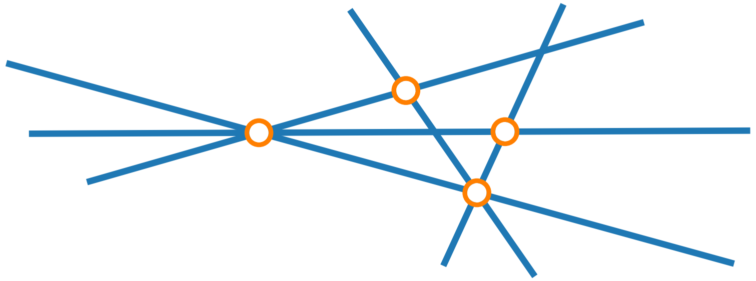
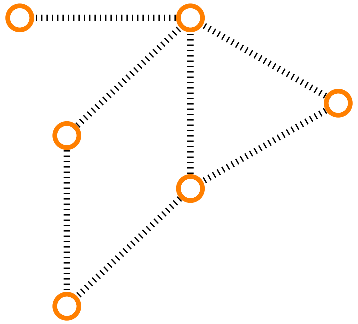
# Visualization of Graphs

## Lecture 11:
## The Crossing Lemma
### and its Applications





## Part I:
## Definition and Hanani–Tutte

Jonathan Klawitter

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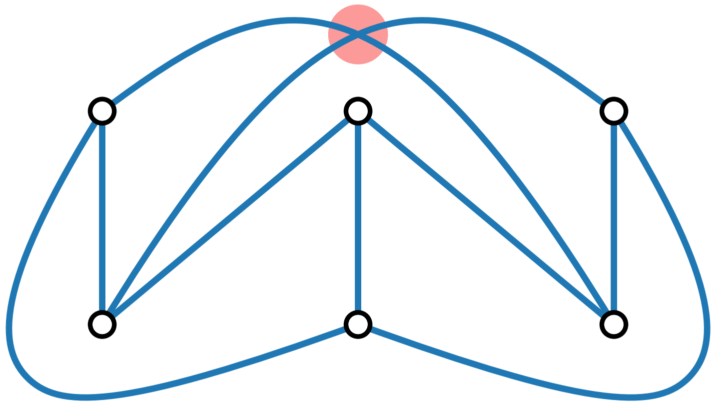
# Crossing Number and Topological Graphs

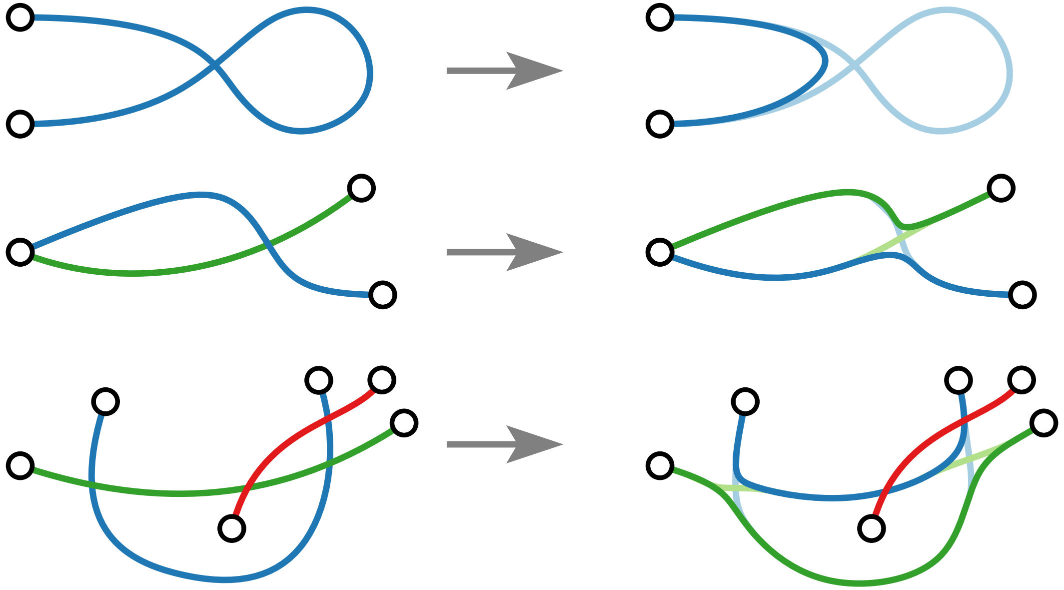
For a graph $G$, the **crossing number** $\text{cr}(G)$ is the smallest number of crossings in a drawing of $G$ (in the plane).

**Example.**
$\text{cr}(K_{3,3}) = 1$



In a crossing-minimal drawing of $G$

- no edge is self-intersecting,
- edges with common endpoints do not intersect,
- two edges intersect at most once, ?
- and wlog, at most two edges intersect at the same point.



# crossings reduced, so terminates

Such a drawing is called a **topological drawing** of $G$.

# Hanani–Tutte Theorem

**Theorem.** [Hanani'43, Tutte'70]
A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.

**Proof Sketch.**
Hanani showed that every drawing of $K_5$ and $K_{3,3}$ must have a pair of edges that crosses an odd number of times.

Every non-planar graph has $K_5$ or $K_{3,3}$ as minor, so there are two paths that cross an odd number of times.

Hence, there must be two edges on these paths that cross an odd number of times. $\square$

# Hanani–Tutte Theorem

**Theorem.** [Hanani'43, Tutte'70]
A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.

The **odd crossing number** ocr$(G)$ of $G$ is the smallest number of pairs of edges that cross oddly in a drawing of $G$.

**Corollary.** ocr$(G) = 0 \Rightarrow$ cr$(G) = 0$

Is ocr$(G) =$ cr$(G)$? No!

**Theorem.** [Pelsmajer, Schaefer & Štefankovič '08, Tóth '08]
There is a graph $G$ with ocr$(G) <$ cr$(G) \leq 10$

**Theorem.** [Pach & Tóth '00]
If Γ is a drawing of G and $E_0$ is the set of edges with only even numbers of crossings in Γ, then $G$ can be drawn such that no edge in $E_0$ is involved in any crossings and no new p

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Is $\text{ocr}(G) = \text{cr}(G)$? No!

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# Hanani–Tutte Theorem

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A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.

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Is $\text{ocr}(G) = \text{cr}(G)$? No!

**Theorem.** [Pelsmajer, Schaefer & Štefankovič '08, Tóth '08]
There is a graph $G$ with $\text{ocr}(G) < \text{cr}(G) \leq 10$

The **pairwise crossing number** $\text{pcr}(G)$ of $G$ is the smallest number of pairs of edges that cross in a drawing of $G$.

By definition $\text{ocr}(G) \leq \text{pcr}(G) \leq \text{cr}(G)$
Is $\text{pcr}(G) = \text{cr}(G)$? Open!

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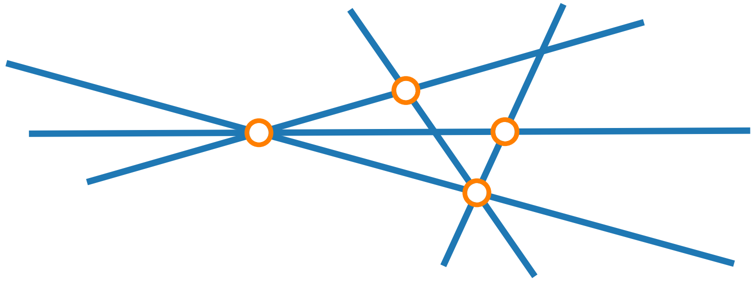

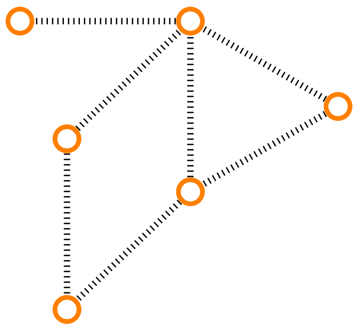

## Part II:
## Computation & Variations

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# Computing the Crossing Number

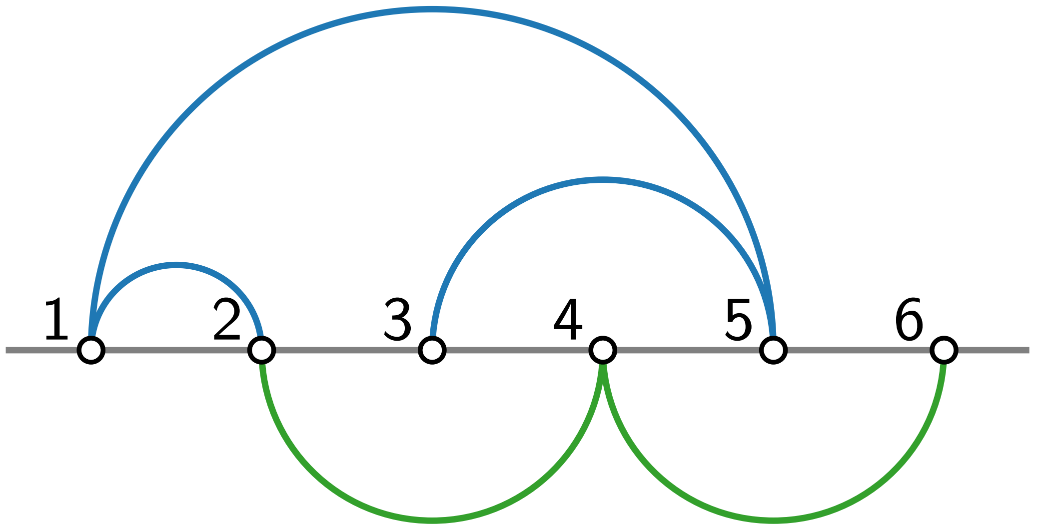
- Computing cr($G$) is NP-hard. [Garey & Johnson '83]
  ... even if $G$ is a planar graph plus one edge! [Cabello & Mohar '08]
- cr($G$) often unknown, only conjectures exist
  - for $K_n$ it is only known for up to $\sim 12$ vertices
- In practice, cr($G$) is often not computed directly but rather drawings of $G$ are optimized with
  - force-based methods,
  - multidimensional scaling,
  - heuristics, ...

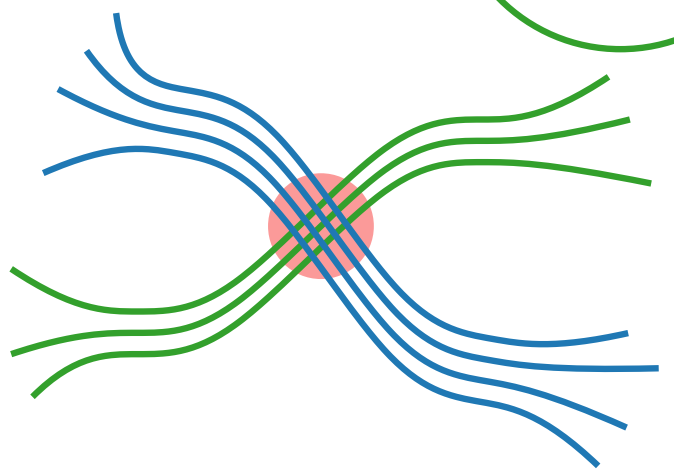
For exact computations, check out `http://crossings.uos.de`!

- cr($G$) is a measure of how far $G$ is from being planar
- Planarization, where we replace crossings with dummy vertices, also uses only heuristics

# Other Crossing Numbers

- Schaefer [Schae20] offers a huge survey on different crossings numbers (and more precise definitions)
- One-sided crossing minimization . . .
- Fixed Linear Crossing Number
- In book embeddings
- Crossings of edge bundles
- On other surfaces, like on donuts
- Weighted crossings
- Crossing minimization is NP-hard for most of the variants

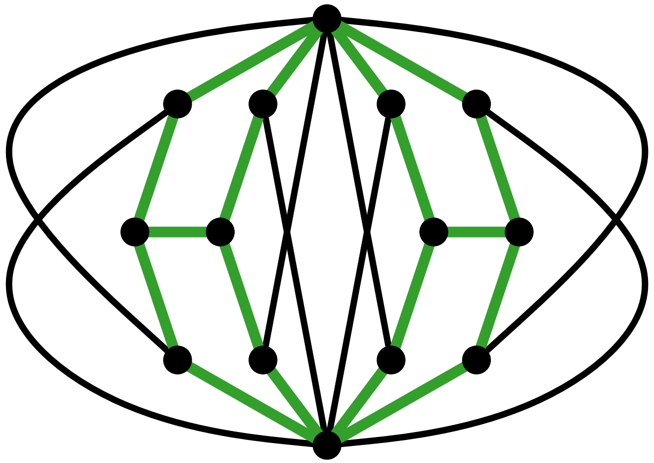


# Rectilinear Crossing Number

**Definition.**
For a graph $G$, the **rectilinear (straight-line) crossing number** $\overline{\text{cr}}(G)$ is the smallest number of crossings in a straight-line drawing of $G$.

**Separation.**
$\text{cr}(K_8) = 18$, but $\overline{\text{cr}}(K_8) = 19$.

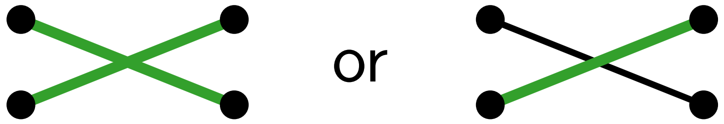


$G_1$

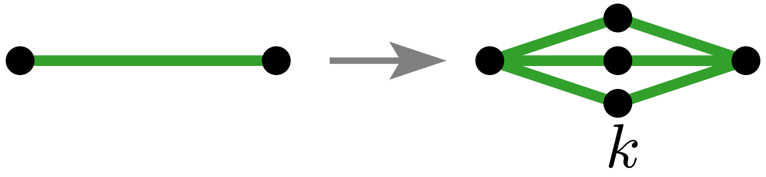
Even more . . .

**Lemma 1.** [Bienstock, Dean '93]
For $k \geq 4$, there exists a graph $G_k$ with $\text{cr}(G_k) = 4$ and $\overline{\text{cr}}(G_k) \geq k$.

- Each straight-line drawing of $G_1$ has at least one crossing of the following types:



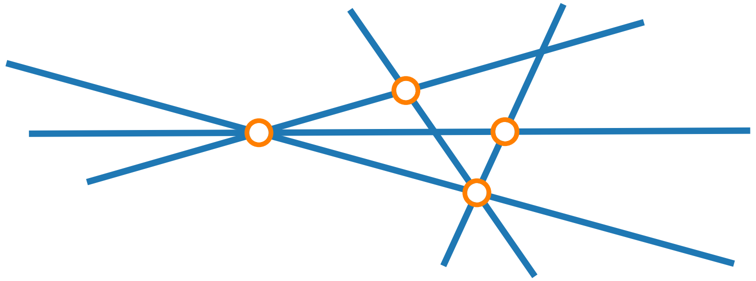

- From $G_1$ to $G_k$ do



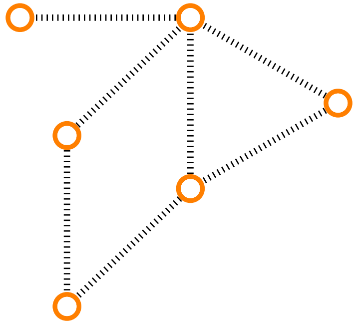
# Visualization of Graphs

## Lecture 11: The Crossing Lemma

### and its Applications



## Part III: First Bounds



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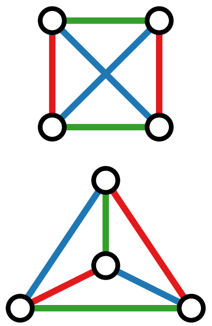
# Bounds for Complete Graphs

**Theorem.** **Conjecture.** [Guy '60]

$$\mathrm{cr}(K_n) \overset{\not\leq}{=} \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{n-2}{2}\right\rfloor\left\lfloor\frac{n-3}{2}\right\rfloor = \frac{3}{8}\binom{n}{4} + O(n^3)$$

Bound is sharp for $n \leq 12$.



Sylvester's four-point problem

**Theorem.** **Conjecture.** [Zarankiewicz '54, Urbaník '55]

$$\mathrm{cr}(K_{m,n}) \overset{\not\leq}{=} \frac{1}{4}\left\lfloor\frac{n}{2}\right\rfloor\left\lfloor\frac{n-1}{2}\right\rfloor\left\lfloor\frac{m}{2}\right\rfloor\left\lfloor\frac{m-1}{2}\right\rfloor$$

Turán's brick factory problem (1944)

Pál Turán
*1910 – 1976
Budapest, Hungary

© TruckinTim

# Bounds for Complete Graphs

**Theorem.** **Conjecture.** [Guy '60]

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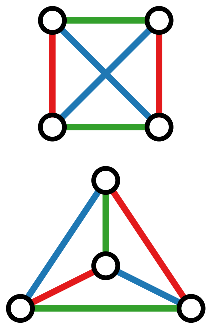

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**Theorem.** [Lovász et al. '04, Aichholzer et al. '06]

$$\left(\frac{3}{8}+\varepsilon\right)\binom{n}{4} + O(n^3) < \overline{\mathrm{cr}}(K_n) < 0.3807\binom{n}{4} + O(n^3)$$

Exact numbers are known for $n \leq 27$.

Check out `http://www.ist.tugraz.at/staff/aichholzer/crossings.html`!

# First Lower Bounds on cr$(G)$

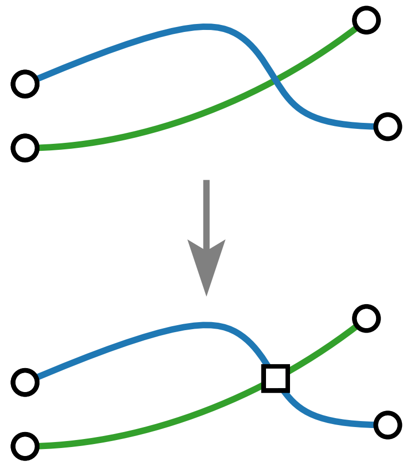
**Lemma 2.**
For a graph $G$ with $n$ vertices and $m$ edges,

$$\text{cr}(G) \geq m - 3n + 6.$$

**Proof.**

- Consider a drawing of $G$ with cr$(G)$ crossings.
- Obtain a graph $H$ by turning crossings into dummy vertices.
- $H$ has $n + \text{cr}(G)$ vertices and $m + 2\text{cr}(G)$ edges.
- $H$ is planar, so

$$m + 2\text{cr}(G) \leq 3(n + \text{cr}(G)) - 6.$$



# First Lower Bounds on cr($G$)

**Lemma 3.**
For a graph $G$ with $n$ vertices and $m$ edges,

$$\text{cr}(G) \geq r\binom{\lfloor m/r \rfloor}{2} \in \Omega\left(\frac{m^2}{n}\right)$$

where $r \leq 3n - 6$ is the maximum number of edges in a planar subgraph of $G$.

Consider this bound for graphs with $\Theta(n)$ and $\Theta(n^2)$ many edges.

**Proof.**

- Take $\lfloor m/r \rfloor$ edge-disjoint subgraphs of $G$ with $r$ edges.
- In the best case, they are all planar.
- For each pair $G_i, G_j$, any edge of $G_j$ induces at least one crossings with $G_i$.
  (If not, swap edges to reduce cr($G_i$).)

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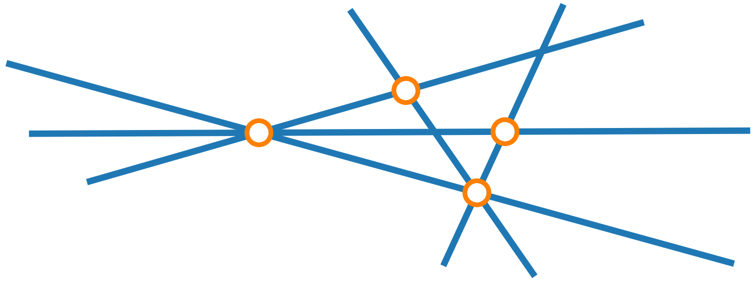

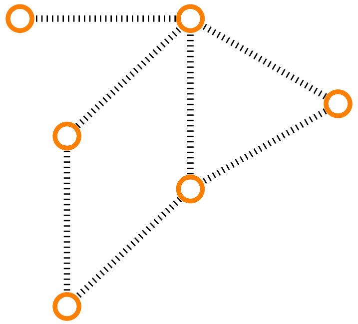

## Part IV:
## The Crossing Lemma

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# The Crossing Lemma

- 1973 Erdős and Guy conjectured that $\text{cr}(G) \in \Omega(\frac{m^3}{n^2})$.
- In 1982 Leighton and, indepedently, Ajtai, Chávtal, Newborn and Szemerédi showed that

$$\text{cr}(G) \geq \frac{1}{64} \frac{m^3}{n^2}.$$

- Bound is asymptotically sharp.
- Result stayed hardly known until Székely in 1997 demonstrated its usefulness.
- We look at a proof "from THE BOOK" by Chazelle, Sharir and Welz.

- Factor $\frac{1}{64}$ was later (with intermediate steps) improved to $\frac{1}{29}$ by Ackerman in 2013.

# The Crossing Lemma

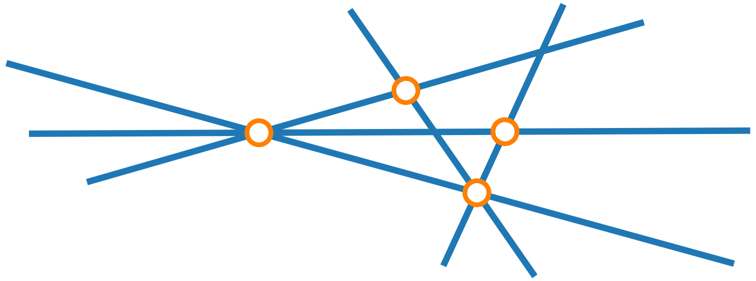
**Crossing Lemma.**
For a graph $G$ with $n$ vertices and $m$ edges, $m \geq 4n$,

$$\text{cr}(G) \geq \frac{1}{64}\frac{m^3}{n^2}.$$

**Proof.**

- Consider a minimal embedding of $G$.
- Let $p$ be a number in $(0, 1)$.
- Keep every vertex of $G$ independently with probability $p$.
- Let $G_p$ be the remaining graph.
- Let $n_p, m_p, X_p$ be the random variables counting the number of vertices/edges/crossings of $G_p$.
- By Lem 2, $\mathbb{E}(X_p - m_p + 3n_p) \geq 0$.
- $\mathbb{E}(n_p) = pn$ and $\mathbb{E}(m_p) = p^2m$
- $\mathbb{E}(X_p) = p^4\text{cr}(G)$
- $0 \leq \mathbb{E}(X_p) - \mathbb{E}(m_p) + 3\mathbb{E}(n_p)$
  $= p^4\text{cr}(G) - p^2m + 3pn$
- $\text{cr}(G) \geq \frac{p^2m - 3pn}{p^4} = \frac{m}{p^2} - \frac{3n}{p^3}$
- Set $p = \frac{4n}{m}$.
- $\text{cr}(G) \geq \frac{m^3}{16n^2} - \frac{3m^3}{64n^2} = \frac{1}{64}\frac{m^3}{n^2}$

# Visualization of Graphs



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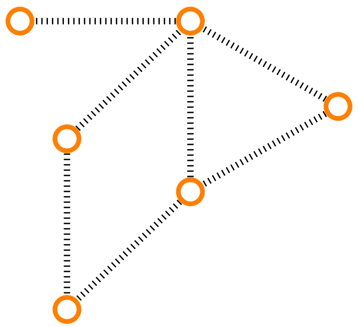

### Part V:
### Applications

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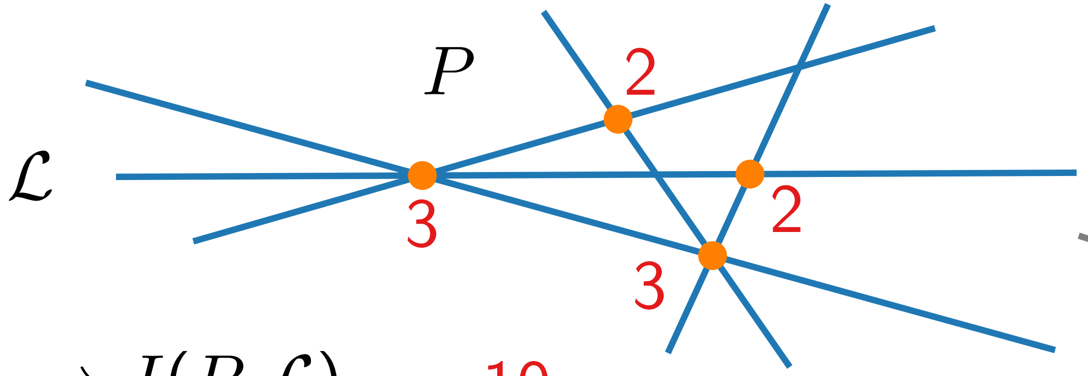
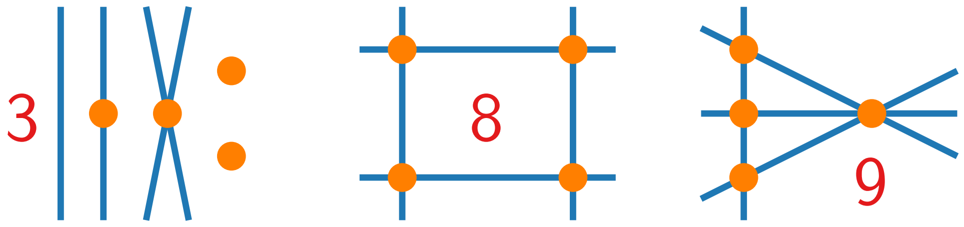
# Application 1: Point-Line Incidences

- For points $P \subset \mathbb{R}^2$ and lines $\mathcal{L}$, $I(P, \mathcal{L})$ = number of point-line incidences in $(P, \mathcal{L})$.



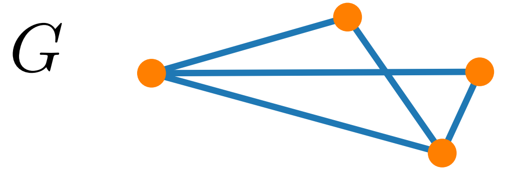

$\Rightarrow I(P, \mathcal{L}) =$ 10

- Define $I(n,k) = \max_{|P|=n, |\mathcal{L}|=k} I(P, \mathcal{L})$.
- For example: $I(4,4) =$ 9




**Theorem 1.**
[Szemerédi, Trotter '83, Székely '97]
$I(n,k) \leq c(n^{2/3}k^{2/3} + n + k)$.

**Proof.**




- cr$(G) \leq k^2$
- \# points on $l$ $= 1+$ \# edges on $l$
- $I(n,k) - k \leq m$
- Crossing Lemma: $\frac{1}{64}\frac{m^3}{n^2} \leq$ cr$(G)$
- $c'(I(n,k) - k)^3 / n^2 \leq$ cr$(G)$ $\leq k^2$
- if $m \not\geq 4n$, then $I(n,k) - k \leq 4n$

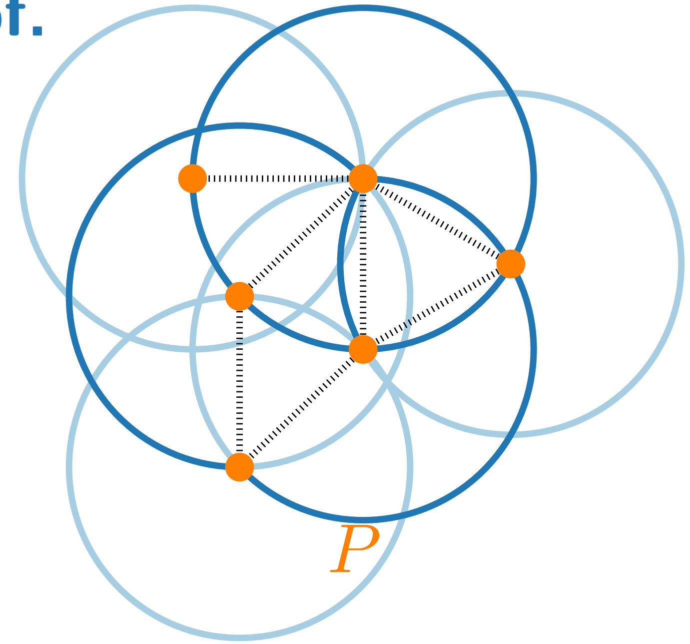
# Application 2: Unit Distances

For points $P \subset \mathbb{R}^2$ define

- $U(P) =$ number of pairs in $P$ at unit distance and
- $U(n) = \max_{|P|=n} U(P)$.

**Theorem 2.**
[Spencer, Szemerédi, Trotter '84, Székely '97]
$U(n) < 6.7n^{4/3}$

**Proof.**



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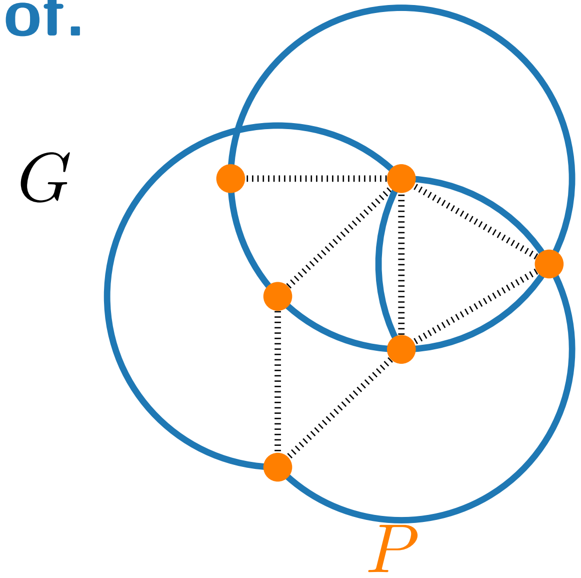
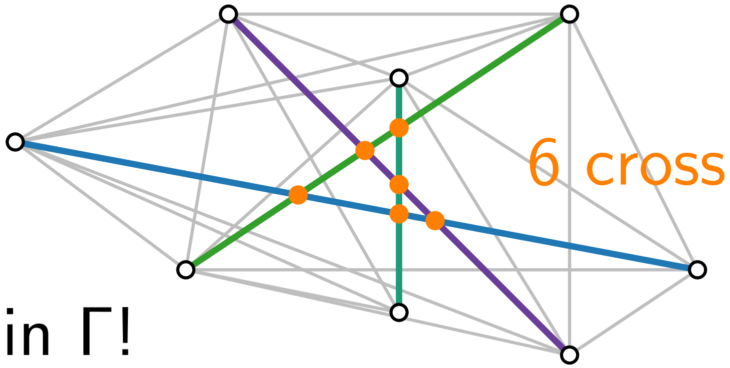



- $U(P) \leq c' m$
- $\text{cr}(G) \leq 2n^2$
- $c \frac{U(P)^3}{n^2} \leq \text{cr}(G) \leq 2n^2$

# Application 3: Max. Num. of Crossings in Matchings




Given point set of $n$ points

What is the max. number of crossings in any matching?

Point set spans drawing $\Gamma$ of $K_n$

We will analyze the number of crossings in a **random** matching in $\Gamma$!

Number of crossings in $\Gamma \geq \overline{\text{cr}}(K_n) > \frac{3}{8}\binom{n}{4}$

Number of edges in $K_n$: $\binom{n}{2}$

Number of *potential crossings* (all pairs of edges): $\text{pot}(K_n) = \binom{\binom{n}{2}}{2} \approx 3\binom{n}{4}$

Pick two random edges $e_1, e_2$

$\Pr[e_1 \text{ and } e_2 \text{ cross}] \geq \overline{\text{cr}}(K_n)/\text{pot}(K_n) > \frac{1}{8}$

Fix matching $M$; it has $\leq n/2$ edges, so $\binom{n/2}{2} = \frac{1}{8}n(n-2)$ pairs of edges

By linearity of expectations,
exp. number of crossings in $M$ is $> \frac{1}{8}\binom{n/2}{2} = \frac{1}{64}n(n-2)$

# Literature

- [Aigner, Ziegler] Proofs from THE BOOK
- [Schaefer ’20] The Graph Crossing Number and its Variants: A Survey
- Terrence Tao blog post “The crossing number inequality” from 2007
- [Garey, Johnson ’83] Crossing number is NP-complete
- [Bienstock, Dean ’93] Bounds for rectilinear crossing numbers
- [Székely ’97] Crossing Numbers and Hard Erdös Problems in Discrete Geometry
- Documentary/Biography “N Is a Number: A Portrait of Paul Erdös”
- Exact computations of crossing numbers: `http://crossings.uos.de`