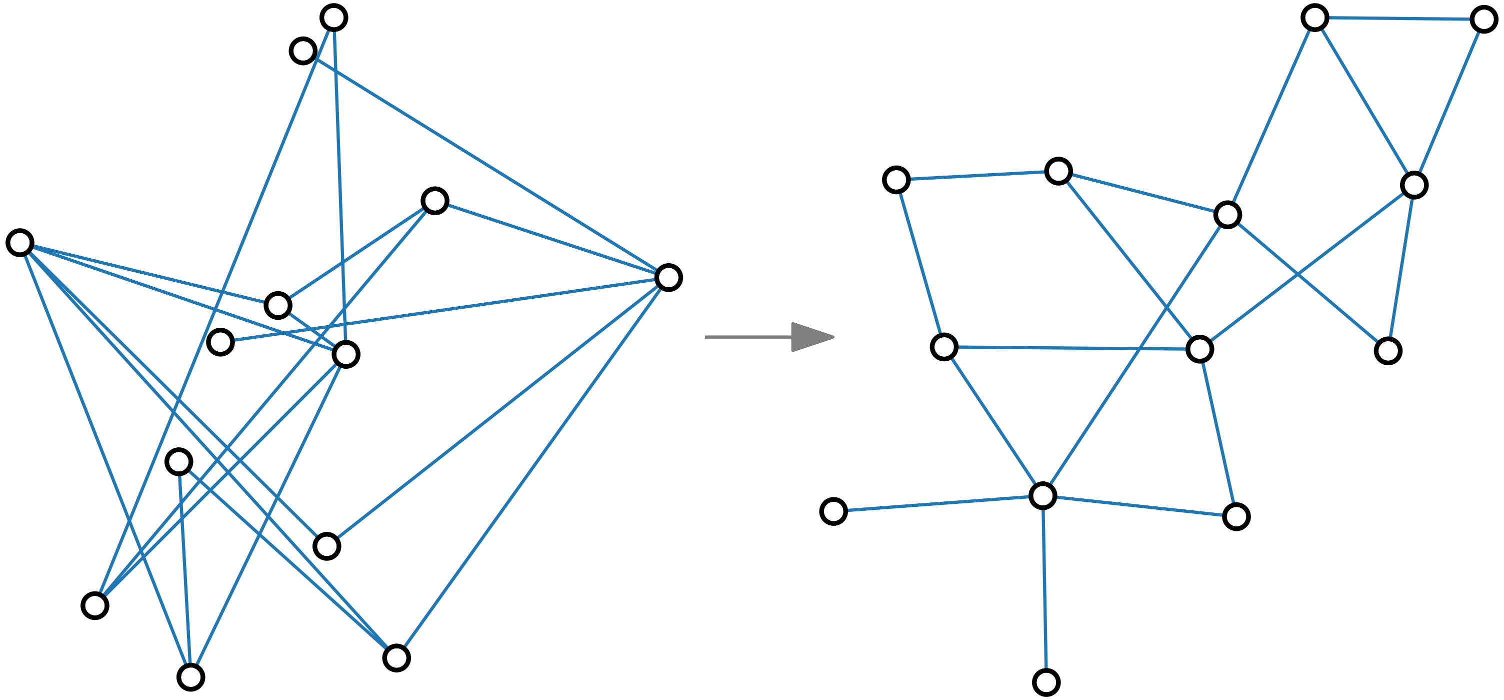

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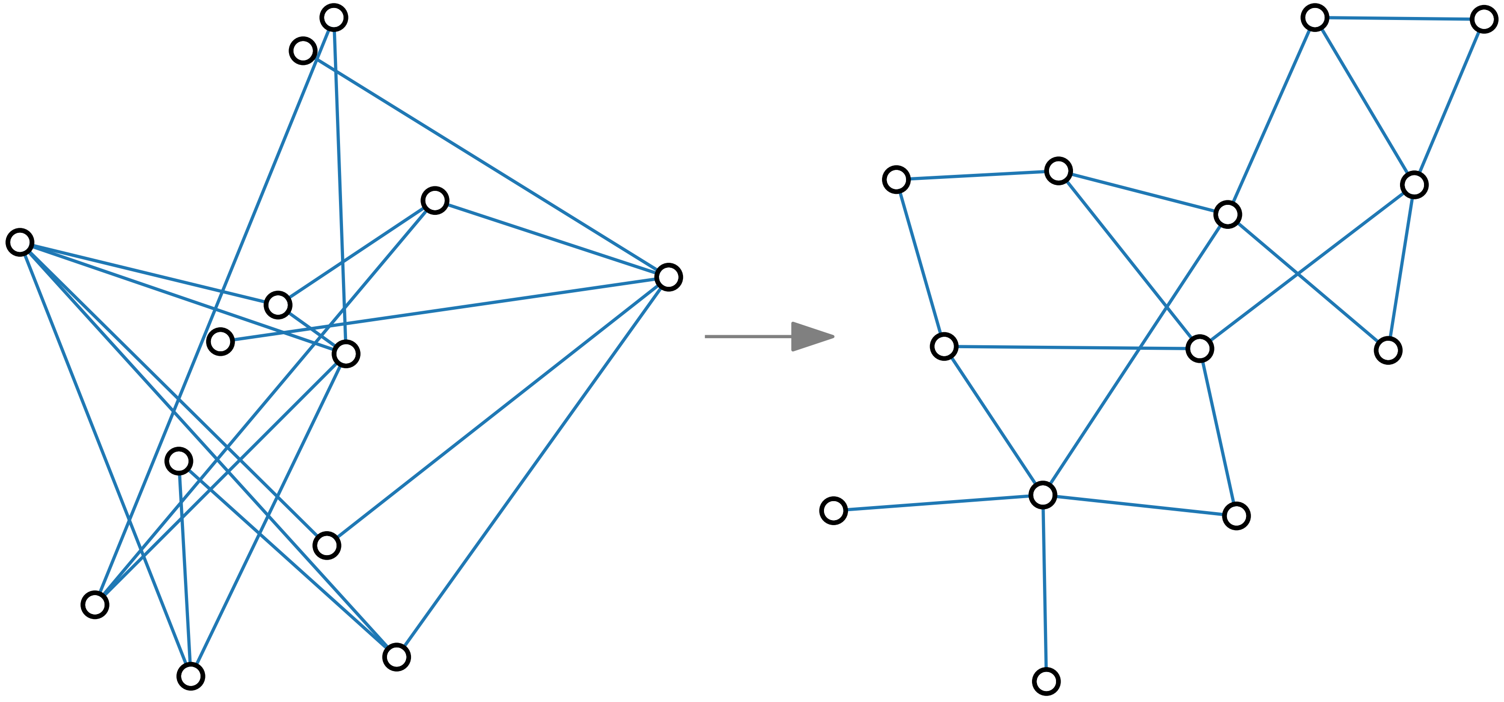
Attractive forces.



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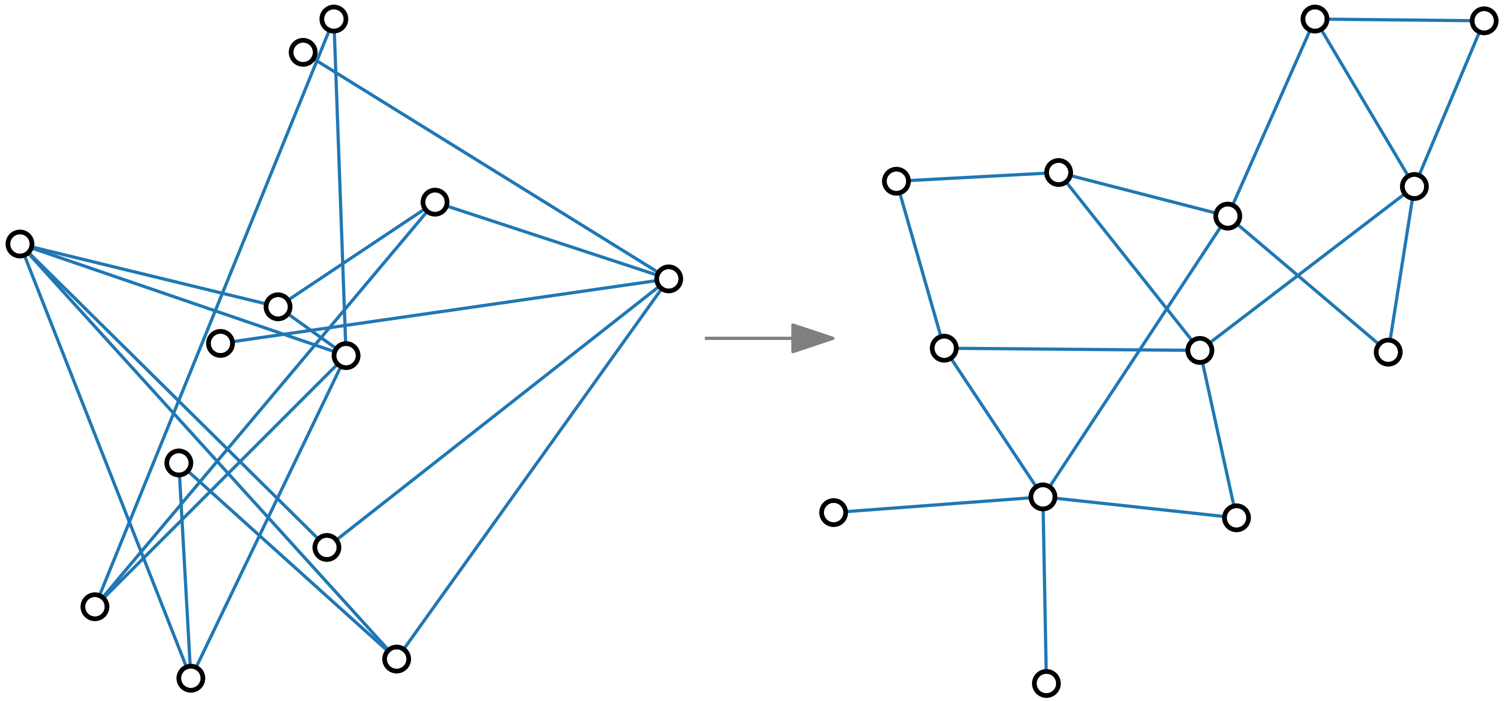

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adjacent vertices u and v:

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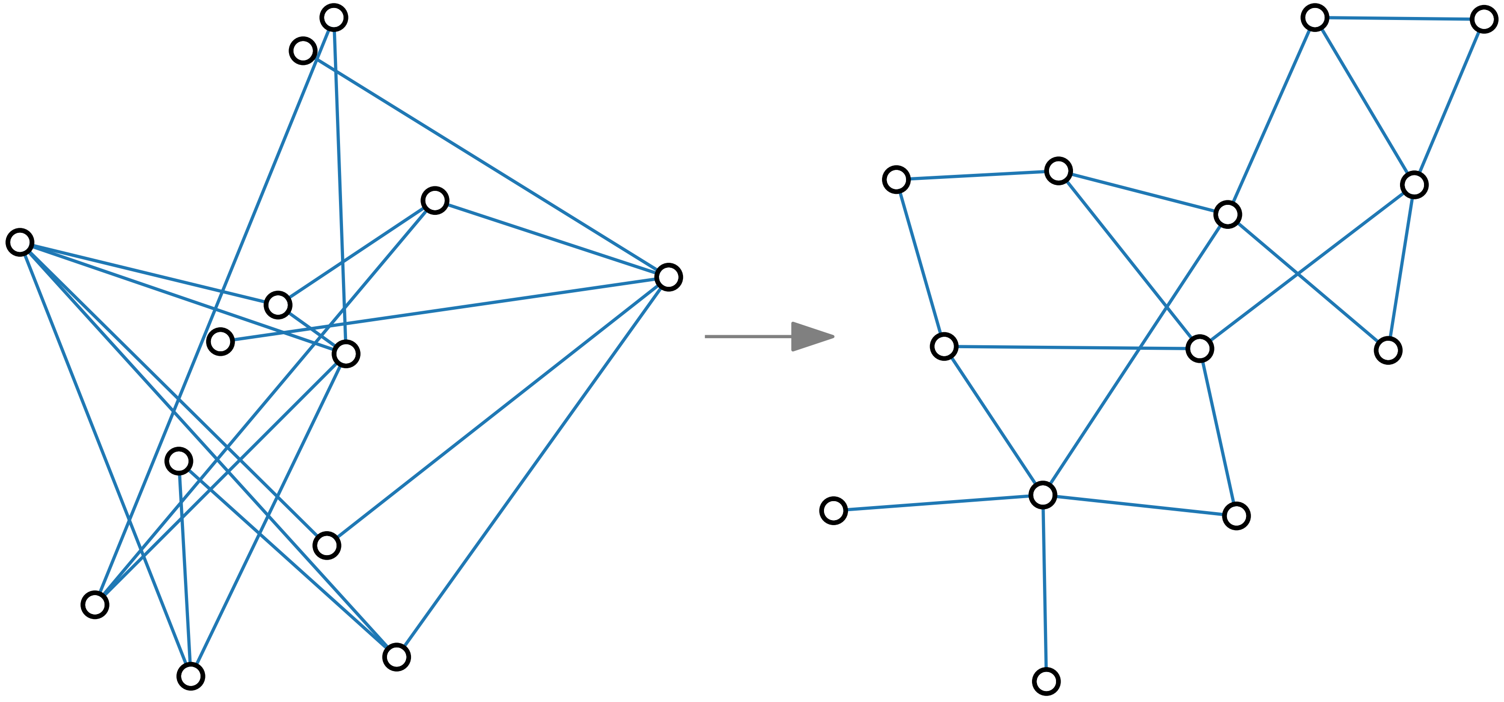

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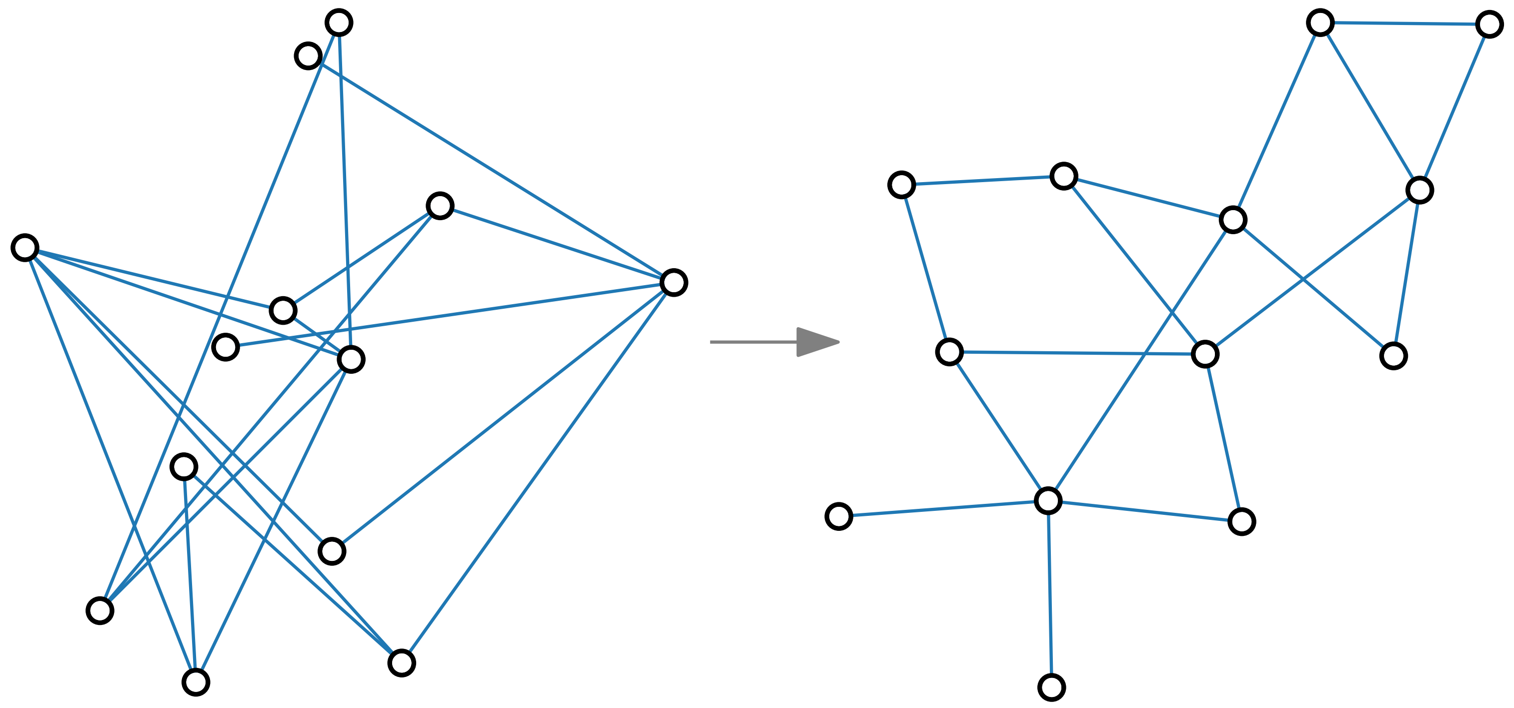

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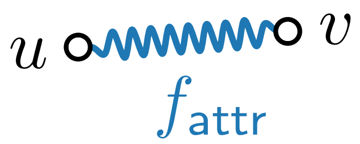

Repulsive forces.

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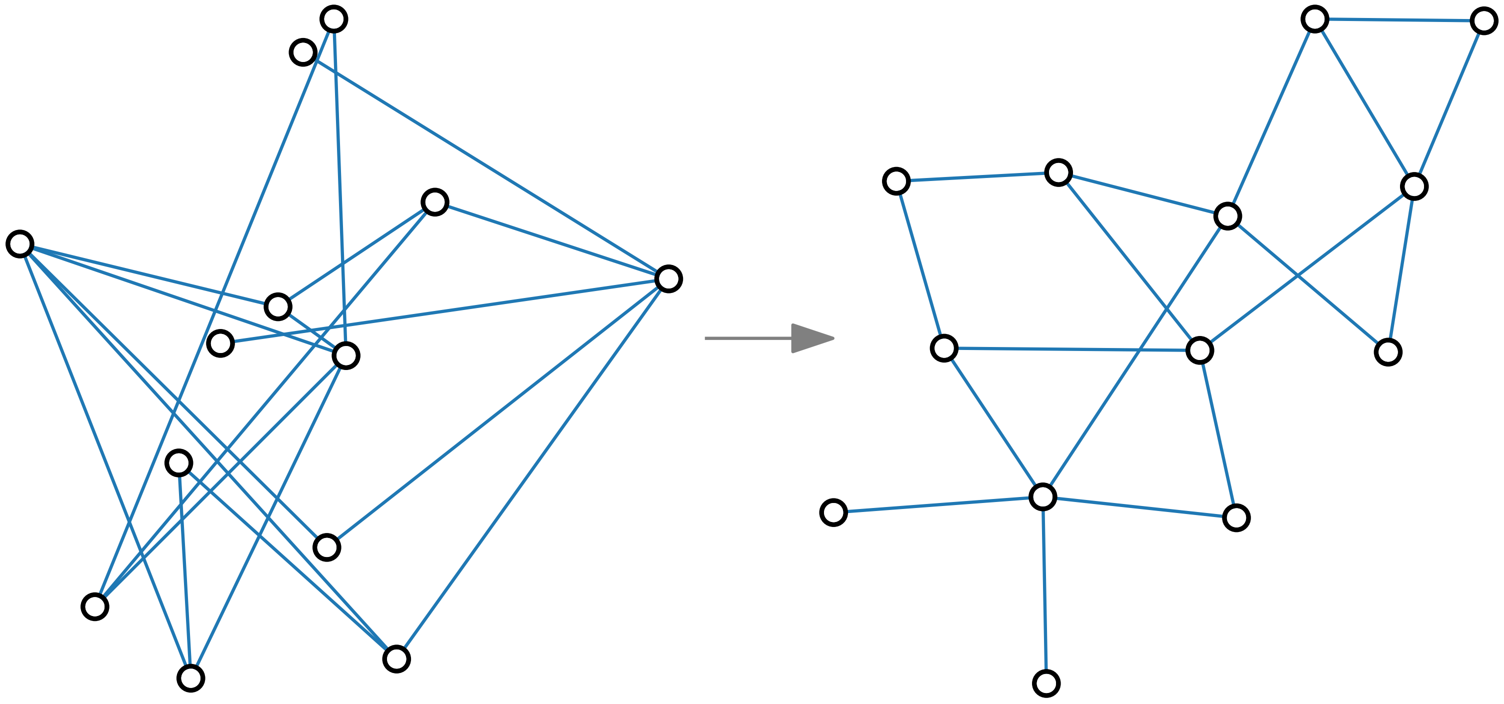

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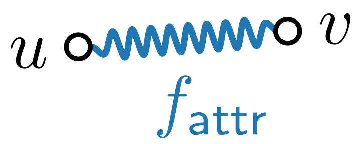

Repulsive forces.

all vertices x and y:

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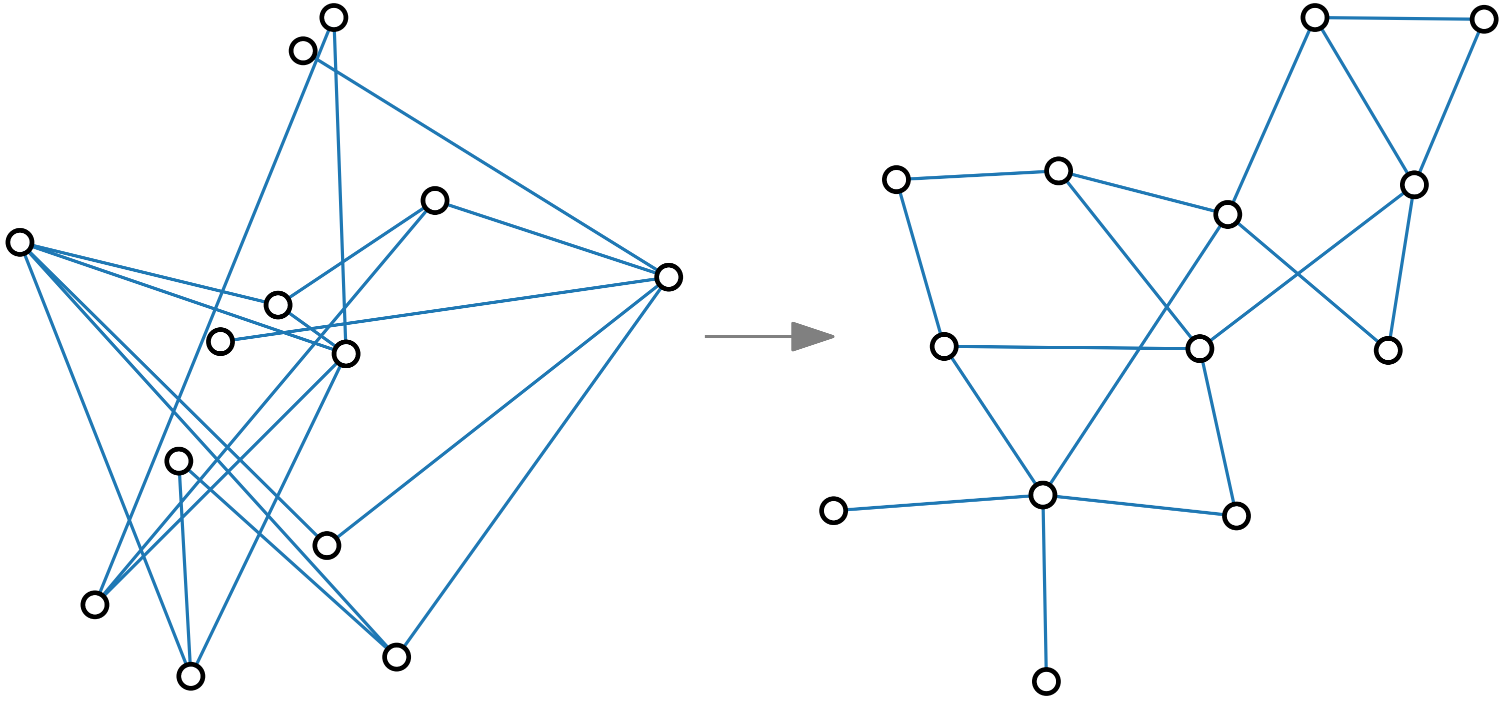

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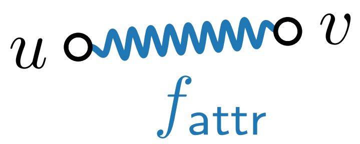

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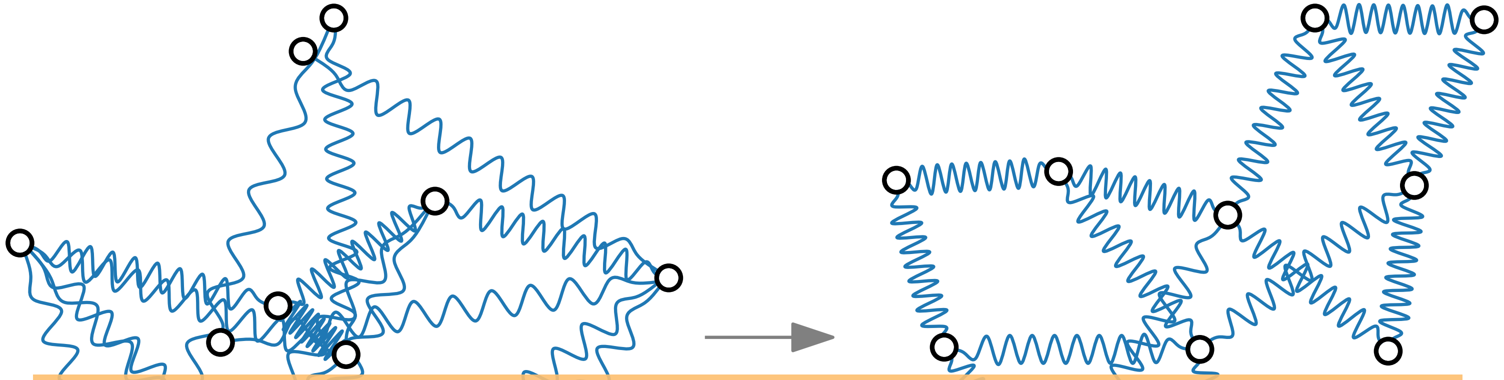

So-called **spring-embedder** algorithms that work according to this or similar principles are among the most frequently used graph-drawing methods in practice.

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adjacent vertices u and v:

Repulsive forces.

all vertices x and y:

Force-Directed Algorithms

ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

return p

Force-Directed Algorithms

initial layout

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Force-Directed Algorithms

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ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

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end layout

Force-Directed Algorithms

initial layout

threshold

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max # iterations

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ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

$t \leftarrow 1$

while $t < K$ **and** $\max_{v \in V} \|F_v(t)\| > \varepsilon$ **do**

$t \leftarrow t + 1$

return p

end layout

Force-Directed Algorithms

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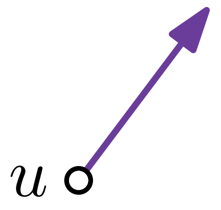

Force-Directed Algorithms

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      $F_u(t) \leftarrow \sum_{v \in V} f_{\text{rep}}(u, v) +$
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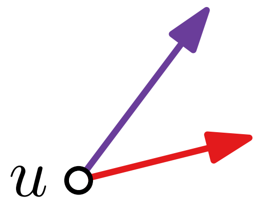
end layout



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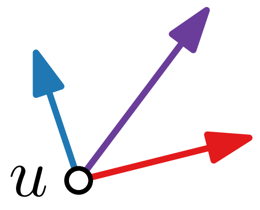
end layout



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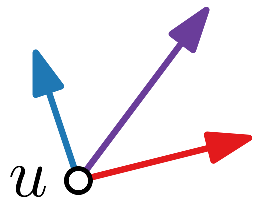

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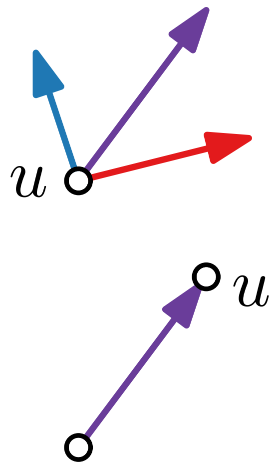

Force-Directed Algorithms

initial layout

threshold

max # iterations

ForceDirected$(G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N})$

$t \leftarrow 1$

while $t < K$ **and** $\max_{v \in V} \|F_v(t)\| > \varepsilon$ **do**

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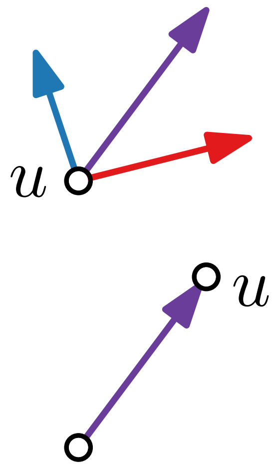

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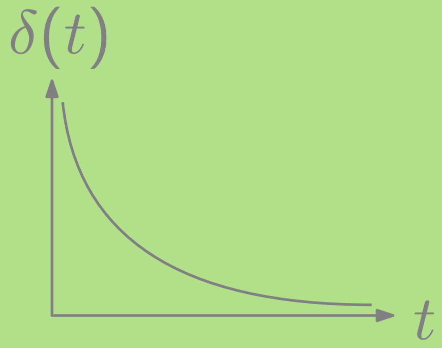
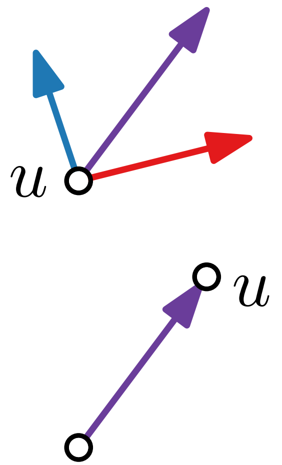


cooling factor

end layout



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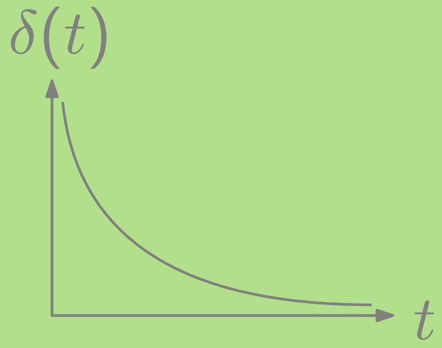

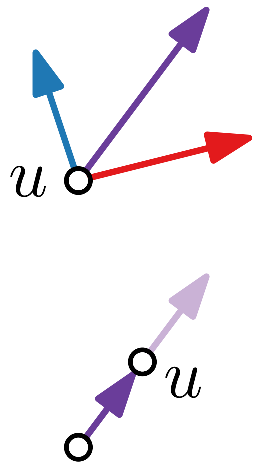

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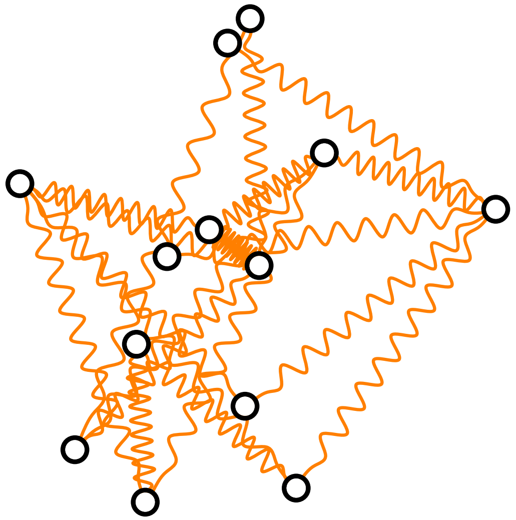
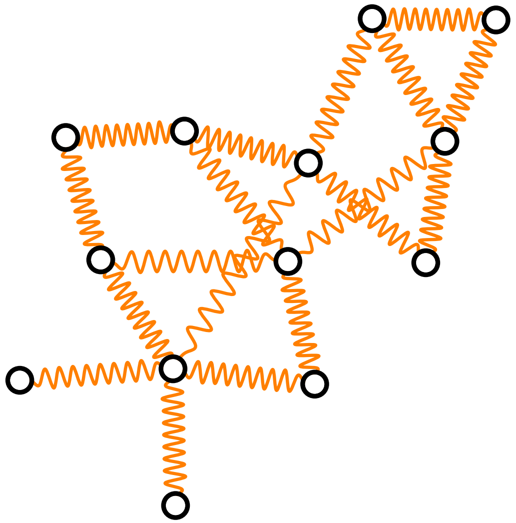
WÜRZBURG

Visualization of Graphs

Lecture 2:
Force-Directed Drawing Algorithms



Part II:
Spring Embedders by Eades and Fruchterman & Reingold



Jonathan Klawitter

Spring Embedder by Eades – Model

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- **Repulsive forces**

- **Attractive forces**

- **Resulting displacement vector**

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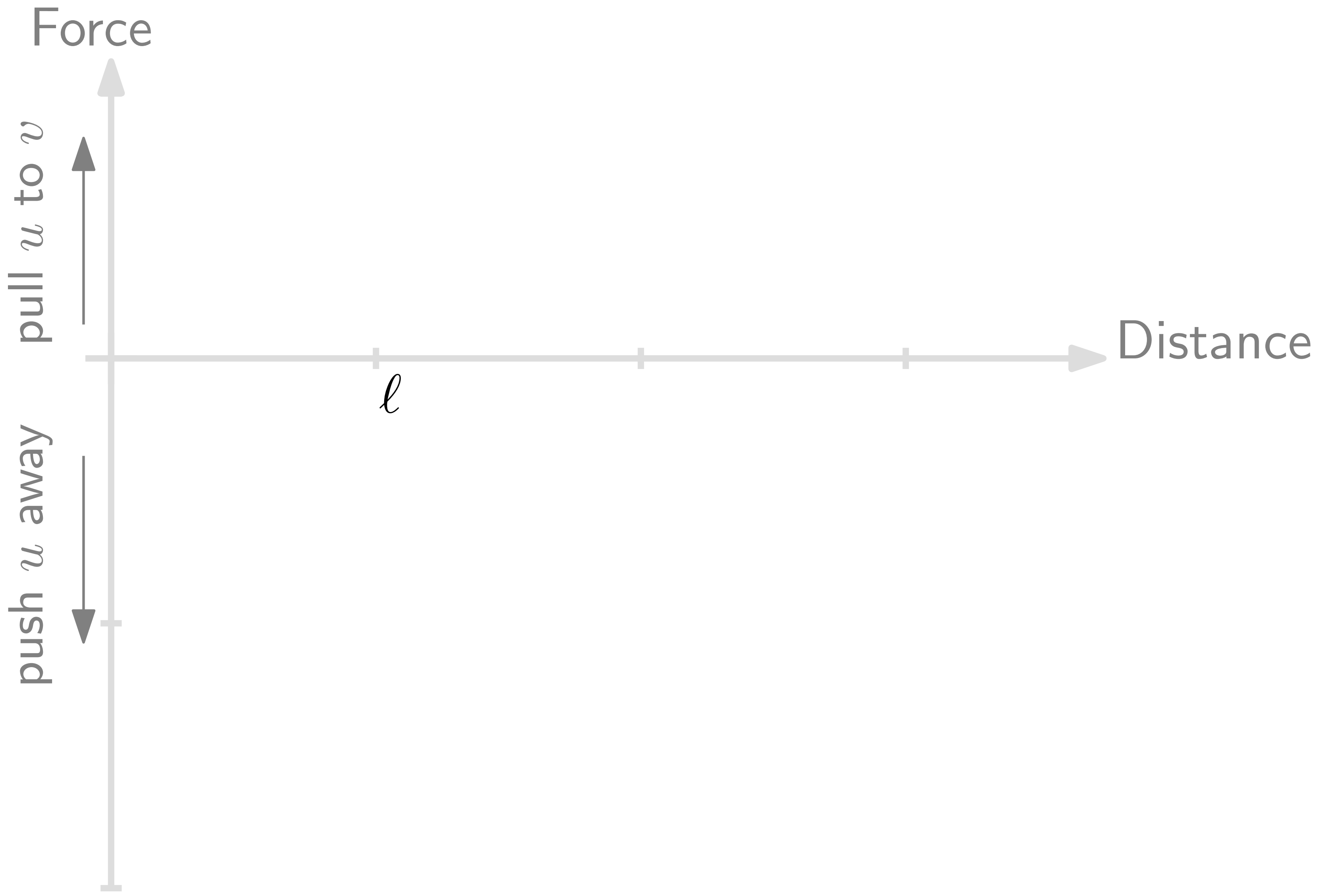
Spring Embedder by Eades – Force Diagram



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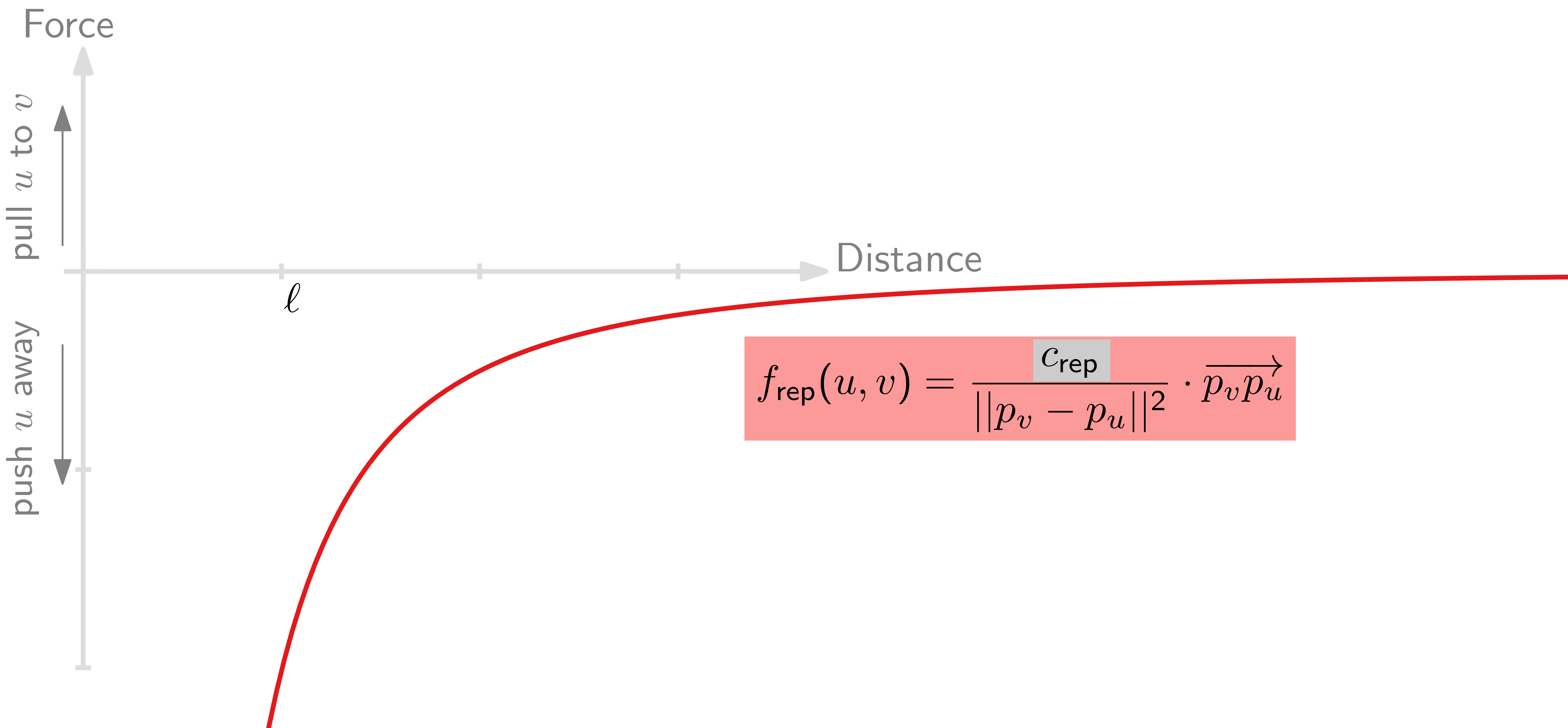

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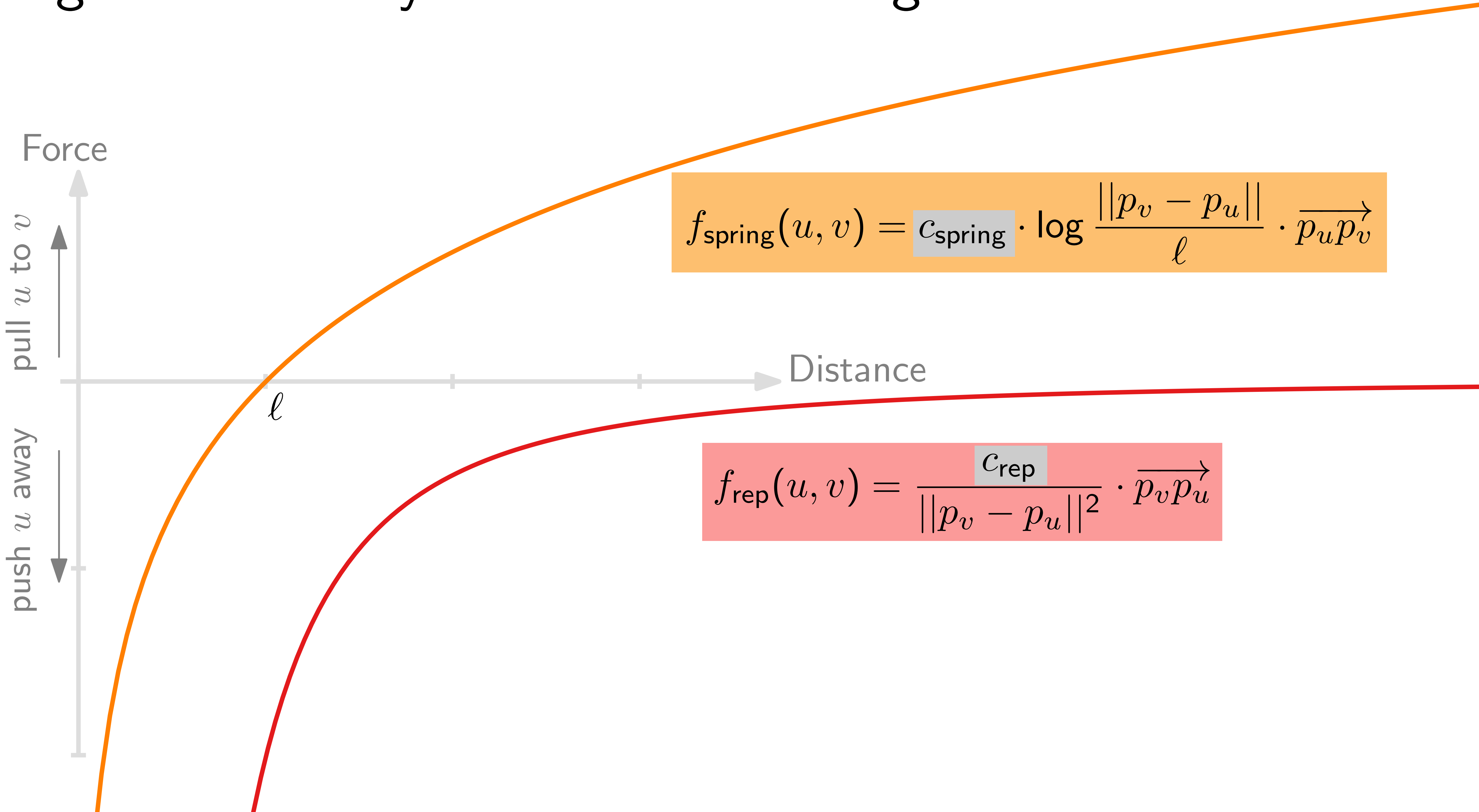

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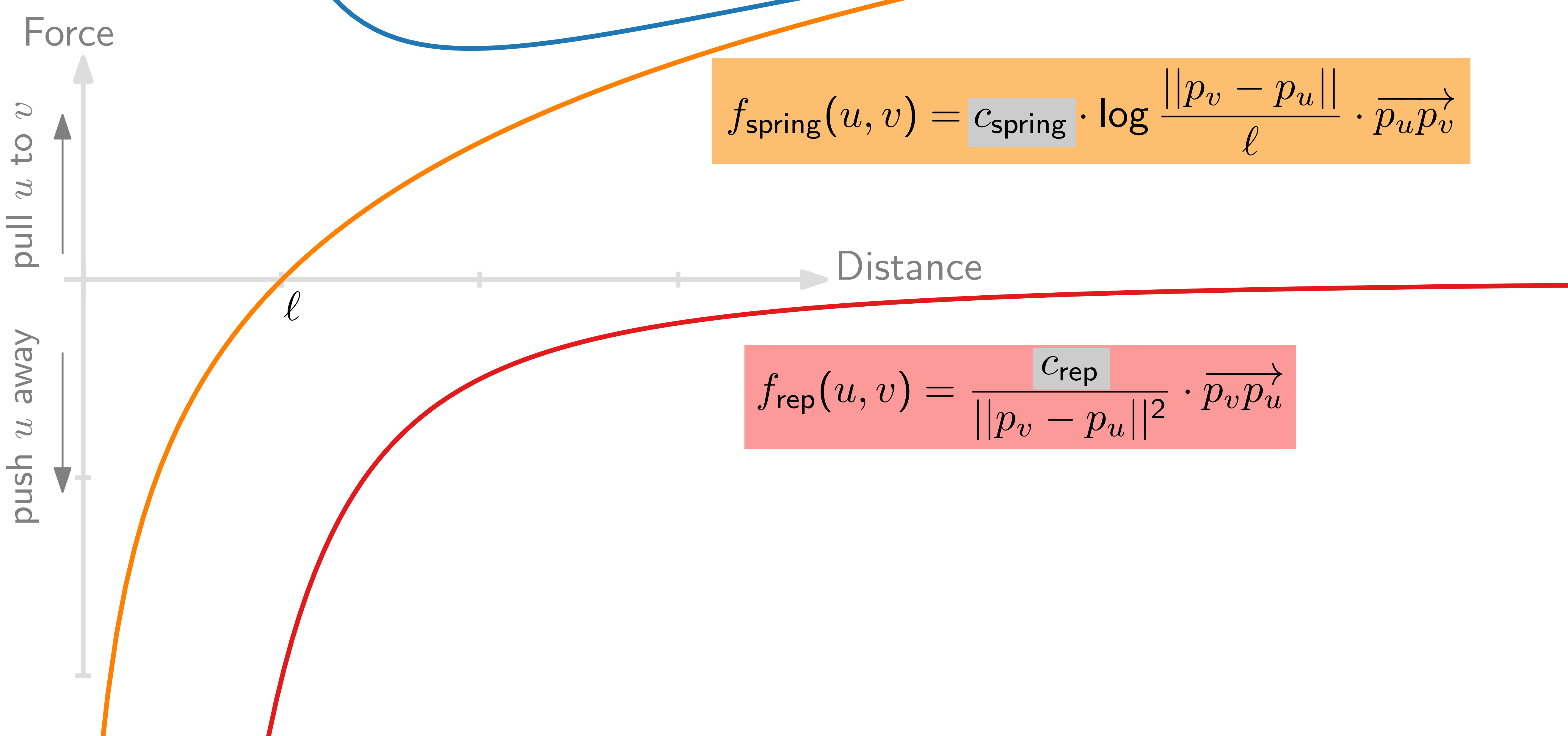

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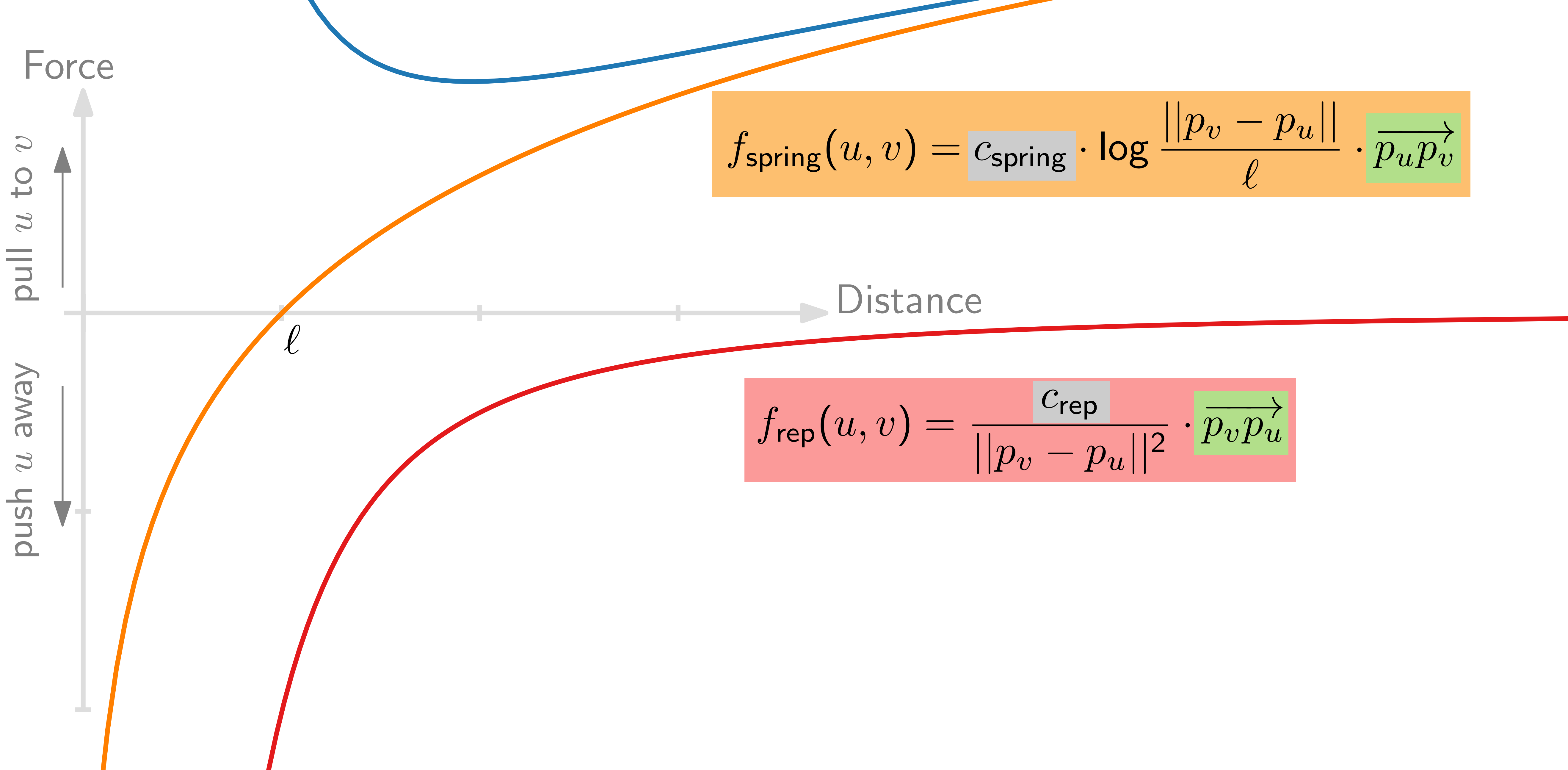

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- basis for many further ideas

Variant by Fruchterman & Reingold

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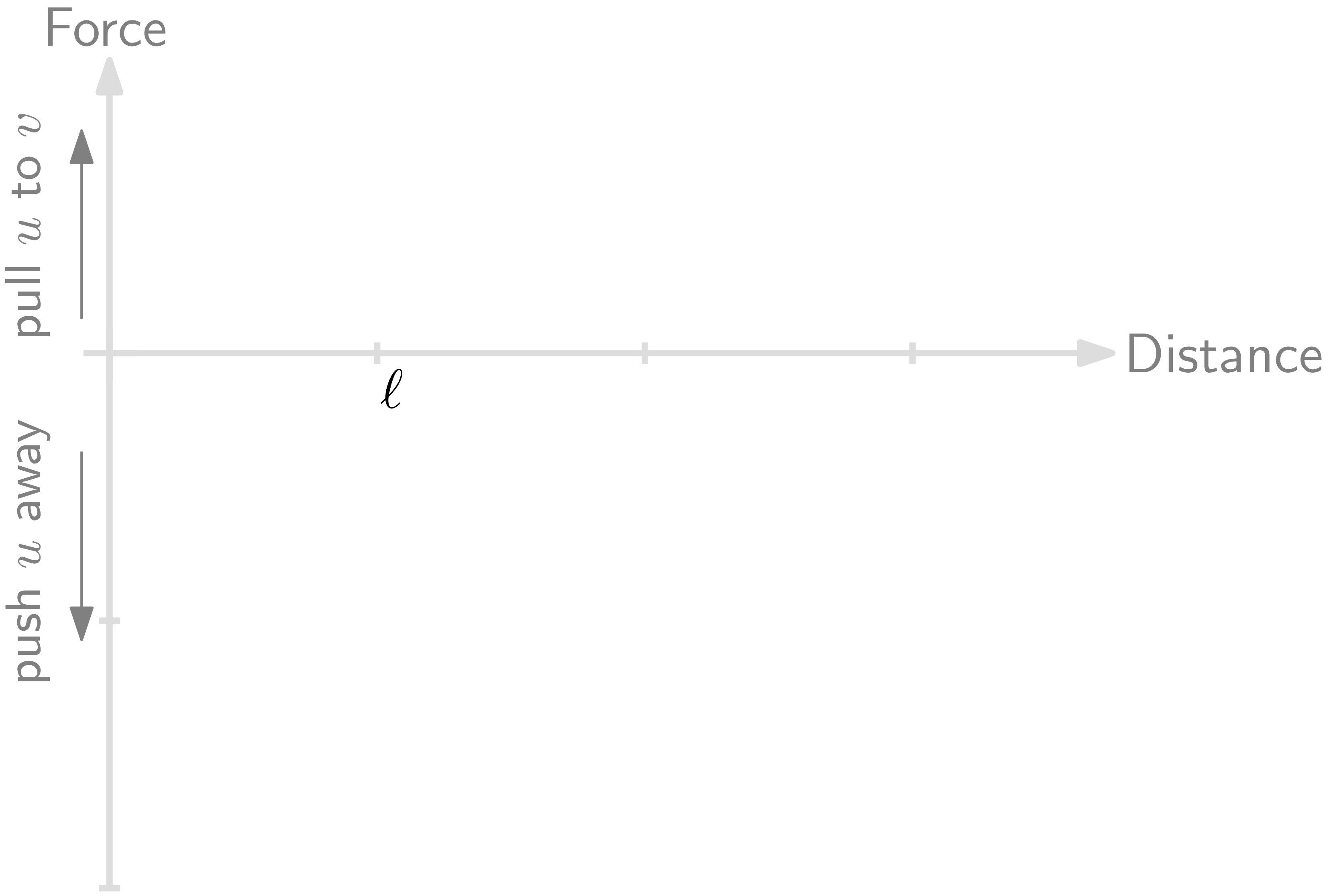
      $F_u(t) \leftarrow \sum_{v \in V} f_{\text{rep}}(u,v) + \sum_{uv \in E} f_{\text{attr}}(u,v)$
    foreach $u \in V$ **do**
      $p_u \leftarrow p_u + \delta(t) \cdot F_u(t)$
    $t \leftarrow t + 1$
  return p

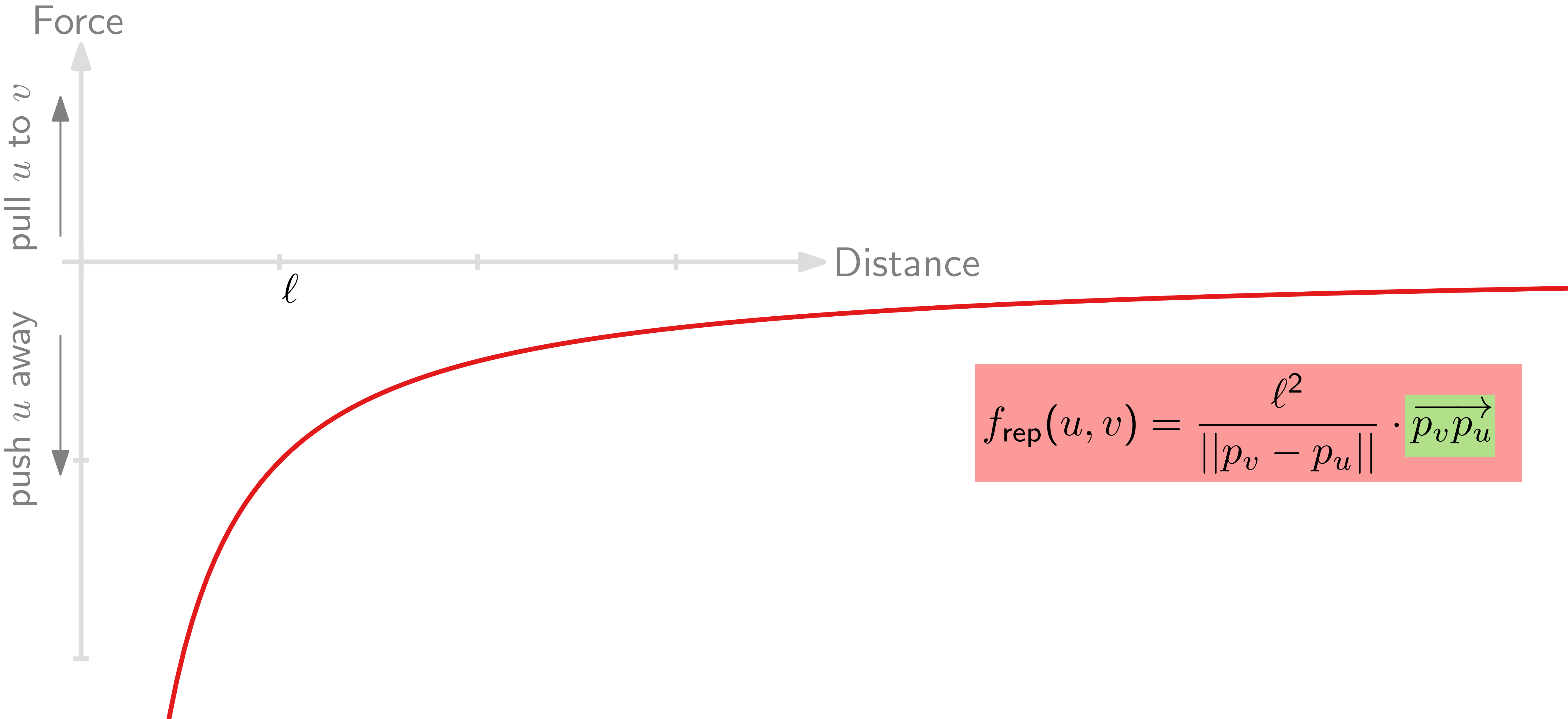
Notation.

- $||p_u - p_v||$ = Euclidean distance between u and v
- $\overrightarrow{p_u p_v}$ = unit vector pointing from u to v
- ℓ = ideal spring length for edges

Fruchterman & Reingold – Force Diagram



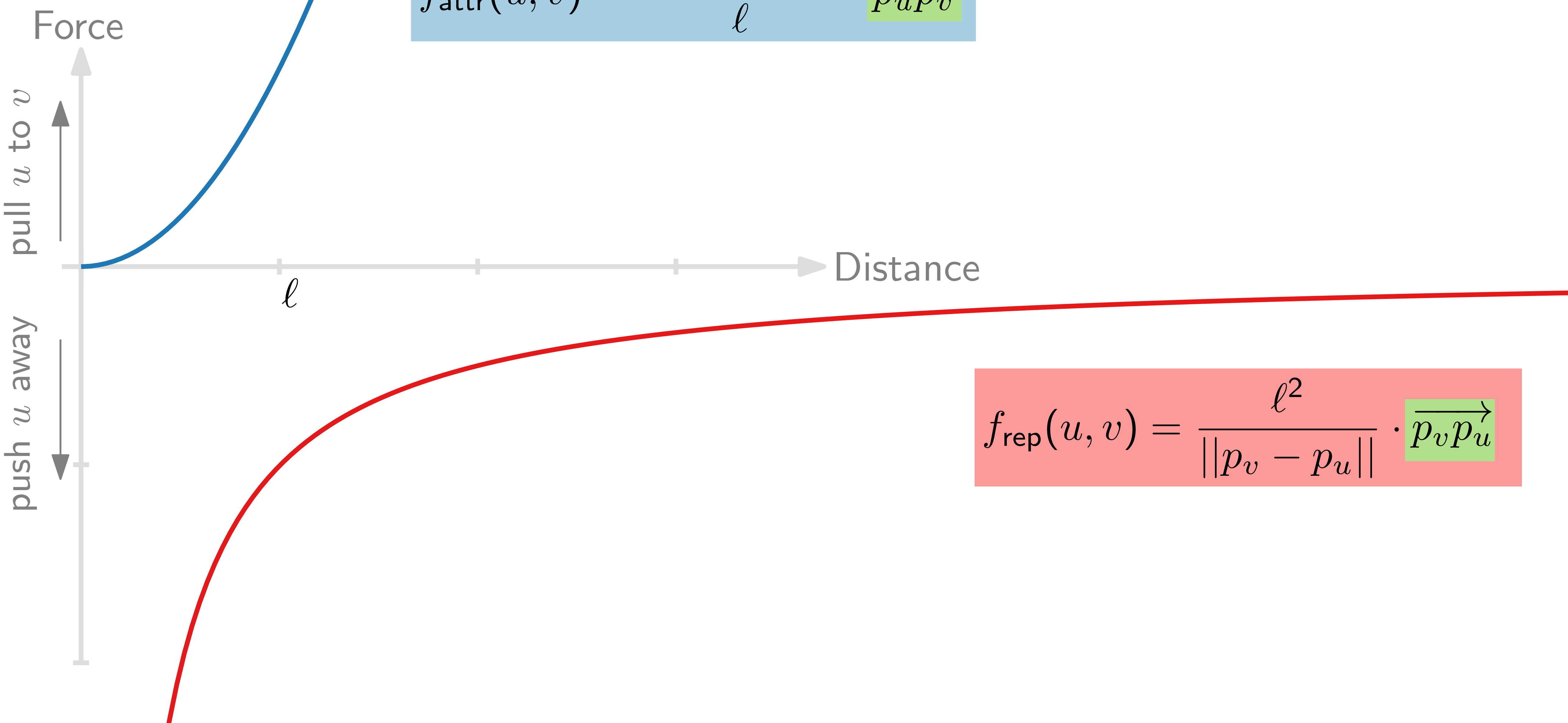
Fruchterman & Reingold – Force Diagram



$$f_{\mathsf{rep}}(u,v) = \frac{\ell^2}{||p_v - p_u||} \cdot \overrightarrow{p_v p_u}$$

Fruchterman & Reingold – Force Diagram

$$f_{\text{attr}}(u,v) = \frac{||p_v - p_u||^2}{\ell} \cdot \overrightarrow{p_u p_v}$$

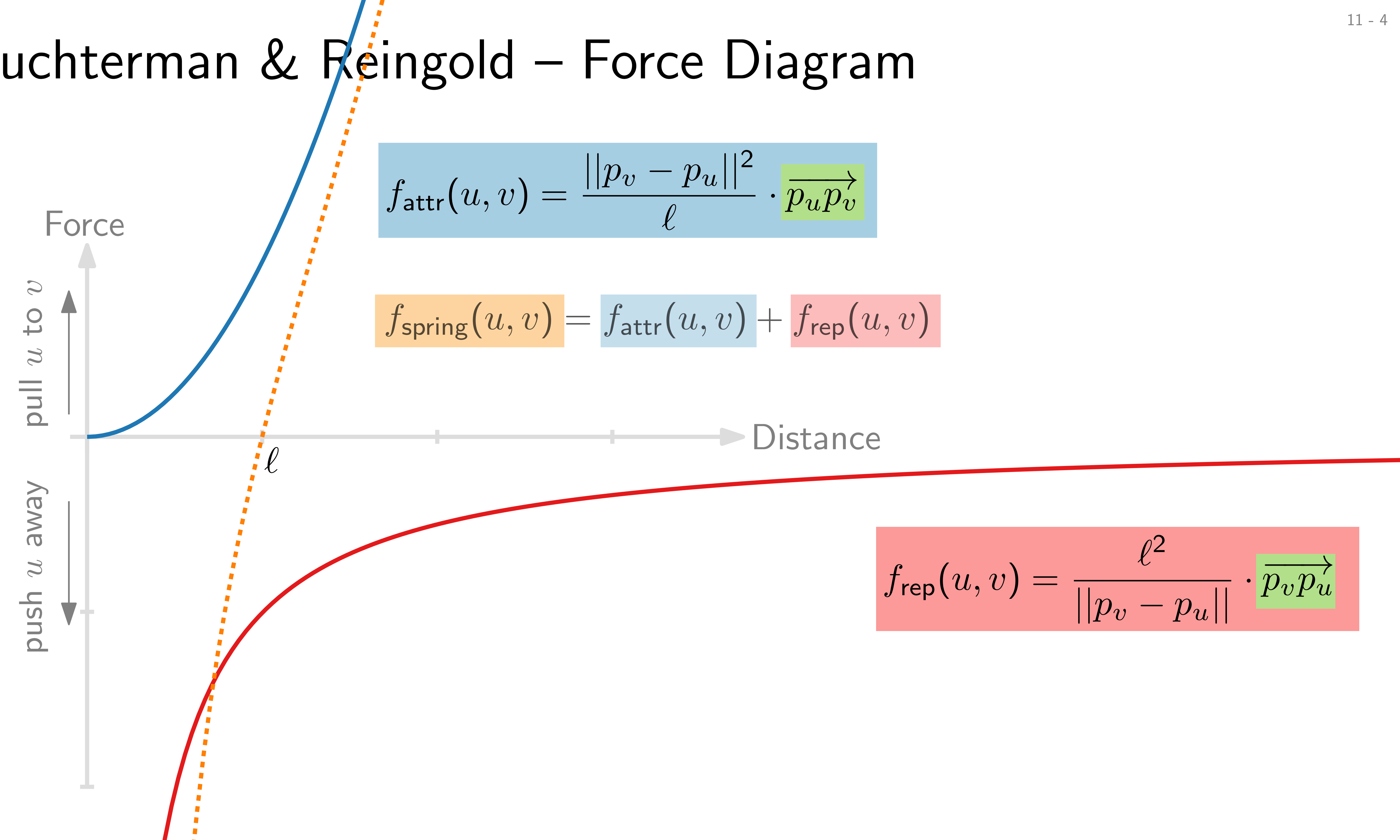


$$f_{\text{rep}}(u,v) = \frac{\ell^2}{||p_v - p_u||} \cdot \overrightarrow{p_v p_u}$$

Fruchterman & Reingold – Force Diagram

$$f_{\text{attr}}(u,v) = \frac{||p_v - p_u||^2}{\ell} \cdot \overrightarrow{p_u p_v}$$

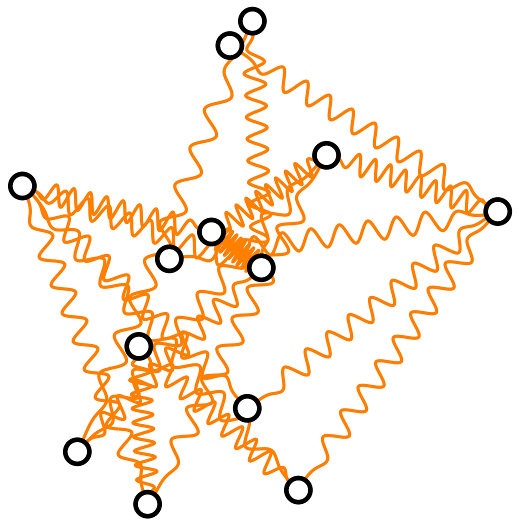
$$f_{\text{spring}}(u,v) = f_{\text{attr}}(u,v) + f_{\text{rep}}(u,v)$$



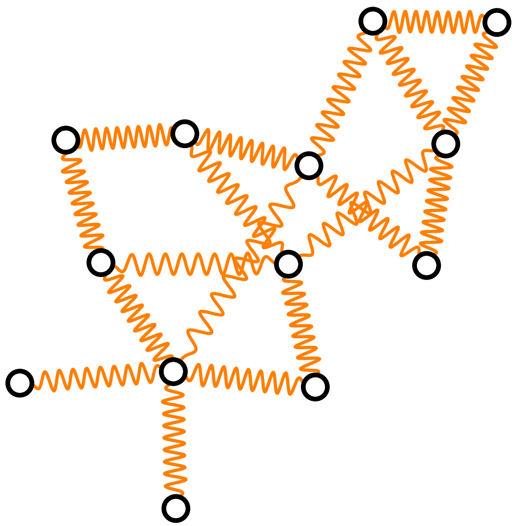
$$f_{\text{rep}}(u,v) = \frac{\ell^2}{||p_v - p_u||} \cdot \overrightarrow{p_v p_u}$$

Visualization of Graphs

Lecture 2:
Force-Directed Drawing Algorithms



Part III:
Variants & Improvements



Jonathan Klawitter

Adaptability

Inertia.

- Define vertex mass $\Phi(v) = 1 + \mathrm{deg}(v)/2$
- Set $f_{\mathrm{attr}}(p_u, p_v) \leftarrow f_{\mathrm{attr}}(p_u, p_v) \cdot 1/\Phi(v)$

Adaptability

Inertia.

- Define vertex mass $\Phi(v) = 1 + \deg(v)/2$
- Set $f_{\text{attr}}(p_u, p_v) \leftarrow f_{\text{attr}}(p_u, p_v) \cdot 1/\Phi(v)$

Gravitation.

- Define centroid $p_{\text{bary}} = 1/|V| \cdot \sum_{v \in V} p_v$
- Add force $f_{\text{grav}}(p_v) = c_{\text{grav}} \cdot \Phi(v) \cdot \overrightarrow{p_v p_{\text{bary}}}$

Adaptability

Inertia.

- Define vertex mass $\Phi(v) = 1 + \deg(v)/2$
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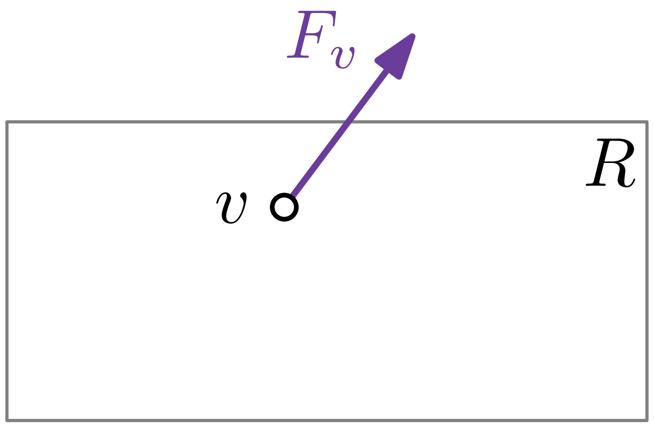
- Add force $f_{\text{grav}}(p_v) = c_{\text{grav}} \cdot \Phi(v) \cdot \overrightarrow{p_v p_{\text{bary}}}$

Restricted drawing area.

If F_v points beyond area R, clip vector appropriately at the border of R.



Adaptability

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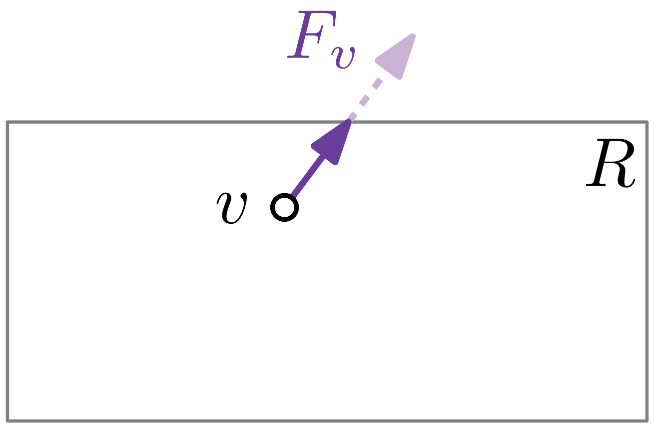

Adaptability

Inertia.

- Define vertex mass $\Phi(v) = 1 + \deg(v)/2$
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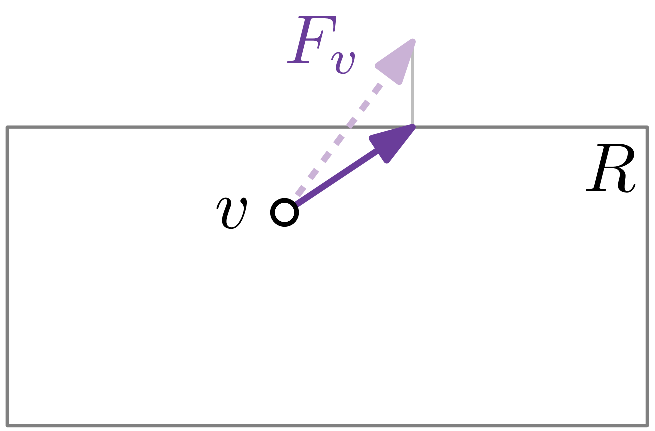

Adaptability

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- Set $f_{\text{attr}}(p_u, p_v) \leftarrow f_{\text{attr}}(p_u, p_v) \cdot 1/\Phi(v)$

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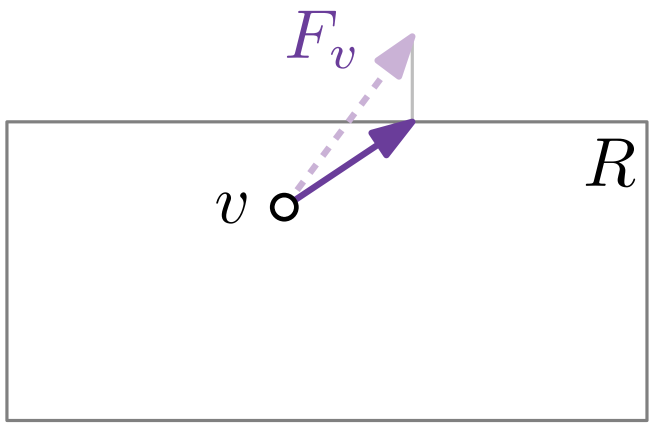
Restricted drawing area.

If F_v points beyond area R, clip vector appropriately at the border of R.



And many more...

- magnetic orientation of edges [GD Ch. 10.4]
- other energy models
- planarity preserving
- speedups

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

```
t ← 1
while t < K and max_{v∈V} ‖F_v(t)‖ > ε do
    foreach u ∈ V do
        F_u(t) ← Σ_{v∈V} f_rep(u, v) + Σ_{uv∈E} f_attr(u, v)
    foreach u ∈ V do
        p_u ← p_u + δ(t) · F_u(t)
    t ← t + 1
return p
```

- $t \leftarrow 1$
- **while** $t < K$ **and** $\max_{v \in V} \|F_v(t)\| > \varepsilon$ **do**
  - **foreach** $u \in V$ **do**
    - $F_u(t) \leftarrow \sum_{v \in V} f_{\text{rep}}(u, v) + \sum_{uv \in E} f_{\text{attr}}(u, v)$
  - **foreach** $u \in V$ **do**
    - $p_u \leftarrow p_u + \delta(t) \cdot F_u(t)$
  - $t \leftarrow t + 1$
- **return** p

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

$t \leftarrow 1$

while $t < K$ **and** $\max_{v \in V} \|F_v(t)\| > \varepsilon$ **do**

  foreach $u \in V$ **do**

    $F_u(t) \leftarrow \sum_{v \in V} f_{\text{rep}}(u, v) + \sum_{uv \in E} f_{\text{attr}}(u, v)$

  foreach $u \in V$ **do**

    $p_u \leftarrow p_u + \cancel{\delta(t)} \cdot F_u(t)$

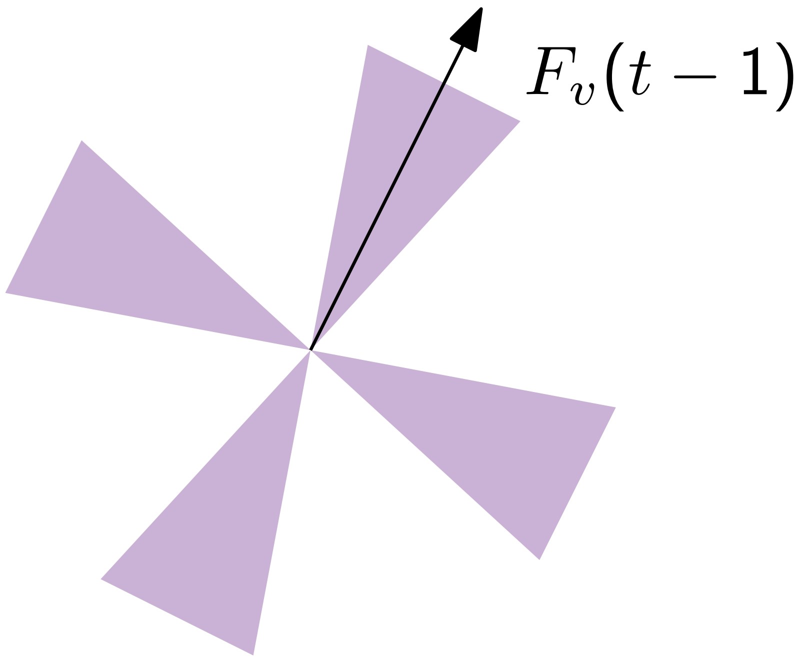
    $\delta_v(t)$

  $t \leftarrow t + 1$

return p

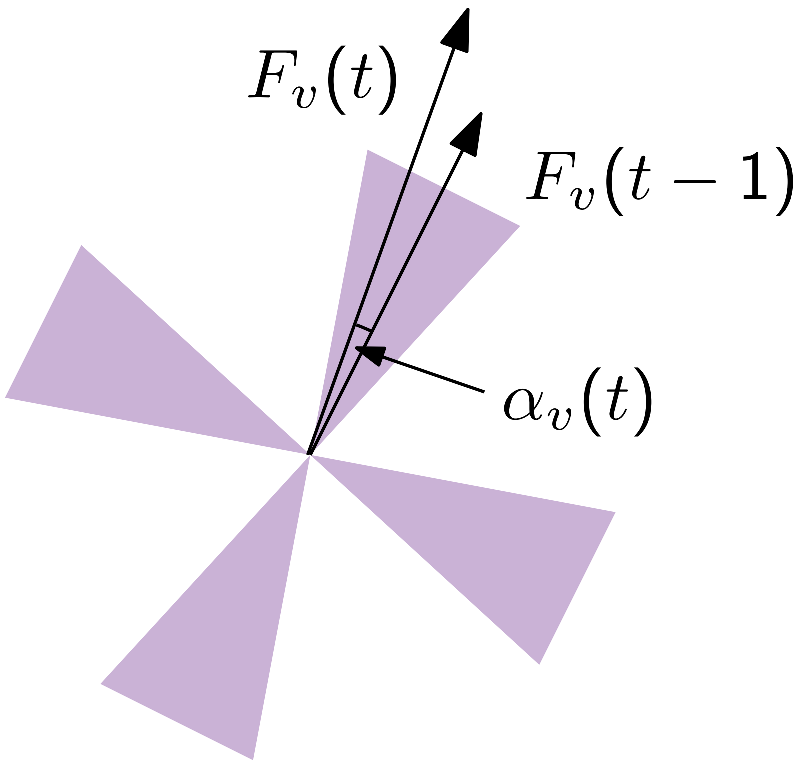
Speeding up “Convergence” by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau ’95]



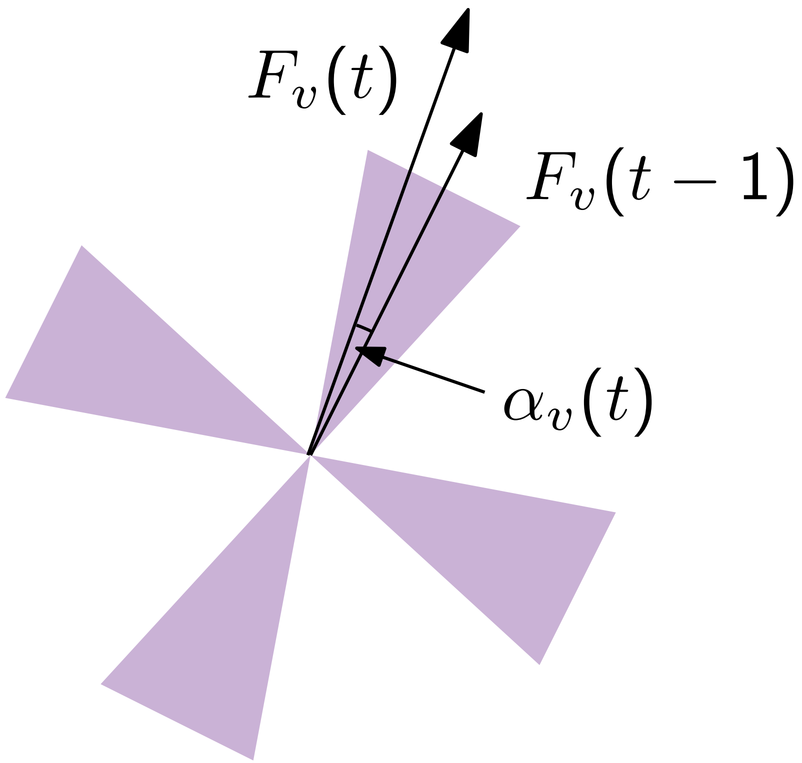
Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]



Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]

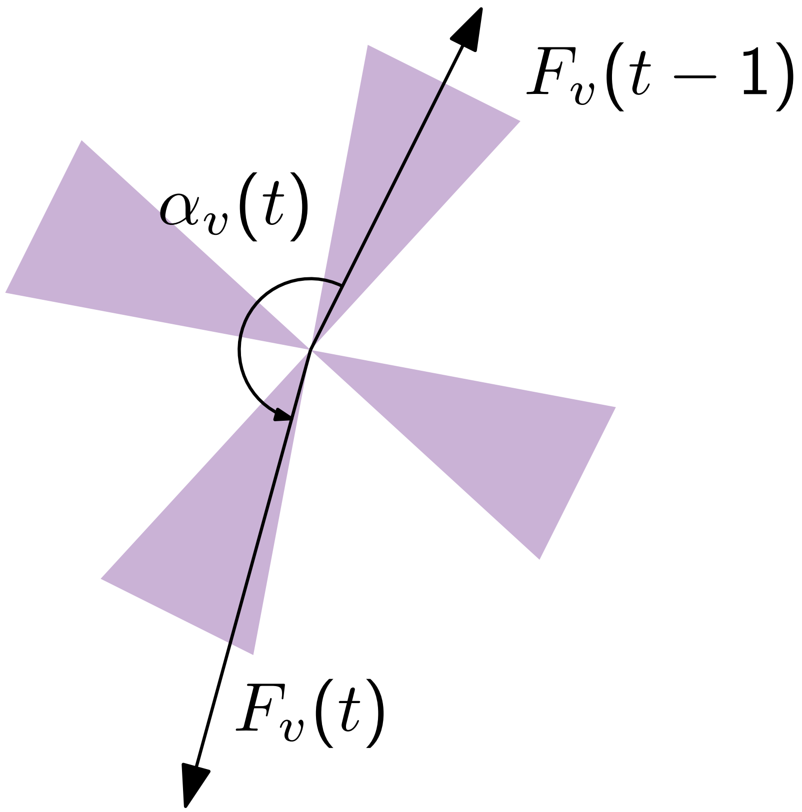


Same direction.

$\rightarrow$ increase temperature $\delta_v(t)$

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]

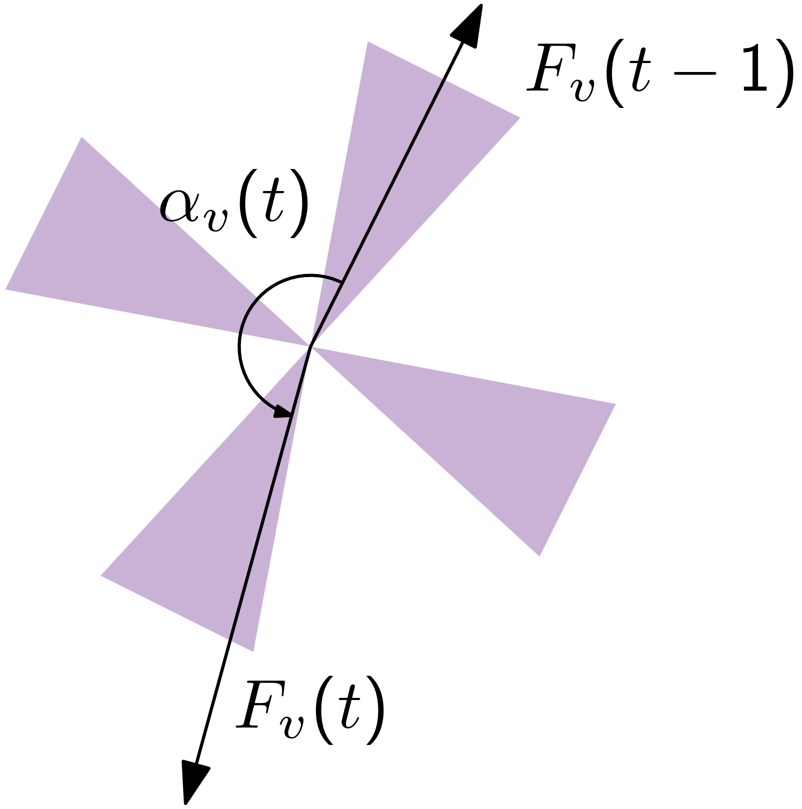


Same direction.

$\rightarrow$ increase temperature $\delta_v(t)$

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]

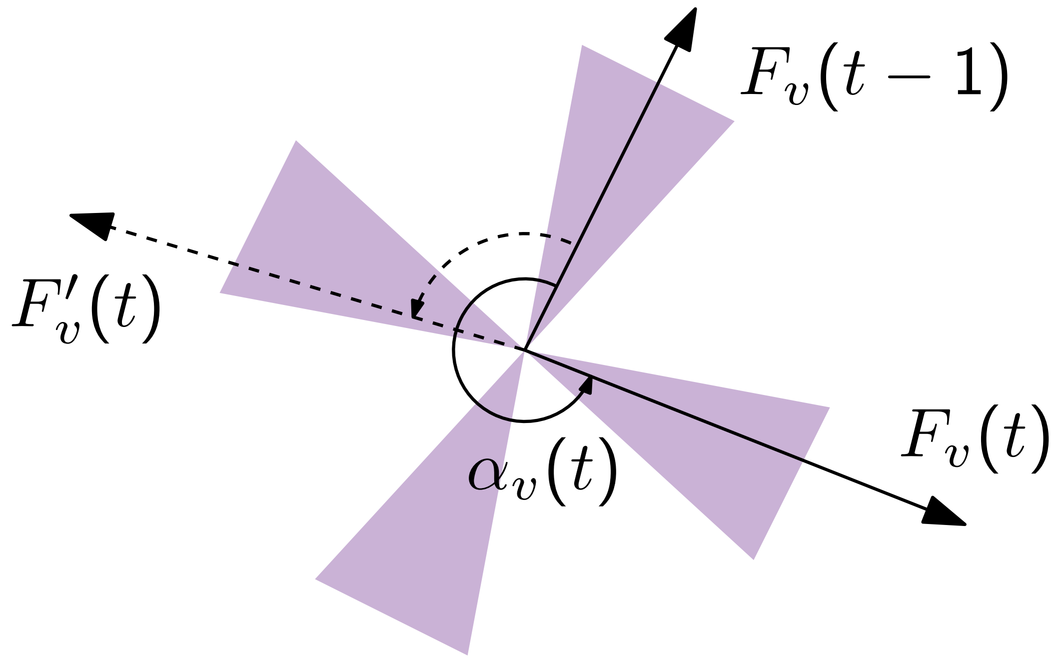


Same direction.
$\rightarrow$ increase temperature $\delta_v(t)$

Oszillation.
$\rightarrow$ decrease temperature $\delta_v(t)$

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]



Same direction.

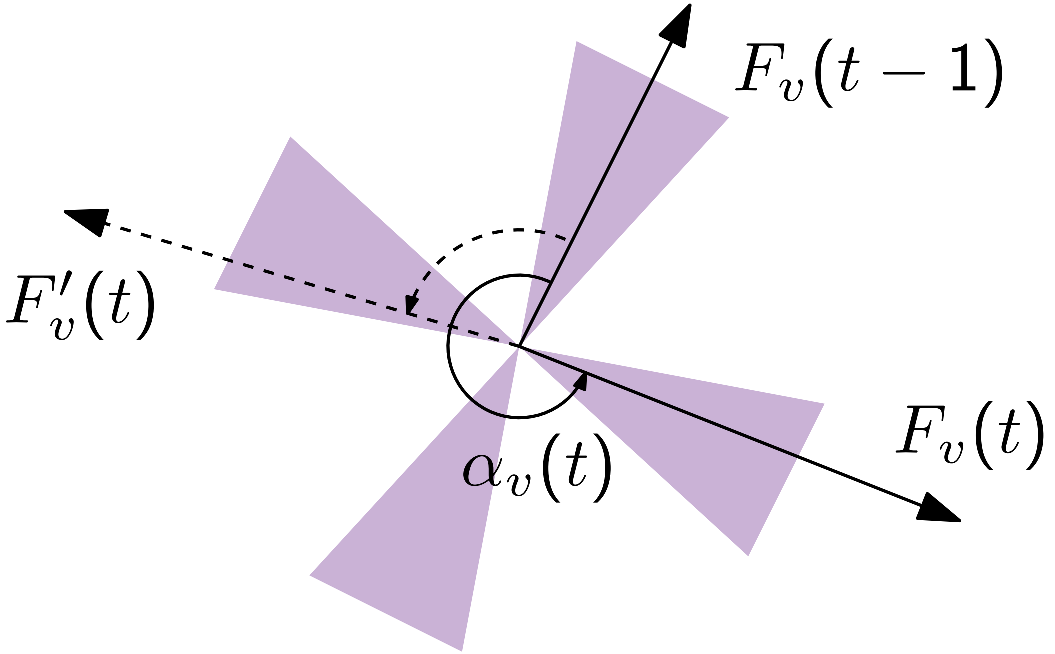
$\rightarrow$ increase temperature $\delta_v(t)$

Oszillation.

$\rightarrow$ decrease temperature $\delta_v(t)$

Speeding up "Convergence" by Adaptive Displacement $\delta_v(t)$

[Frick, Ludwig, Mehldau '95]



Same direction.

$\rightarrow$ increase temperature $\delta_v(t)$

Oszillation.

$\rightarrow$ decrease temperature $\delta_v(t)$

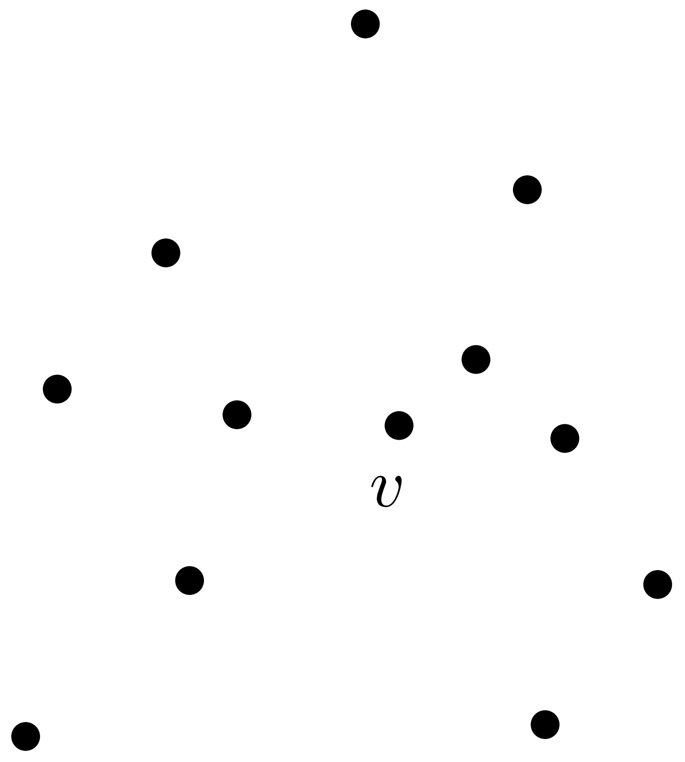
Rotation.

- count rotations
- if applicable

$\rightarrow$ decrease temperature $\delta_v(t)$

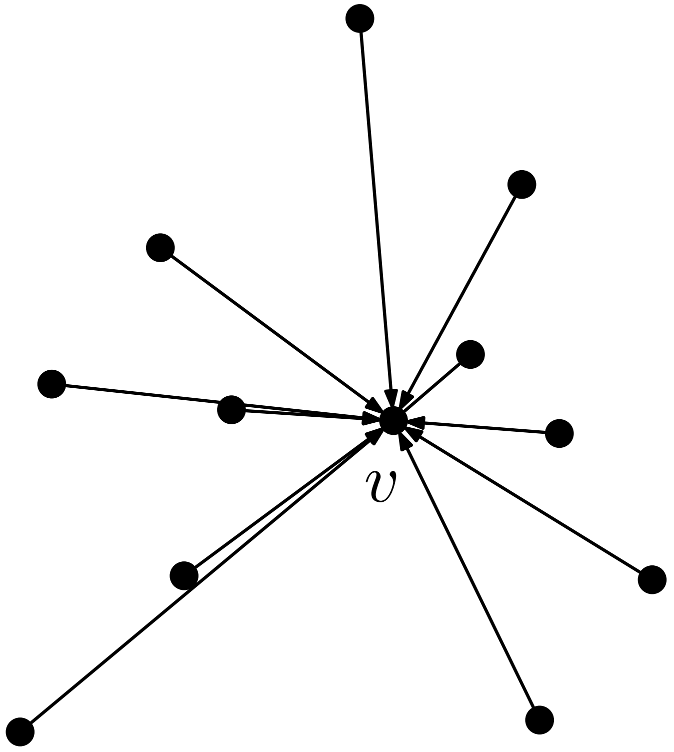
Speeding up "Convergence" via Grids

[Fruchterman & Reingold '91]



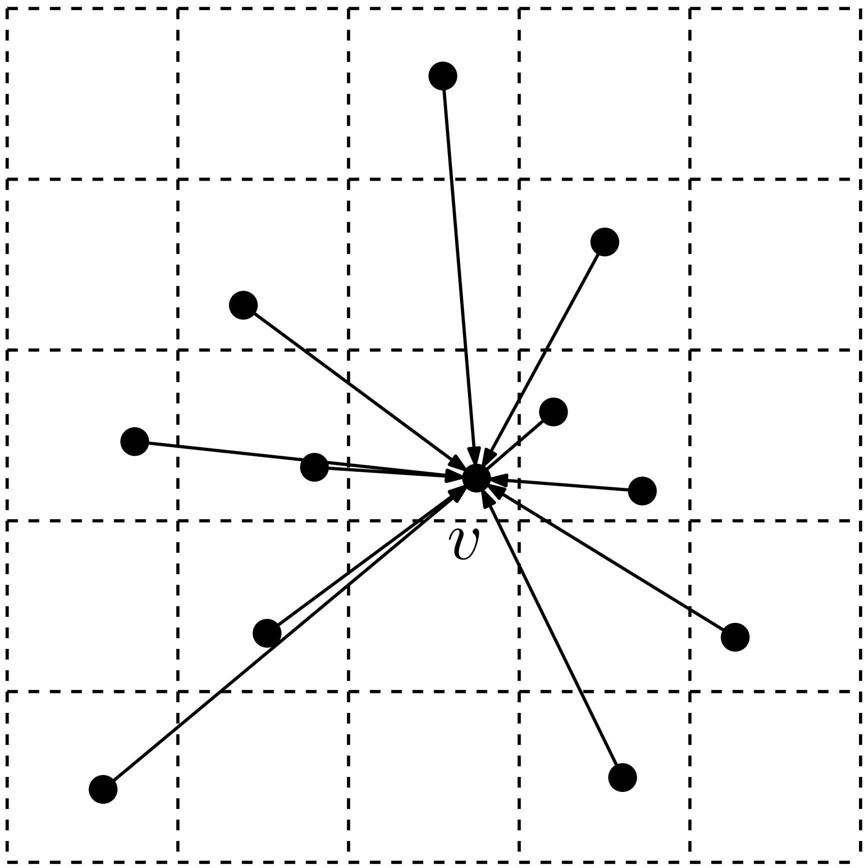
Speeding up "Convergence" via Grids

[Fruchterman & Reingold '91]



Speeding up "Convergence" via Grids

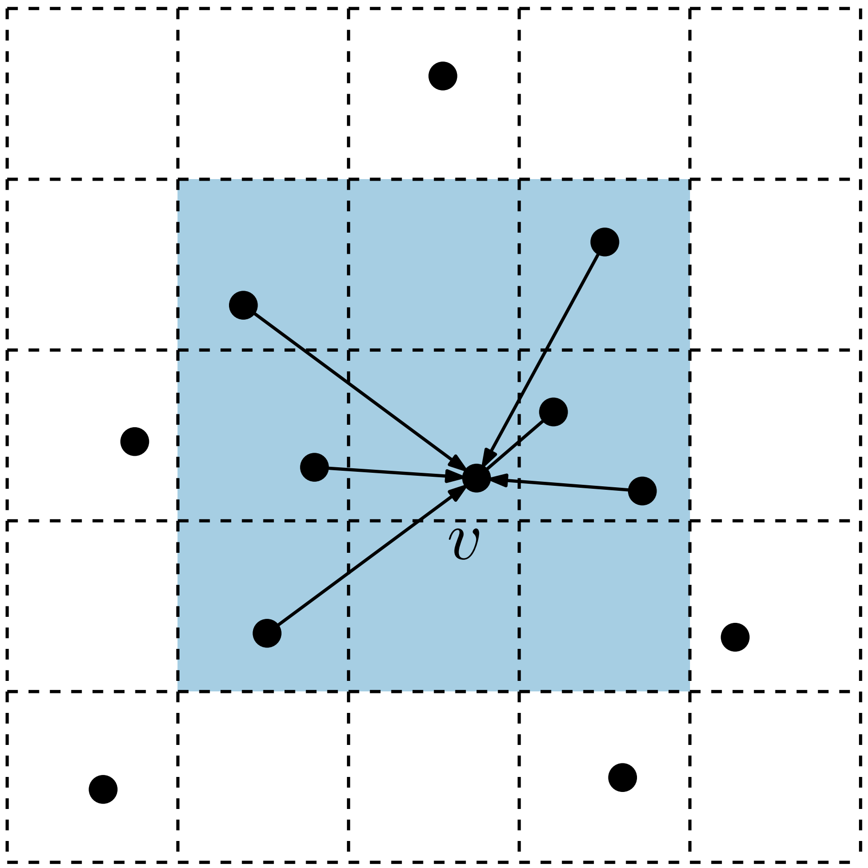
[Fruchterman & Reingold '91]



- divide plane into grid

Speeding up "Convergence" via Grids

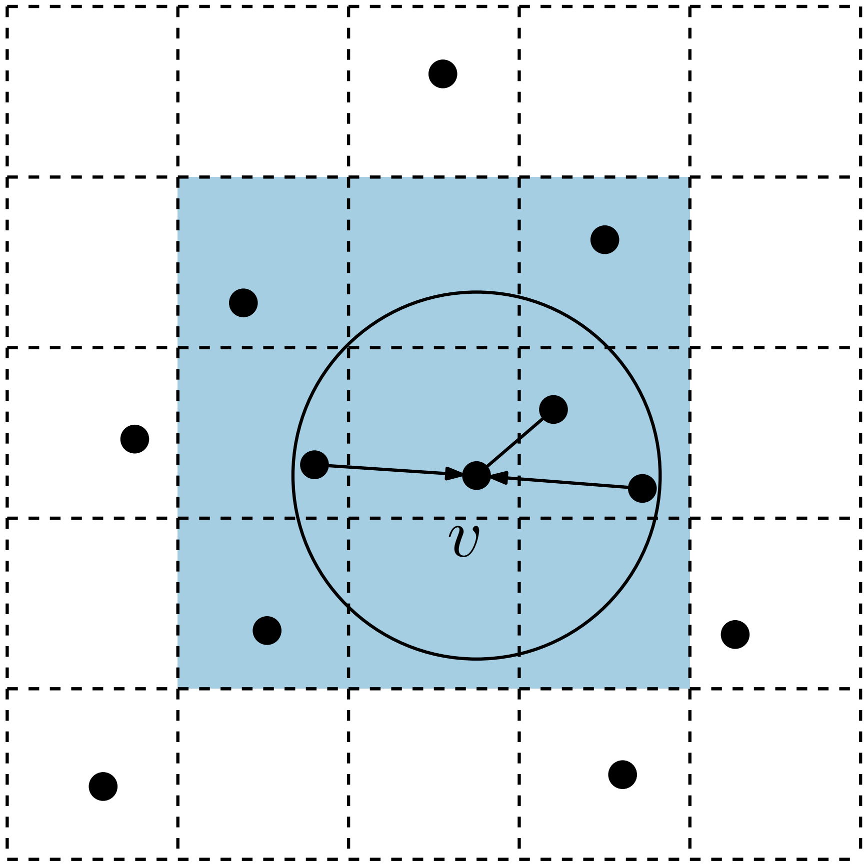
[Fruchterman & Reingold '91]



- divide plane into grid
- consider repelling forces only to vertices in neighboring cells

Speeding up "Convergence" via Grids

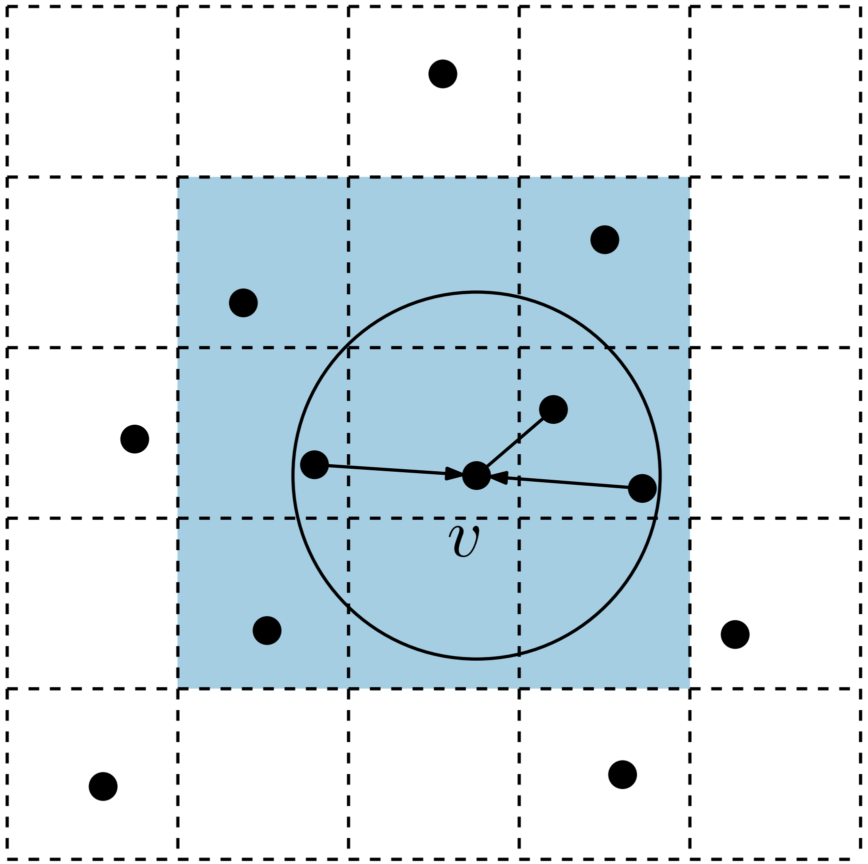
[Fruchterman & Reingold '91]



- divide plane into grid
- consider repelling forces only to vertices in neighboring cells
- and only if distance is less than some max distance

Speeding up "Convergence" via Grids

[Fruchterman & Reingold '91]



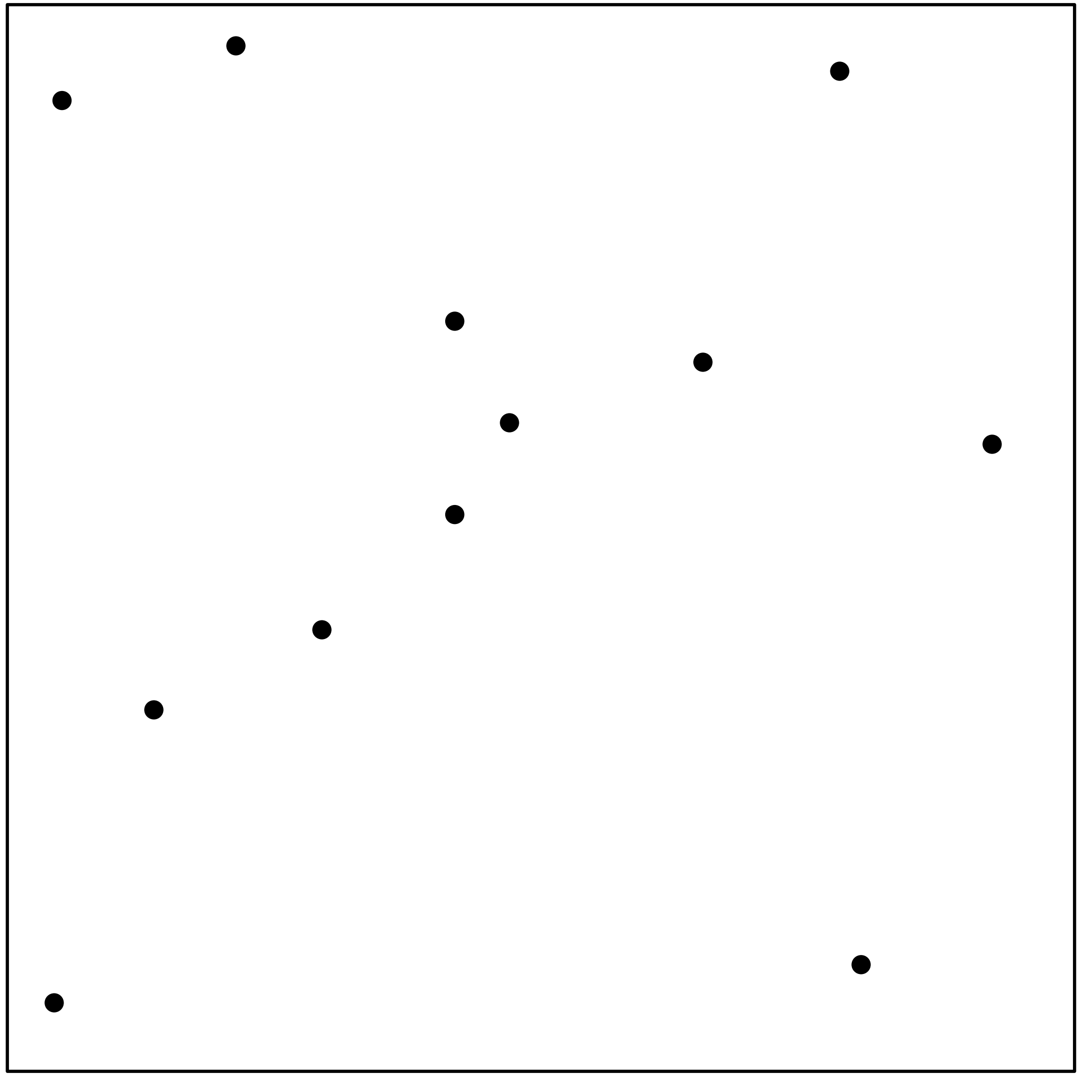
- divide plane into grid
- consider repelling forces only to vertices in neighboring cells
- and only if distance is less than some max distance

Discussion.

- good idea to improve runtime
- worst-case has not improved
- might introduce oszillation and thus a quality loss

Speeding up with Quad Trees

[Barnes, Hut '86]

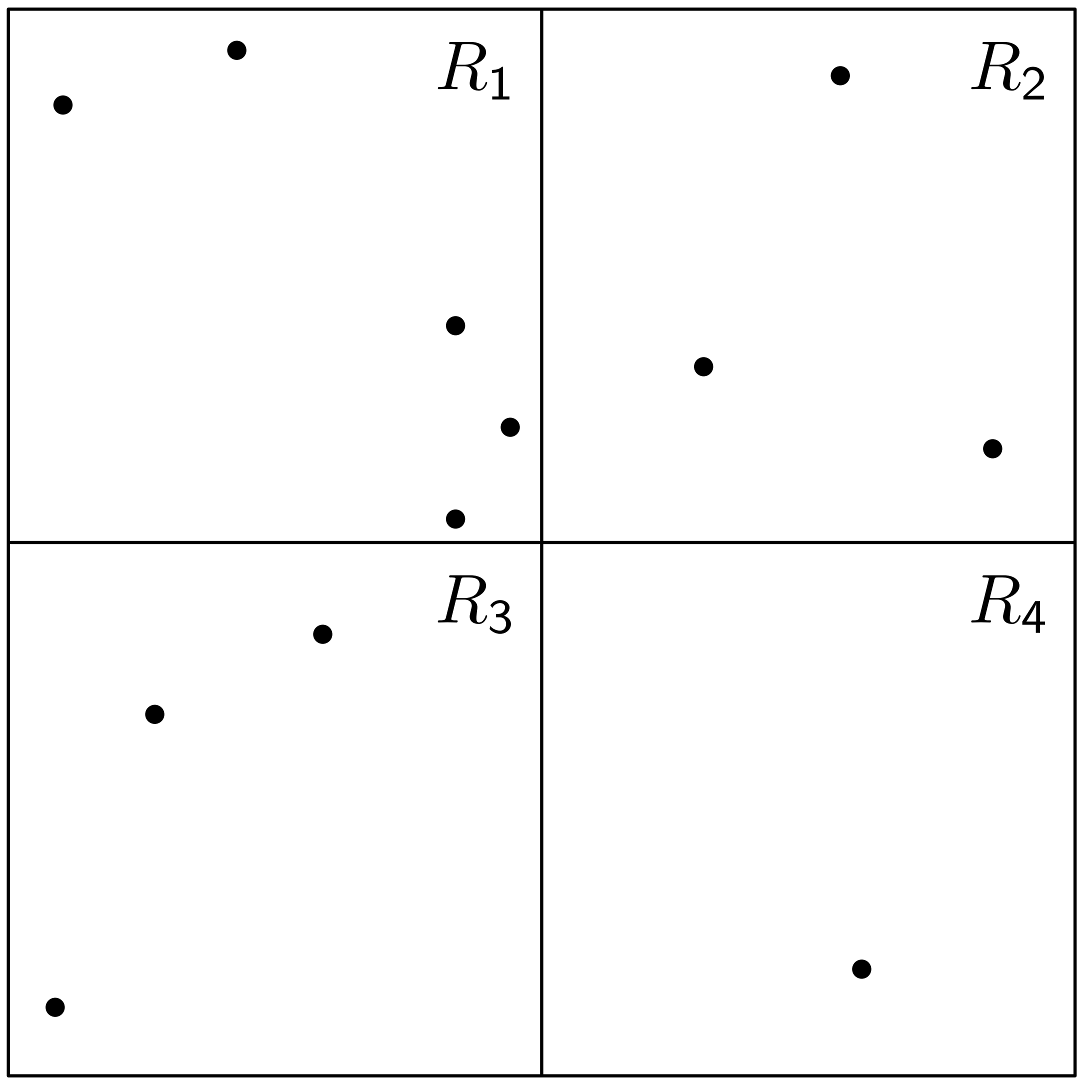


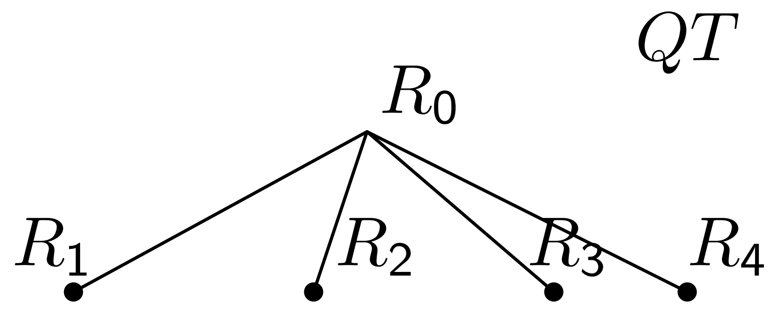
QT

R_0

Speeding up with Quad Trees

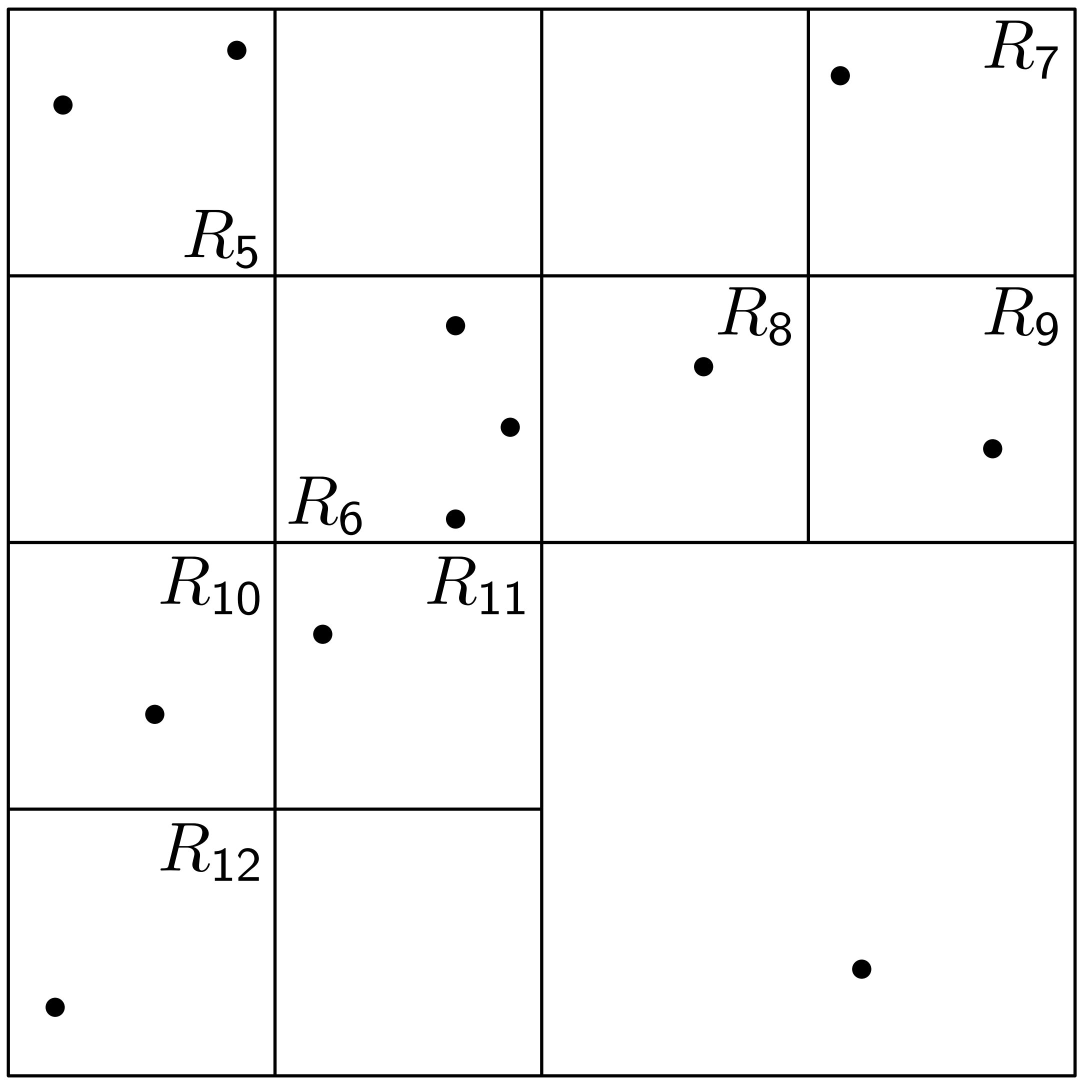
[Barnes, Hut '86]

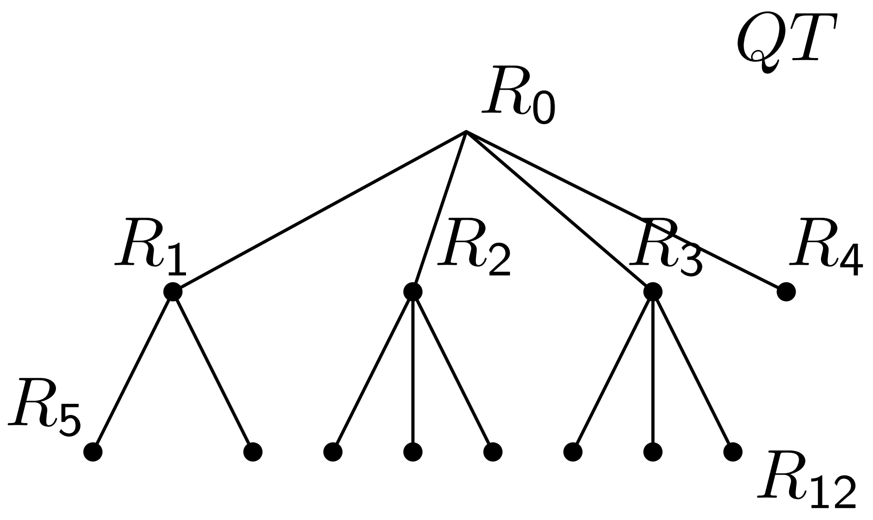




Speeding up with Quad Trees

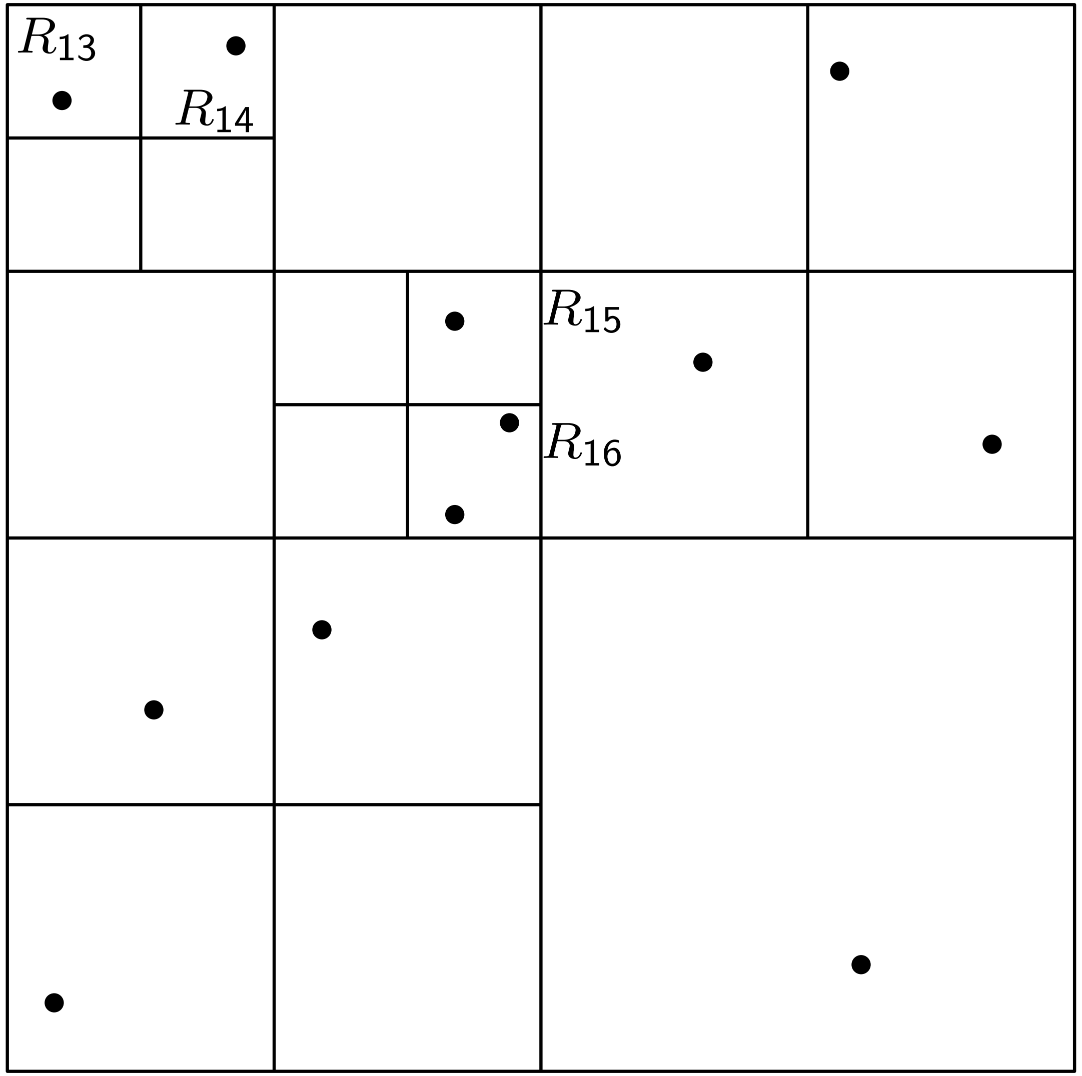
[Barnes, Hut '86]

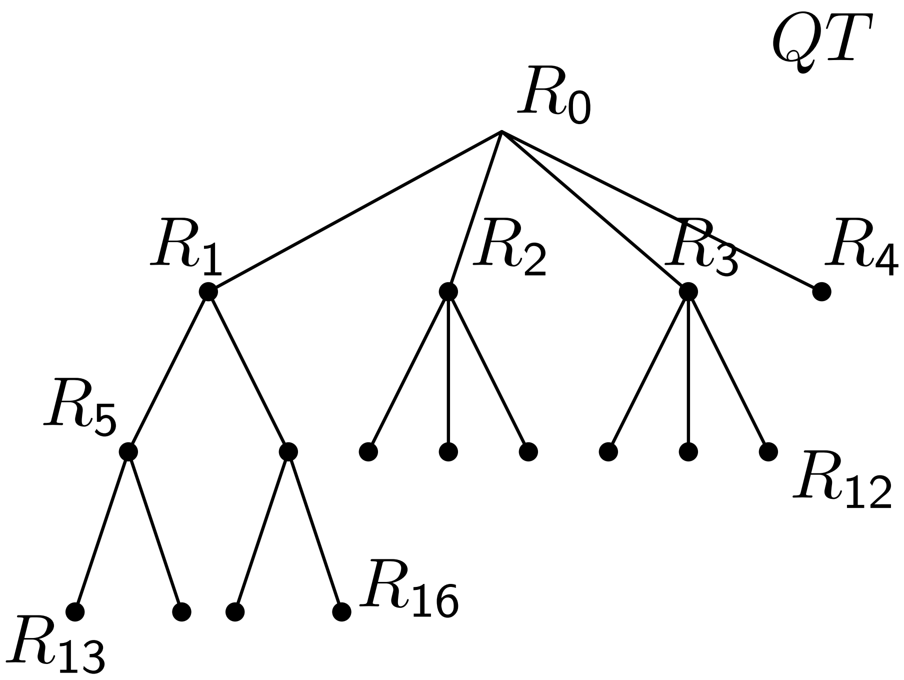




Speeding up with Quad Trees

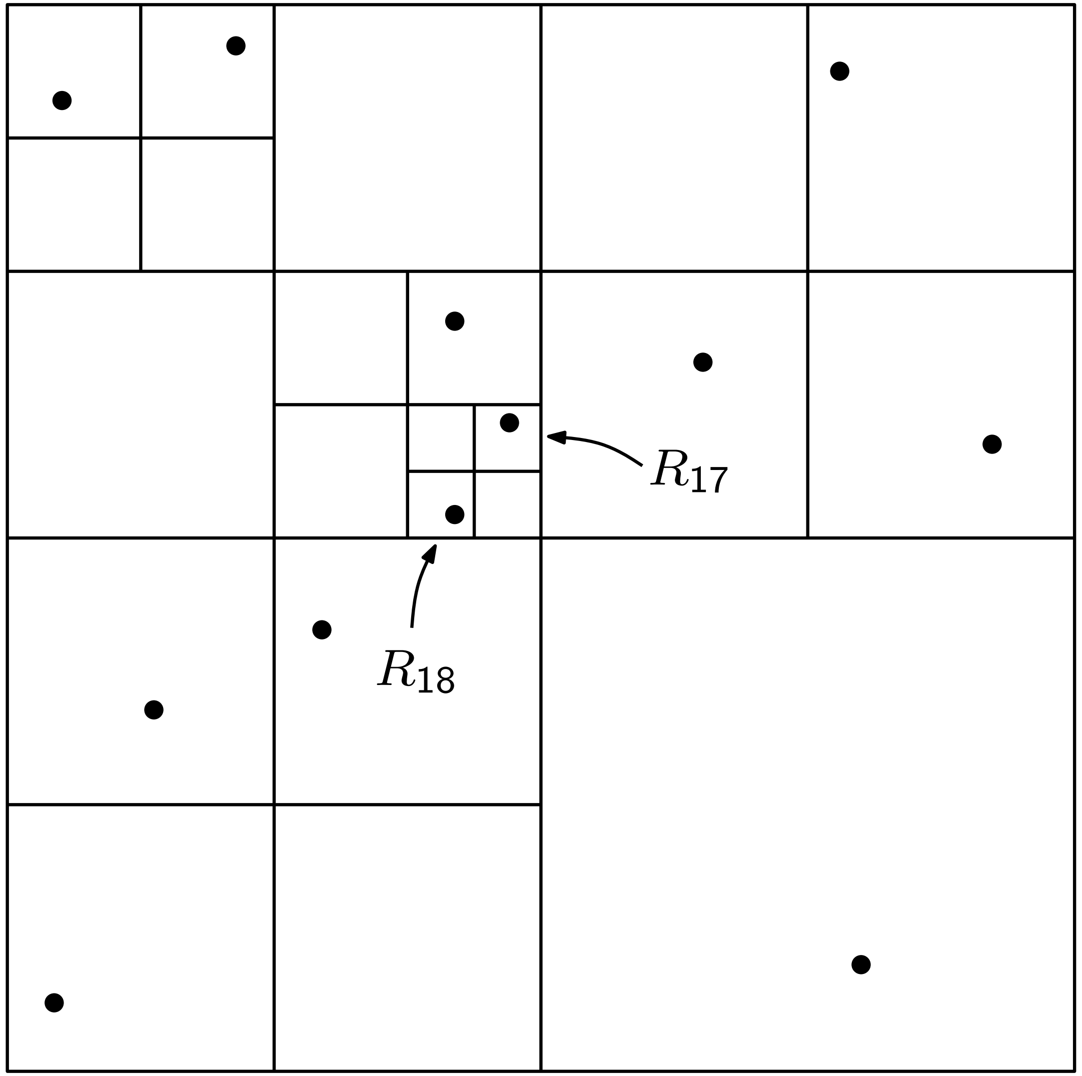
[Barnes, Hut '86]

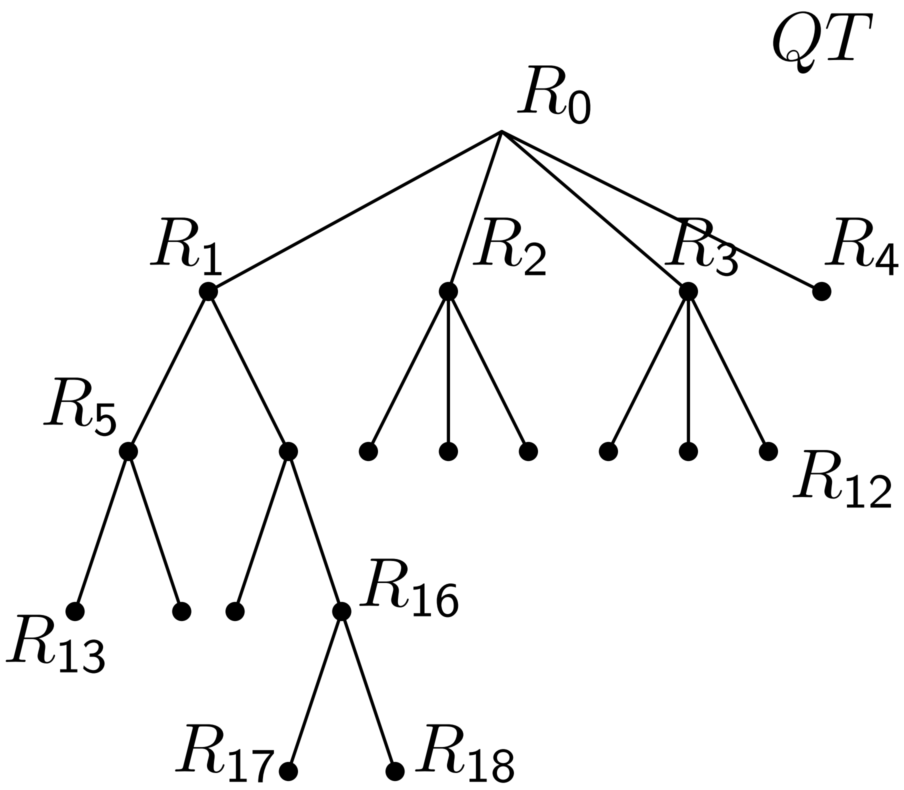




Speeding up with Quad Trees

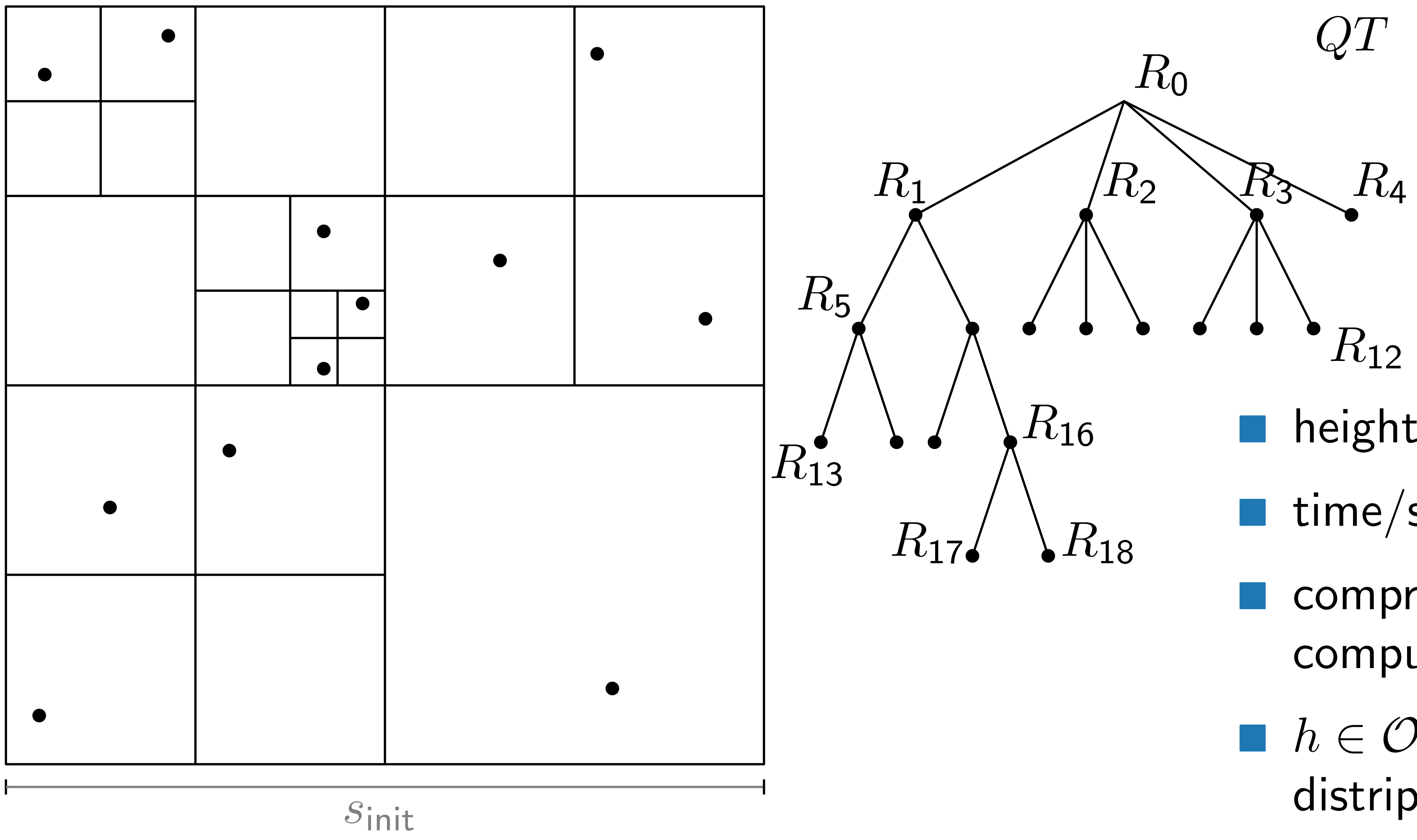
[Barnes, Hut '86]





Speeding up with Quad Trees

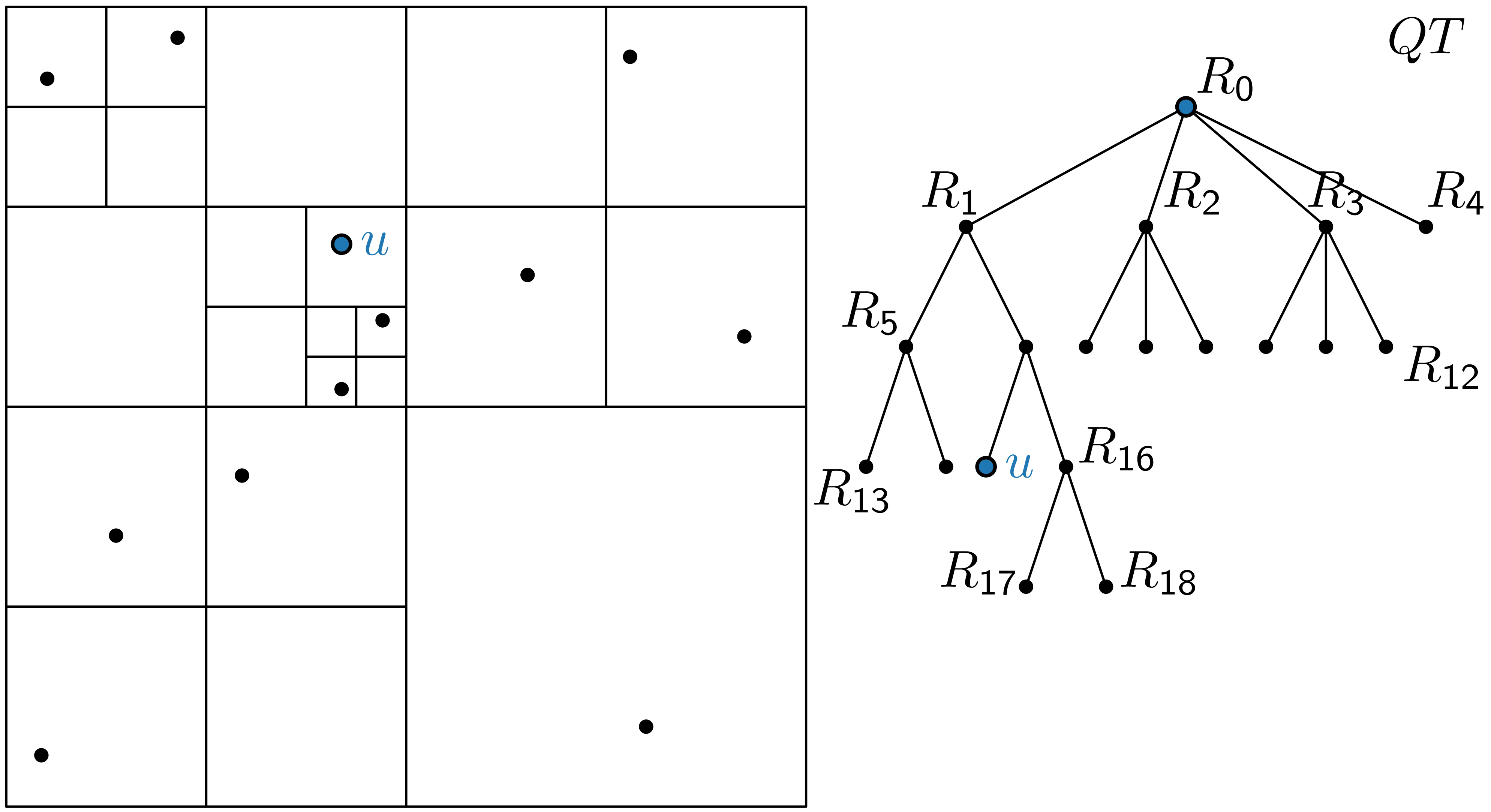
[Barnes, Hut '86]



- height $h \leq \log \frac{s_{\text{init}}}{d_{\text{min}}} + \frac{3}{2}$
- time/space in $\mathcal{O}(hn)$
- compressed quad tree can be computed in $\mathcal{O}(n \log n)$ time
- $h \in \mathcal{O}(\log n)$ if vertices evenly distriputed

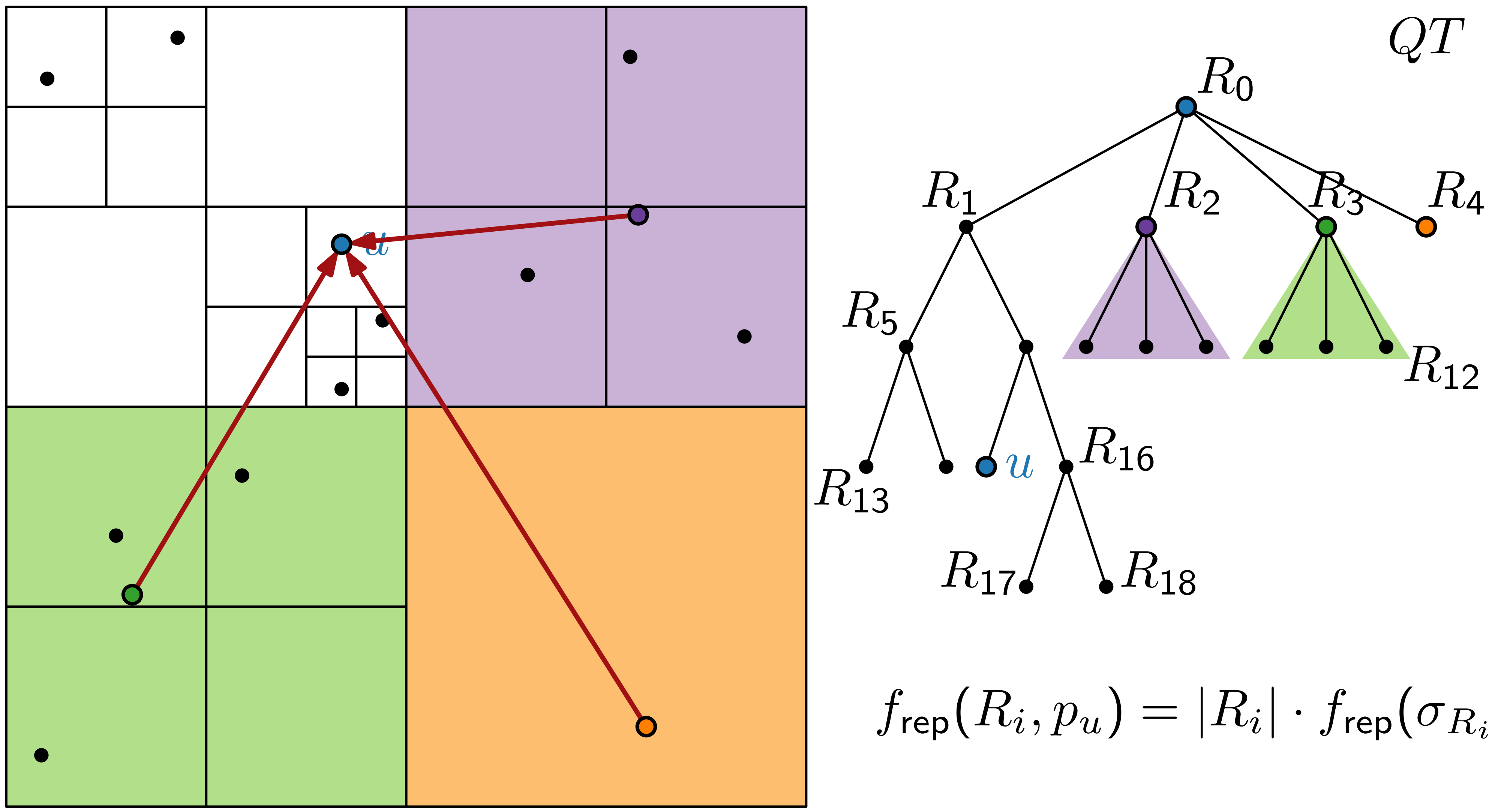
Speeding up with Quad Trees

[Barnes, Hut '86]



Speeding up with Quad Trees

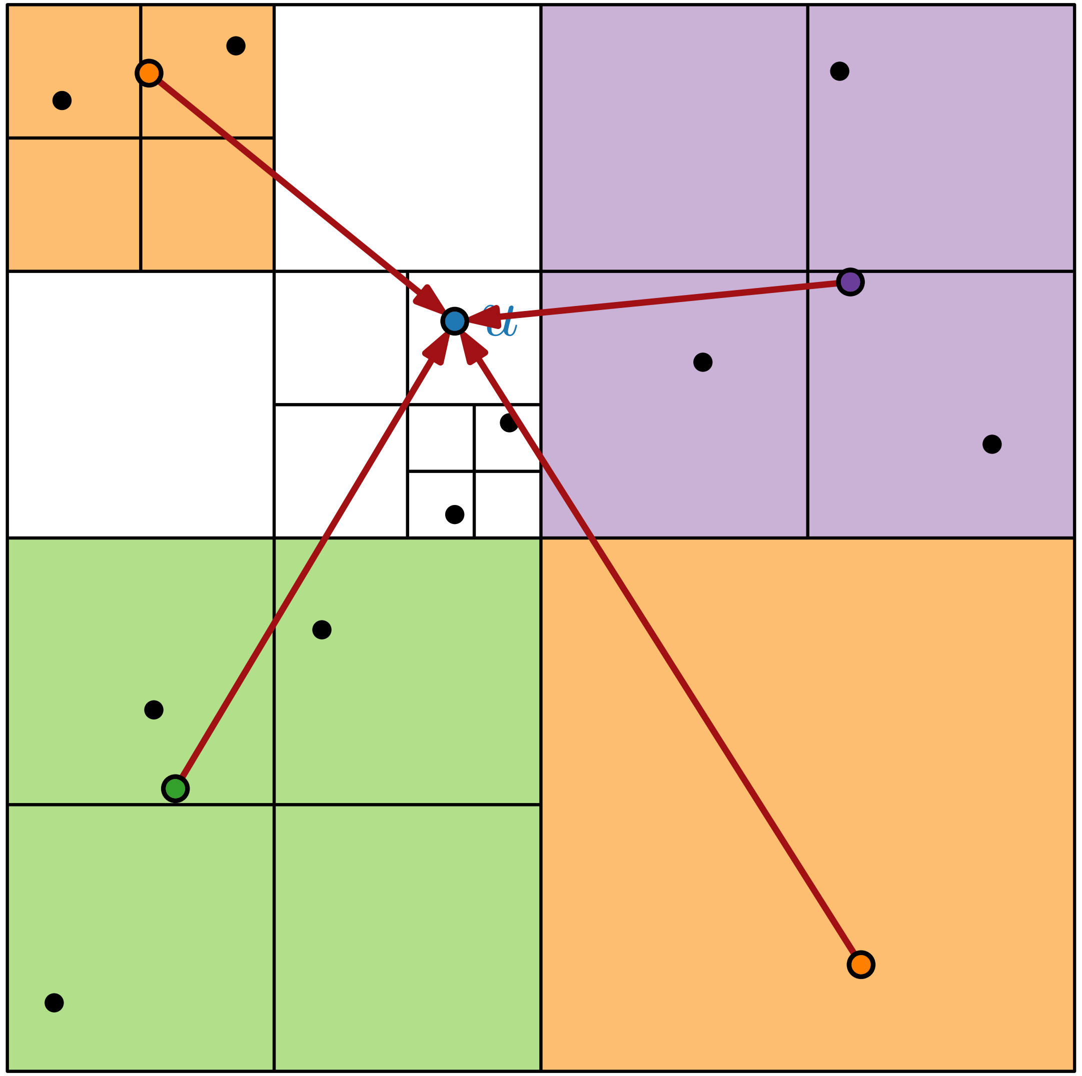
[Barnes, Hut '86]

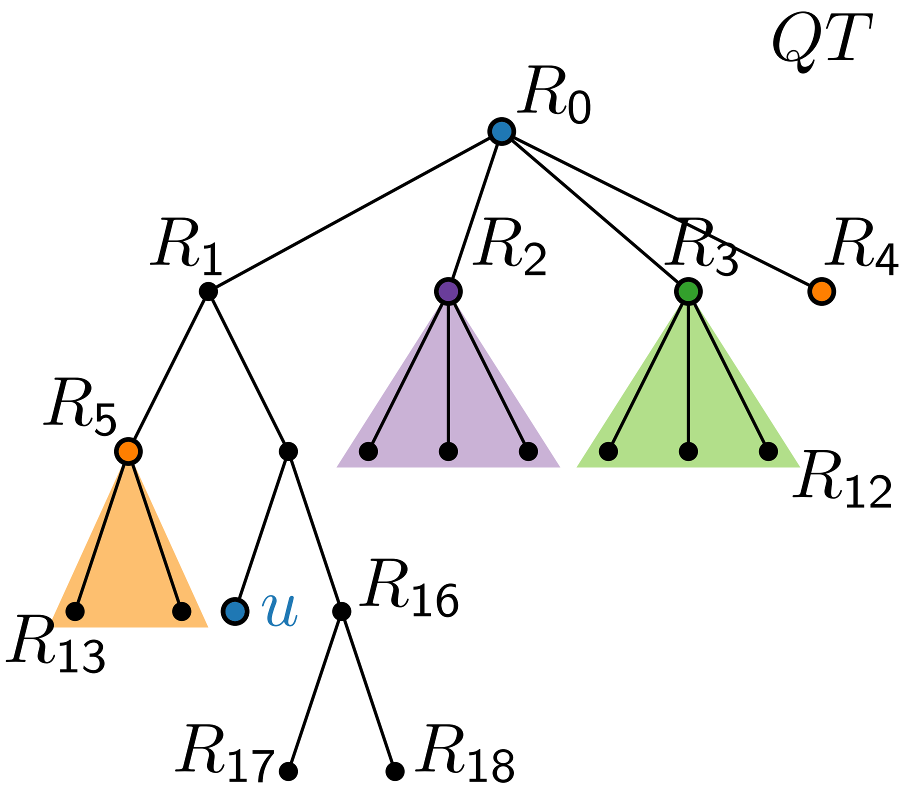


$$f_{\text{rep}}(R_i, p_u) = |R_i| \cdot f_{\text{rep}}(\sigma_{R_i}, p_u)$$

Speeding up with Quad Trees

[Barnes, Hut '86]

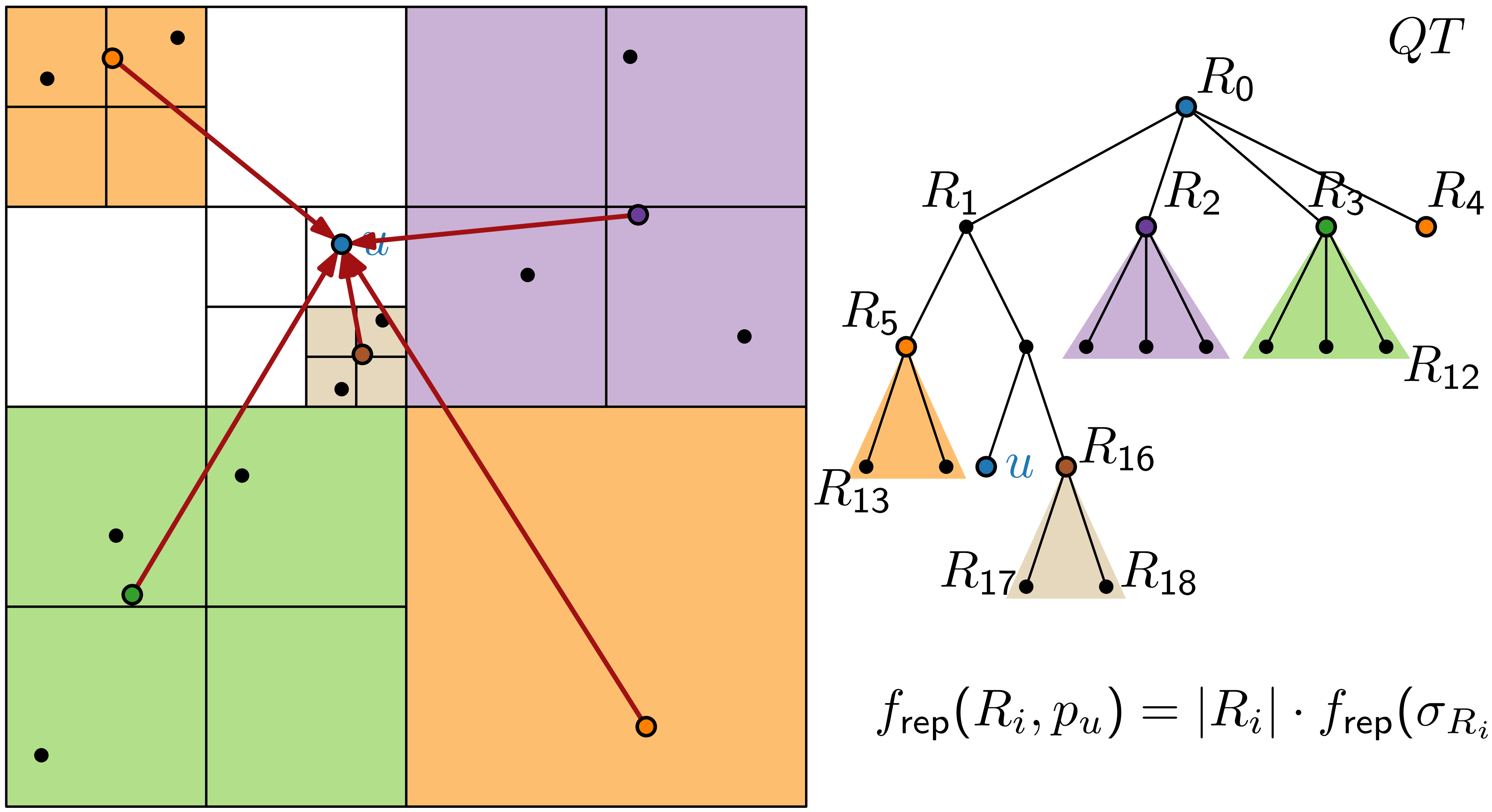




$$f_{\mathrm{rep}}(R_i, p_u) = |R_i| \cdot f_{\mathrm{rep}}(\sigma_{R_i}, p_u)$$

Speeding up with Quad Trees

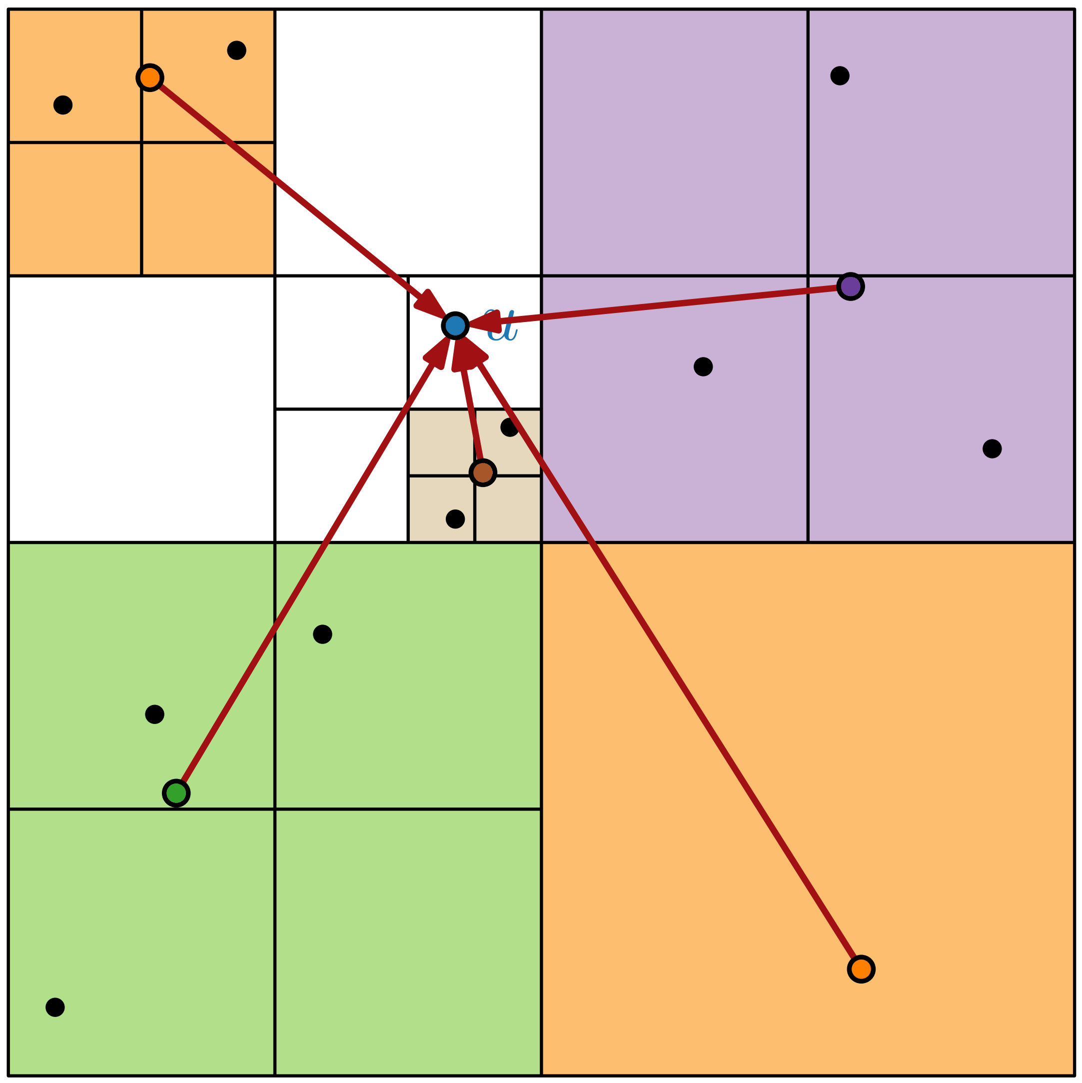
[Barnes, Hut '86]

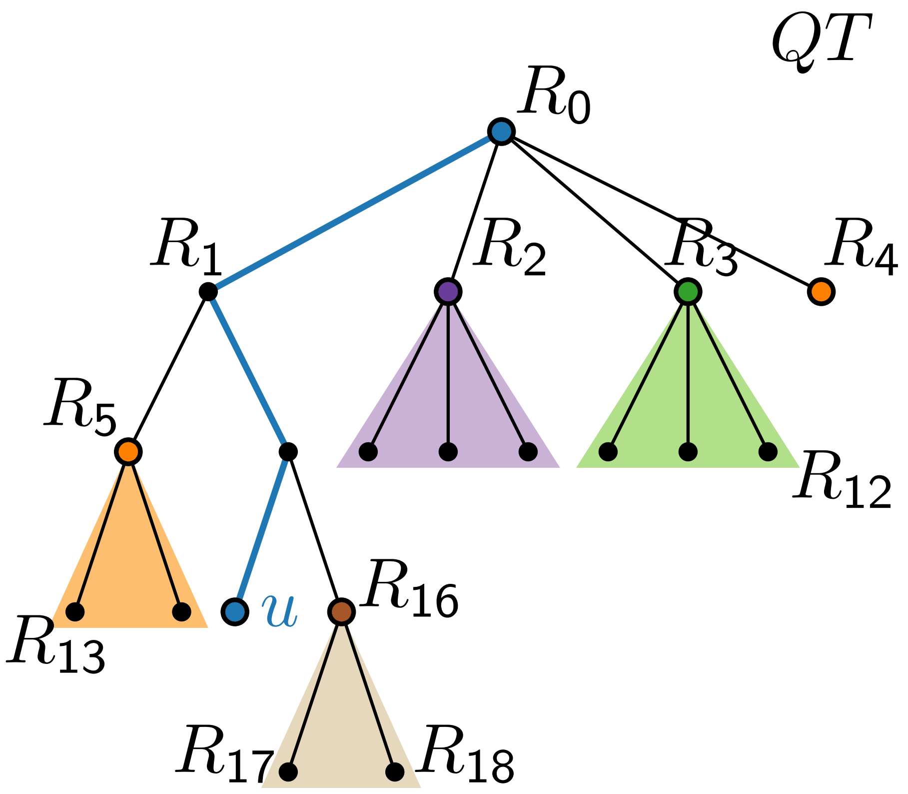


$$f_{\mathsf{rep}}(R_i, p_u) = |R_i| \cdot f_{\mathsf{rep}}(\sigma_{R_i}, p_u)$$

Speeding up with Quad Trees

[Barnes, Hut ’86]





$$f_{\mathsf{rep}}(R_i, p_u) = |R_i| \cdot f_{\mathsf{rep}}(\sigma_{R_i}, p_u)$$

for each child R_i of a vertex on path from u to R_0

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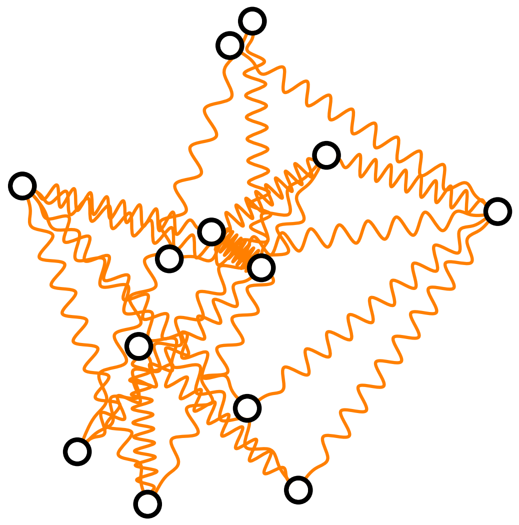
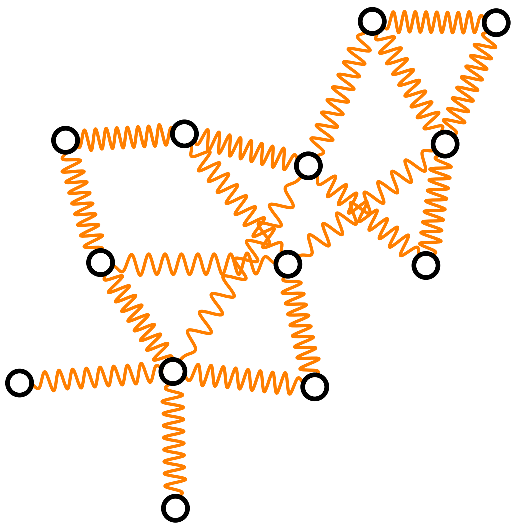
UNIVERSITÄT
WÜRZBURG

Visualization of Graphs

Lecture 2:
Force-Directed Drawing Algorithms

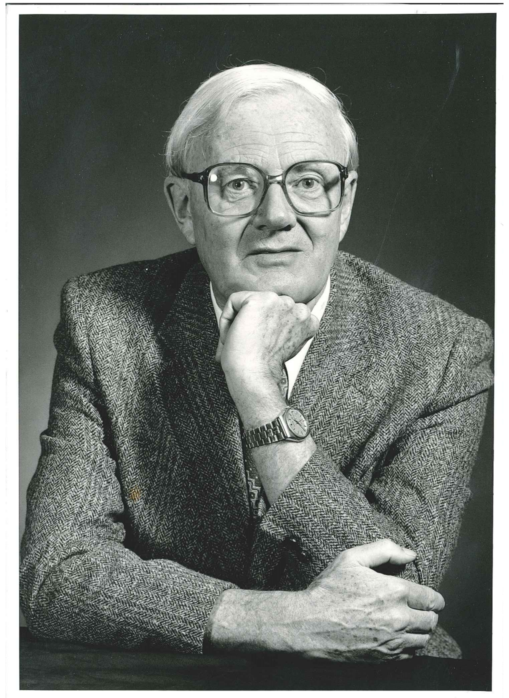


Part IV:
Tutte Embedding



Jonathan Klawitter

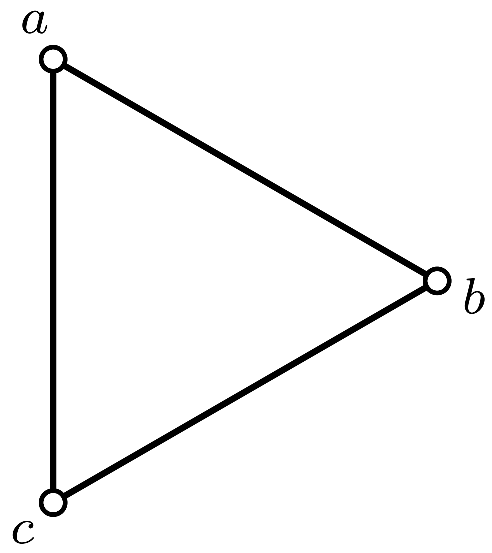
Idea

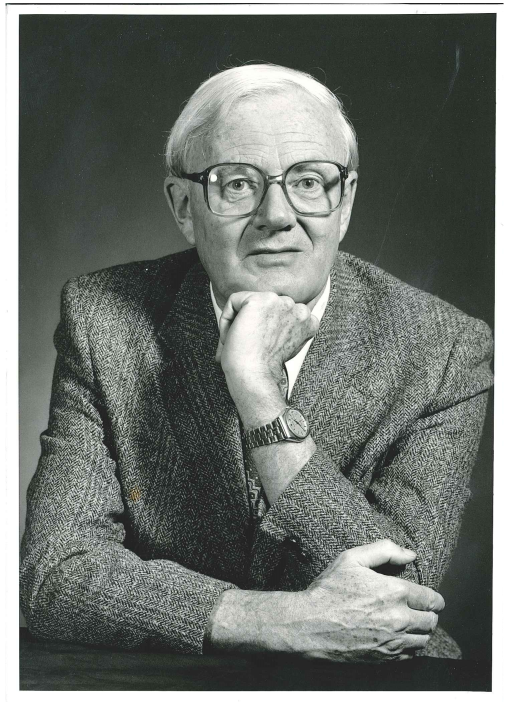


William T. Tutte
1917 – 2002

Idea

Consider a fixed triangle (a, b, c)

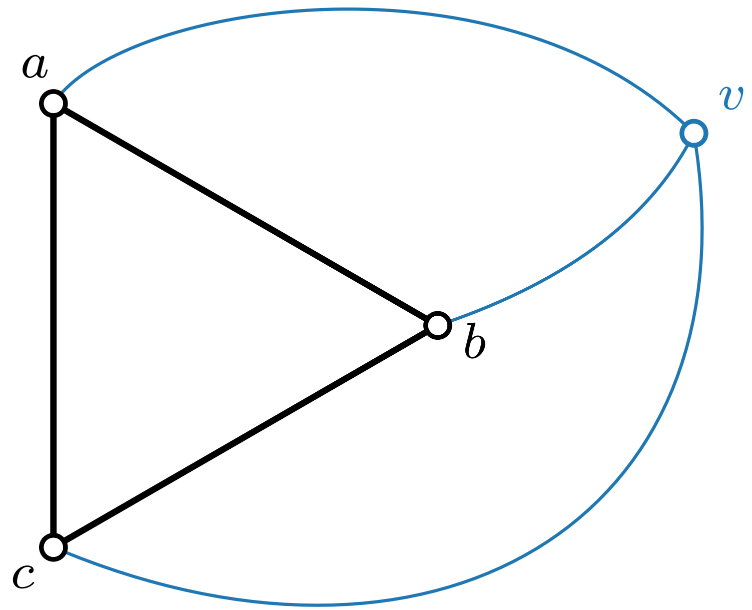


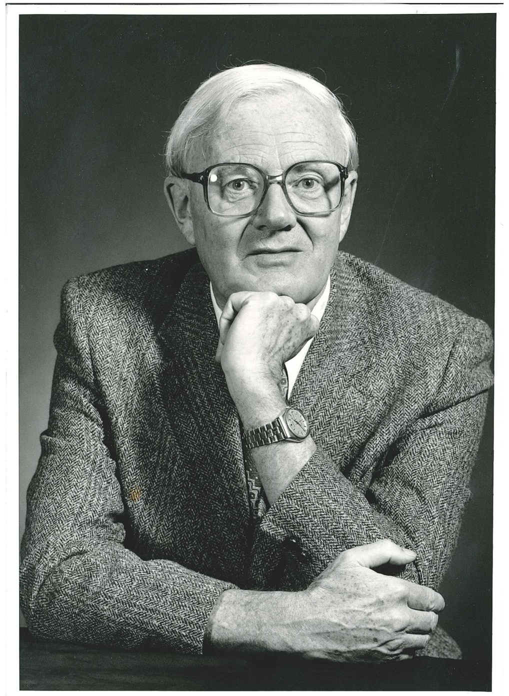


William T. Tutte
1917 – 2002

Idea

Consider a fixed triangle (a, b, c) with one common neighbor v





William T. Tutte
1917 – 2002

Idea

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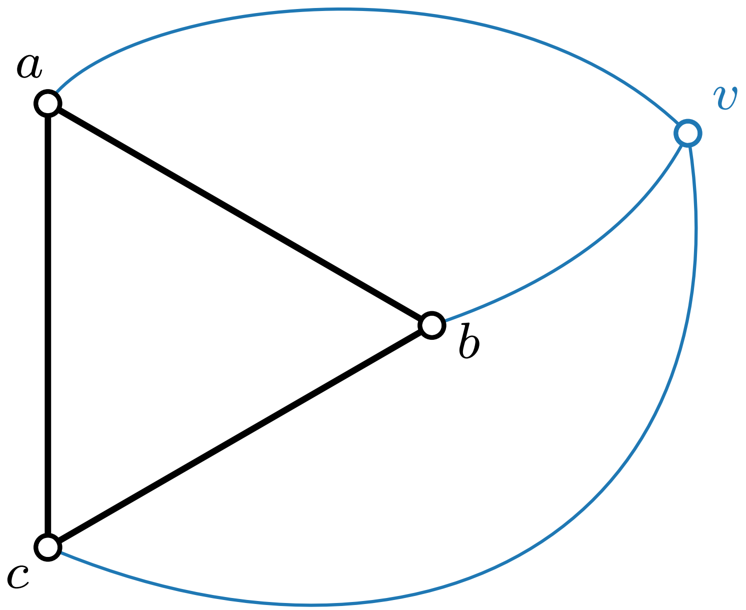
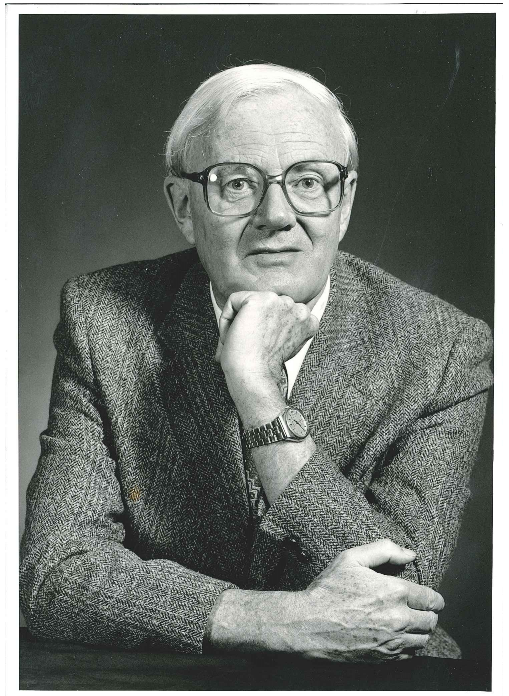
Where would you place v?





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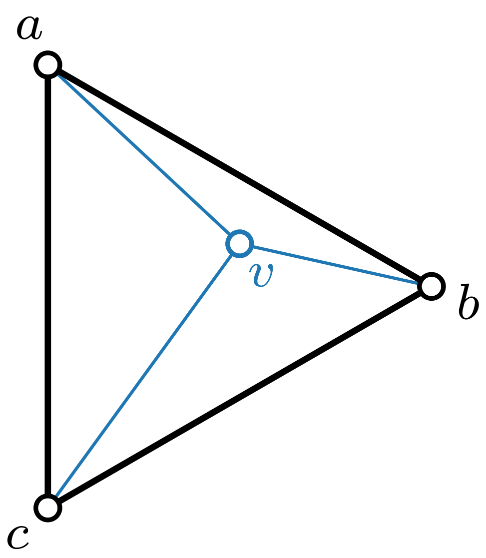

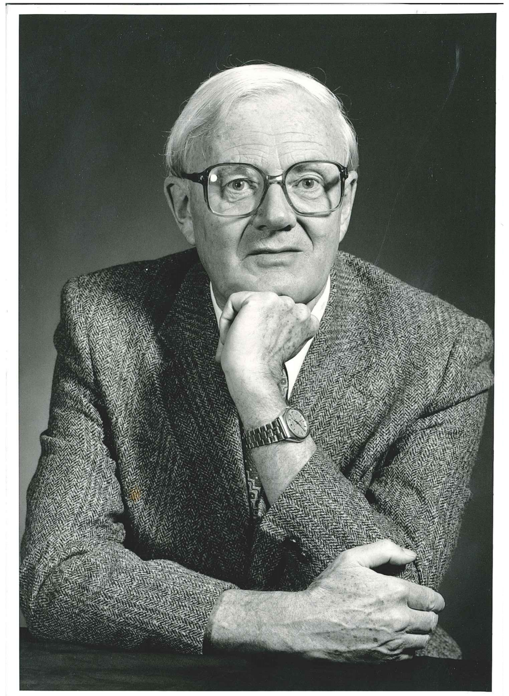

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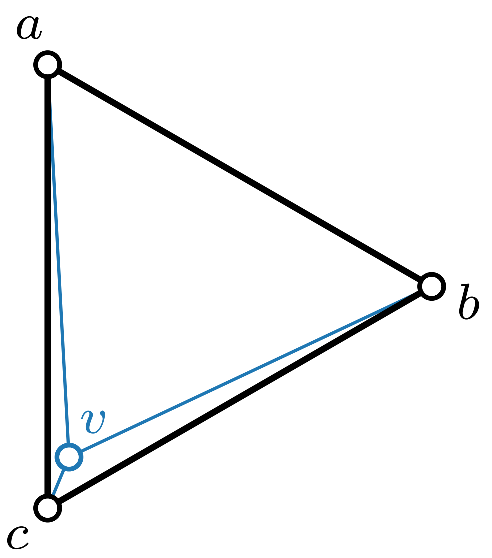

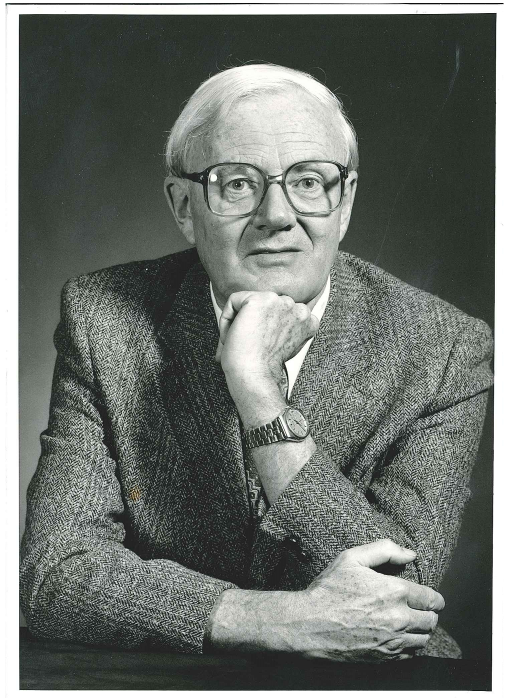

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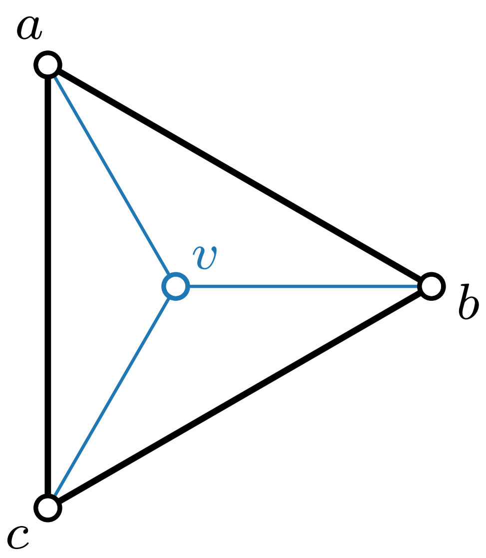

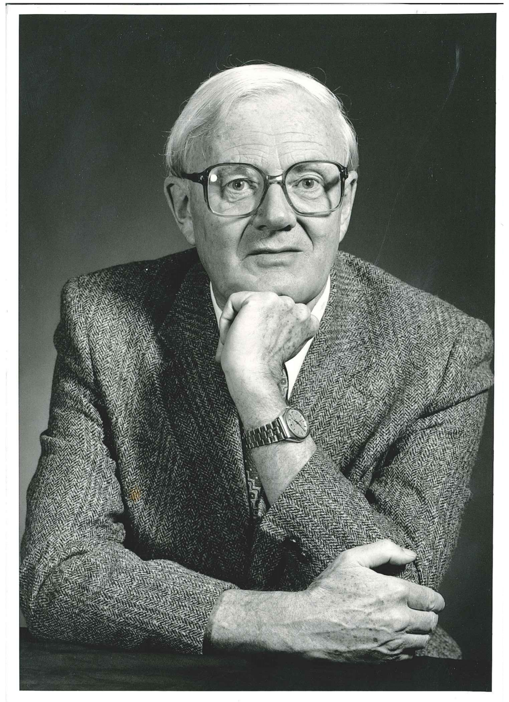

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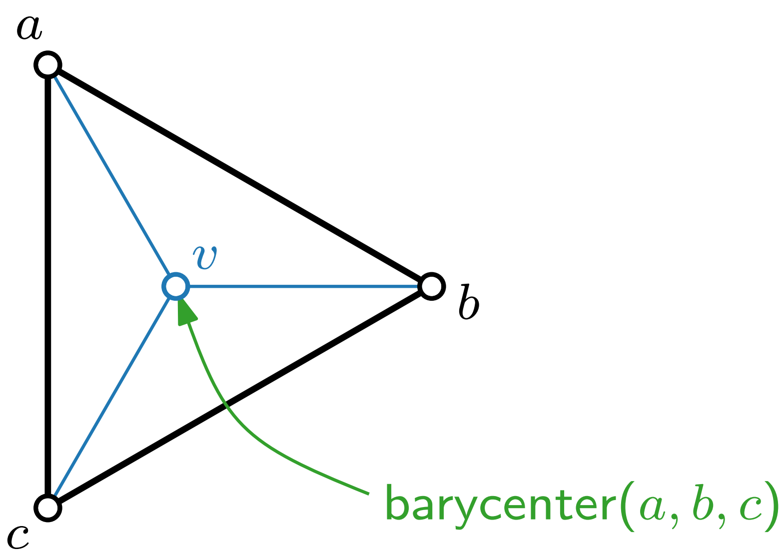
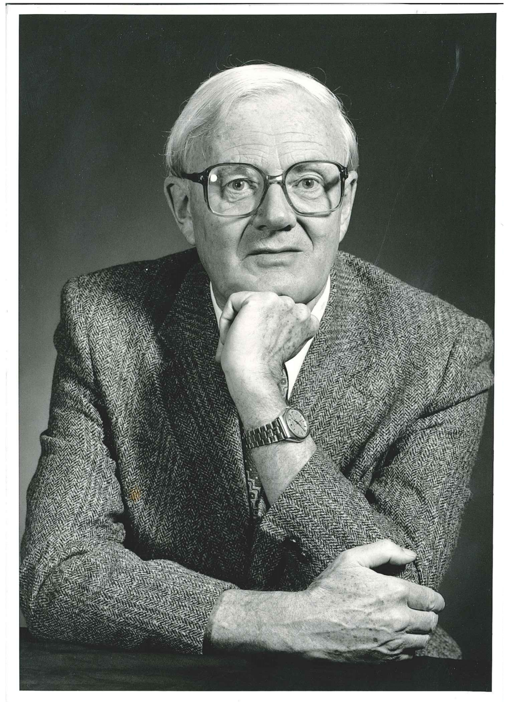




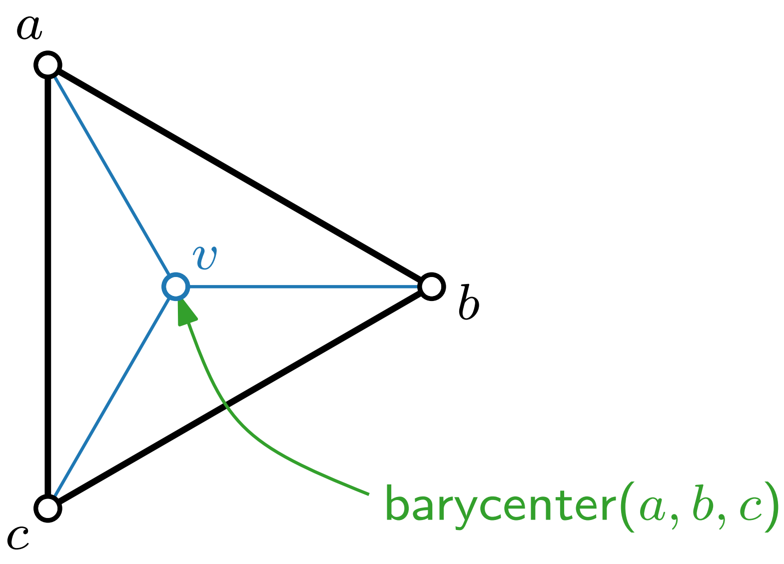
William T. Tutte
1917 – 2002

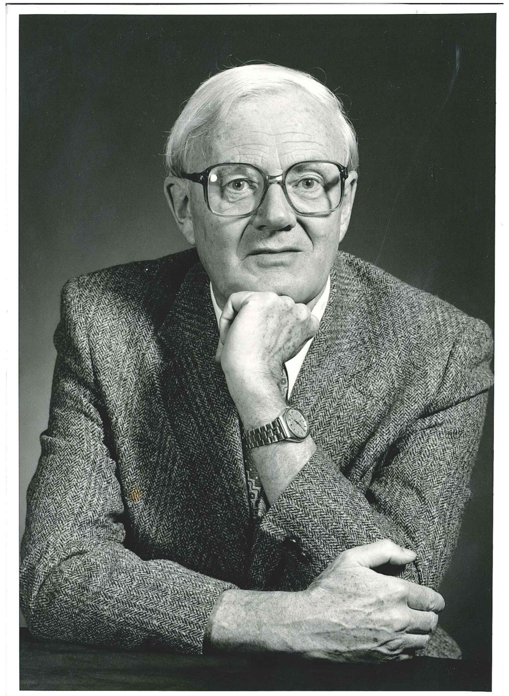
Idea

Consider a fixed triangle (a, b, c)
with one common neighbor v

Where would you place v?

barycenter$(x_1, \ldots, x_k) =$?





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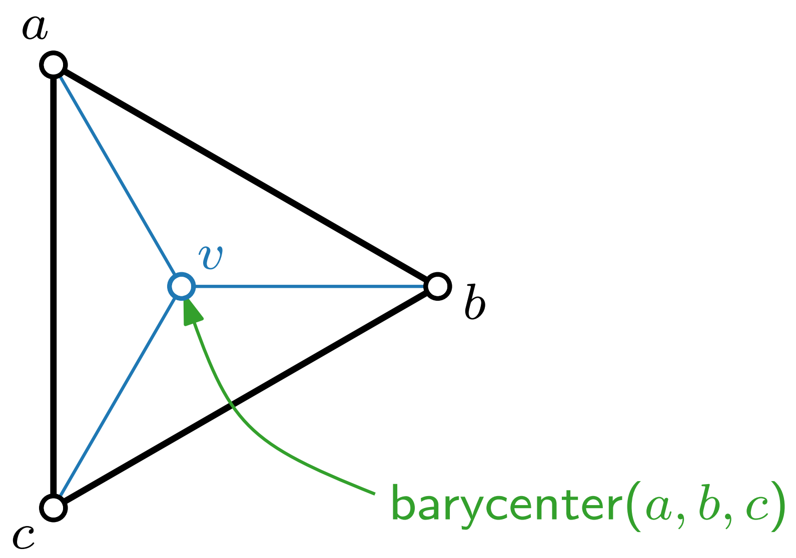

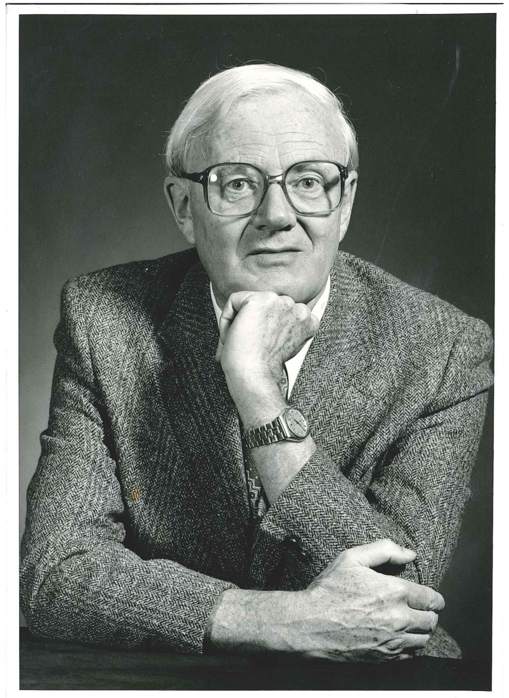

William T. Tutte
1917 – 2002

$$\text{barycenter}(x_1, \ldots, x_k) = \sum_{i=1}^{k} x_i / k$$

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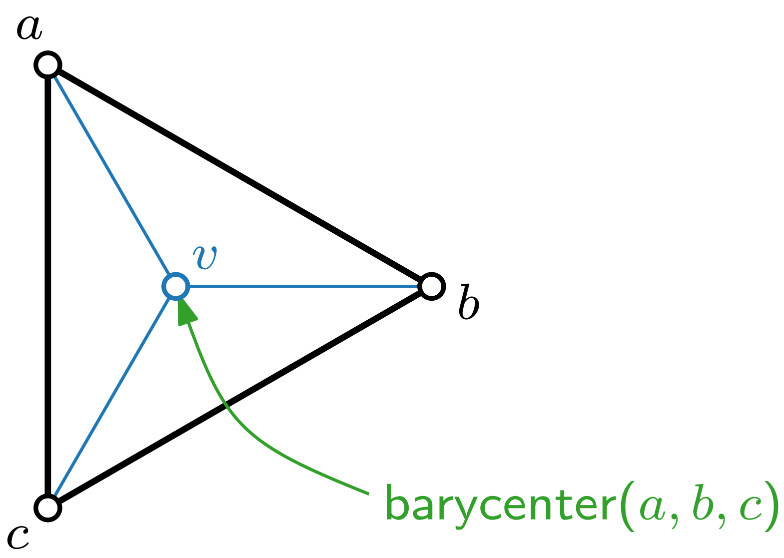

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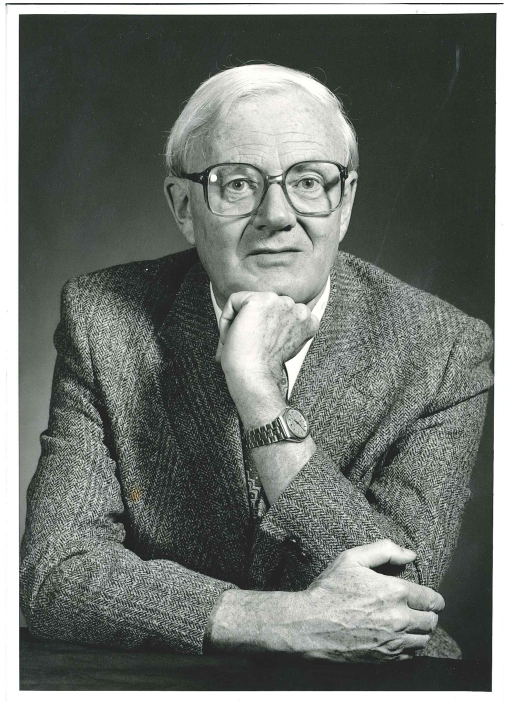

William T. Tutte
1917 – 2002

Idea.
Repeatedly place every vertex at barycenter of neighbors.

Tutte's Forces

ForceDirected($G = (V, E)$, $p = (p_v)_{v \in V}$, $\varepsilon > 0$, $K \in \mathbb{N}$)

$t \leftarrow 1$

while $t < K$ **and** $\max_{v \in V} \|F_v(t)\| > \varepsilon$ **do**

  foreach $u \in V$ **do**

    $F_u(t) \leftarrow \sum_{v \in V} f_{\mathsf{rep}}(u, v) + \sum_{uv \in E} f_{\mathsf{attr}}(u, v)$

  foreach $u \in V$ **do**

    $p_u \leftarrow p_u + \delta(t) \cdot F_u(t)$

  $t \leftarrow t + 1$

return p

Tutte's Forces

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Tutte's Forces

Goal.

$p_u = \text{barycenter}(\bigcup_{uv \in E} v)$

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Tutte's Forces

Goal.

$$p_u = \text{barycenter}\left(\bigcup_{uv \in E} v\right)$$
$$= \sum_{uv \in E} p_v /$$

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Tutte's Forces

Goal.

$p_u = \text{barycenter}(\bigcup_{uv \in E} v)$
$= \sum_{uv \in E} p_v / \deg(u)$

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- **Attractive forces**

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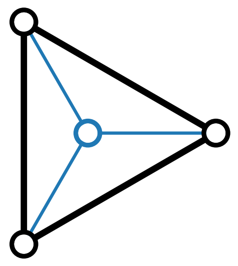

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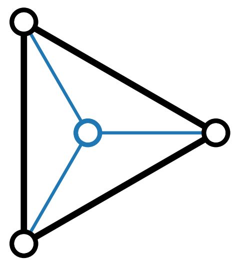

Fix coordinates of outer face!

Tutte's Forces

Goal.

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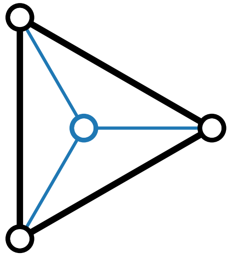
- **Repulsive forces**

$$f_{\text{rep}}(u, v) = 0$$

- **Attractive forces**

$$f_{\text{attr}}(u, v) = \begin{cases} 0 & u \text{ fixed} \\ \frac{1}{\deg(u)} \cdot ||p_u - p_v|| & \text{else} \end{cases}$$

Solution: $p_u = (0, 0)\ \forall u \in V$



Fix coordinates of outer face!

Linear System of Equations

Goal.

$p_u = \text{barycenter}(\bigcup_{uv \in E} v) \;\; = \sum_{uv \in E} p_v / \deg(u)$

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

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$x_u = \sum_{uv \in E} x_v / \deg(u) \quad \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v$

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Linear System of Equations

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$$y_u = \sum_{uv \in E} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{uv \in E} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{uv \in E} y_v = 0$$

Linear System of Equations

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$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

2 Systems of linear equations

$x_u = \sum_{uv \in E} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$

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Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\bigcup_{uv \in E} v) = \sum_{uv \in E} p_v / \deg(u)$

$Ax = b$

2 Systems of linear equations

$x_u = \sum_{uv \in E} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$

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$$x_u = \sum_{uv \in E} x_v / \deg(u) \;\Leftrightarrow\; \deg(u) \cdot x_u = \sum_{uv \in E} x_v \;\Leftrightarrow\; \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$$

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2 Systems of linear equations

Linear System of Equations

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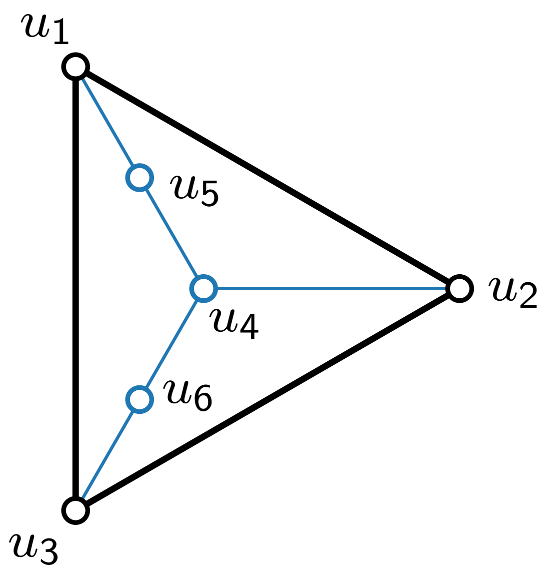

Linear System of Equations

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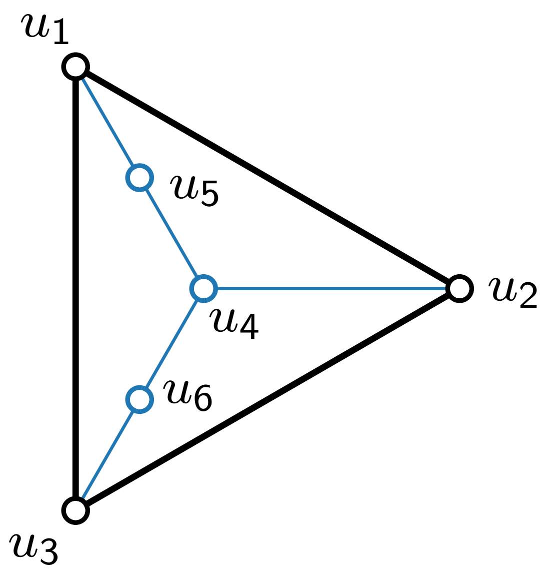
A



$$\begin{pmatrix} \quad \end{pmatrix}$$

Linear System of Equations

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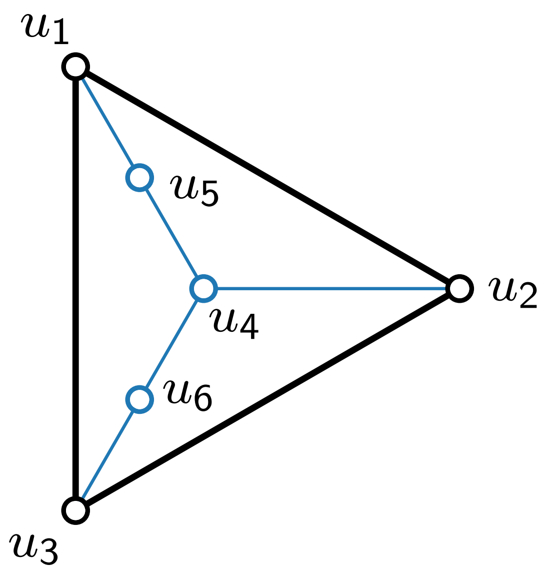

$$\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \left(\begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array} \right)$$

Linear System of Equations

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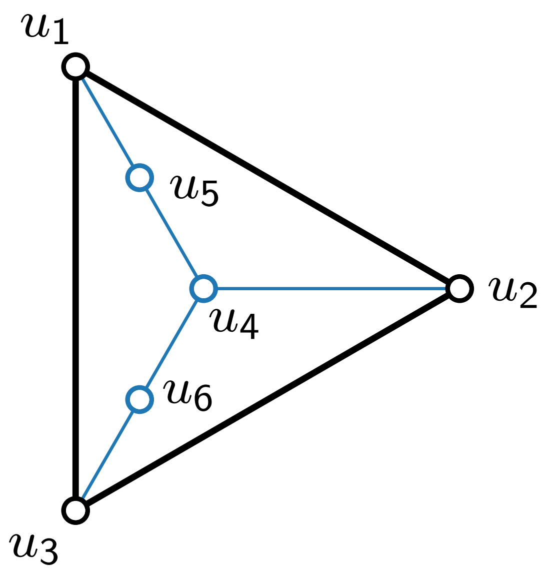

$$A = \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & & & & & & \\ u_2 & & & & & & \\ u_3 & & & & & & \\ u_4 & & & & & & \\ u_5 & & & & & & \\ u_6 & & & & & & \end{array}$$

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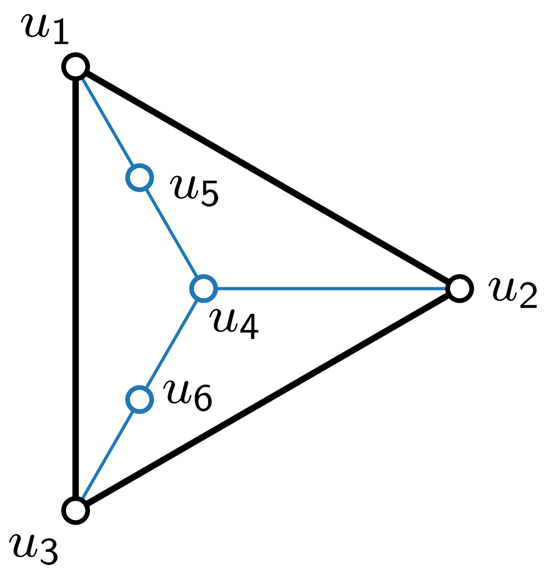

$$A = \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 3 & & & & & \\ u_2 & & & & & & \\ u_3 & & & & & & \\ u_4 & & & & & & \\ u_5 & & & & & & \\ u_6 & & & & & & \end{array}$$

Linear System of Equations

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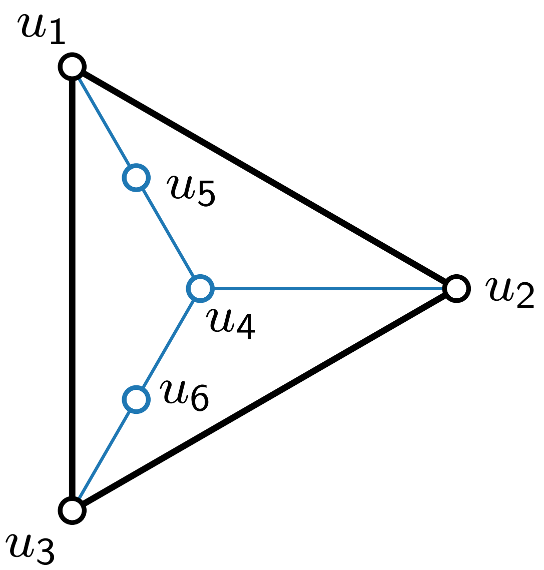

$$A \quad \begin{array}{c c} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & -1 & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

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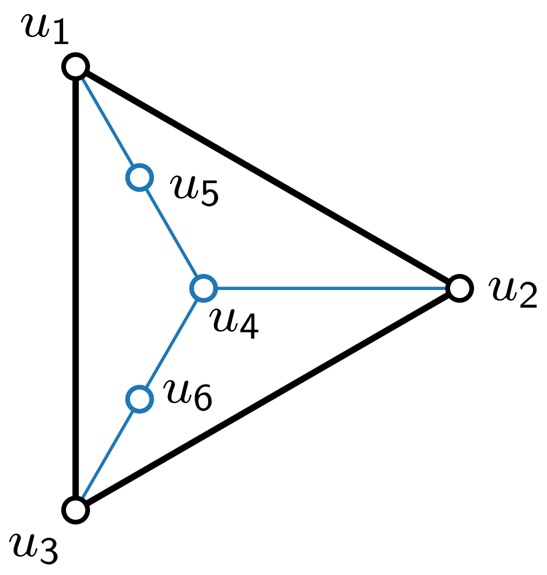

$$A = \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 3 & -1 & -1 & & & \\ u_2 & & & & & & \\ u_3 & & & & & & \\ u_4 & & & & & & \\ u_5 & & & & & & \\ u_6 & & & & & & \end{array}$$

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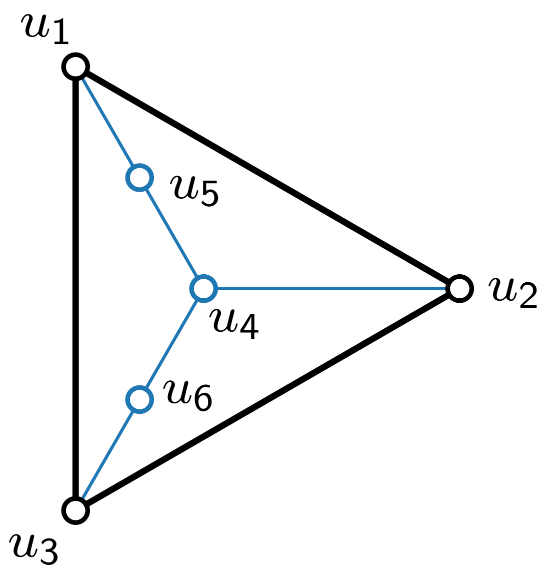

$$A$$

$$\begin{array}{c|c}
 & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\
\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & -1 & -1 & 0 & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix}
\end{array}$$

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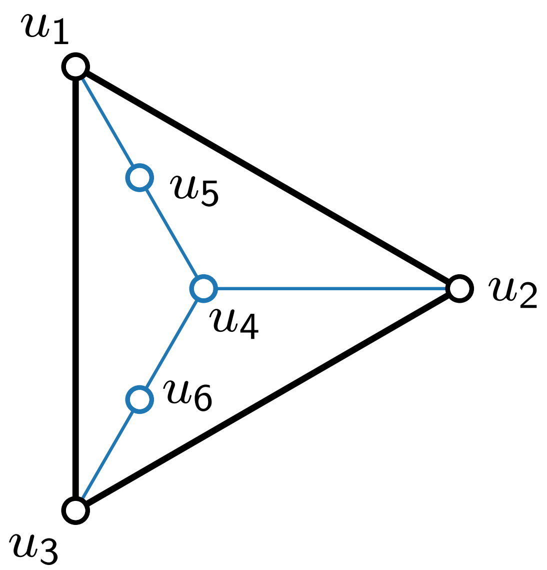

$$\begin{array}{c|c}
 & \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{matrix} \\
\begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix}
\end{array} \quad A$$

Linear System of Equations

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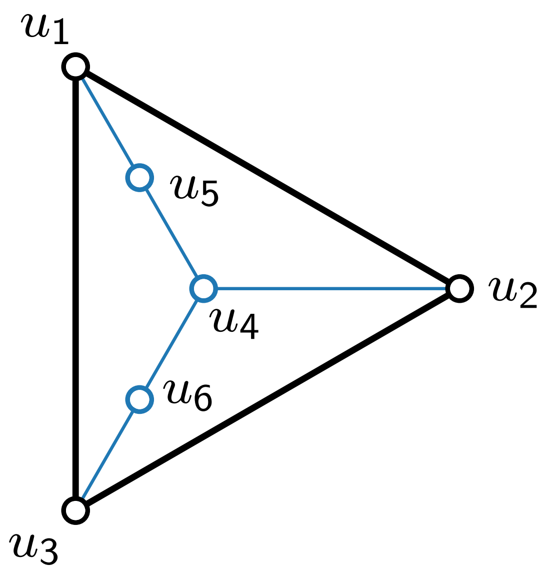

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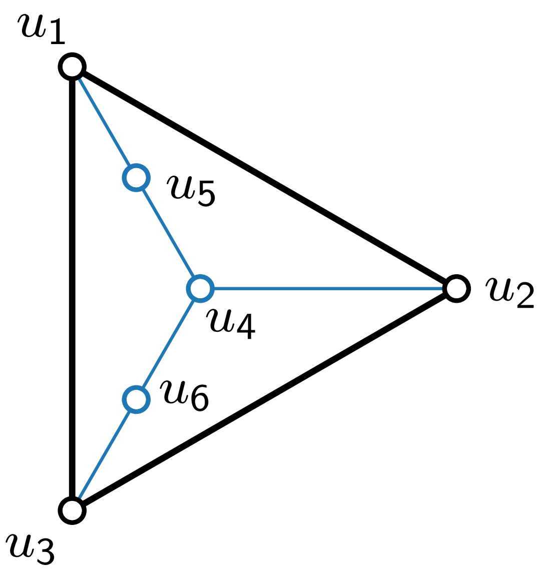

$$A: \quad \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ & 3 & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

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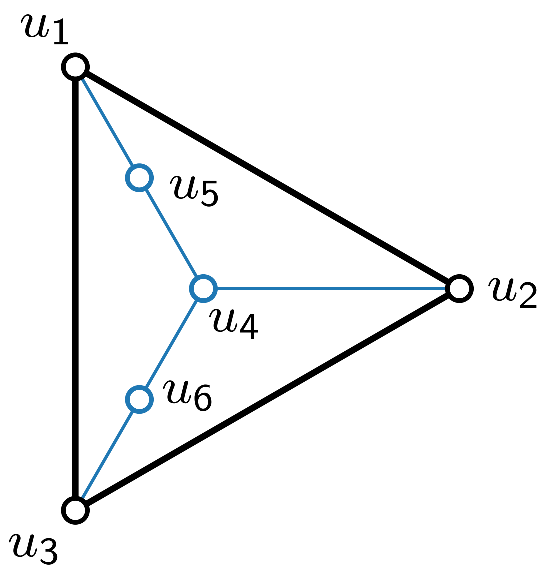

$$A \qquad \begin{array}{cc} & \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

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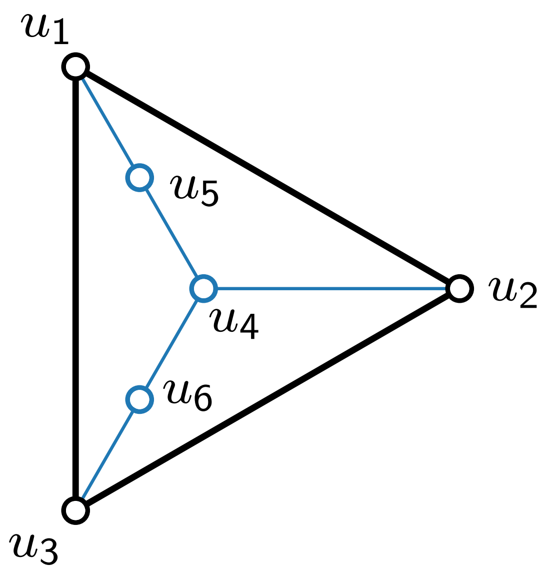

$$A = \begin{array}{c|c} & \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ & & & & & \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

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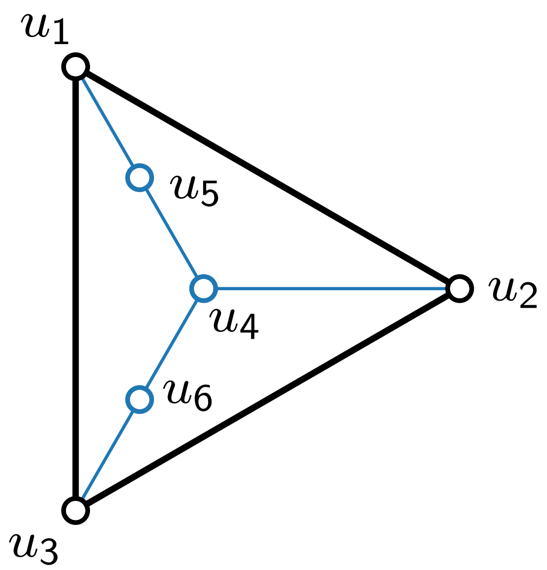
$y_u = \sum_{uv \in E} y_v / \deg(u) \Leftrightarrow \deg(u) \cdot y_u = \sum_{uv \in E} y_v \Leftrightarrow \deg(u) \cdot y_u - \sum_{uv \in E} y_v = 0$



$$A = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ & & & & & \\ & & & & & \end{pmatrix} \end{array}$$

Linear System of Equations

Goal. $p_u = (x_u, y_u)$

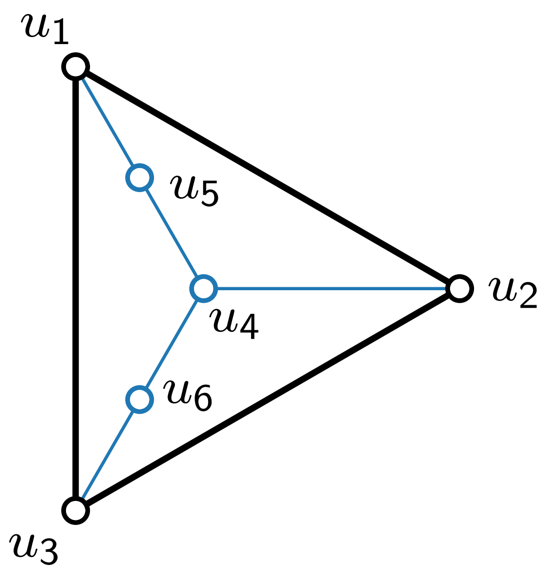
$p_u = \text{barycenter}(\bigcup_{uv\in E} v) \;= \sum_{uv\in E} p_v / \deg(u)$

$Ax = b \qquad Ay = b \qquad b = (0)_n$

2 Systems of linear equations

$$x_u = \sum_{uv\in E} x_v / \deg(u) \;\Leftrightarrow\; \deg(u)\cdot x_u = \sum_{uv\in E} x_v \;\Leftrightarrow\; \deg(u)\cdot x_u - \sum_{uv\in E} x_v = 0$$

$$y_u = \sum_{uv\in E} y_v / \deg(u) \;\Leftrightarrow\; \deg(u)\cdot y_u = \sum_{uv\in E} y_v \;\Leftrightarrow\; \deg(u)\cdot y_u - \sum_{uv\in E} y_v = 0$$



$$\begin{array}{cc} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ & & & & & \end{pmatrix} \end{array} \quad A$$

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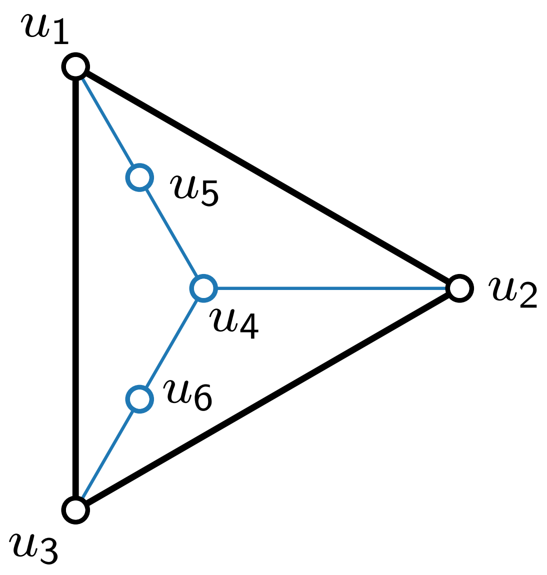
$$x_u = \sum_{uv \in E} x_v / \deg(u) \;\Leftrightarrow\; \deg(u) \cdot x_u = \sum_{uv \in E} x_v \;\Leftrightarrow\; \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$$

$$y_u = \sum_{uv \in E} y_v / \deg(u) \;\Leftrightarrow\; \deg(u) \cdot y_u = \sum_{uv \in E} y_v \;\Leftrightarrow\; \deg(u) \cdot y_u - \sum_{uv \in E} y_v = 0$$



$$A = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

Linear System of Equations

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$$x_u = \sum_{uv \in E} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v \Leftrightarrow \boxed{\deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0}$$

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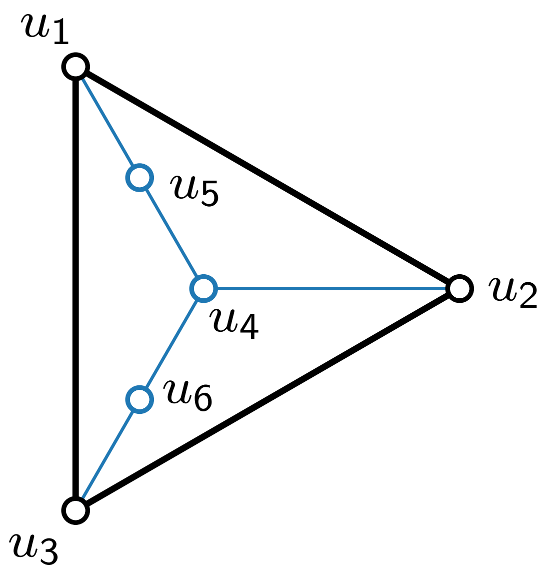

$$A = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

$A_{ii} = \deg(u_i)$

Linear System of Equations

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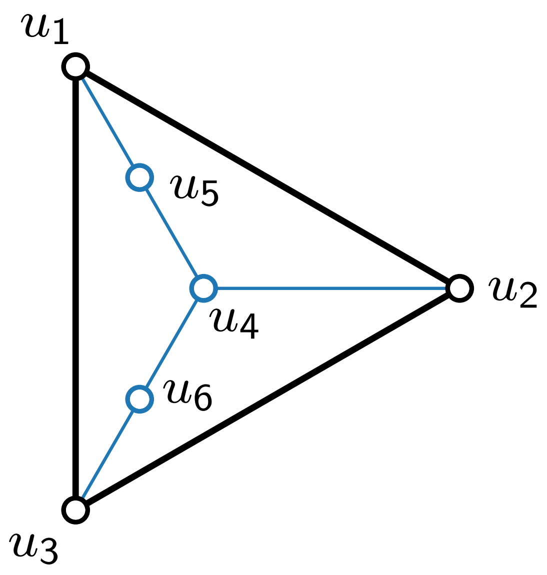

$$A = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

$$A_{ii} = \deg(u_i)$$

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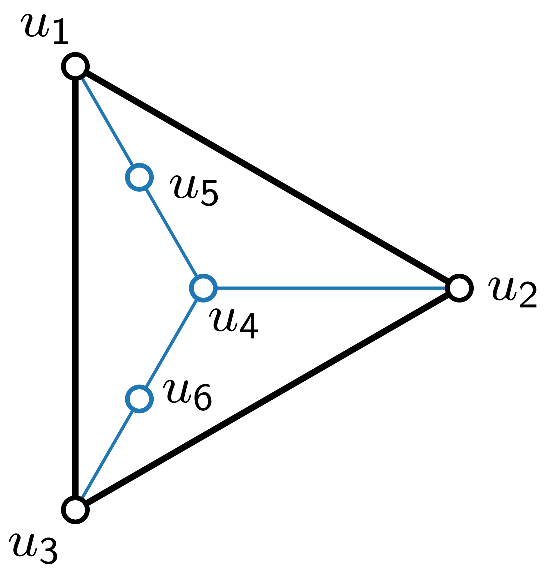

$$A = \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 3 & -1 & -1 & 0 & -1 & 0 \\ u_2 & -1 & 3 & -1 & -1 & 0 & 0 \\ u_3 & -1 & -1 & 3 & 0 & 0 & -1 \\ u_4 & 0 & -1 & 0 & 3 & -1 & -1 \\ u_5 & -1 & 0 & 0 & -1 & 2 & 0 \\ u_6 & 0 & 0 & -1 & -1 & 0 & 2 \end{array}$$

Laplacian matrix of G

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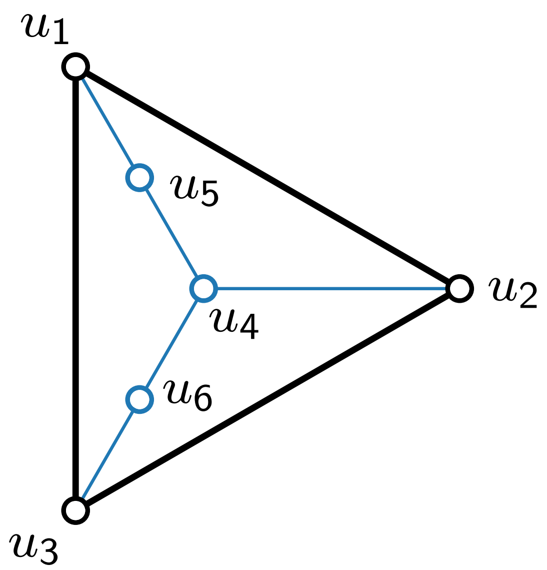

$$A = \begin{array}{c} \\ u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} \begin{array}{c} \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

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unique solution

Linear System of Equations

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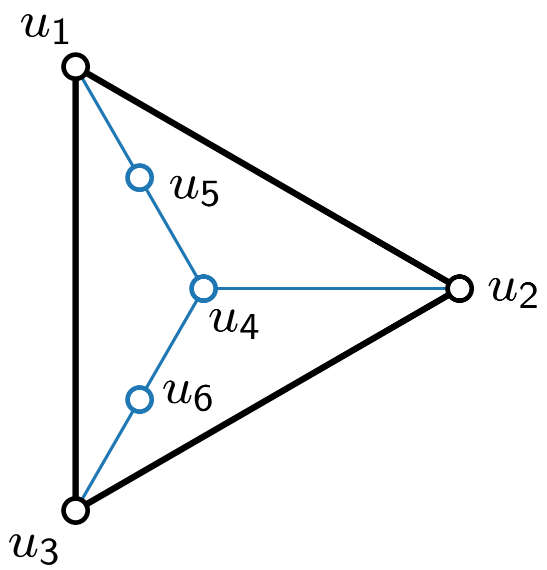

$$A = \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \multicolumn{6}{c}{\begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix}} \end{array}$$

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variables, constraints, $\det(A) =$

unique solution

Linear System of Equations

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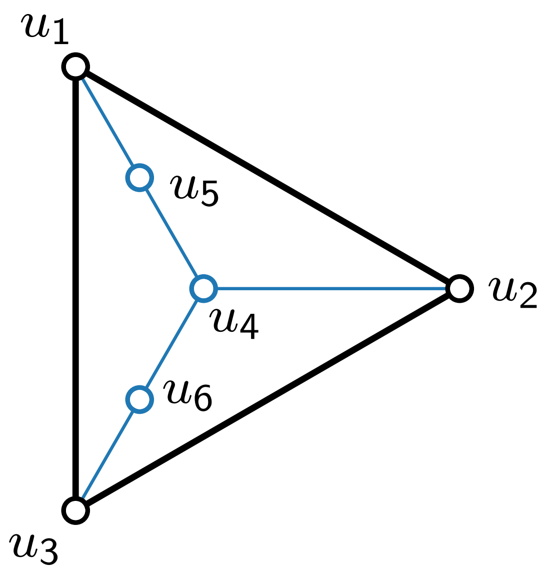

$$A = \begin{array}{c|c} & \begin{matrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{matrix} \\ \begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array}$$

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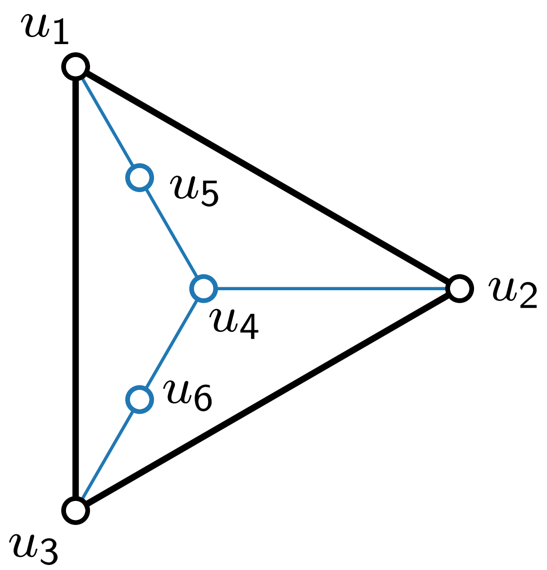

$$
\begin{array}{c c}
 & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\
\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} &
\begin{pmatrix}
3 & -1 & -1 & 0 & -1 & 0 \\
-1 & 3 & -1 & -1 & 0 & 0 \\
-1 & -1 & 3 & 0 & 0 & -1 \\
0 & -1 & 0 & 3 & -1 & -1 \\
-1 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & -1 & -1 & 0 & 2
\end{pmatrix}
\end{array}
\quad A
$$

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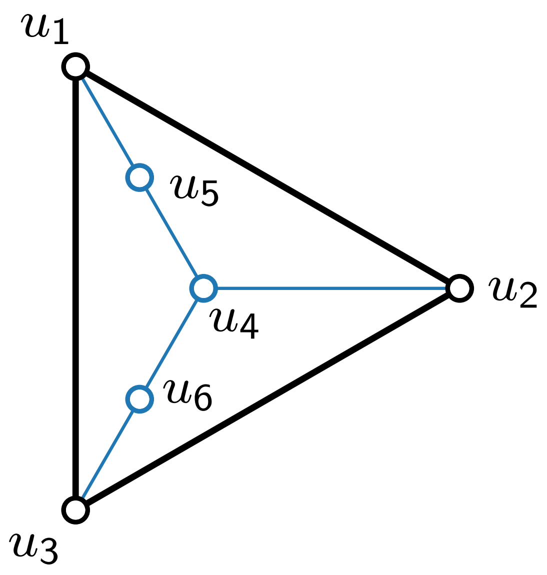

$$
\begin{array}{c c}
 & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\
\begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} &
\begin{pmatrix}
3 & -1 & -1 & 0 & -1 & 0 \\
-1 & 3 & -1 & -1 & 0 & 0 \\
-1 & -1 & 3 & 0 & 0 & -1 \\
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$$

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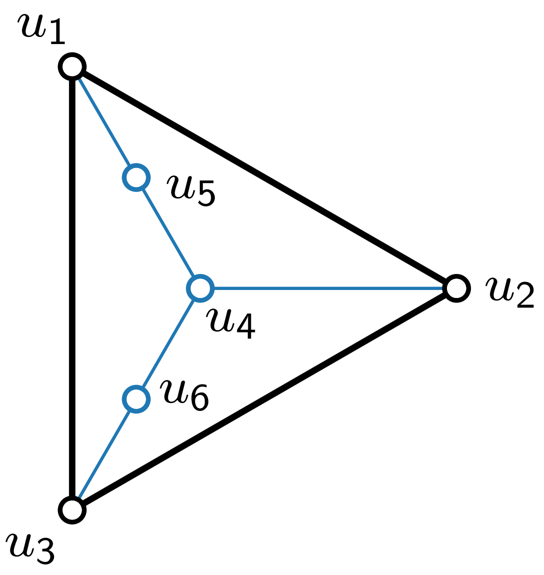

$$\begin{array}{c c} & \begin{array}{cccccc} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \end{array} \\ \begin{array}{c} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{array} & \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 & 0 \\ -1 & -1 & 3 & 0 & 0 & -1 \\ 0 & -1 & 0 & 3 & -1 & -1 \\ -1 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix} \end{array} \; A$$

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$\Rightarrow$ no unique solution

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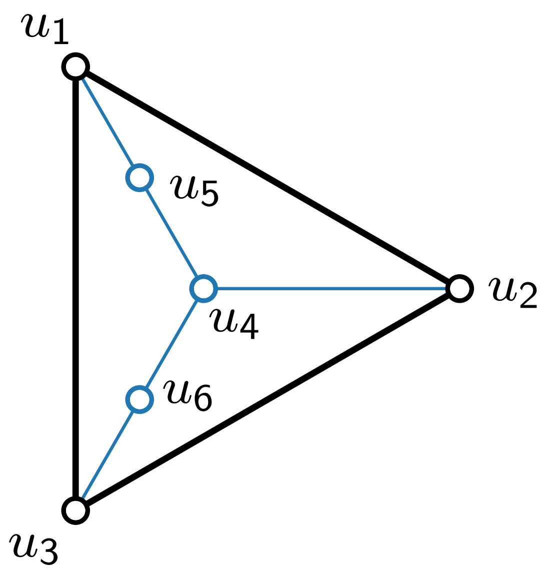

$$A = \begin{array}{c|cccccc} & u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\ \hline u_1 & 3 & -1 & -1 & 0 & -1 & 0 \\ u_2 & -1 & 3 & -1 & -1 & 0 & 0 \\ u_3 & -1 & -1 & 3 & 0 & 0 & -1 \\ u_4 & 0 & -1 & 0 & 3 & -1 & -1 \\ u_5 & -1 & 0 & 0 & -1 & 2 & 0 \\ u_6 & 0 & 0 & -1 & -1 & 0 & 2 \end{array}$$

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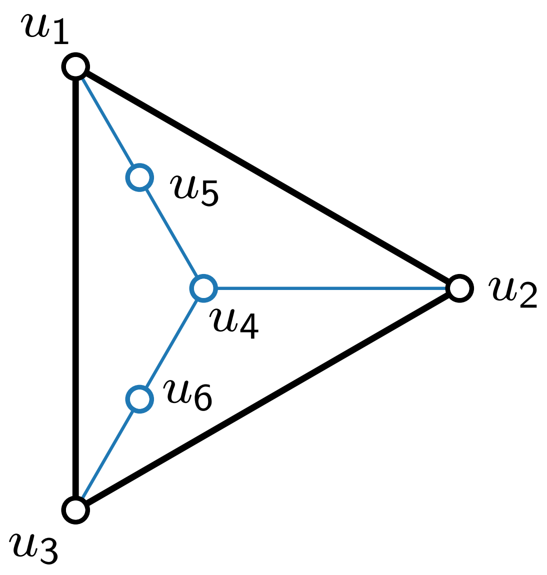

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n variables, k constraints, $\det(A) = 0$

$k = \#\text{free vertices}$

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2 Systems of linear equations

$x_u = \sum_{uv \in E} x_v / \deg(u) \;\Leftrightarrow\; \deg(u) \cdot x_u = \sum_{uv \in E} x_v \;\Leftrightarrow\; \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$

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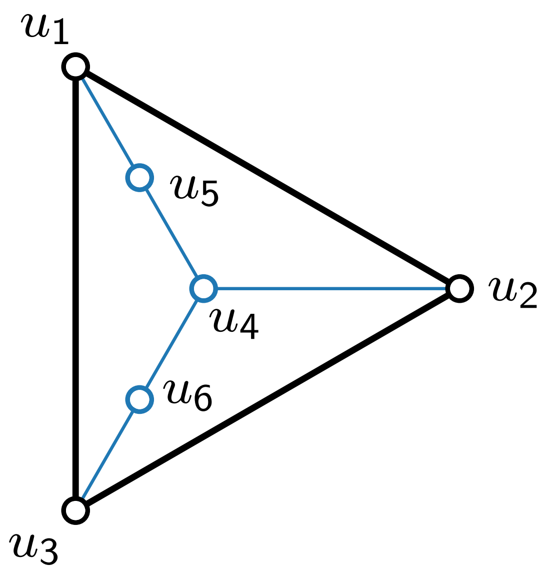

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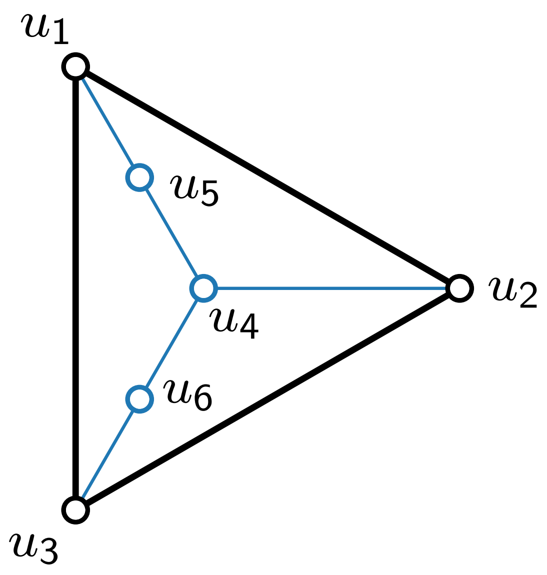

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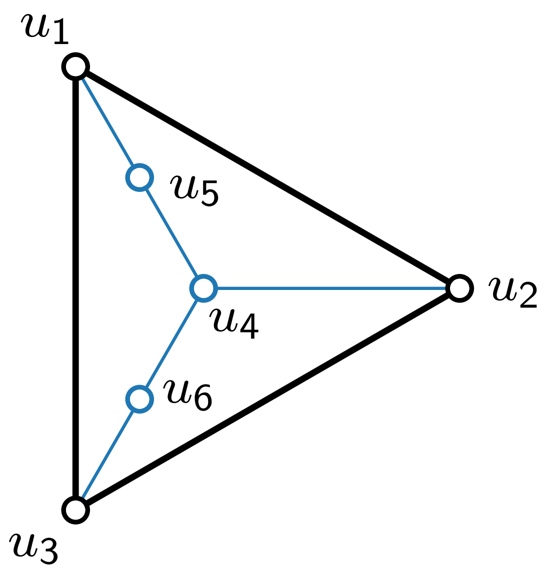

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\begin{pmatrix}
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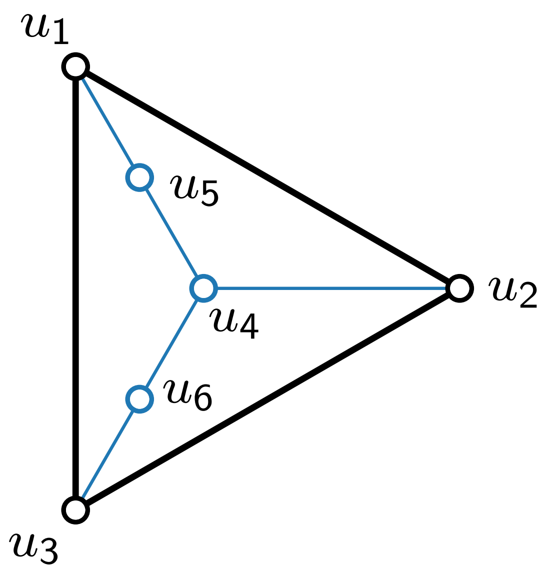

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Linear System of Equations

Theorem.
Tutte's barycentric algorithm admits a unique solution.
It can be computed in polynomial time.

Goal. $p_u = (x_u, y_u)$

$p_u = \text{barycenter}(\bigcup_{uv \in E} v)$

$$x_u = \sum_{uv \in E} x_v / \deg(u) \Leftrightarrow \deg(u) \cdot x_u = \sum_{uv \in E} x_v \Leftrightarrow \deg(u) \cdot x_u - \sum_{uv \in E} x_v = 0$$

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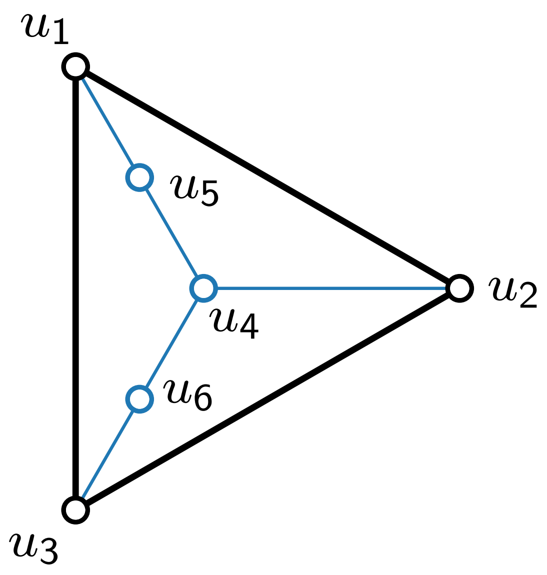

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Linear System of Equations

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Theorem. **Tutte drawing**
Tutte's barycentric algorithm admits a unique solution.
It can be computed in polynomial time.

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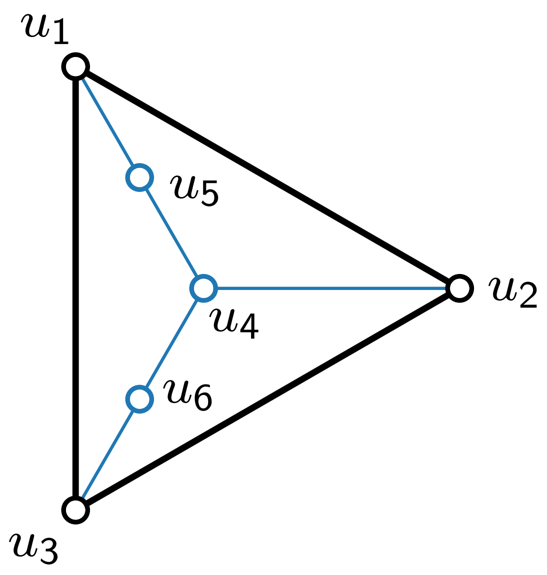

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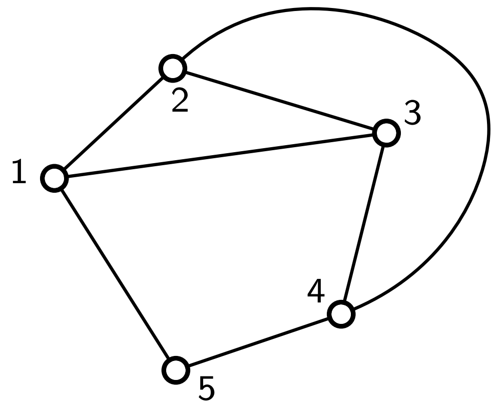
3-Connected Planar Graphs

planar: G can be drawn in such a way that no edges cross each other

connected: There is a u-v-path for every $u, v \in V$



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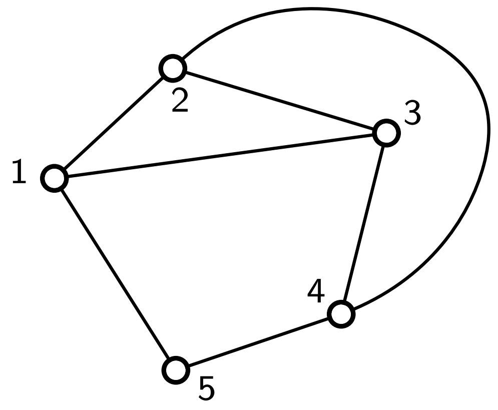

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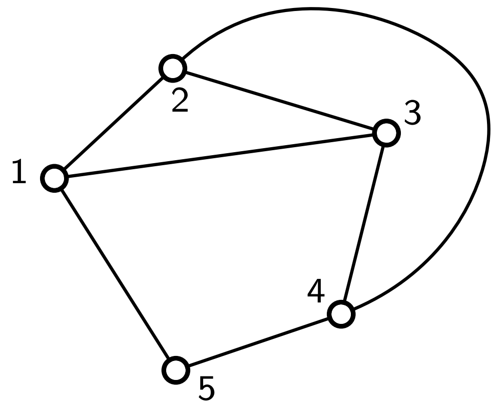

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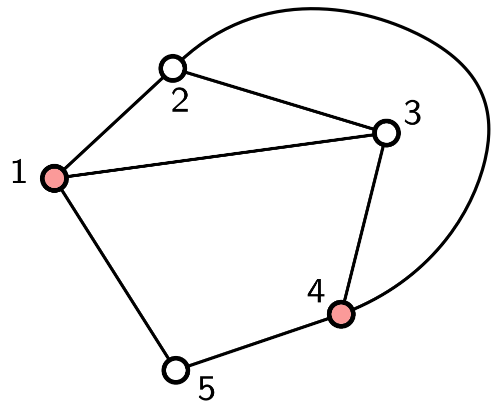

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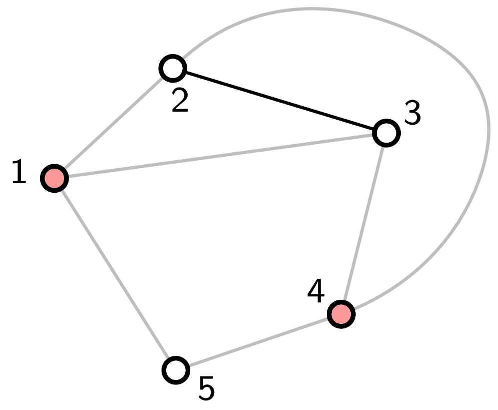

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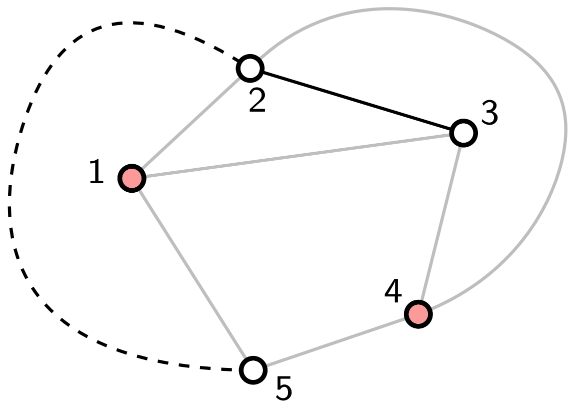

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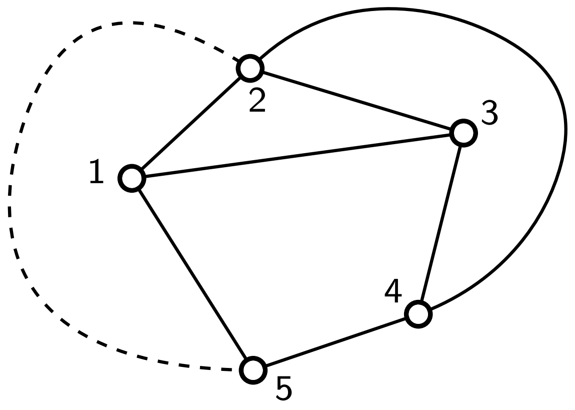

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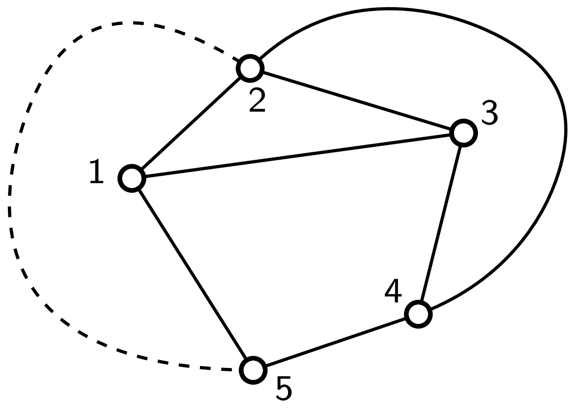

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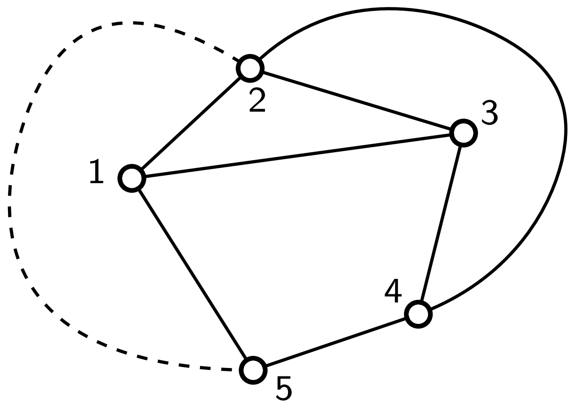

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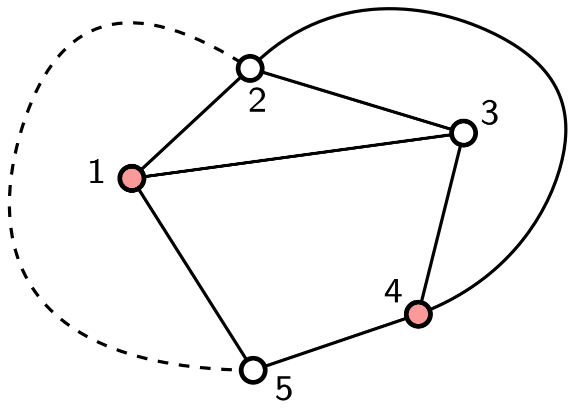

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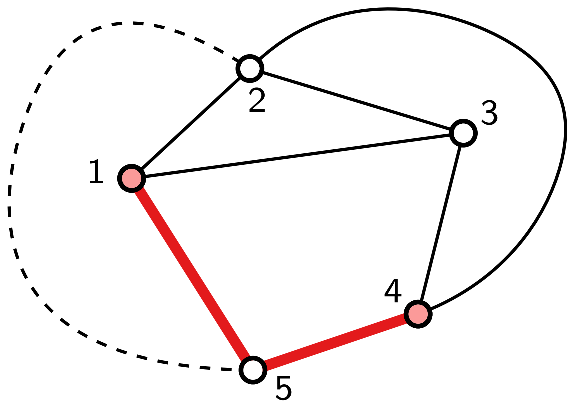

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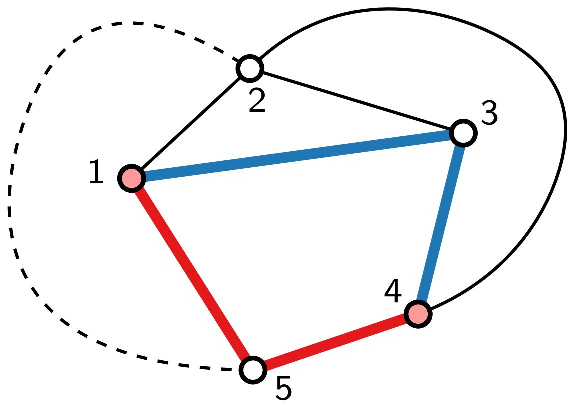

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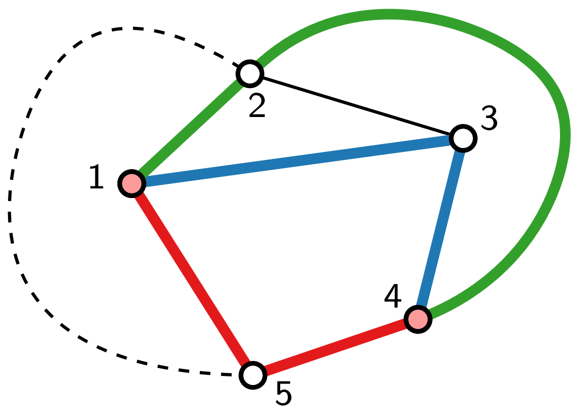

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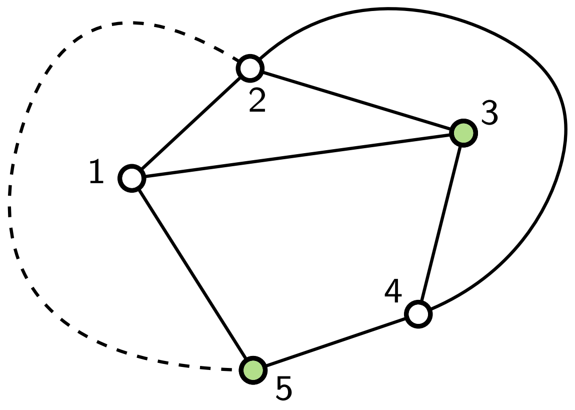

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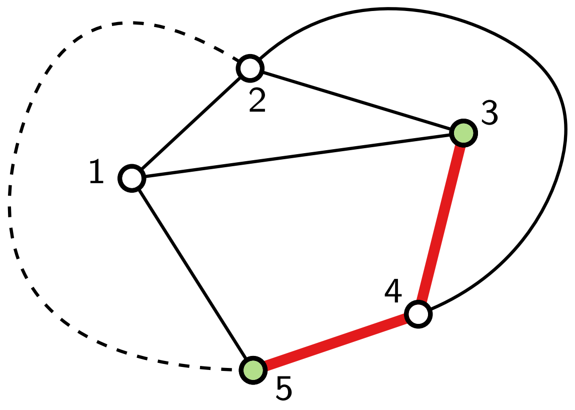

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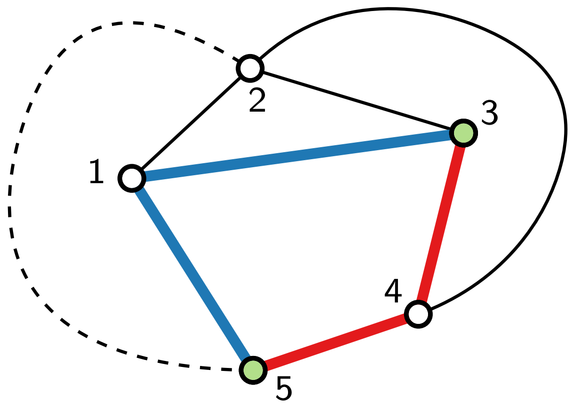

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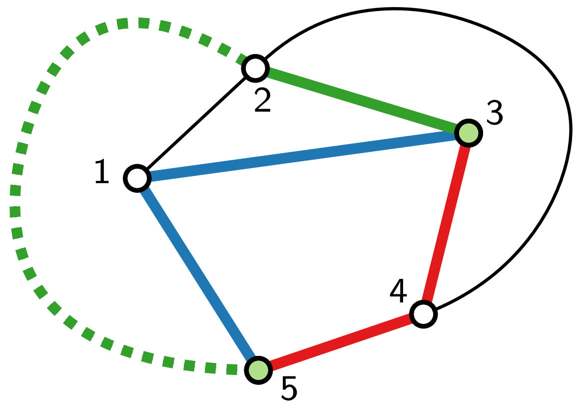

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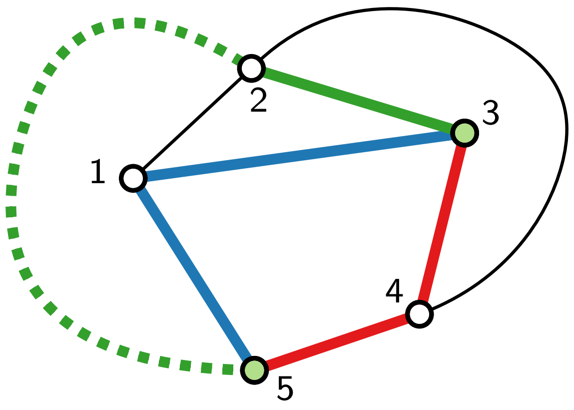
Theorem. [Whitney 1933]
Every 3-connected planar graph has a unique planar embedding.



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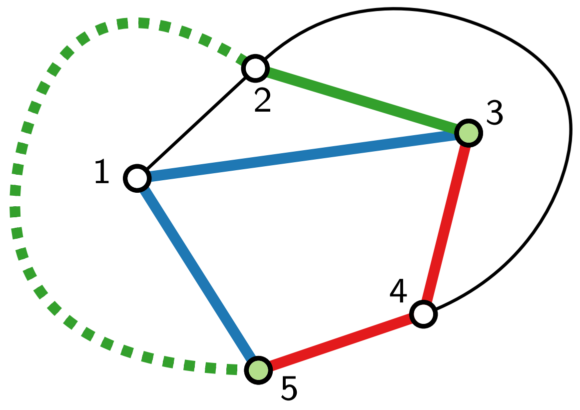
Proof sketch.



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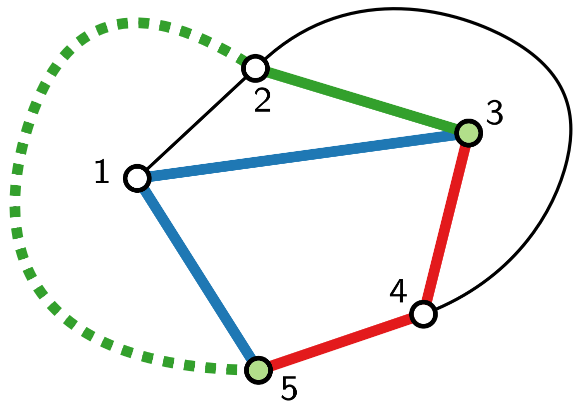

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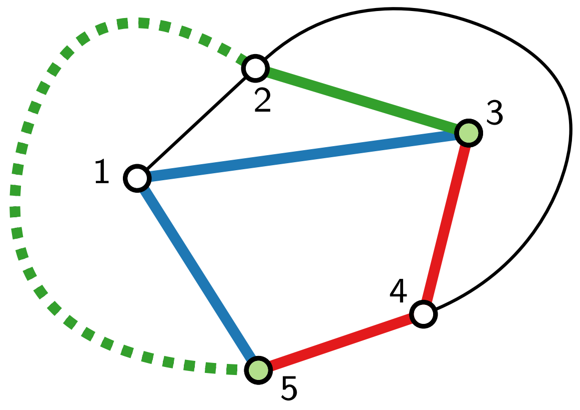

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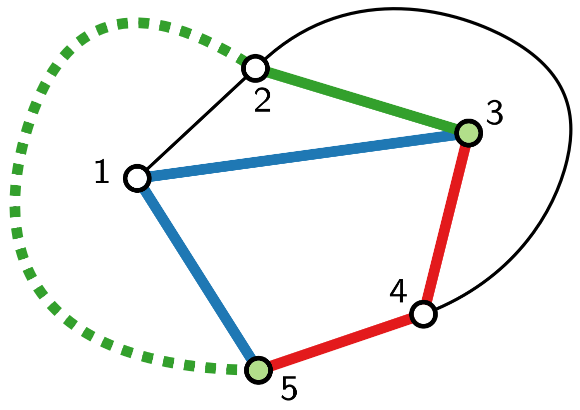

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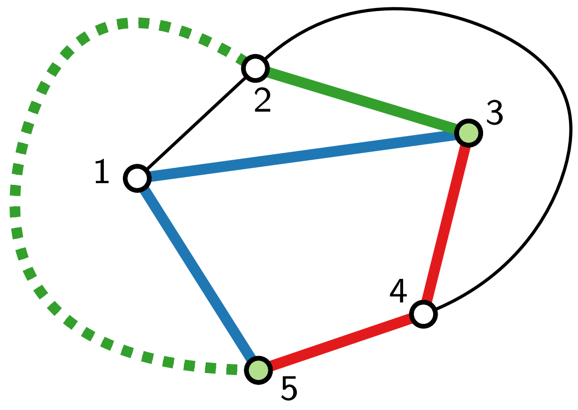

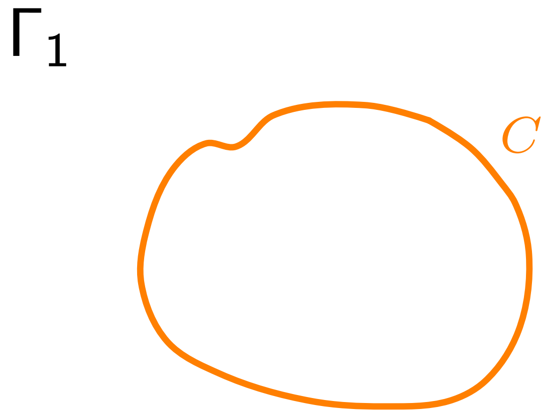

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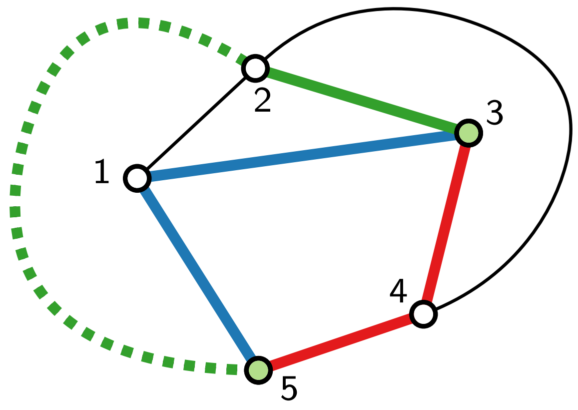

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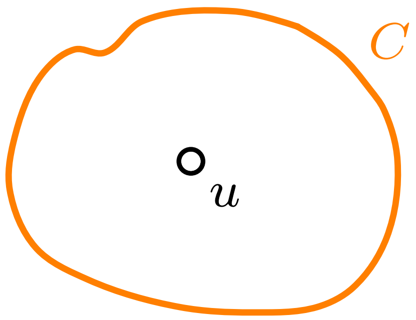

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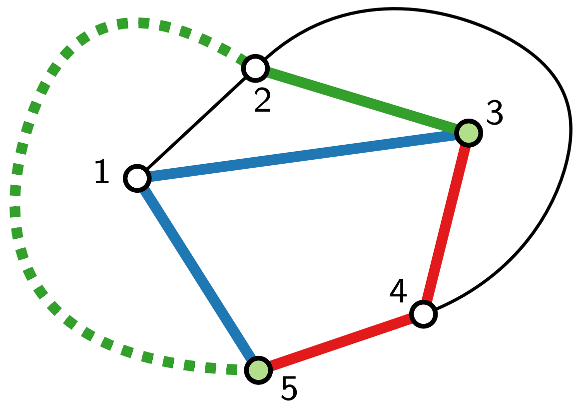

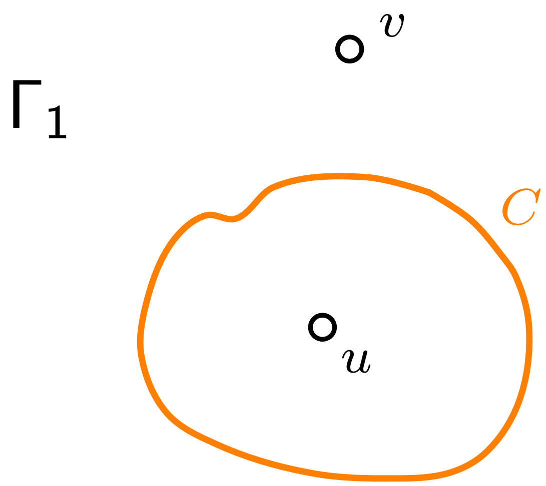

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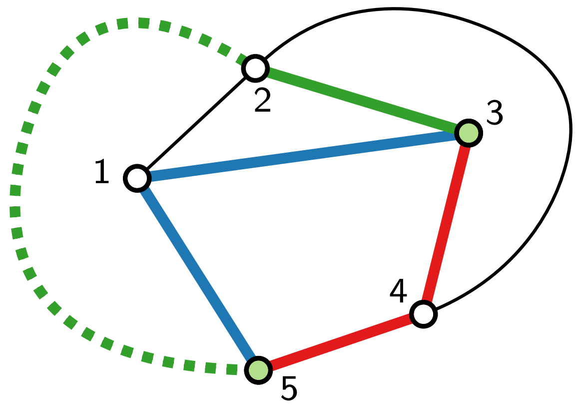

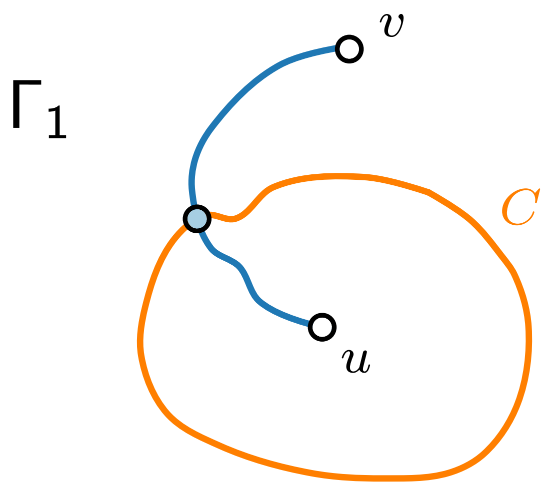

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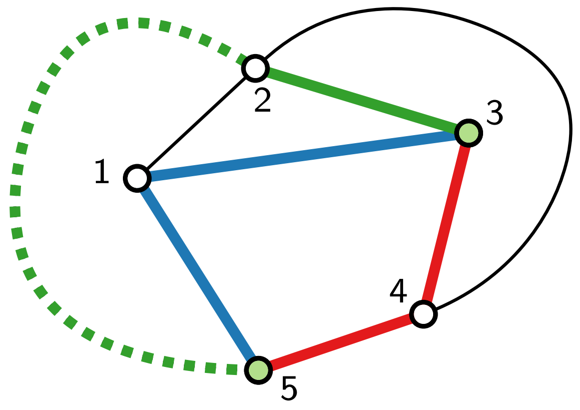

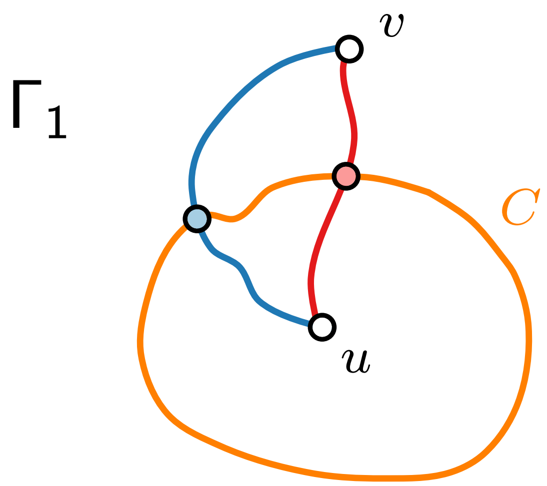

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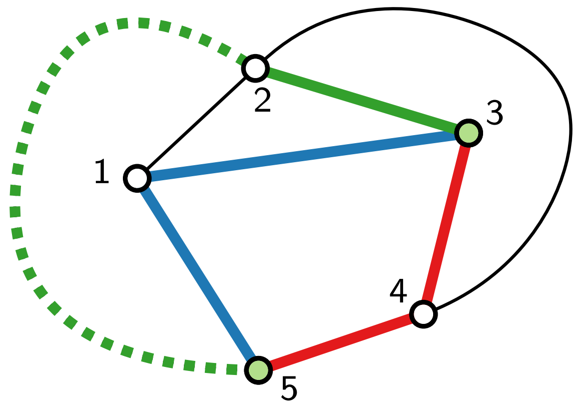

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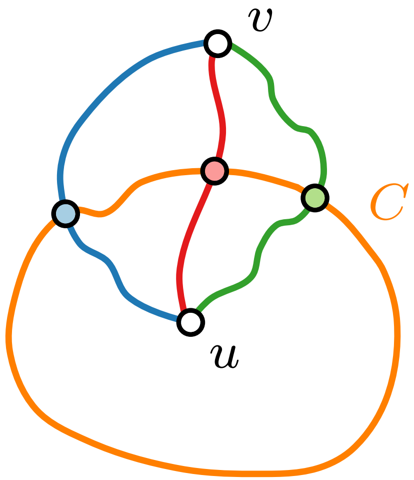

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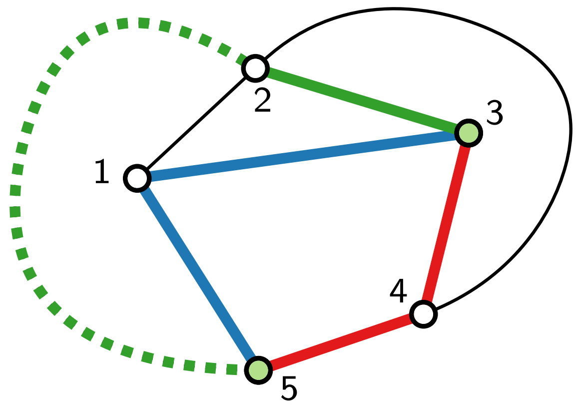

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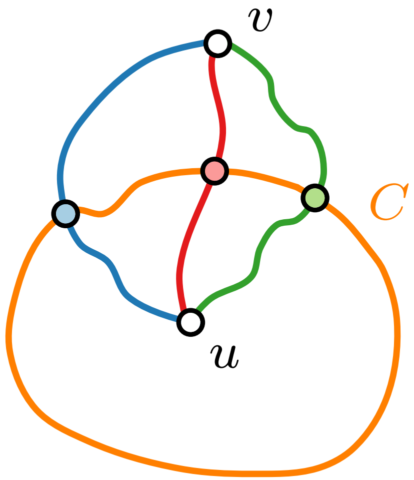

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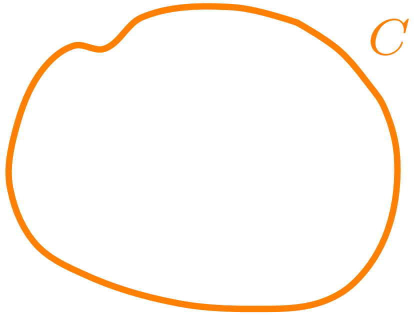

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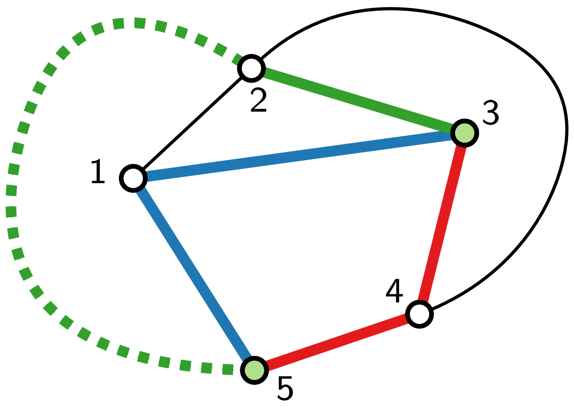

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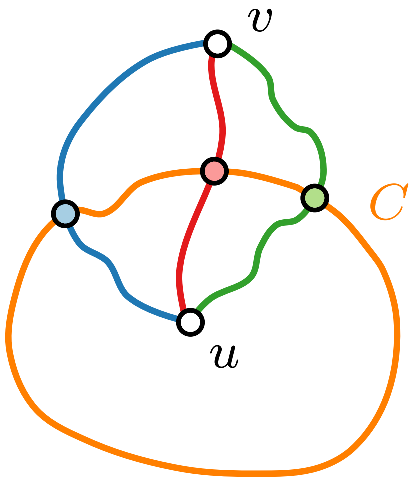

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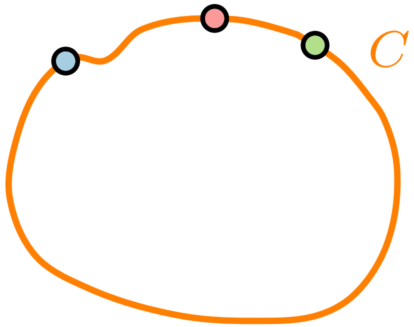

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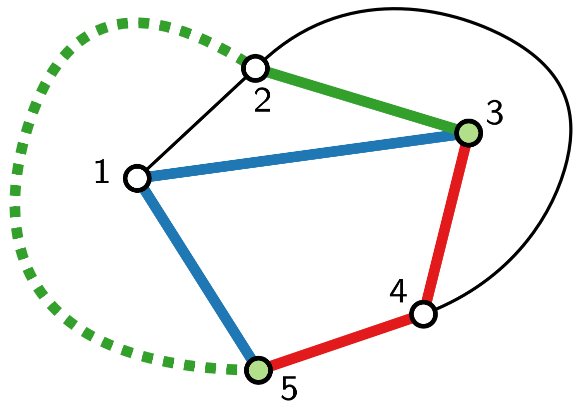

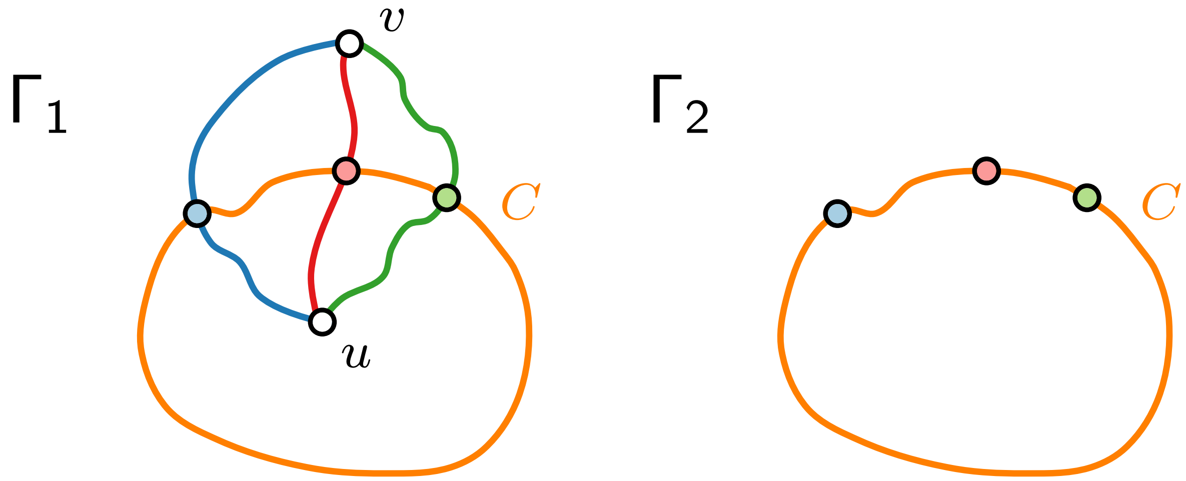

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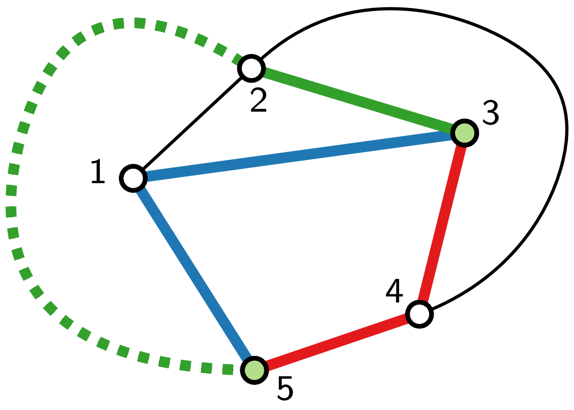

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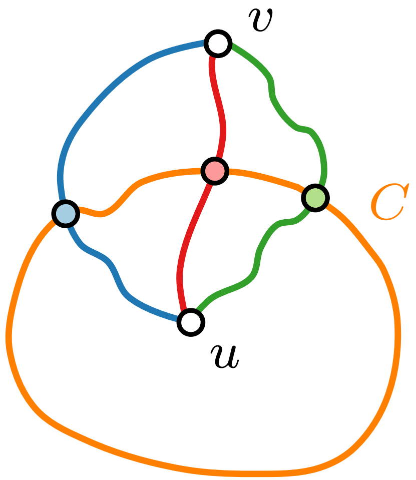

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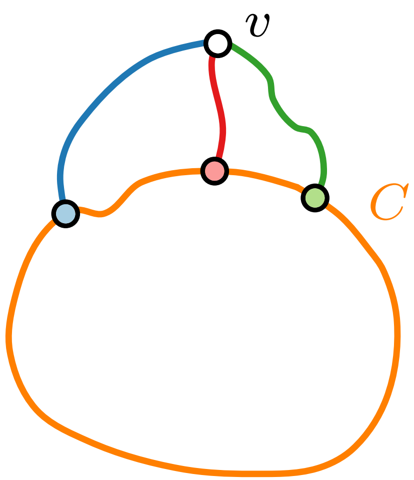

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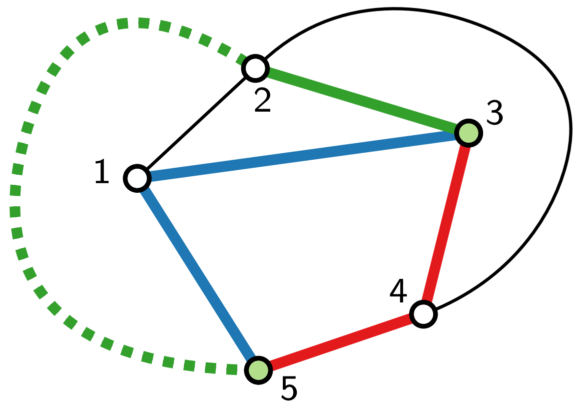

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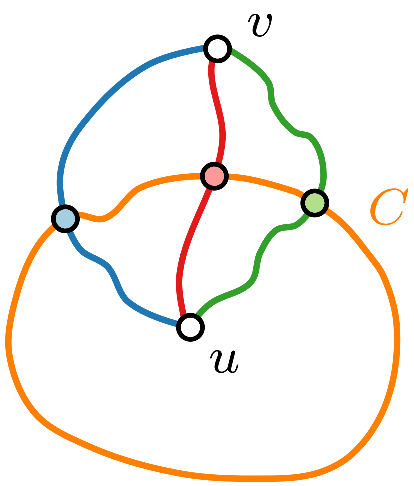

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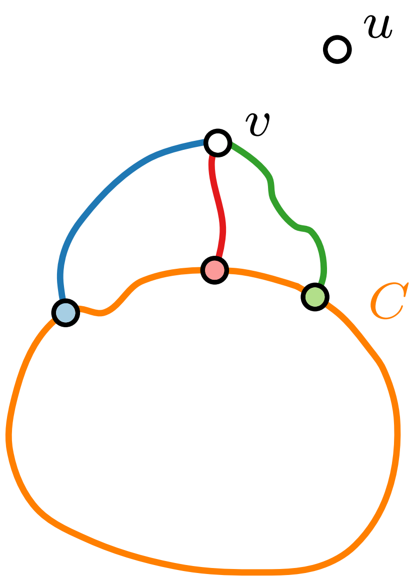

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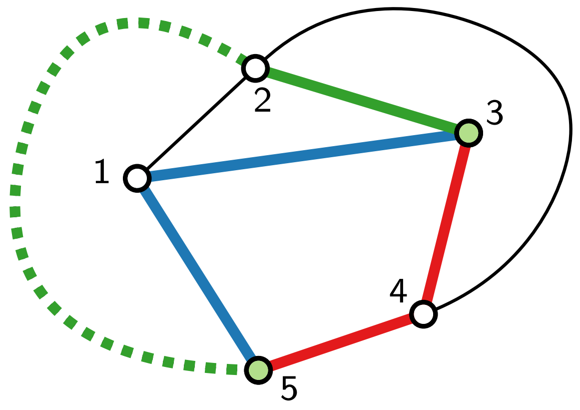

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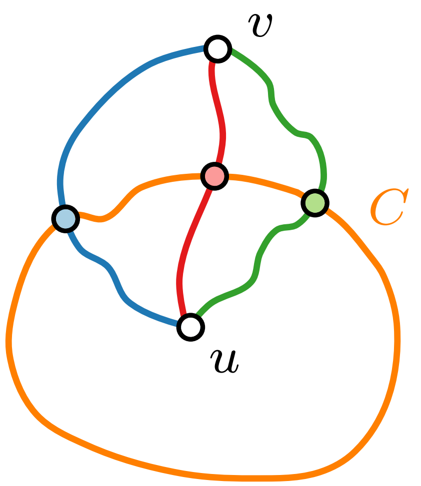

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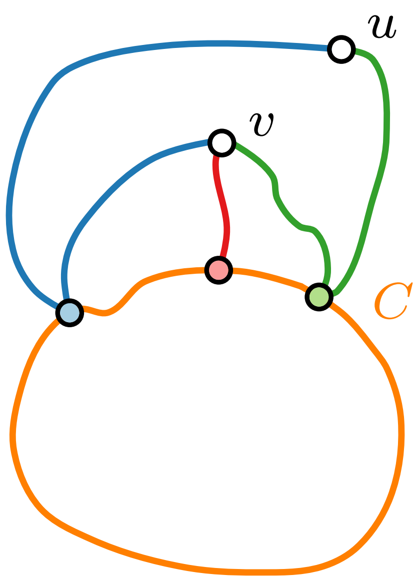

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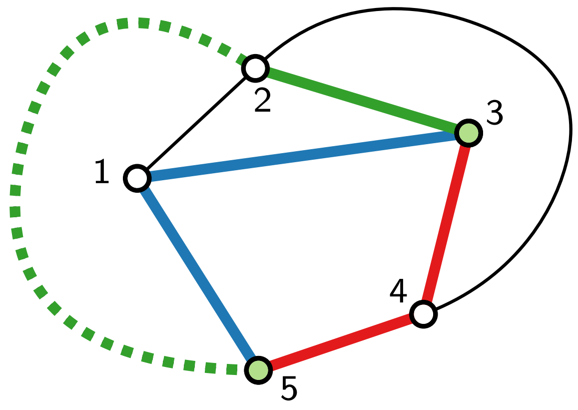

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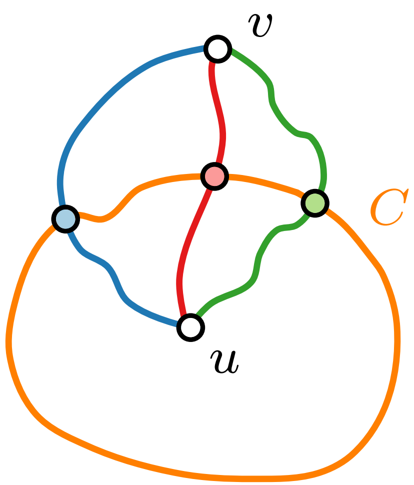

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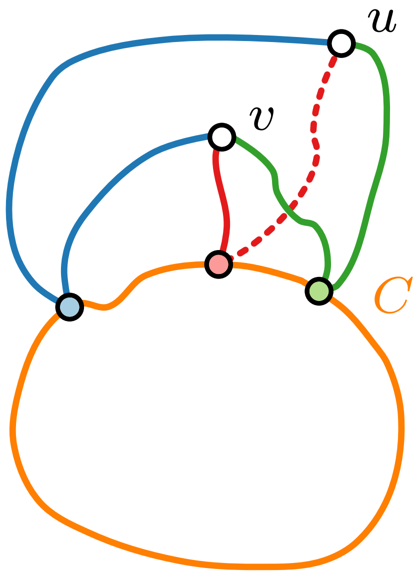

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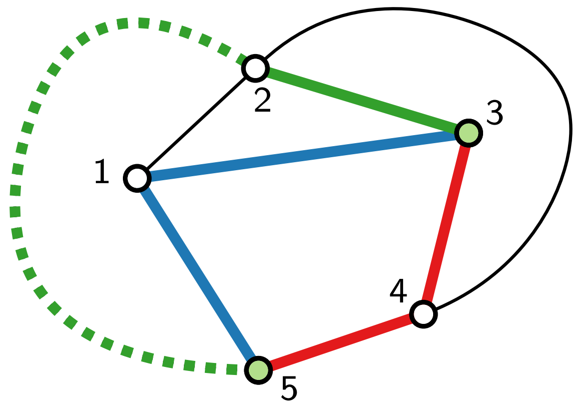

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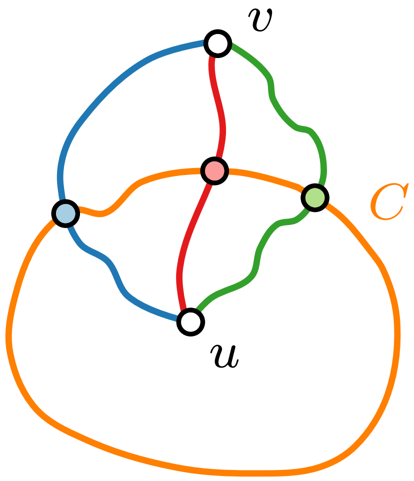

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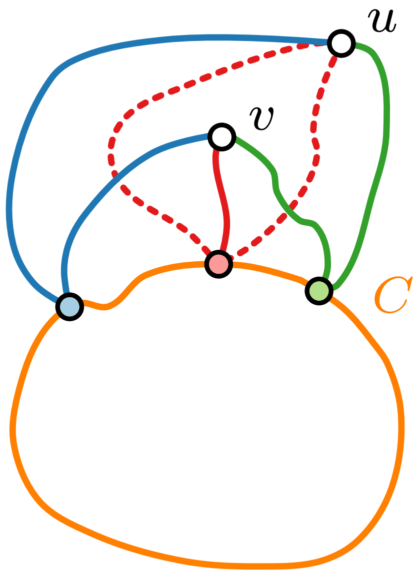

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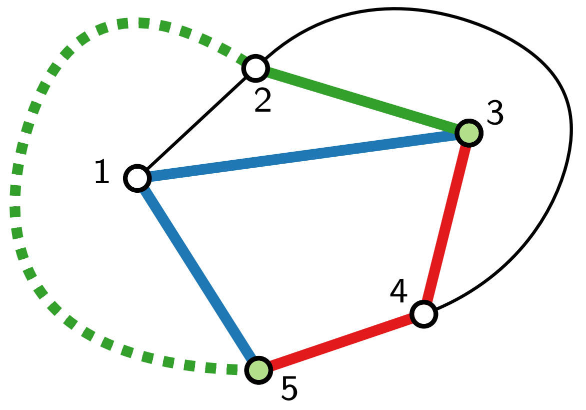
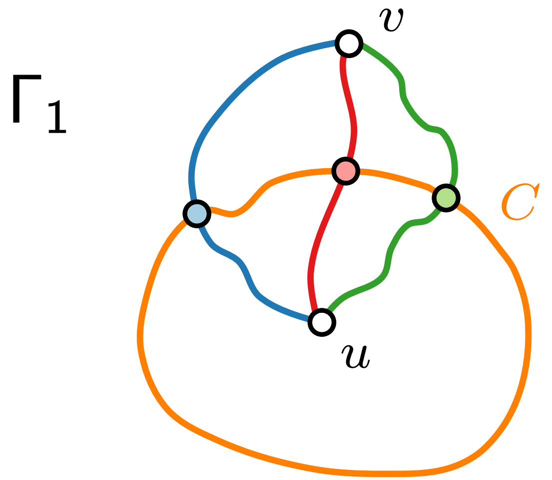
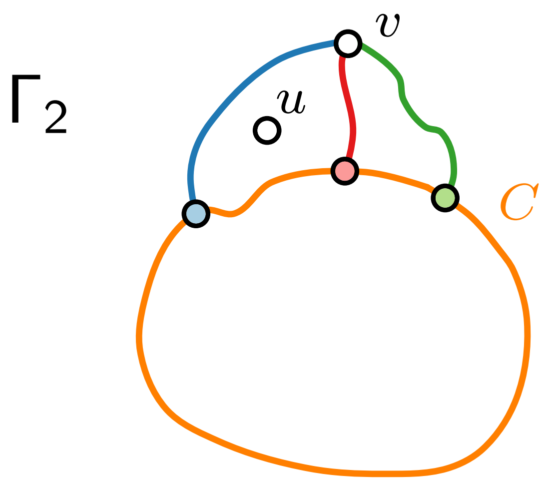






3-Connected Planar Graphs

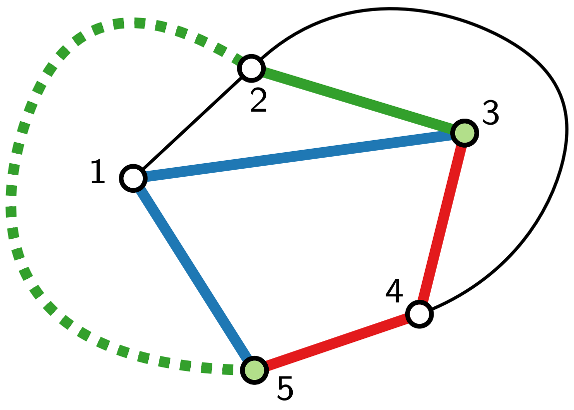
planar: G can be drawn in such a way that no edges cross each other

connected: There is a u-v-path for every $u, v \in V$

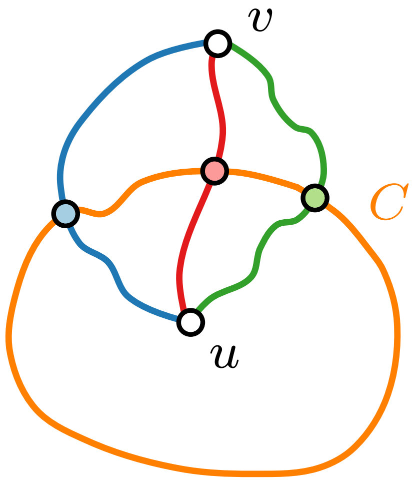
k-**connected:** $G - \{v_1, \ldots, v_{k-1}\}$ is connected for **any** $v_1 \ldots, v_{k-1} \in V$
or (equivalently)
There are at least k vertex-disjoint u-v-paths for every $u, v \in V$

Theorem. [Whitney 1933]
Every 3-connected planar graph has a unique planar embedding.

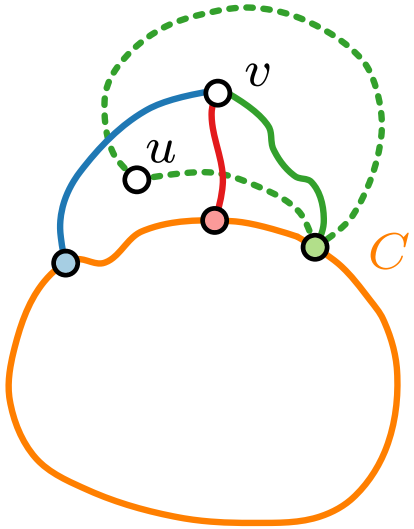
Proof sketch.
Γ_1, Γ_2 embeddings of G
C face of Γ_2, but not Γ_1
u inside C in Γ_1 , v outside C in Γ_1
both on same side in Γ_2



Γ_1



Γ_2



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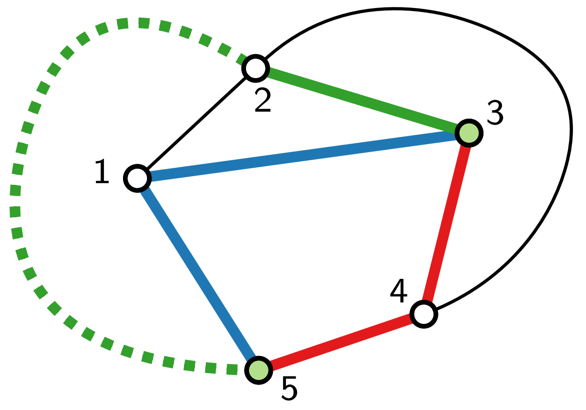

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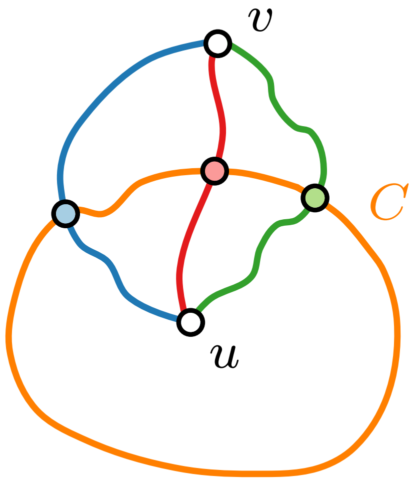
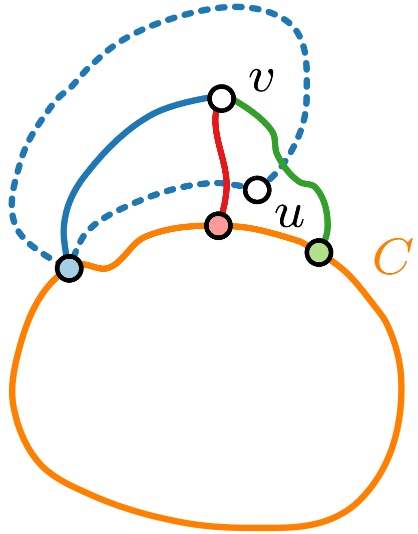


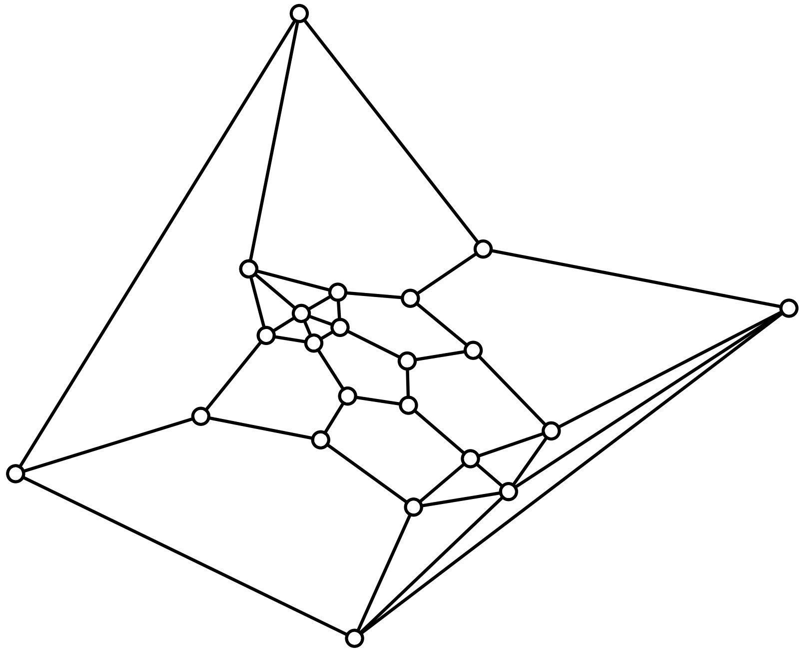
Γ_2



Tutte's Theorem

Theorem. [Tutte 1963]

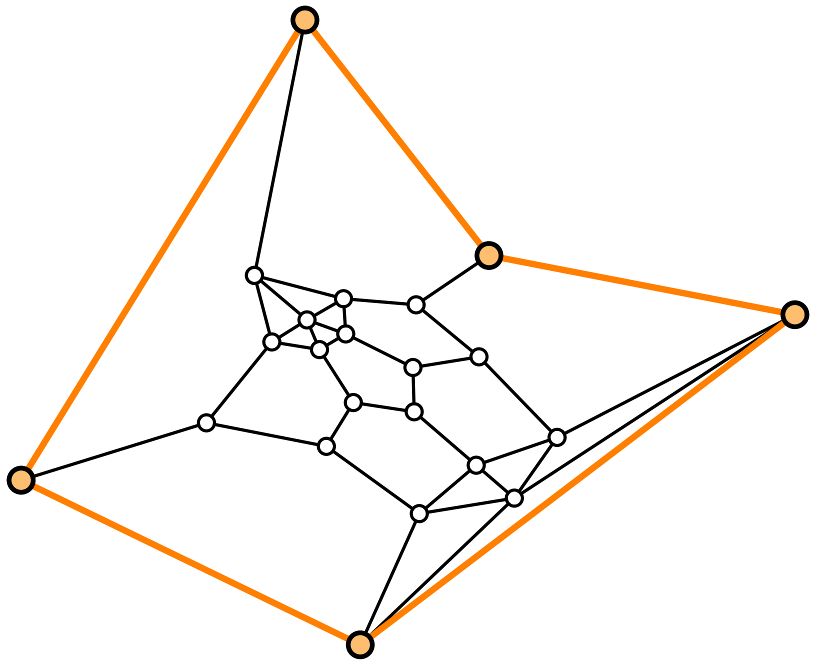
Let G be a 3-connected planar graph,



Tutte's Theorem

Theorem. [Tutte 1963]

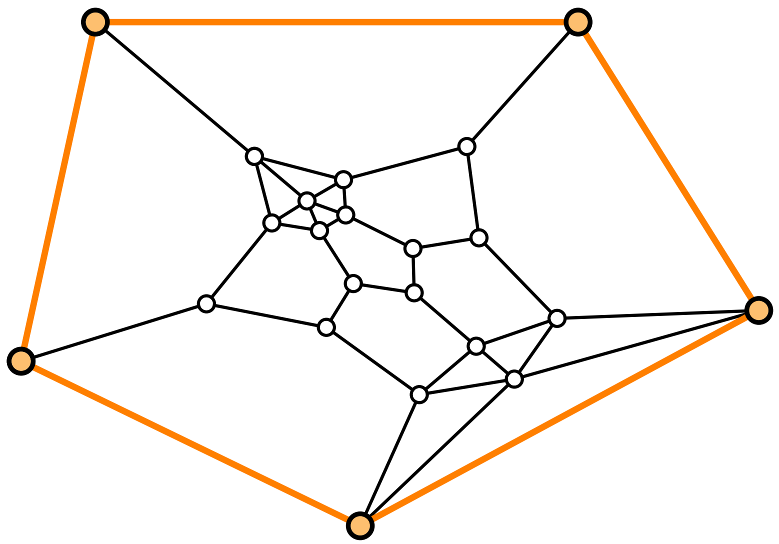
Let G be a 3-connected planar graph, and let C be a face of its unique embedding.



Tutte's Theorem

Theorem. [Tutte 1963]

Let G be a 3-connected planar graph, and let C be a face of its unique embedding. If we fix C on a strictly convex polygon,



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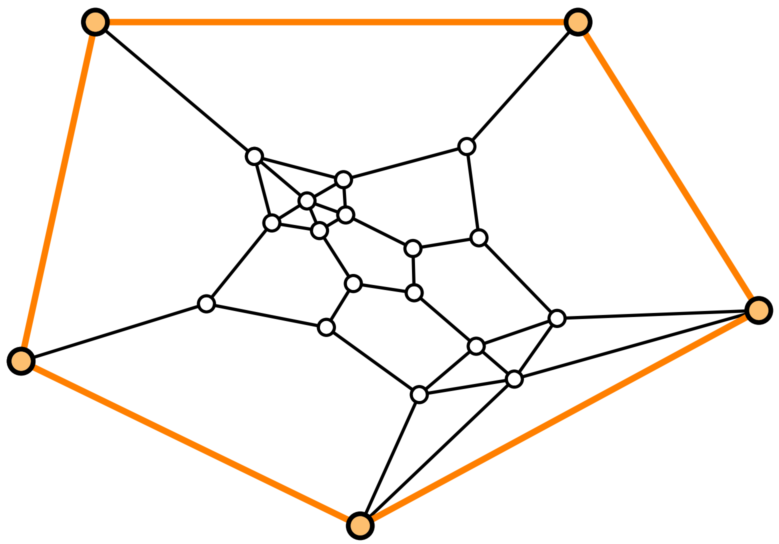
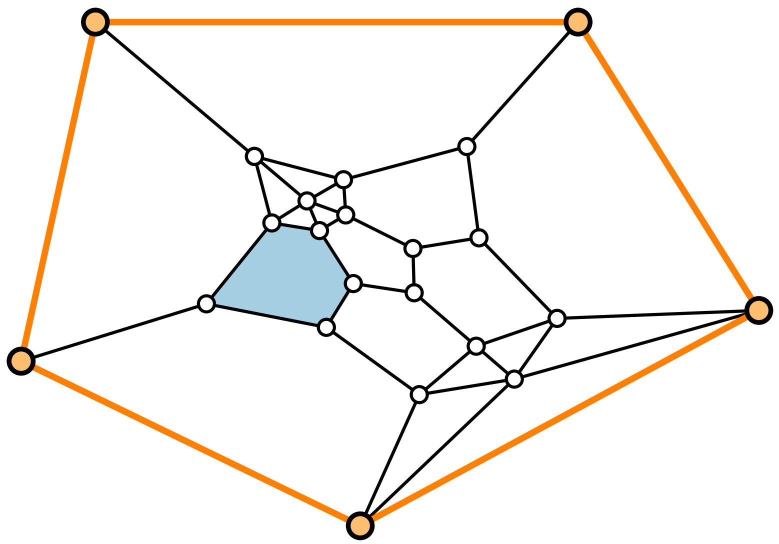
Theorem. [Tutte 1963]
Let G be a 3-connected planar graph, and let C be a face of its unique embedding.
If we fix C on a strictly convex polygon, then the Tutte drawing of G is planar



Tutte's Theorem

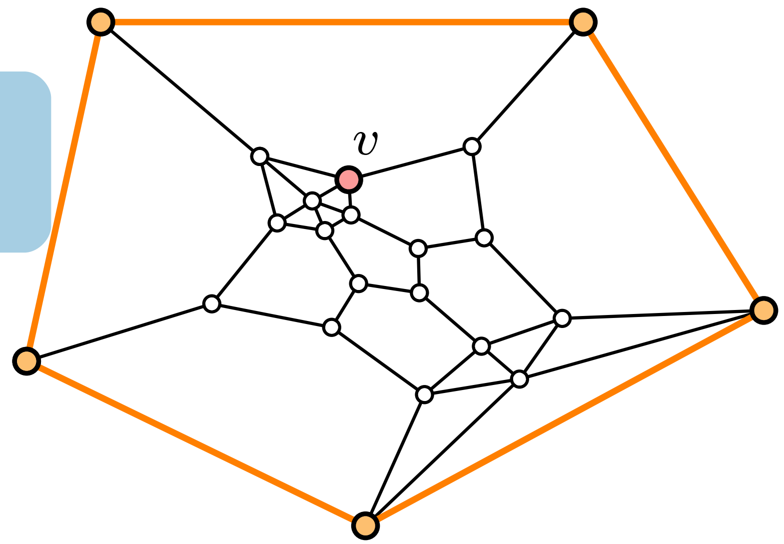
Theorem. [Tutte 1963]

Let G be a 3-connected planar graph, and let C be a face of its unique embedding. If we fix C on a strictly convex polygon, then the Tutte drawing of G is planar and all its faces are strictly convex.



Properties of Tutte Drawings

Property 1. Let $v \in V$ free,



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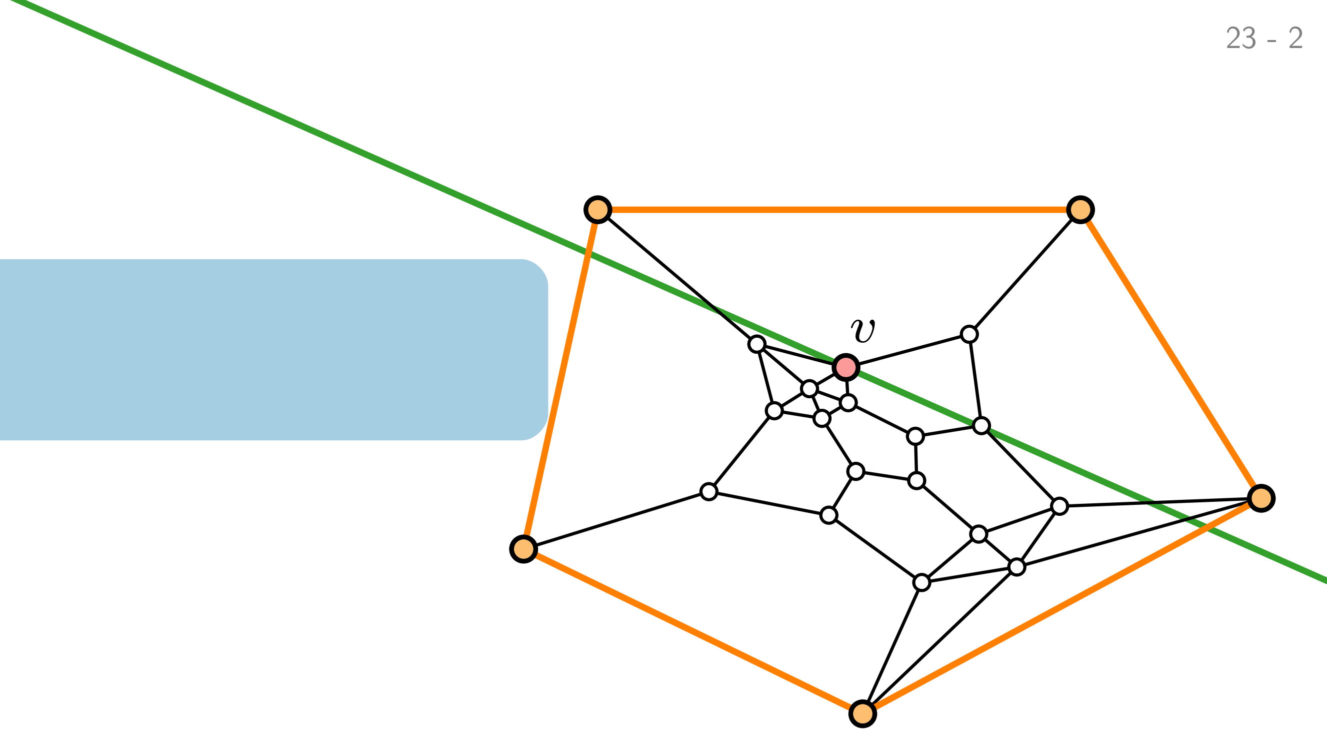

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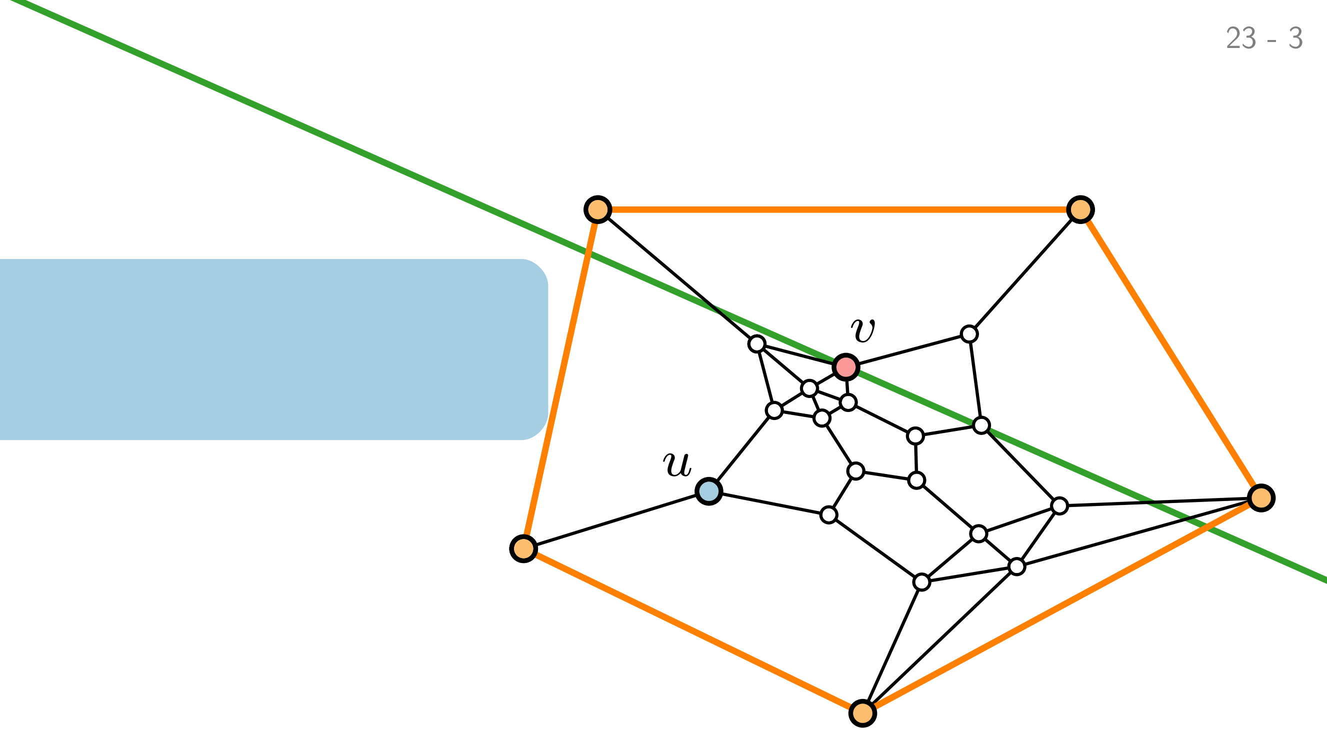

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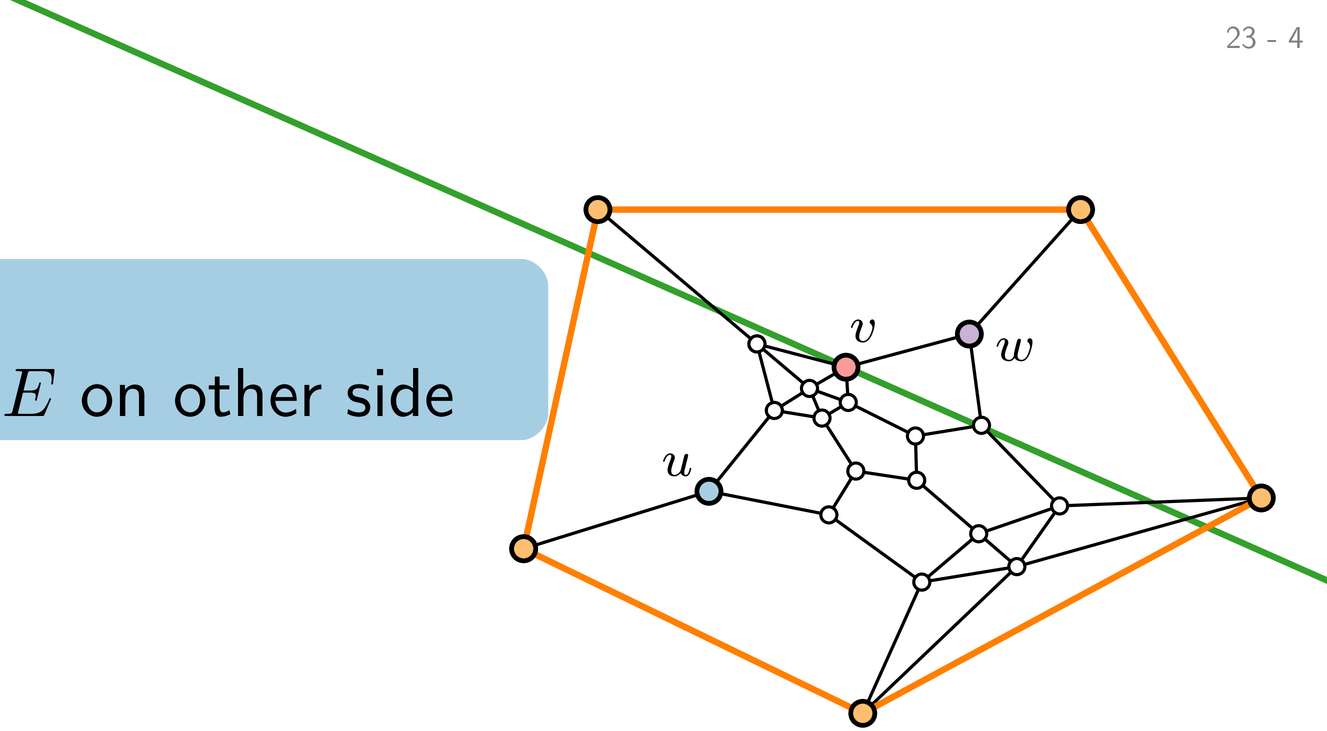

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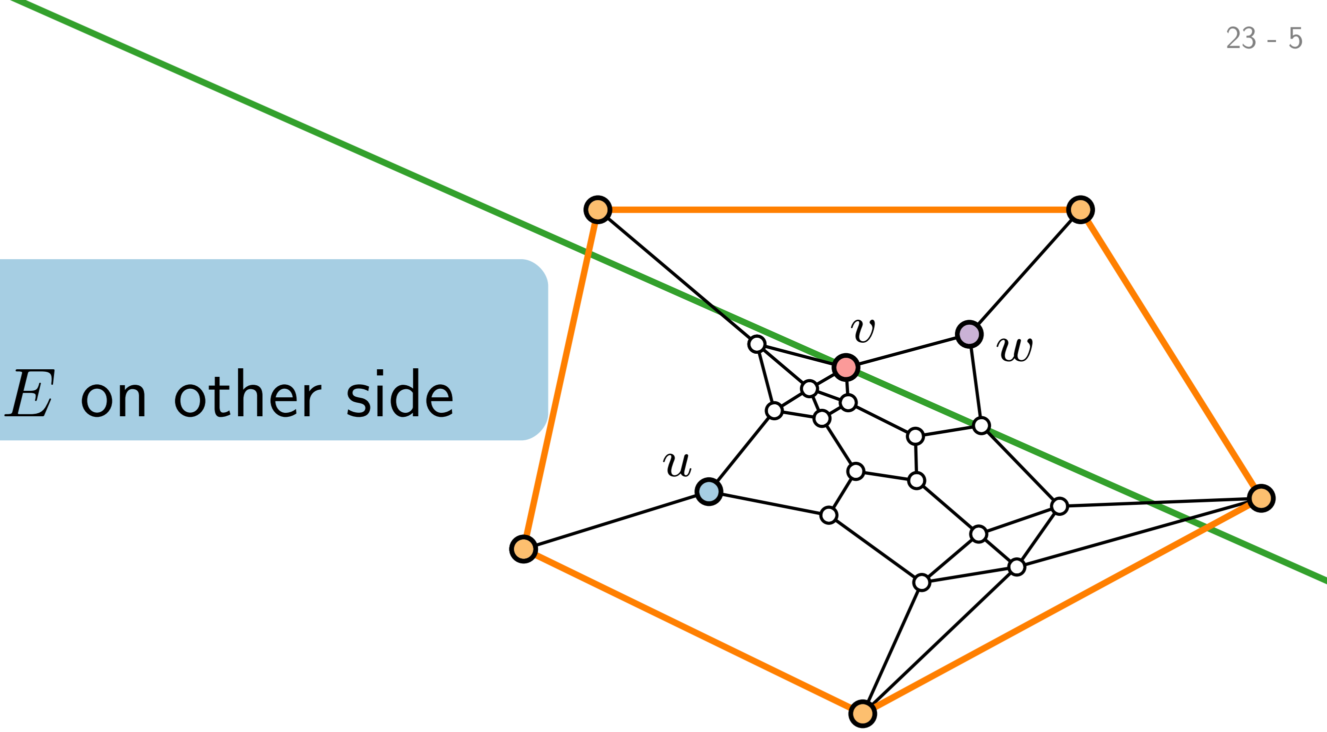

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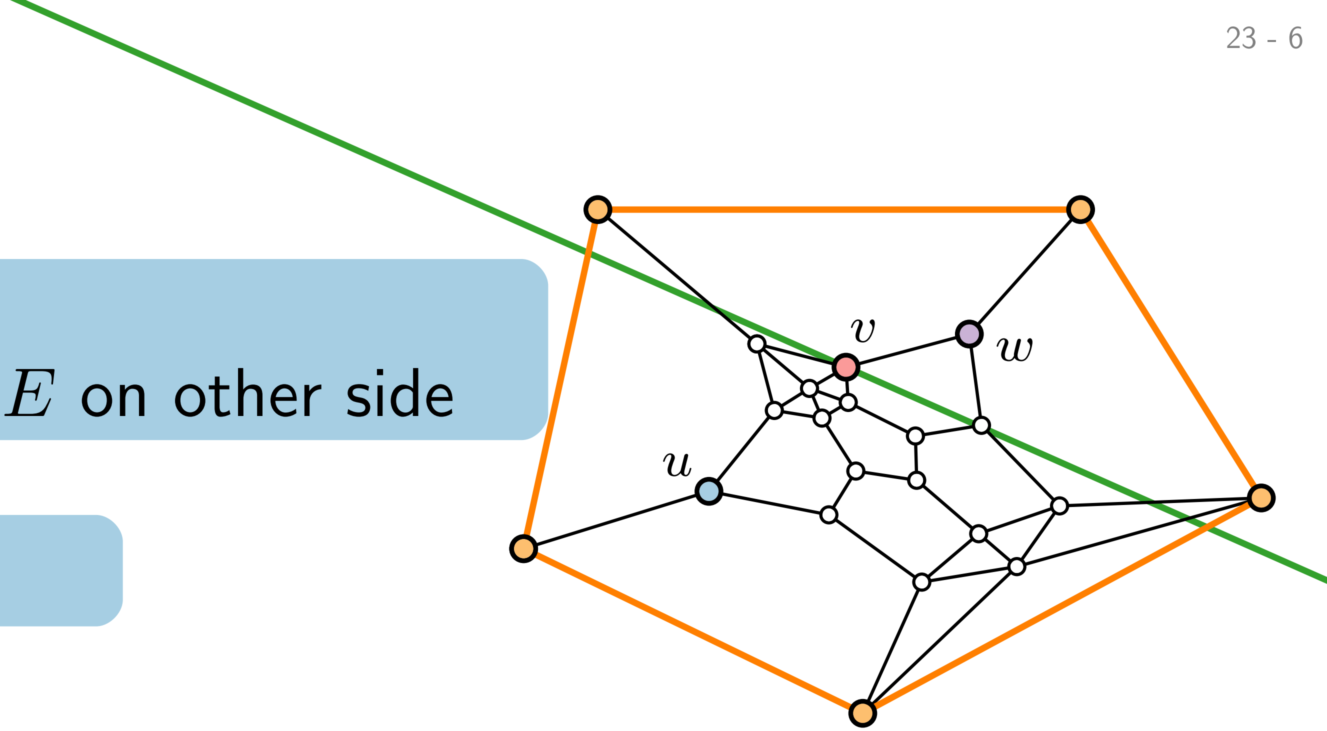

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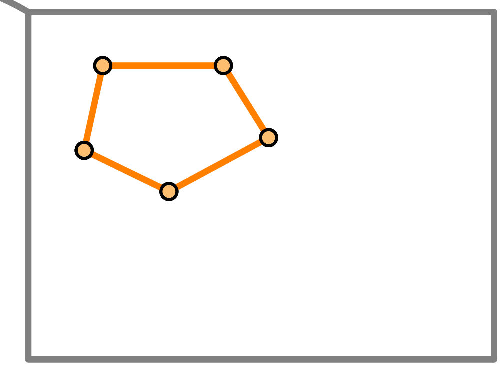
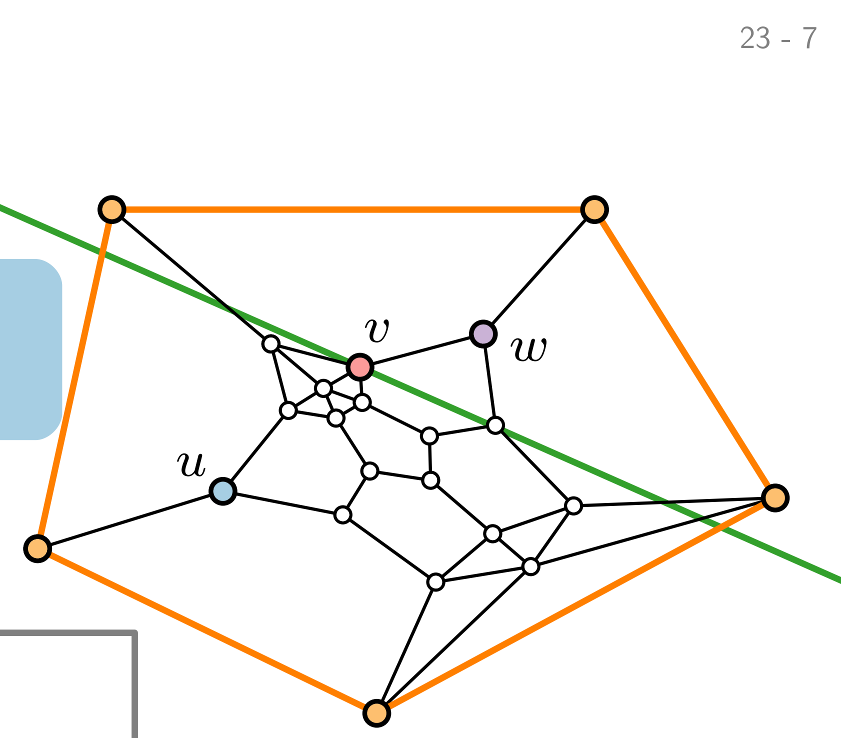

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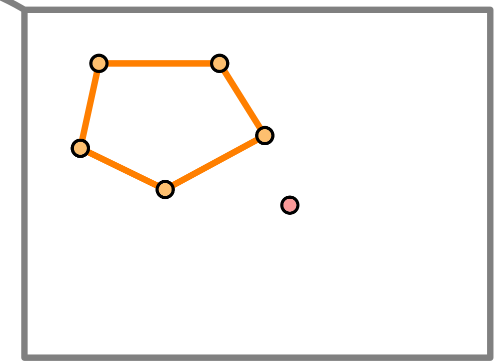

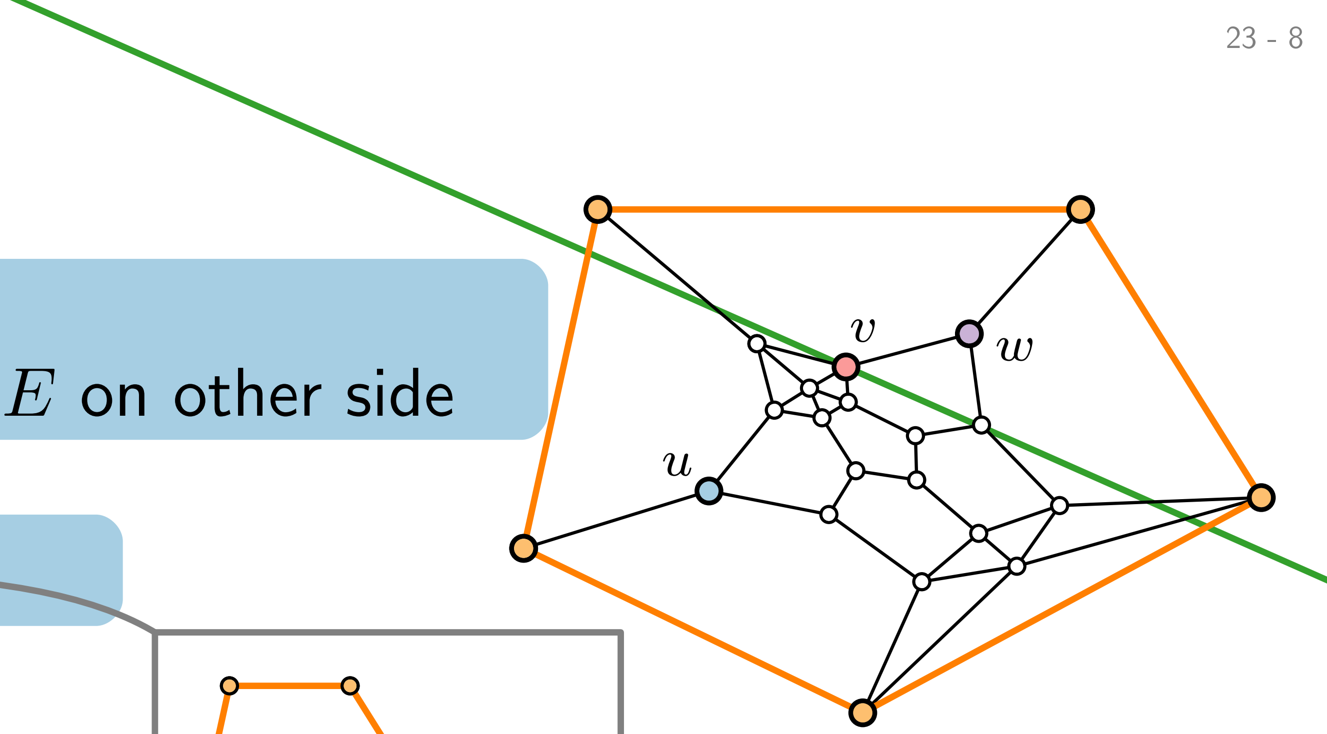

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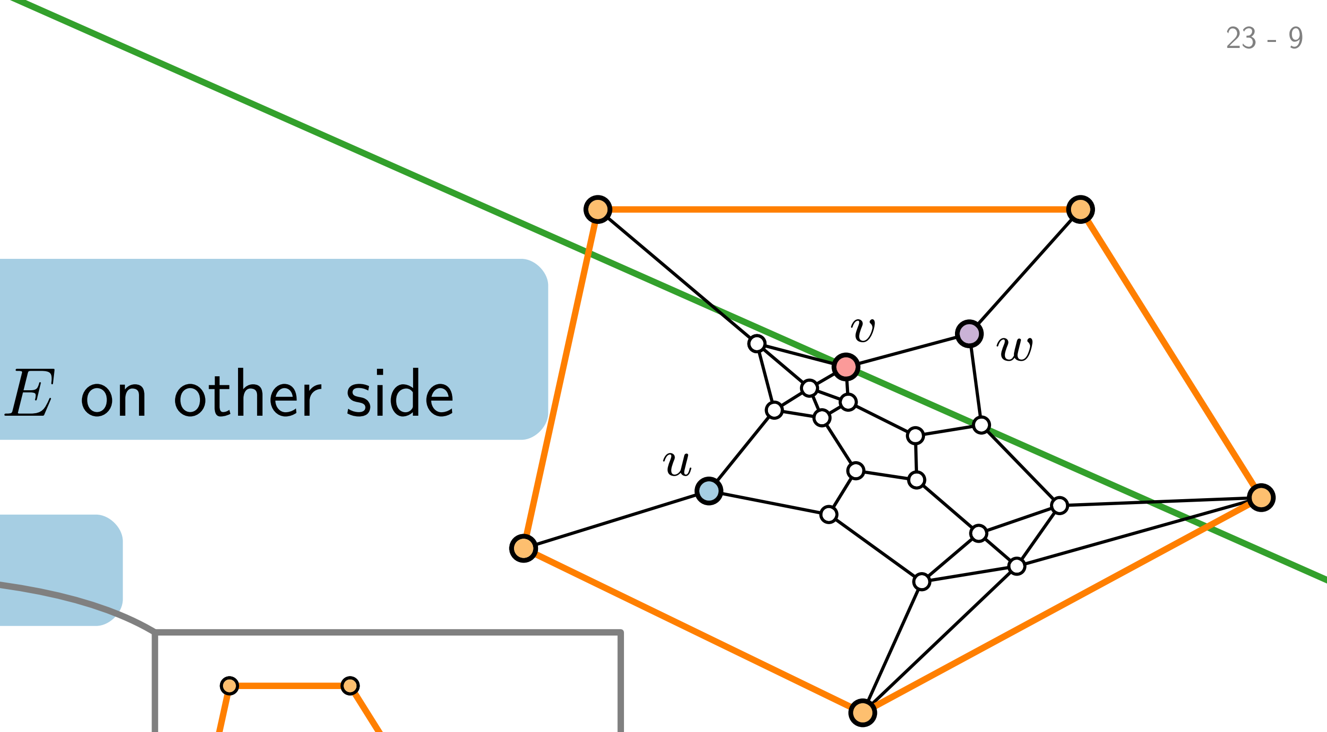

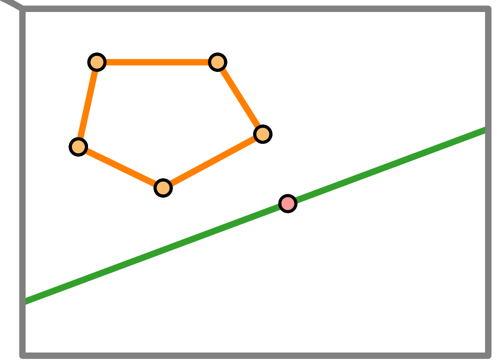

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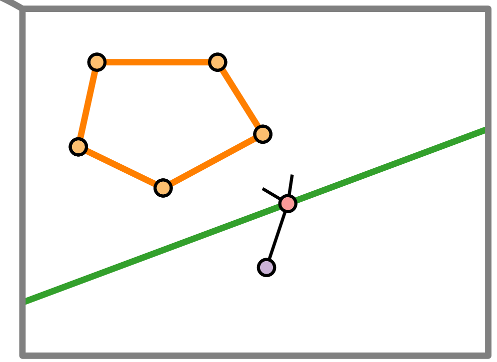

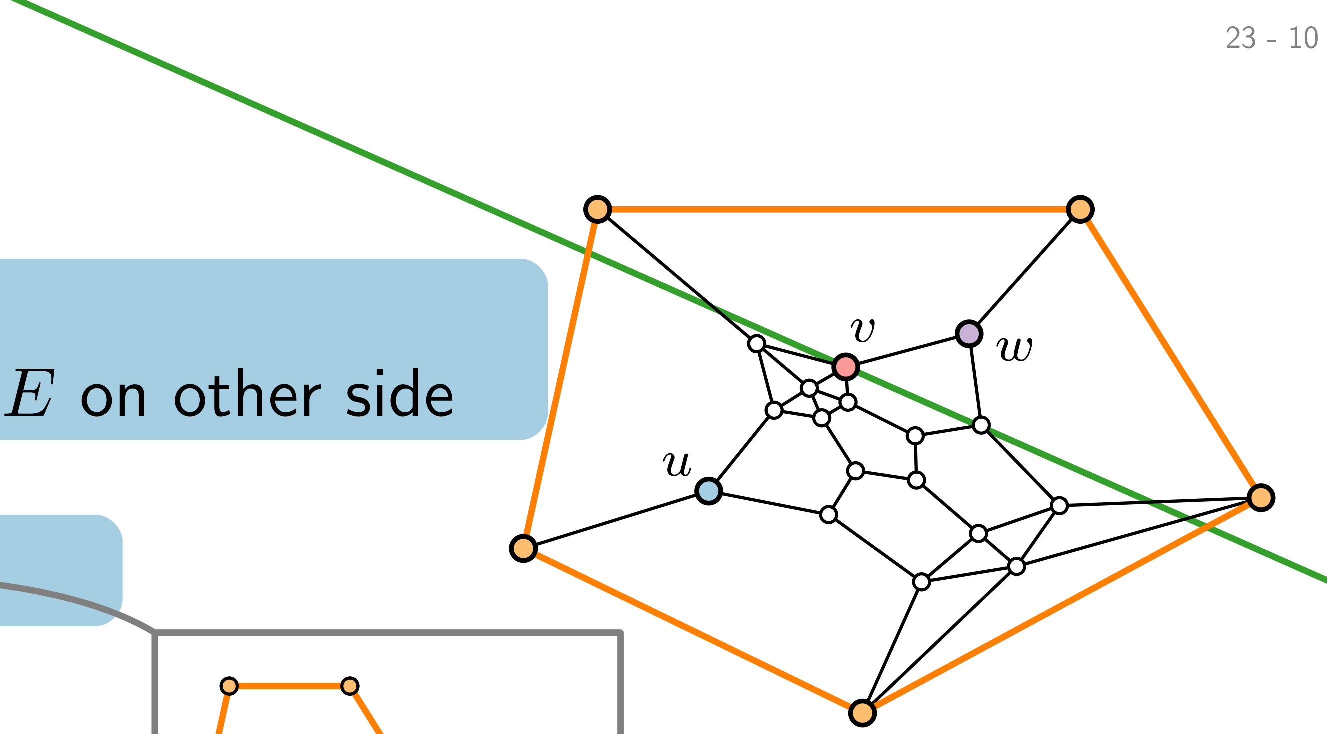

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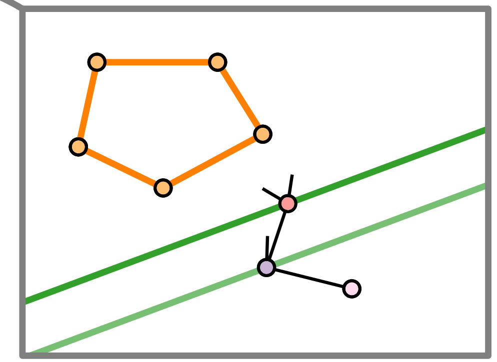

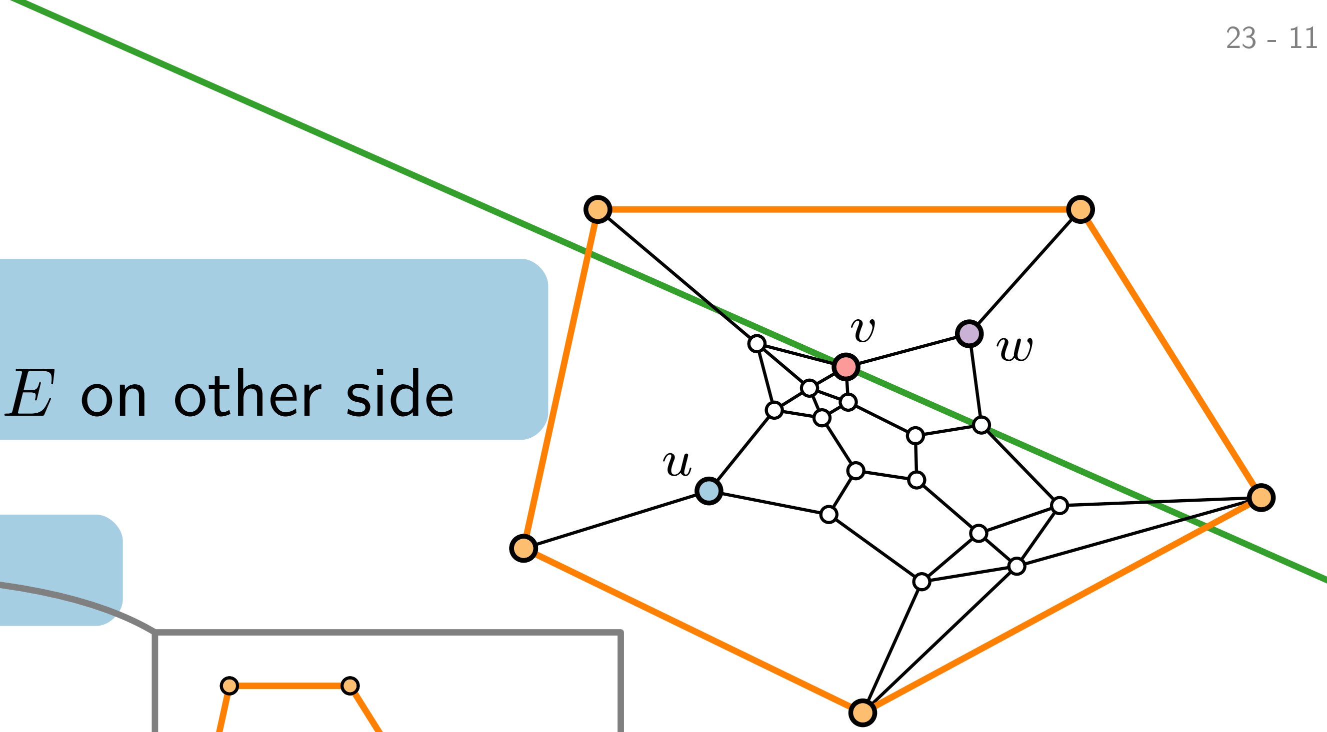

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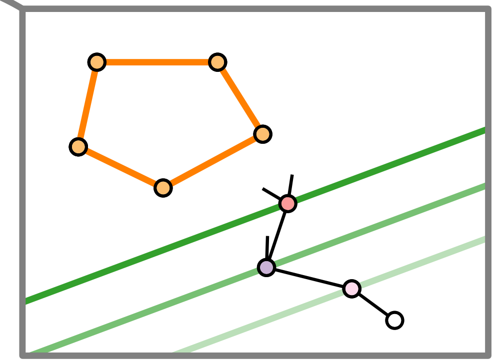

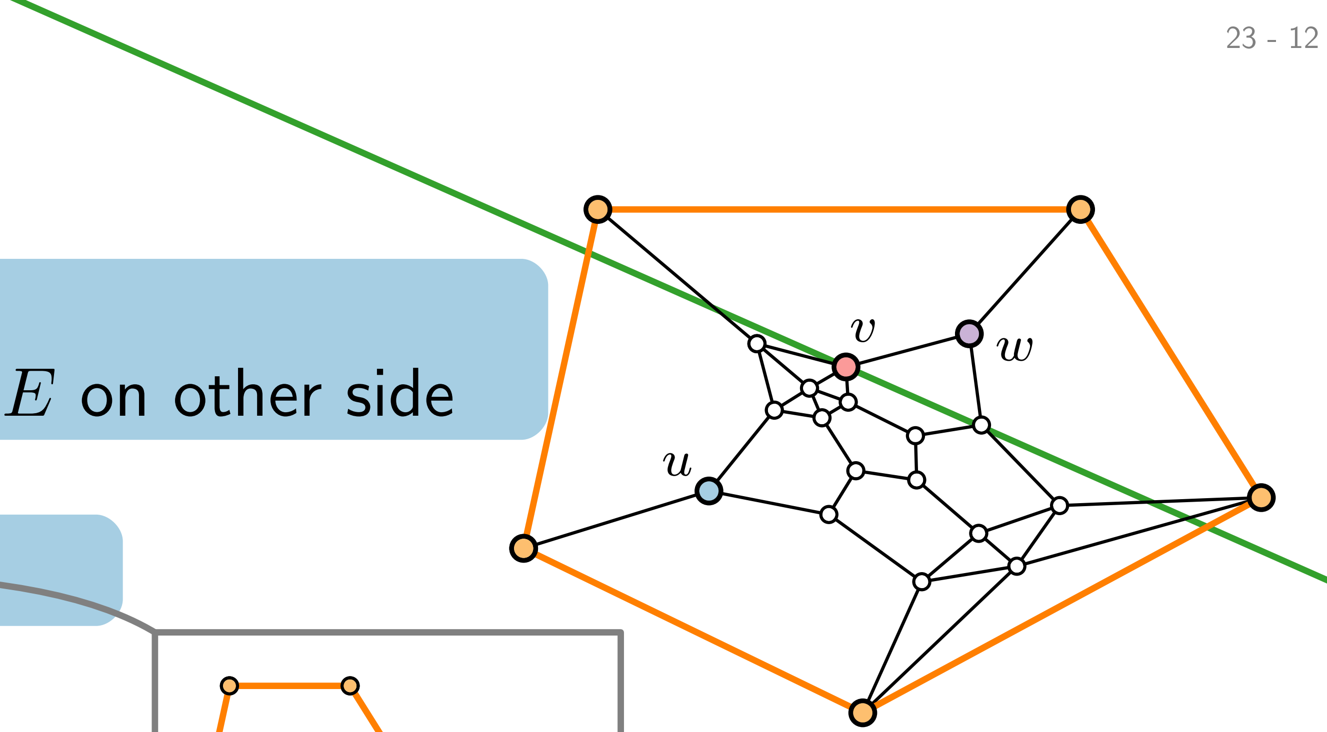

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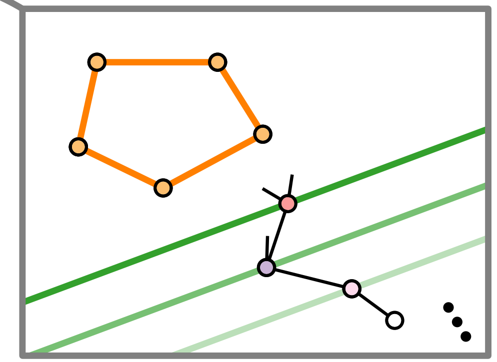

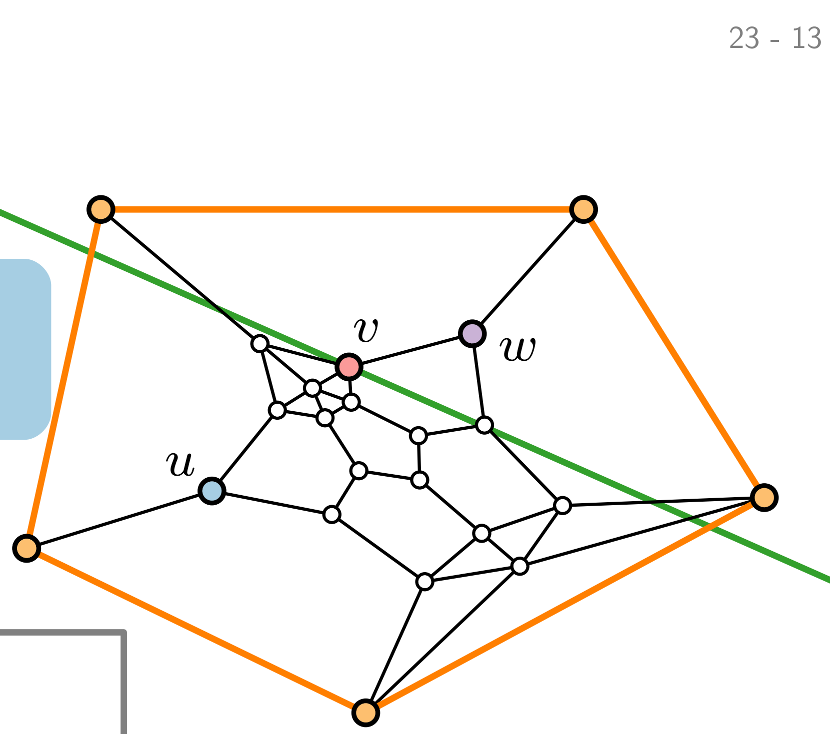

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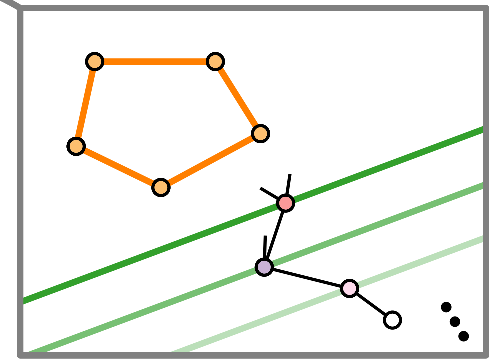

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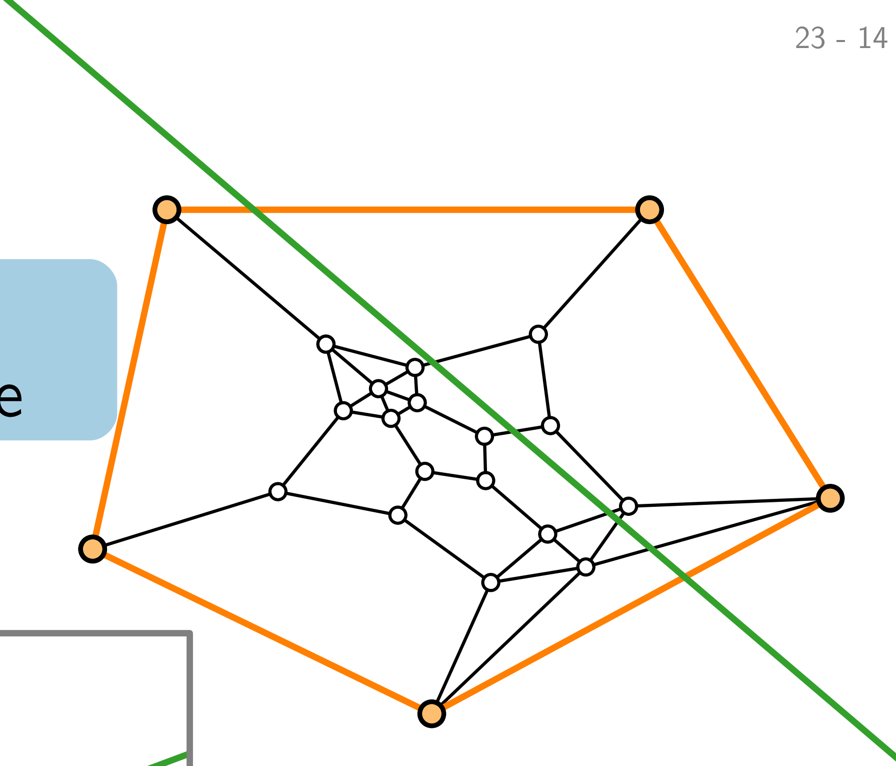

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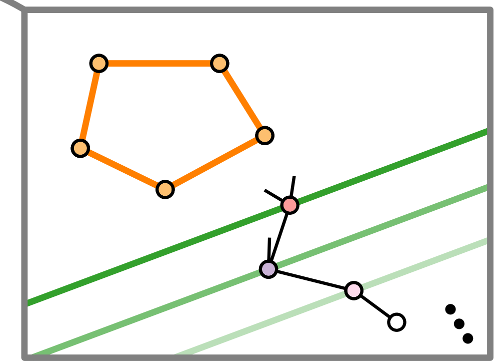

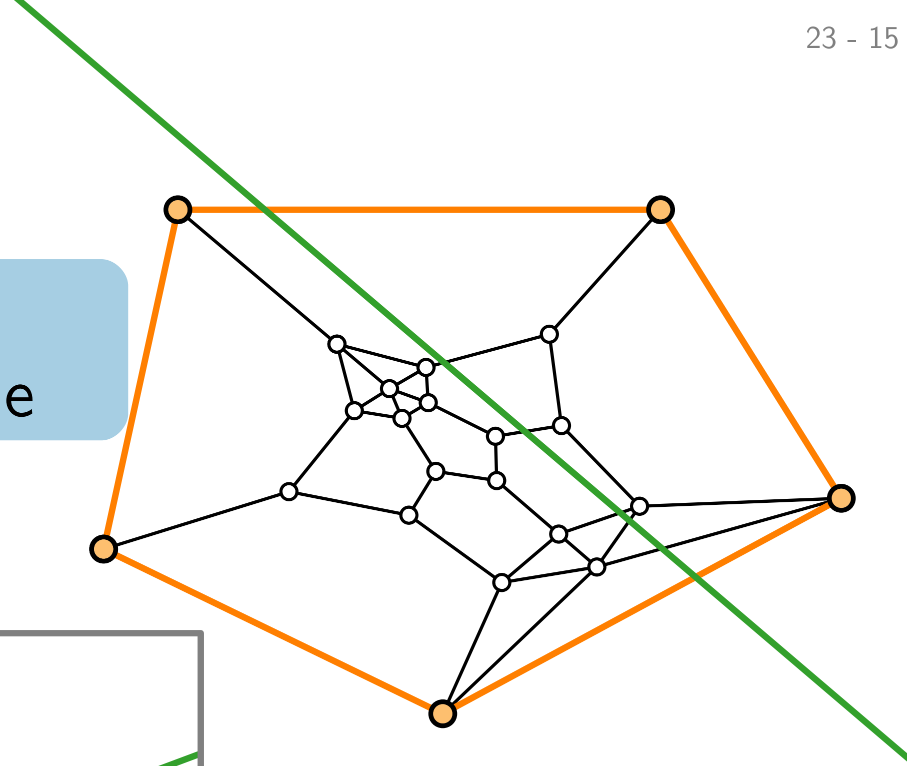

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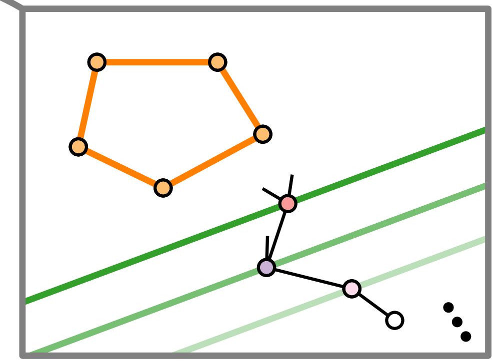

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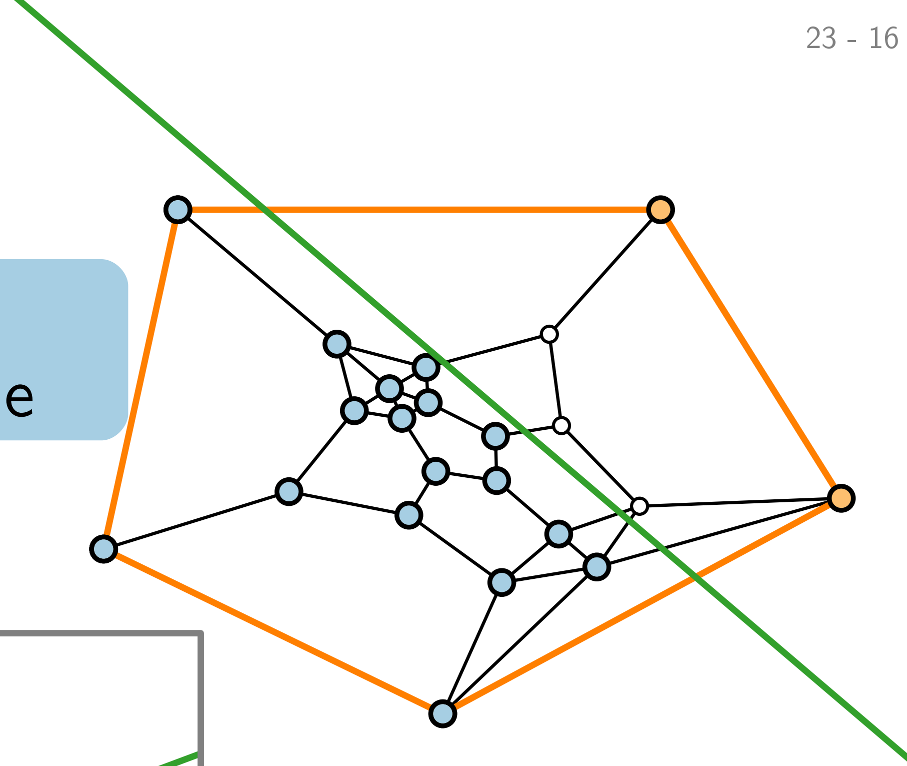

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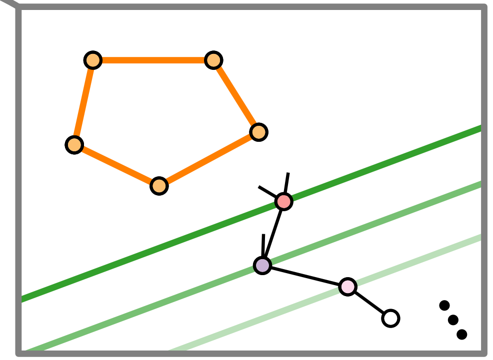

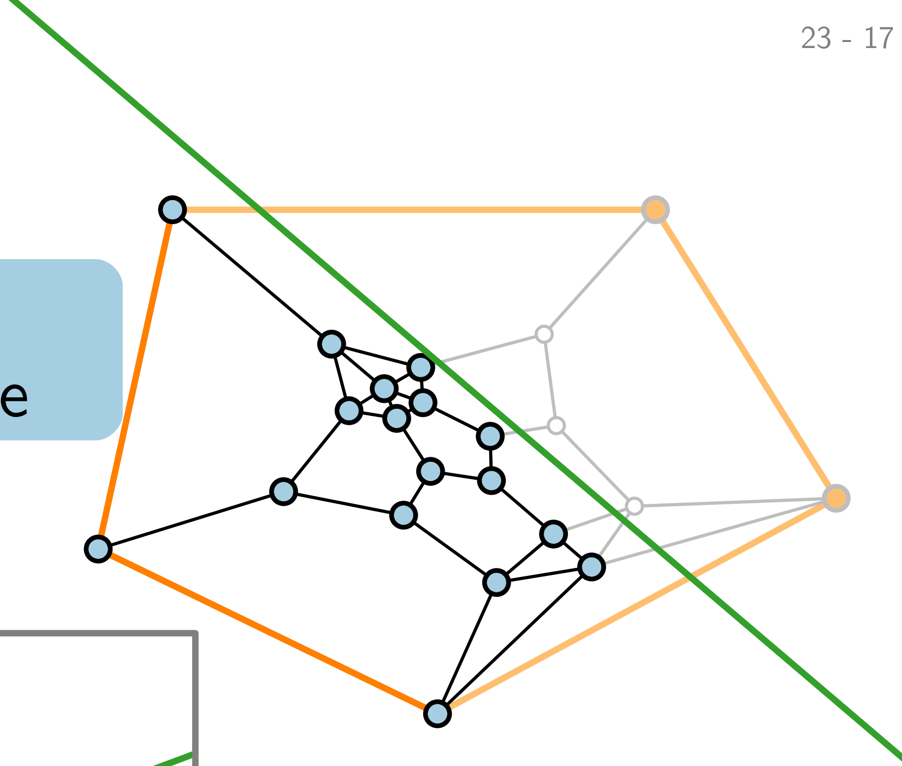

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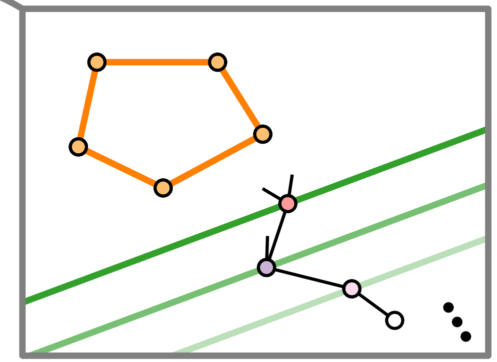

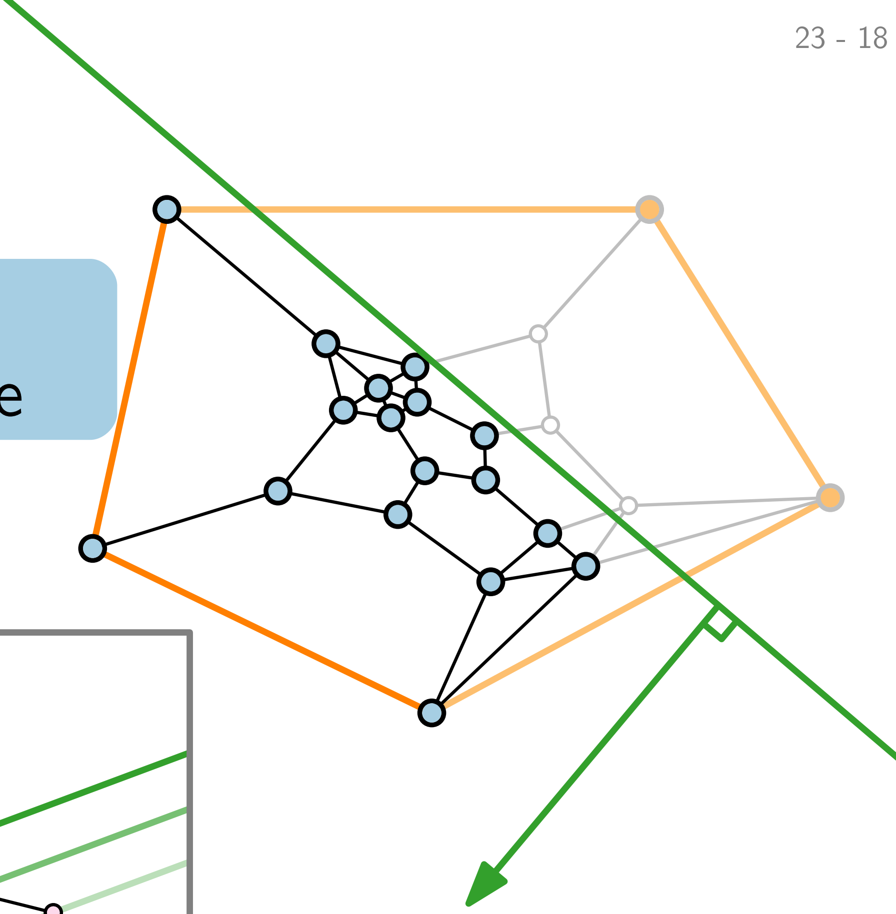

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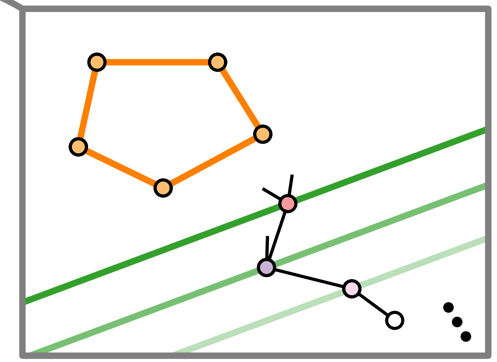

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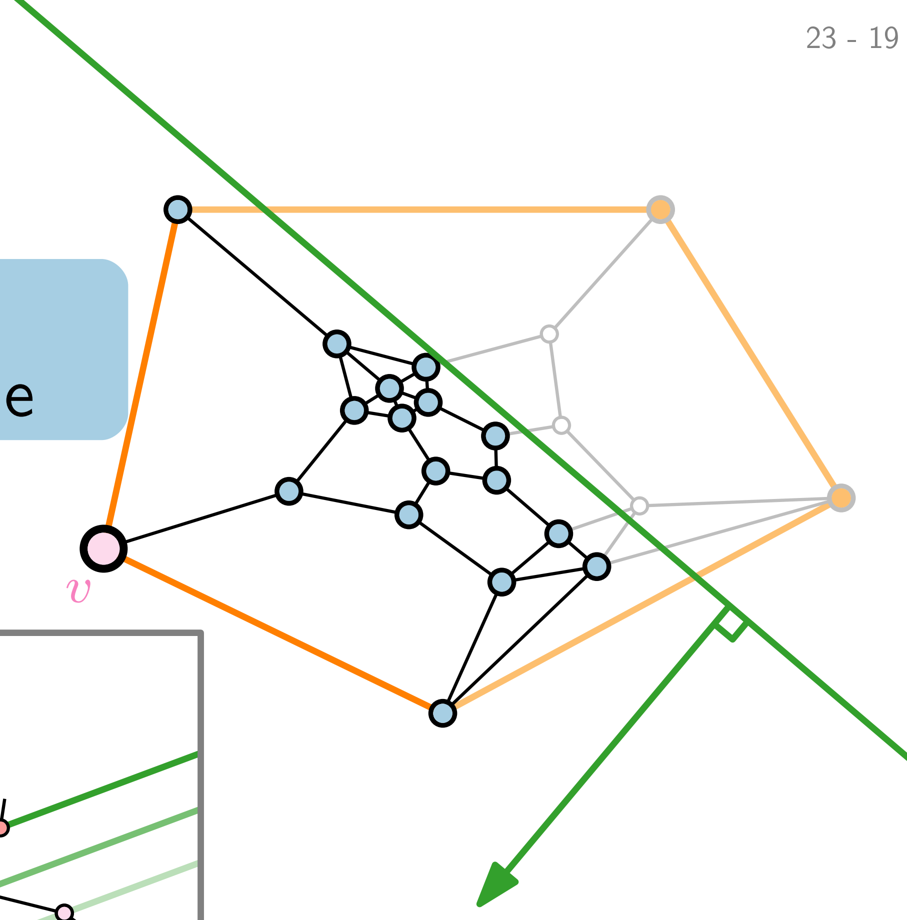
v furthest away from ℓ



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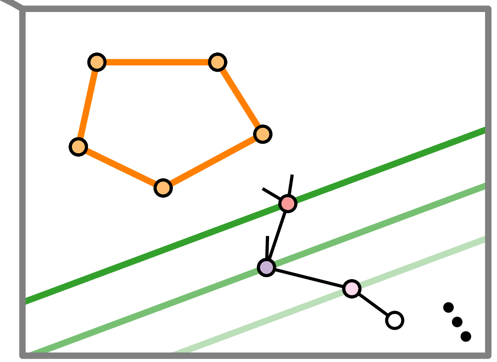

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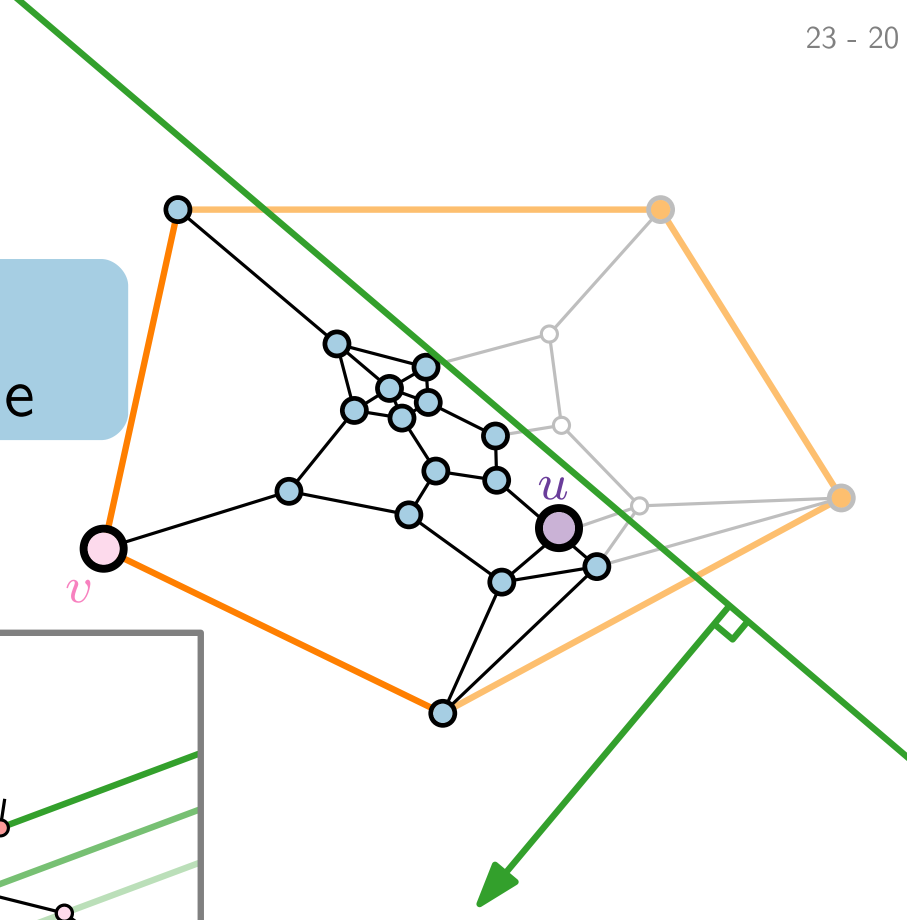
Pick any vertex u



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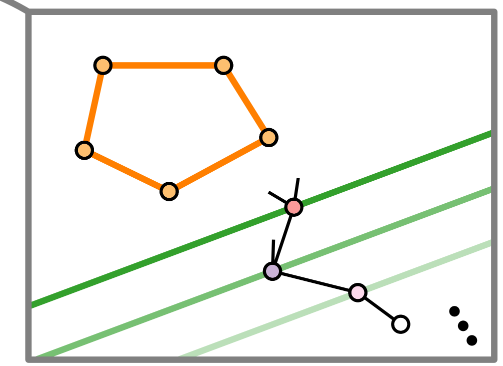

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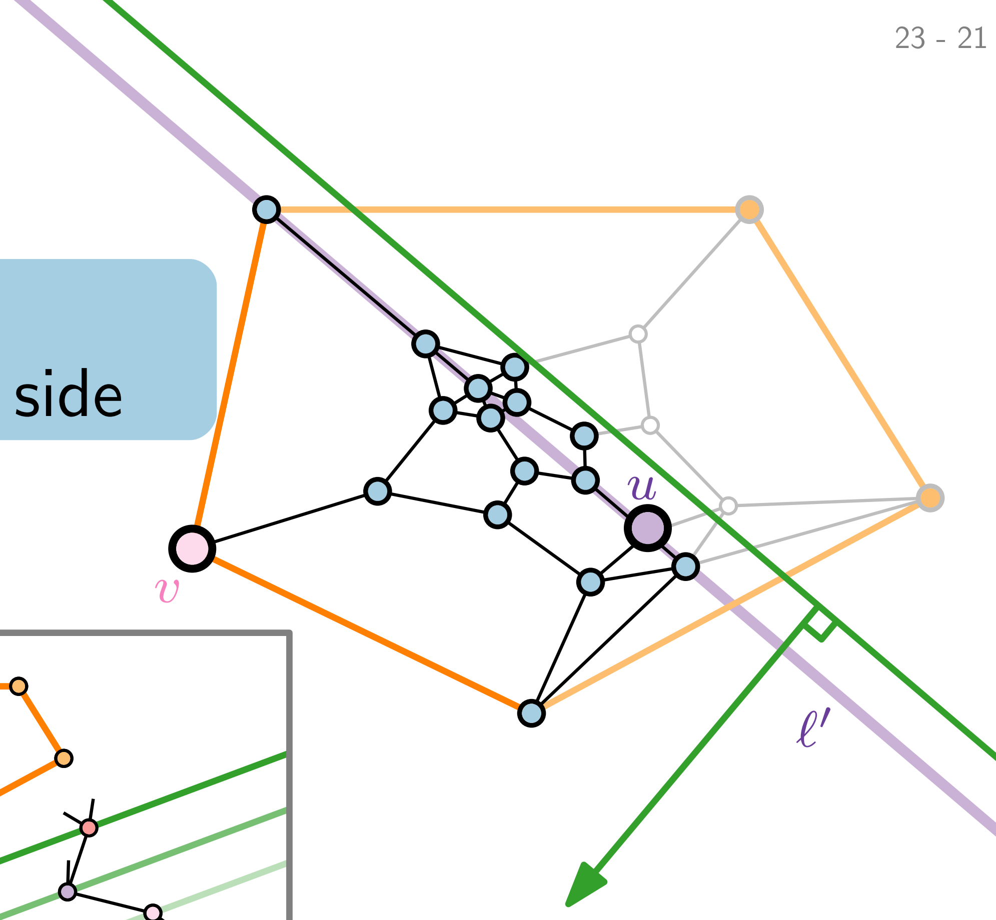
v furthest away from ℓ
Pick any vertex u, ℓ' parallel to ℓ throught u



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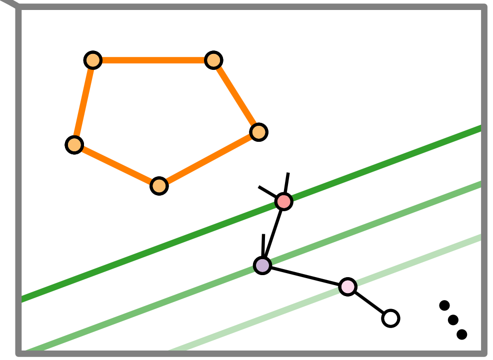
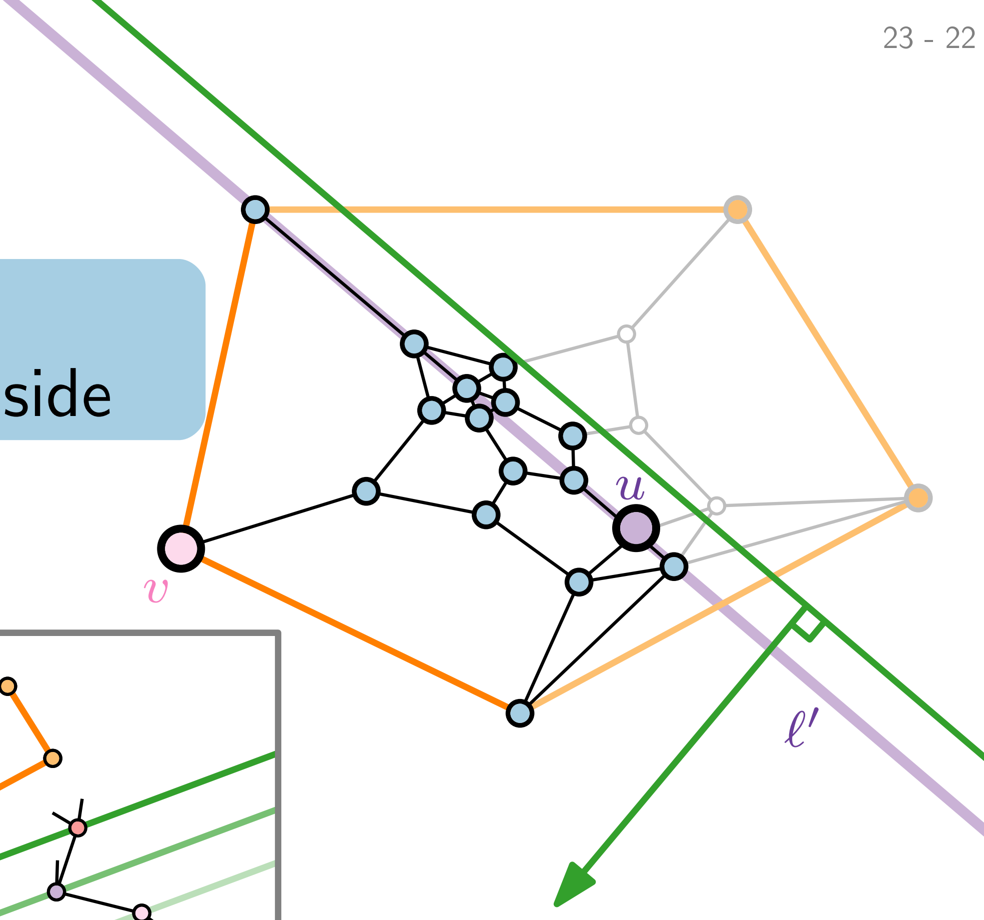
Pick any vertex u, ℓ' parallel to ℓ throught u
G connected, v not on ℓ'





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Otherwise, all forces to same side . . .

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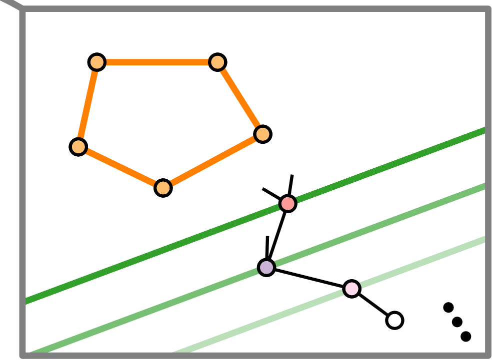

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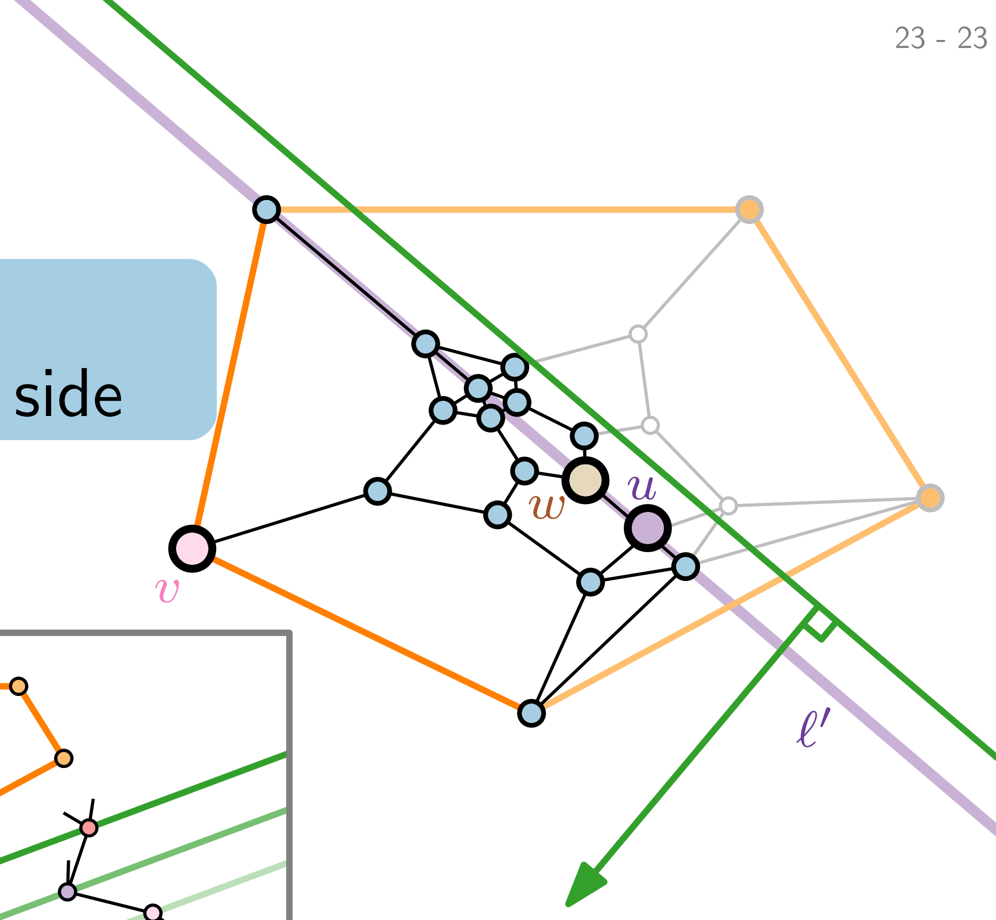

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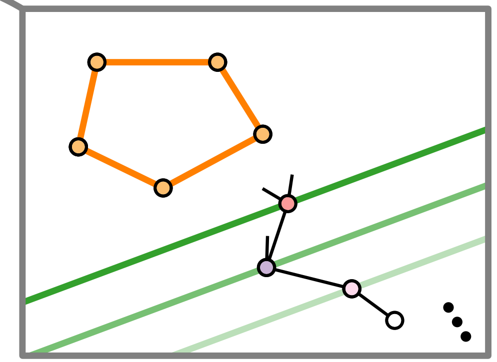

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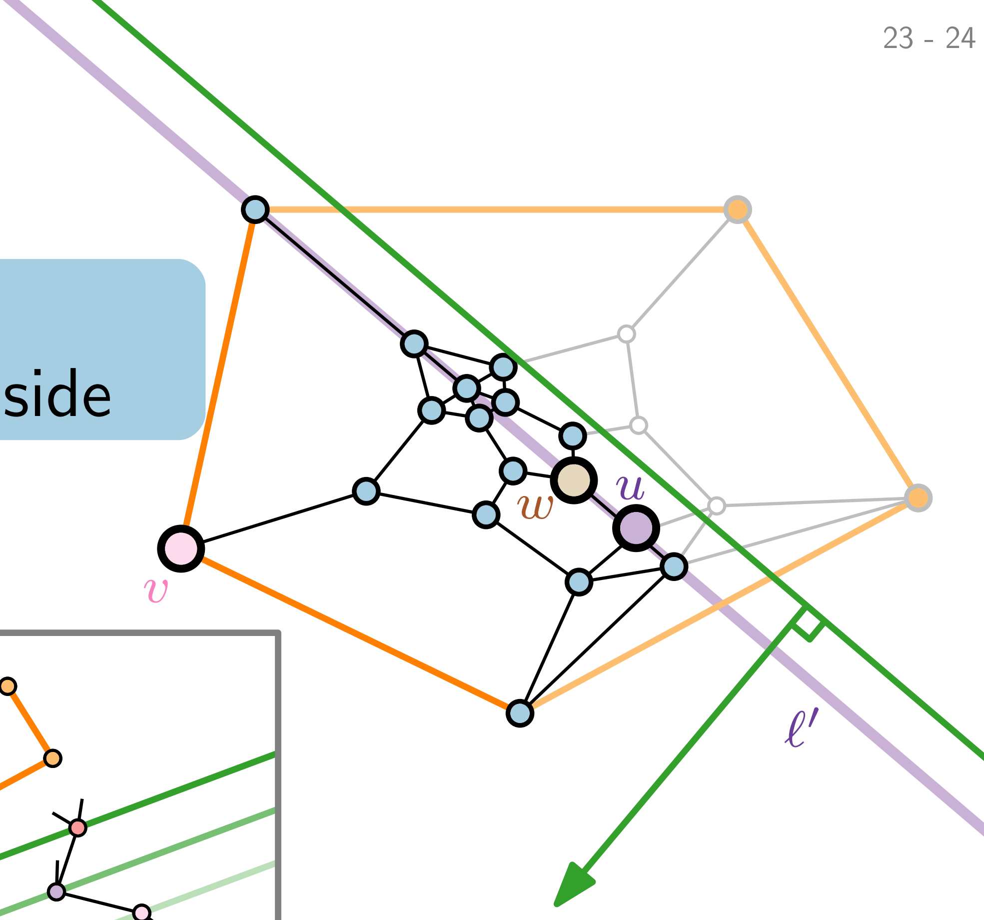

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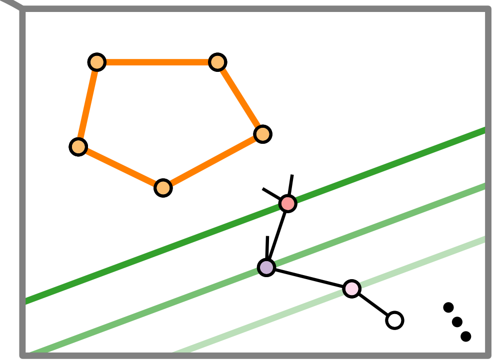

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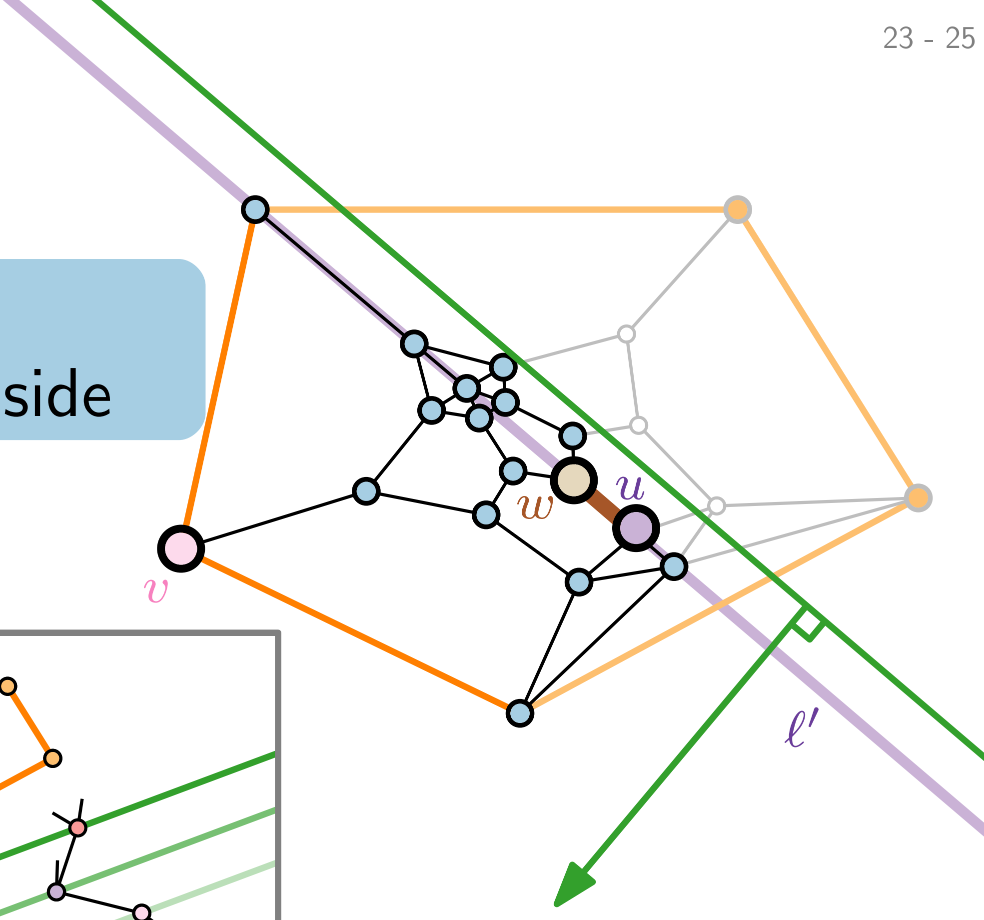

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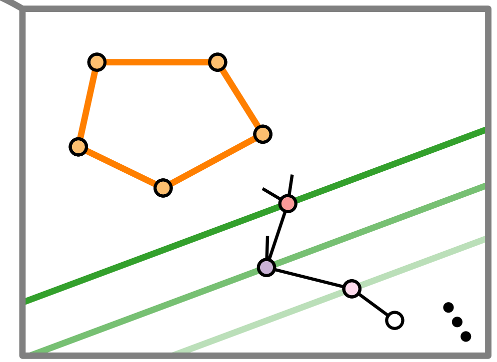

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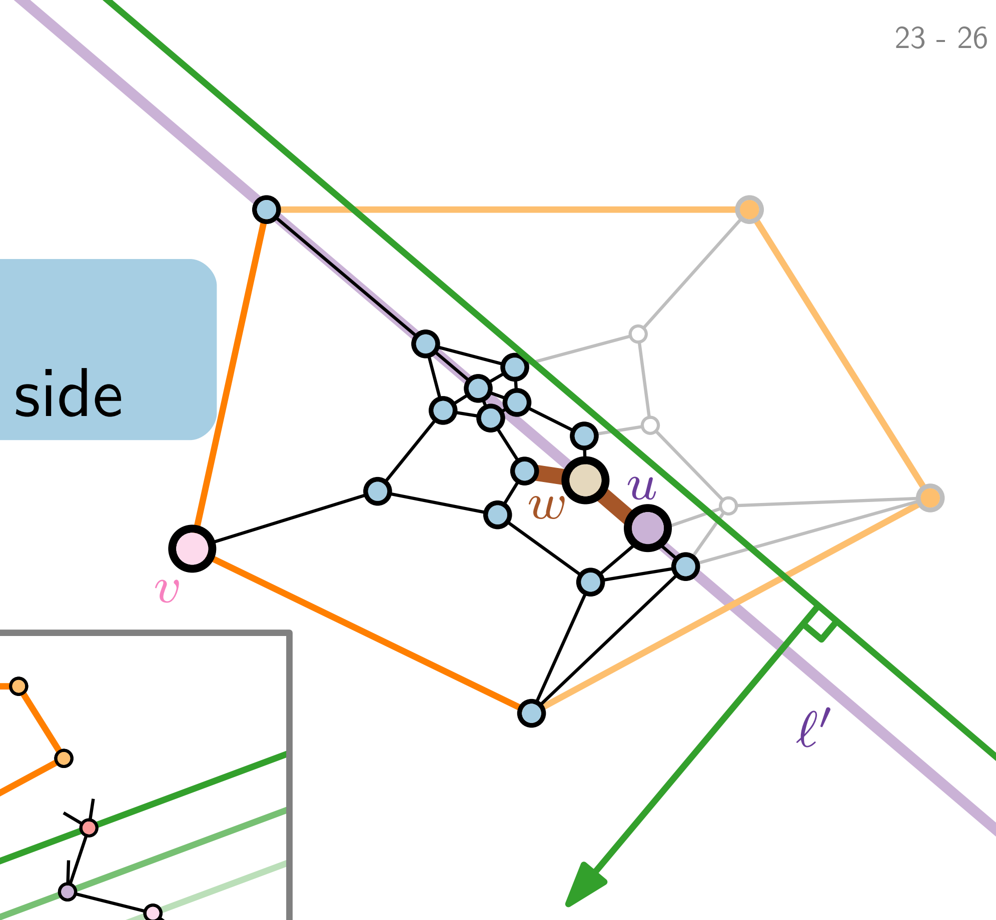

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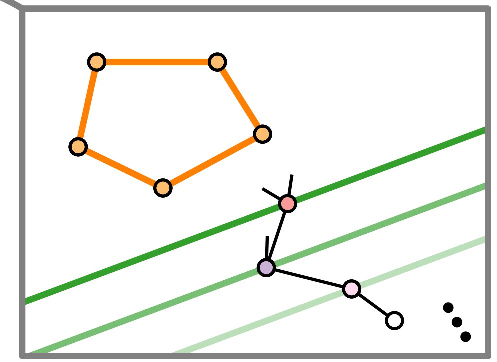

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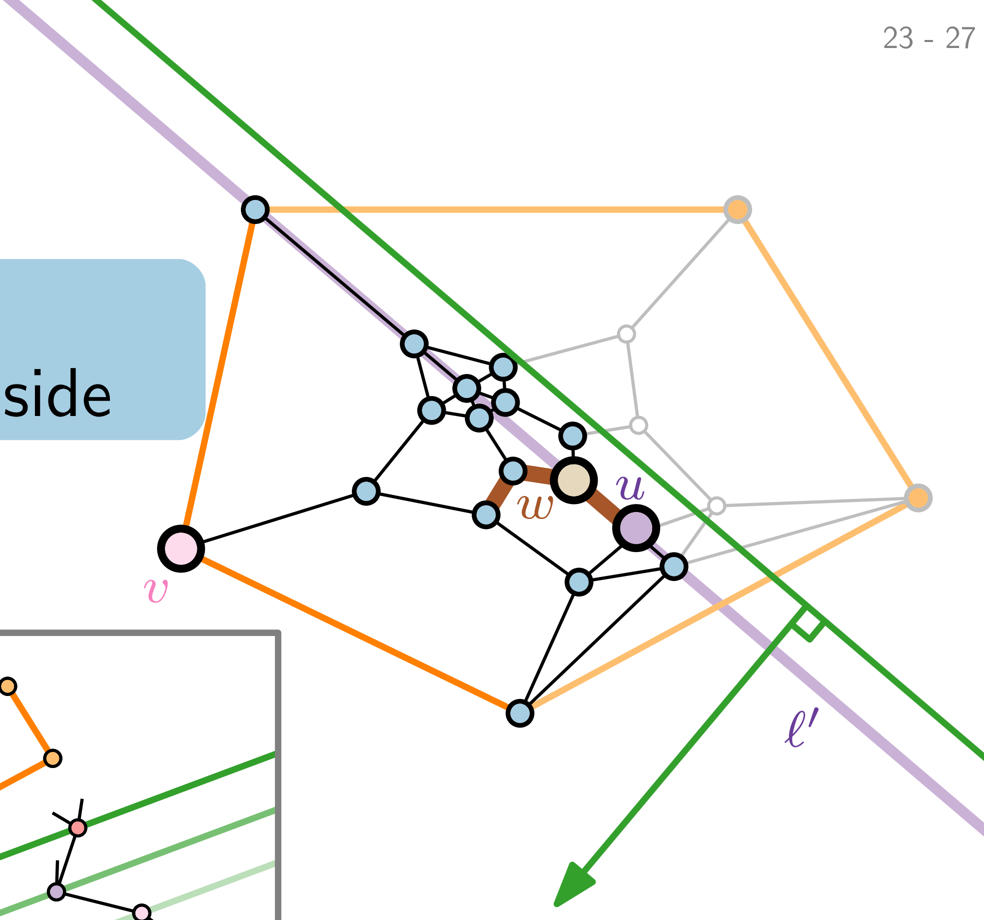

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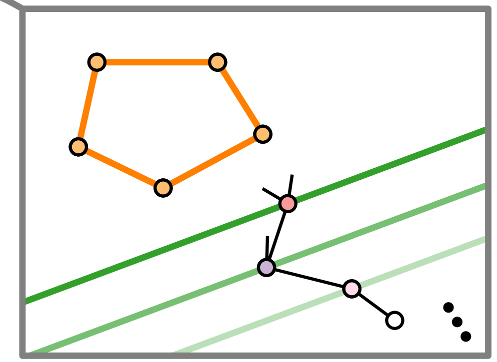

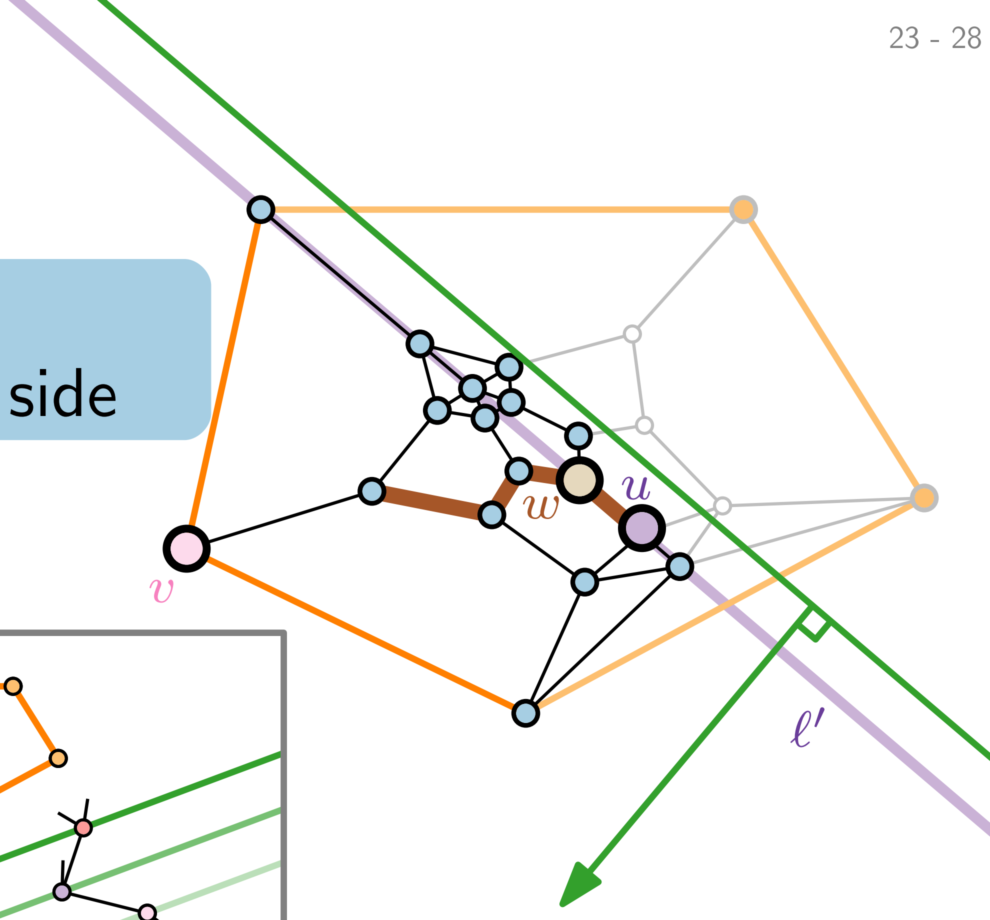

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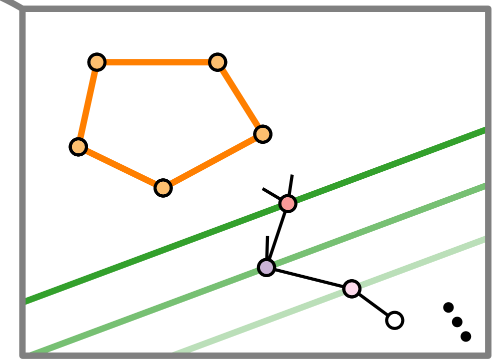

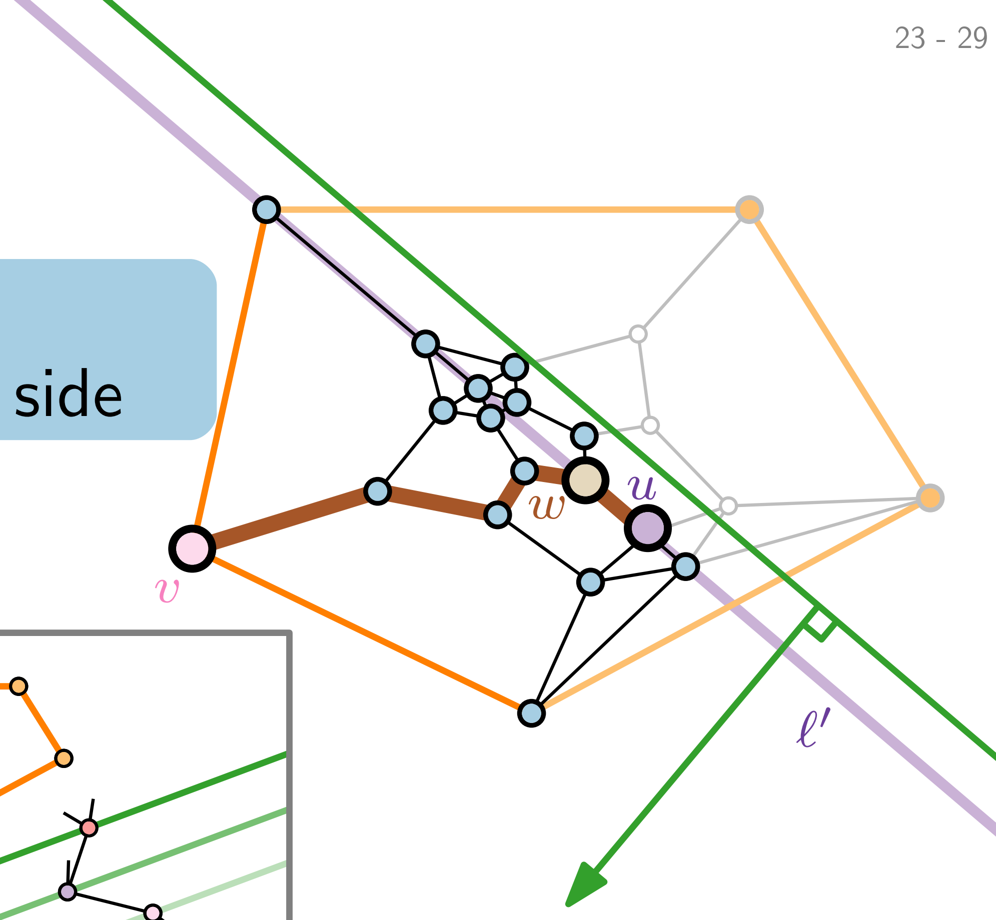

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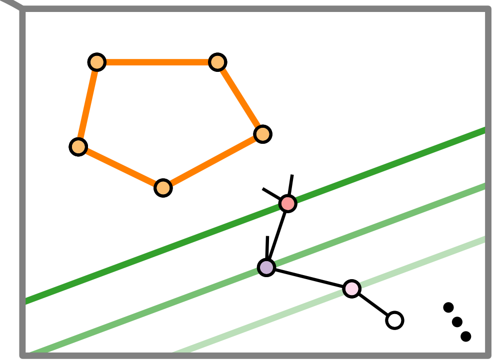

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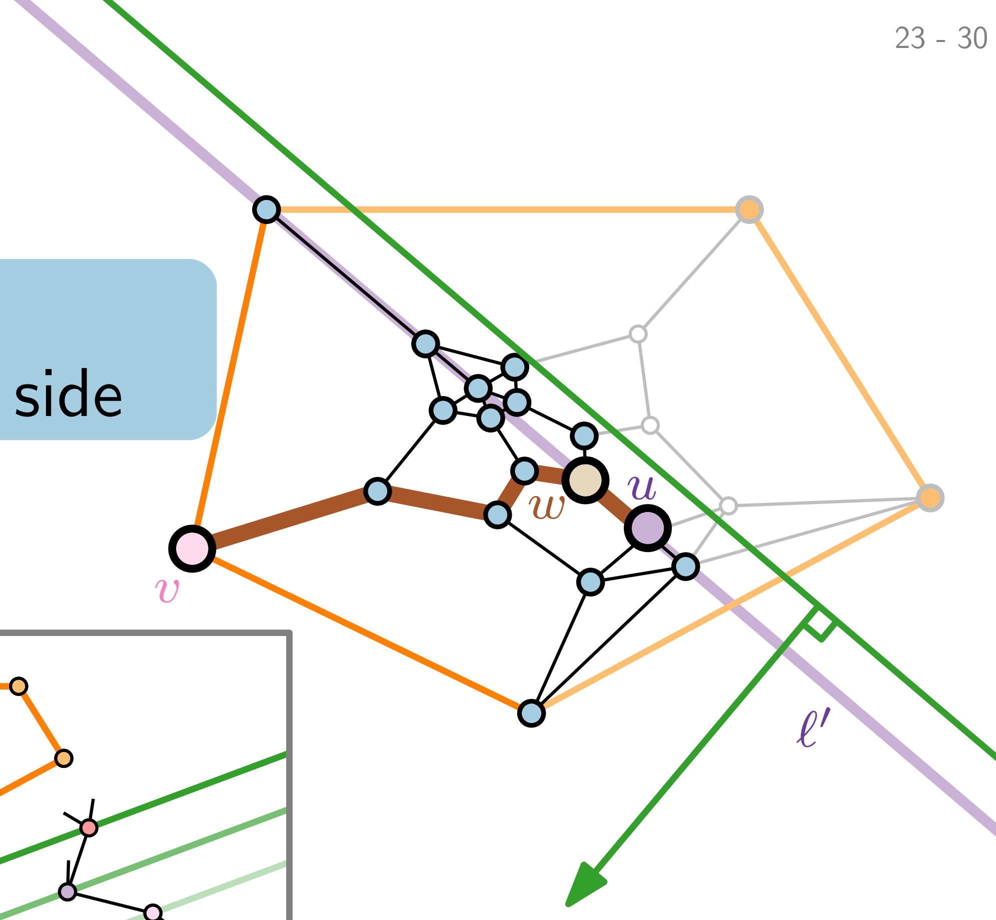

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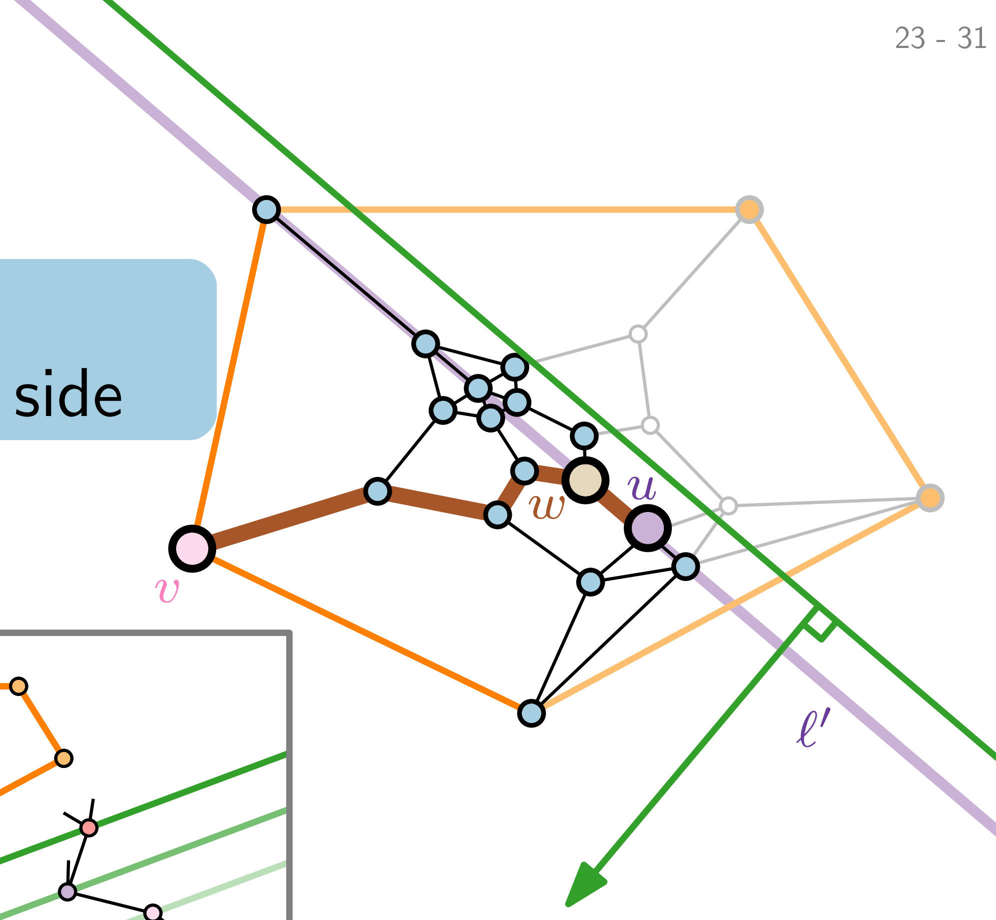

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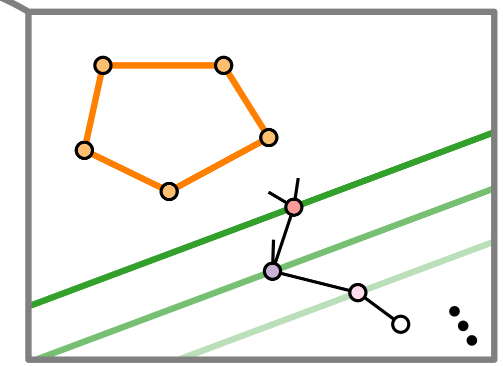

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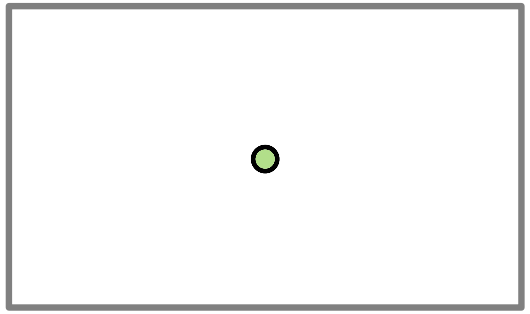

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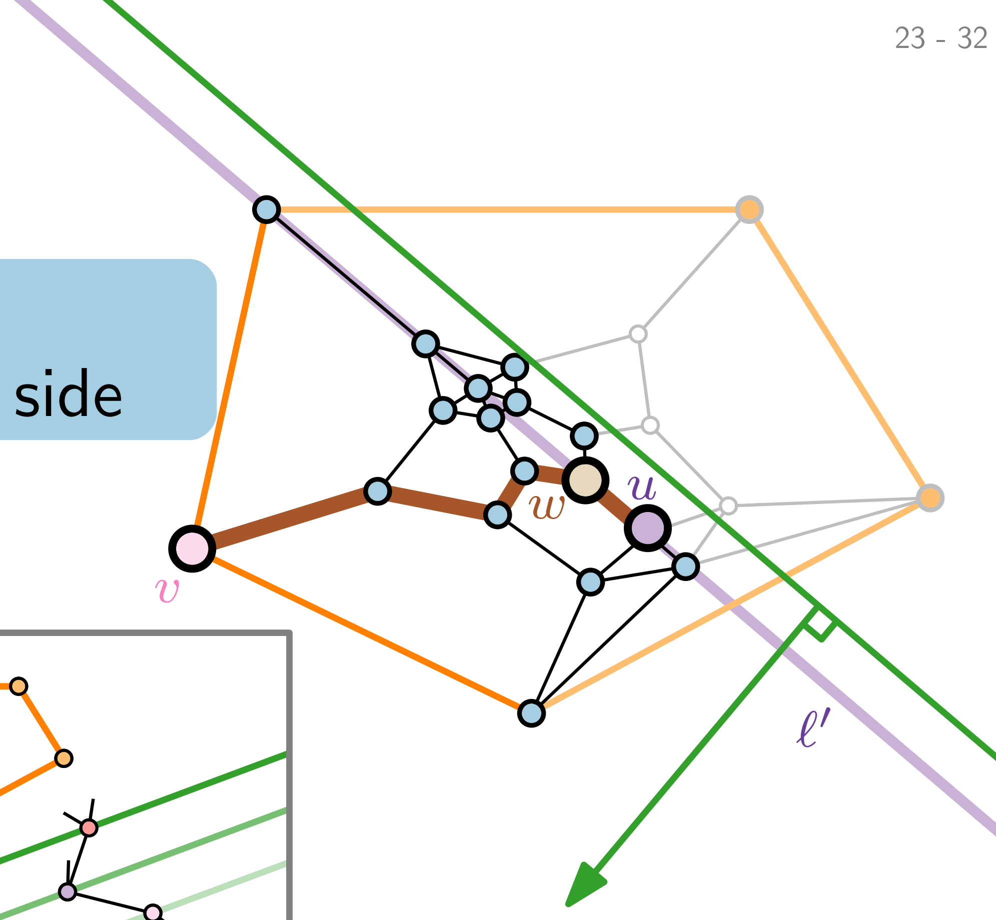

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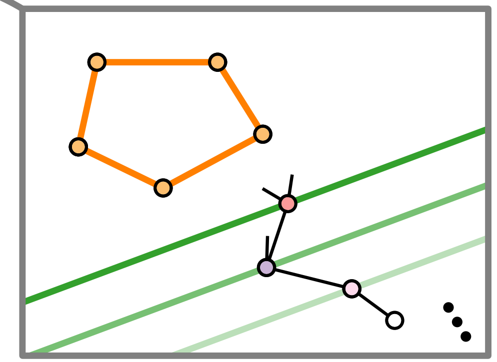

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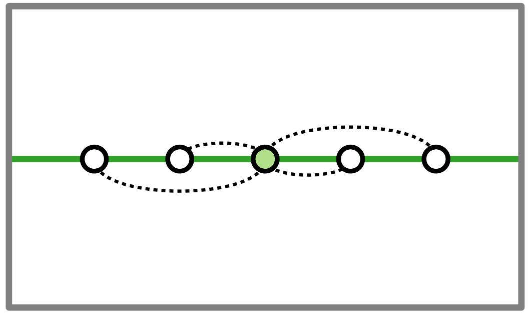

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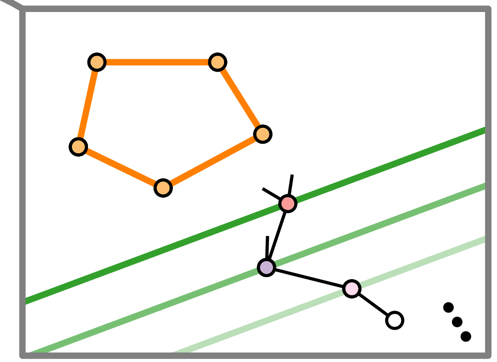

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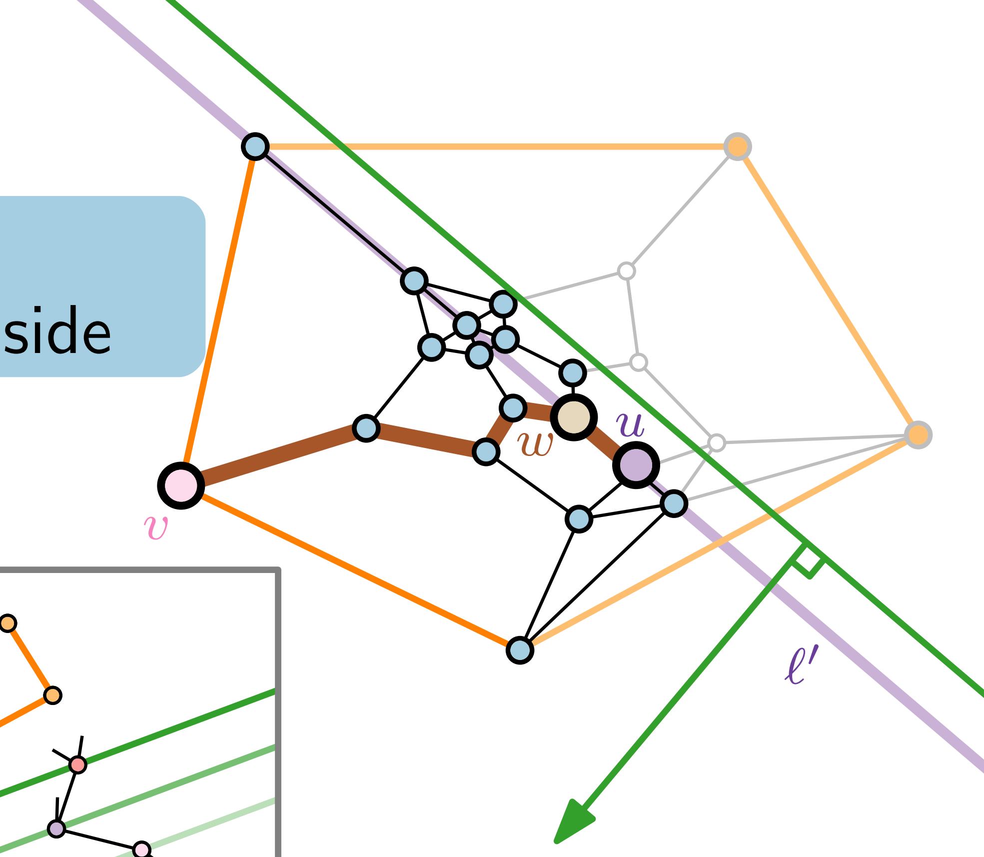

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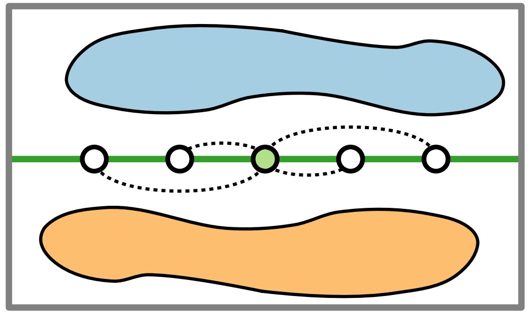

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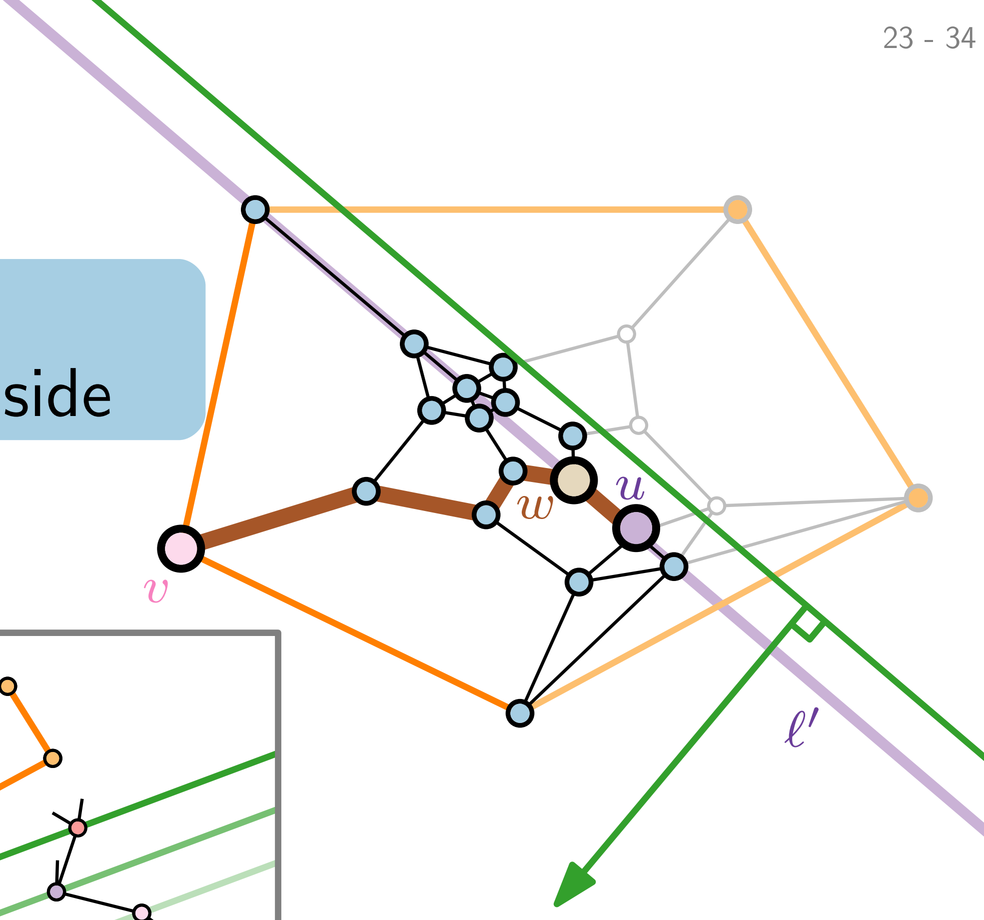

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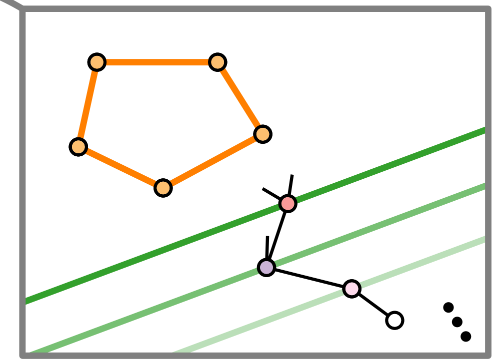

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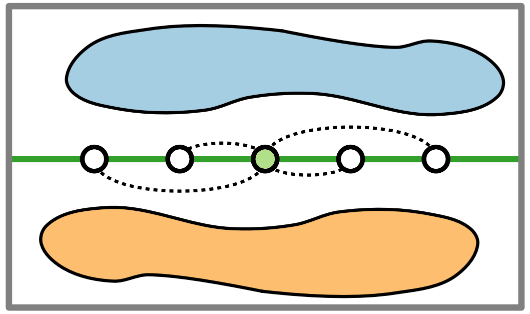

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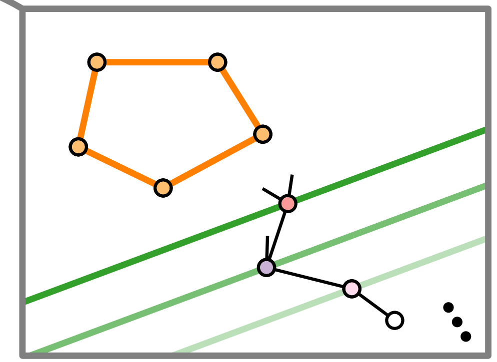

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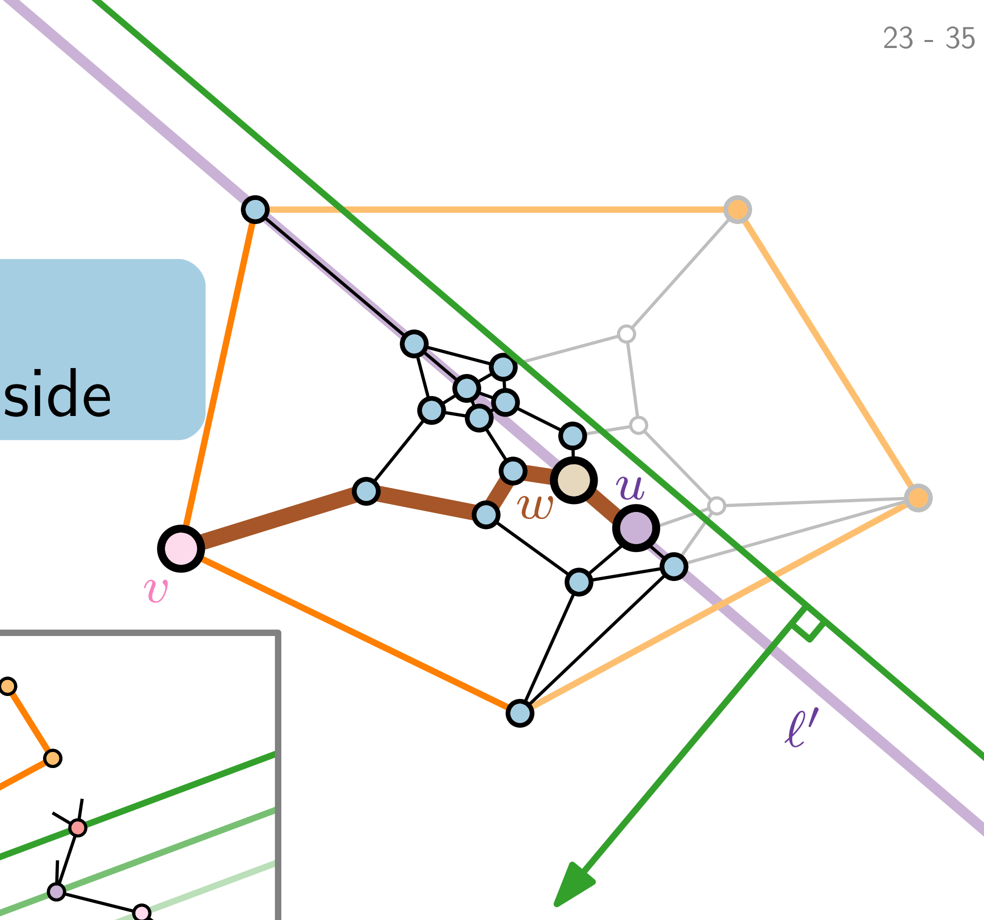

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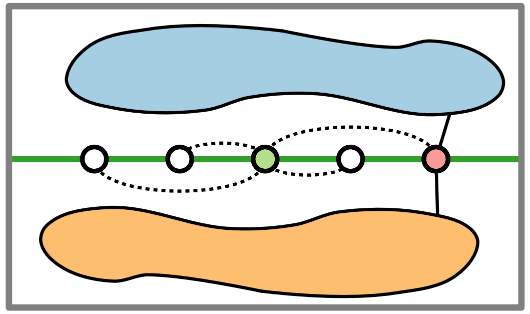

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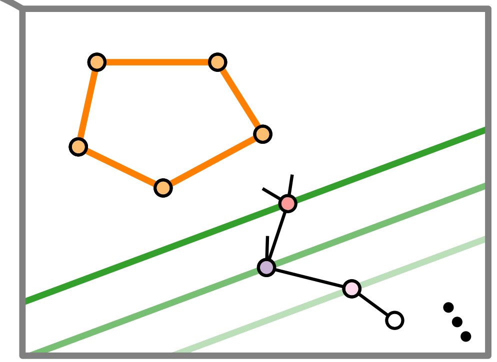

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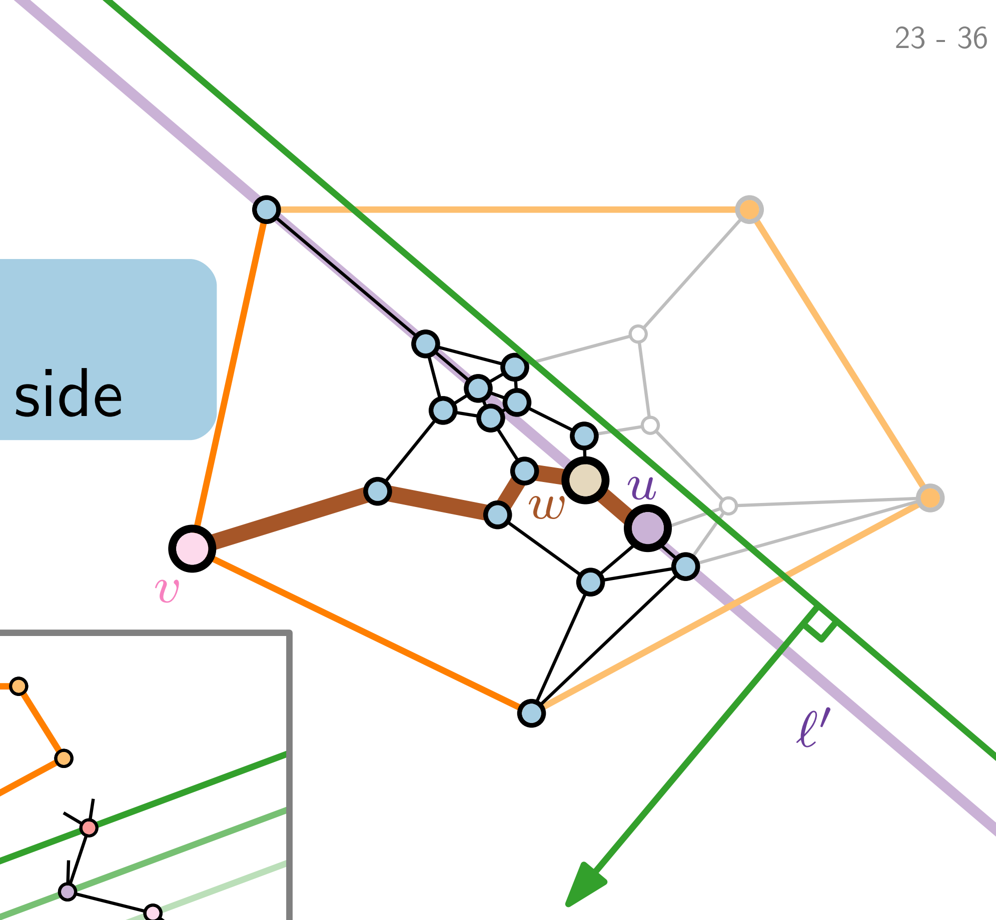

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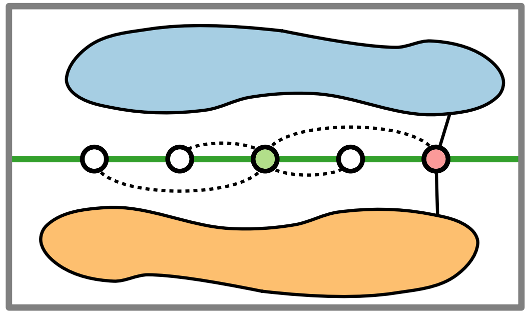
G 3-connected



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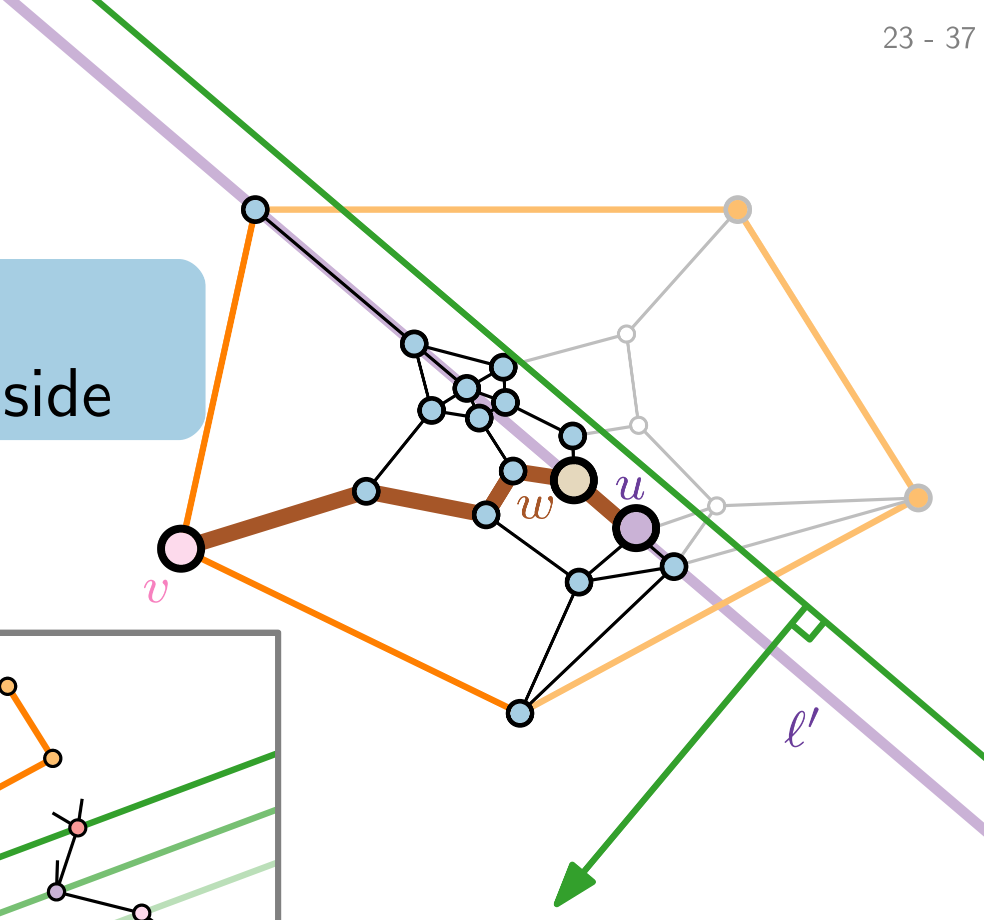

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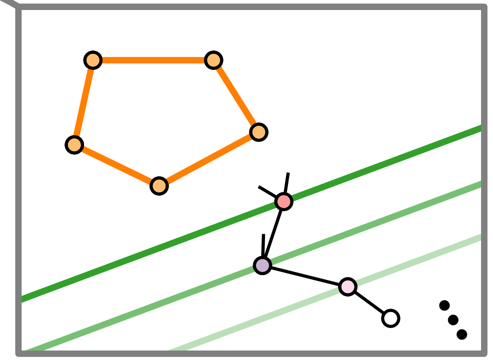

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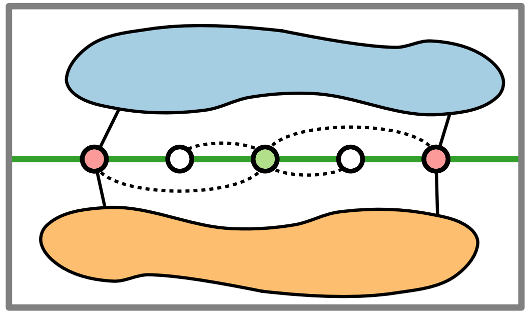

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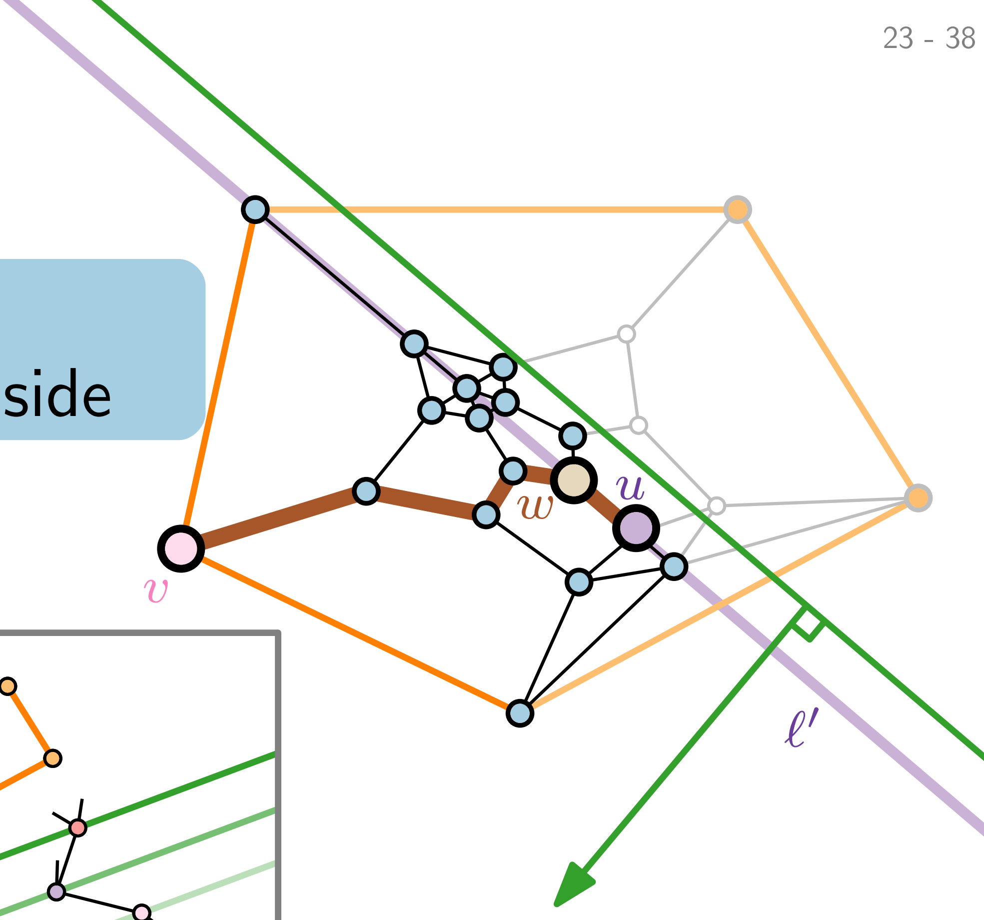

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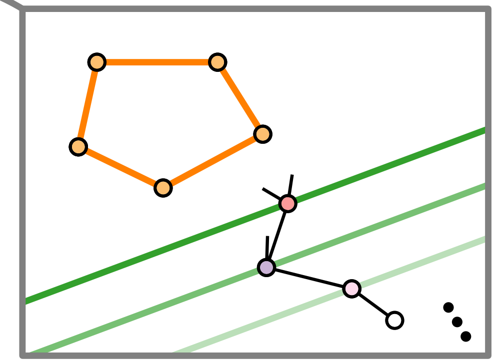

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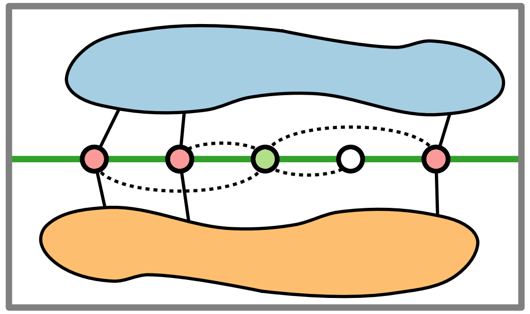

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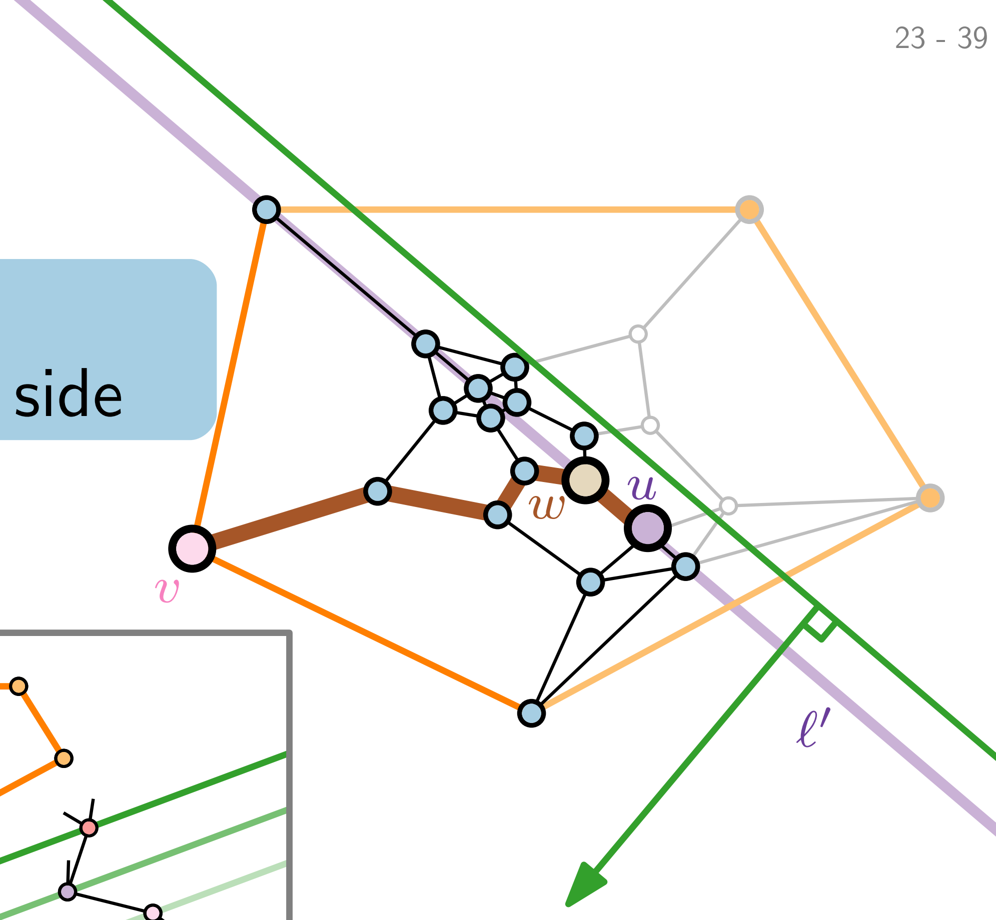

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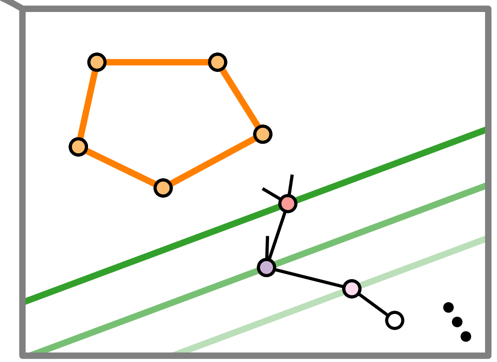

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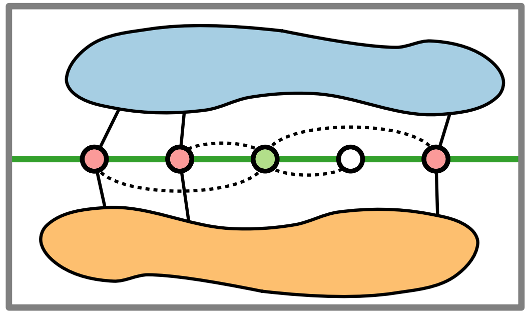
$\Rightarrow K_{3,3}$ minor



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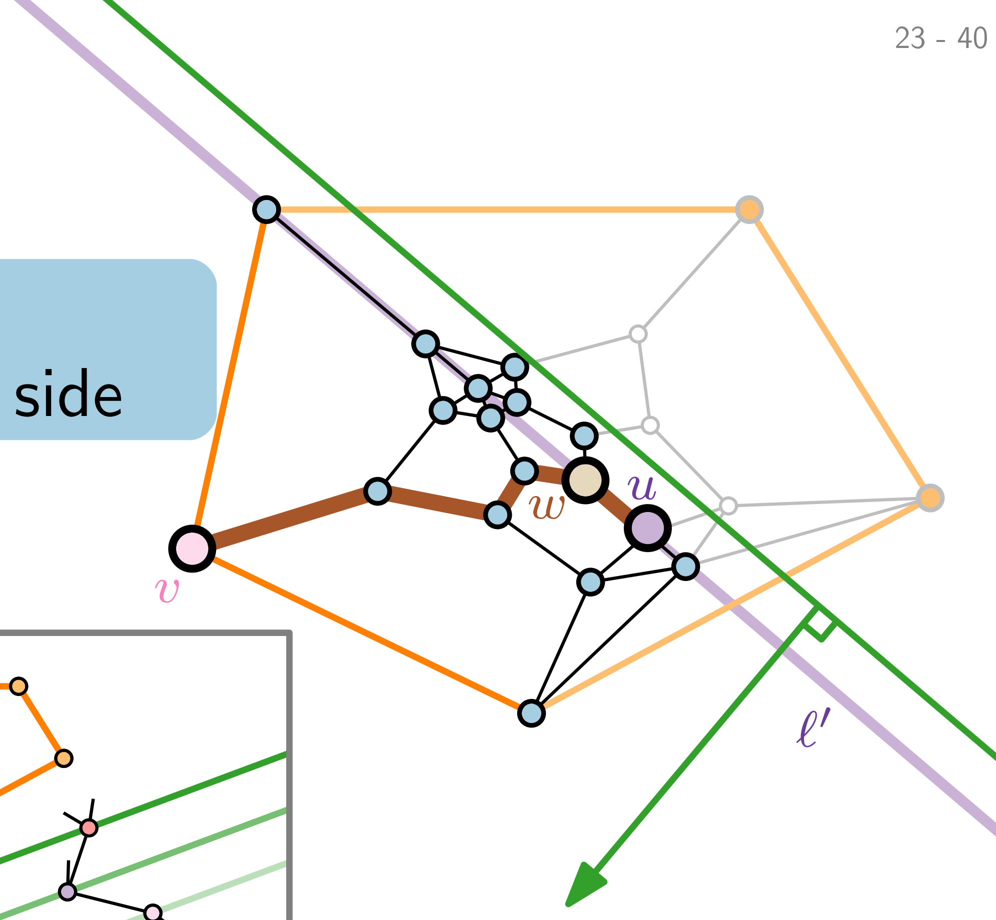

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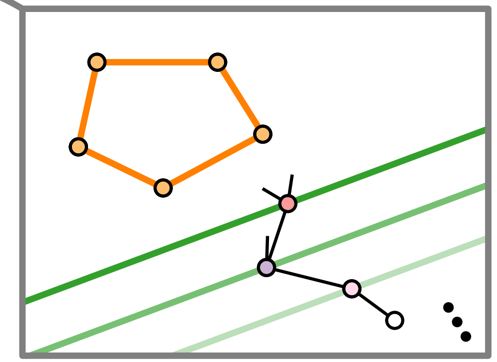

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v furthest away from ℓ
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Not all vertices collinear
G 3-connected
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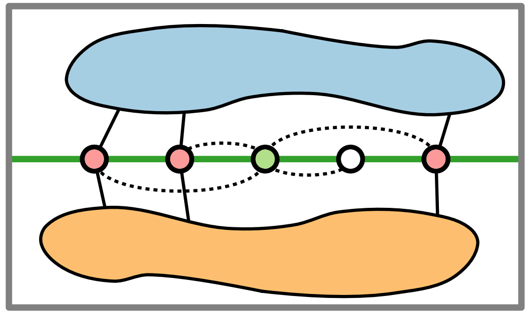
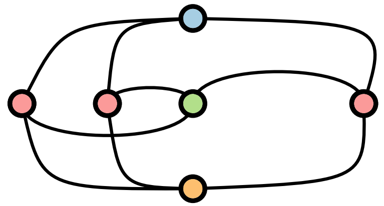
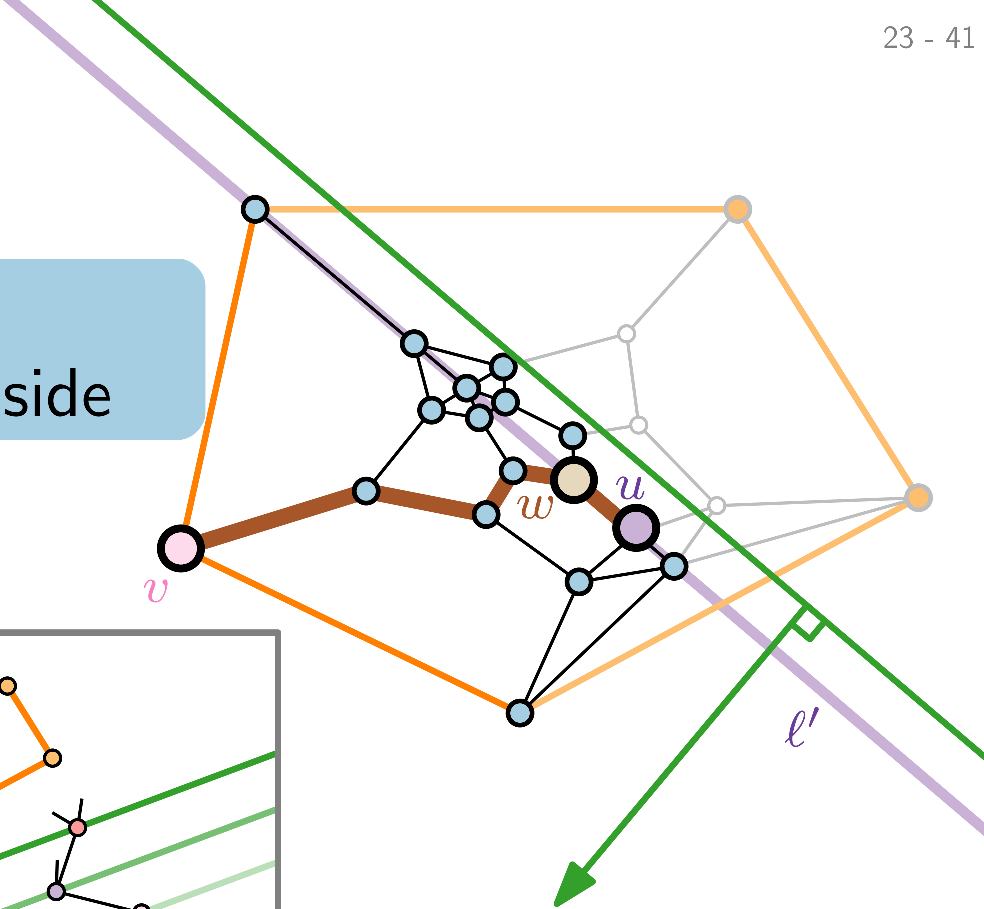




Properties of Tutte Drawings



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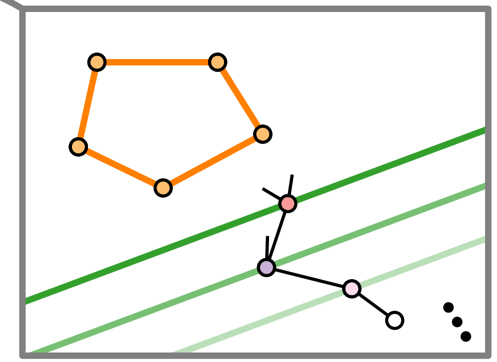

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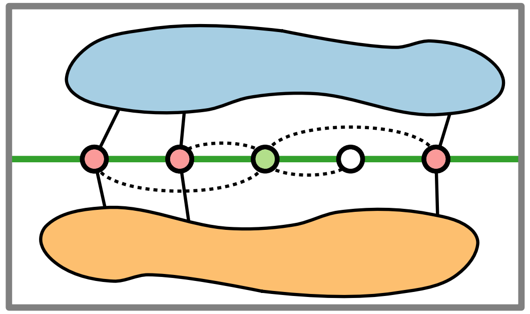

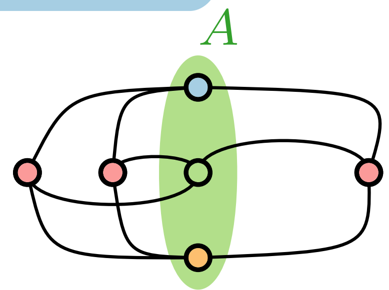

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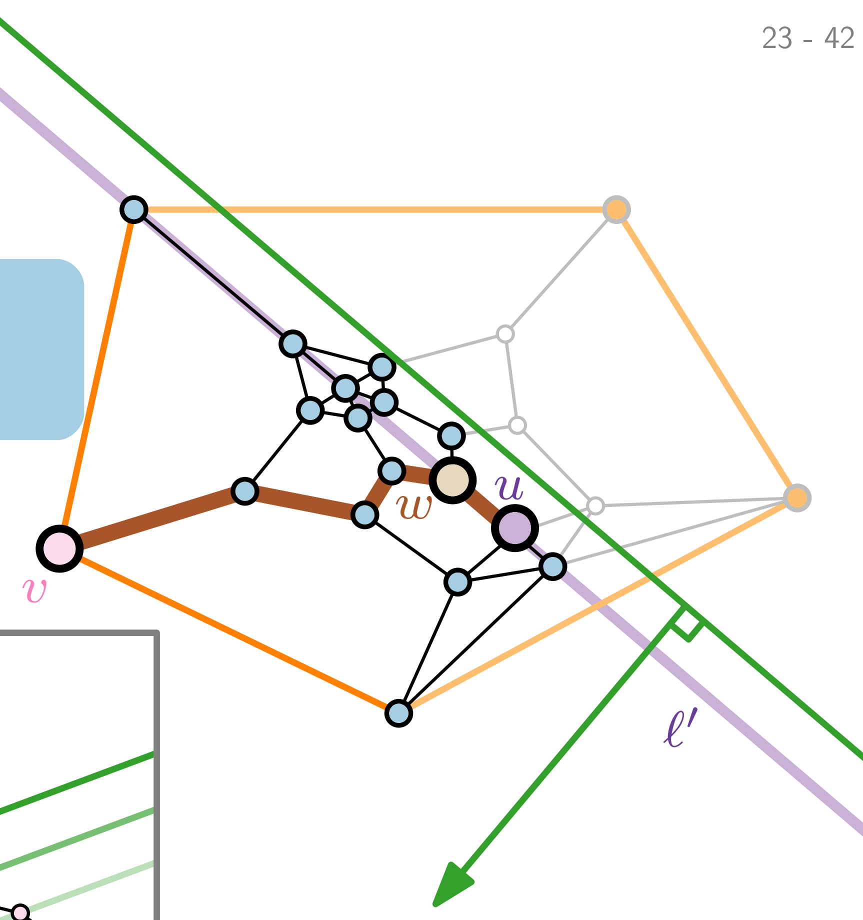

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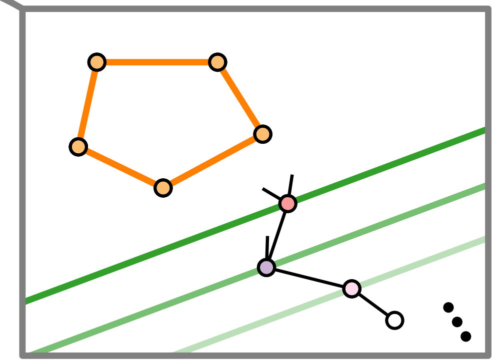

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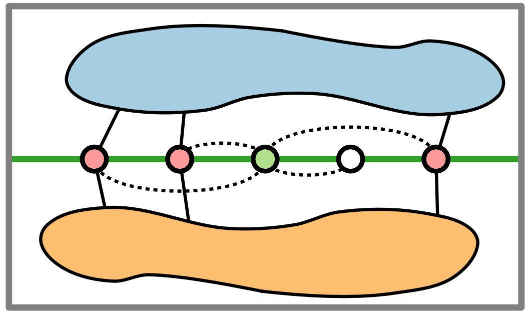

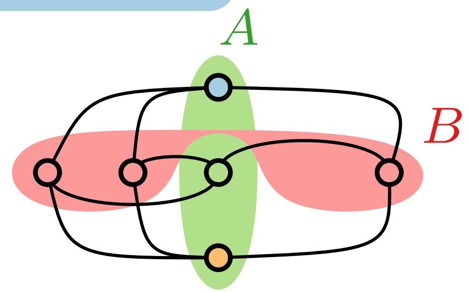

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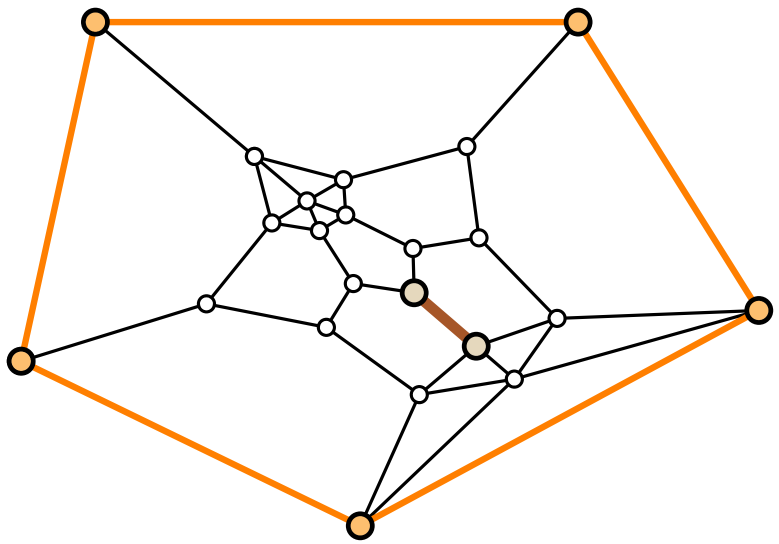

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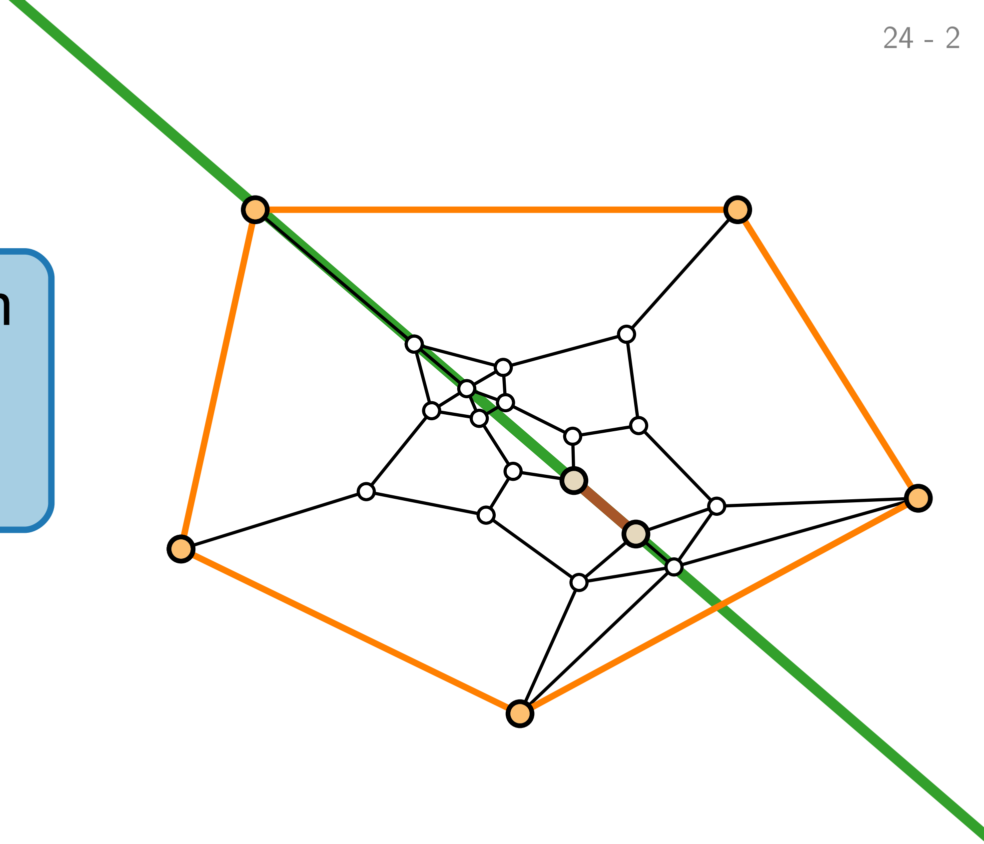

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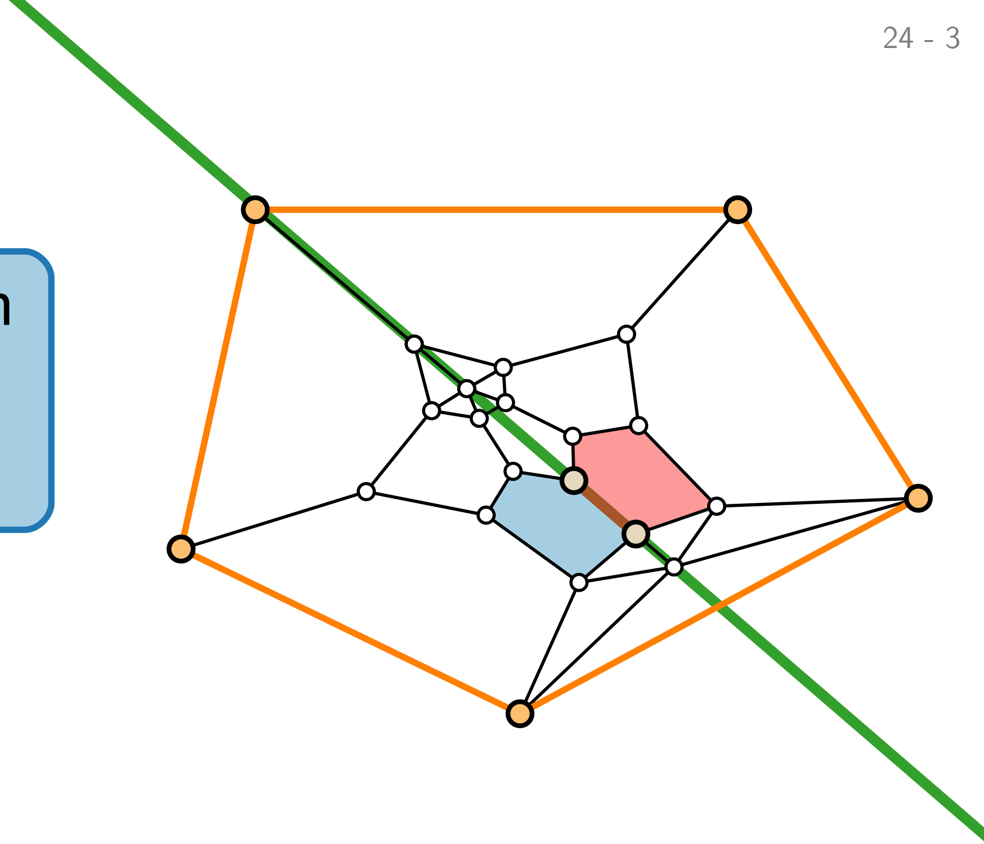

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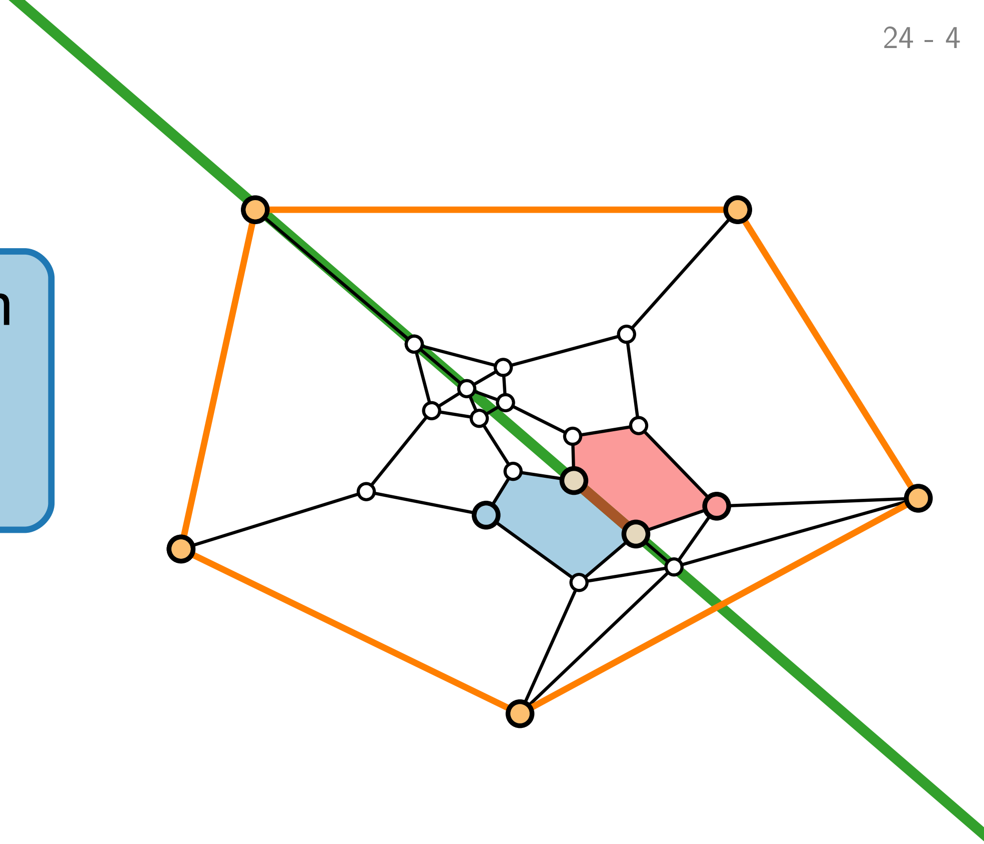

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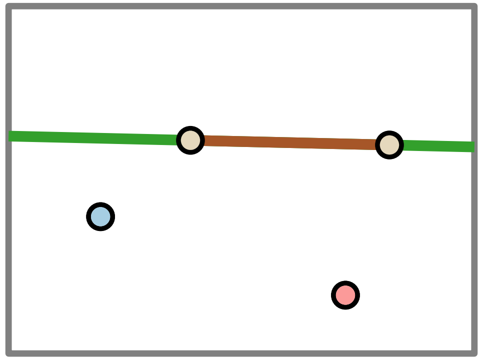

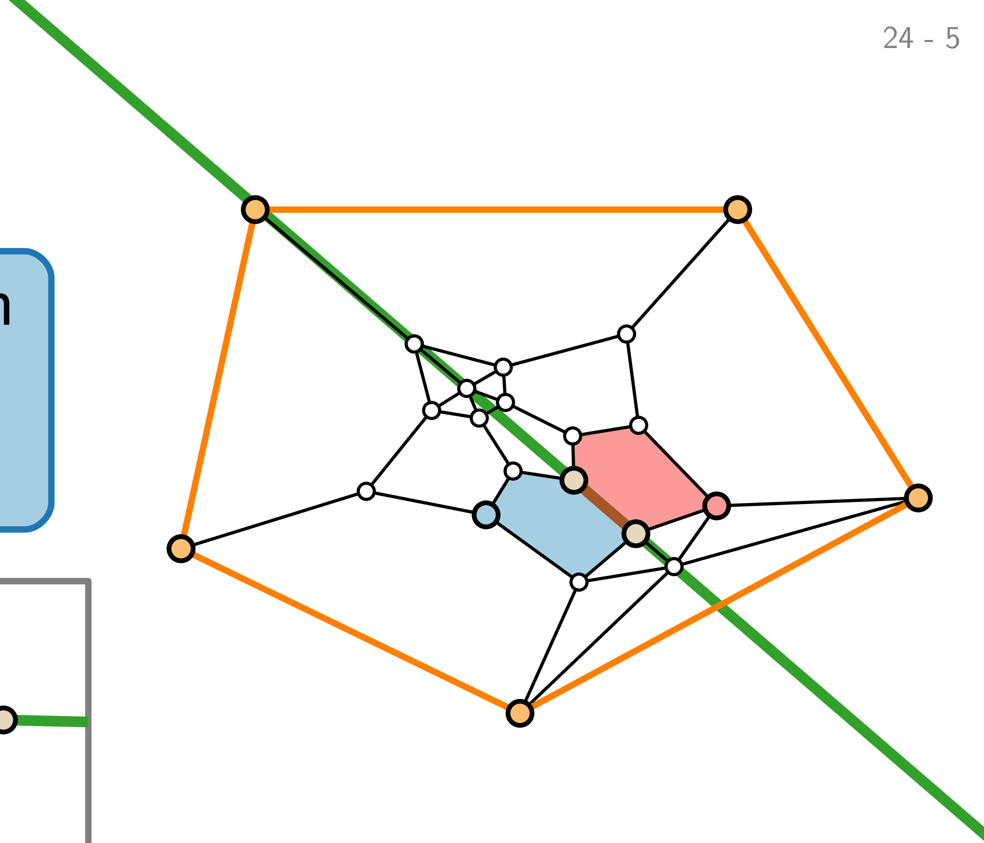

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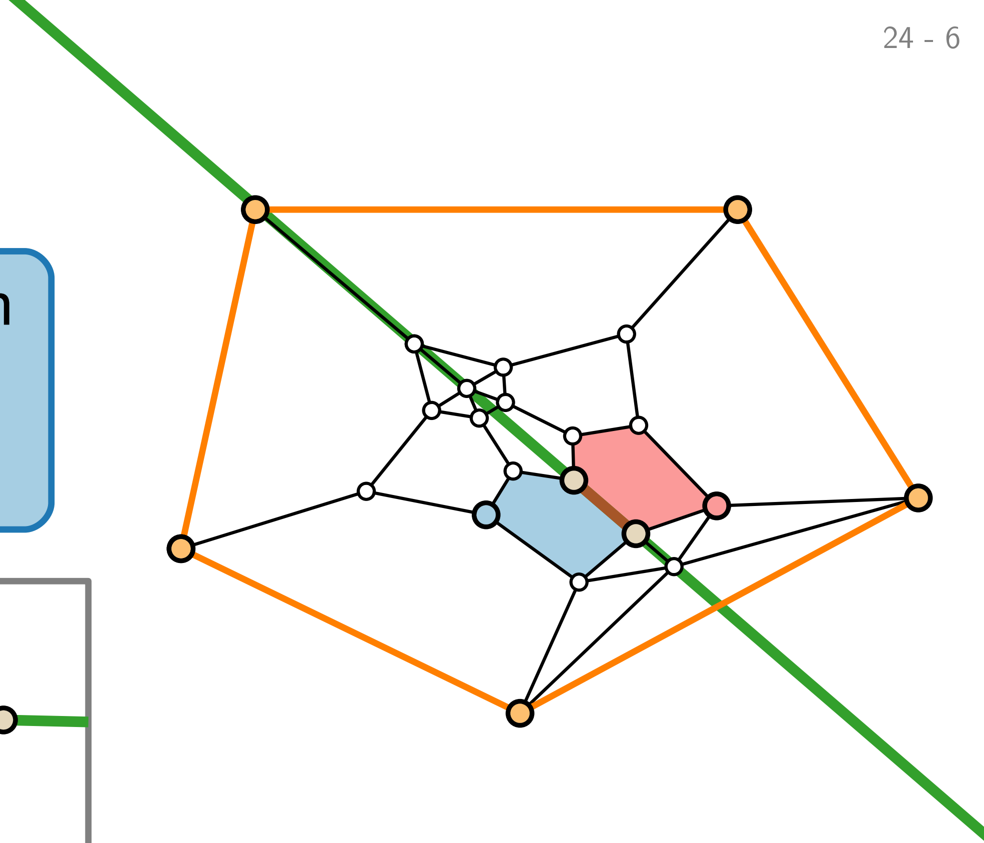

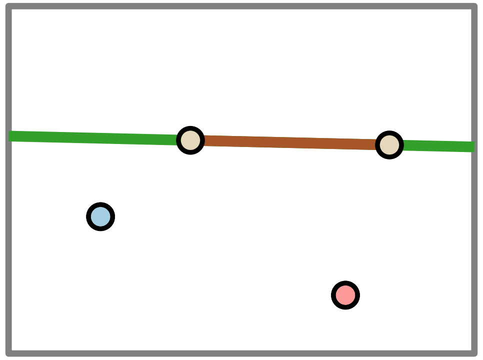

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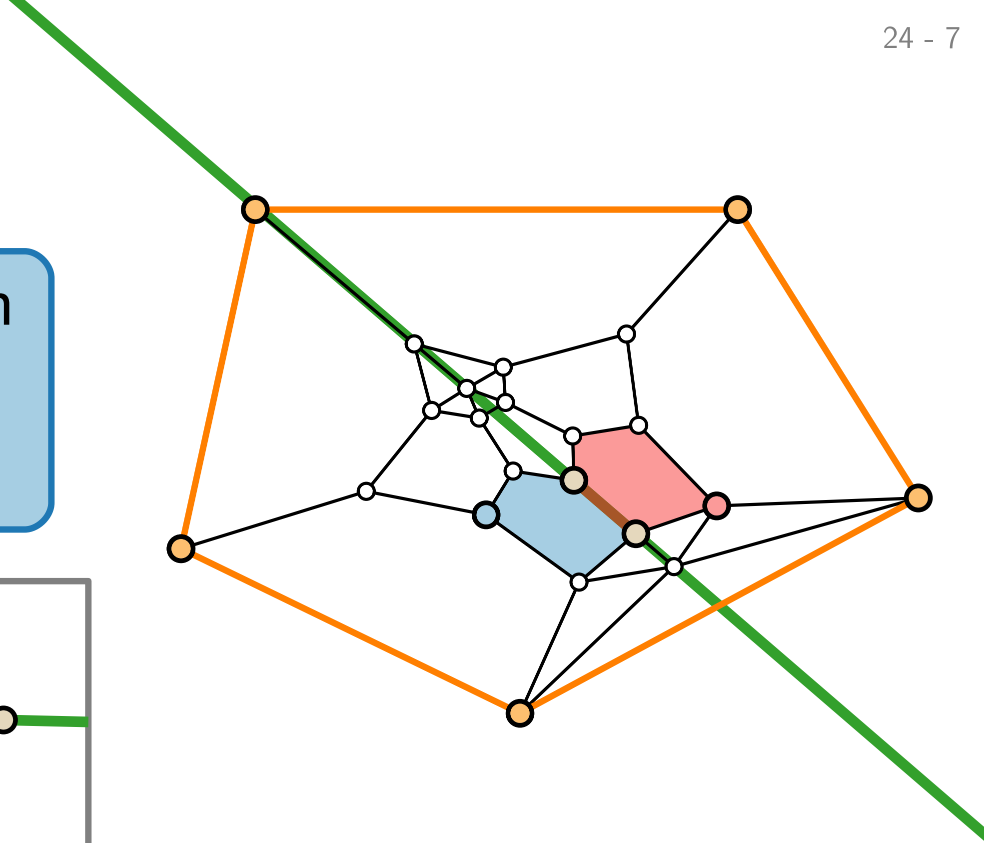

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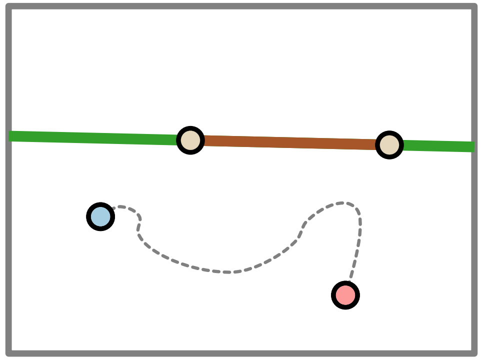

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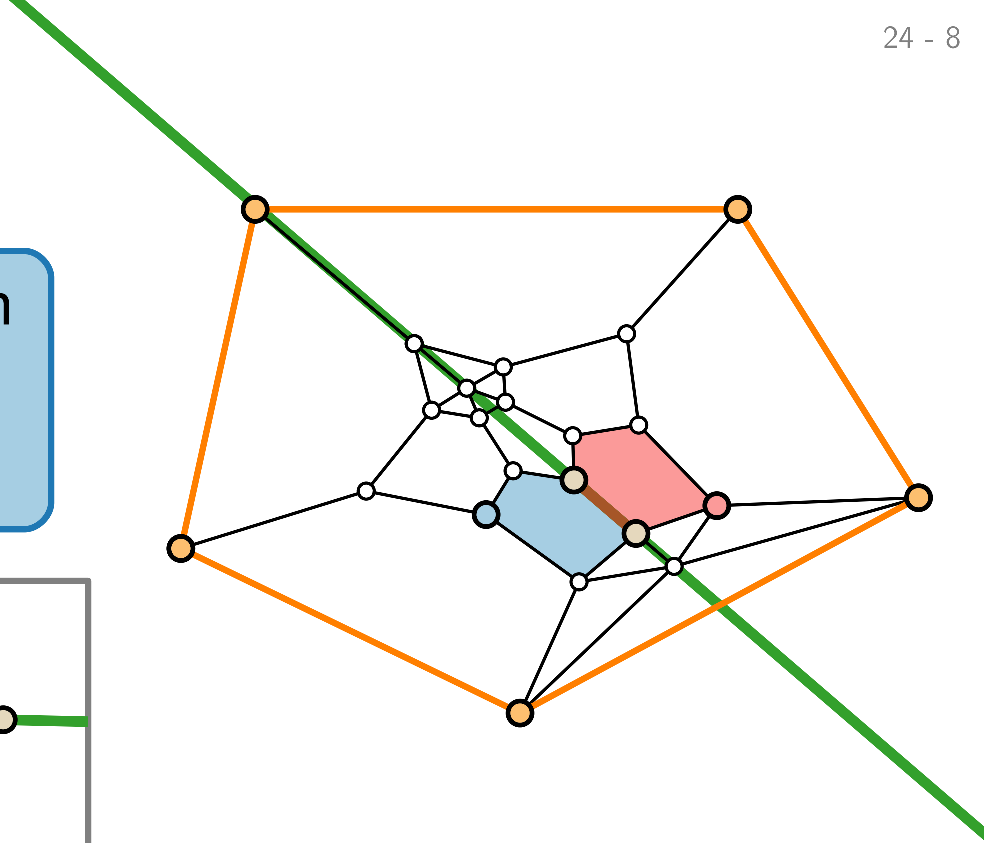

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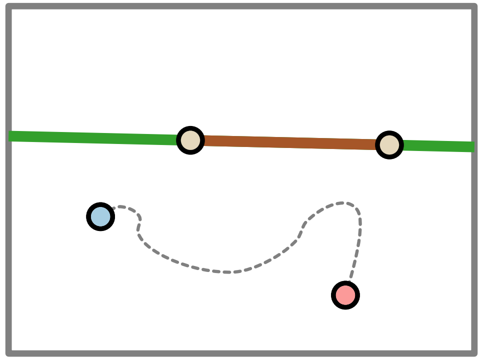

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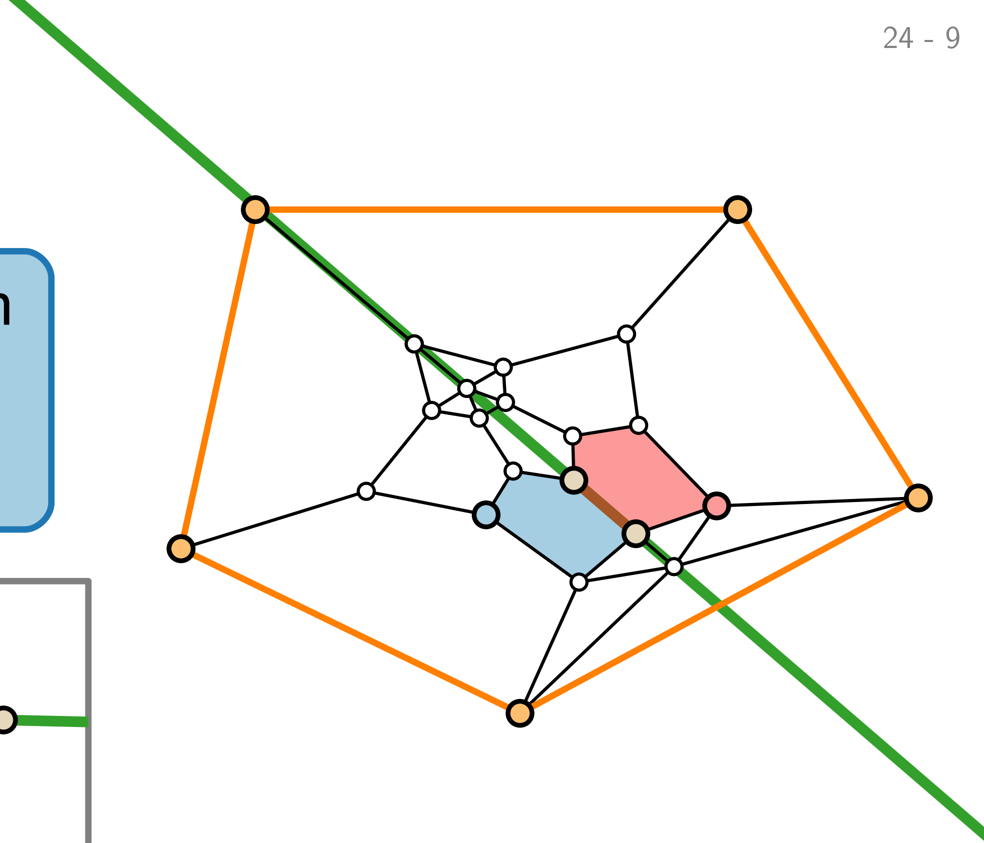

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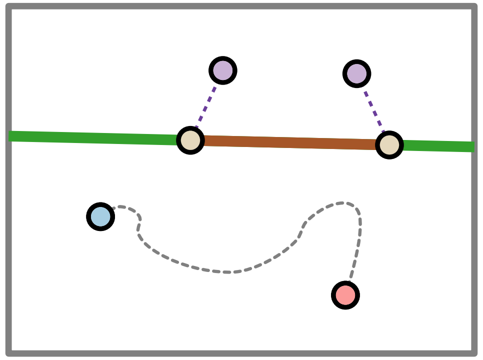

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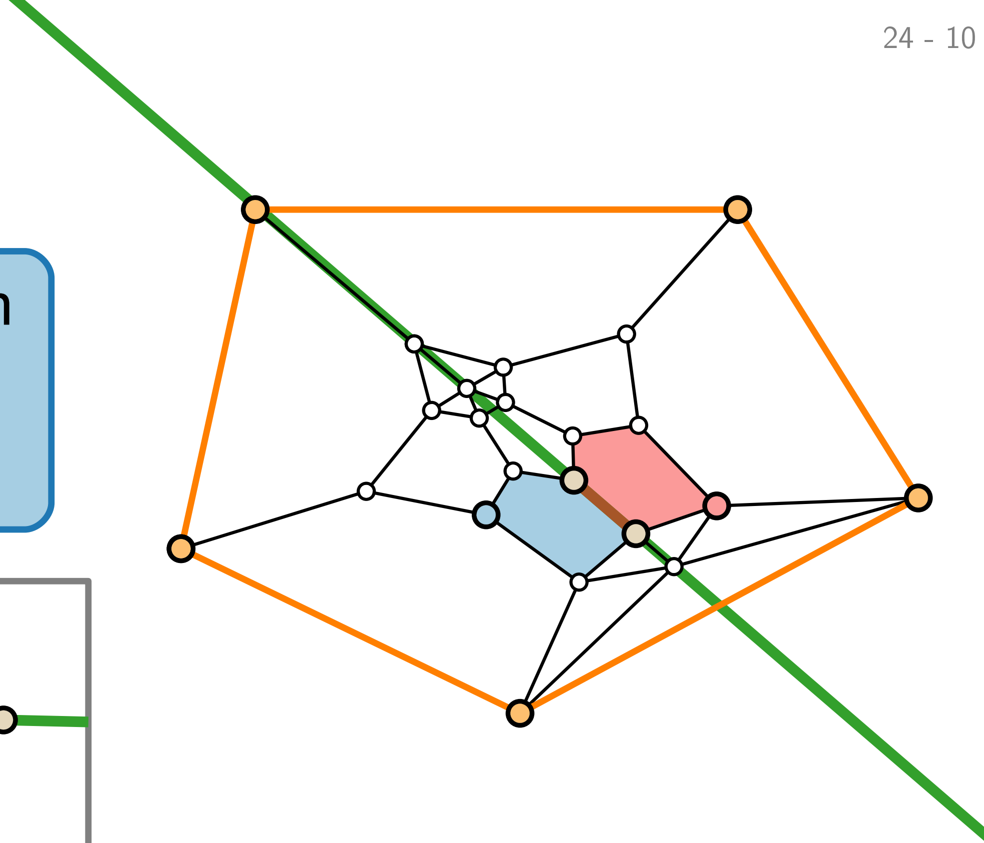

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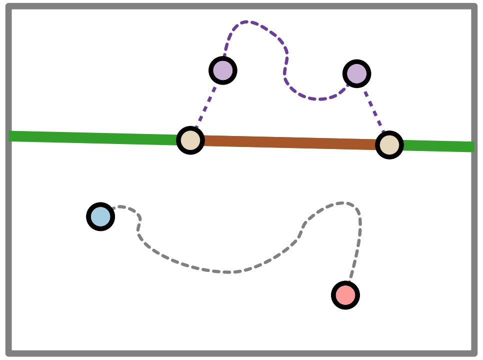

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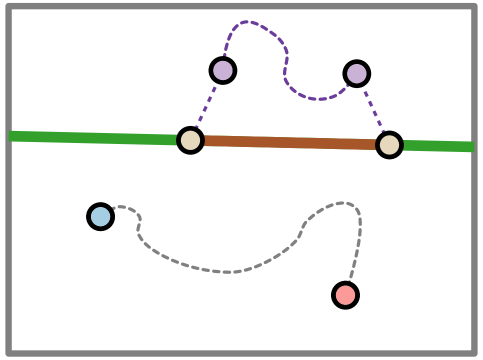

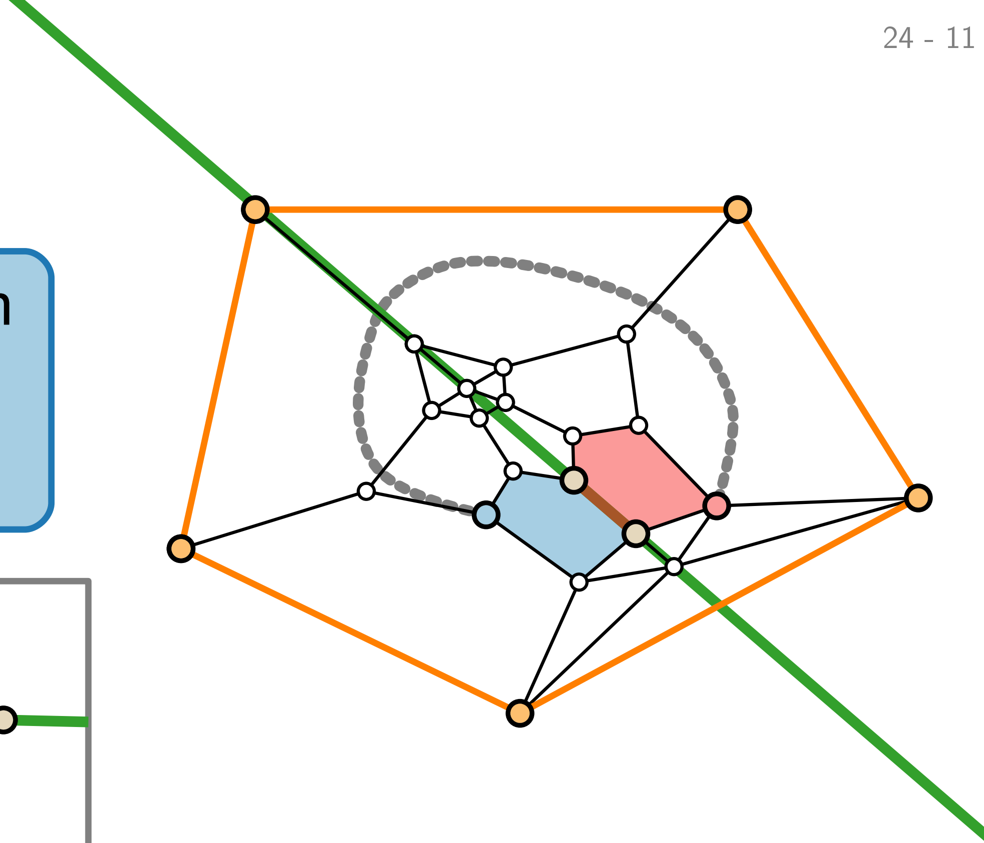

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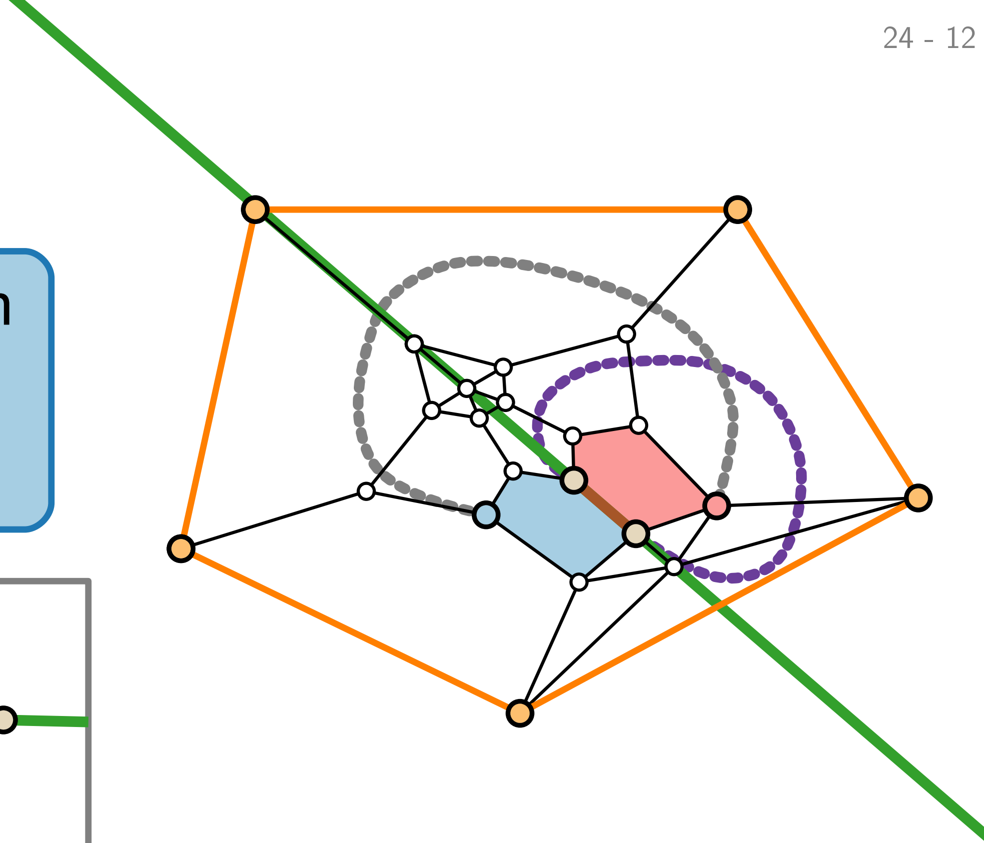

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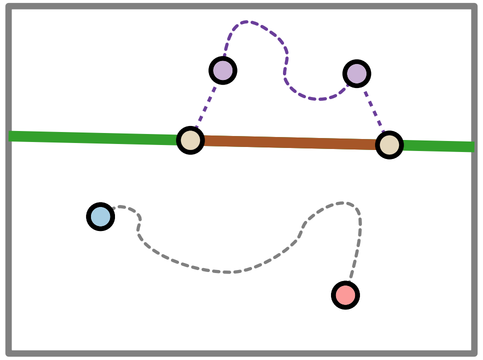

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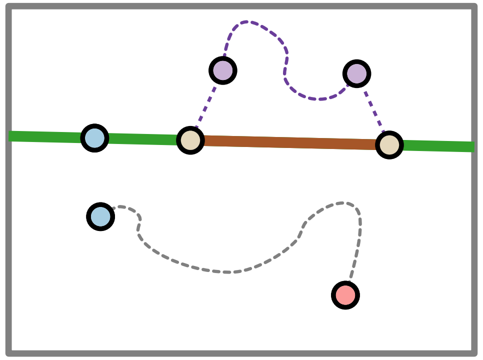

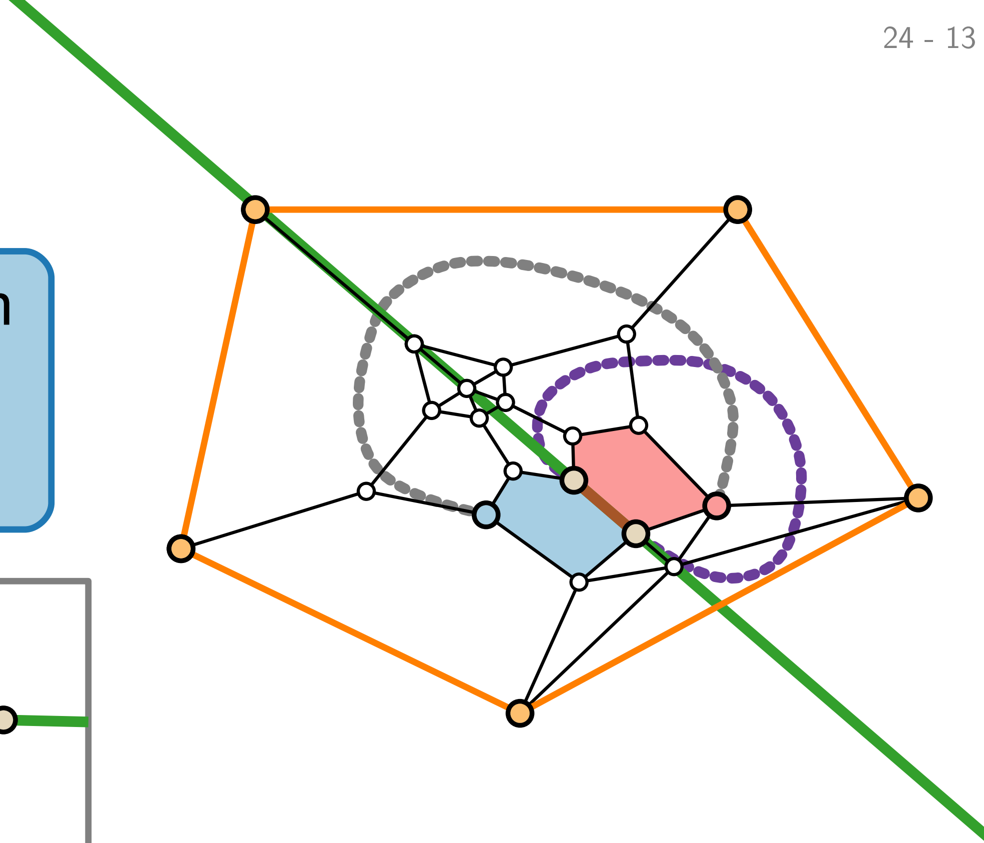

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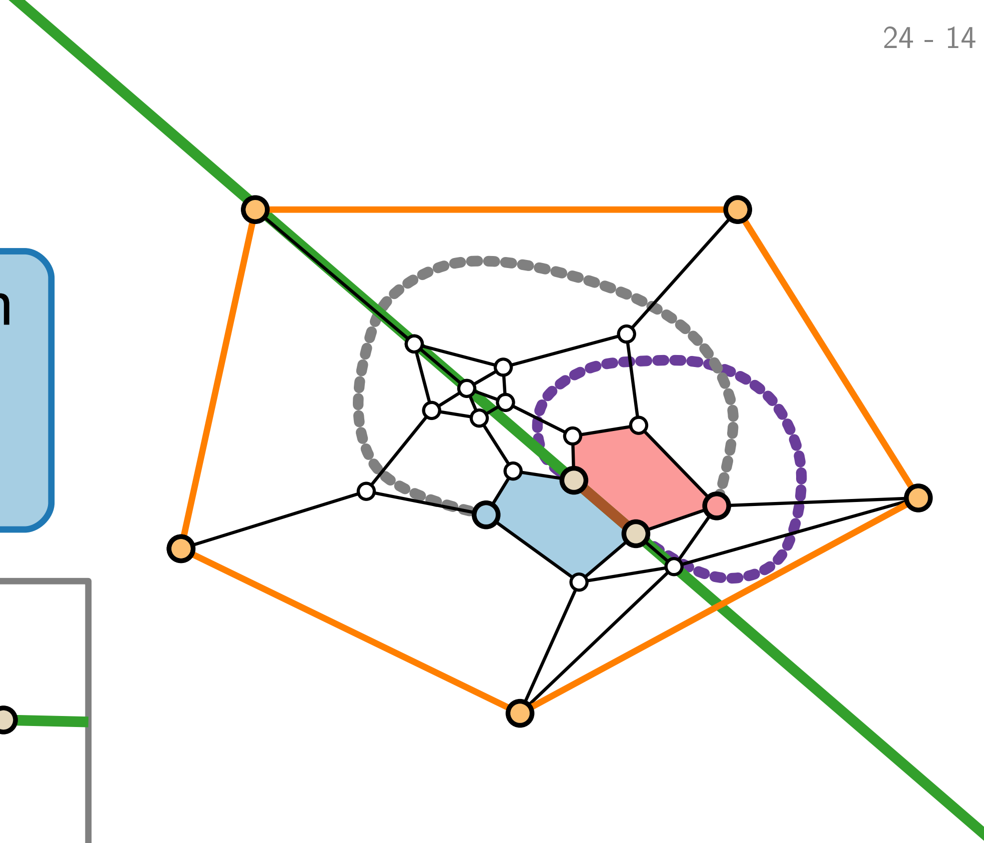

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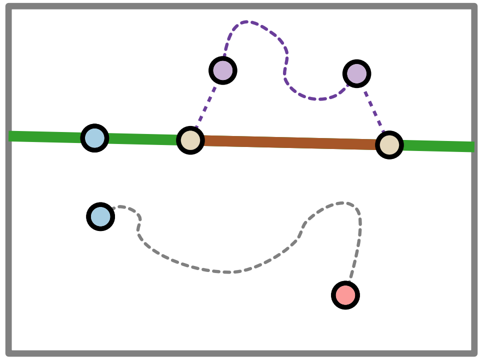

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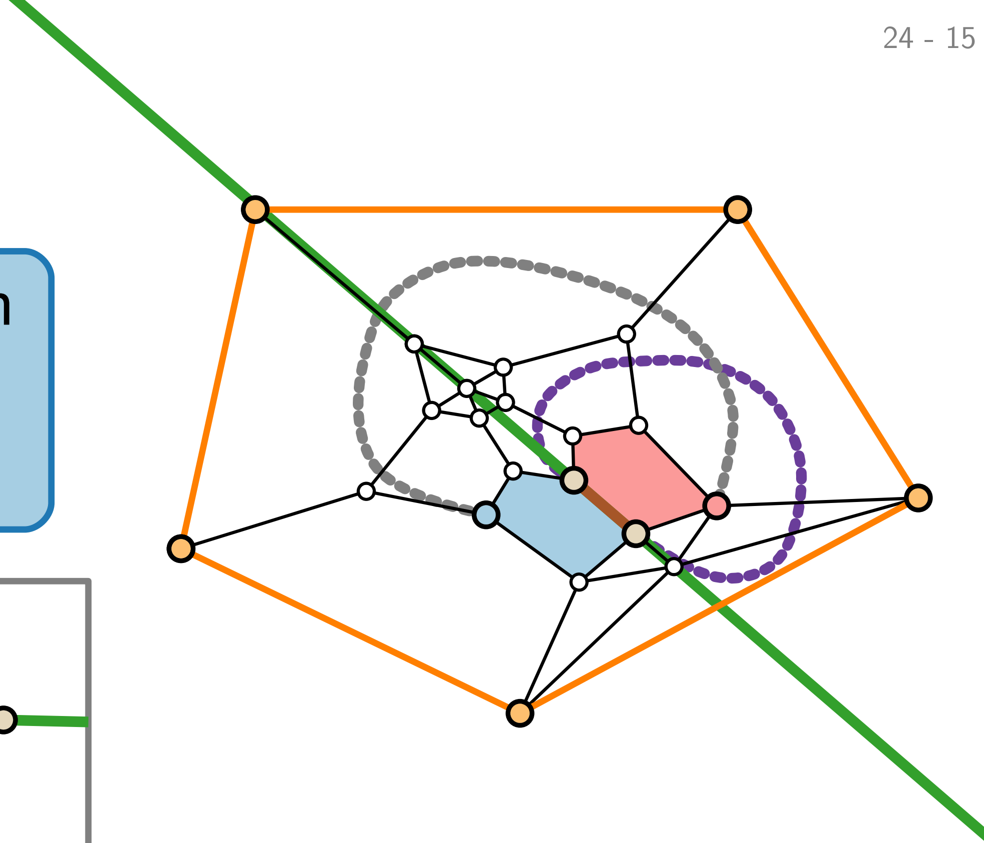

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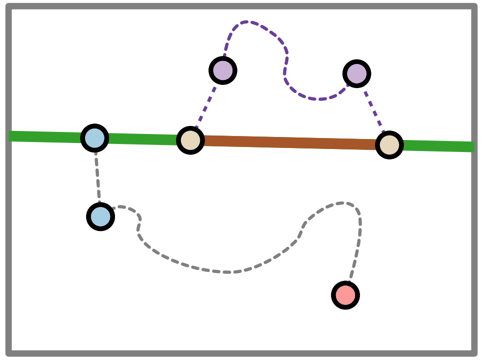

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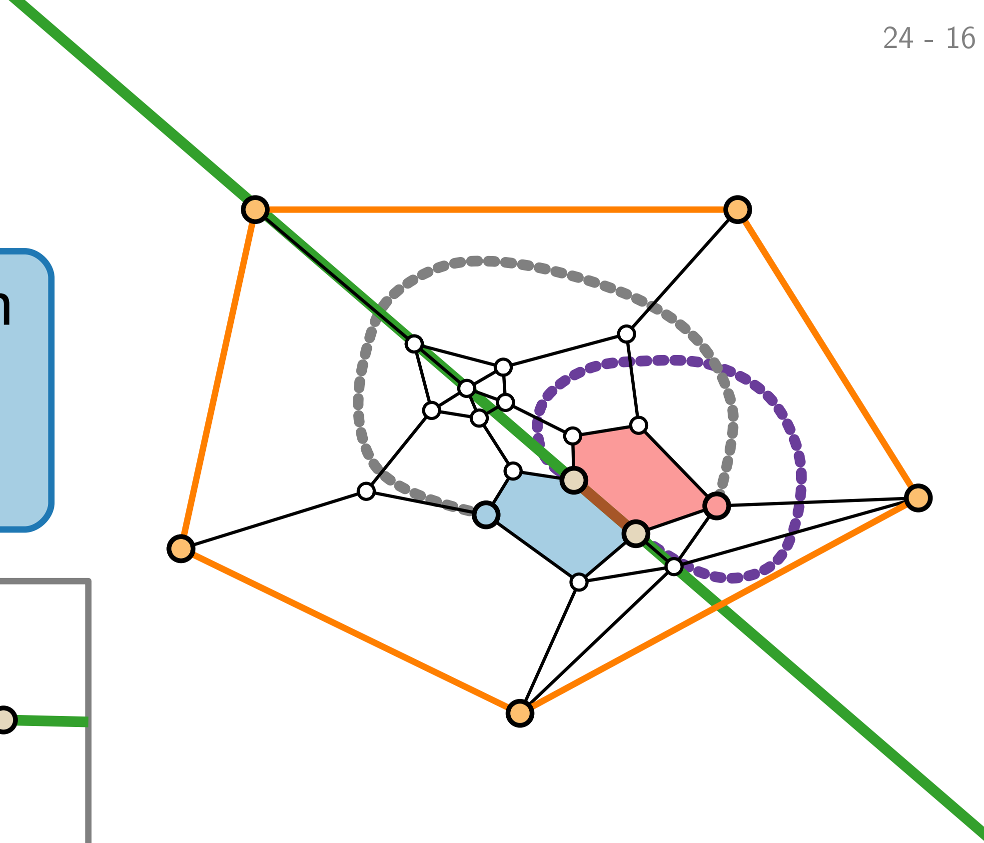

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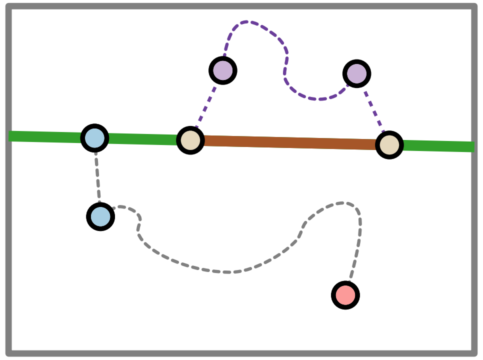

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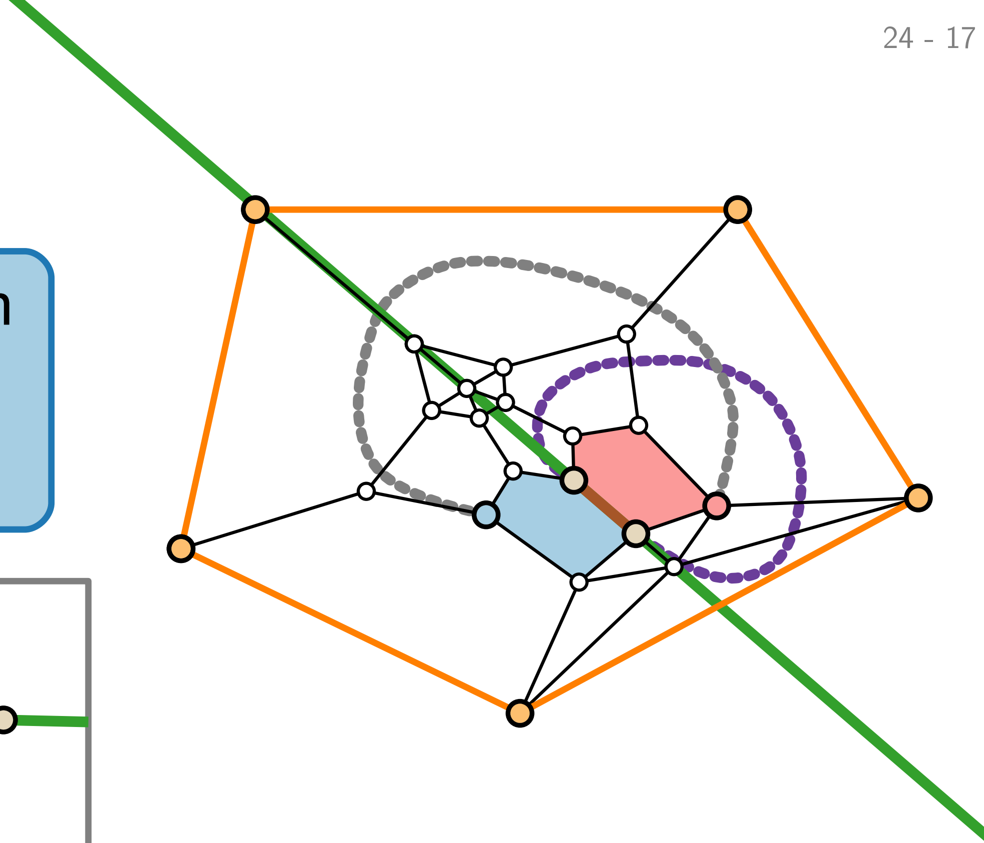

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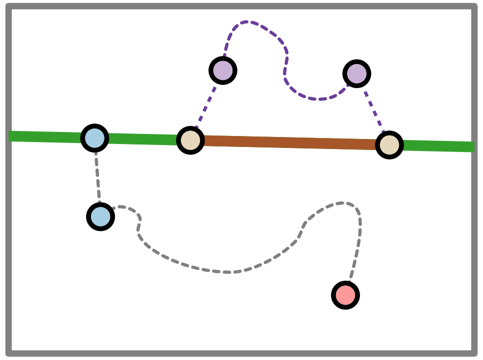

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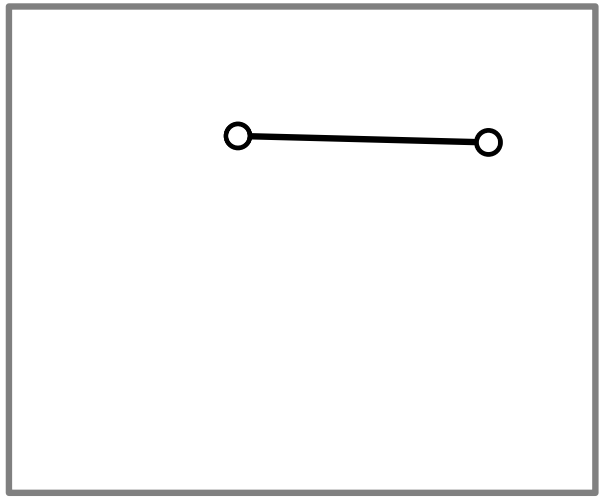

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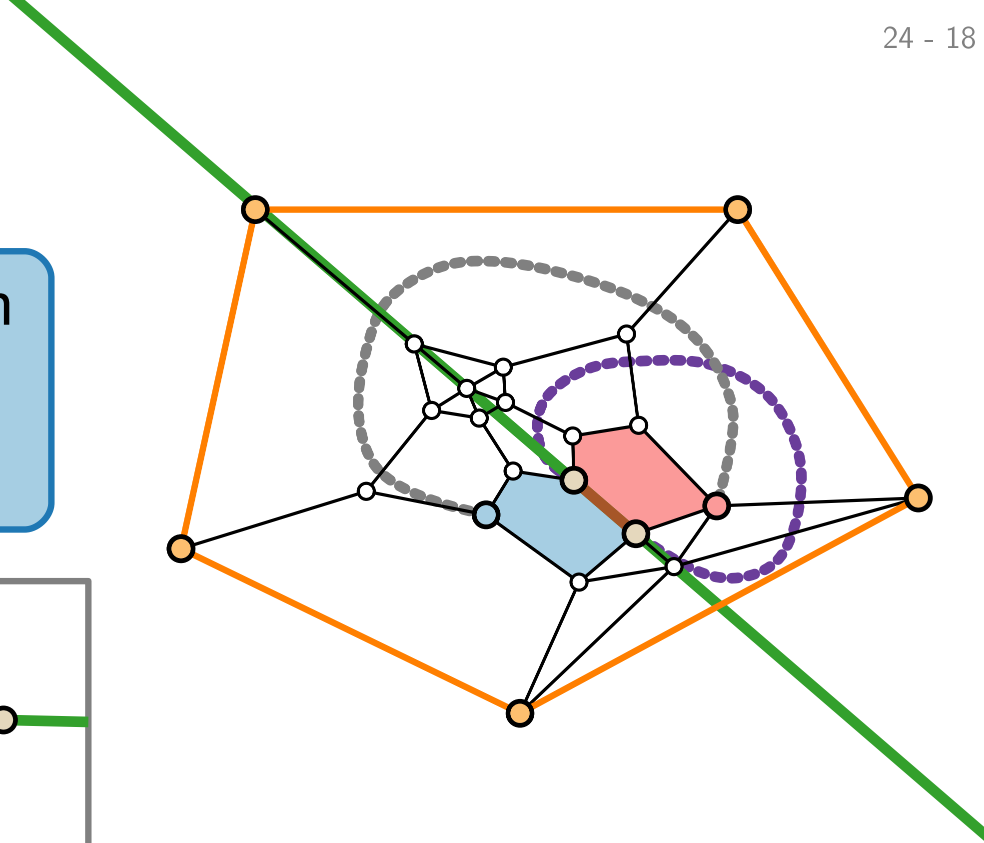

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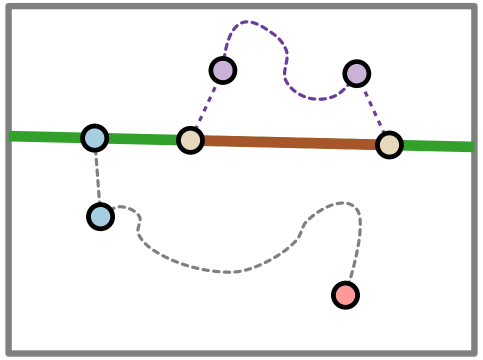

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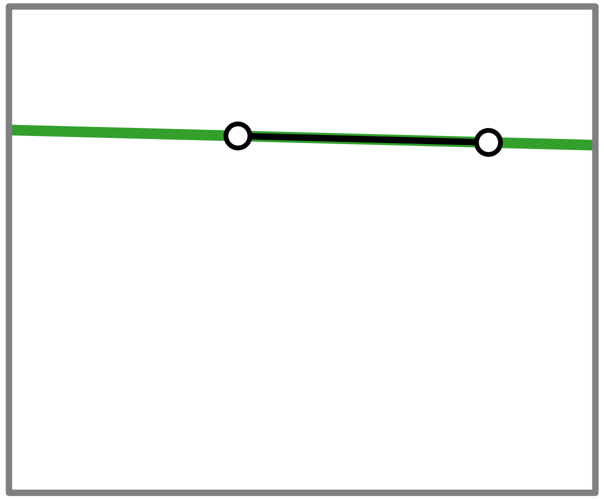

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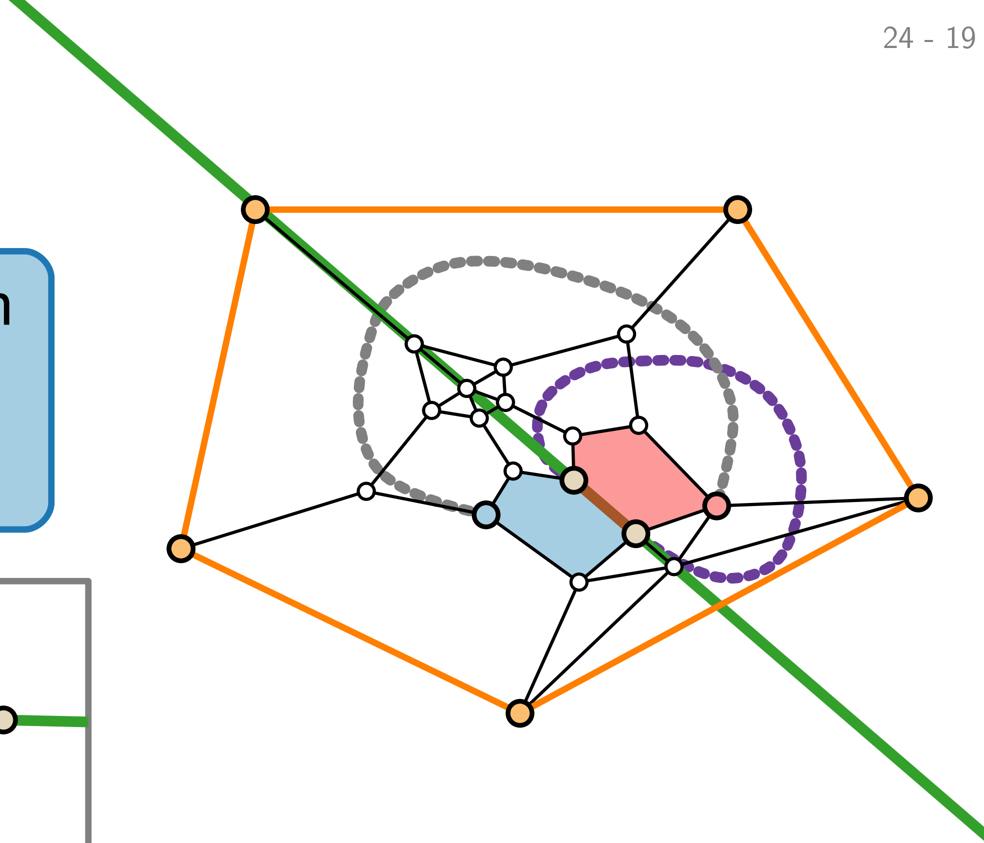

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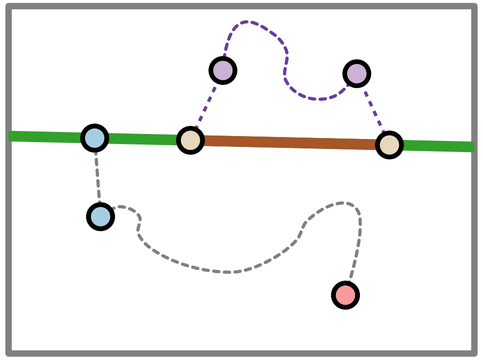

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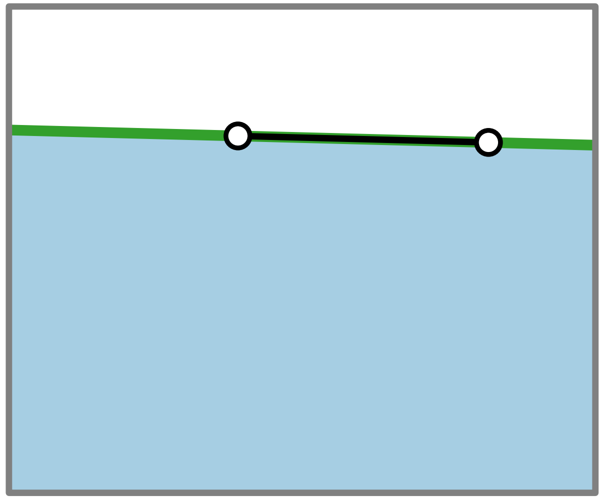

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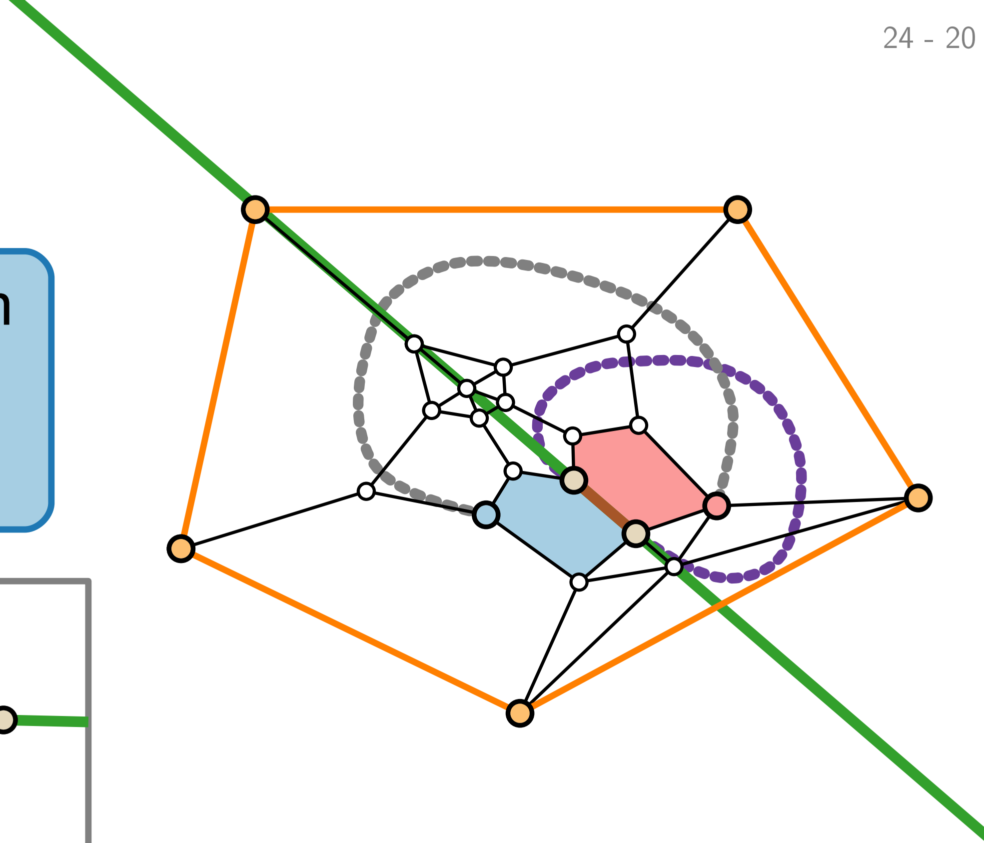

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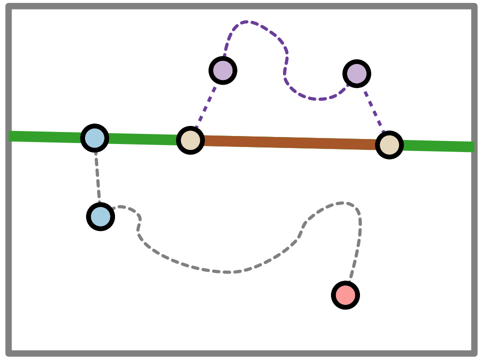

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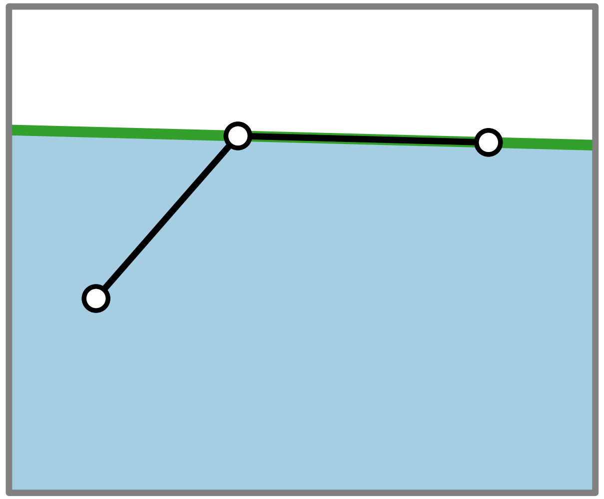

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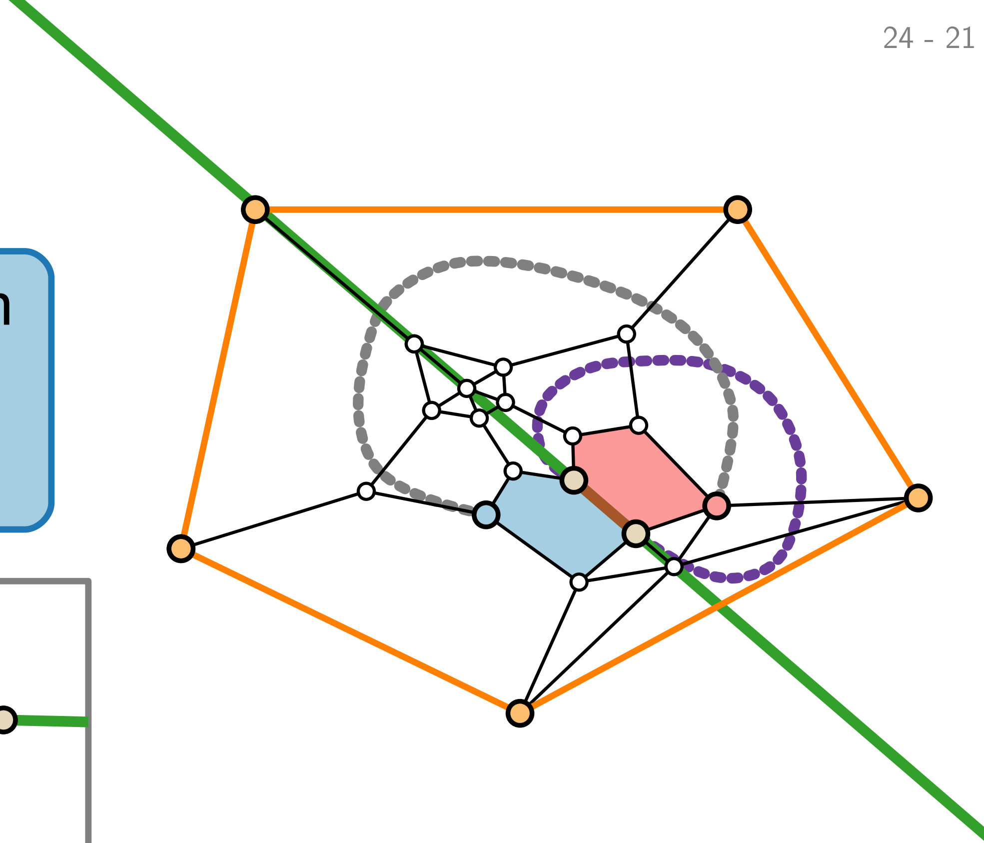

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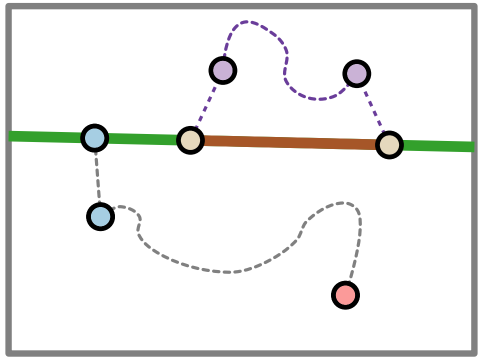

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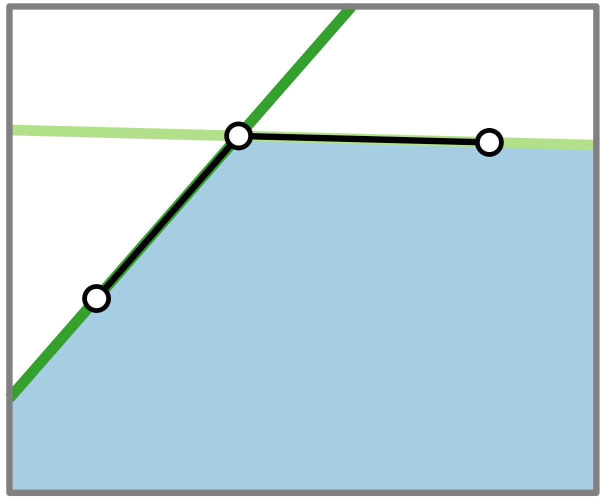

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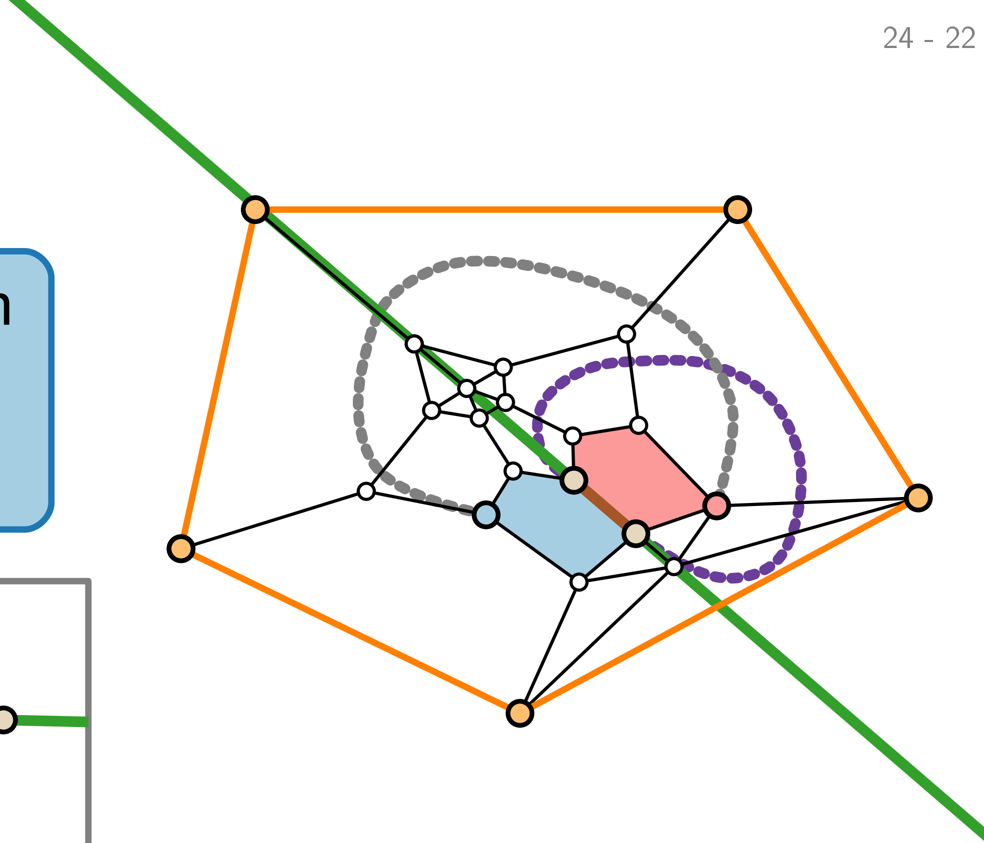

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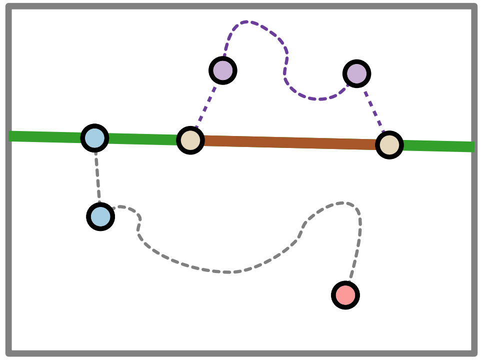

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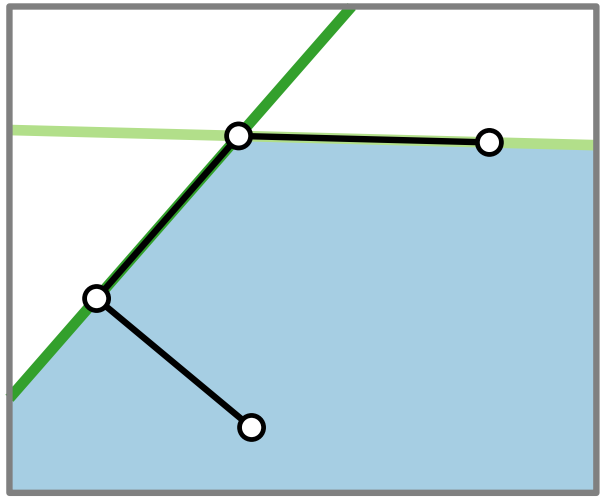

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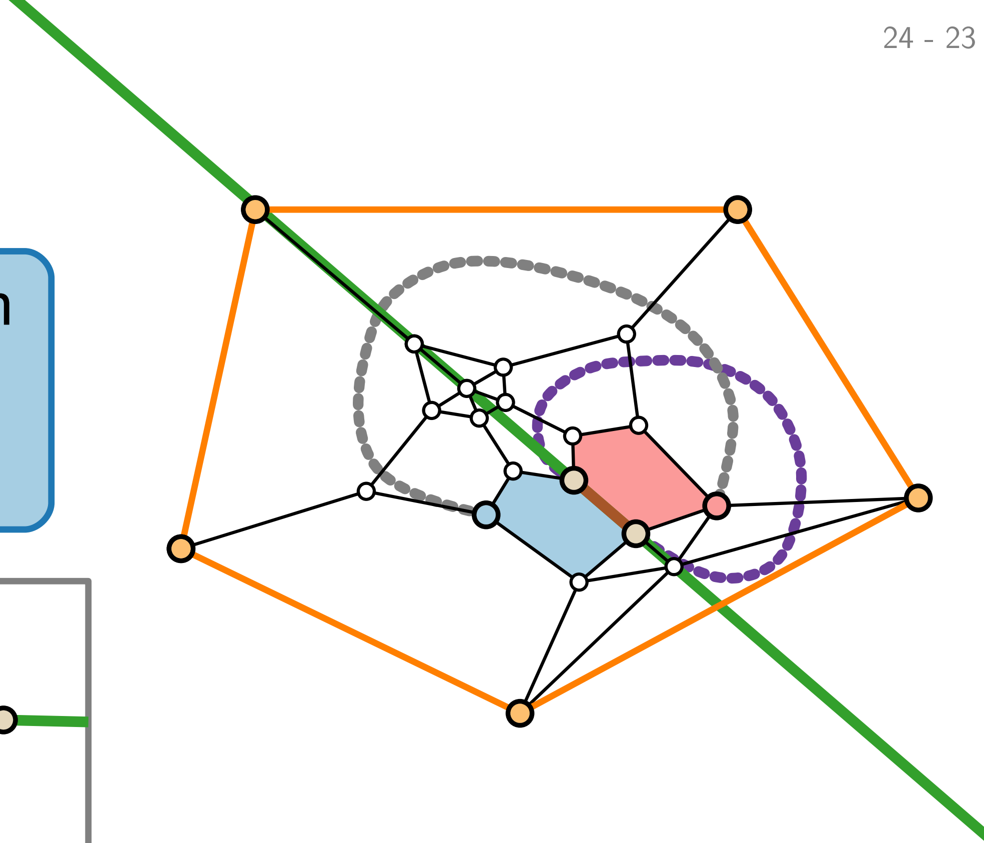

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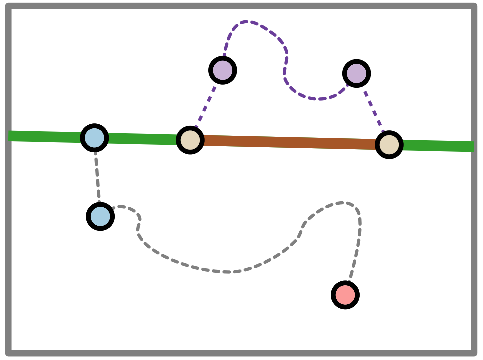

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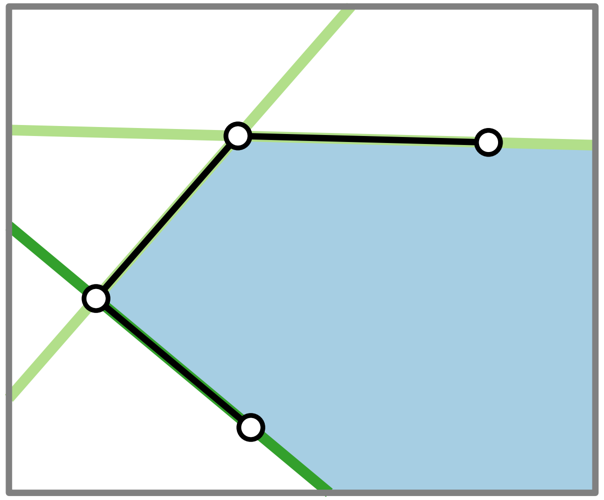

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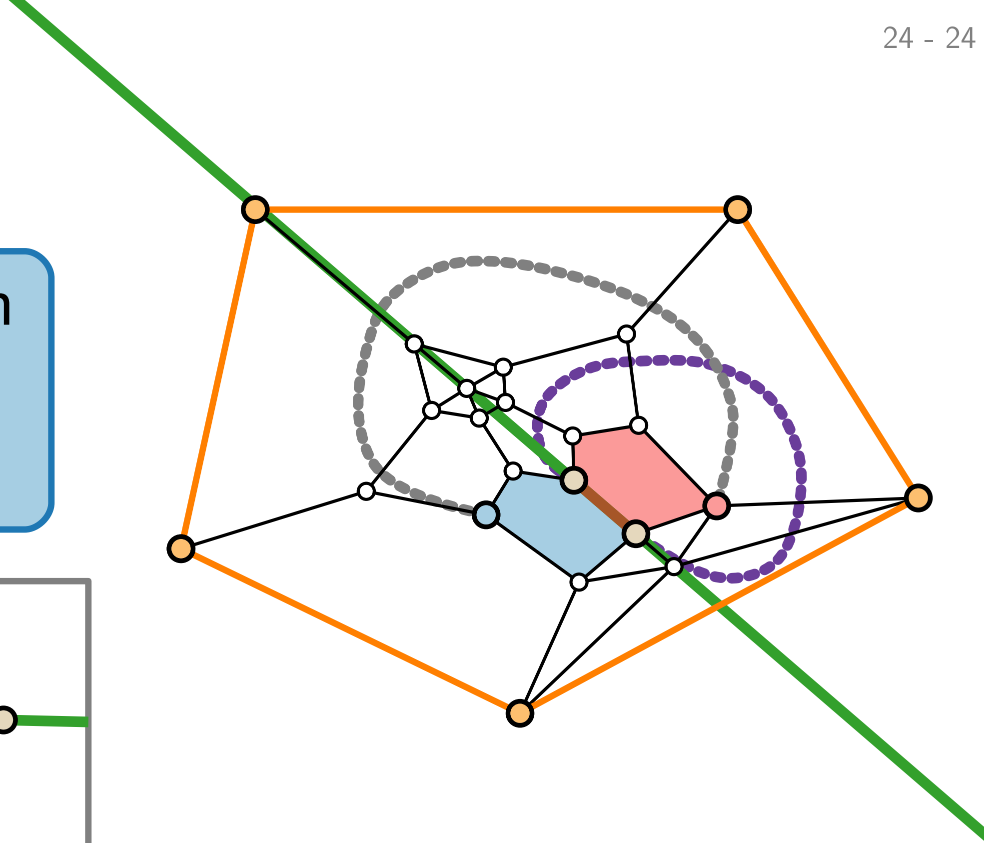

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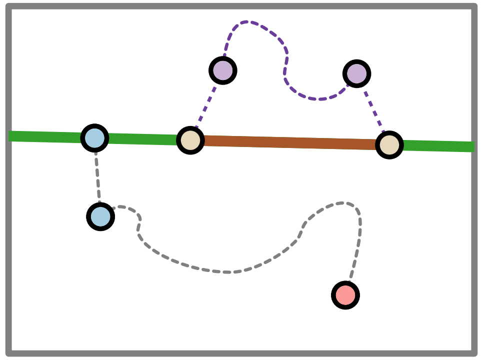

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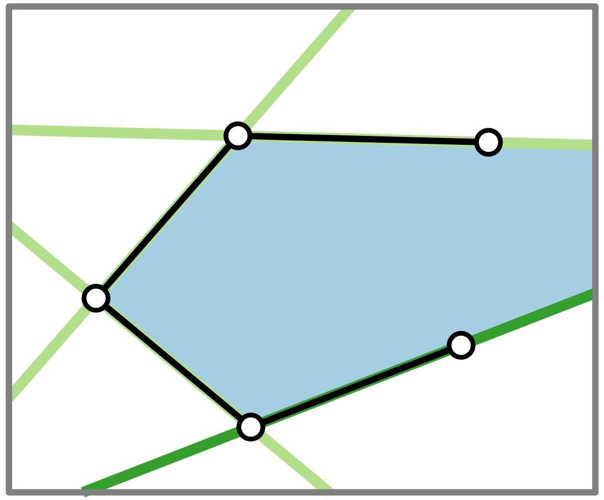

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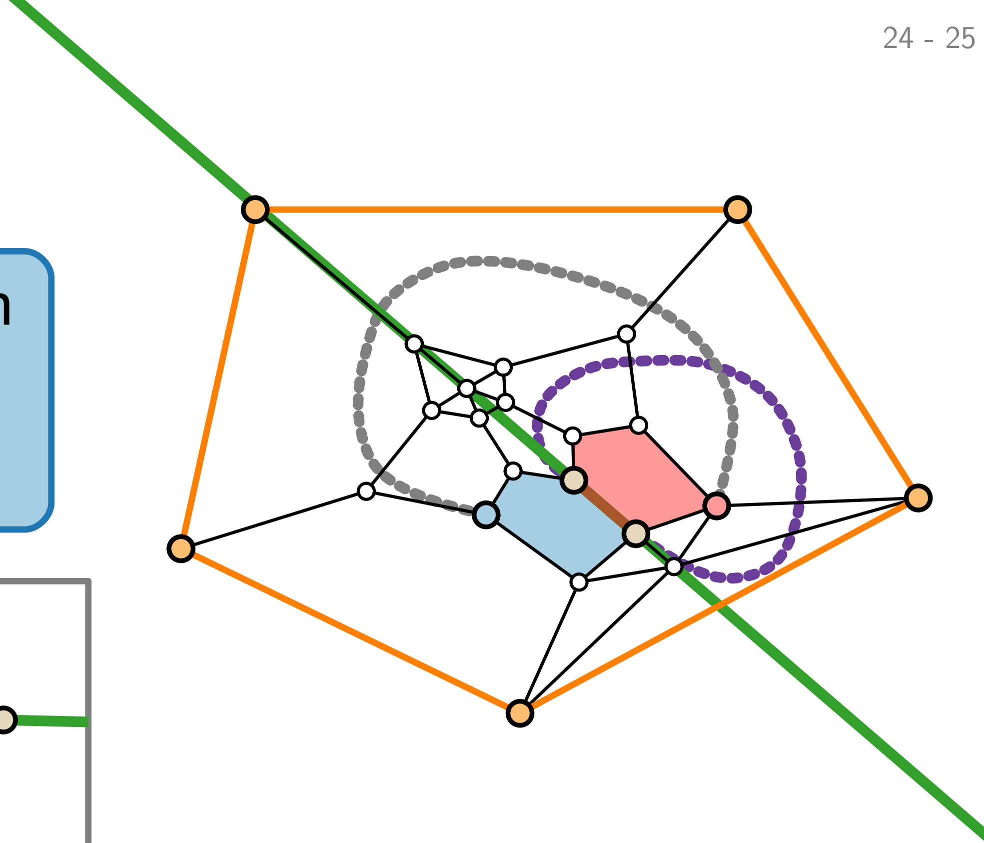

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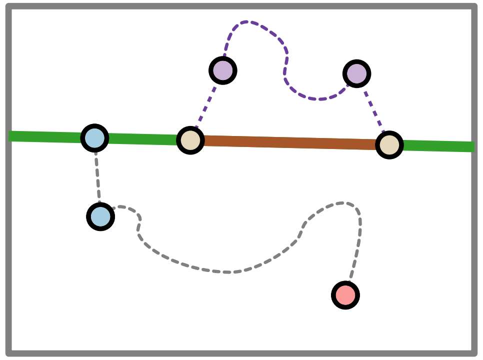

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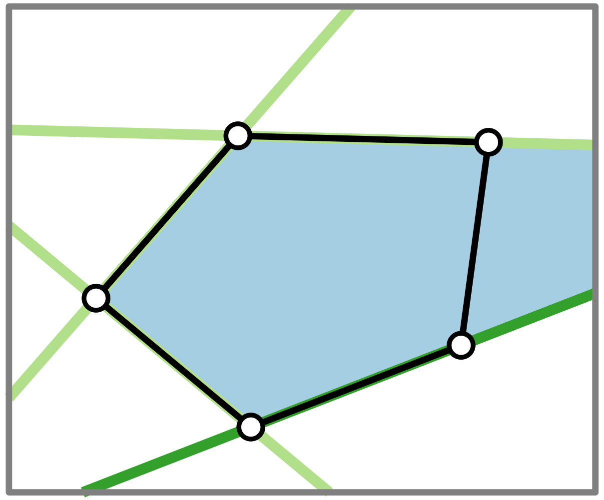

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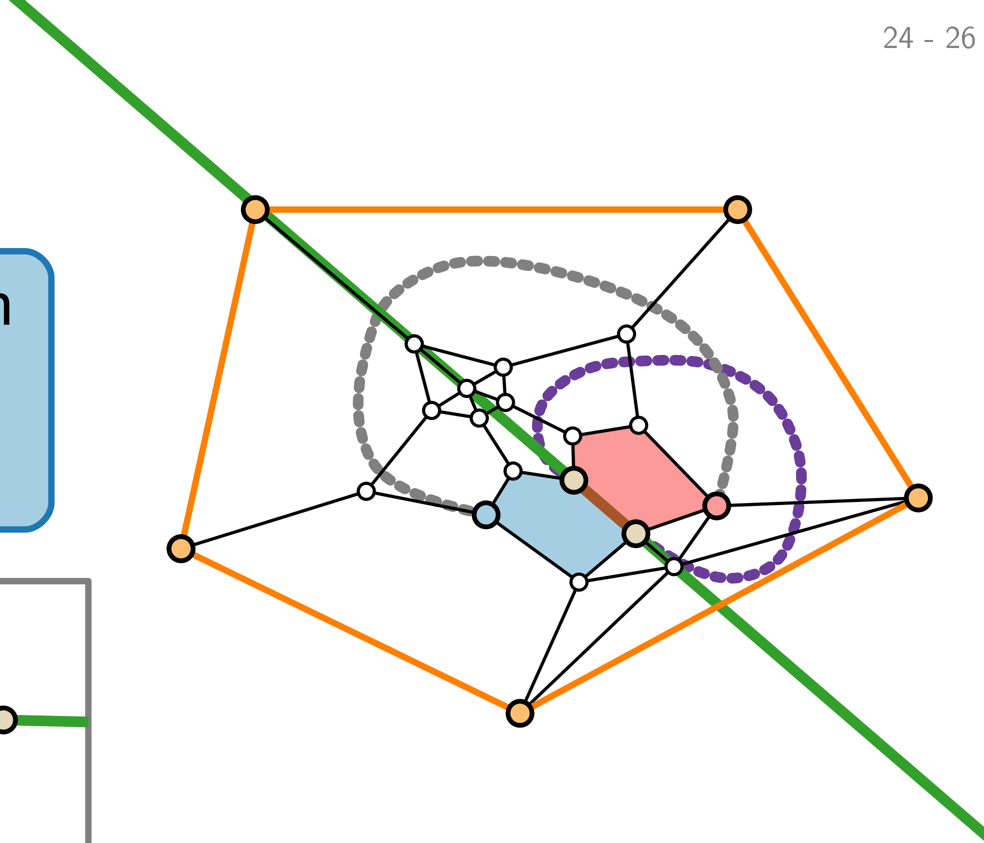

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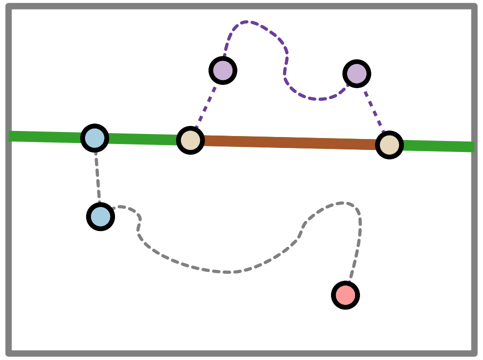

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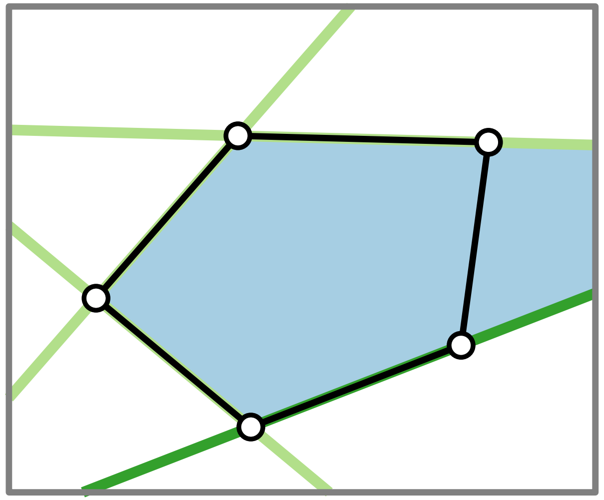

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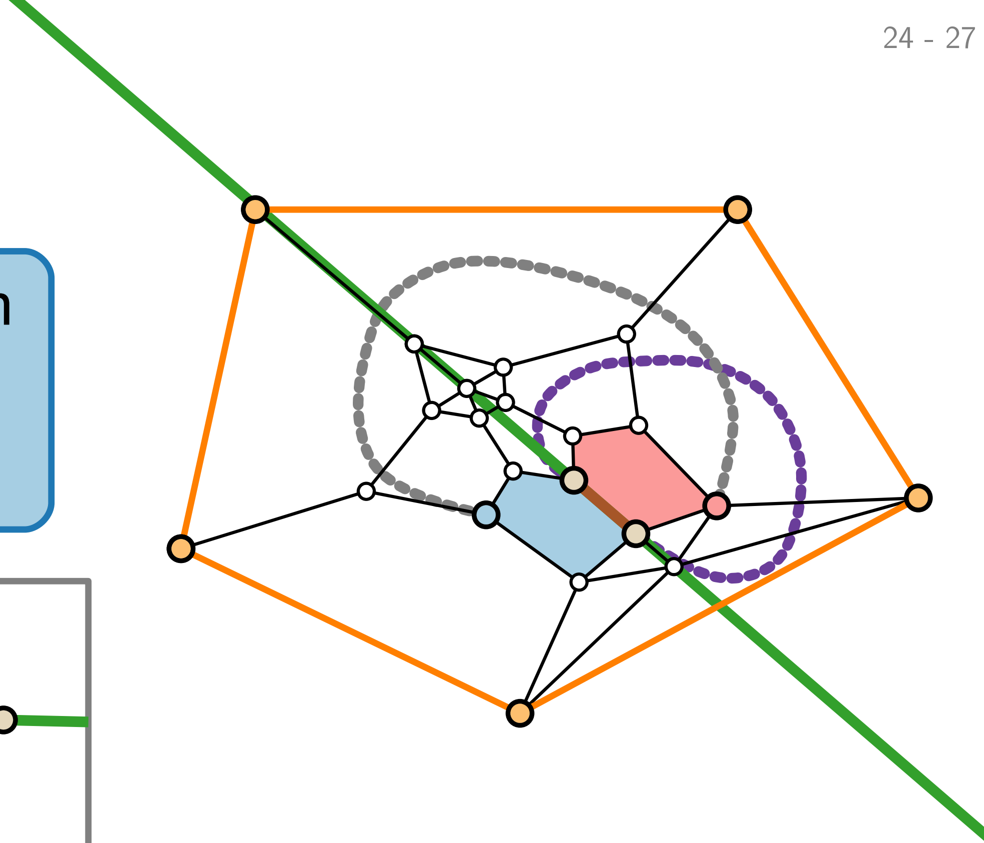

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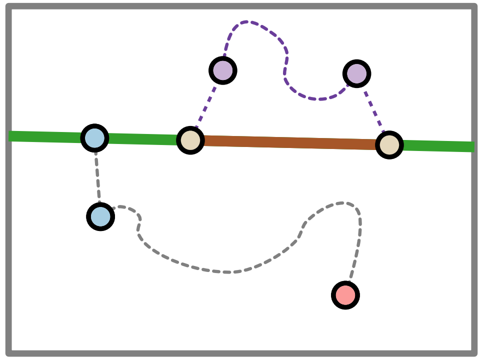

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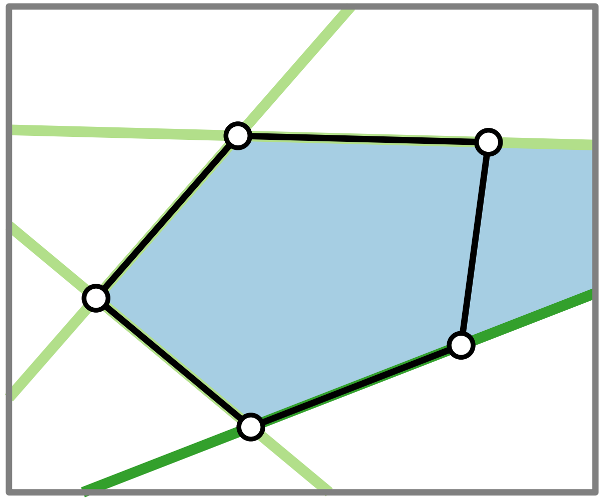

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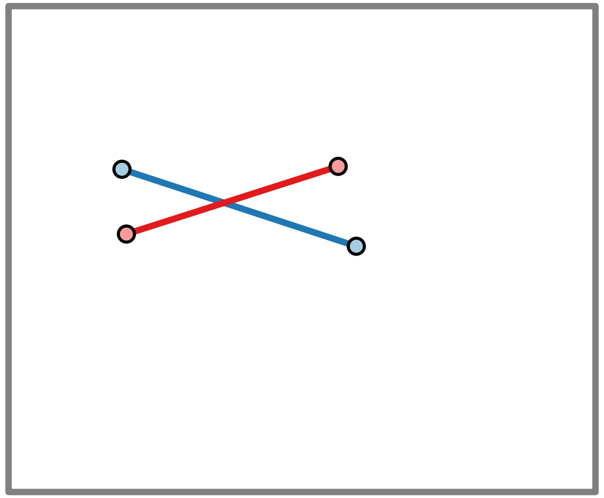

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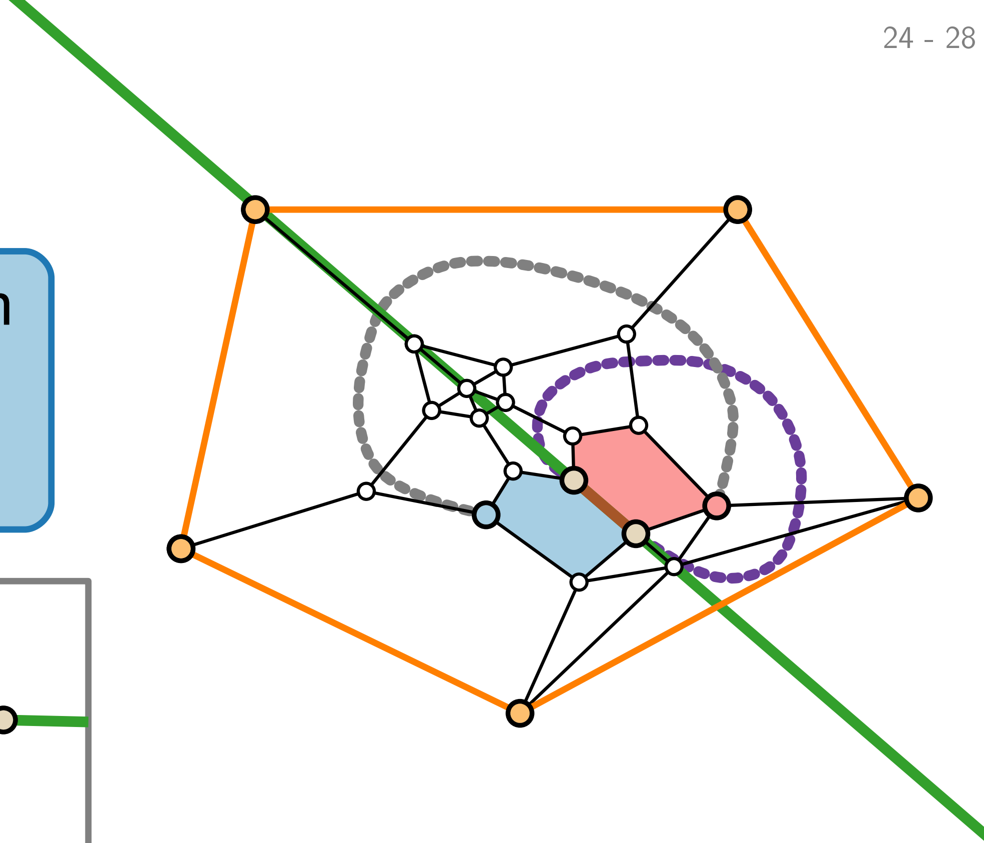

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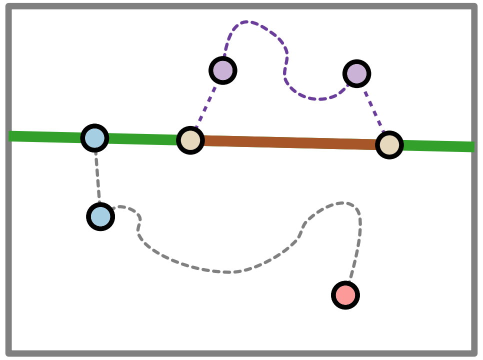

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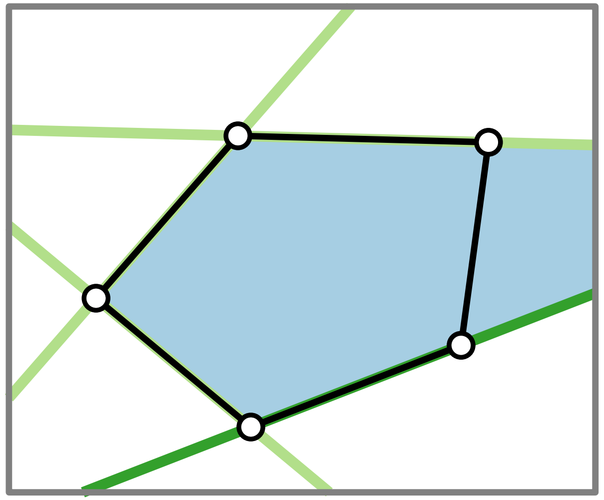

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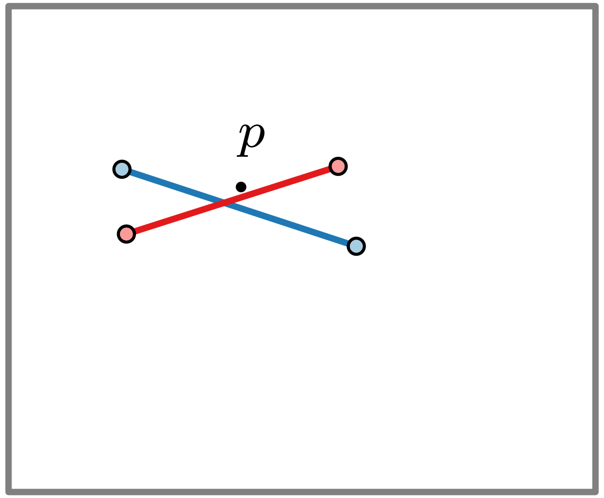

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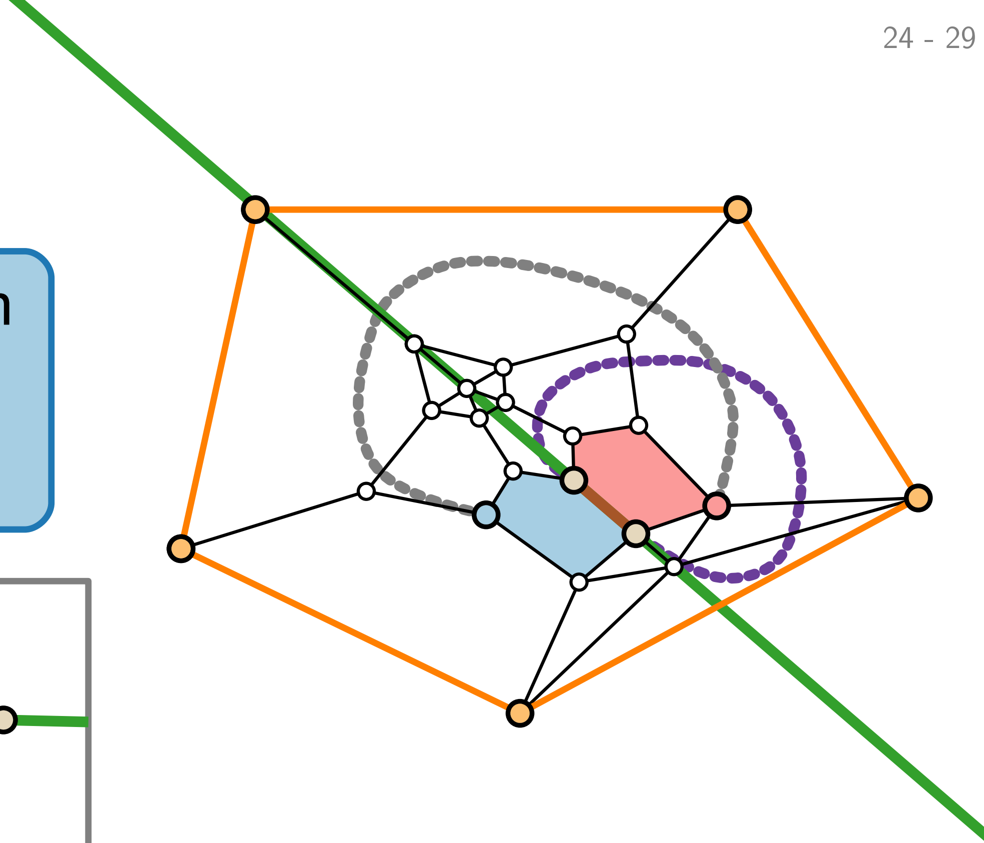

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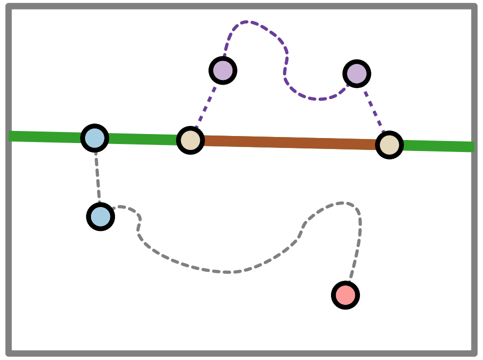

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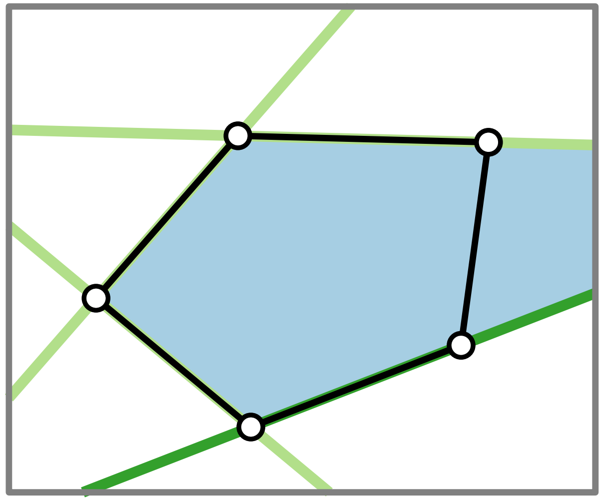

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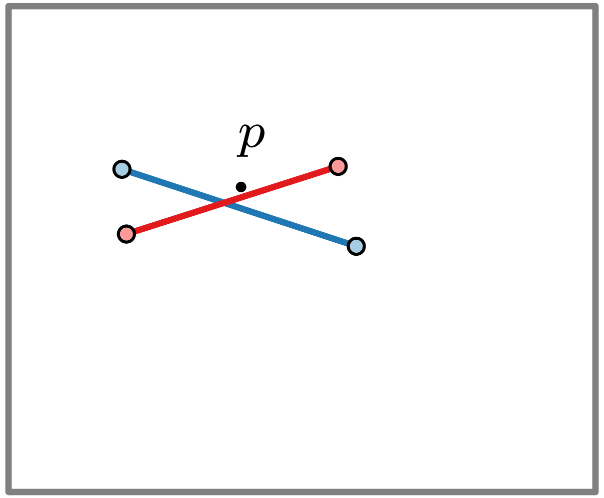
p inside two faces



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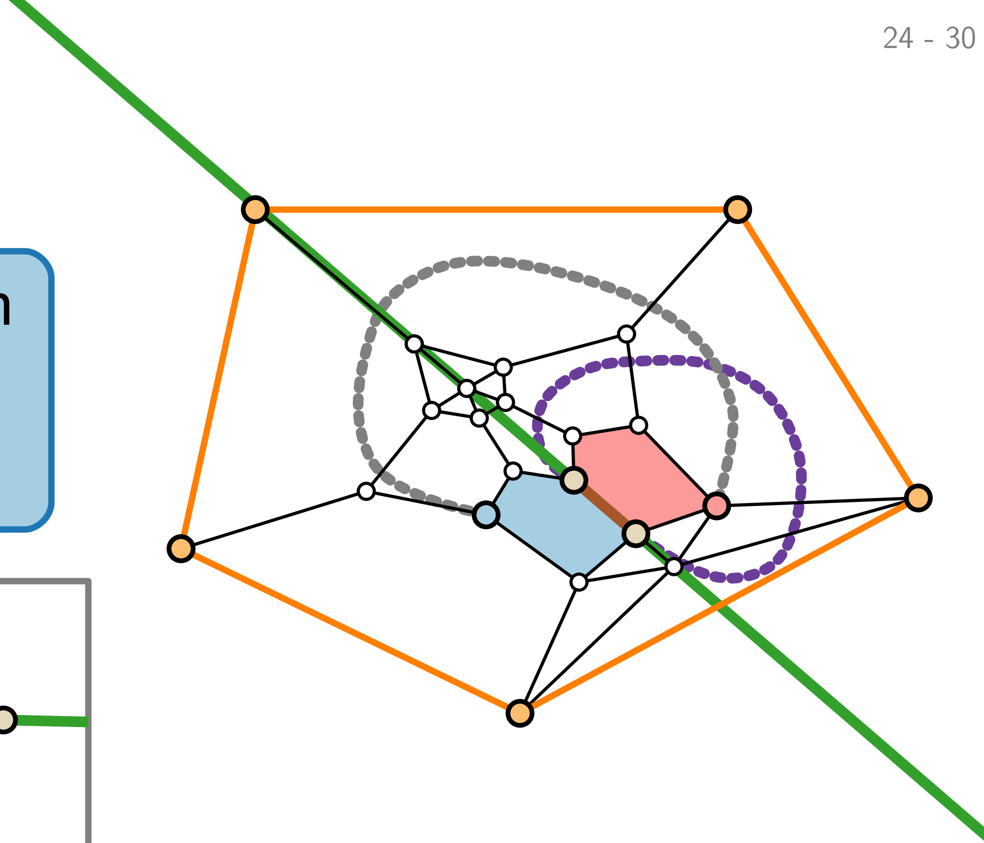

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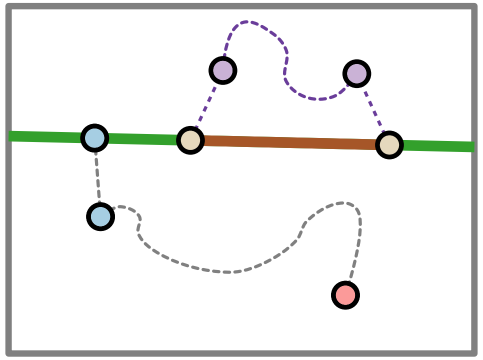

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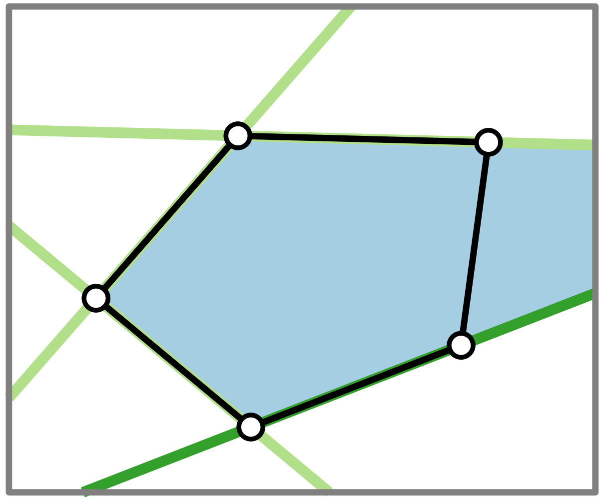

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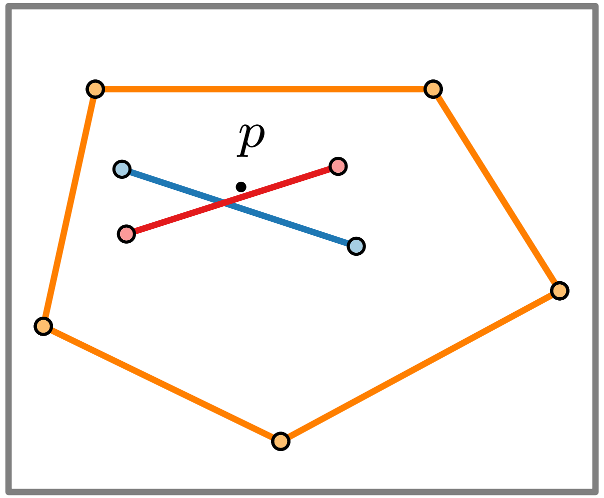

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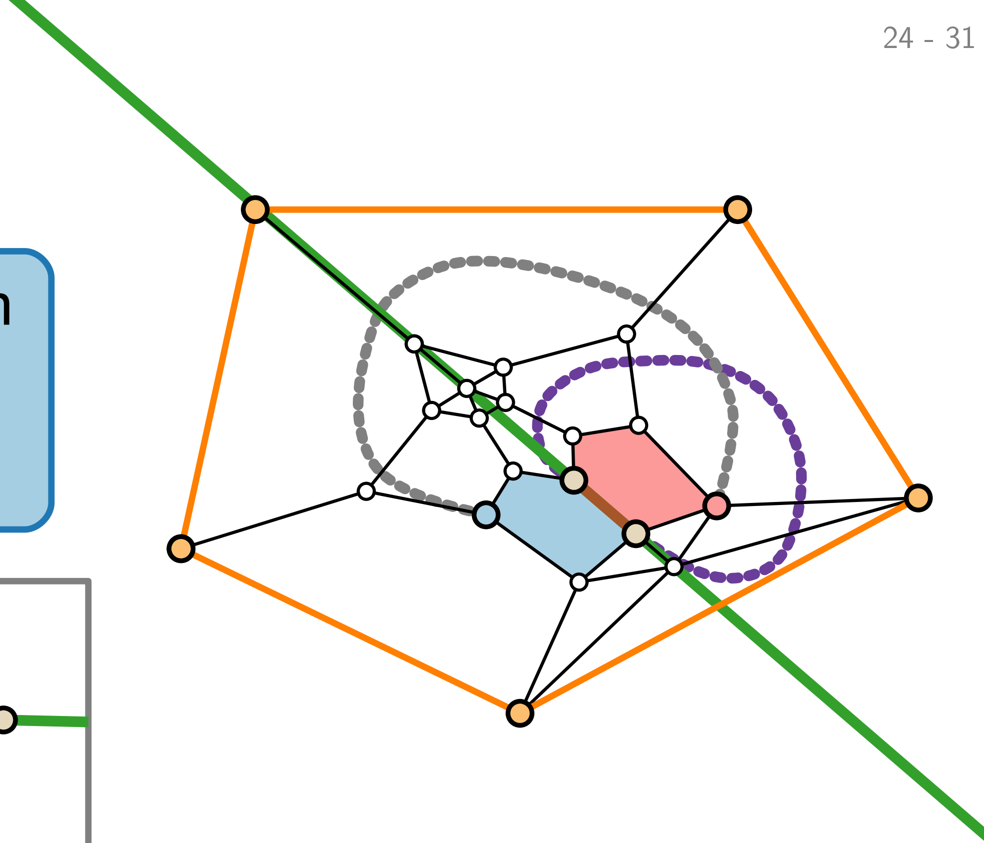

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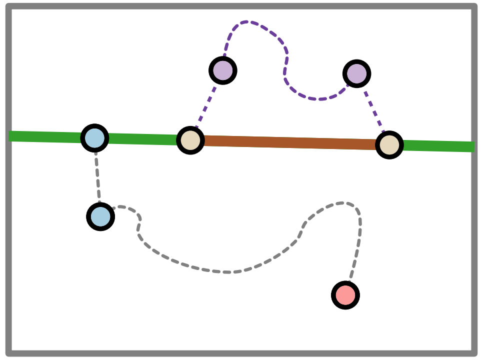

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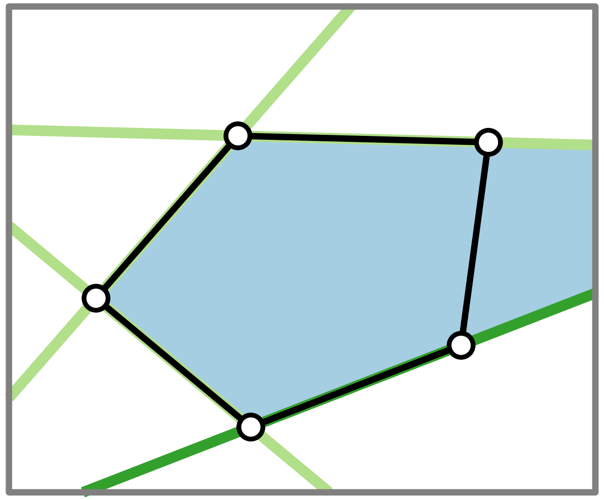

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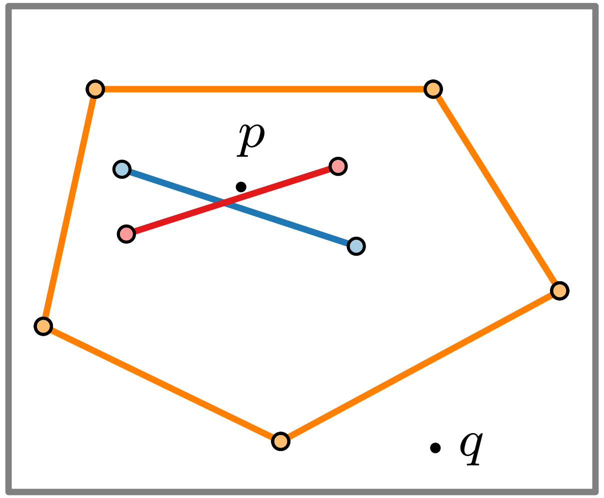

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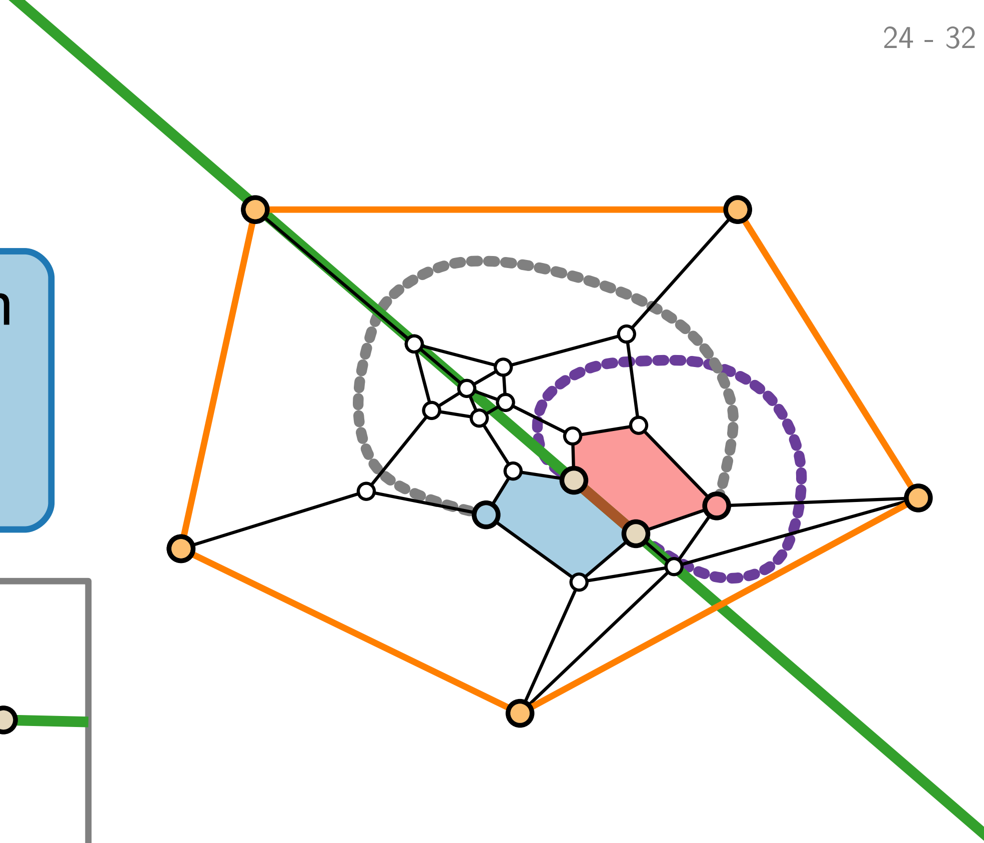

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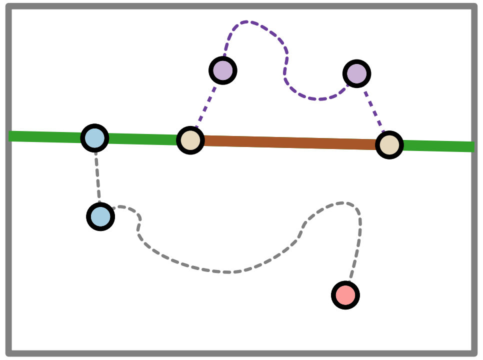

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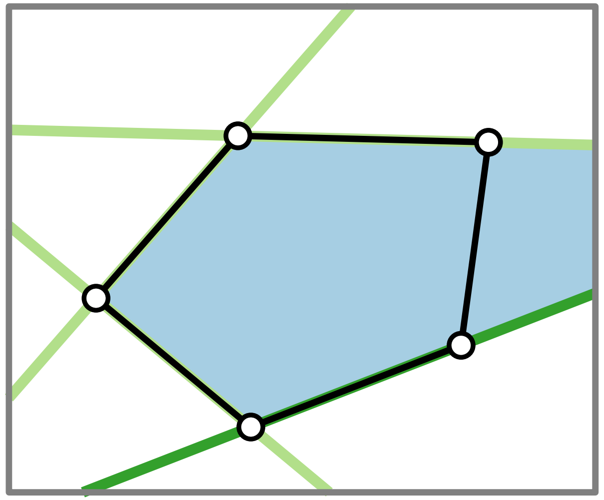

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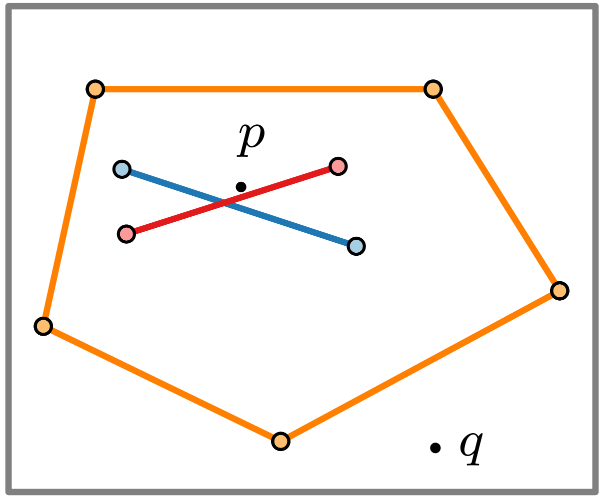
Property 2. All free vertices lie inside C.



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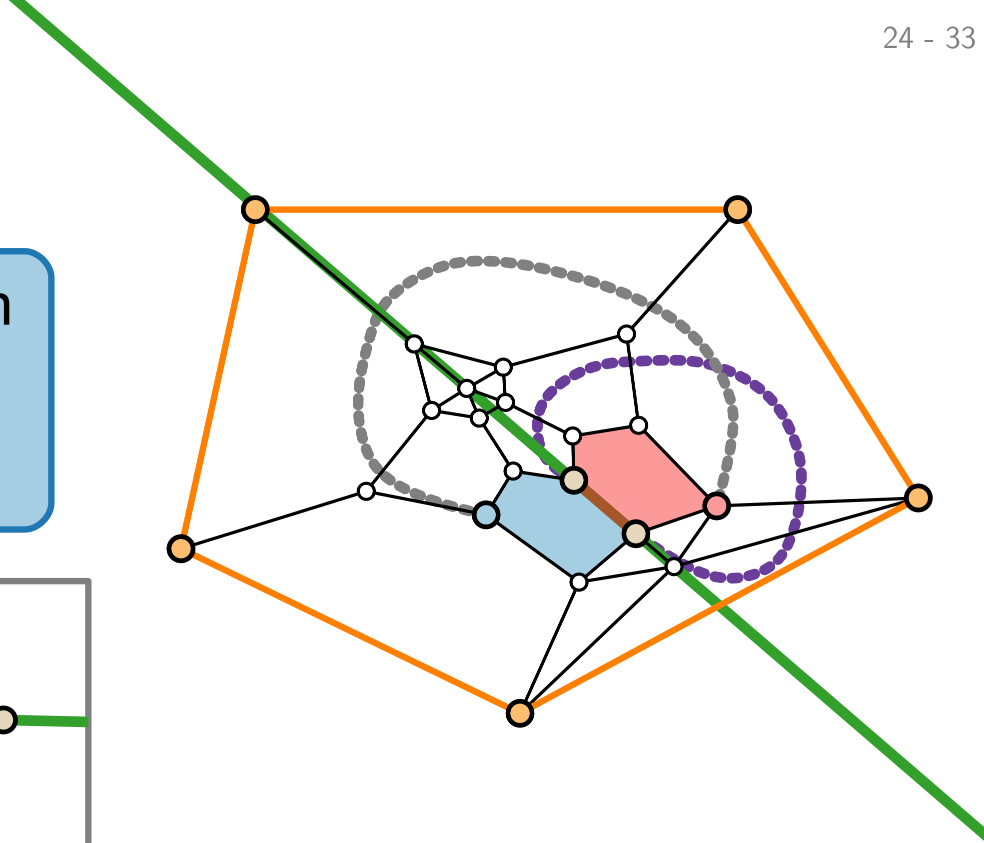

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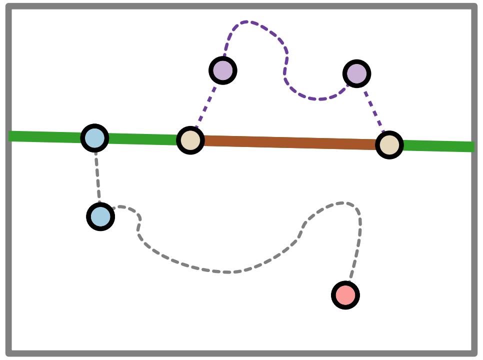

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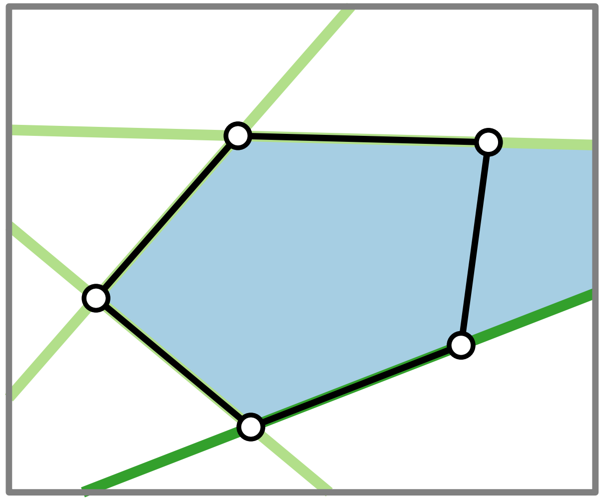

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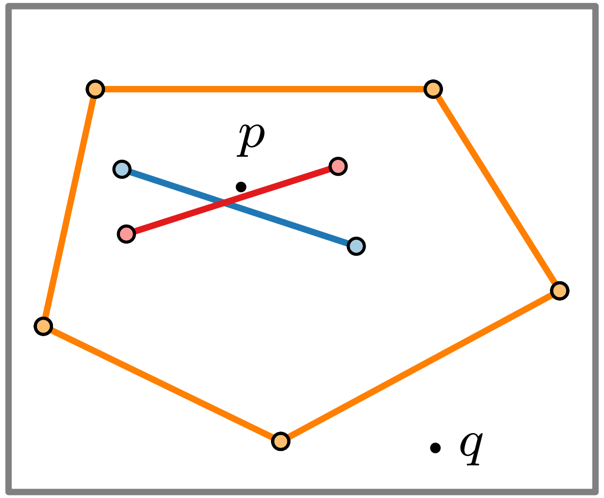

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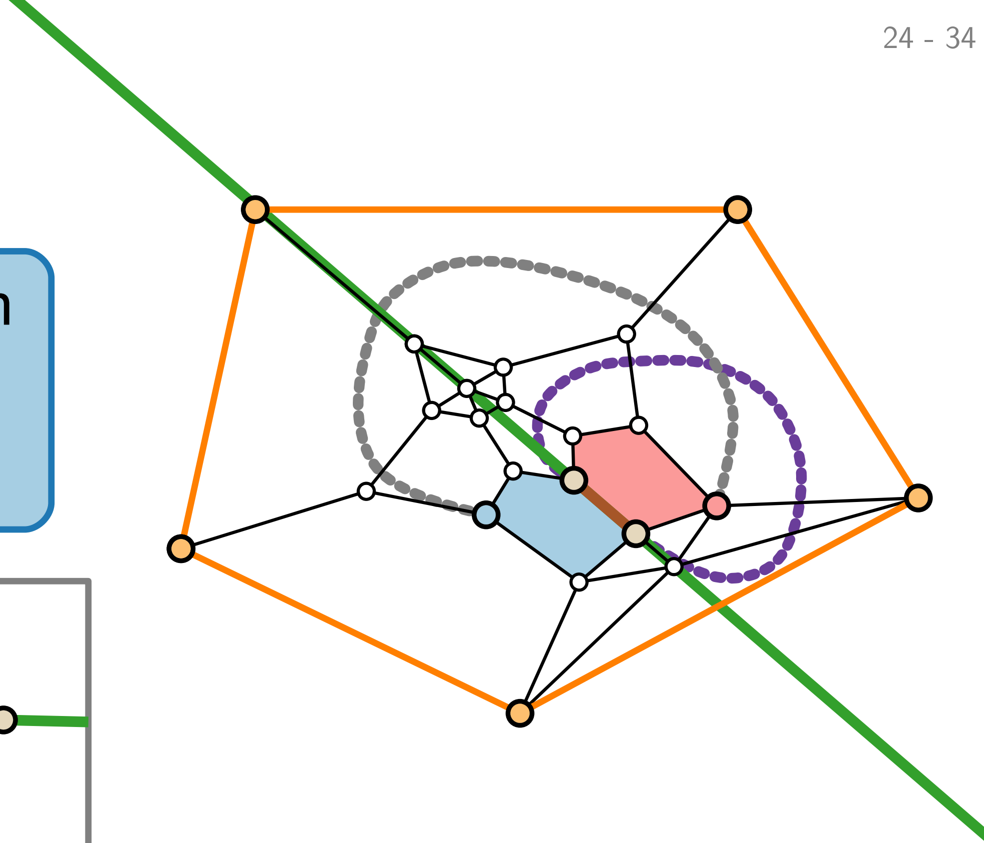

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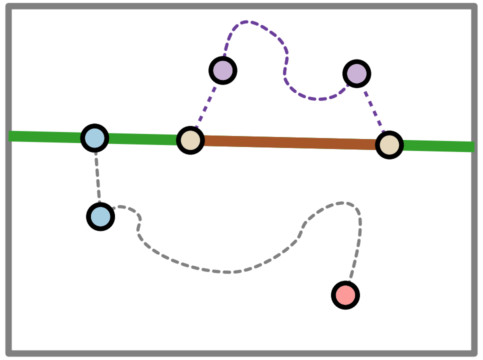

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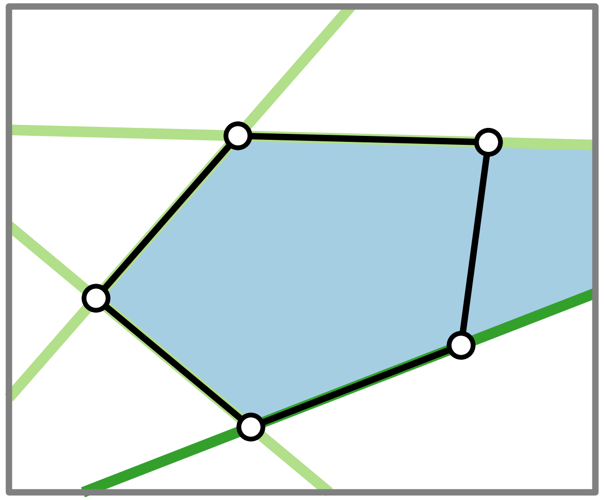

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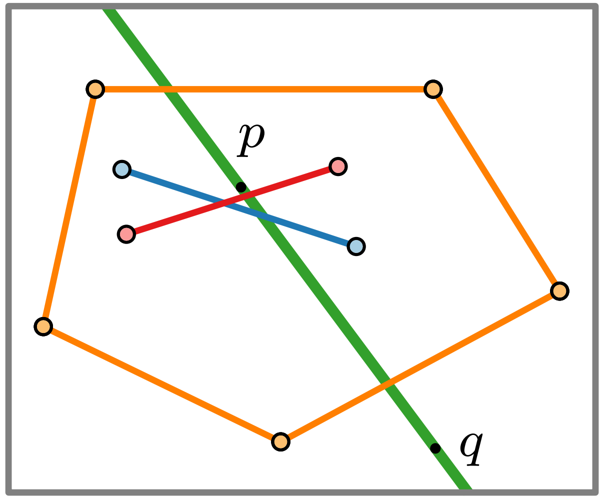
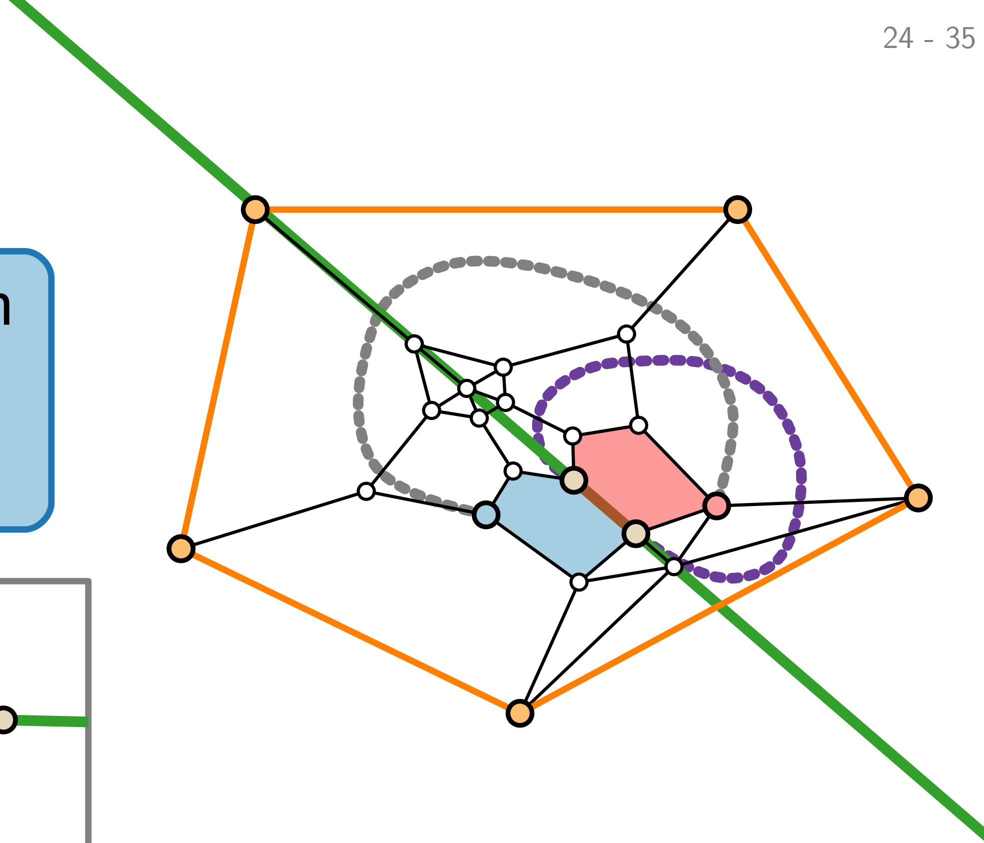


Proof of Tutte's Theorem



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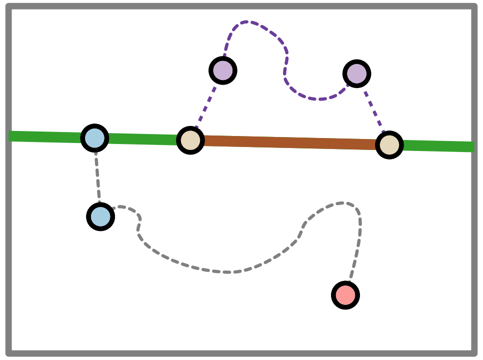
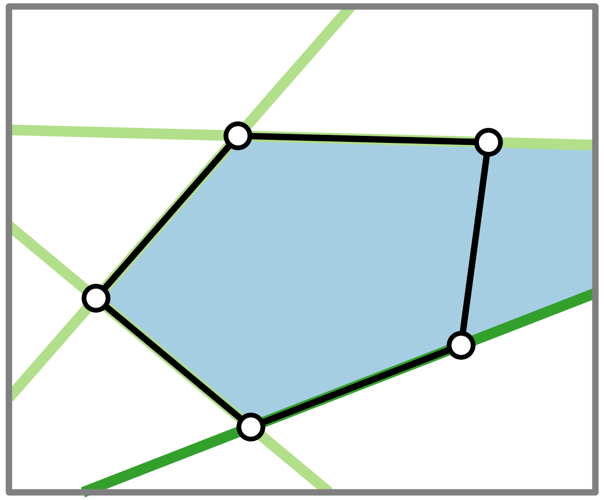


Lemma. All faces are strictly convex.



Lemma. The drawing is planar.

p inside two faces

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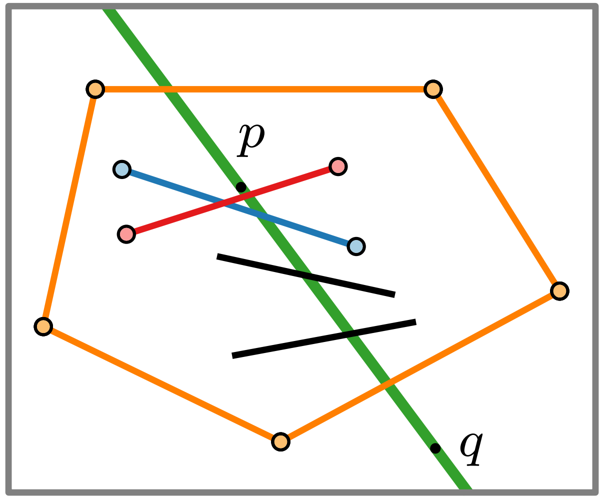

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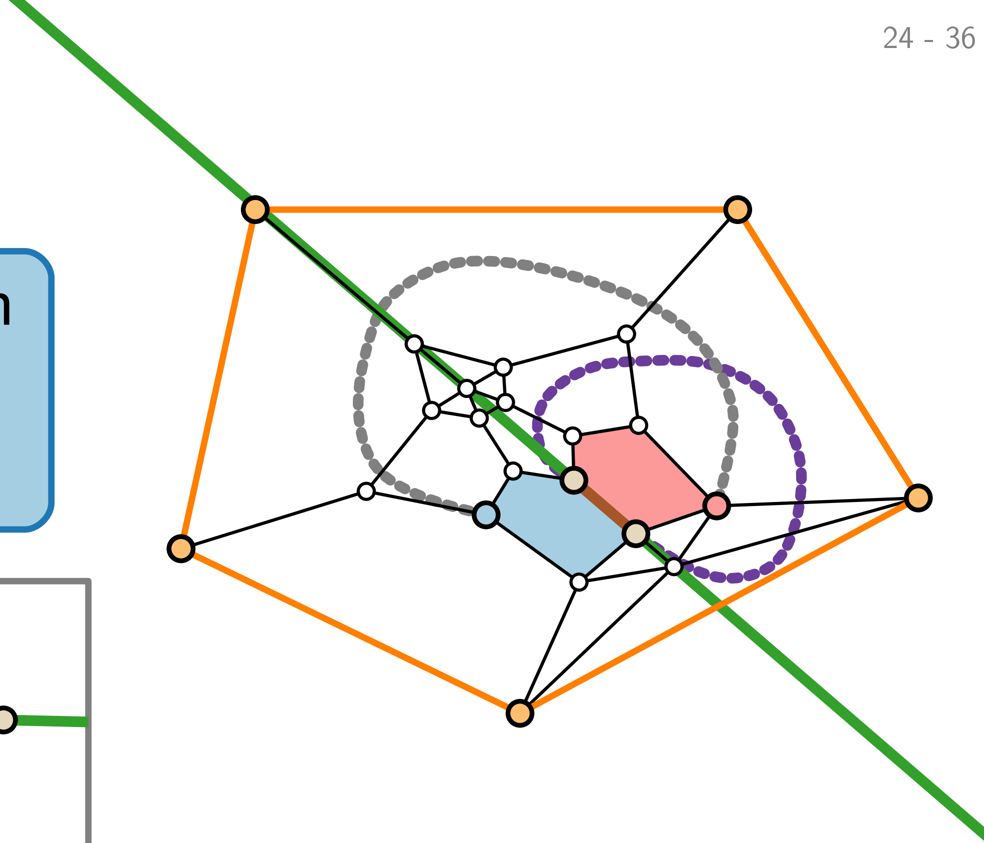

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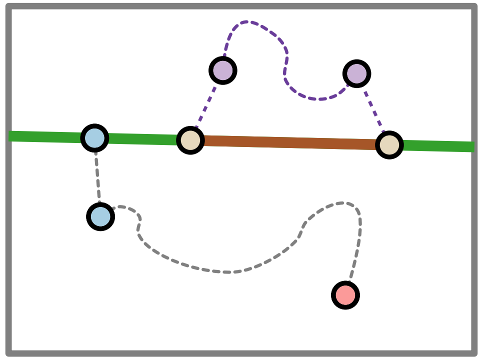

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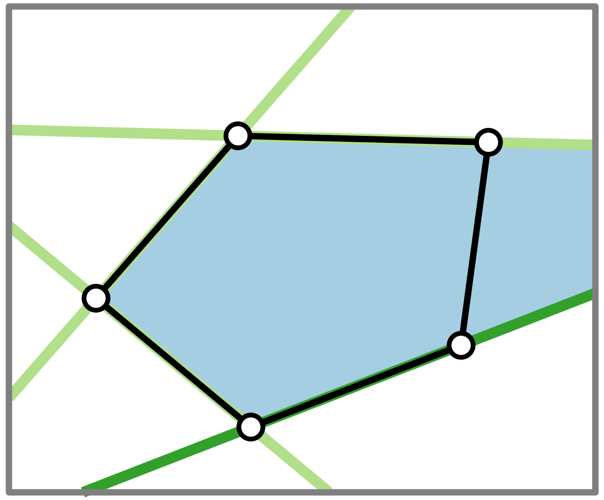

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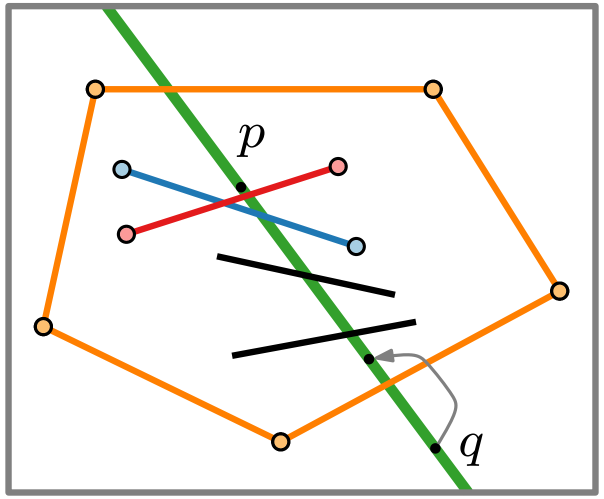

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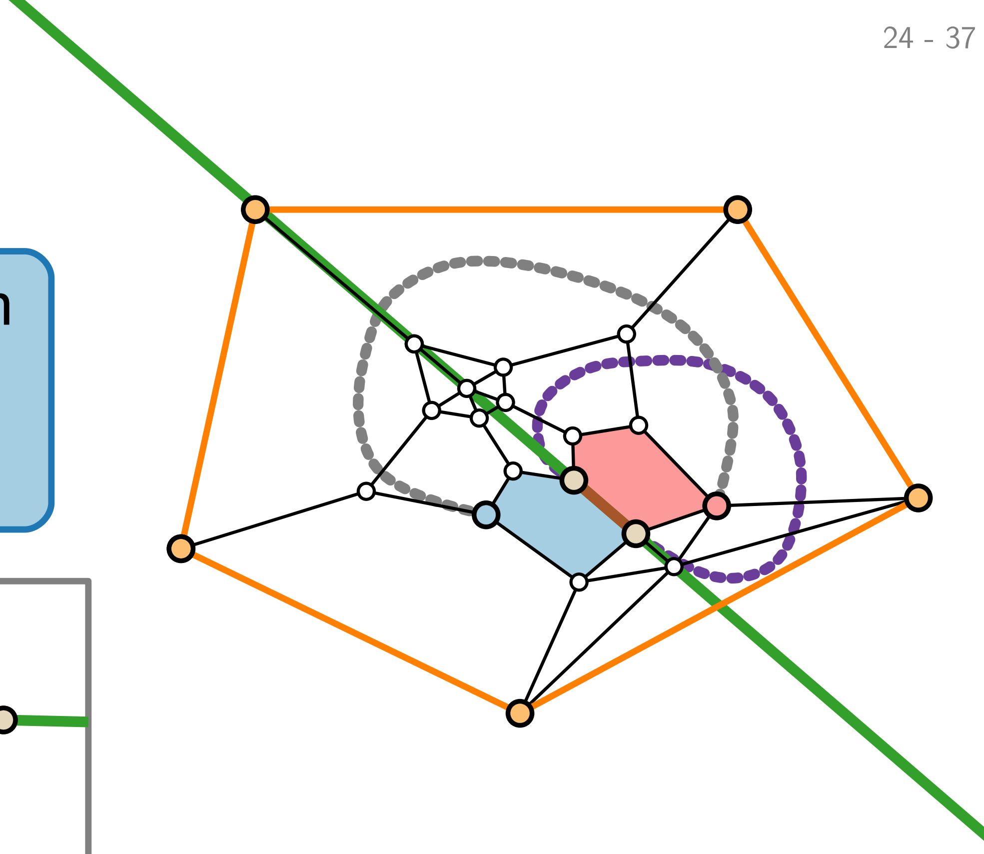

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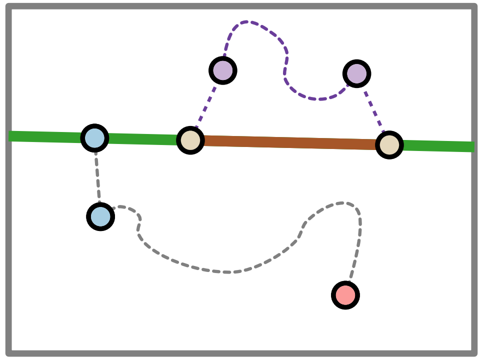

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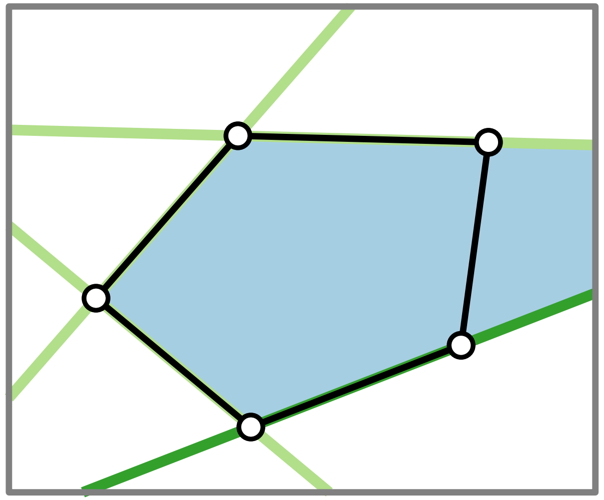

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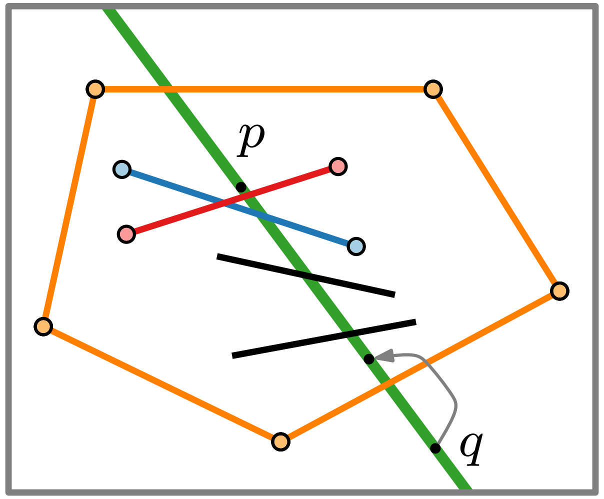
jumping over edge
$\rightarrow$ #faces the same



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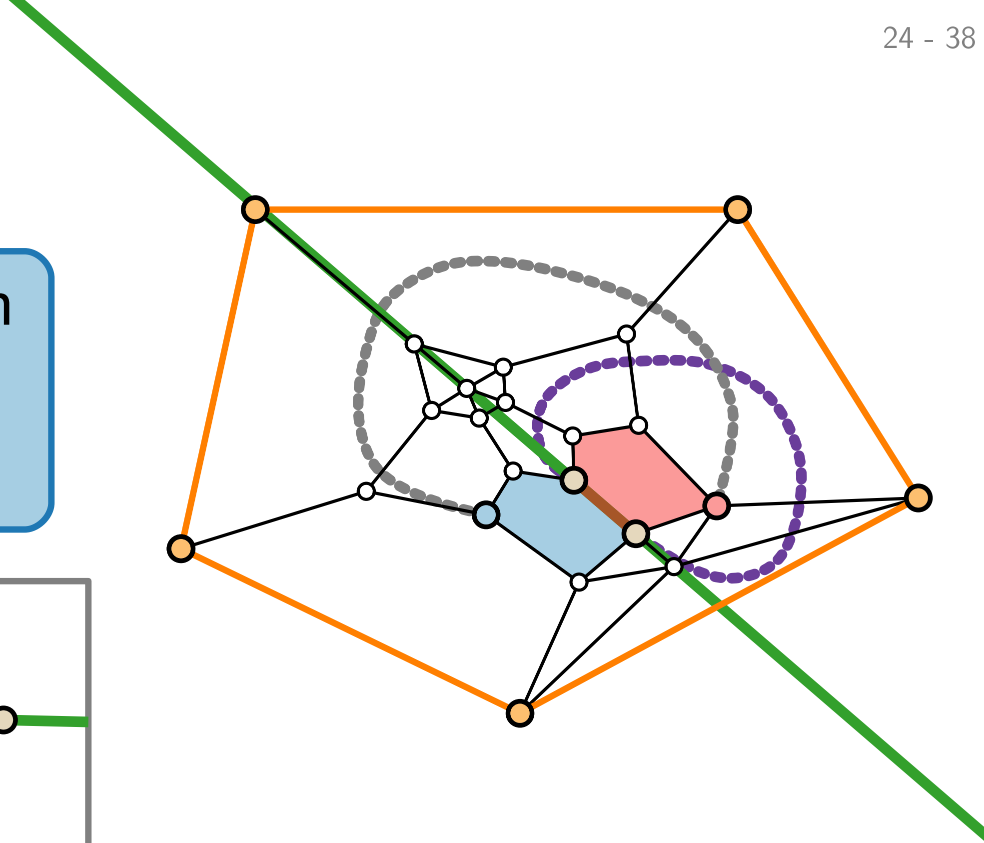

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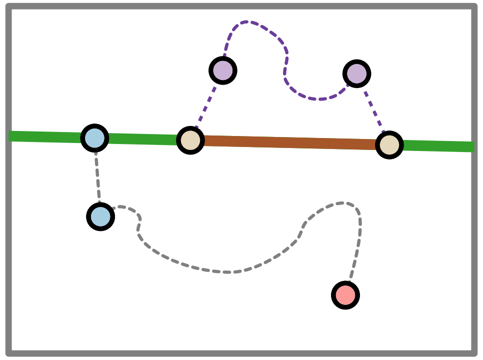

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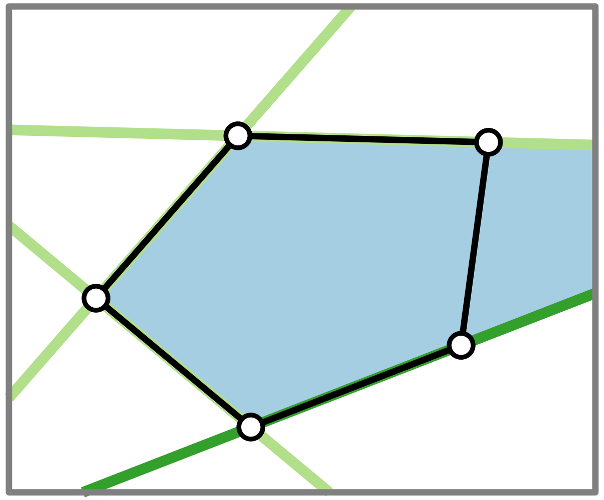

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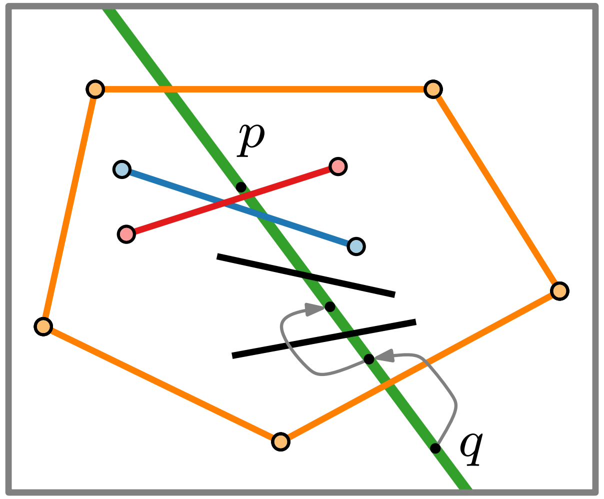

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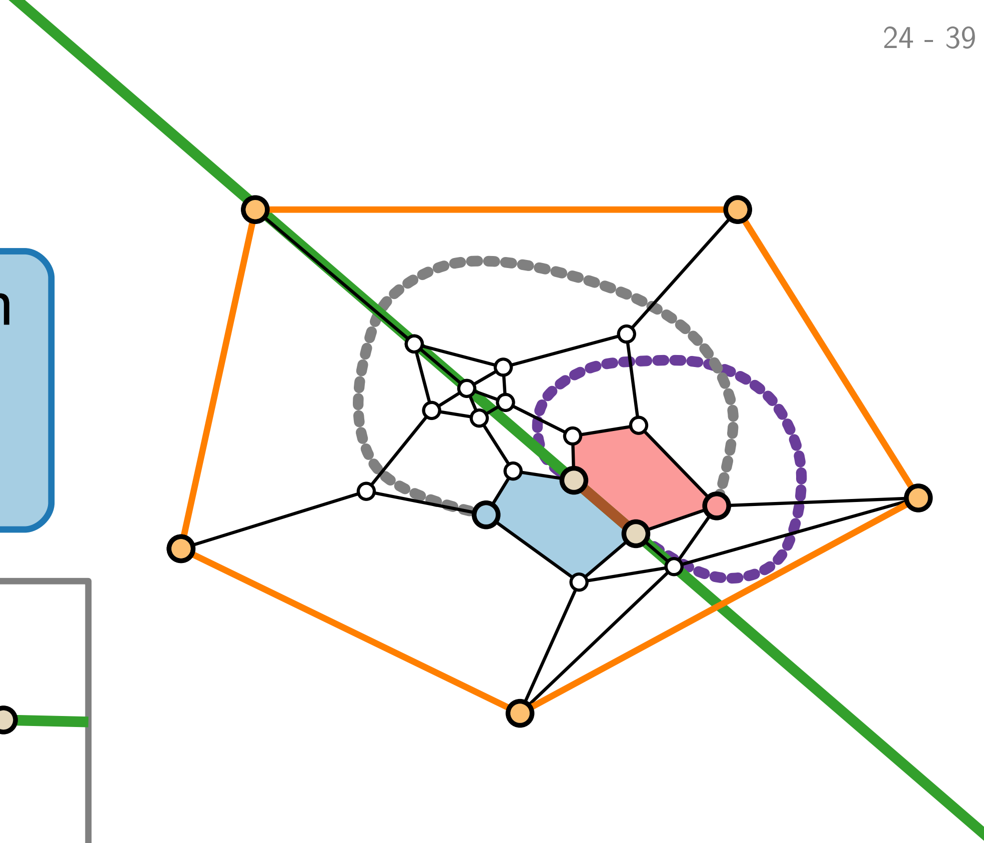

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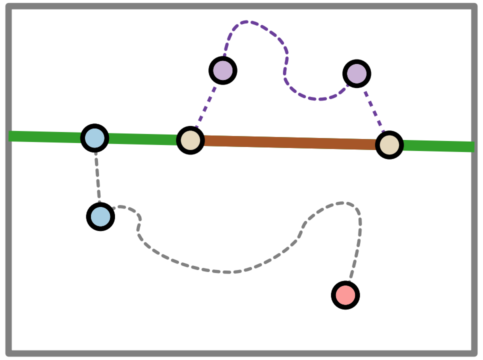

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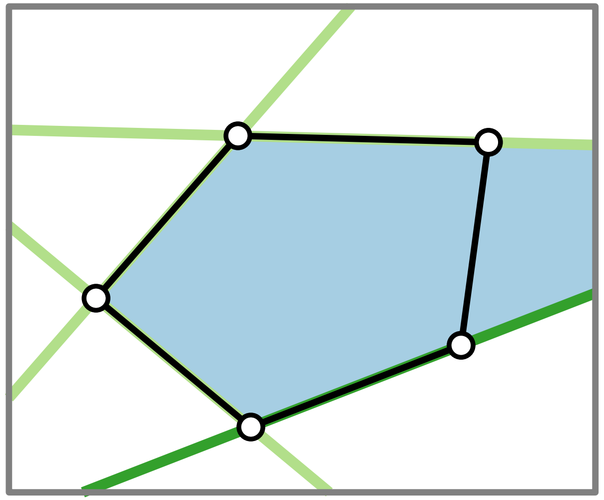

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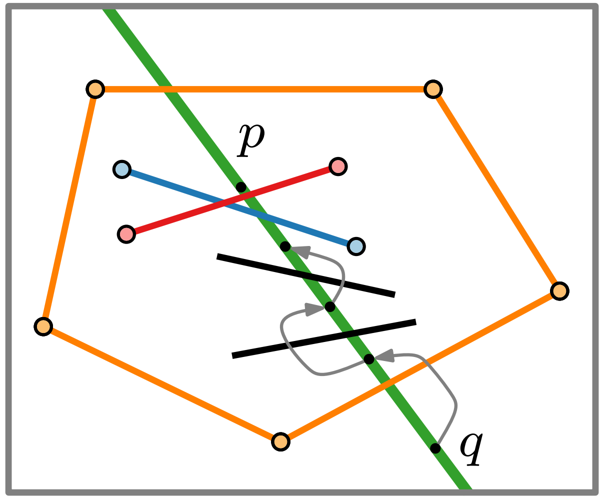

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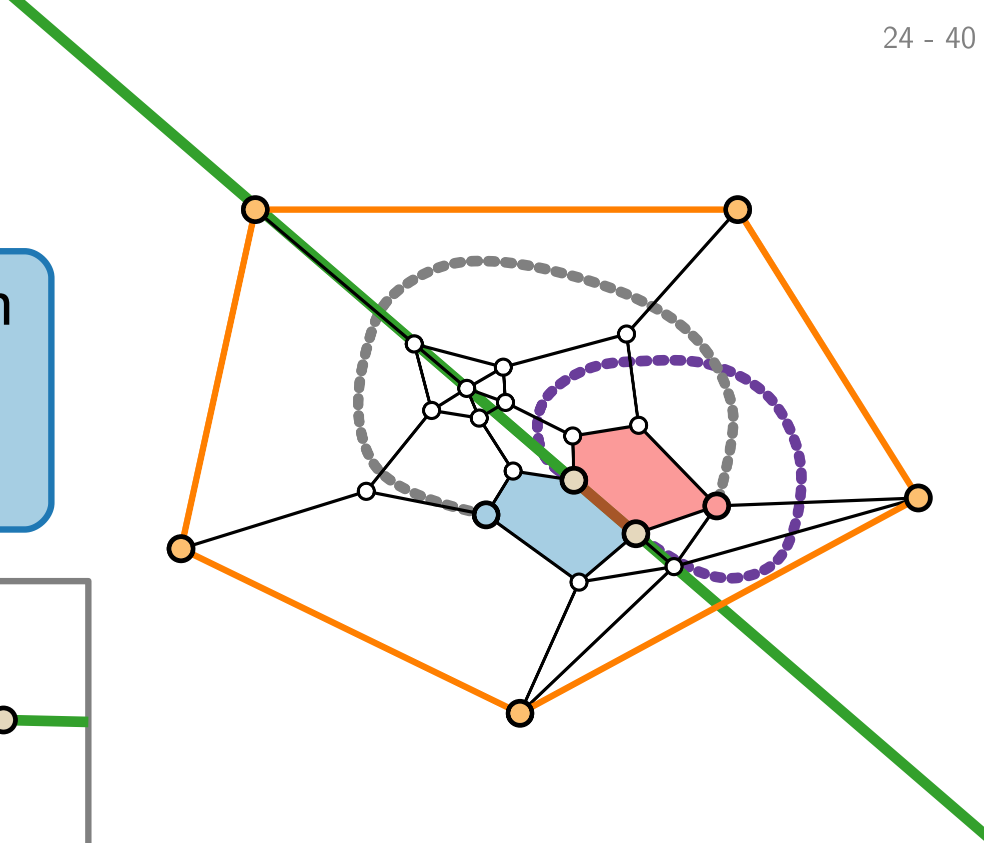

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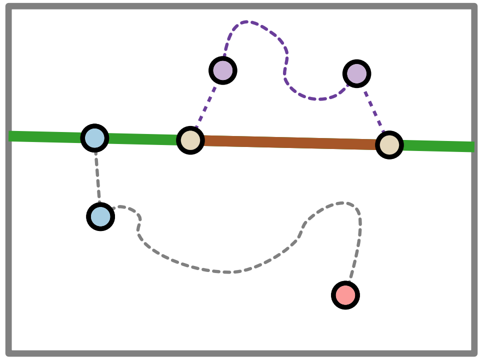

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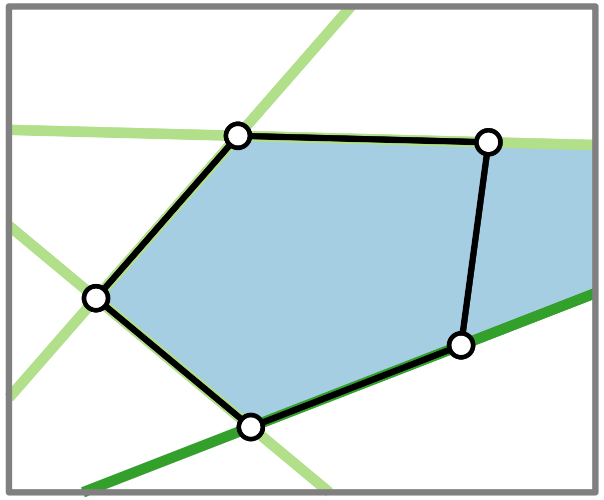

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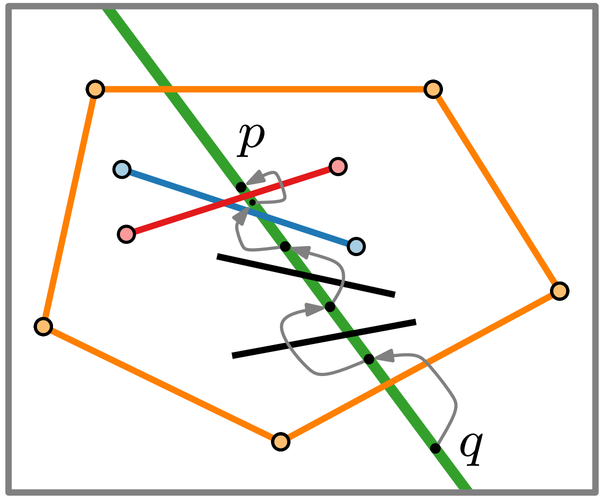

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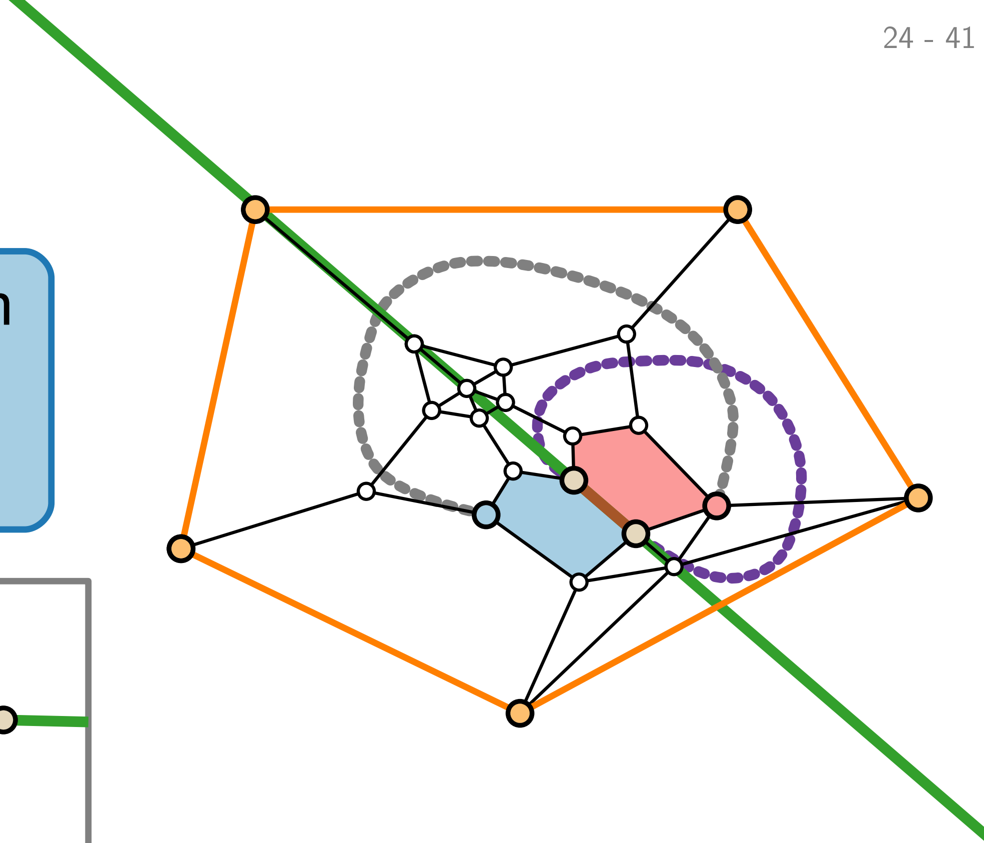

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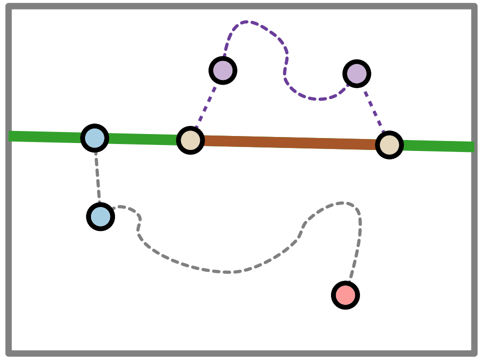

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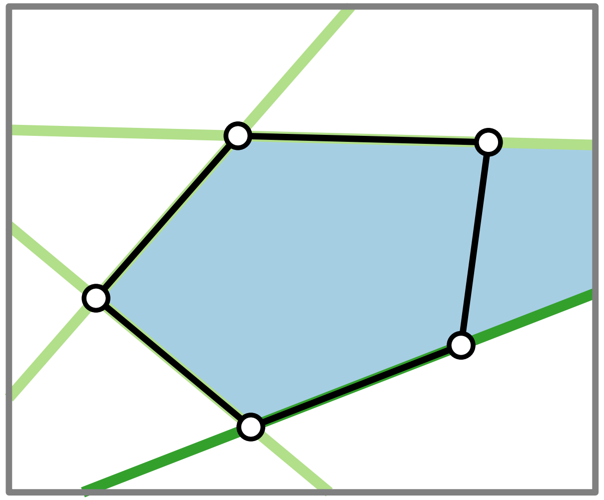

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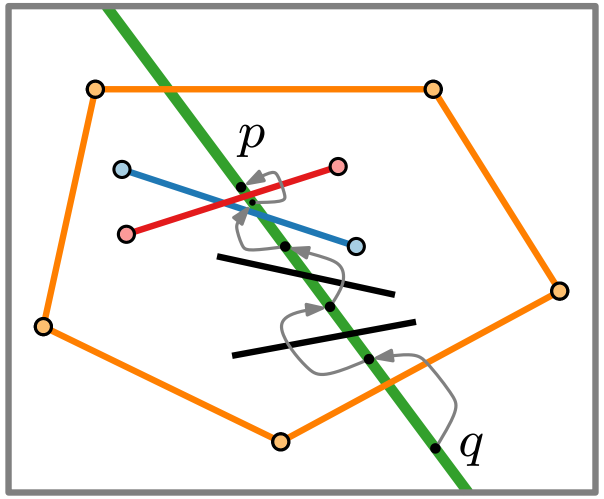

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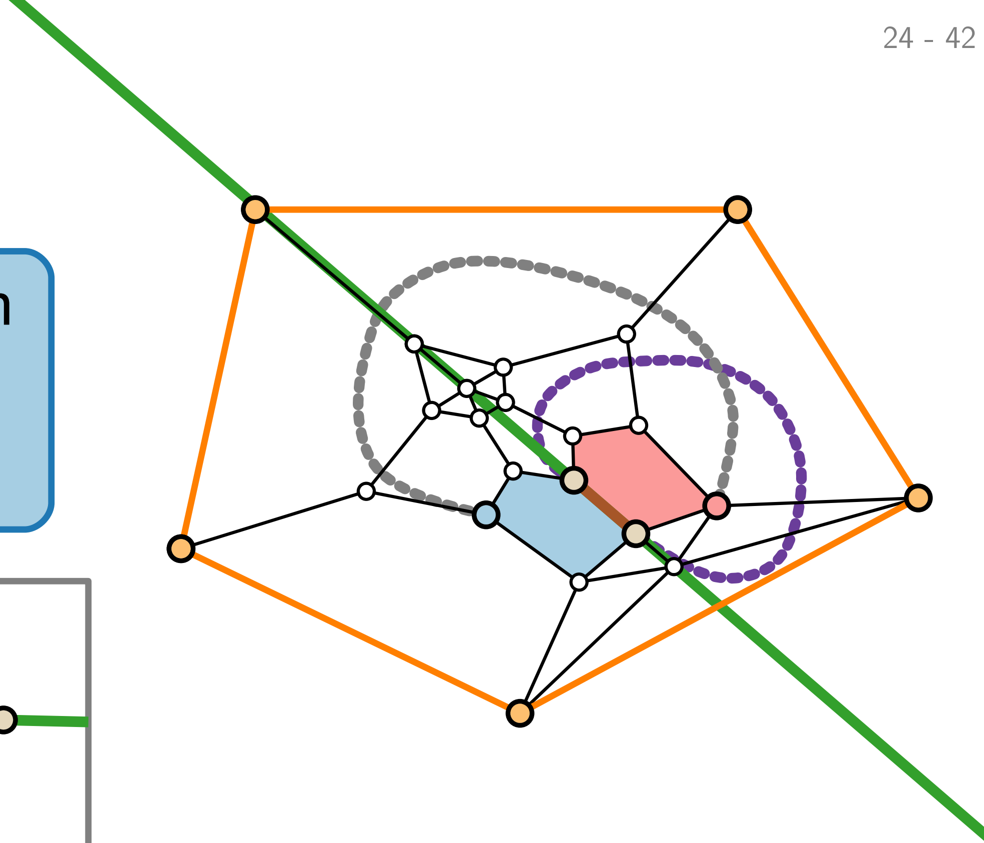

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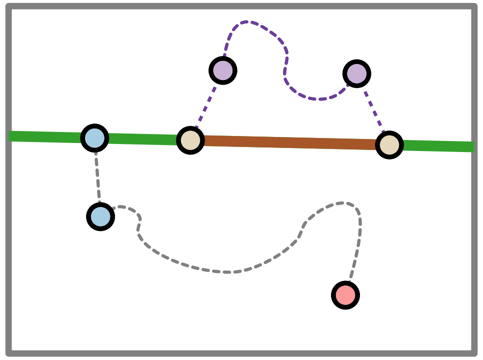

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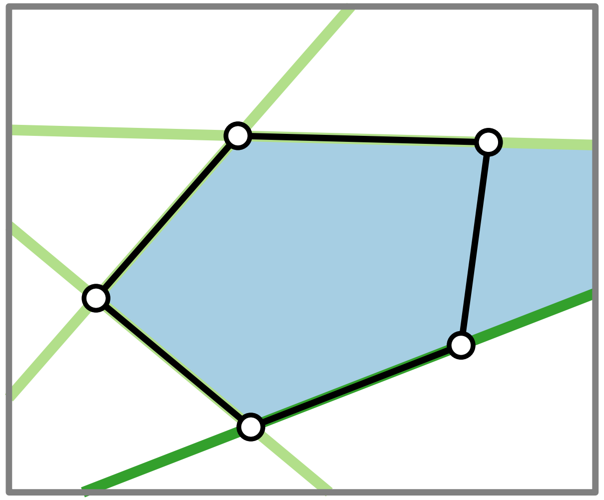

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p inside two faces
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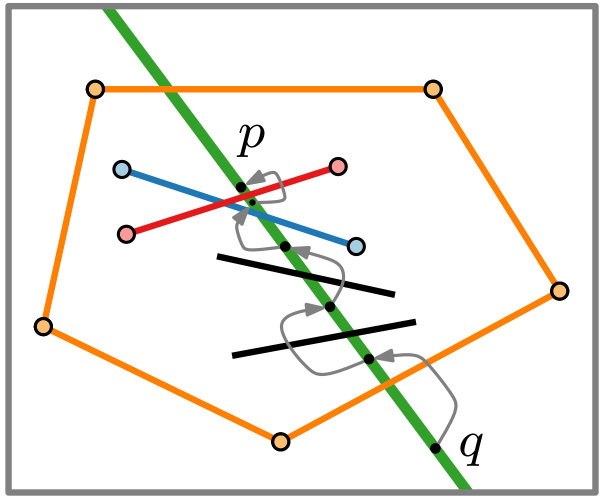

Literature

Main sources:

- [GD Ch. 10] Force-Directed Methods
- [DG Ch. 4] Drawing on Physical Analogies

Original papers:

- [Eades 1984] A heuristic for graph drawing
- [Fruchterman, Reingold 1991] Graph drawing by force-directed placement
- [Tutte 1963] How to draw a graph