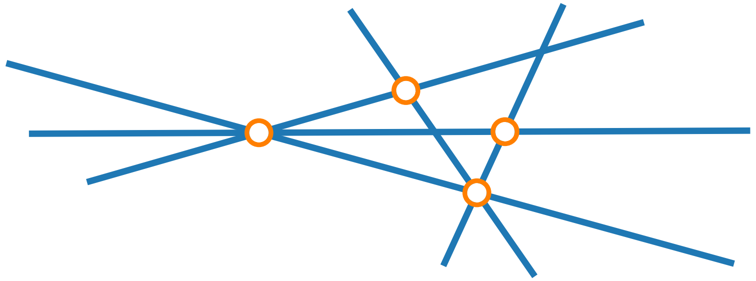
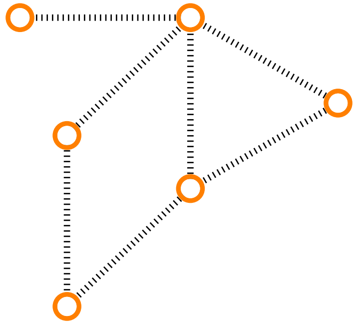
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Visualization of Graphs

Lecture 11:
The Crossing Lemma
and its Applications





Part I:
Definition and Hanani–Tutte

Jonathan Klawitter

Crossing Number and Topological Graphs

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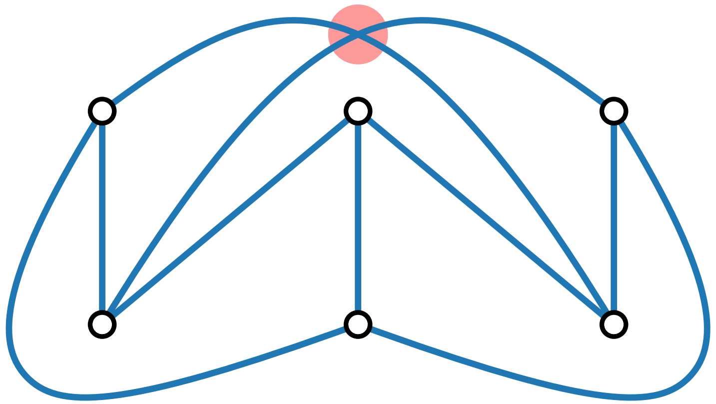
$\mathrm{cr}(K_{3,3}) = 1$



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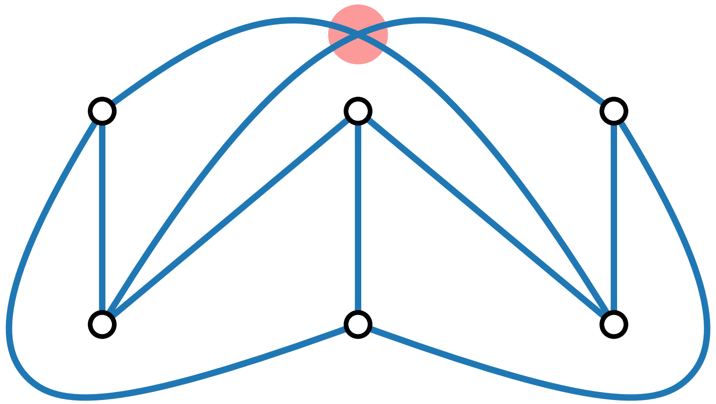

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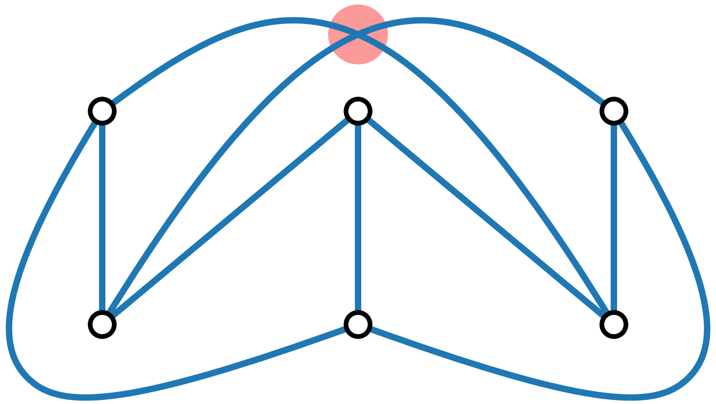

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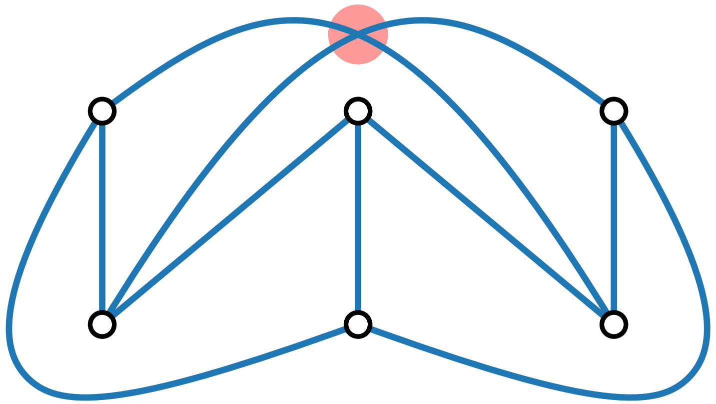

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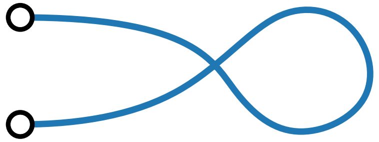

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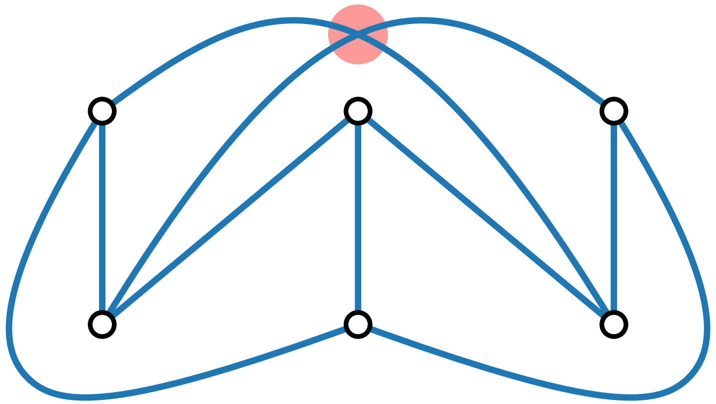

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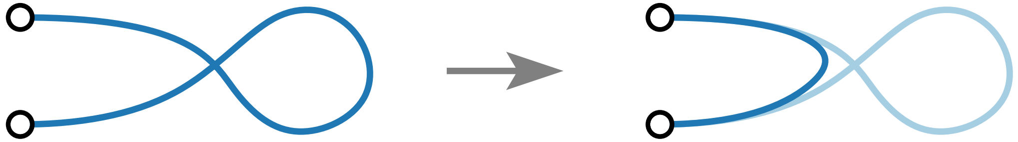

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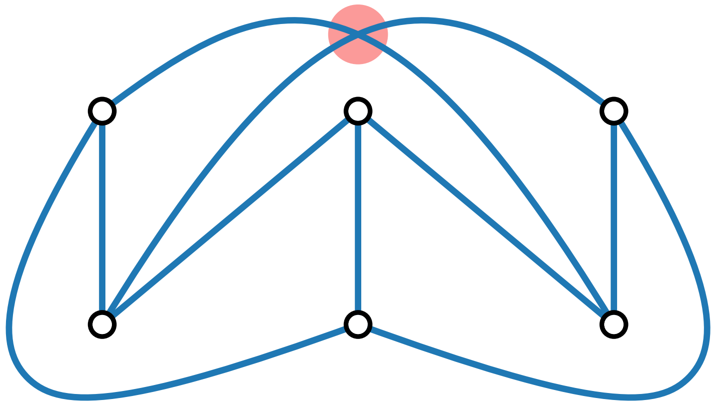

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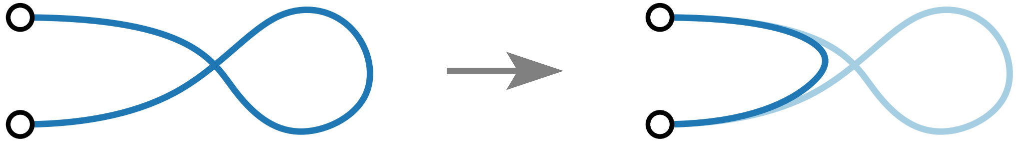

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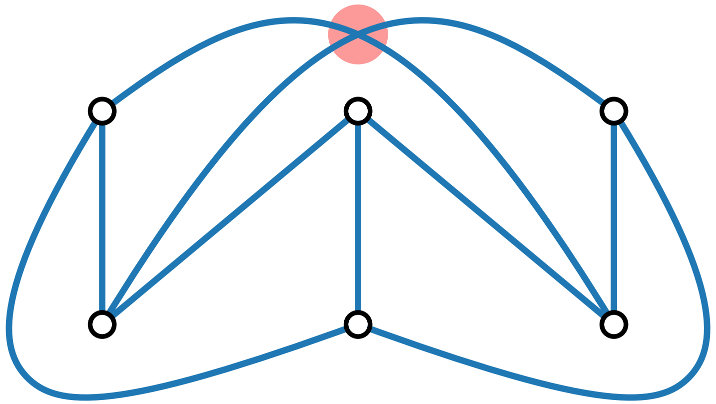

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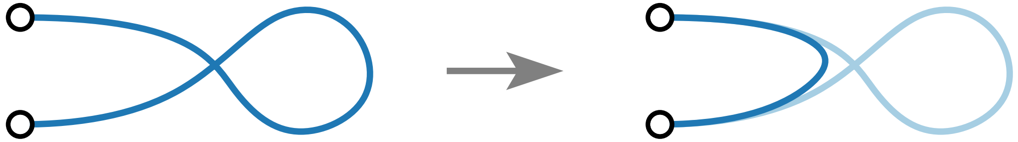

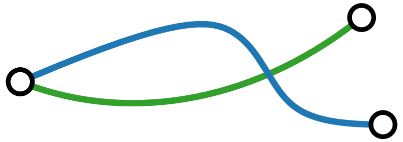

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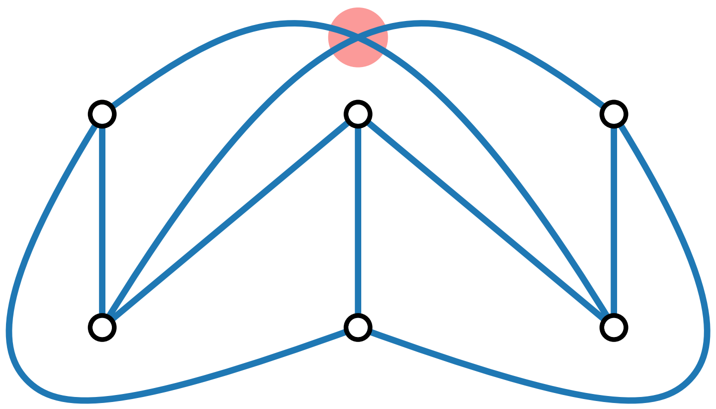

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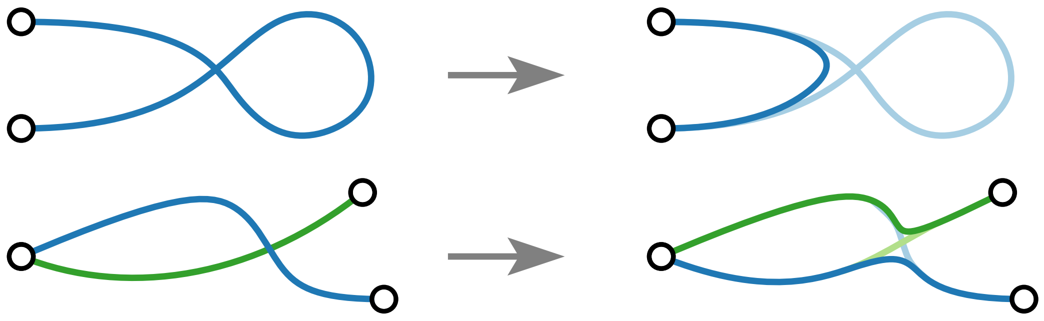

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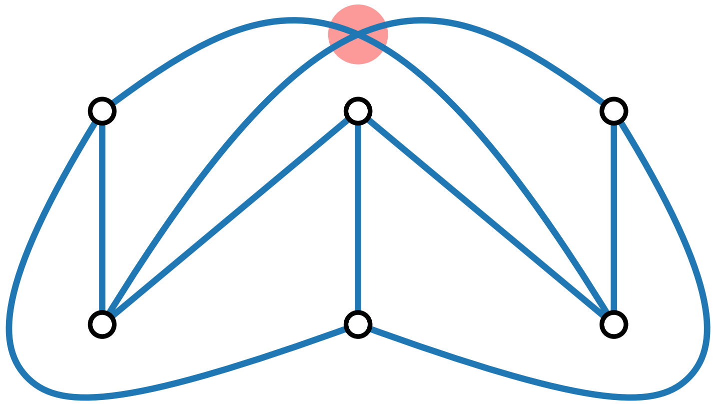

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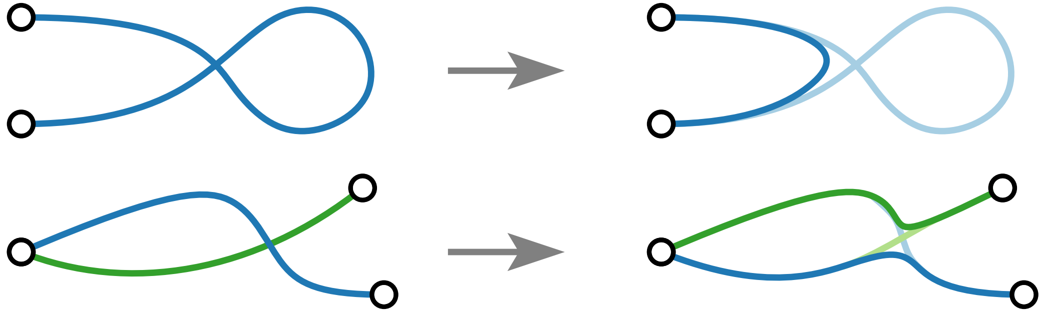

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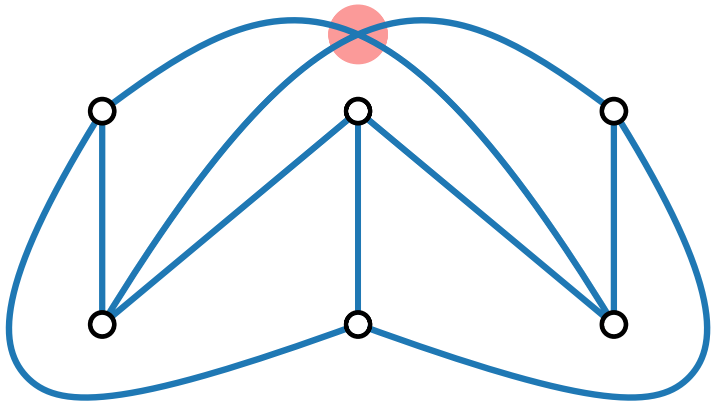

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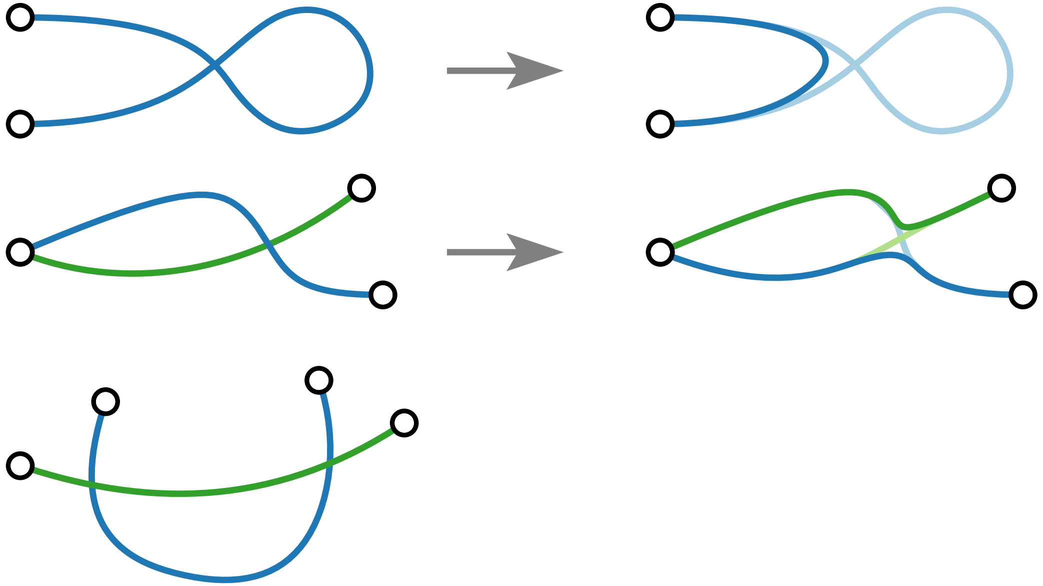

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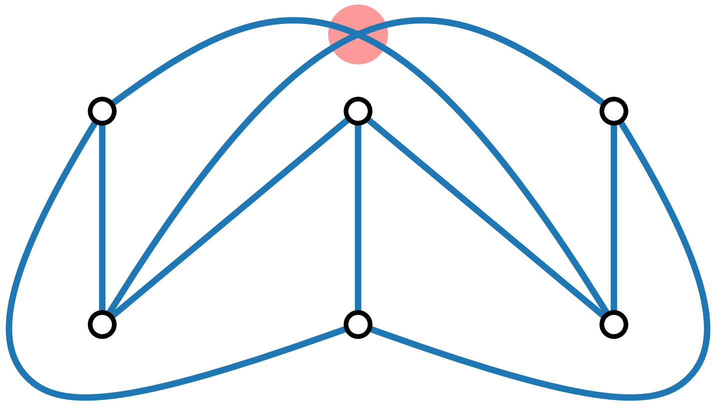

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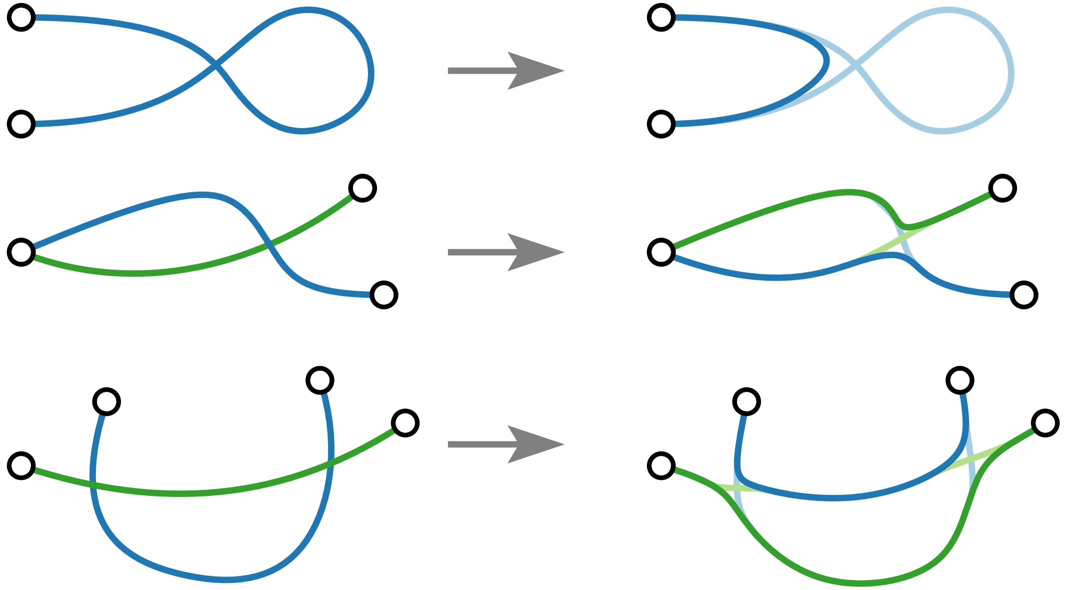

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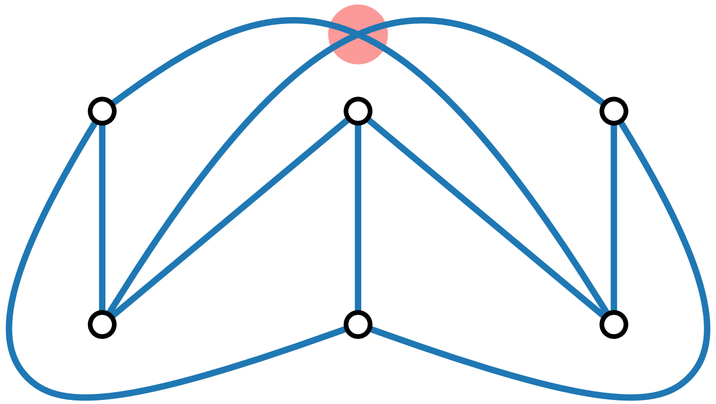

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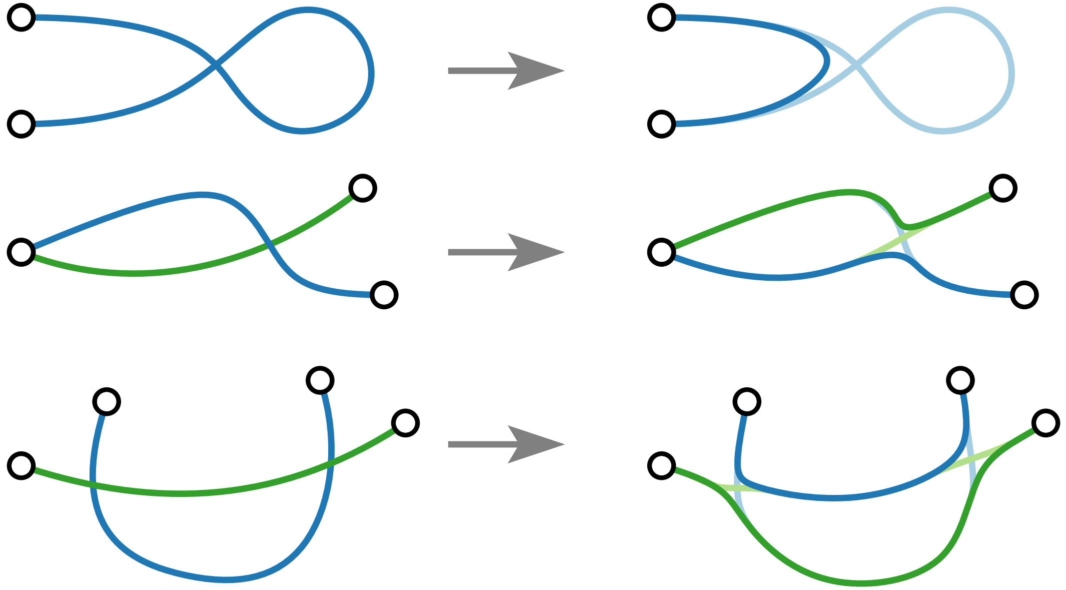

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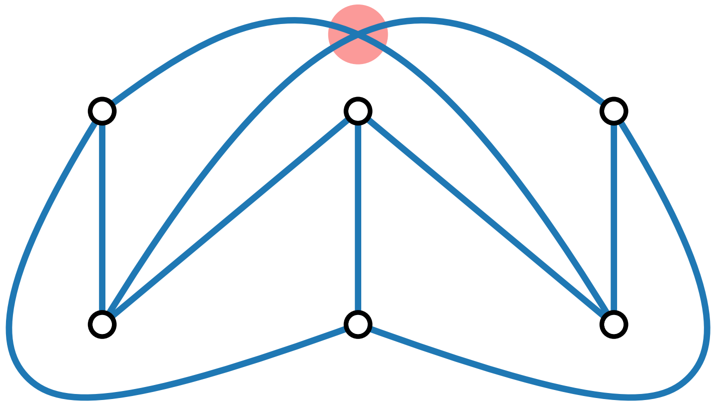

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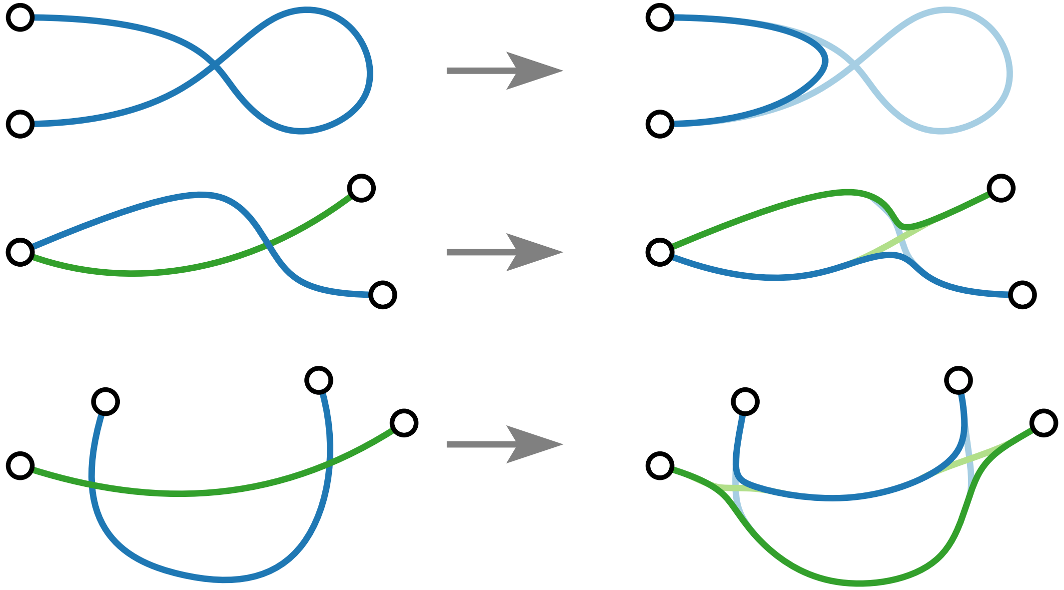

Such a drawing is called a **topological drawing** of G.

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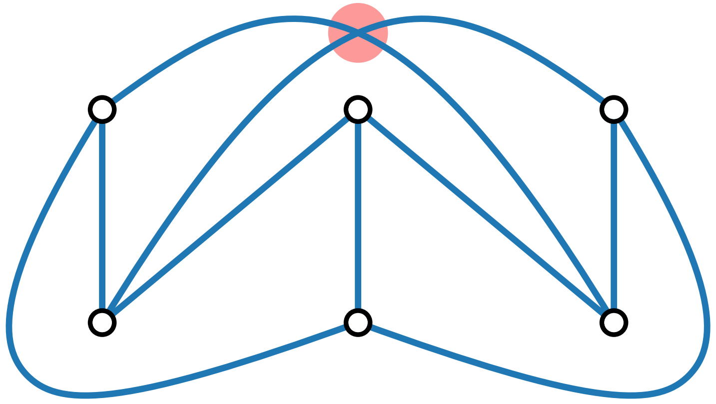

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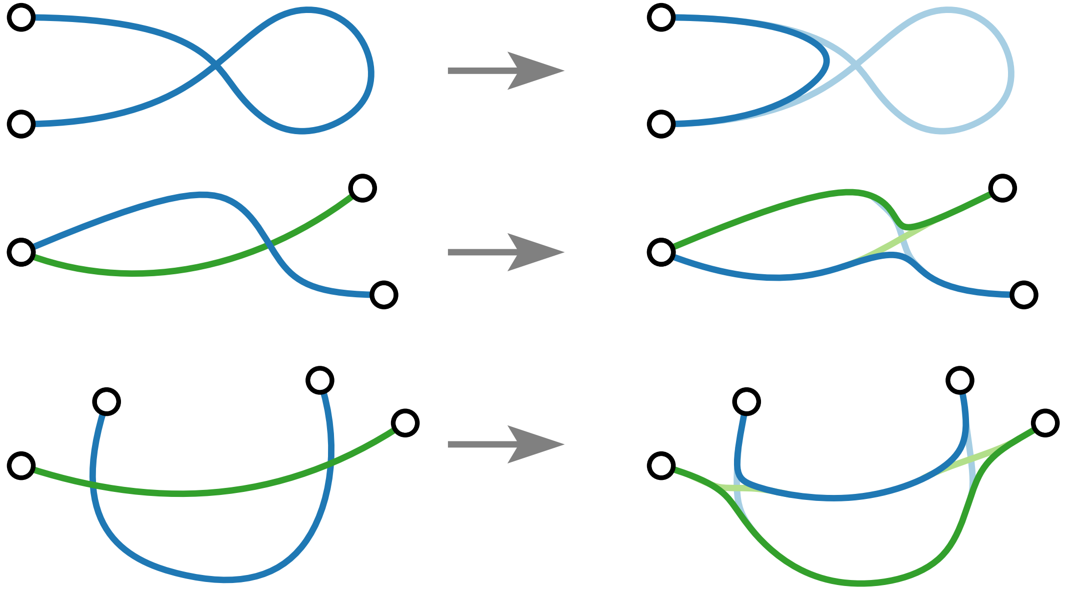

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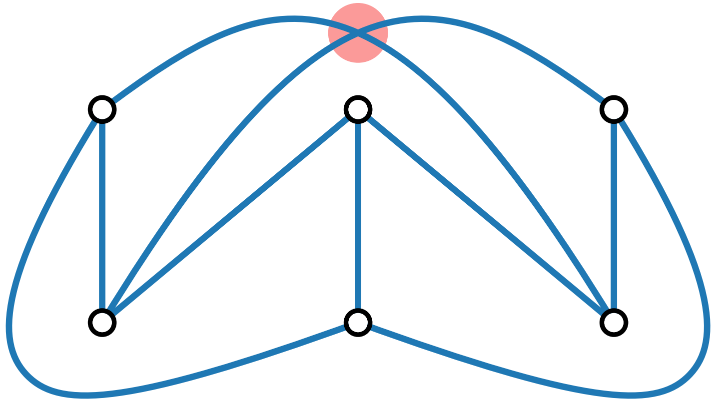

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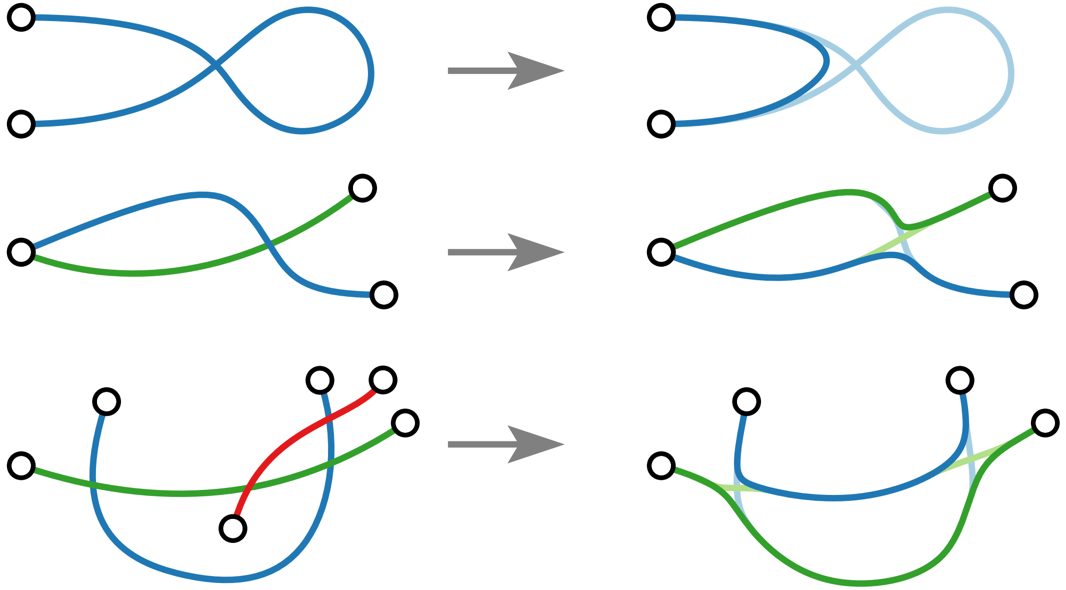

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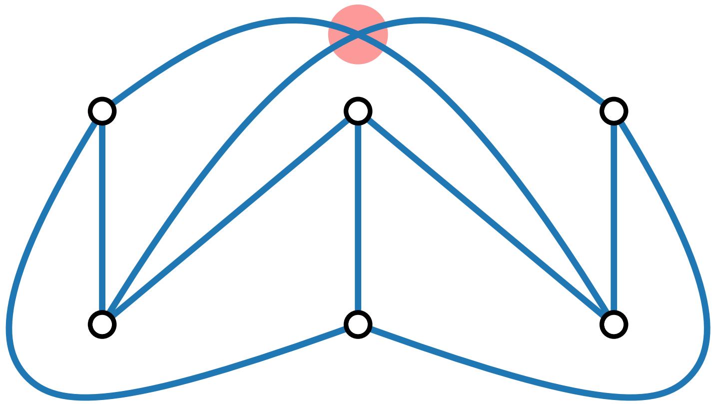

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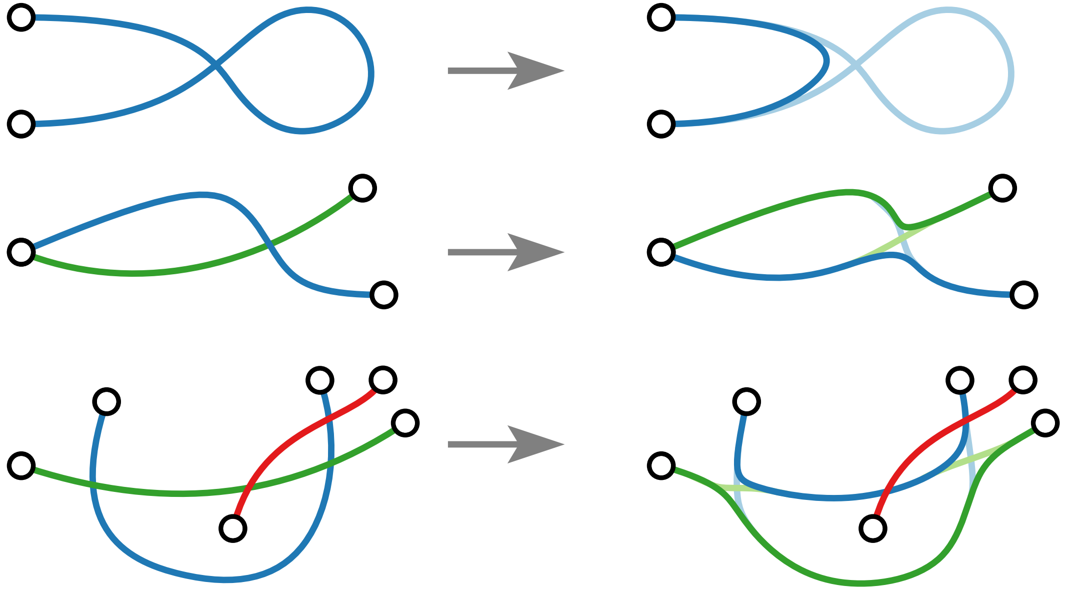

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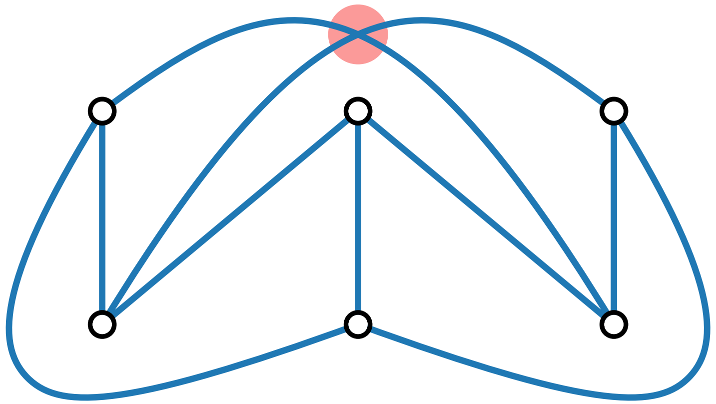

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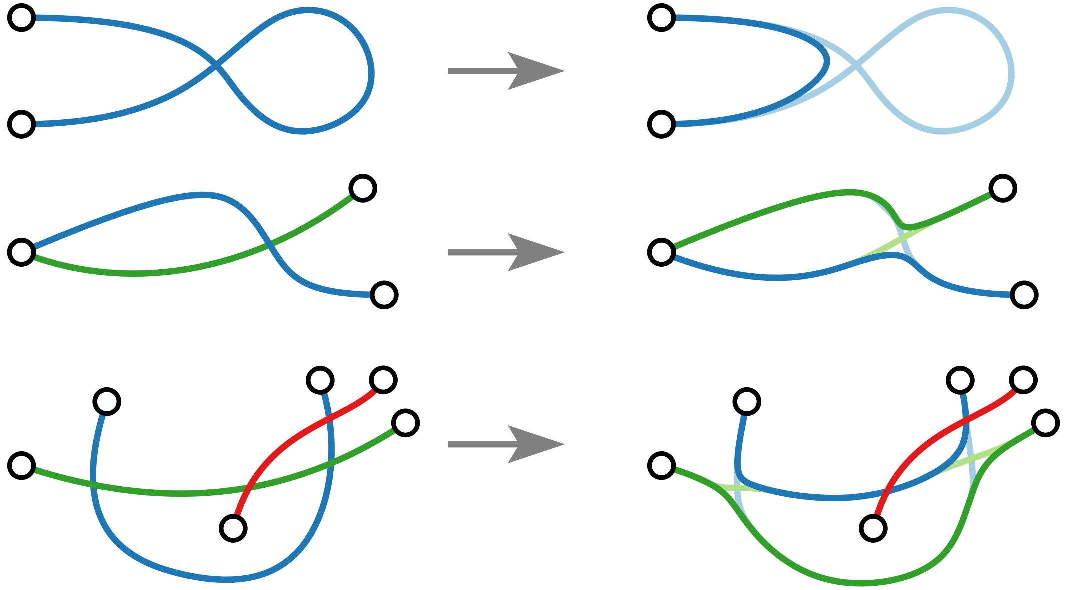

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Hanani–Tutte Theorem

Theorem. [Hanani'43, Tutte'70]
A graph is planar if and only if it has a drawing in which all pairs of vertex-disjoint edges cross an even number of times.

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Hanani showed that every drawing of K_5 and $K_{3,3}$ must have a pair of edges that crosses an odd number of times.

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Every non-planar graph has K_5 or $K_{3,3}$ as minor, so there are two paths that cross an odd number of times.

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Hence, there must be two edges on these paths that cross an odd number of times. $\square$

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Corollary. $\text{ocr}(G) = 0 \Rightarrow \text{cr}(G) = 0$

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Theorem. [Pelsmajer, Schaefer & Štefankovič '08, Tóth '08]
There is a graph G with $\text{ocr}(G) < \text{cr}(G) \leq 10$

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If Γ is a drawing of G and E_0 is the set of edges with only even numbers of crossings in Γ, then G can be drawn such that no edge in E_0 is involved in any crossings and no new p

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Theorem. [Pelsmajer, Schaefer & Štefankovič '08, Tóth '08]
There is a graph G with ocr$(G) <$ cr$(G) \leq 10$

The **pairwise crossing number** pcr(G) of G is the smallest number of pairs of edges that cross in a drawing of G.

By definition ocr$(G) \leq$ pcr$(G) \leq$ cr(G)

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The Crossing Lemma
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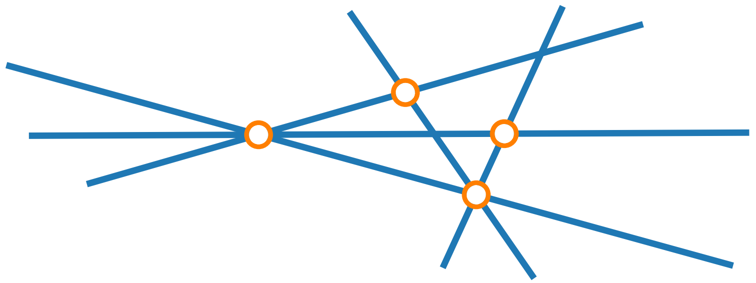

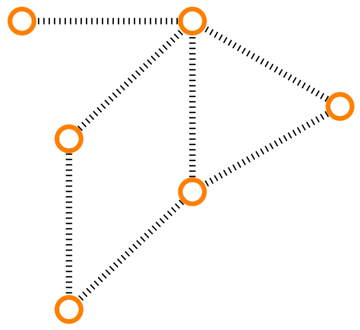

Part II:
Computation & Variations

Jonathan Klawitter

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- cr(G) is a measure of how far G is from being planar
- Planarization, where we replace crossings with dummy vertices, also uses only heuristics

Other Crossing Numbers

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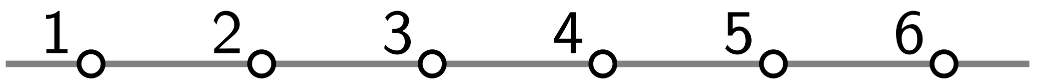

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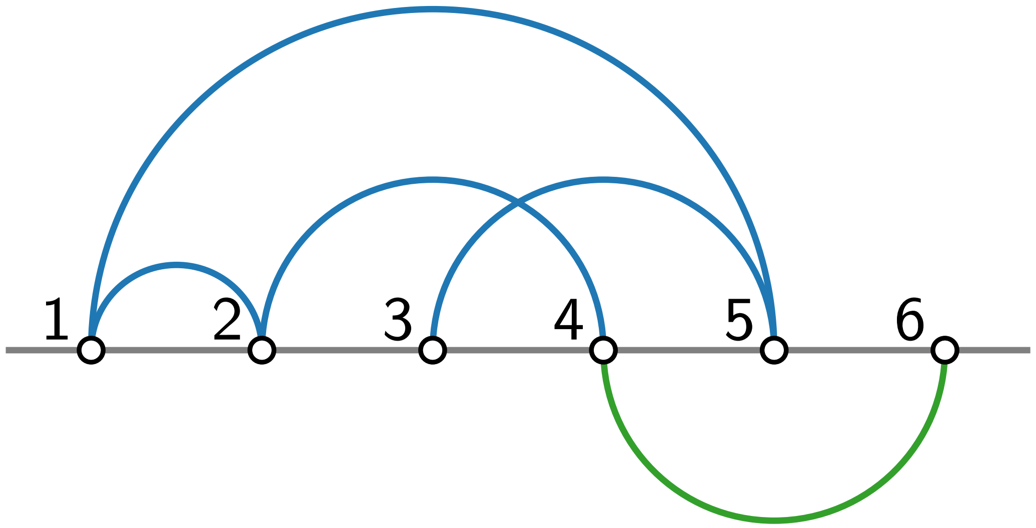

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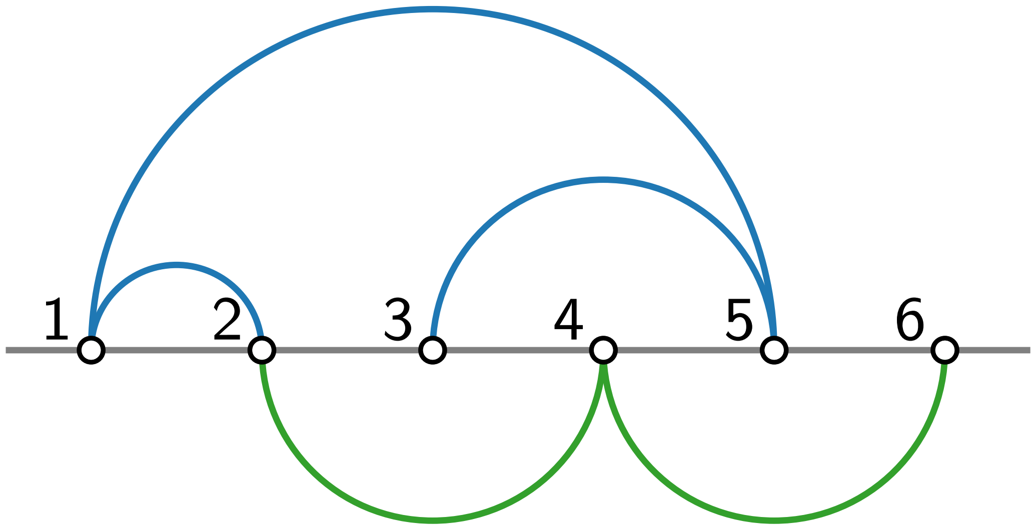

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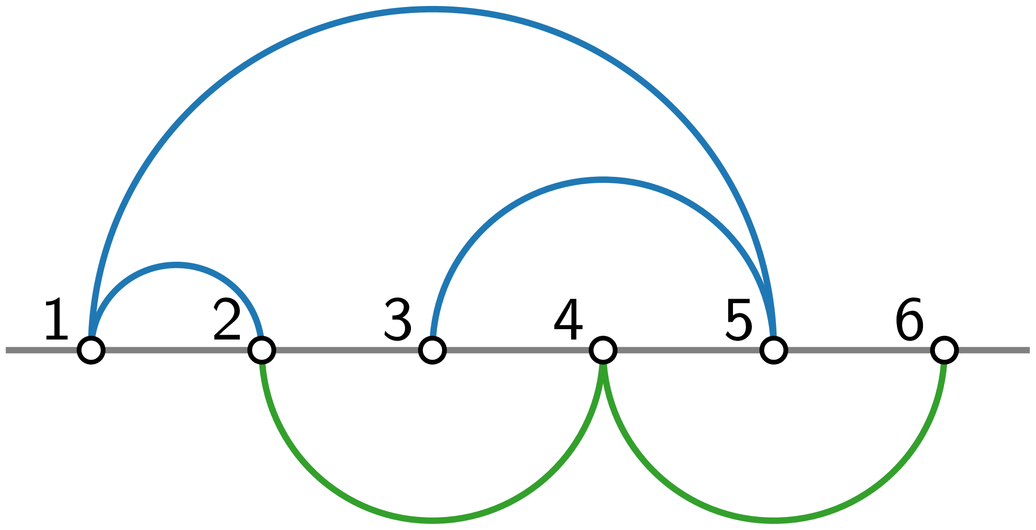

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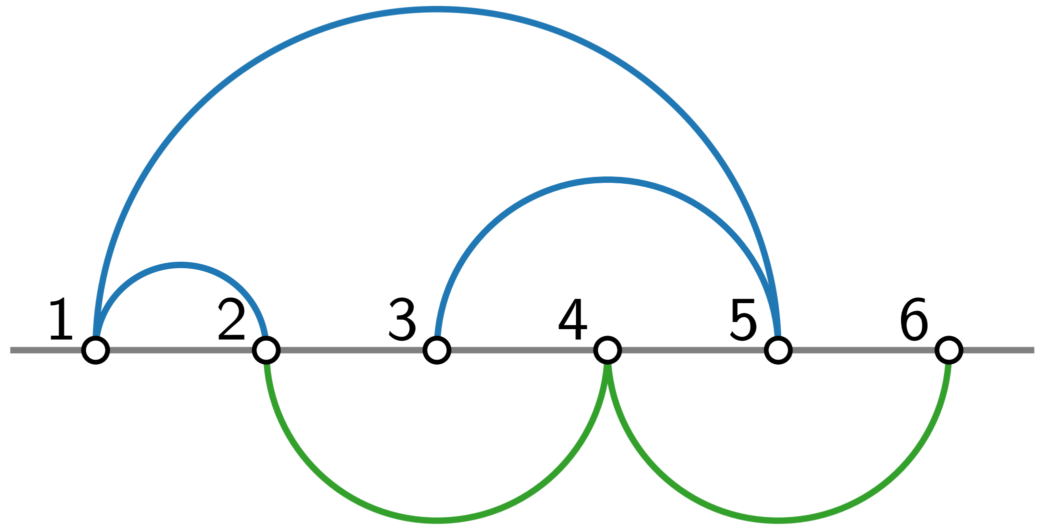

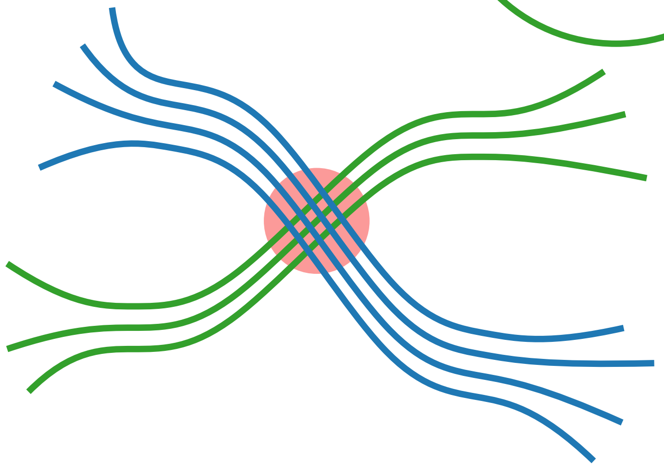

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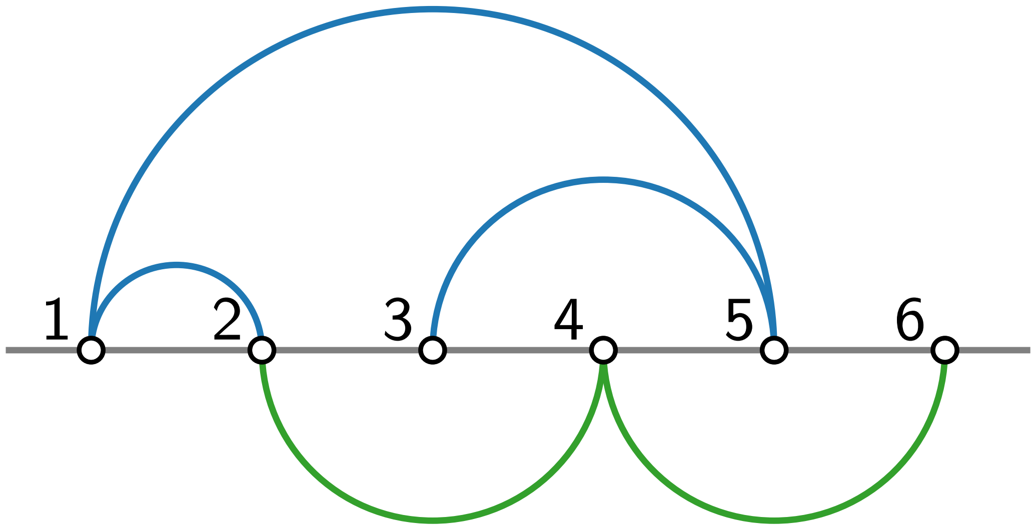

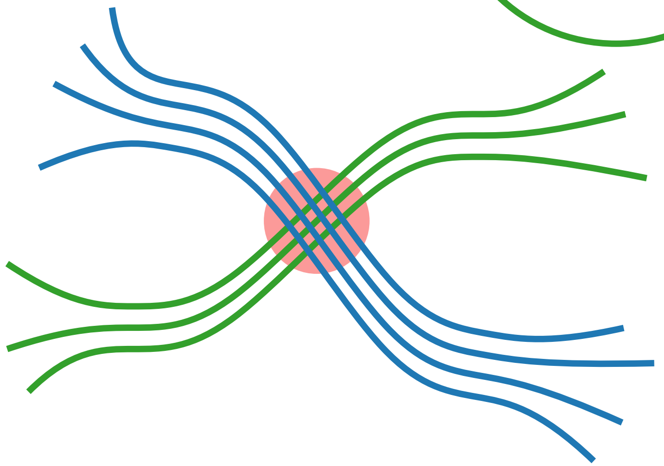

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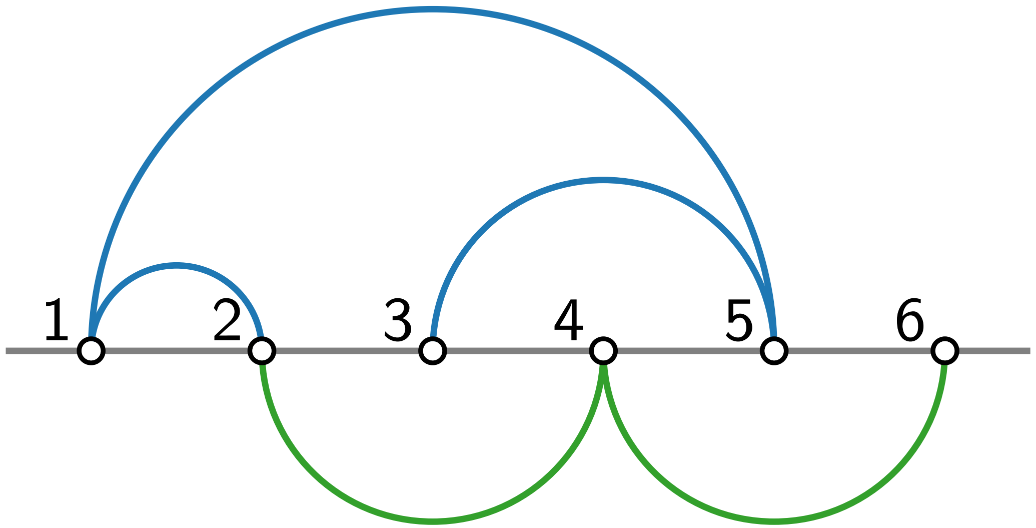

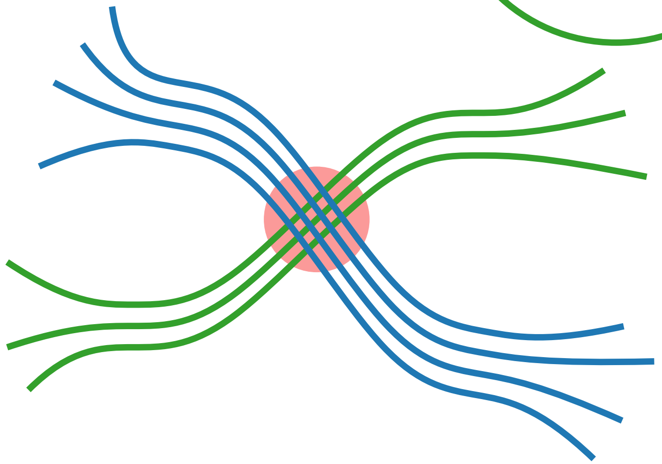

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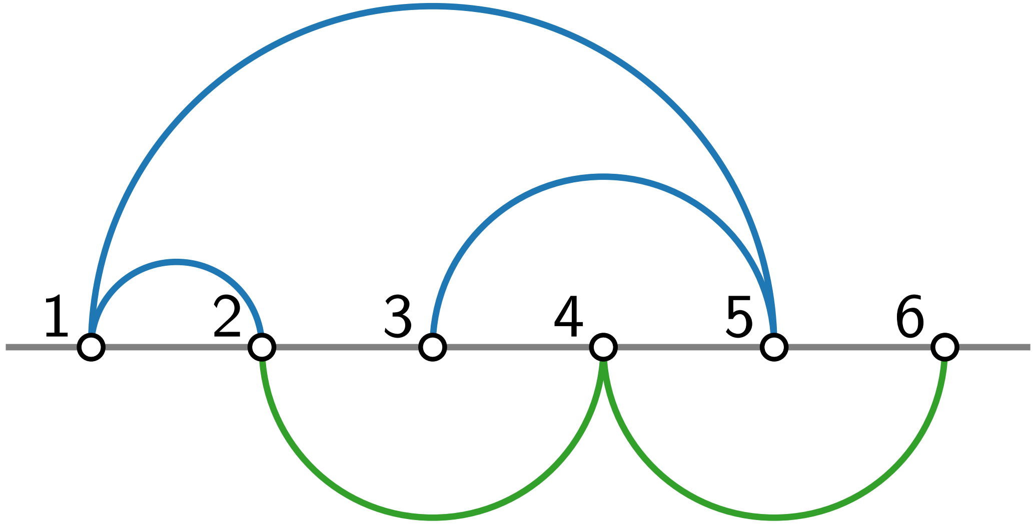
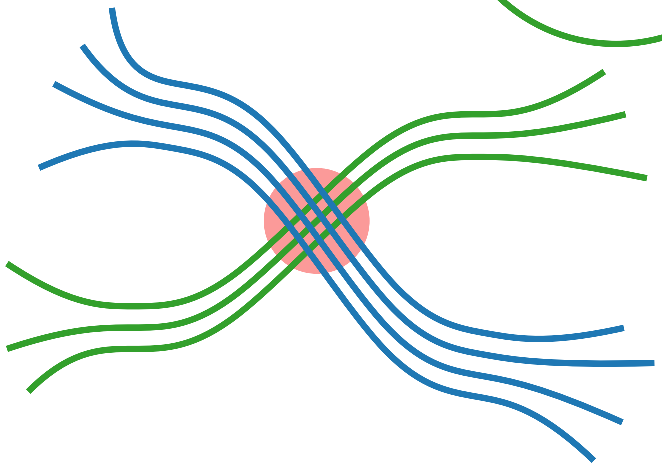
- Crossing minimization is NP-hard for most of the variants





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For a graph G, the **rectilinear (straight-line) crossing number** $\overline{\mathrm{cr}}(G)$ is the smallest number of crossings in a straight-line drawing of G.

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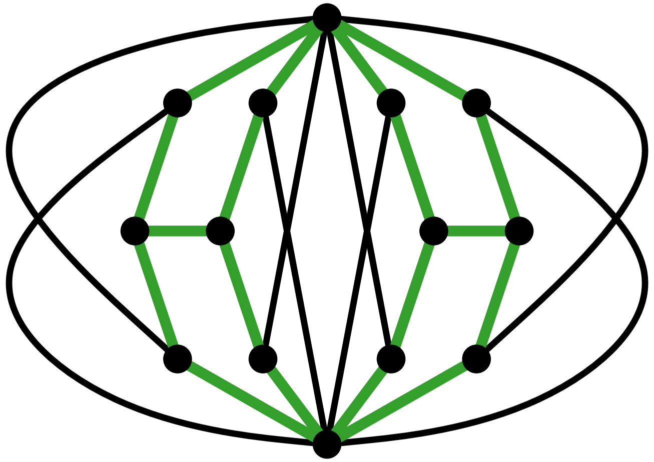

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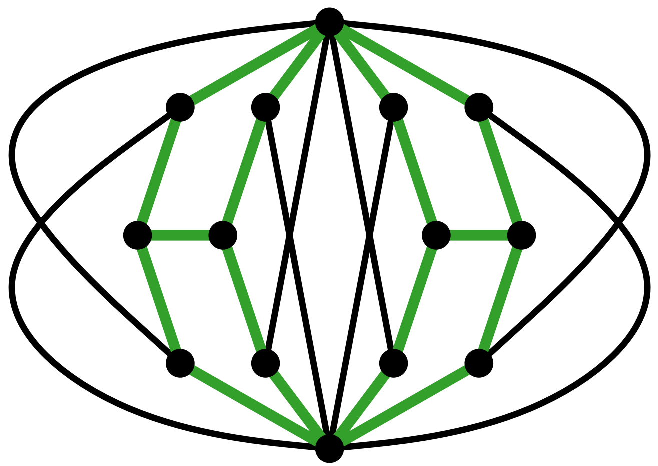

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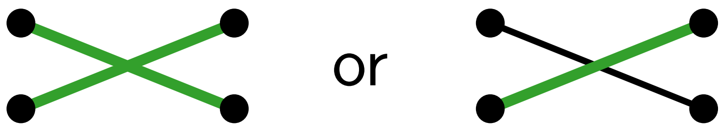

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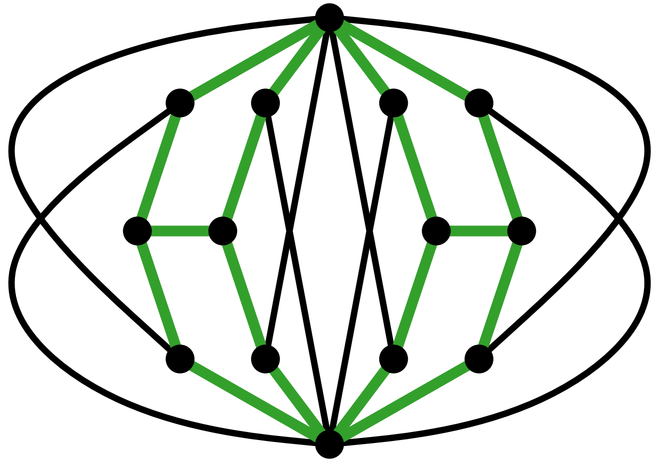

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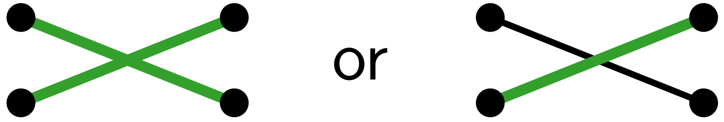
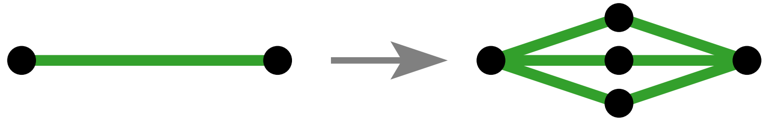


- From G_1 to G_k do



Visualization of Graphs

Lecture 11: The Crossing Lemma
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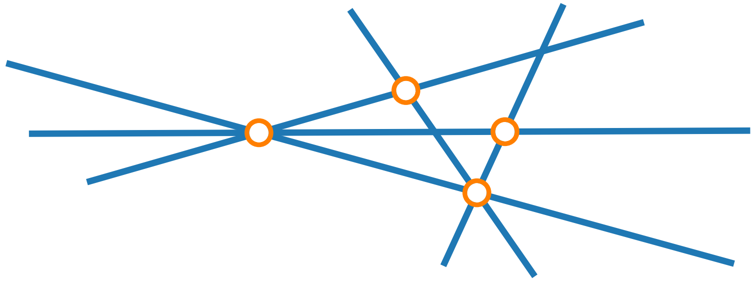

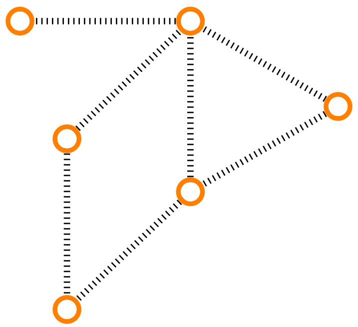

Part III: First Bounds

Jonathan Klawitter

Bounds for Complete Graphs

Theorem. [Guy '60]

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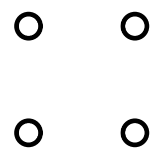

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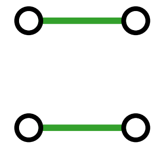

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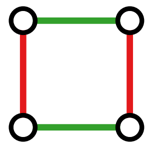

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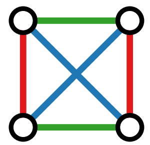

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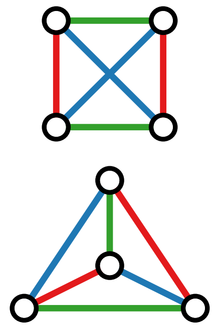

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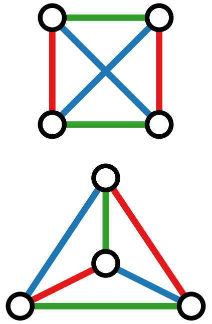

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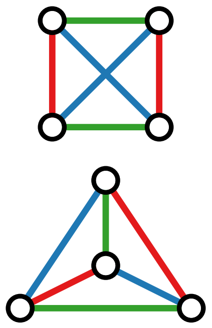

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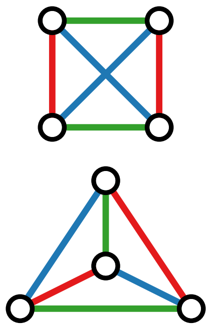

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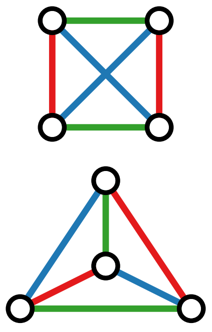

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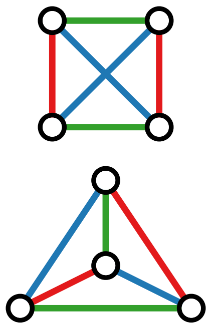

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Pál Turán
*1910 – 1976
Budapest, Hungary

Turán's brick factory problem (1944)

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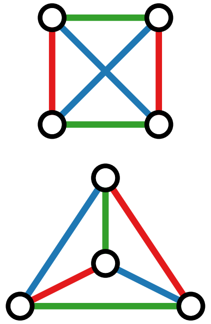

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$$\mathrm{cr}(K_n) \not\leq \underline{=} \frac{1}{4}\left\lceil\frac{n}{2}\right\rceil\left\lceil\frac{n-1}{2}\right\rceil\left\lceil\frac{n-2}{2}\right\rceil\left\lceil\frac{n-3}{2}\right\rceil = \frac{3}{8}\binom{n}{4} + O(n^3)$$

Bound is sharp for $n \leq 12$.

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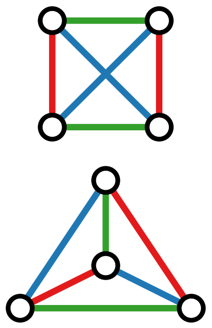

Sylvester's four-point problem

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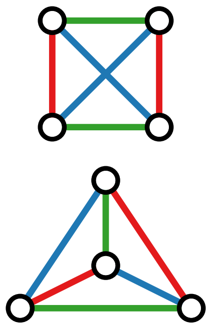

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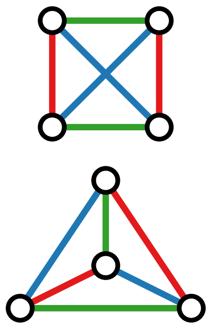

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Check out `http://www.ist.tugraz.at/staff/aichholzer/crossings.html`!

First Lower Bounds on cr(G)

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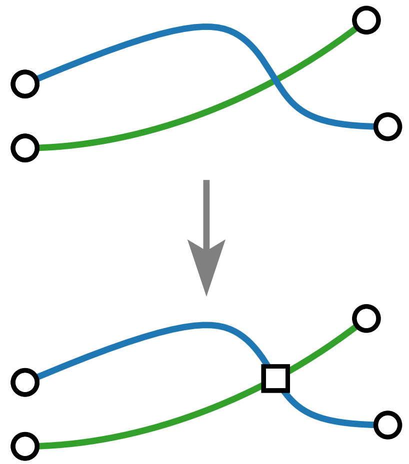

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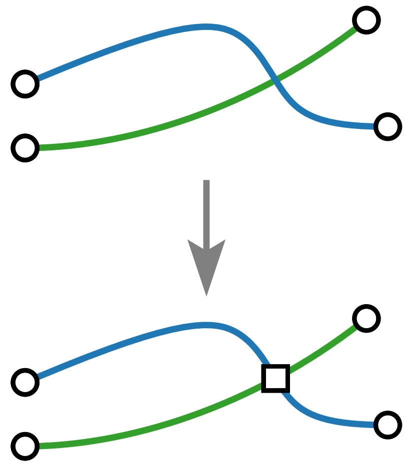

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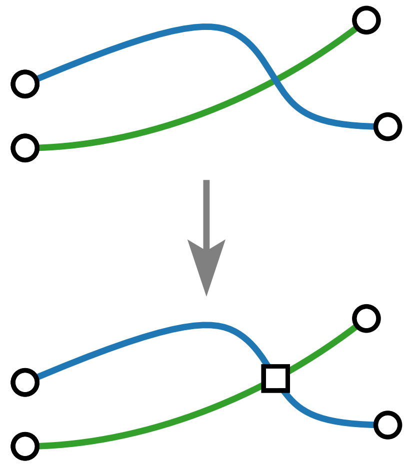
$$m + 2\text{cr}(G) \leq 3(n + \text{cr}(G)) - 6.$$



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Consider this bound for graphs with $\Theta(n)$ and $\Theta(n^2)$ many edges.

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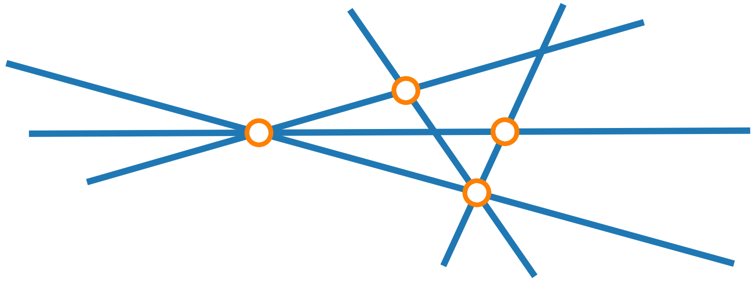
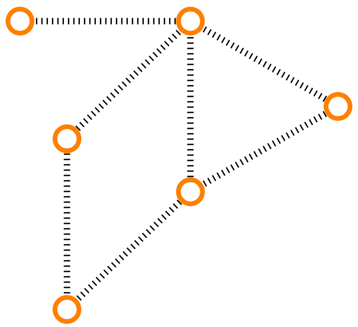
Julius-Maximilians-UNIVERSITÄT WÜRZBURG

Visualization of Graphs

Lecture 11: The Crossing Lemma

and its Applications





Part IV: The Crossing Lemma

Jonathan Klawitter

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- We look at a proof "from THE BOOK" by Chazelle, Sharir and Welz.
- Factor $\frac{1}{64}$ was later (with intermediate steps) improved to $\frac{1}{29}$ by Ackerman in 2013.

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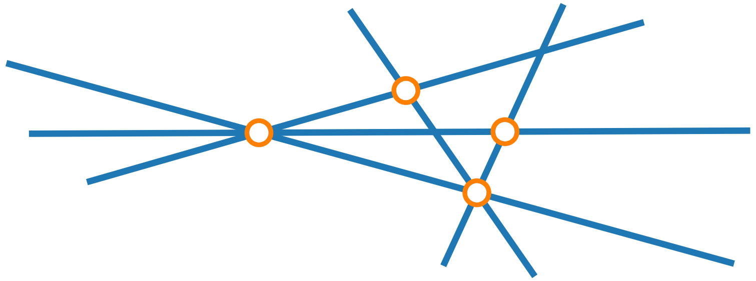
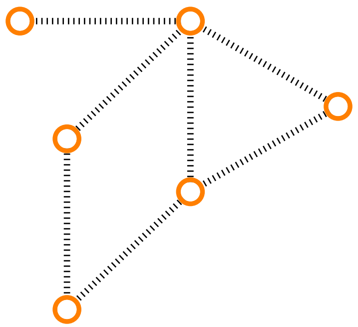
- $\mathbb{E}(X_p) = p^4 \text{cr}(G)$
- $0 \leq \mathbb{E}(X_p) - \mathbb{E}(m_p) + 3\mathbb{E}(n_p)$ $= p^4 \text{cr}(G) - p^2 m + 3pn$
- $\text{cr}(G) \geq \frac{p^2 m - 3pn}{p^4} = \frac{m}{p^2} - \frac{3n}{p^3}$
- Set $p = \frac{4n}{m}$.
- $\text{cr}(G) \geq \frac{m^3}{16n^2} - \frac{3m^3}{64n^2} = \frac{1}{64}\frac{m^3}{n^2}$

Visualization of Graphs

Lecture 11: The Crossing Lemma
and its Applications





Part V: Applications

Jonathan Klawitter

Application 1: Point-Line Incidences

- For points $P \subset \mathbb{R}^2$ and lines $\mathcal{L}$, $I(P, \mathcal{L})$ = number of point-line incidences in $(P, \mathcal{L})$.

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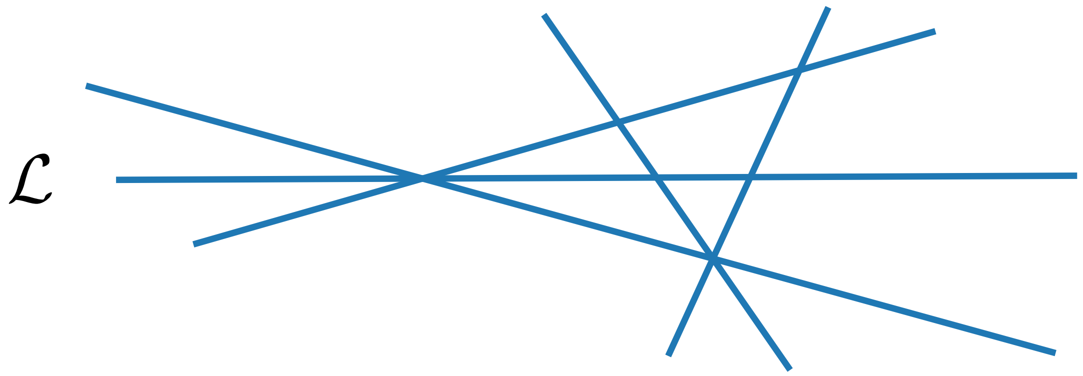

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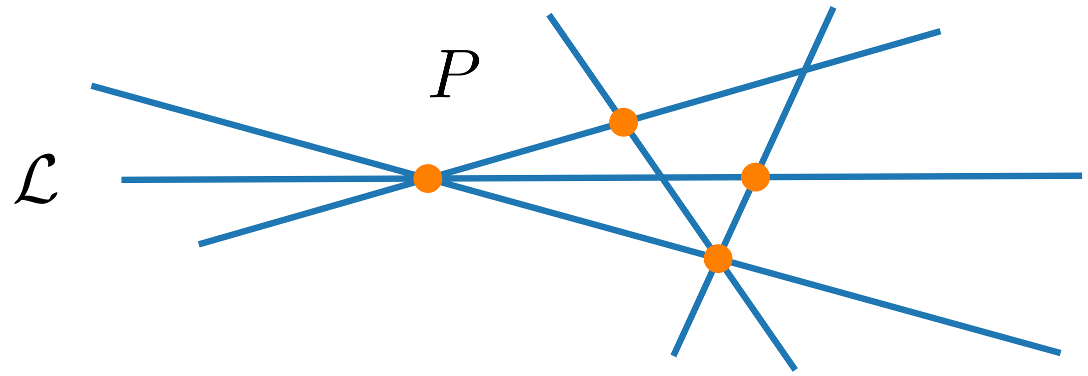

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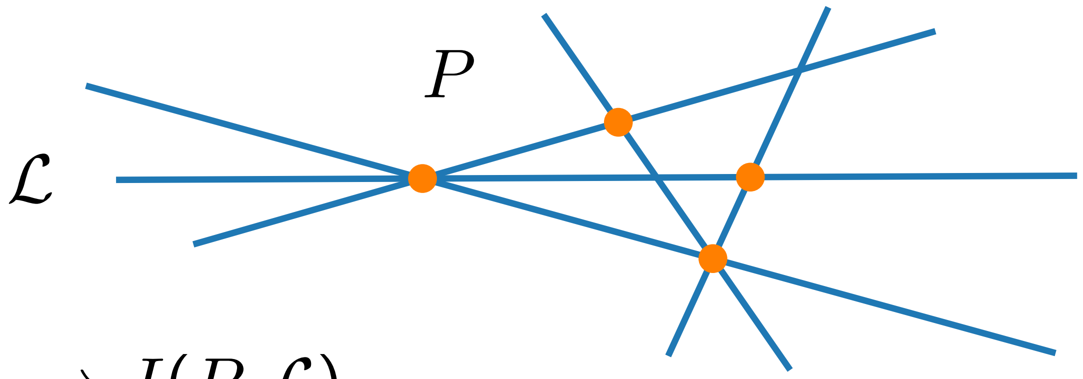

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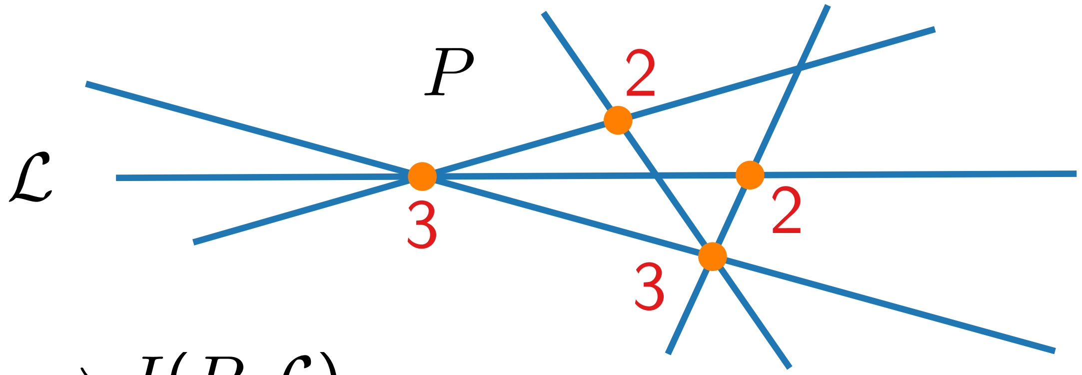

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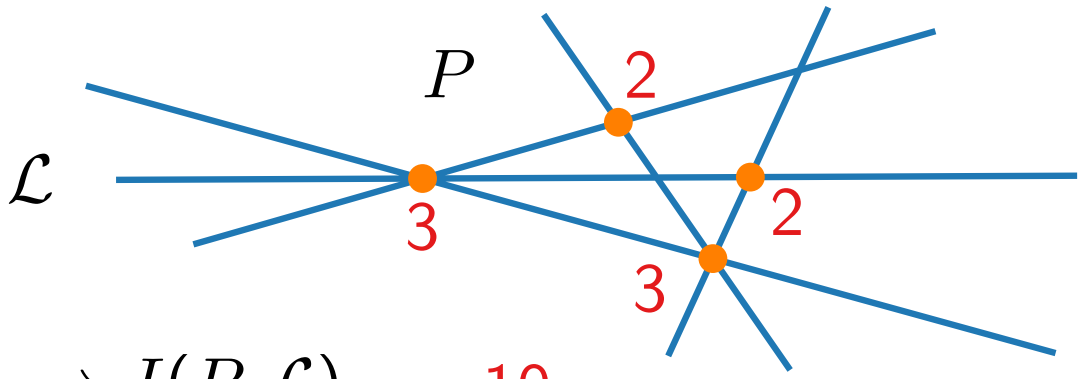

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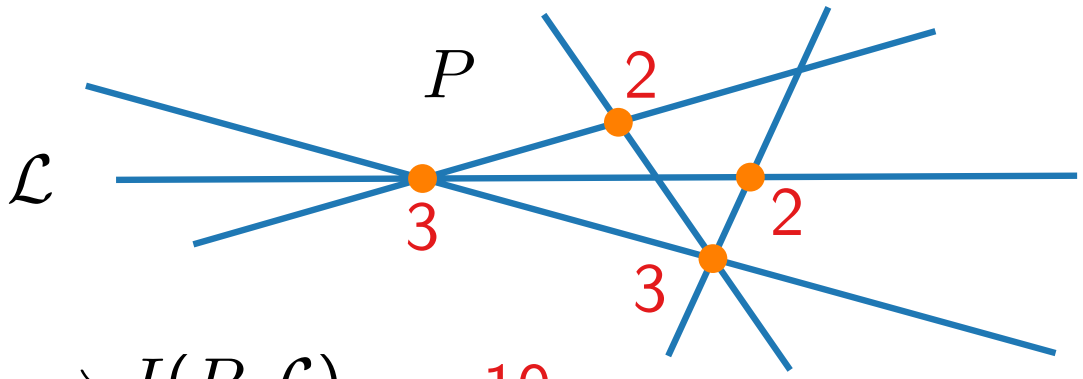

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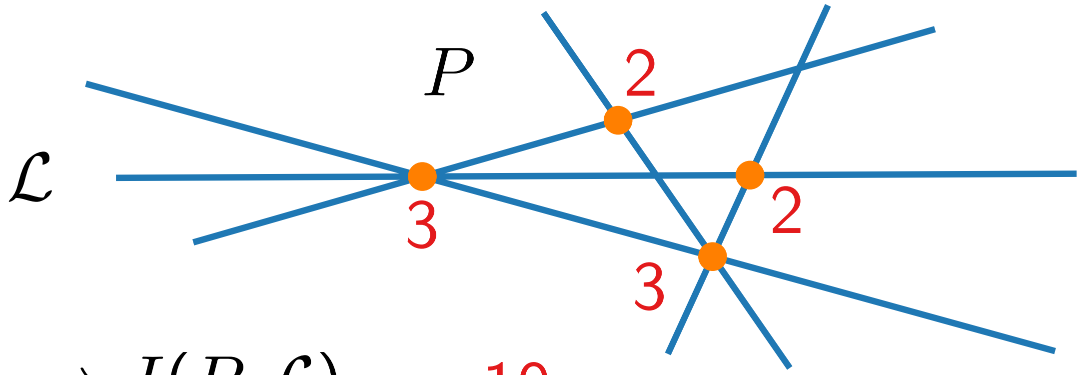

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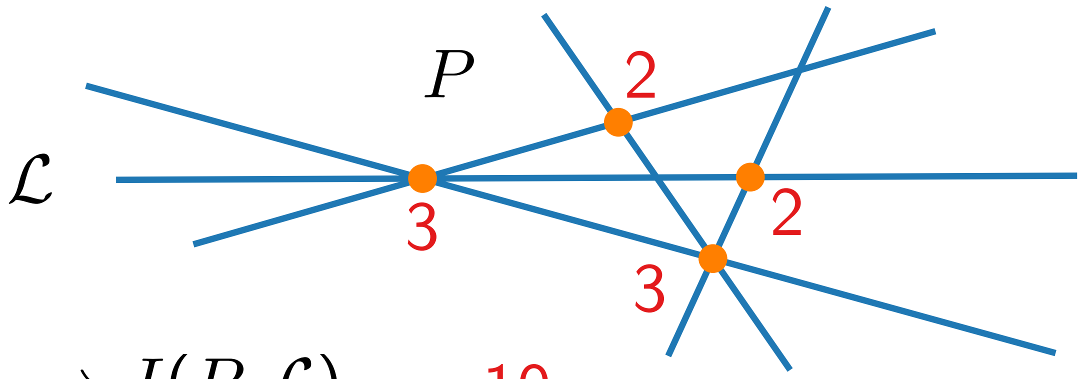

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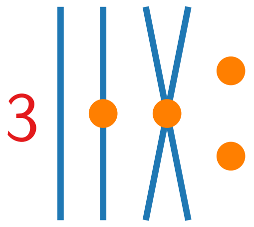

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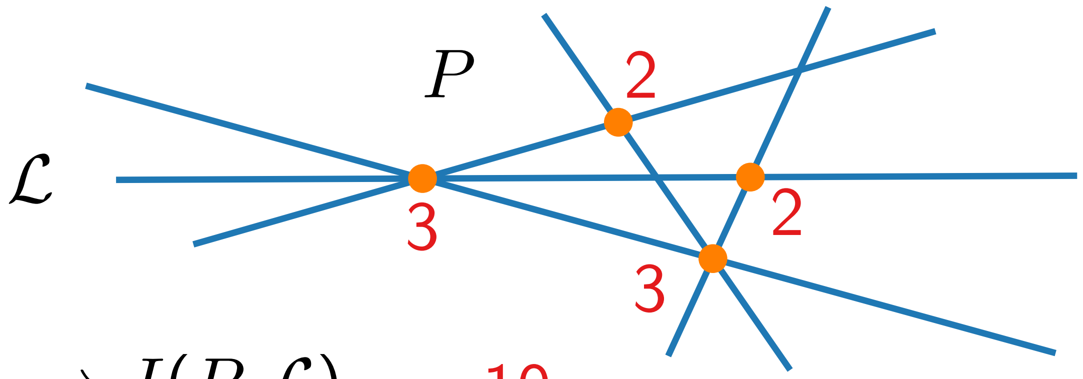

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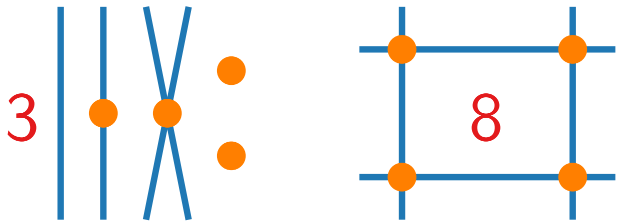

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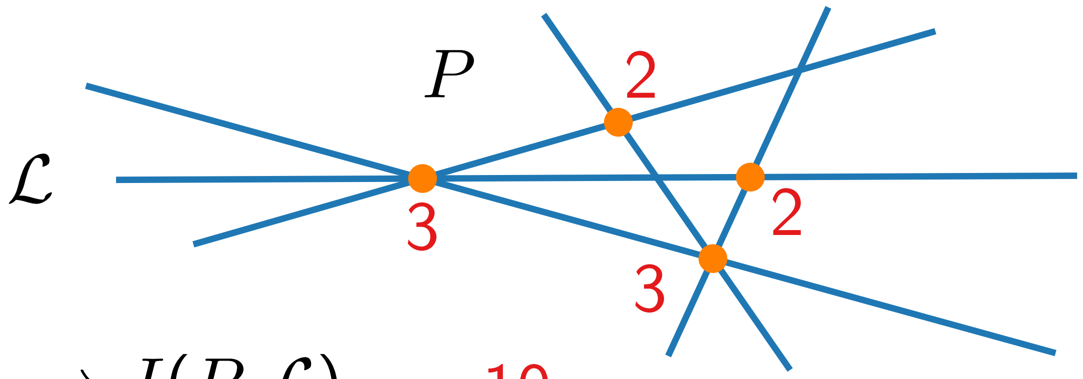

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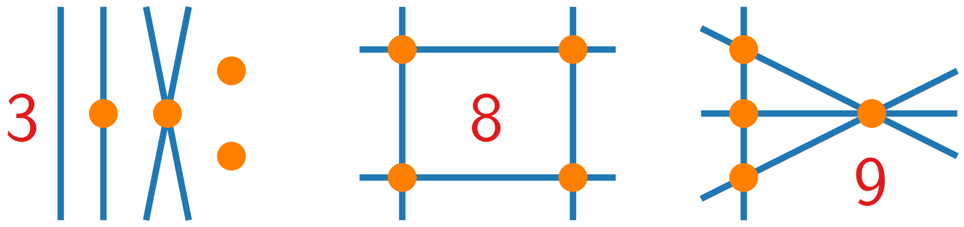

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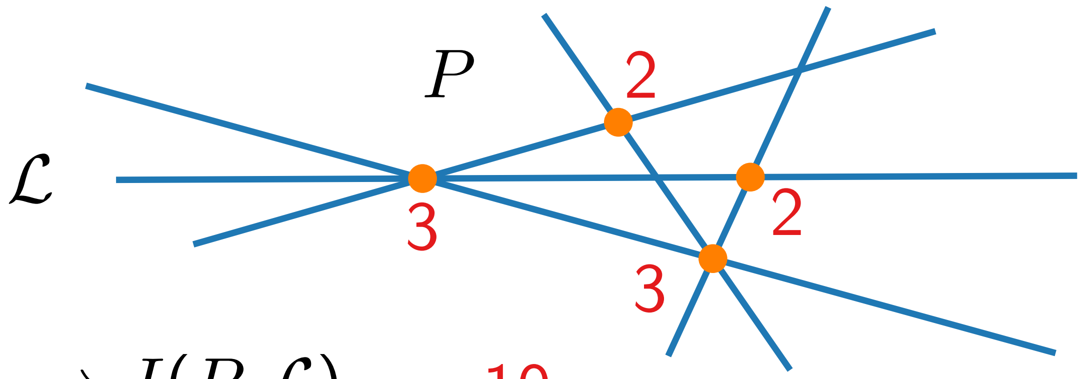

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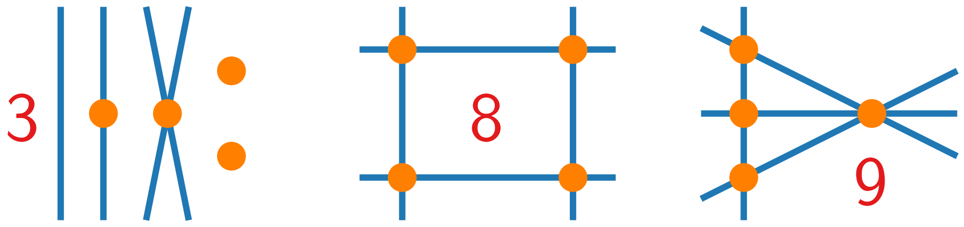

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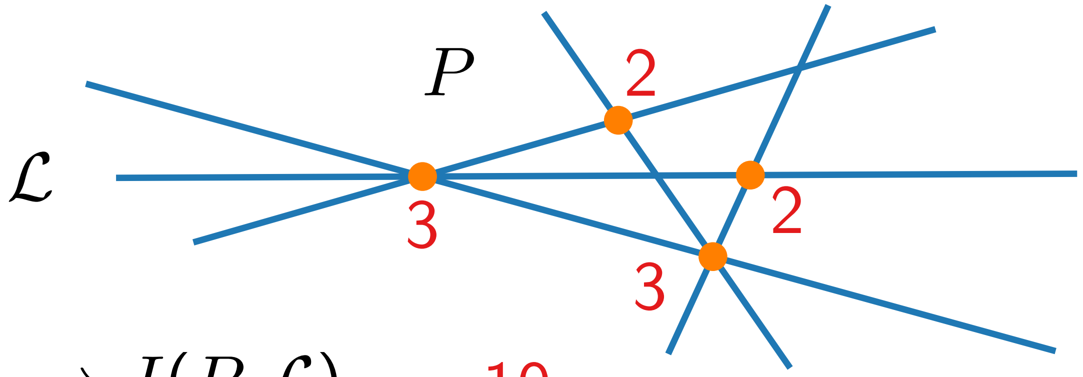

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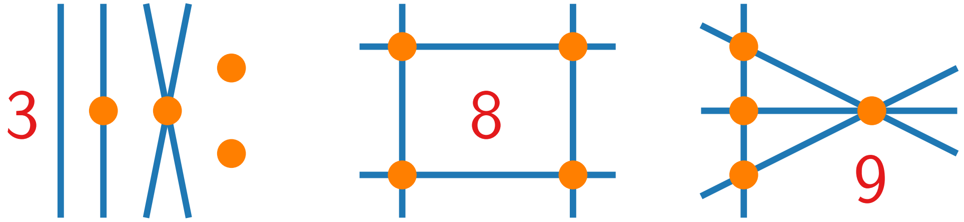

Theorem 1.
[Szemerédi, Trotter '83, Székely '97]
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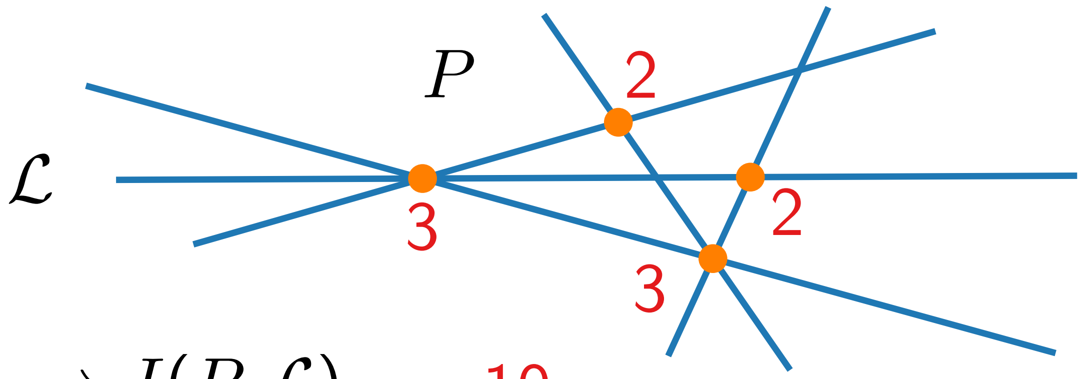

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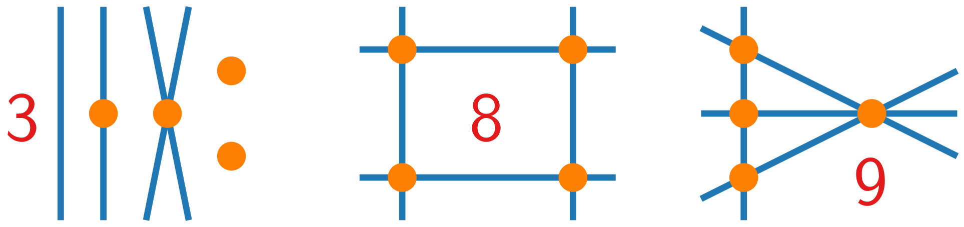

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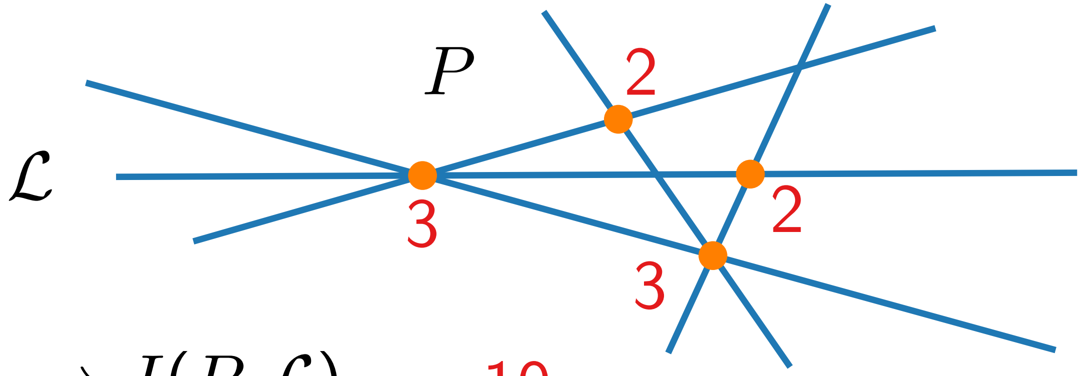

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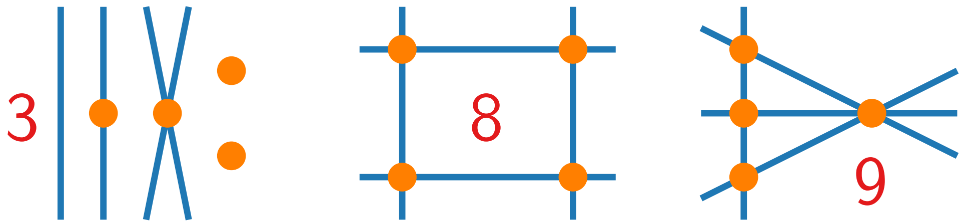

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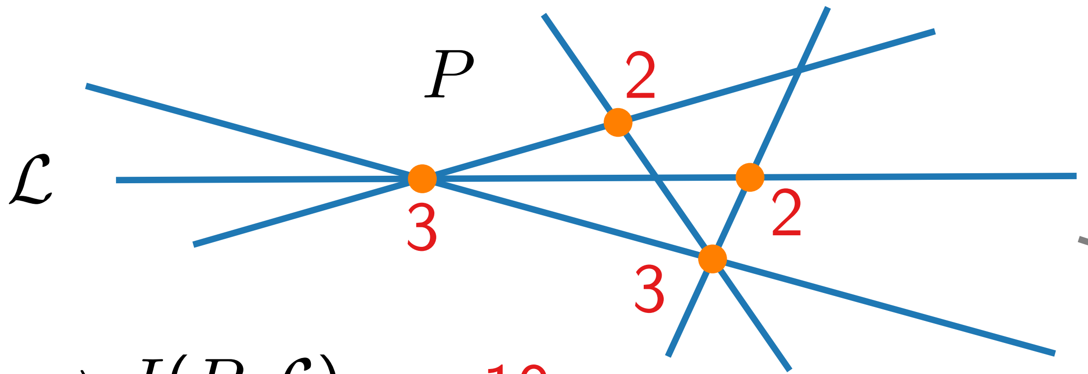

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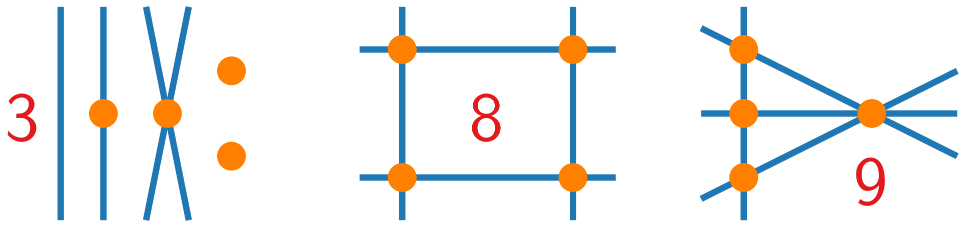

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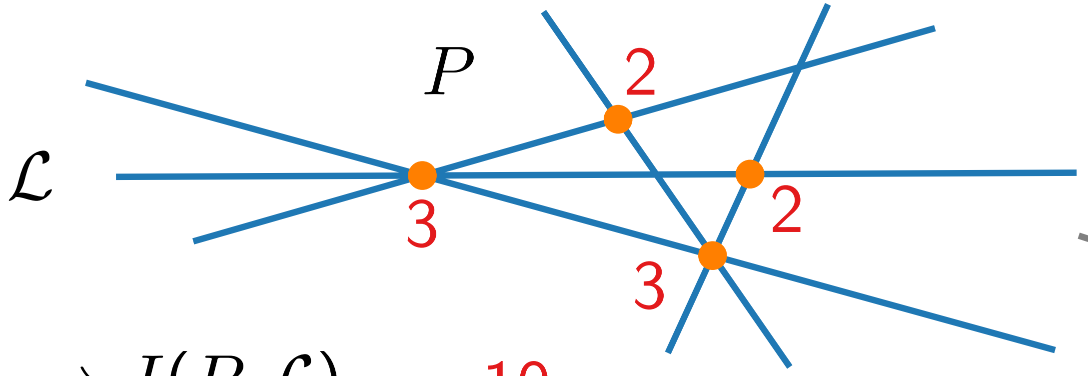

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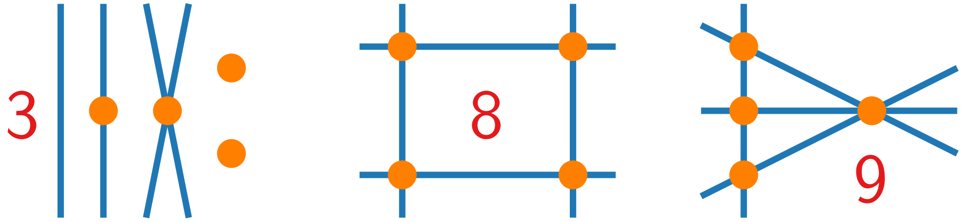

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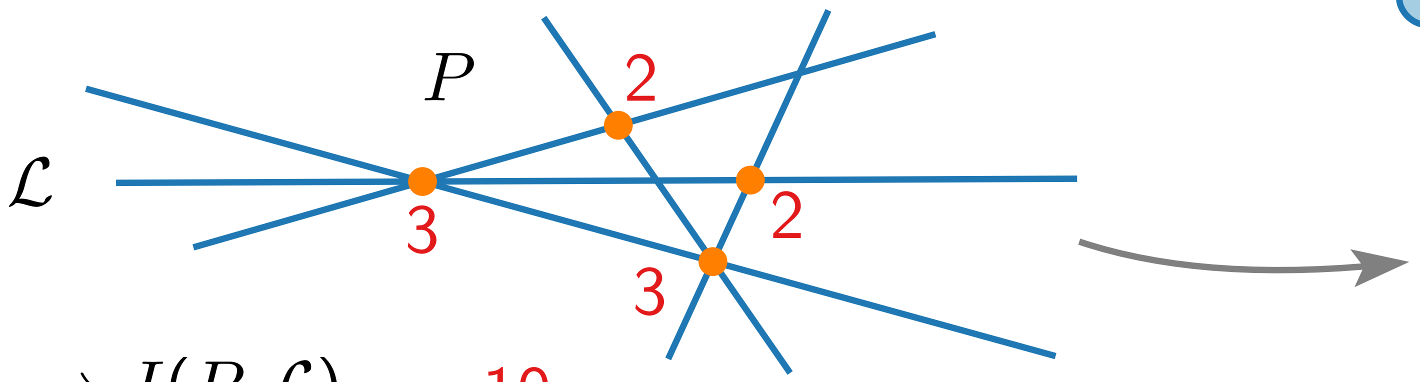

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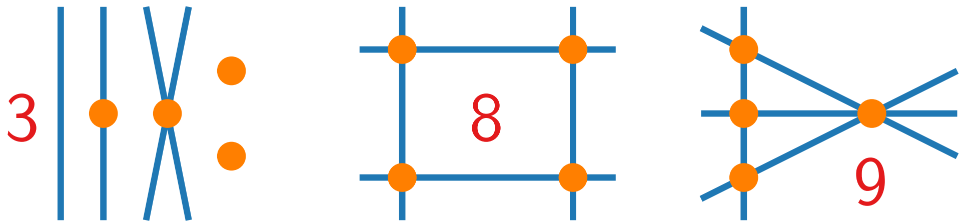

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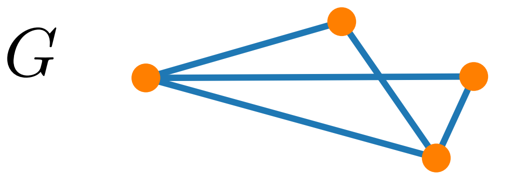

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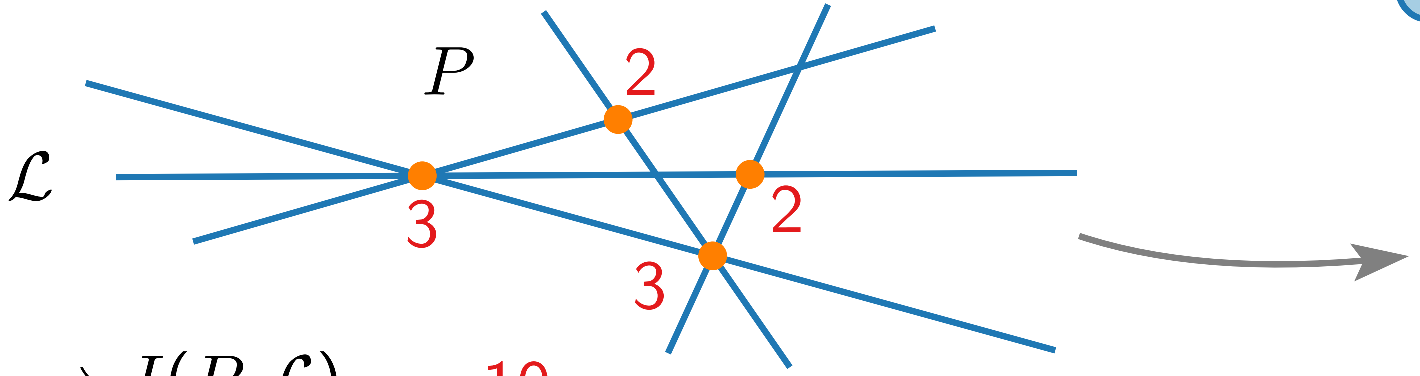

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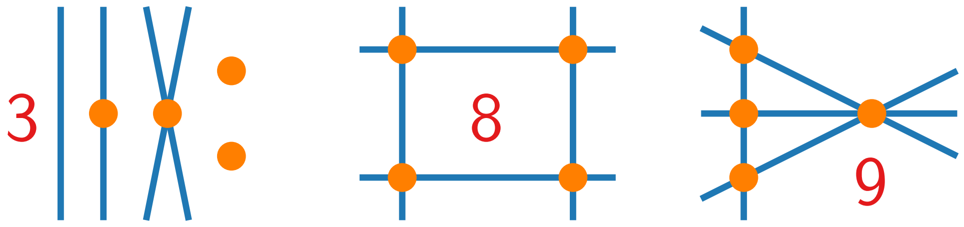

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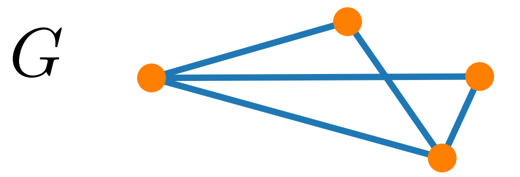

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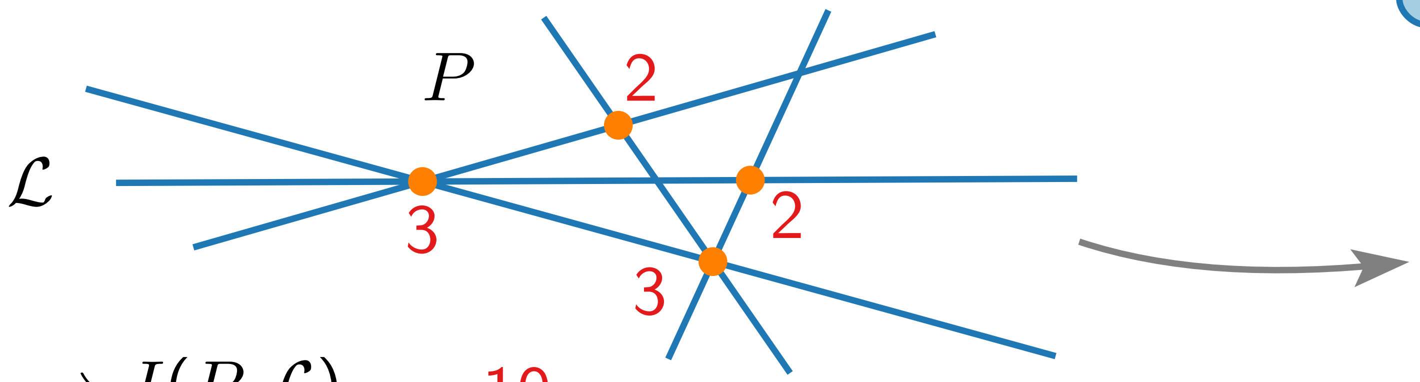

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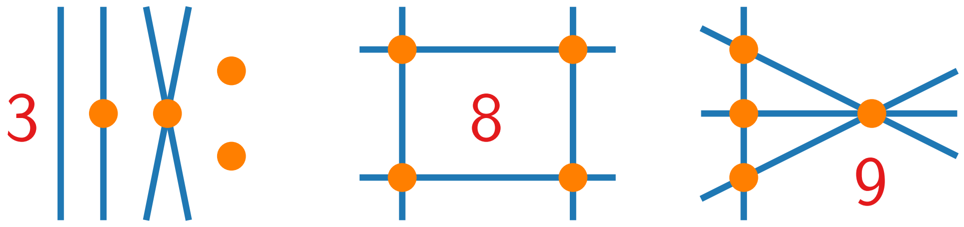

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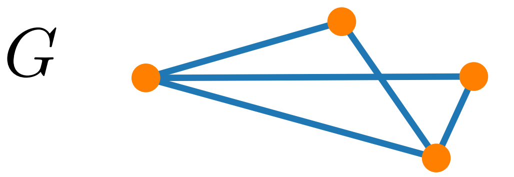

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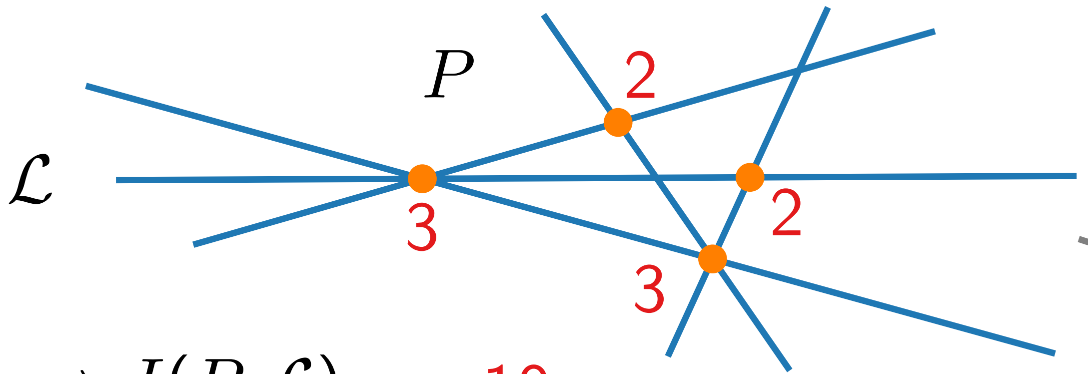

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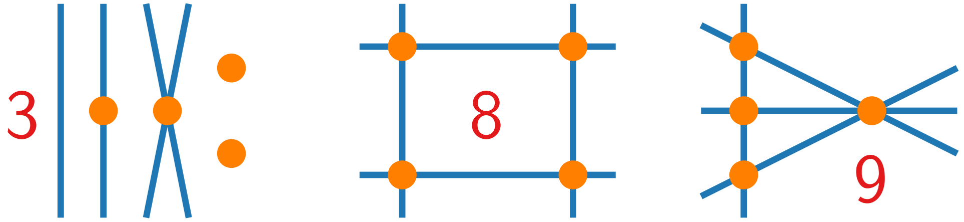

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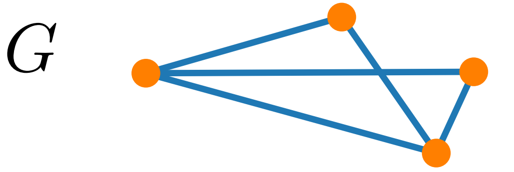

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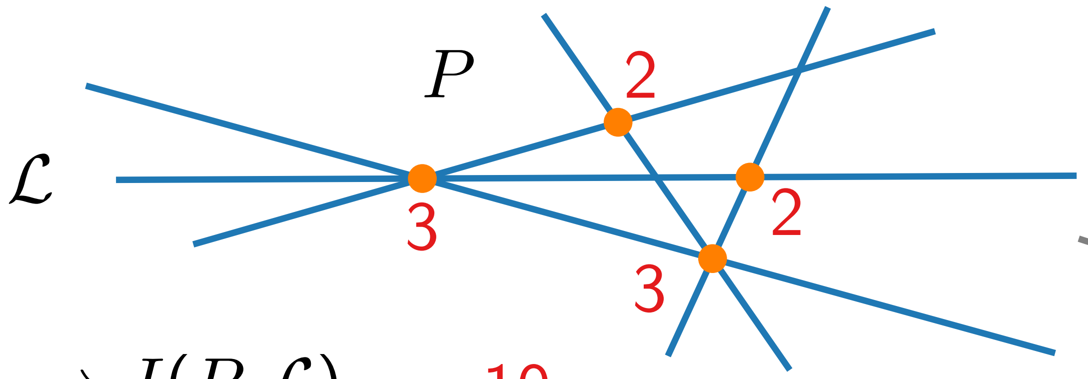

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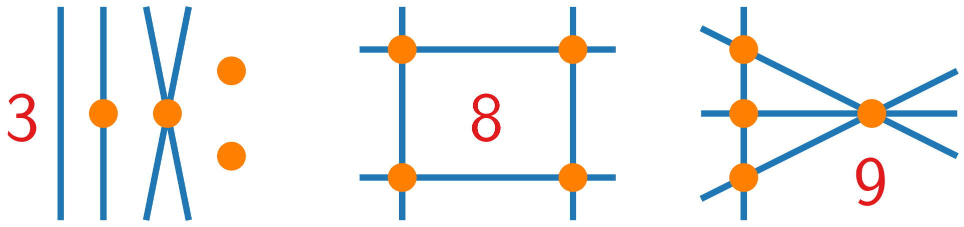

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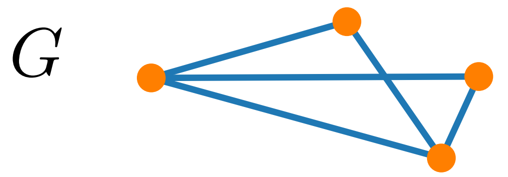

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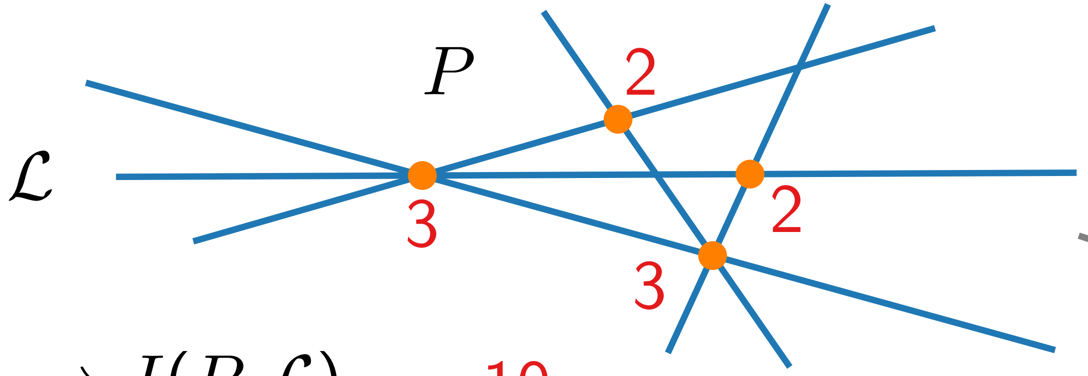
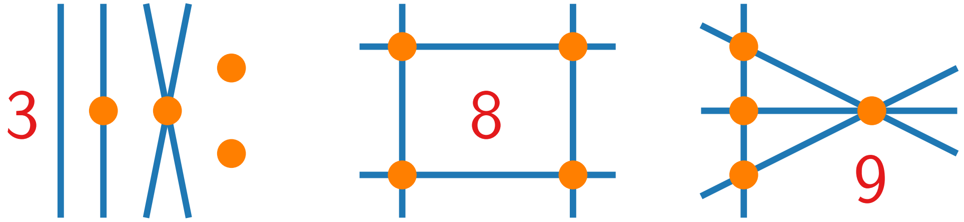
Application 1: Point-Line Incidences

- For points $P \subset \mathbb{R}^2$ and lines $\mathcal{L}$, $I(P,\mathcal{L})$ = number of point-line incidences in $(P,\mathcal{L})$.



$\Rightarrow I(P,\mathcal{L}) =$ 10

- Define $I(n,k) = \max\limits_{|P|=n, |\mathcal{L}|=k} I(P,\mathcal{L})$.
- For example: $I(4,4) =$ 9



Theorem 1.
[Szemerédi, Trotter '83, Székely '97]
$I(n,k) \leq c(n^{2/3}k^{2/3} + n + k)$.

Proof.

- $\mathrm{cr}(G) \leq k^2$
- # points on l = 1+ # edges on l
- $I(n,k) - k \leq m$
- Crossing Lemma: $\frac{1}{64}\frac{m^3}{n^2} \leq \mathrm{cr}(G)$
- $c'(I(n,k)-k)^3/n^2 \leq \mathrm{cr}(G) \quad \leq k^2$
- if $m \not\geq 4n$, then $I(n,k) - k \leq 4n$

Application 2: Unit Distances

For points $P \subset \mathbb{R}^2$ define

- $U(P) =$ number of pairs in P at unit distance and
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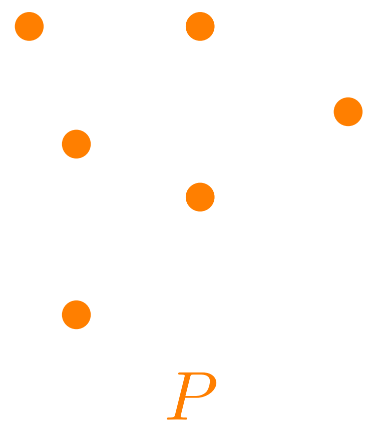

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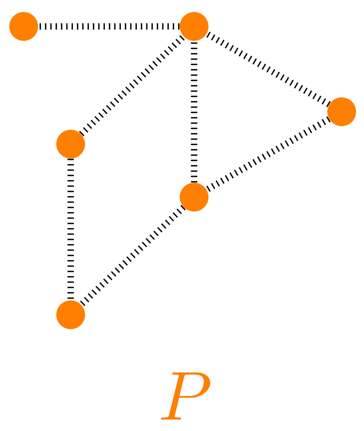

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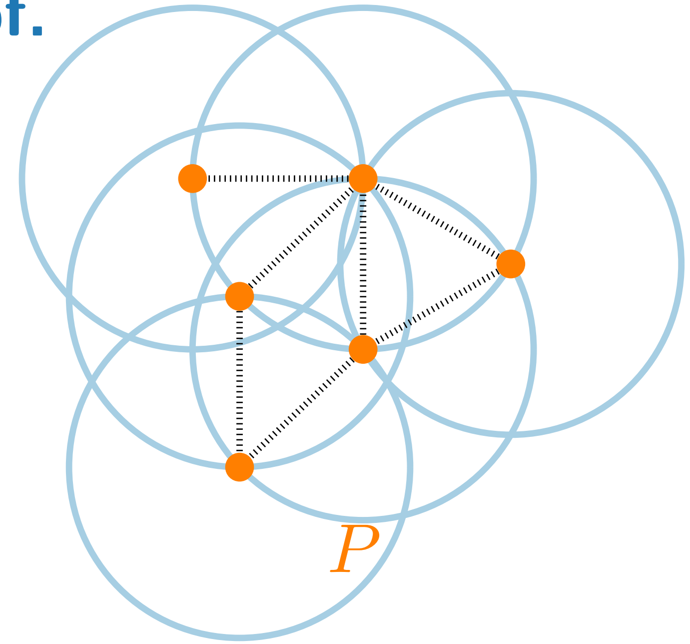

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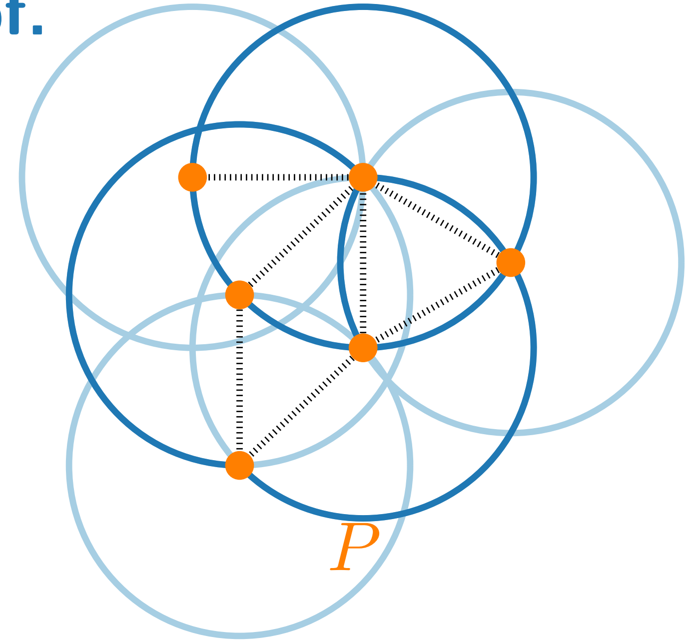

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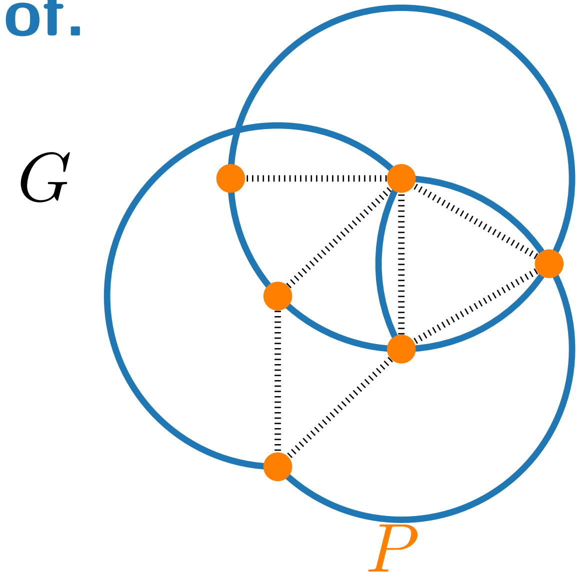

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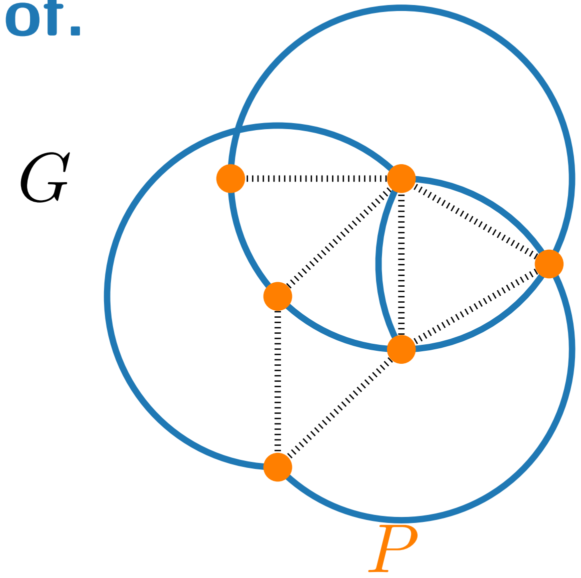

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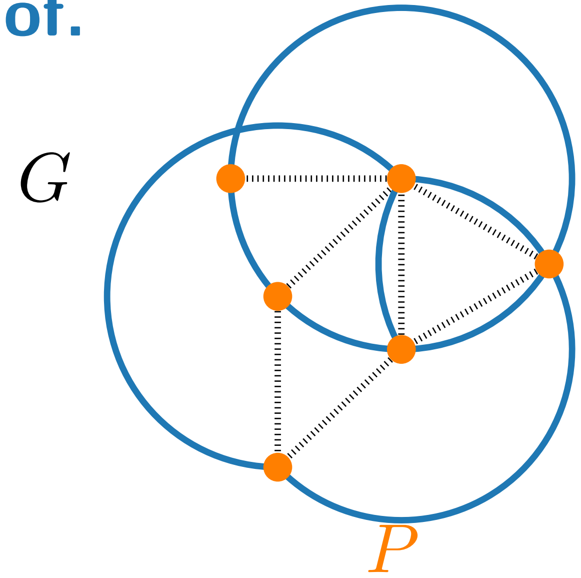

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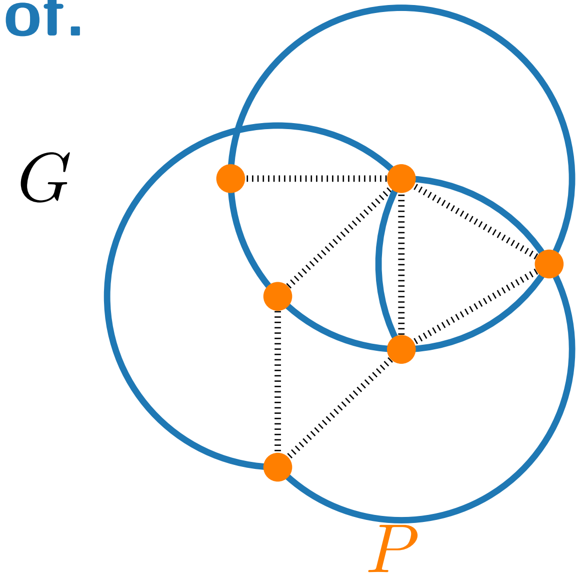

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Application 3: Max. Num. of Crossings in Matchings

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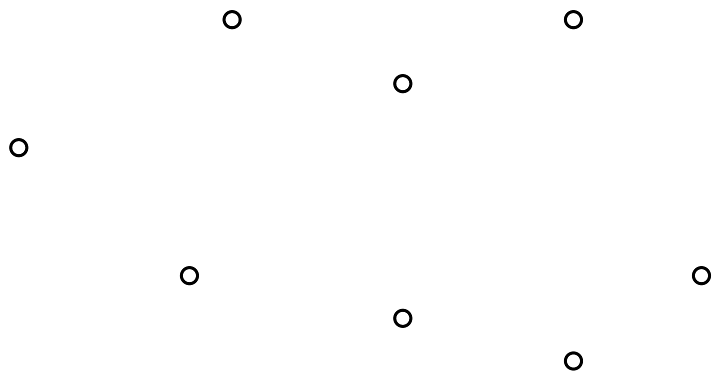

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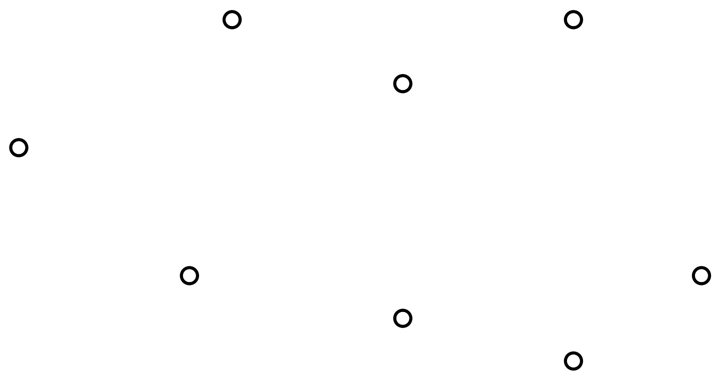

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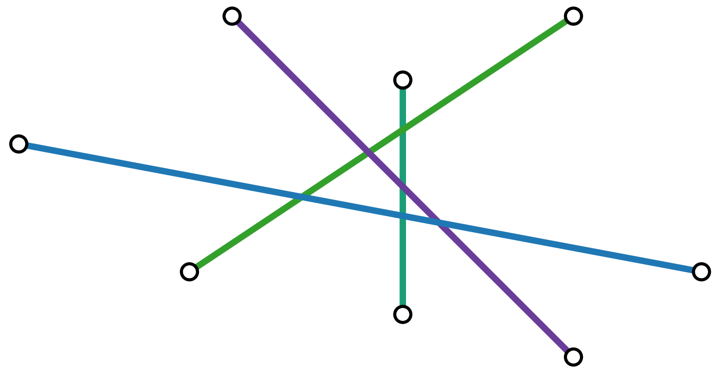

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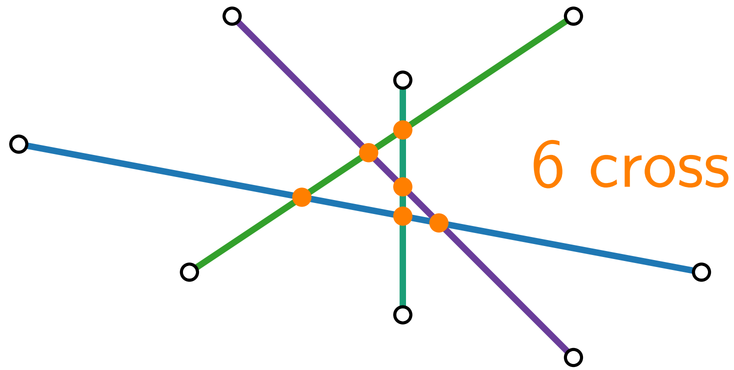

6 crossings

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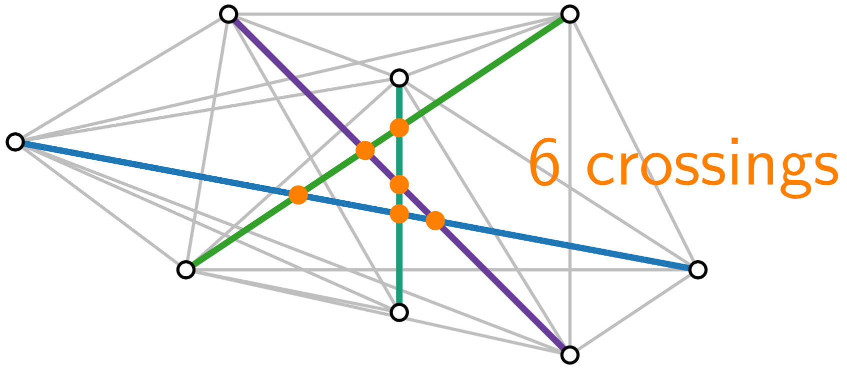
Point set spans drawing Γ of K_n



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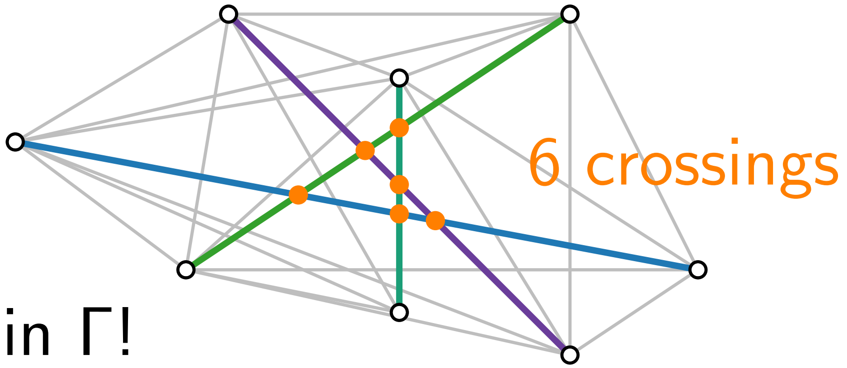

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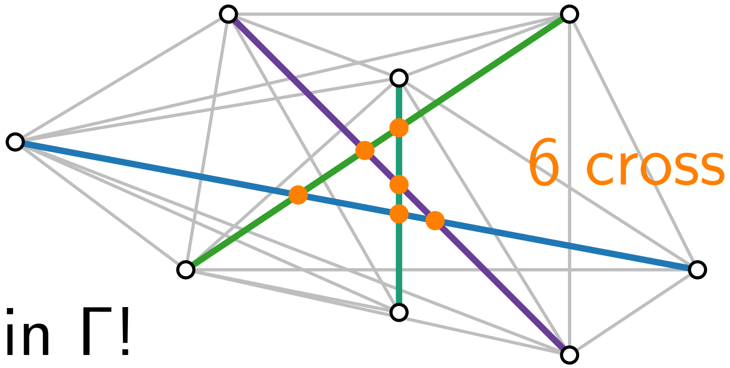

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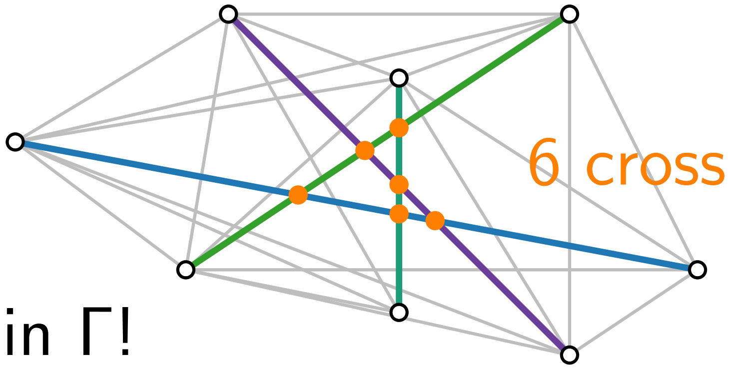

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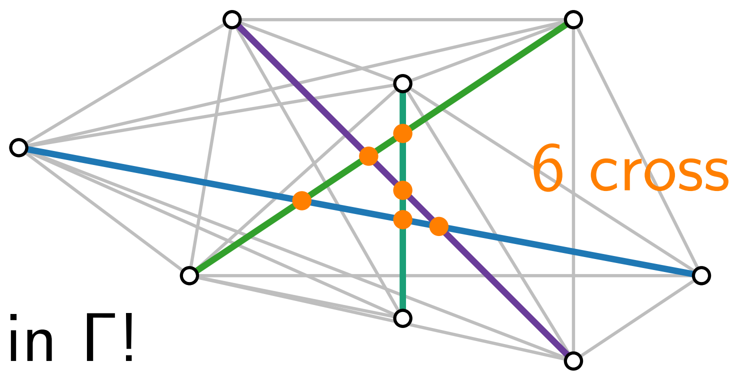

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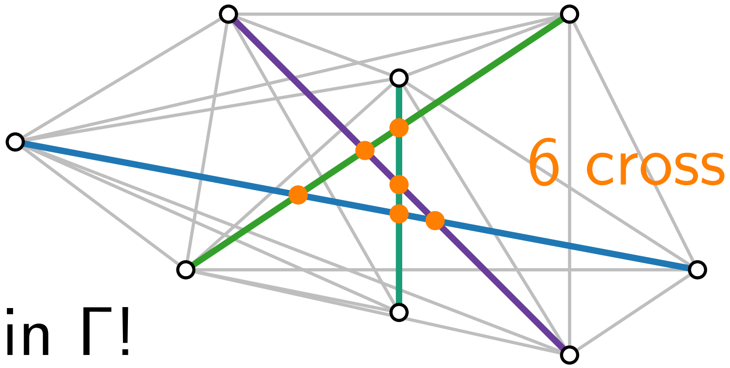

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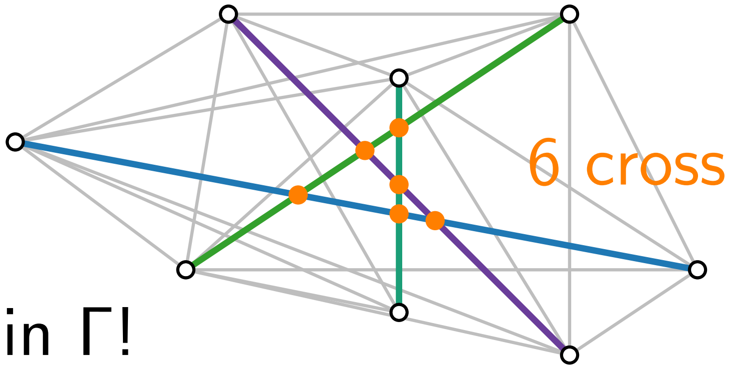

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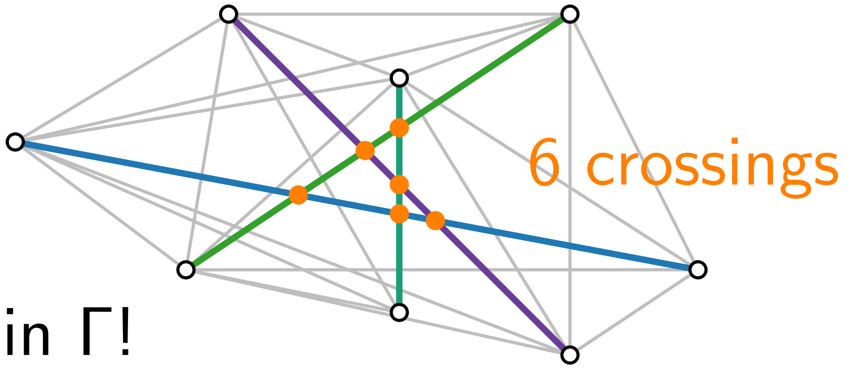

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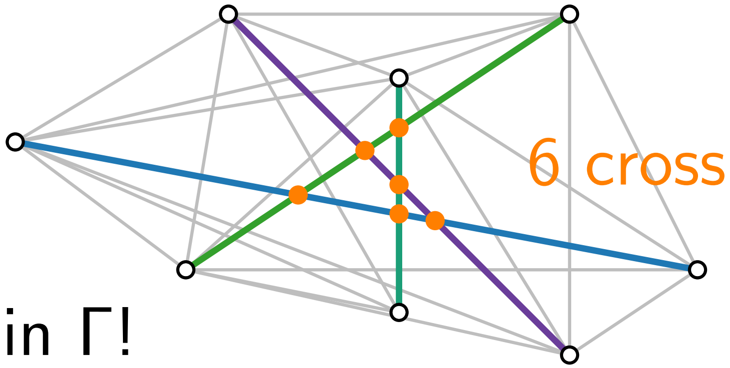

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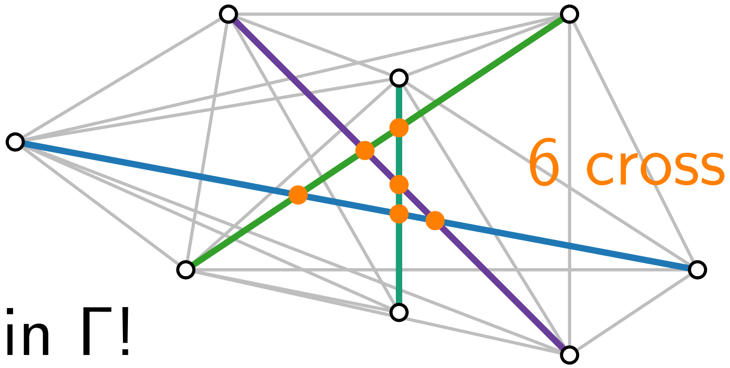

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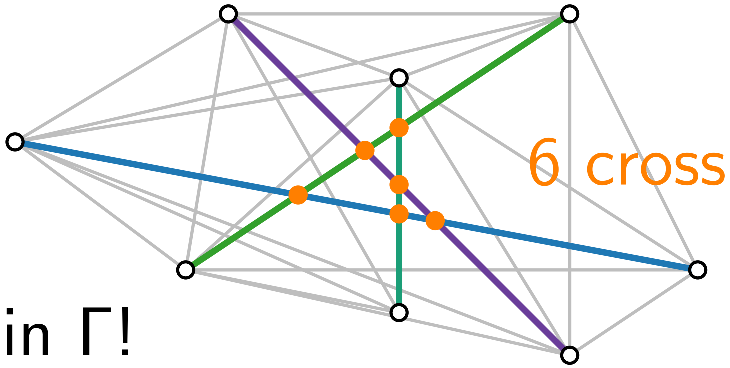

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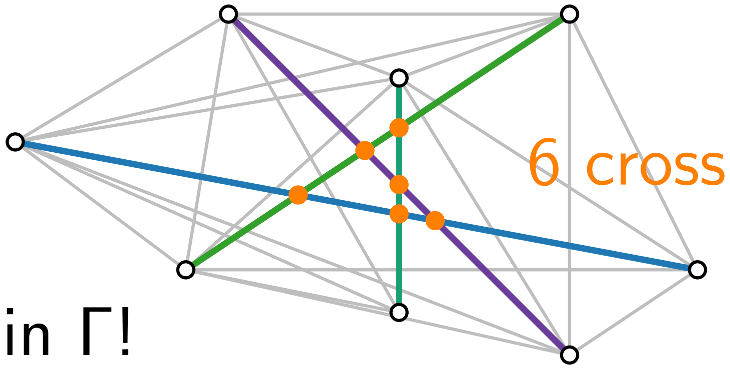

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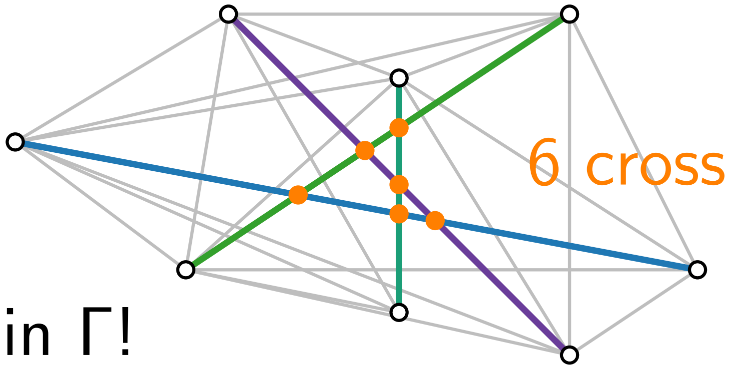

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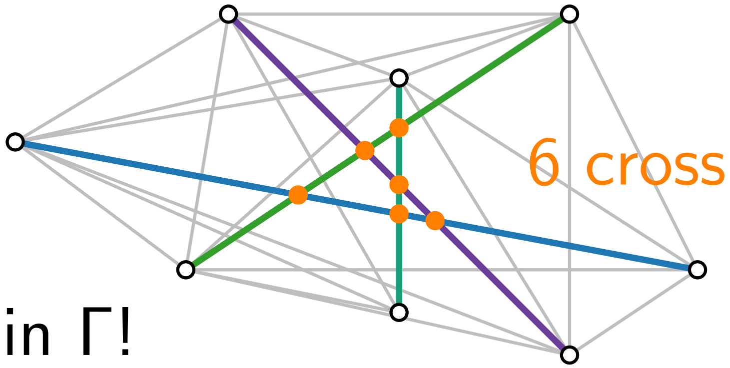

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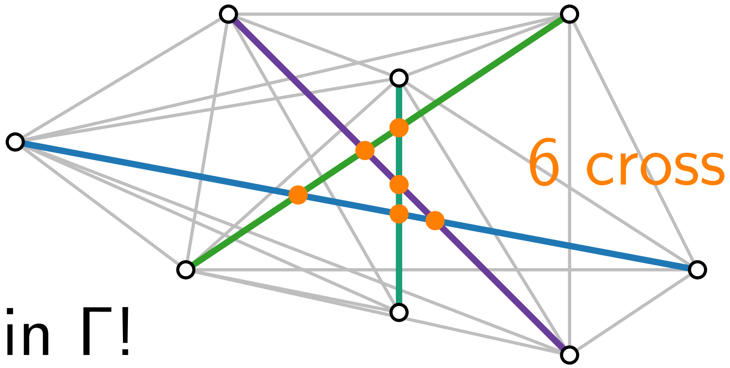

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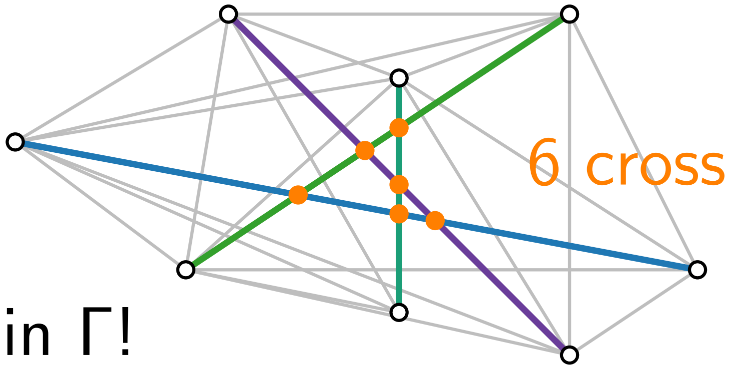

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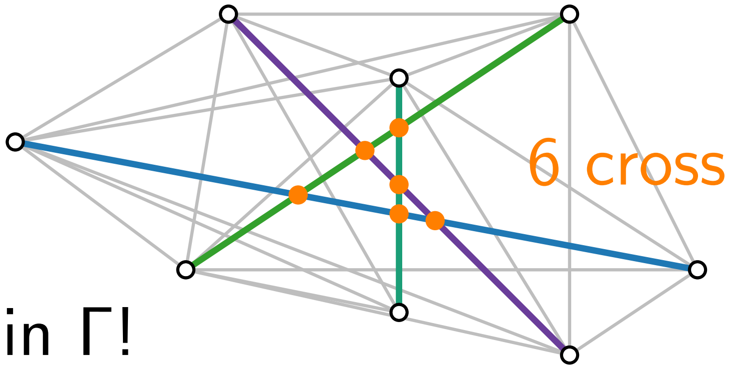

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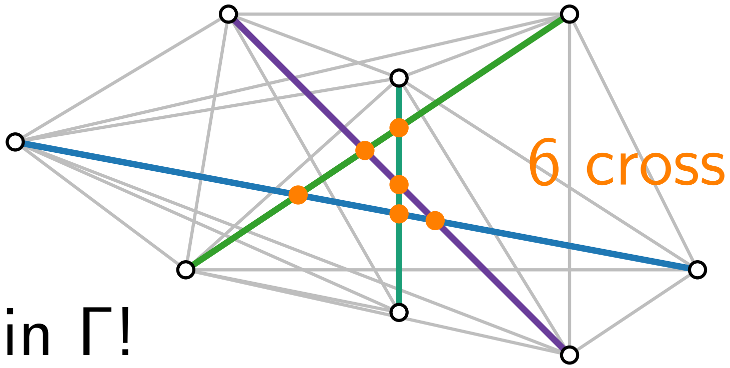
Application 3: Max. Num. of Crossings in Matchings



Given point set of n points

What is the max. number of crossings in any matching?

Point set spans drawing Γ of K_n

We will analyze the number of crossings in a **random** matching in Γ!

Number of crossings in $\Gamma \geq \overline{\text{cr}}(K_n) > \frac{3}{8}\binom{n}{4}$

Number of edges in K_n: $\binom{n}{2}$

Number of *potential crossings* (all pairs of edges): $\text{pot}(K_n) = \binom{\binom{n}{2}}{2} \approx 3\binom{n}{4}$

Pick two random edges e_1, e_2

$\Pr[e_1 \text{ and } e_2 \text{ cross}] \geq \overline{\text{cr}}(K_n)/\text{pot}(K_n) > \frac{1}{8}$

Fix matching M; it has $\leq n/2$ edges, so $\binom{n/2}{2} = \frac{1}{8}n(n-2)$ pairs of edges

By linearity of expectations,
exp. number of crossings in M is $> \frac{1}{8}\binom{n/2}{2} = \frac{1}{64}n(n-2)$

Literature

- [Aigner, Ziegler] Proofs from THE BOOK
- [Schaefer '20] The Graph Crossing Number and its Variants: A Survey
- Terrence Tao blog post "The crossing number inequality" from 2007
- [Garey, Johnson '83] Crossing number is NP-complete
- [Bienstock, Dean '93] Bounds for rectilinear crossing numbers
- [Székely '97] Crossing Numbers and Hard Erdös Problems in Discrete Geometry
- Documentary/Biography "N Is a Number: A Portrait of Paul Erdös"
- Exact computations of crossing numbers: `http://crossings.uos.de`