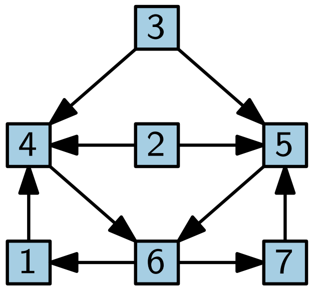
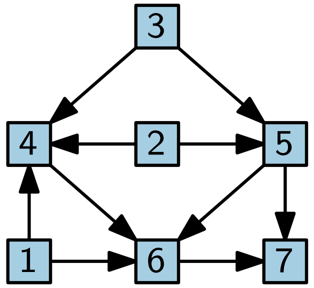
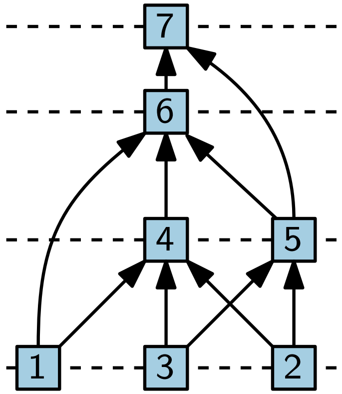
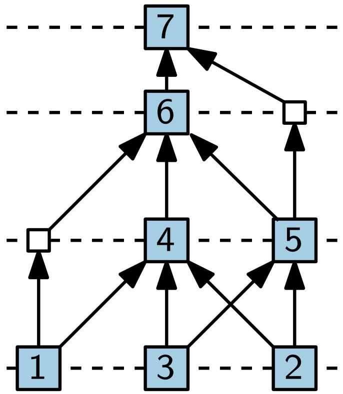
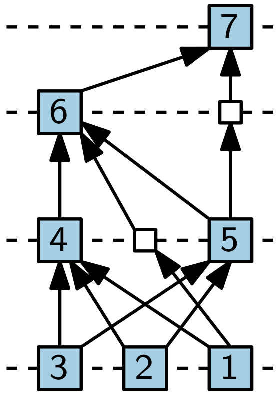
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

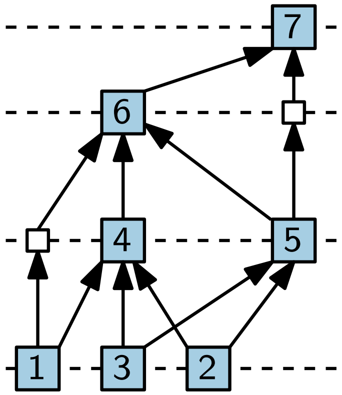
Visualization of Graphs

Lecture 7:
Hierarchical Layouts:
Sugiyama Framework

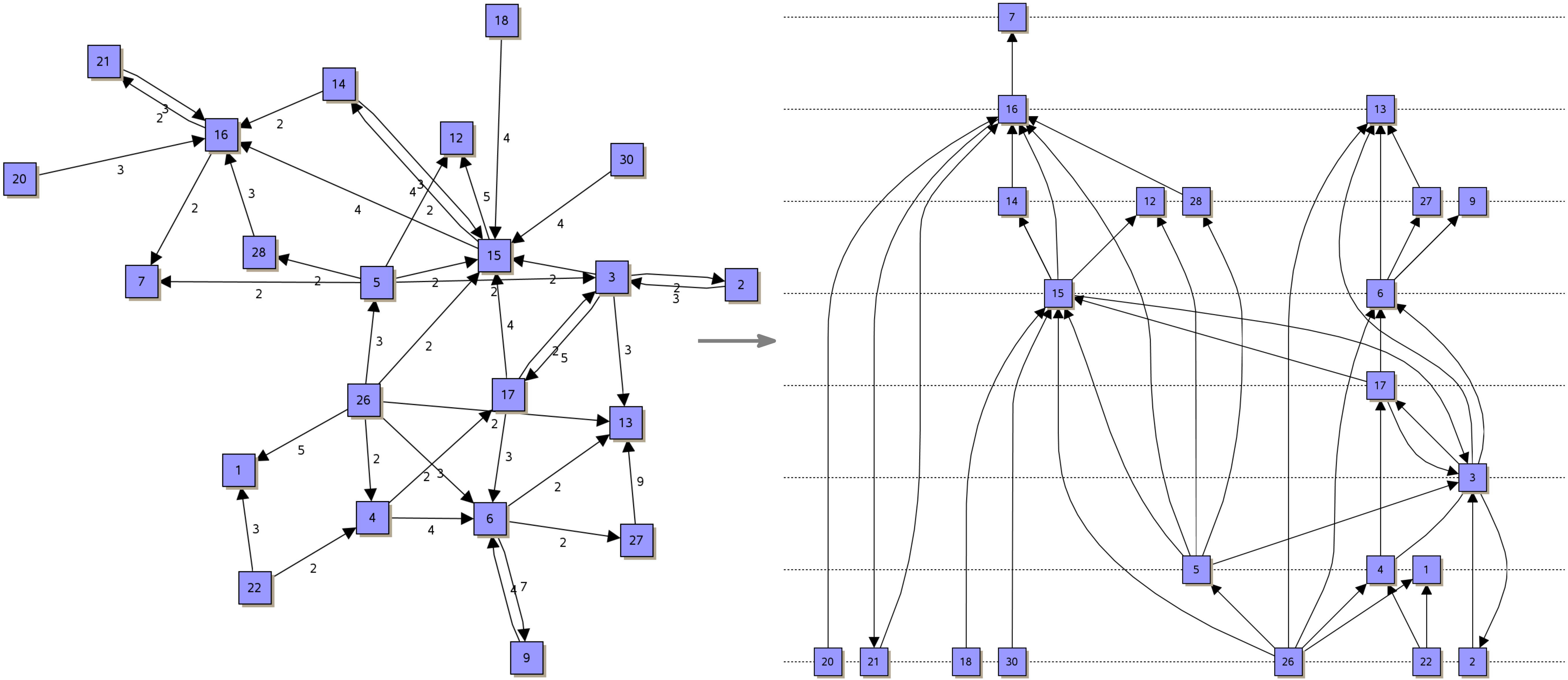


Part I:
The Framework



Jonathan Klawitter

Hierarchical Drawings – Motivation



Hierarchical Drawing

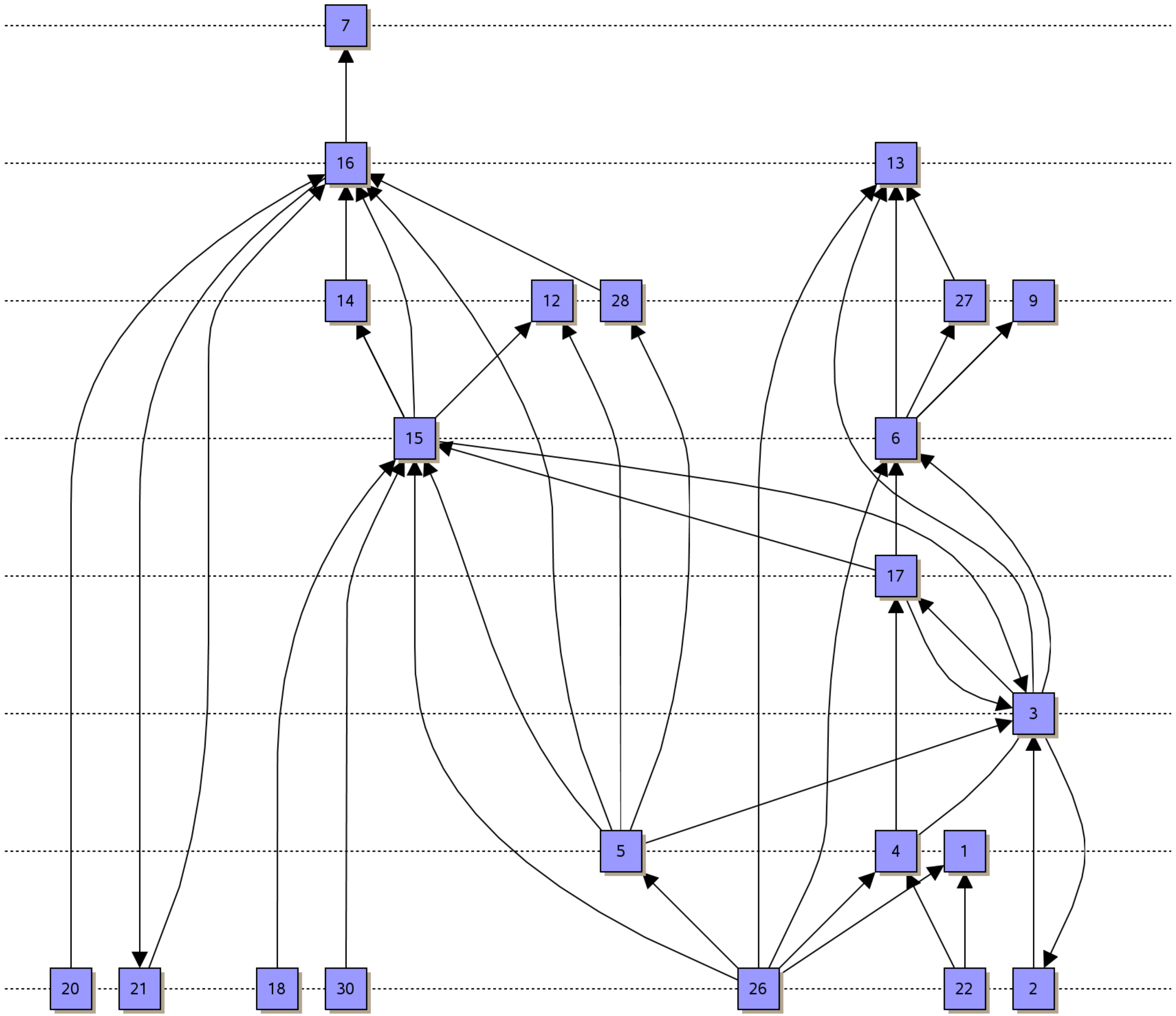
Problem Statement.

- Input: digraph $G = (V, E)$
- Output: drawing of G that "closely" reproduces the hierarchical properties of G

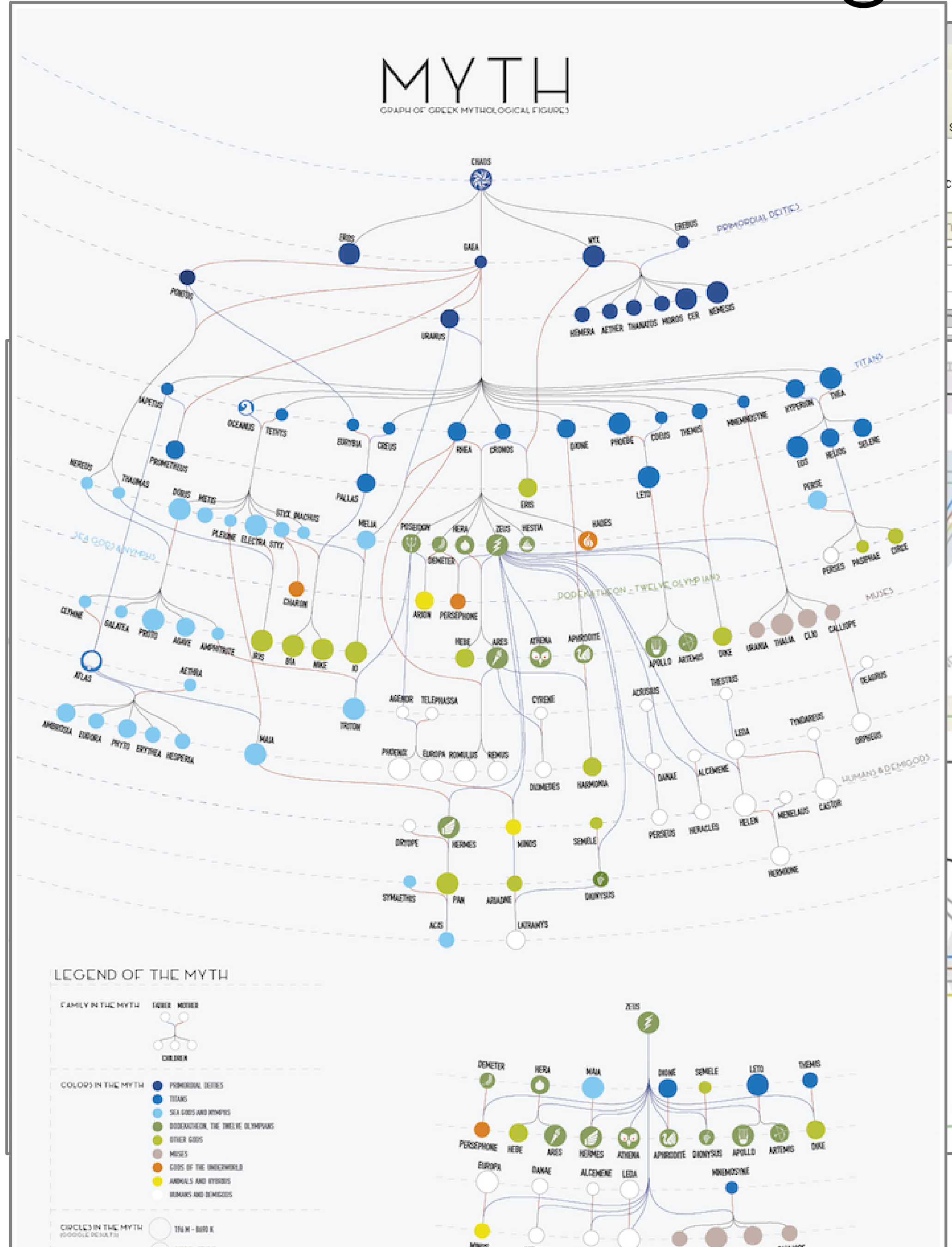
Desirable Properties.

- vertices occur on (few) horizontal lines
- edges directed upwards
- edge crossings minimized
- edges as short as possible
- vertices evenly spaced

Criteria can be contradictory!

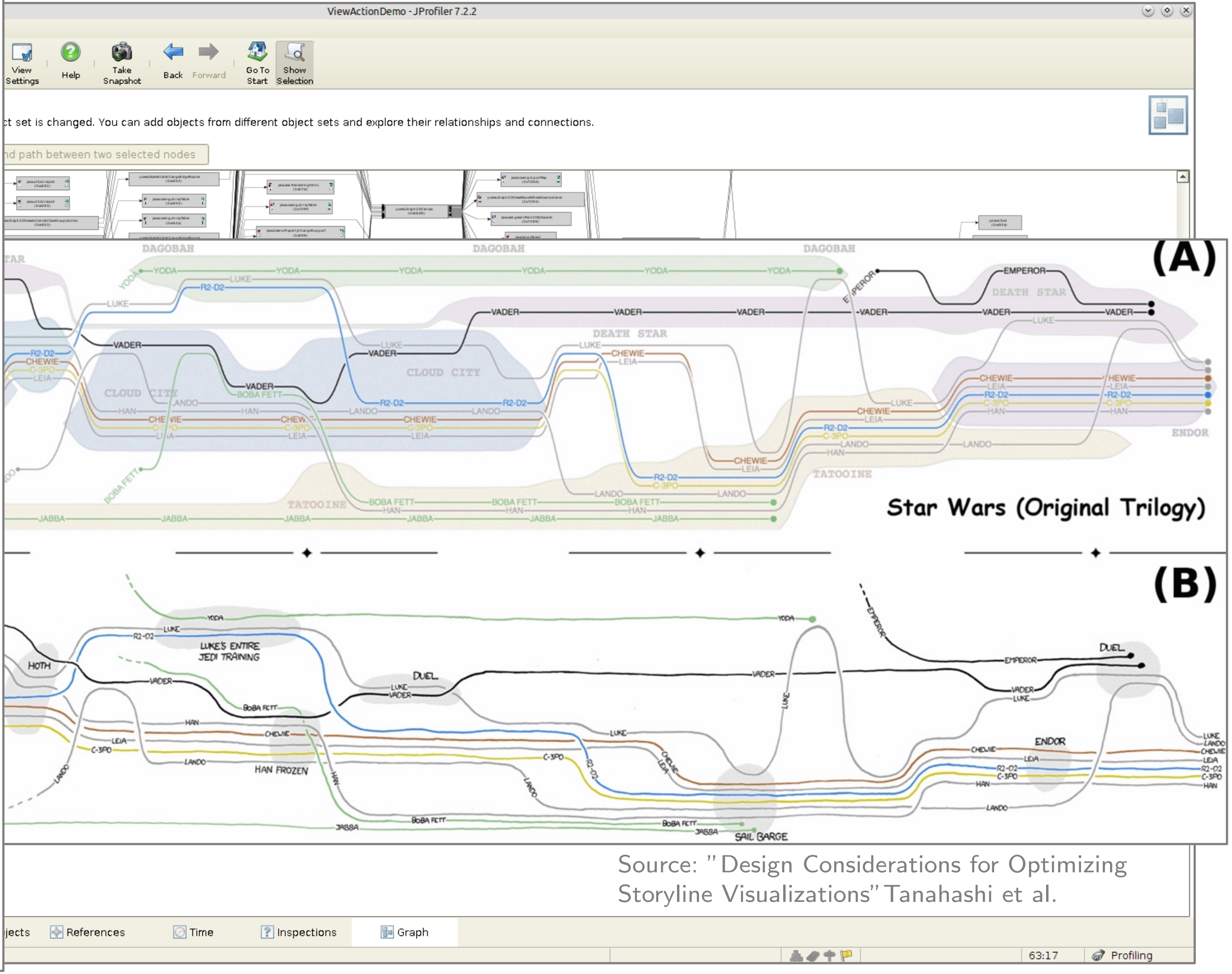


Hierarchical Drawing – Applications



Source: Visualization that won the Graph Drawing Contest 2016. Klawitter & Mchedlidze

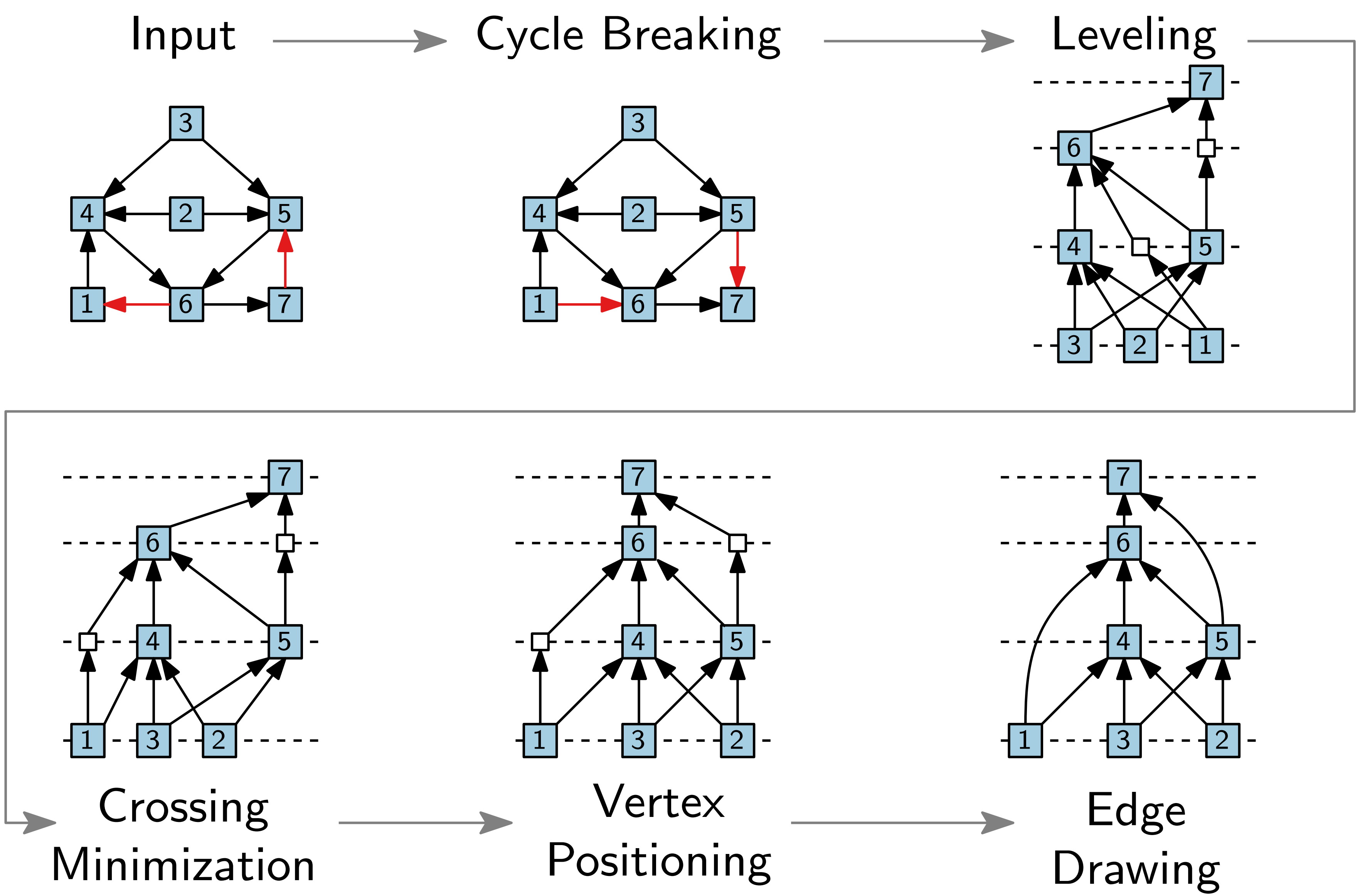
yEd Gallery: Java profiler JProfiler using yFiles



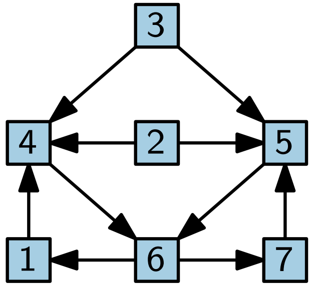
Source: "Design Considerations for Optimizing Storyline Visualizations" Tanahashi et al.

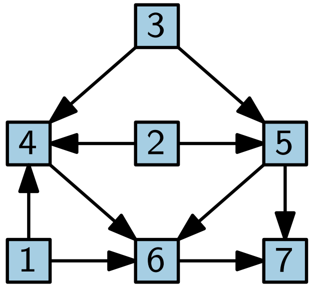
Classical Approach – Sugiyama Framework

[Sugiyama, Tagawa, Toda '81]

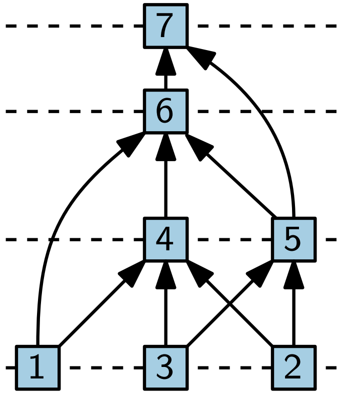


Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

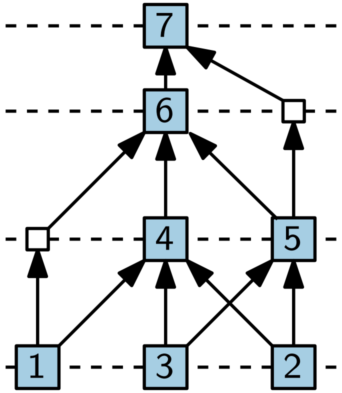


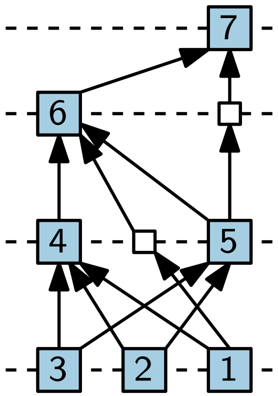


Visualization of Graphs

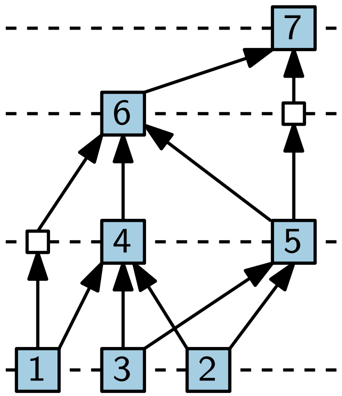


Lecture 7:
Hierarchical Layouts:
Sugiyama Framework



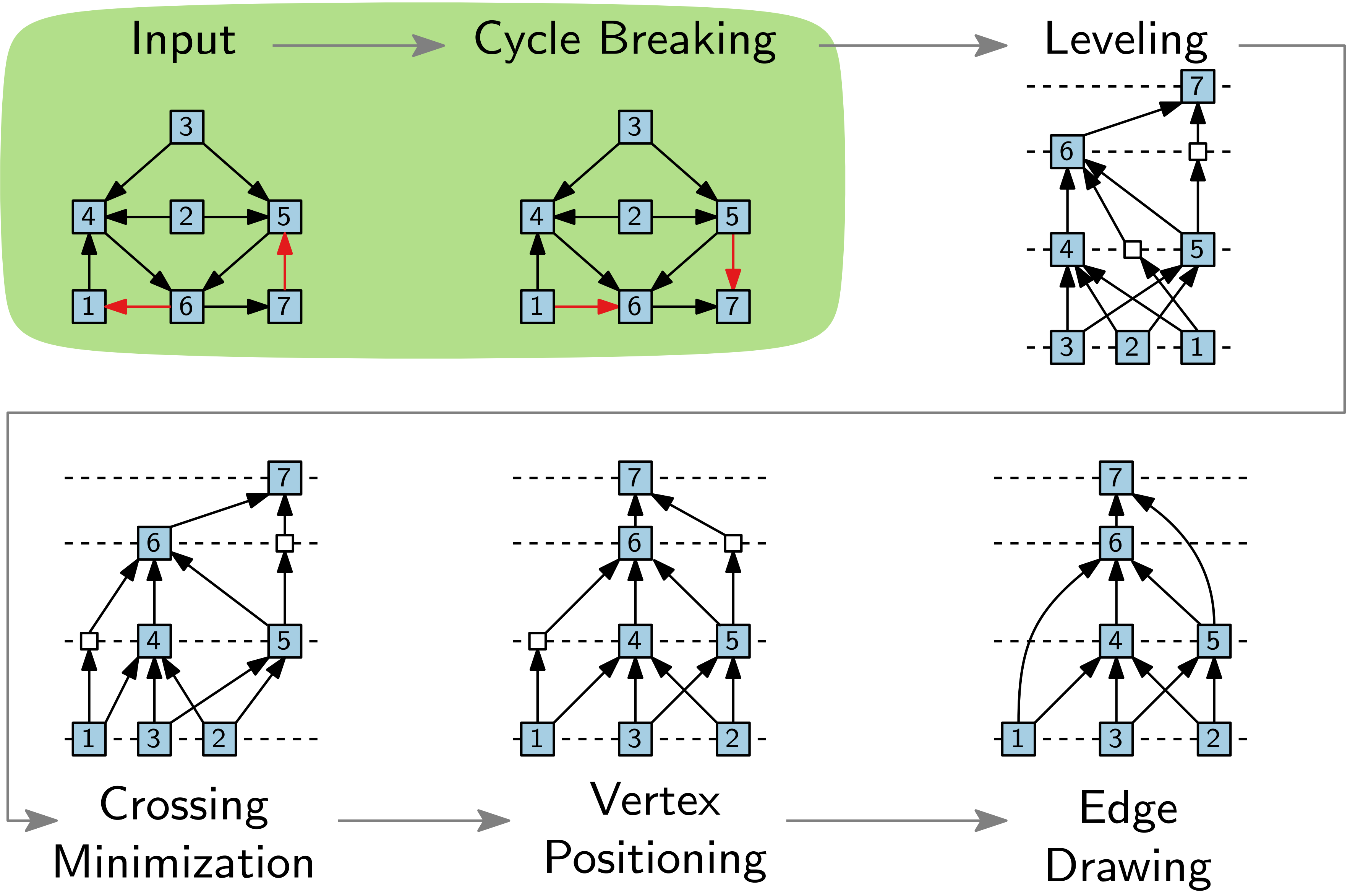


Part II:
Cycle Breaking

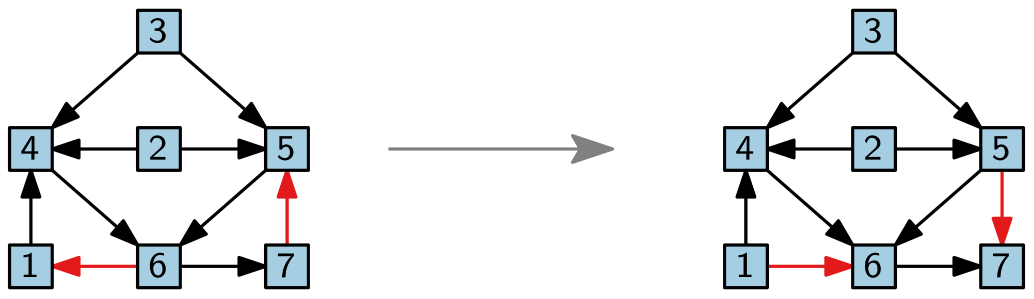


Jonathan Klawitter

Step 1: Cycle breaking



Step 1: Cycle breaking



Approach.

- Find minimum set $E^\star$ of edges which are not upwards.
- Remove $E^\star$ and insert reversed edges.

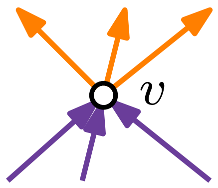
Problem Minimum Feedback ~~Arc~~ Set (F~~A~~S).

- Input: directed graph $G = (V, E)$
- Output: min. set $E^\star \subseteq E$, so that ~~$G - E^\star$~~ $G - E^\star + E^\star_r$ acyclic

. . . NP-hard ☹

Heuristic 1

[Berger, Shor '90]



GreedyMakeAcyclic(Digraph $G = (V, E)$)

$E' \leftarrow \emptyset$

foreach $v \in V$ **do**

  if $|N^{\rightarrow}(v)| \geq |N^{\leftarrow}(v)|$ **then**

    $E' \leftarrow E' \cup N^{\rightarrow}(v)$

  else

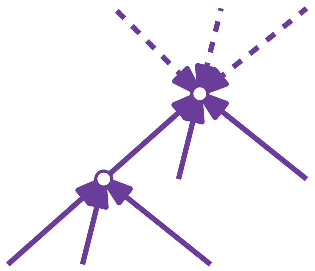
    $E' \leftarrow E' \cup N^{\leftarrow}(v)$

  remove v and $N(v)$ from G.

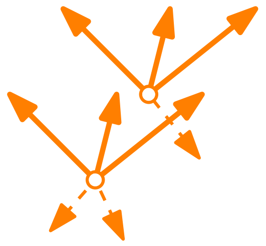
return (V, E')

$$N^{\rightarrow}(v) := \{(v,u) | (v,u) \in E\}$$
$$N^{\leftarrow}(v) := \{(u,v) | (u,v) \in E\}$$
$$N(v) := N^{\rightarrow}(v) \cup N^{\leftarrow}(v)$$

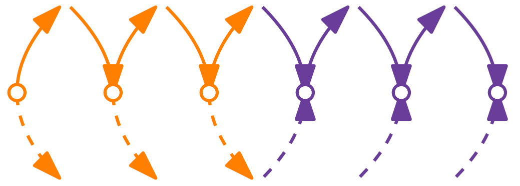
- Time: $\mathcal{O}(n + m)$
- Quality guarantee: $|E'| \geq |E|/2$



- $G' = (V, E')$ is a DAG



- $E \setminus E'$ is a feedback set

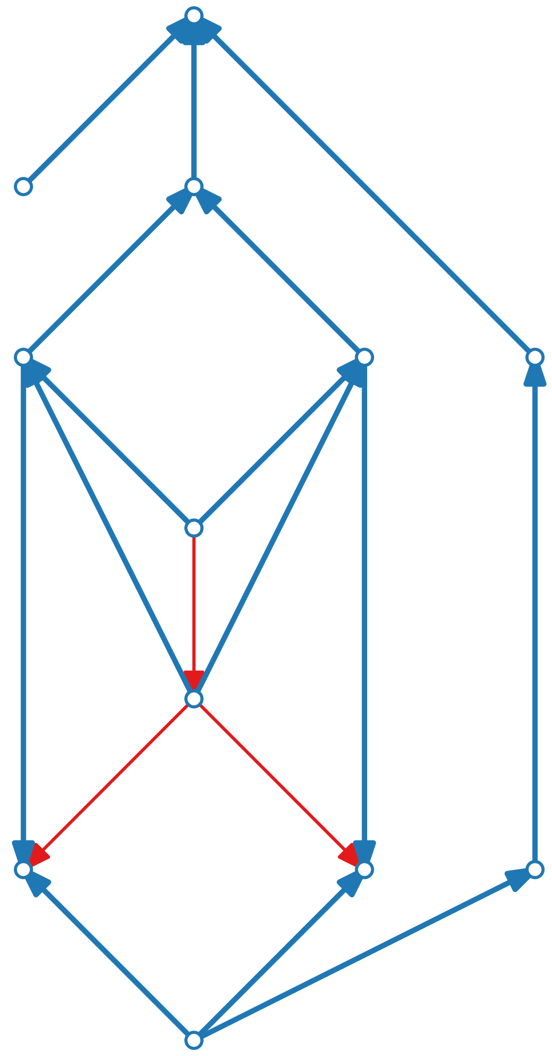


Heuristic 2

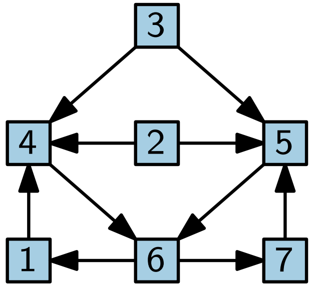
[Eades, Lin, Smyth '93]

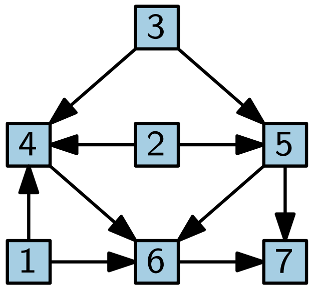
$E' \leftarrow \emptyset$
while $V \neq \emptyset$ **do**
- **while** in V exists a sink v **do**
  - $E' \leftarrow E' \cup N^{\leftarrow}(v)$
  - remove v and $N^{\leftarrow}(v)$
- Remove all isolated vertices from V
- **while** in V exists a source v **do**
  - $E' \leftarrow E' \cup N^{\rightarrow}(v)$
  - remove v and $N^{\rightarrow}(v)$
- **if** $V \neq \emptyset$ **then**
  - let $v \in V$ such that $|N^{\rightarrow}(v)| - |N^{\leftarrow}(v)|$ maximal
  - $E' \leftarrow E' \cup N^{\rightarrow}(v)$
  - remove v and $N(v)$

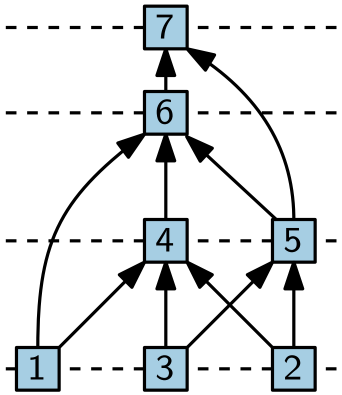
- Time: $\mathcal{O}(n+m)$
- Quality guarantee: $|E'| \geq |E|/2 + |V|/6$

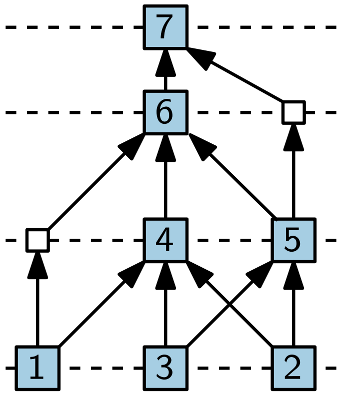


Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG





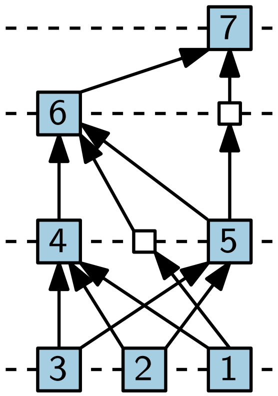


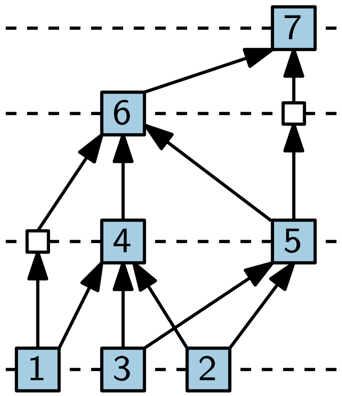


Visualization of Graphs

Lecture 7:
Hierarchical Layouts:
Sugiyama Framework

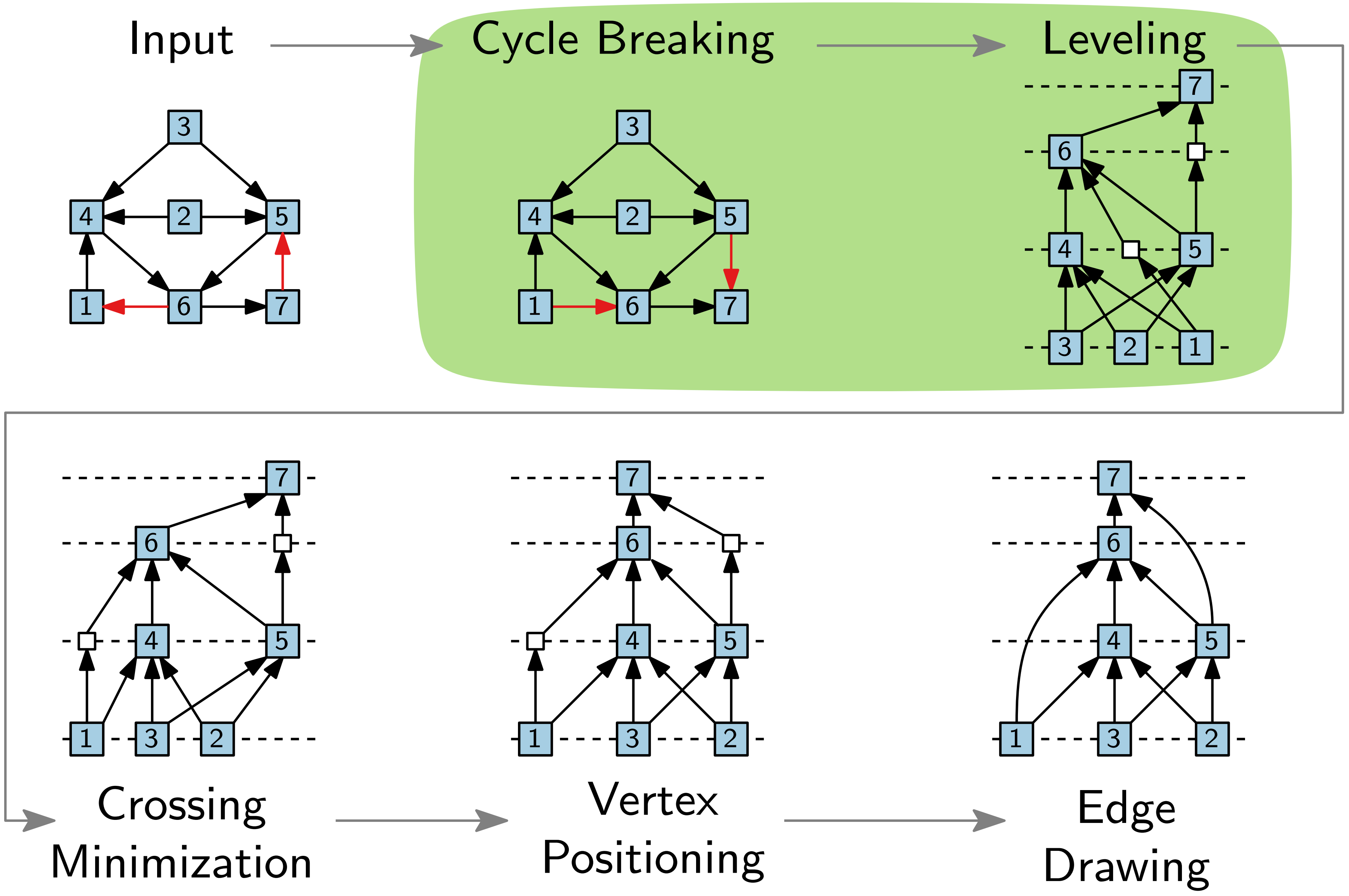
Part III:
Leveling



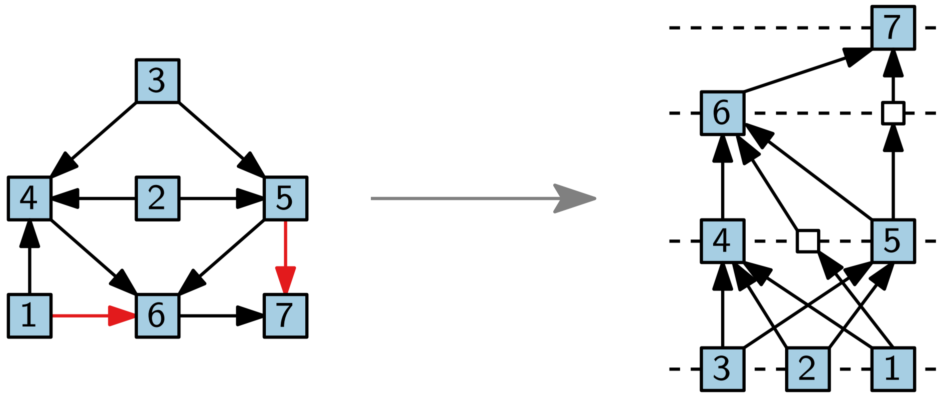


Jonathan Klawitter

Step 2: Leveling



Step 2: Leveling



Problem.

- Input: acyclic digraph $G = (V, E)$
- Output: Mapping $y\colon V \to \{1, \ldots n\}$,
  so that for every $uv \in E$, $y(u) < y(v)$.

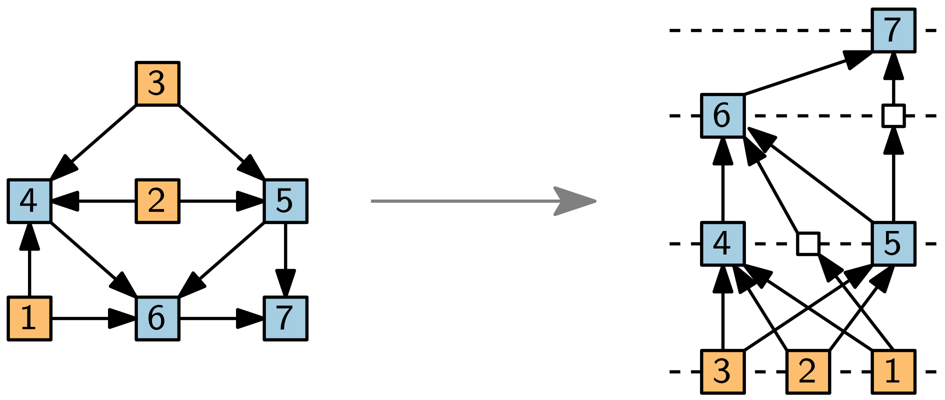
Objective is to *minimize . . .*

- number of layers, i.e. $|y(V)|$
- length of the longest edge, i.e. $\max_{uv \in E} y(v) - y(u)$
- width, i.e. $\max\{|L_i| \mid 1 \leq i \leq h\}$
- total edge length, i.e. number of dummy vertices

Min Number of Layers

Algorithm.

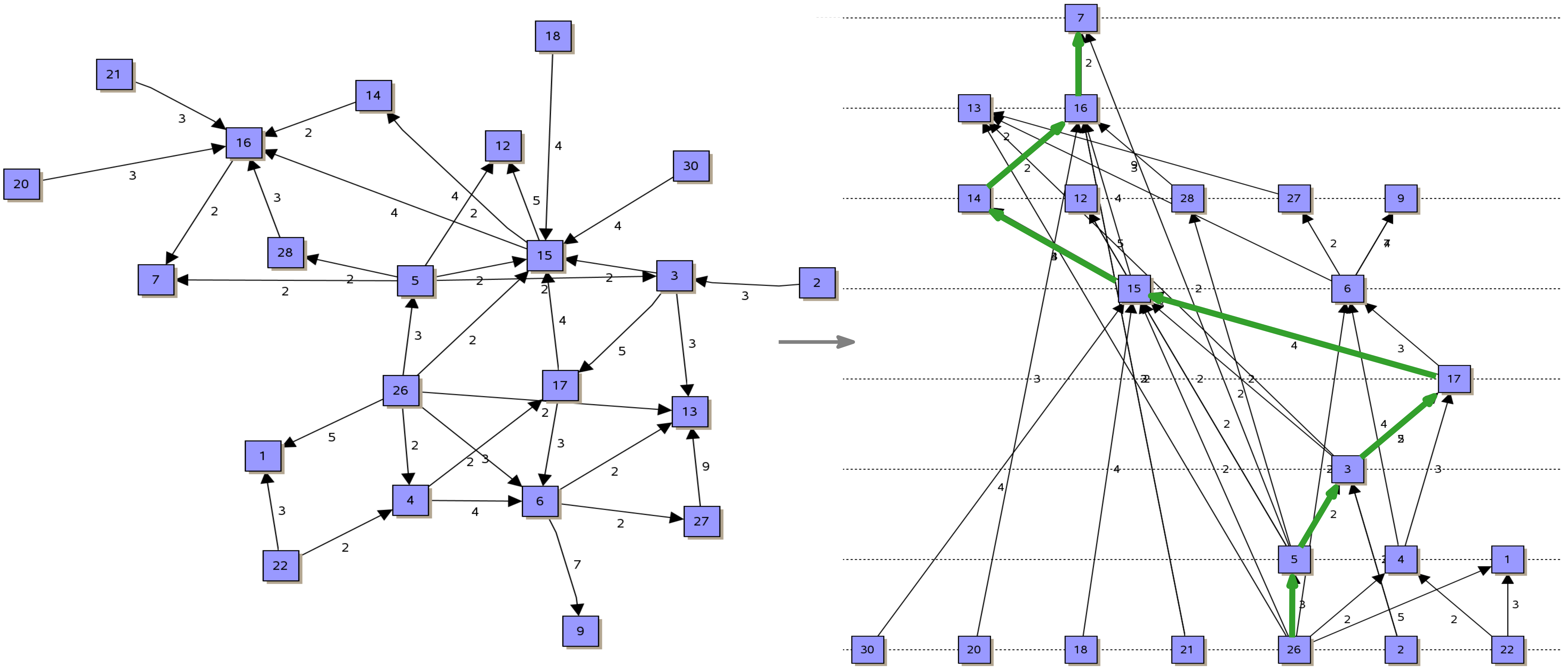
- for each source q
  set $y(q) := 1$
- for each non-source v
  set $y(v) := \max\{y(u) \mid uv \in E\} + 1$



Observation.

- $y(v)$ is length of the longest path from a source to v plus 1.
  ... which is optimal!
- Can be implemented in linear time with recursive algorithm.

Example


21
14
16
20
18
12
30
28
7
5
15
3
2
26
17
13
1
4
6
27
22
9
7
13
16
14
12
28
27
9
15
6
17
3
5
4
1
30
20
18
21
26
2
22

Total Edge Length – ILP

Can be formulated as an integer linear program:

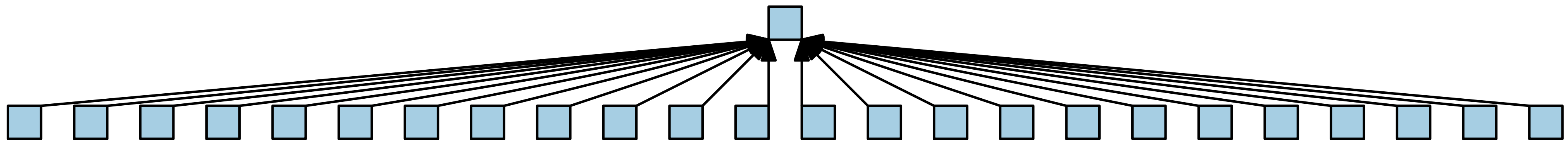
$$
\begin{array}{rll}
\text{min} & \sum_{(u,v)\in E}(y(v)-y(u)) & \\
\text{subject to} & y(v)-y(u)\geq 1 & \forall (u,v)\in E \\
 & y(v)\geq 1 & \forall v\in V \\
 & y(v)\in \mathbb{Z} & \forall v\in V
\end{array}
$$

One can show that:

- Constraint-matrix is **totally unimodular**

  $\Rightarrow$ Solution of the relaxed linear program is integer
- The total edge length can be minimized in polynomial time

Width



Drawings can be very wide.

Narrower Layer Assignment

Problem: Leveling With a Given Width.

- Input: acyclic, digraph $G = (V, E)$, width $W > 0$
- Output: Partition the vertex set into a minimum number of layers such that each layer contains at most W elements.

Problem: Precedence-Constrained Multi-Processor Scheduling

- Input: n jobs with unit (1) processing time, W identical machines, and a partial ordering $<$ on the jobs.
- Output: Schedule respecting $<$ and having minimum processing time.

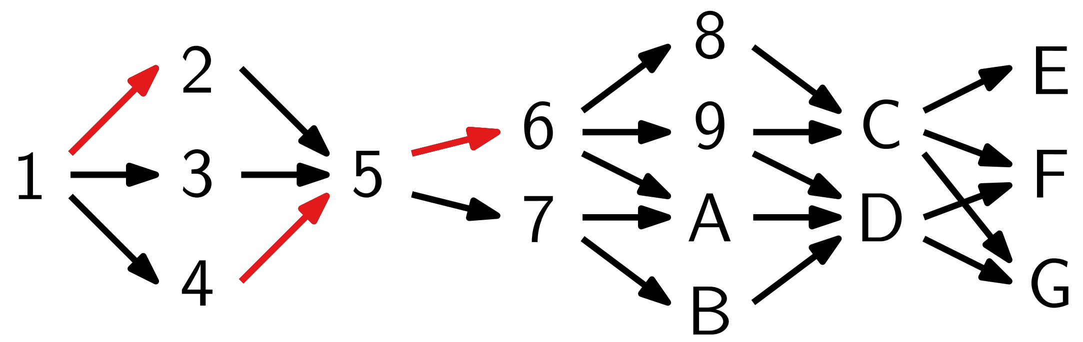
- NP-hard, $(2 - \frac{1}{W})$-Approx., no $(\frac{4}{3} - \varepsilon)$-Approx. ($W \geq 3$).

Approximating PCMPS

- jobs stored in a list L
(in any order, e.g., topologically sorted)
- for each time $t = 1, 2, \ldots$ schedule $\leq W$ available jobs
- a job in L is *available* when all its predecessors have been scheduled
- as long as there are free machines and available jobs, take the first available job and assign it to a free machine

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



Number of Machines is $W = 2$.

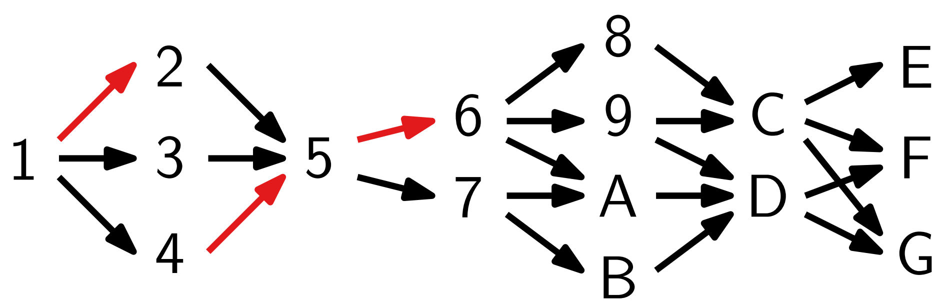
Output: Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

Question: Good approximation factor?

Approximating PCMPS - Analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

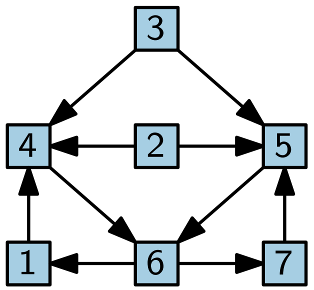
Goal: measure the quality of our algorithm using the lower bounds

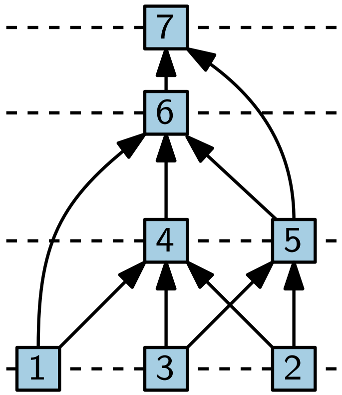
$\leq (2 - 1/W) \cdot$ OPT in general case

Bound. ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx \lceil n/2 \rceil + \ell/2 \leq 3/2 \cdot$ OPT

insertion of pauses (–) in the schedule (except the last) maps to layers of $G_<$

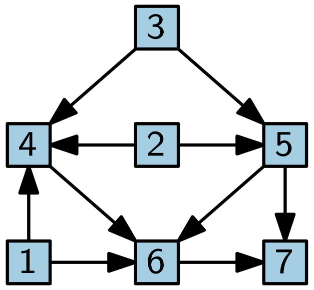
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

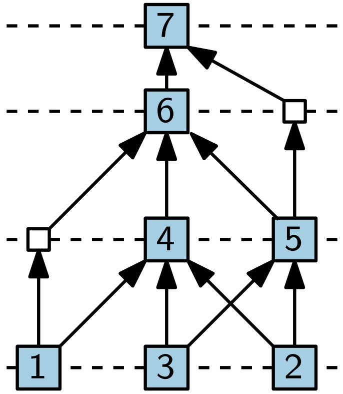


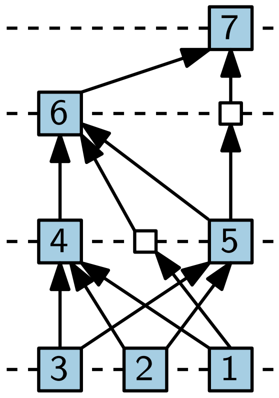


Visualization of Graphs

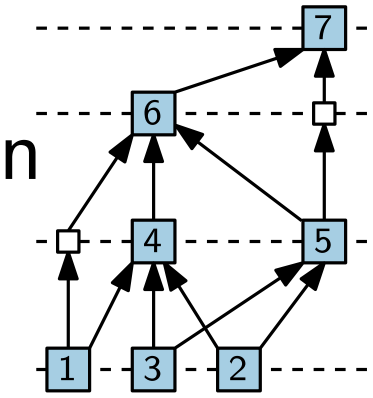
Lecture 7:
Hierarchical Layouts:
Sugiyama Framework





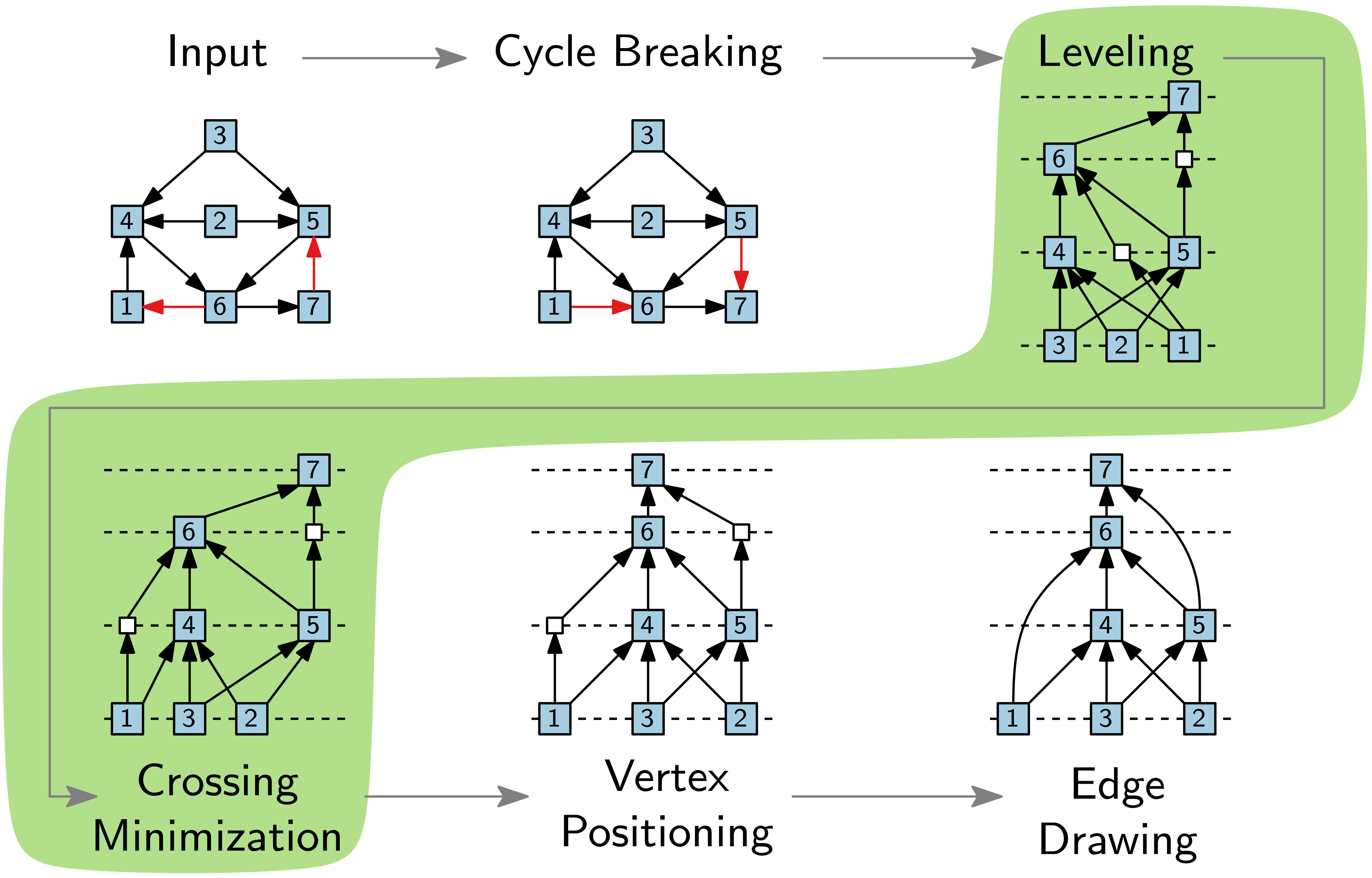


Part IV:
Crossing Minimisation

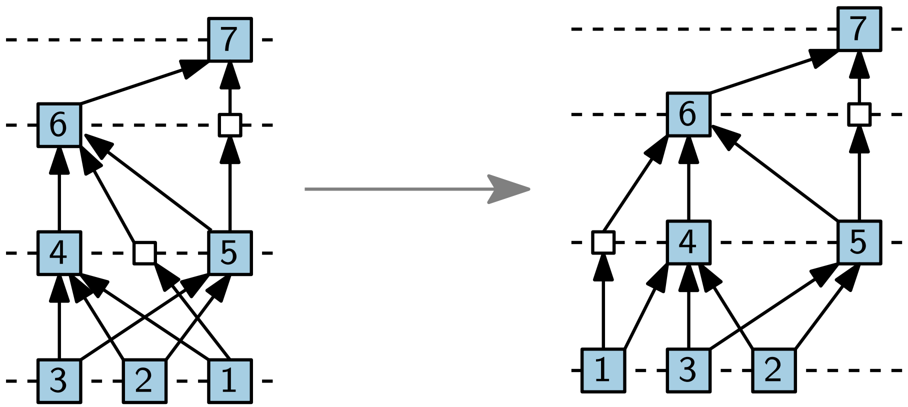


Jonathan Klawitter

Step 3: Crossing Minimization



Step 3: Crossing Minimization



Problem.

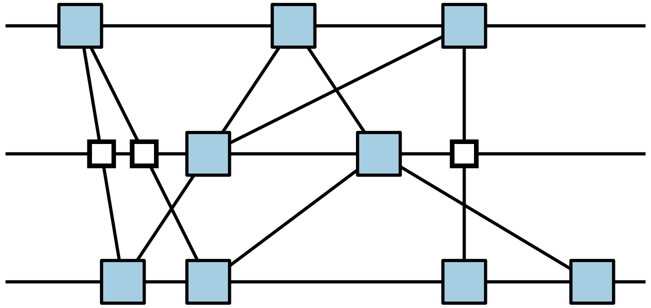
- Input: Graph G, layering $y\colon V \to \{1, \ldots, n\}$
- Output: (Re-)ordering of vertices in each layer so that the number of crossings in minimized.

- NP-hard, even for 2 layers [Garey & Johnson '83]
- hardly any approaches optimize over multiple layers :(

Iterative Crossing Reduction – Idea

Observation.

The number of crossings only depends on permutations of adjacent layers.



- Add dummy-vertices for edges connecting “far” layers.
- Consider adjacent layers $(L_1, L_2), (L_2, L_3), \ldots$ bottom-to-top.
- Minimize crossings by permuting L_{i+1} while keeping L_i fixed.

Iterative Crossing Reduction – Algorithm

one-sided crossing minimization

(1) choose a random permutation of L_1

(2) iteratively consider adjacent layers L_i and L_{i+1}

(3) minimize crossings by permuting L_{i+1} and keeping L_i fixed

(4) repeat steps (2)–(3) in the reverse order (starting from L_h)

(5) repeat steps (2)–(4) until no further improvement is achieved

(6) repeat steps (1)–(5) with different starting permutations

One-Sided Crossing Minimization

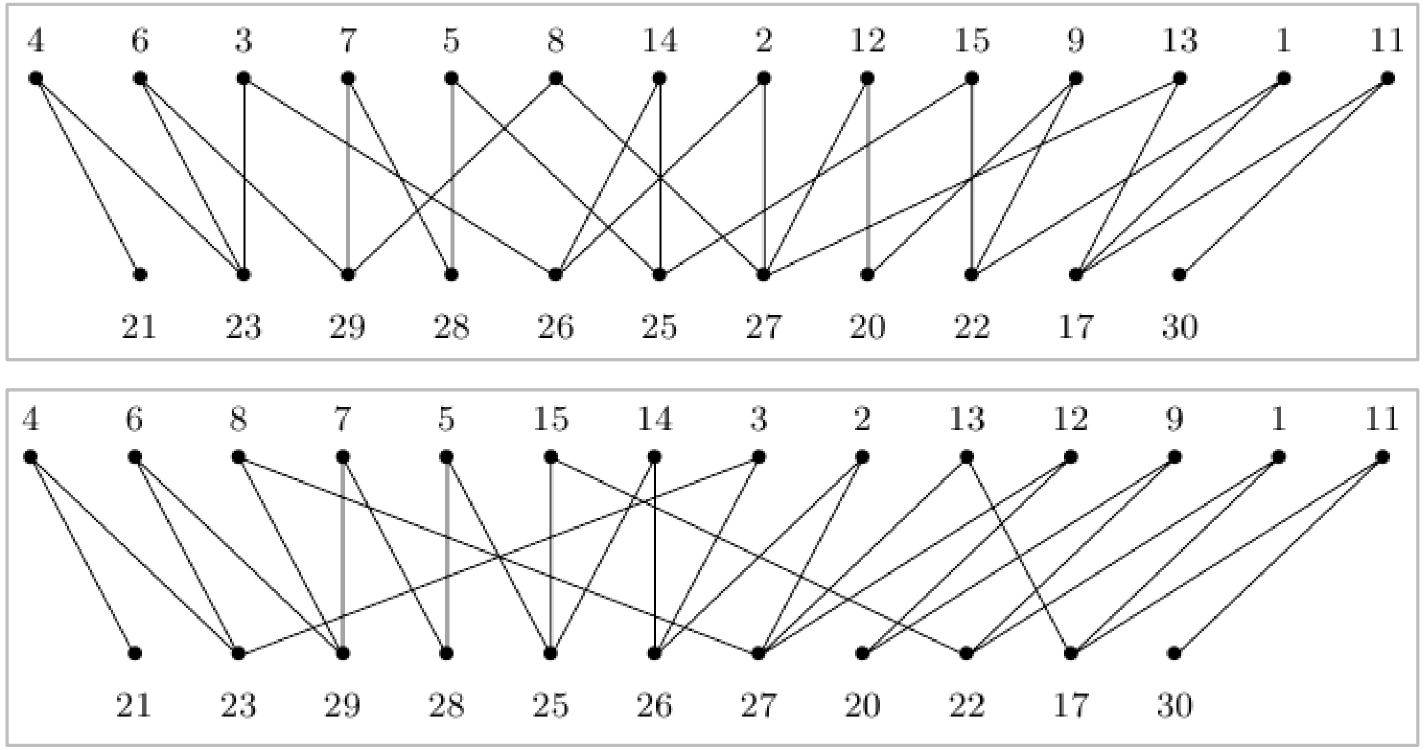
Problem.

- Input: bipartite graph $G = (L_1 \cup L_2, E)$, permutation π_1 on L_1
- Output: permutation π_2 of L_2 minimizing the number of edge crossings.

One-sided crossing minimization is NP-hard.
[Eades & Whitesides '94]

Algorithms.

- barycenter heuristic
- median heuristic
- Greedy-Switch
- ILP
- ...



Barycenter Heuristic

[Sugiyama et al. '81]

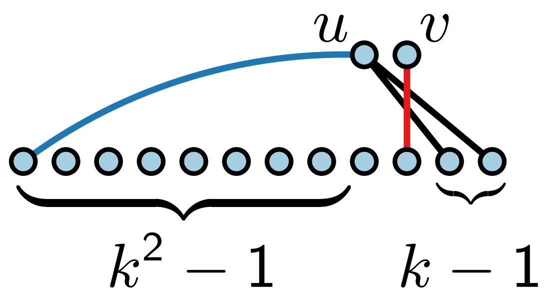
- Intuition: few intersections occur when vertices are close to their neighbors

- The barycentre of u is the mean x-coordinate of the neighbours of u in layer L_1 $[x_1 \equiv \pi_1]$

$$x_2(u) := \text{bary}(u) := \frac{1}{\deg(u)} \sum_{v \in N(u)} x_1(v)$$

- Vertices with the same barycentre are offset by a small δ.
- linear runtime
- relatively good results
- optimal if no crossings are required ← Exercise!
- $O(\sqrt{n})$-approximation factor

Worst case?

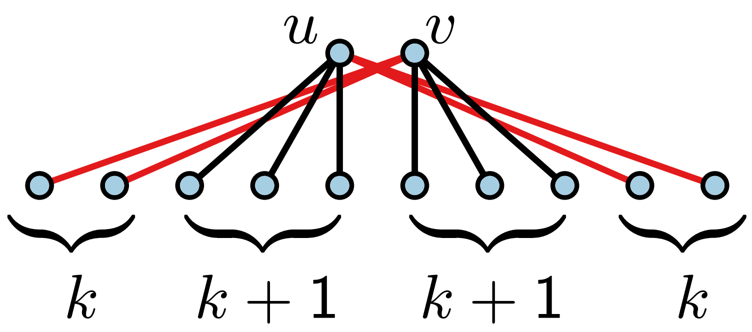


Median Heuristic

[Eades & Wormald '94]

- $\{v_1, \dots, v_k\} := N(u)$ with $\pi_1(v_1) < \pi_1(v_2) < \dots < \pi_1(v_k)$
- $$x_2(u) := \text{med}(u) := \begin{cases} 0 & \text{when } N(u) = \emptyset \\ \pi_1(v_{\lceil k/2 \rceil}) & \text{otherwise} \end{cases}$$
- Move vertices u und v by small δ, when $x_2(u) = x_2(v)$

Worst case?



$2k(k+1) + k^2$ vs. $(k+1)^2$

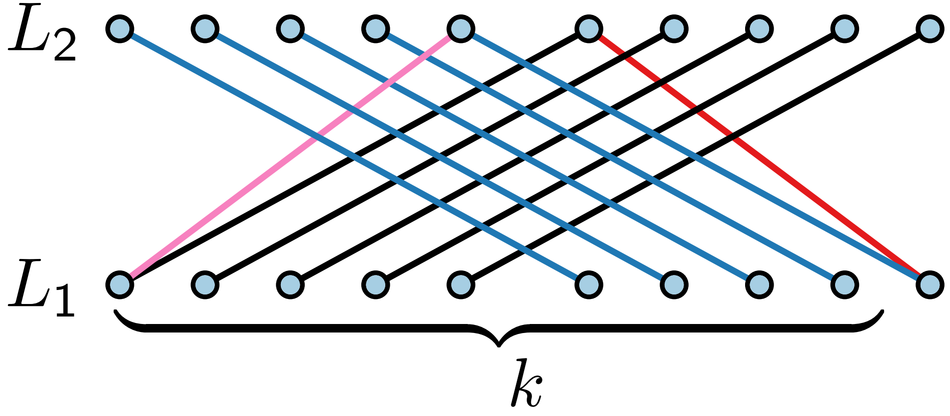
- Linear runtime
- Relatively good results
- Optimal if no crossings are required
- 3-Approximation factor

  Proof in [GD Ch 11]

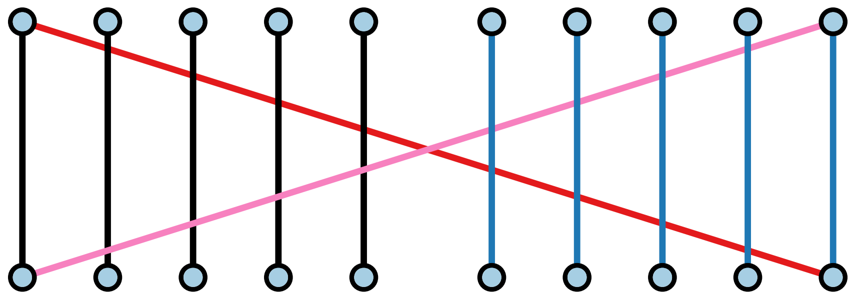
Greedy-Switch Heuristic

- Iteratively swap adjacent nodes as long as crossings decrease
- Runtime $O(L_2)$ per iteration; at most $|L_2|$ iterations
- Suitable as post-processing for other heuristics

Worst case?



$\approx k^2/4$



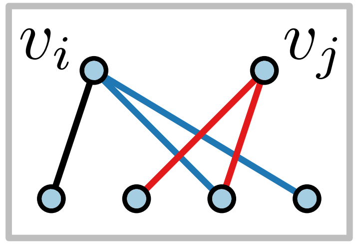
$\approx 2k$

Integer Linear Program

[Jünger & Mutzel, '97]

- Constant c_{ij} := # crossings between edges incident to v_i or v_j when $\pi_2(v_i) < \pi_2(v_j)$

- Variable x_{ij} for each $1 \leq i < j \leq n_2 := |L_2|$



$$x_{ij} = \begin{cases} 1 & \text{when } \pi_2(v_i) < \pi_2(v_j) \\ 0 & \text{otherwise} \end{cases}$$

- The number of crossings of a permutations π_2

$$\text{cross}(\pi_2) = \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} (c_{ij} - c_{ji}) x_{ij} + \underbrace{\sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} c_{ji}}_{\text{constant}}$$

Integer Linear Program

- Minimize the number of crossings:

$$\text{minimize} \sum_{i=1}^{n_2-1} \sum_{j=i+1}^{n_2} (c_{ij} - c_{ji}) x_{ij}$$

- Transitivity constraints:

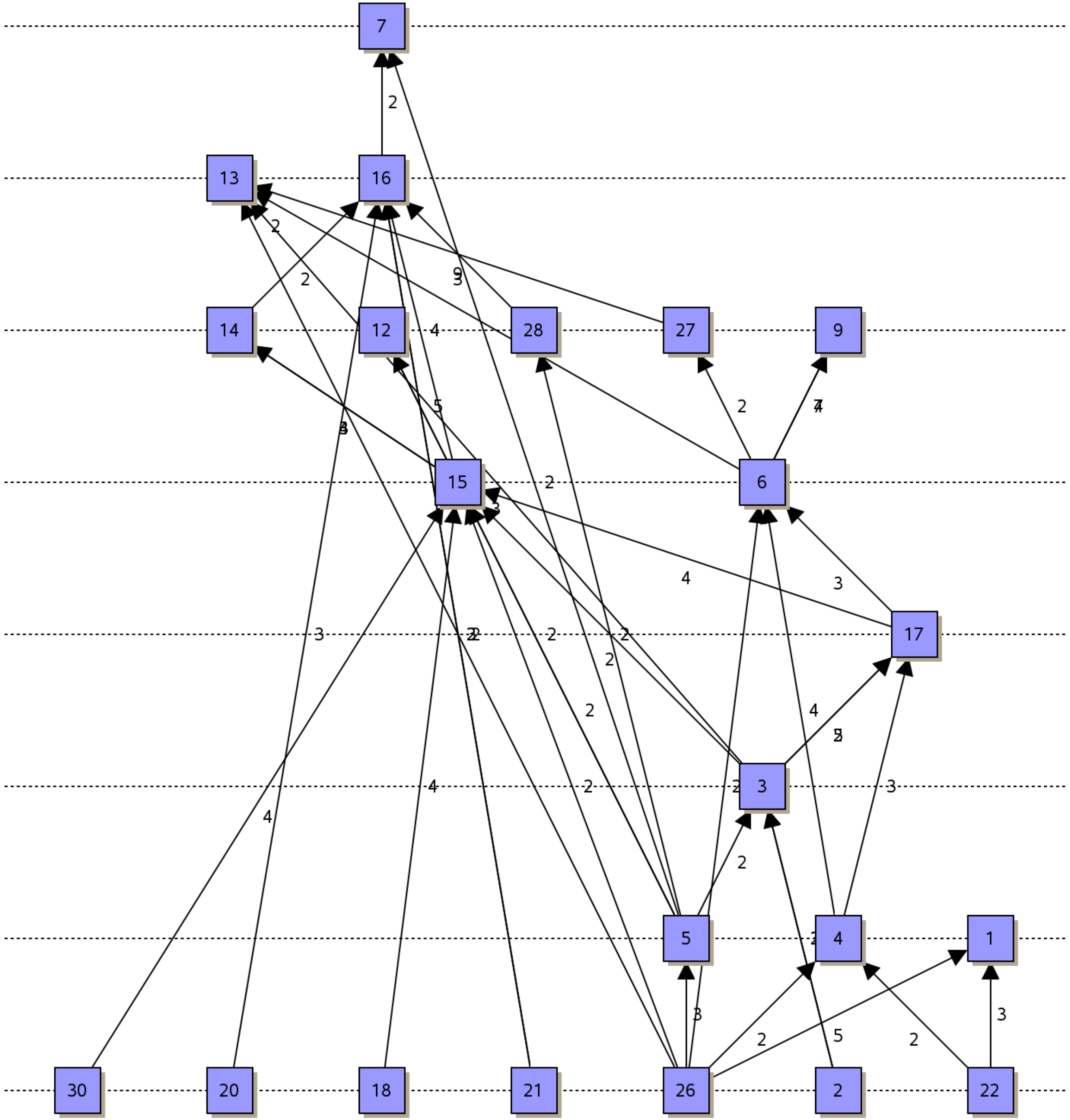
$$0 \leq x_{ij} + x_{jk} - x_{ik} \leq 1 \qquad \text{for } 1 \leq i < j < k \leq n_2$$

i.e., if $x_{ij} = \genfrac{}{}{0pt}{}{1}{0}$ and $x_{jk} = \genfrac{}{}{0pt}{}{1}{0}$, then $x_{ik} = \genfrac{}{}{0pt}{}{1}{0}$

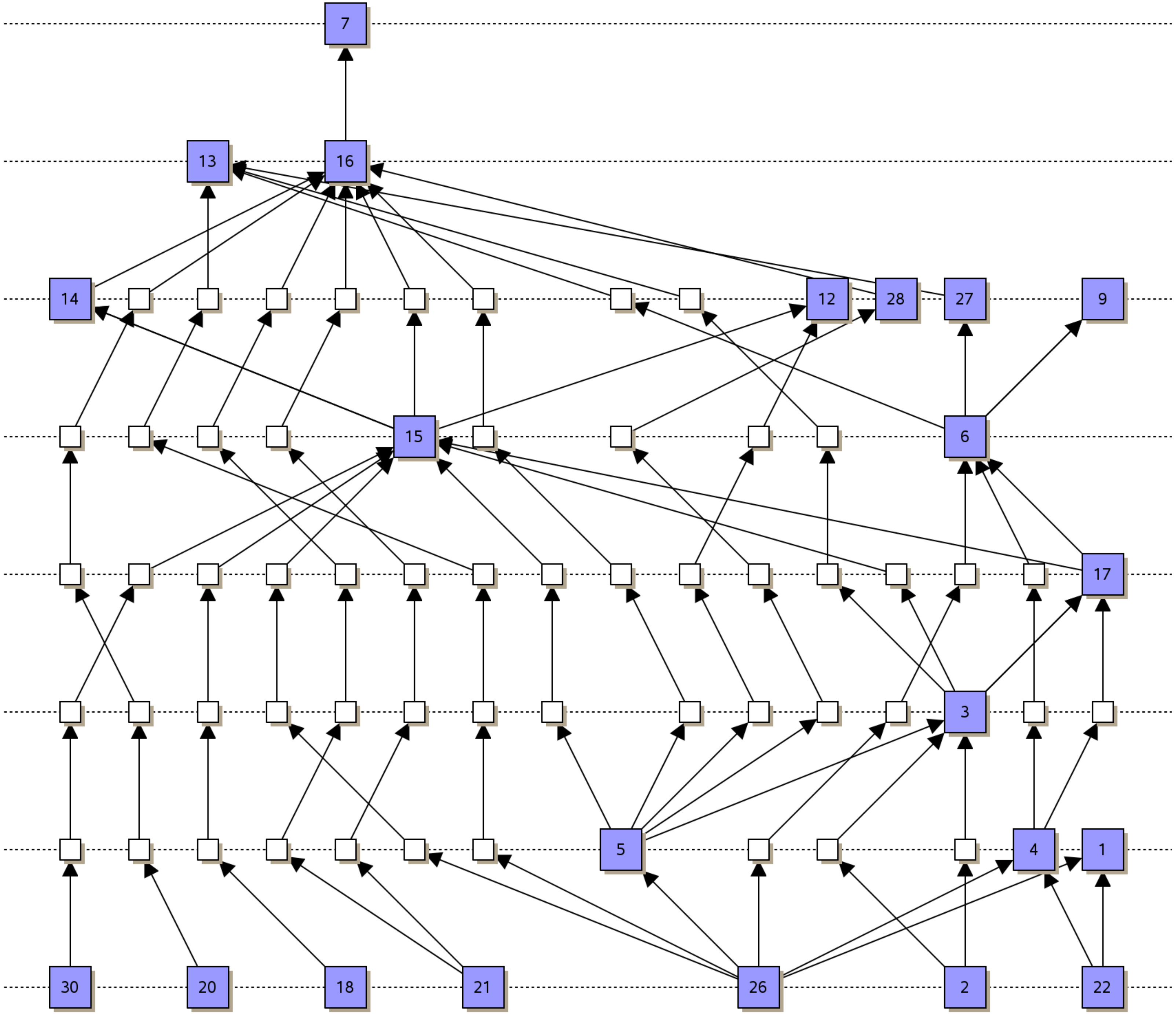
Properties.

- Branch-and-cut technique for DAGs of limited size
- Useful for graphs of small to medium size
- Finds optimal solution
- Solution in polynomial time is not guaranteed

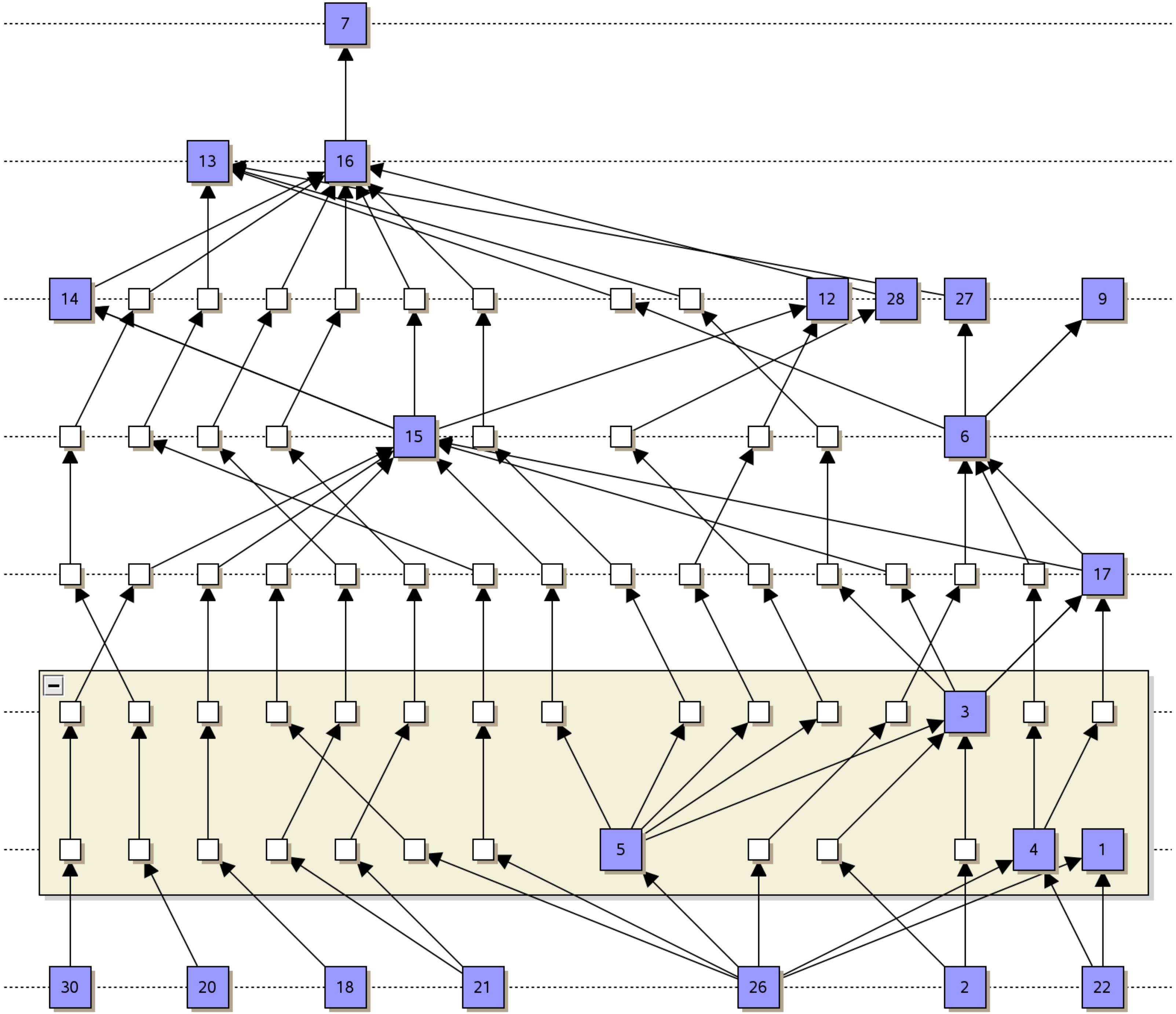
Iterations on Example



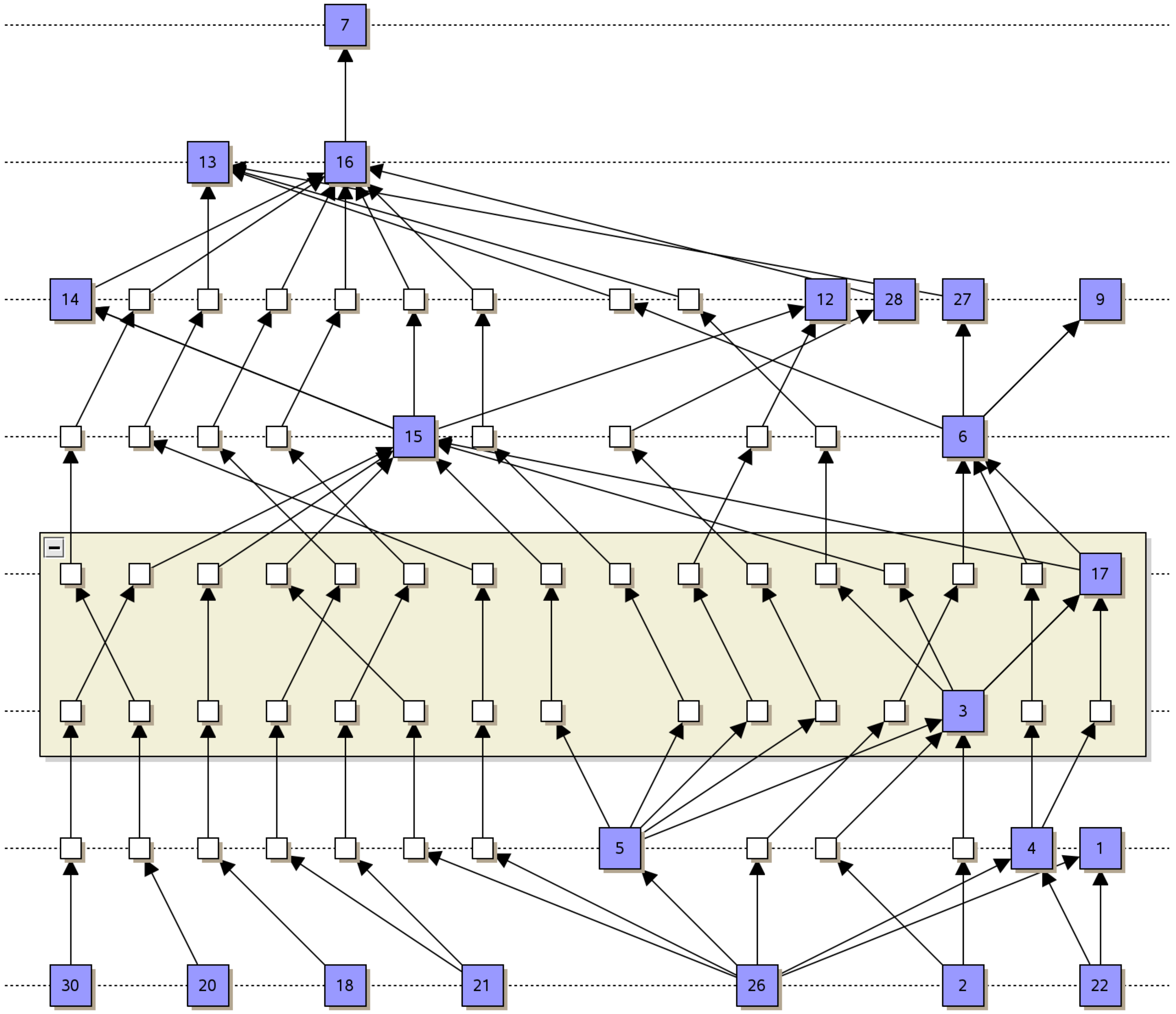
Iterations on Example



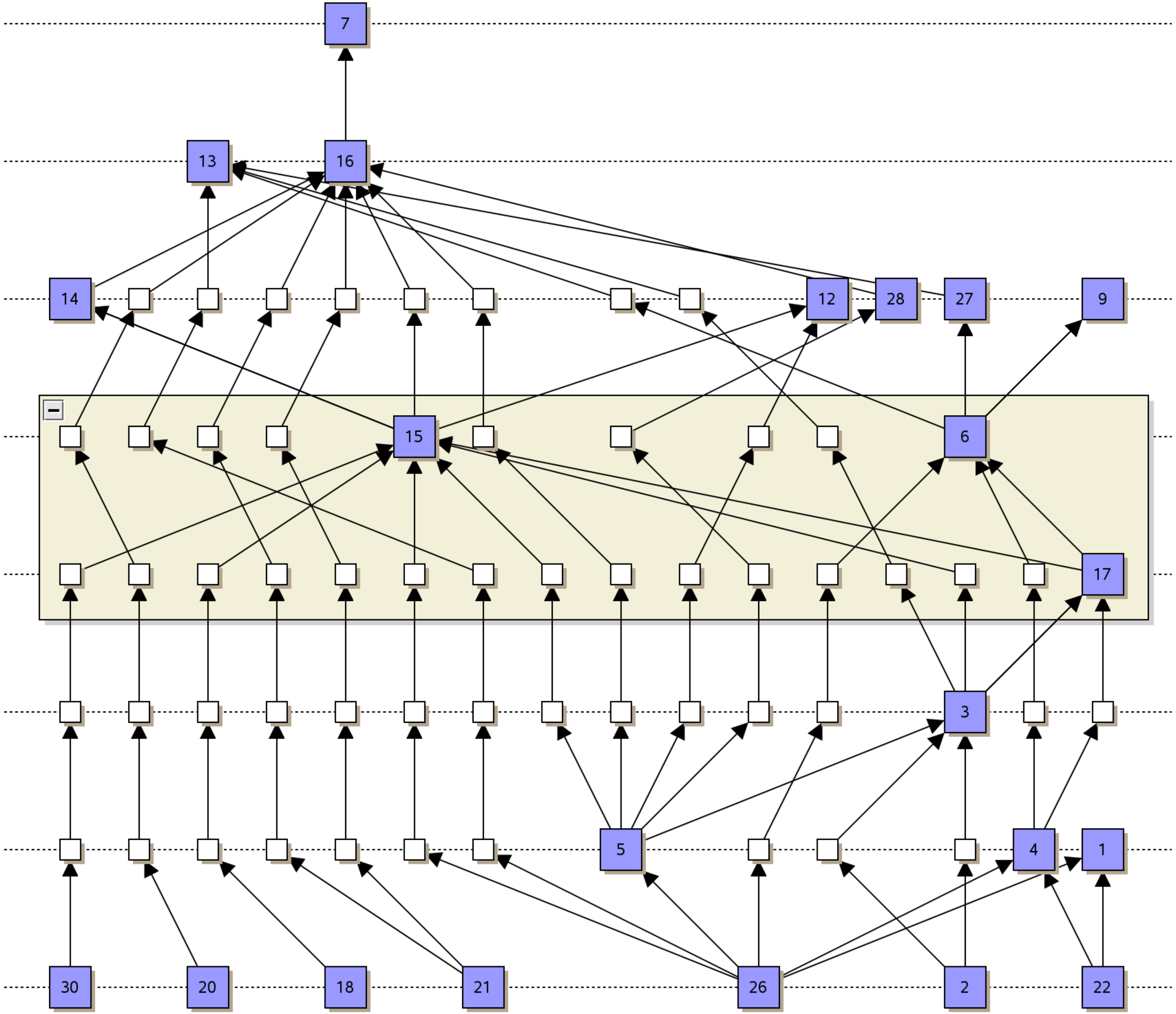
Iterations on Example



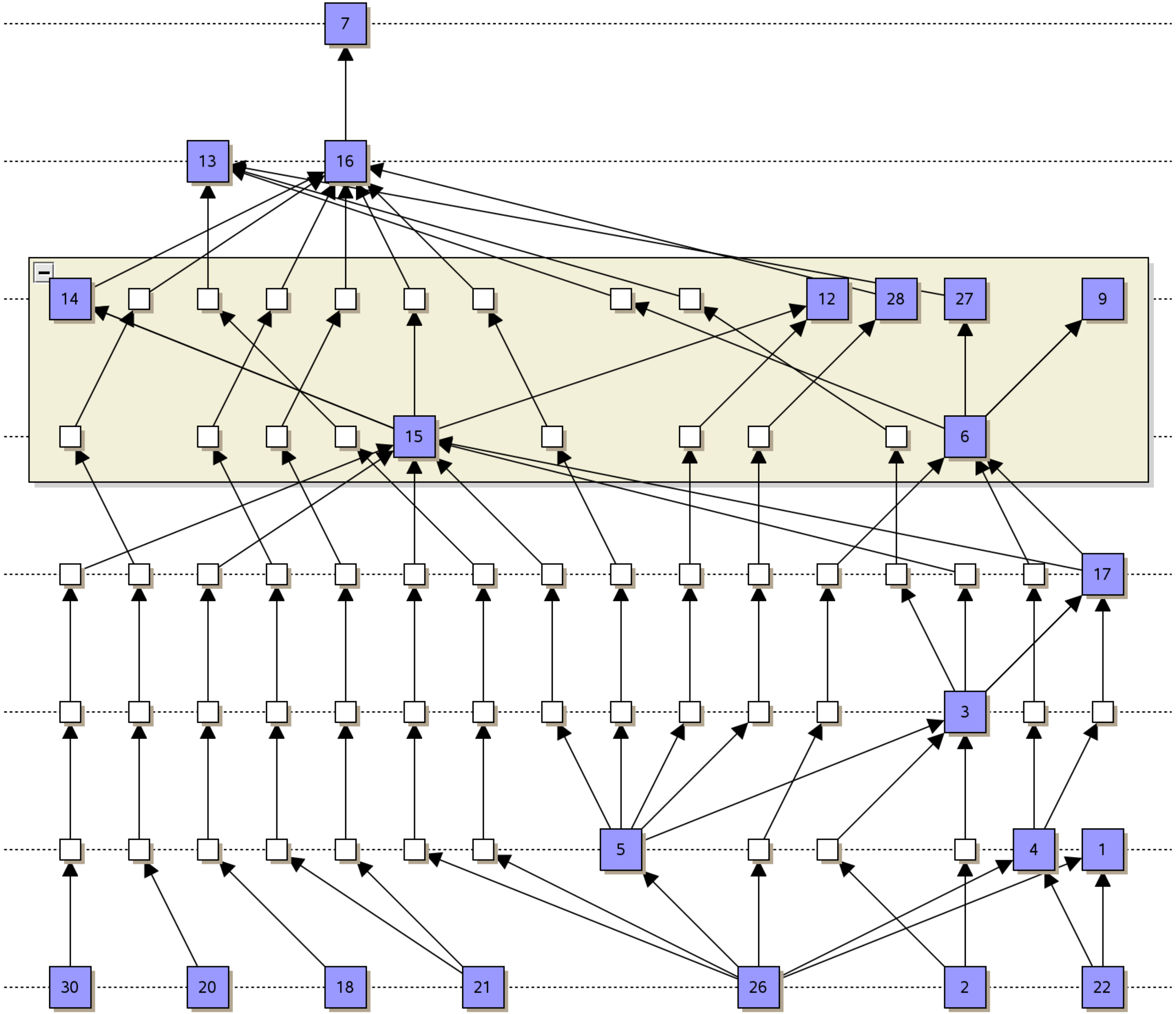
Iterations on Example



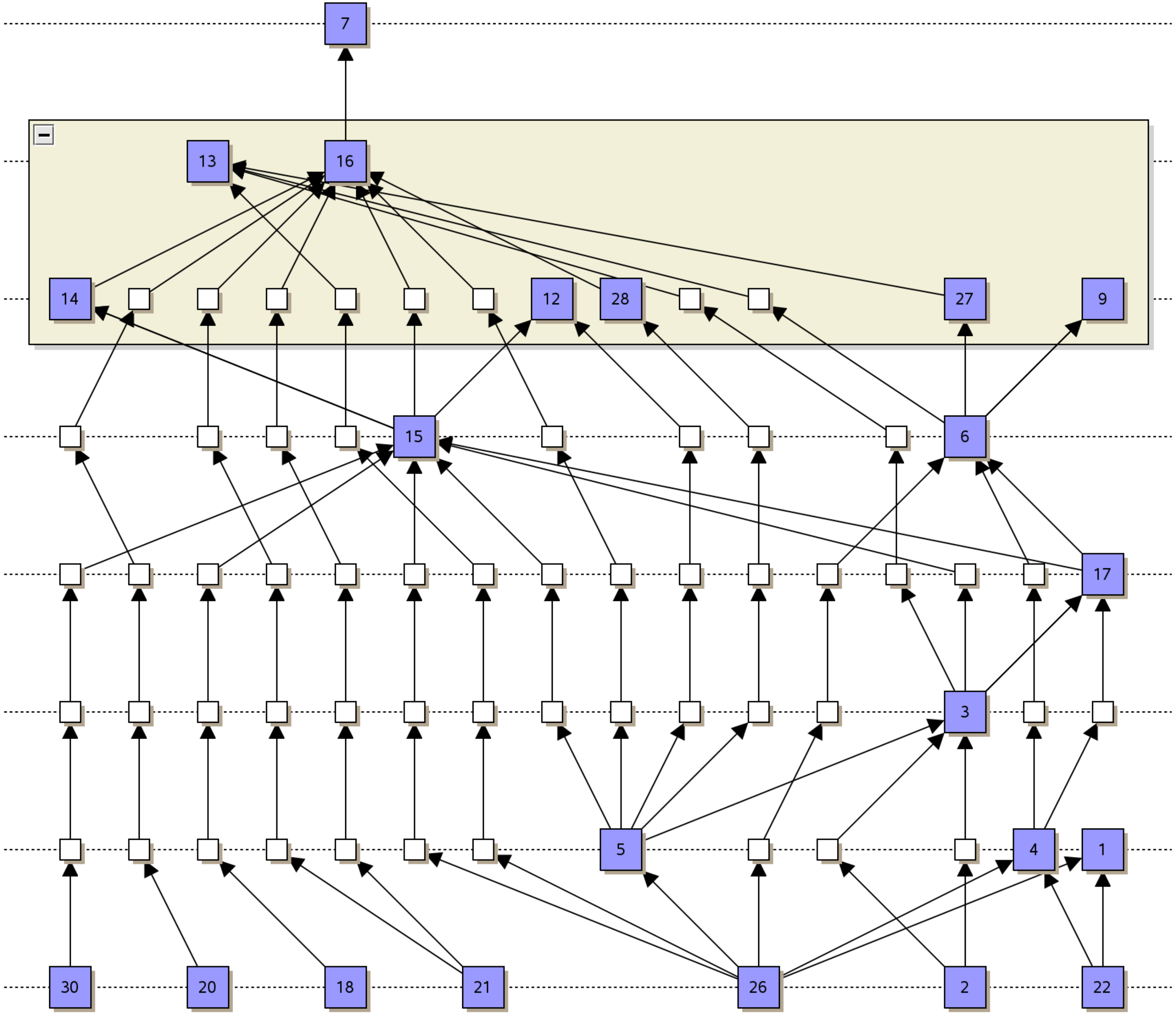
Iterations on Example



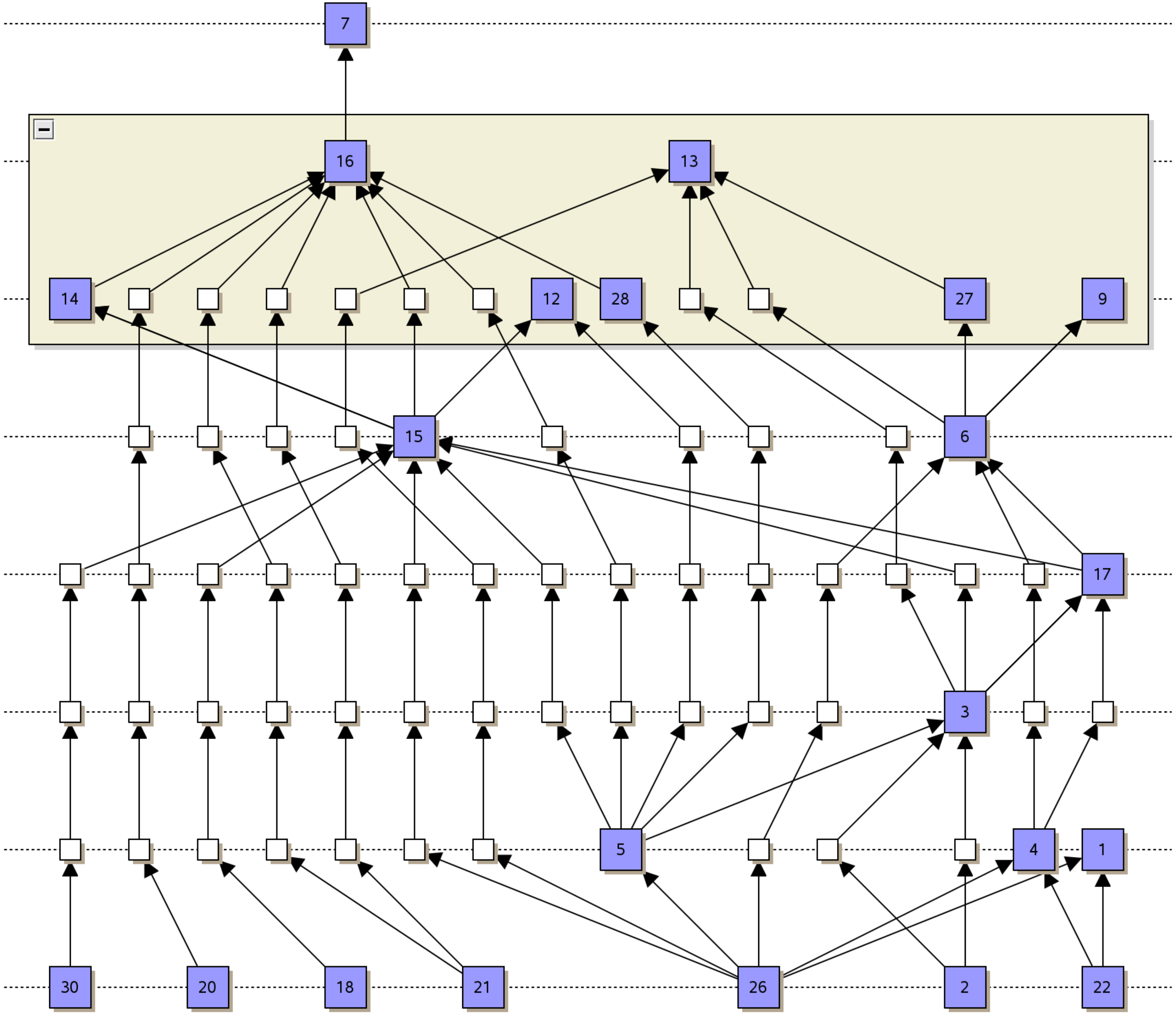
Iterations on Example



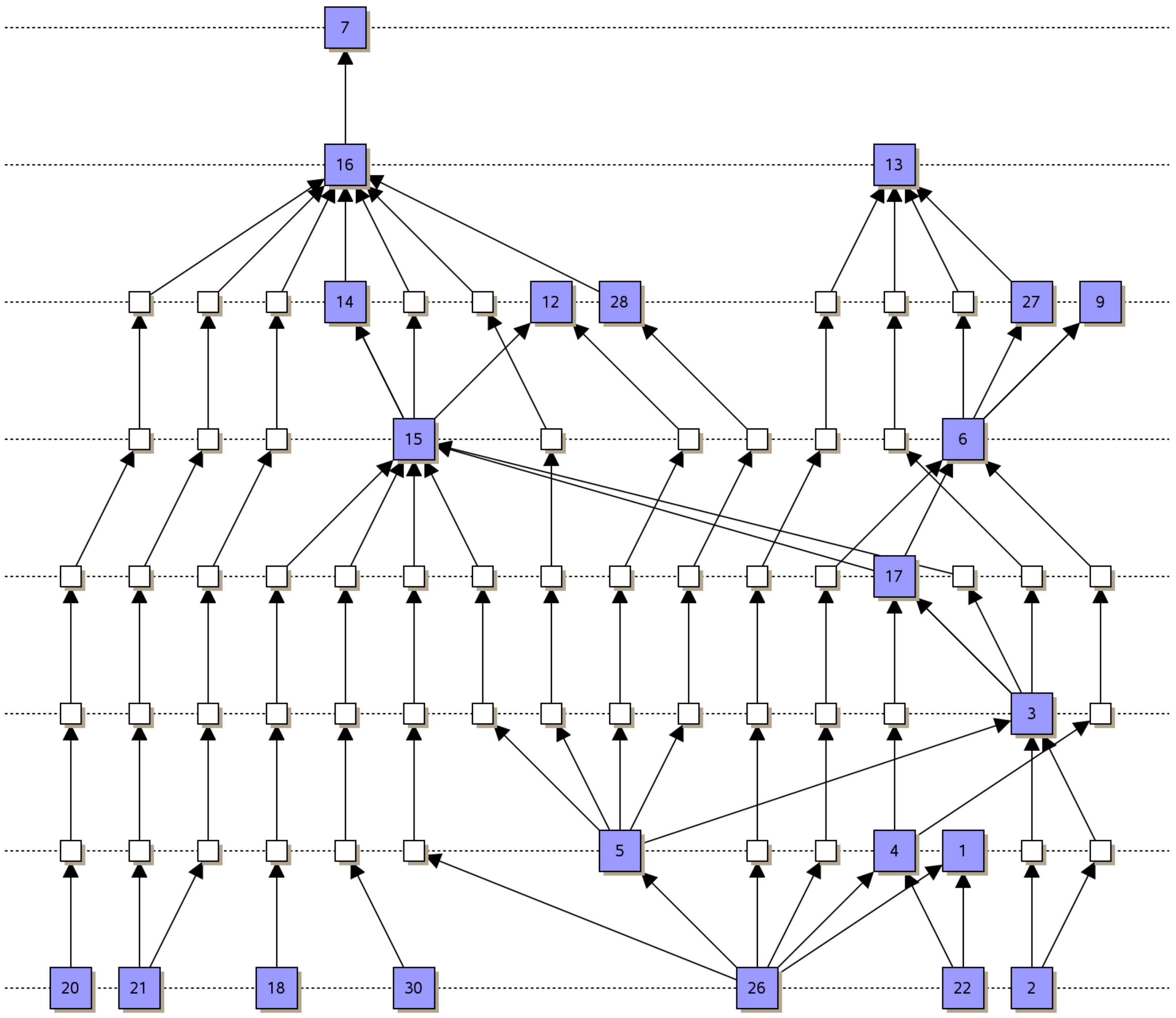
Iterations on Example



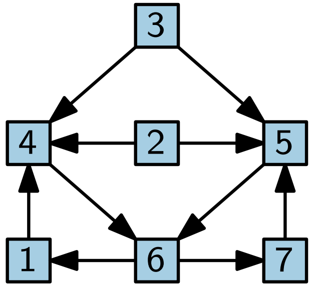
Iterations on Example

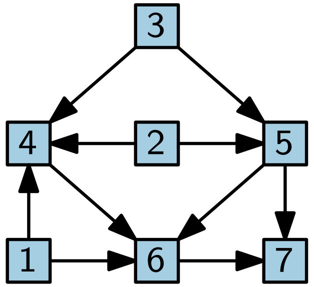


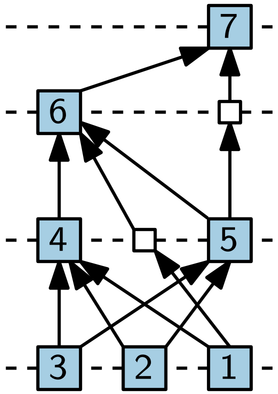
Iterations on Example



Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG





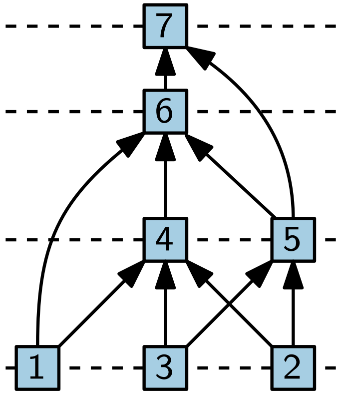


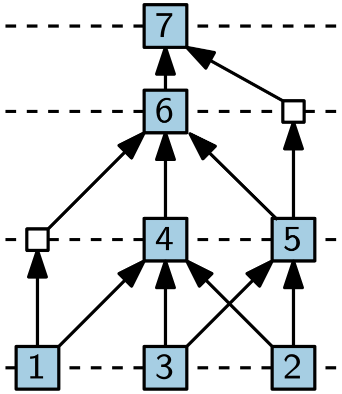
Visualization of Graphs

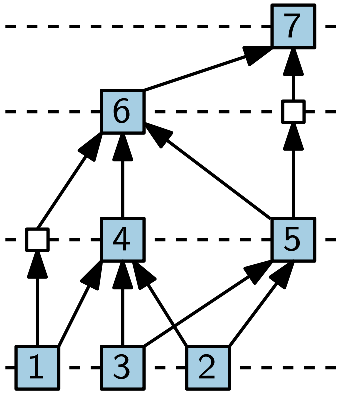
Lecture 7:
Hierarchical Layouts:
Sugiyama Framework

Part V:
Vertex Positioning & Drawing Edges

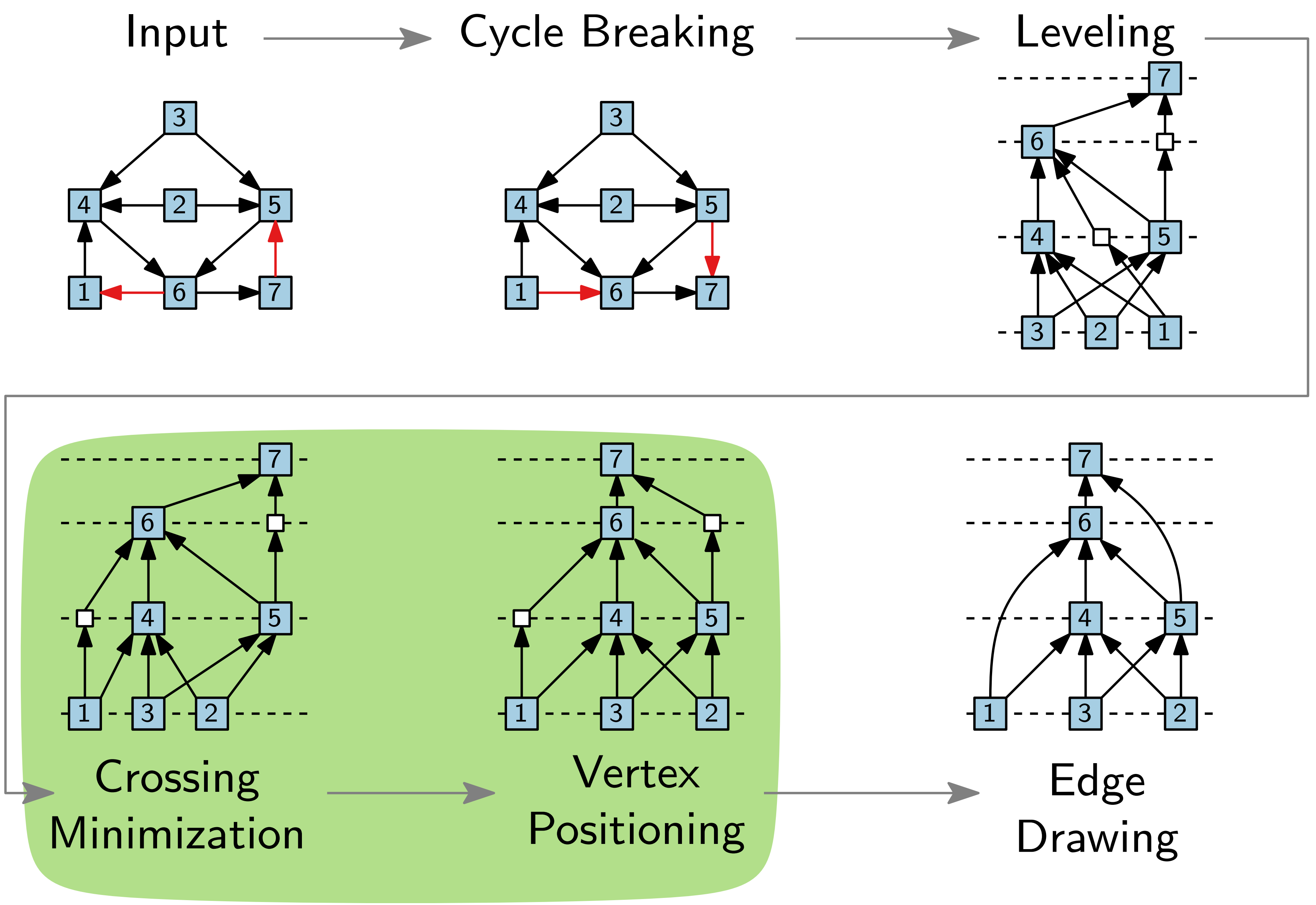
Jonathan Klawitter



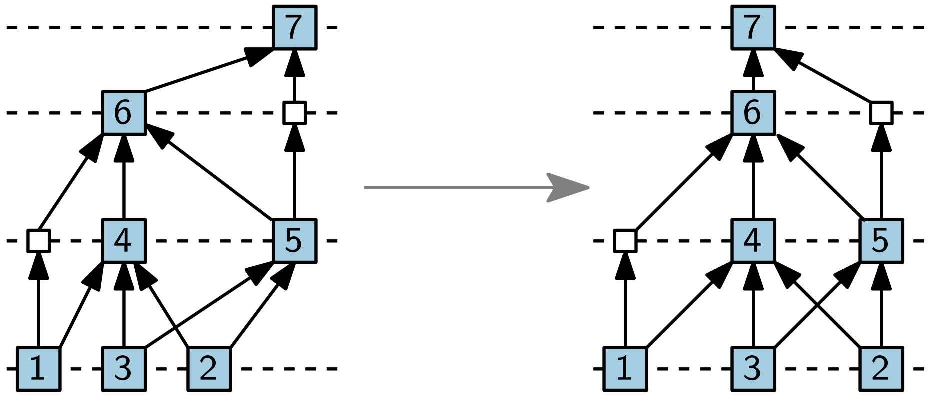




Step 4: Vertex Positioning



Step 4: Vertex Positioning



Goal.

Paths should be close to straight, vertices evenly spaced

- **Exact:** Quadratic Program (QP)
- **Heuristic:** Iterative approach

Quadratic Program

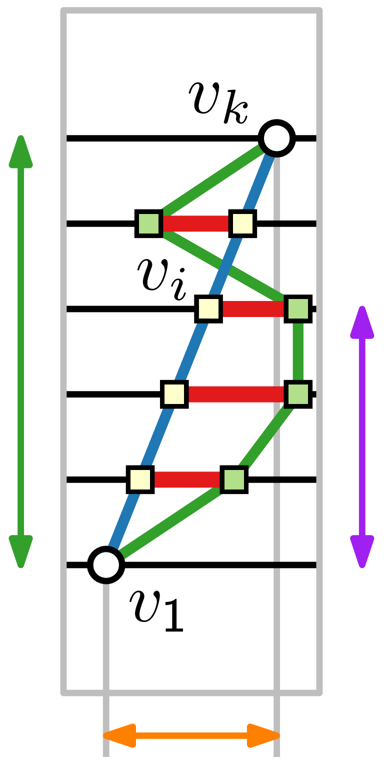
- Consider the path $p_e = (v_1, \ldots, v_k)$ of an edge $e = v_1 v_k$ with dummy vertices: $v_2, \ldots, v_{k-1}$
- x-coordinate of v_i according to the line $\overline{v_1 v_k}$ (with equal spacing):

$$\overline{x(v_i)} = x(v_1) + \frac{i-1}{k-1}\left(x(v_k) - x(v_1)\right)$$

- Define the deviation from the line

$$\mathrm{dev}(p_e) := \sum_{i=2}^{k-1} \left(x(v_i) - \overline{x(v_i)}\right)^2$$

- Objective function: min $\sum_{e \in E} \mathrm{dev}(p_e)$
- Constraints for all vertices v, w in the same layer with w right of v: $x(w) - x(v) \geq \rho(w, v)$ ← min. horizontal distance



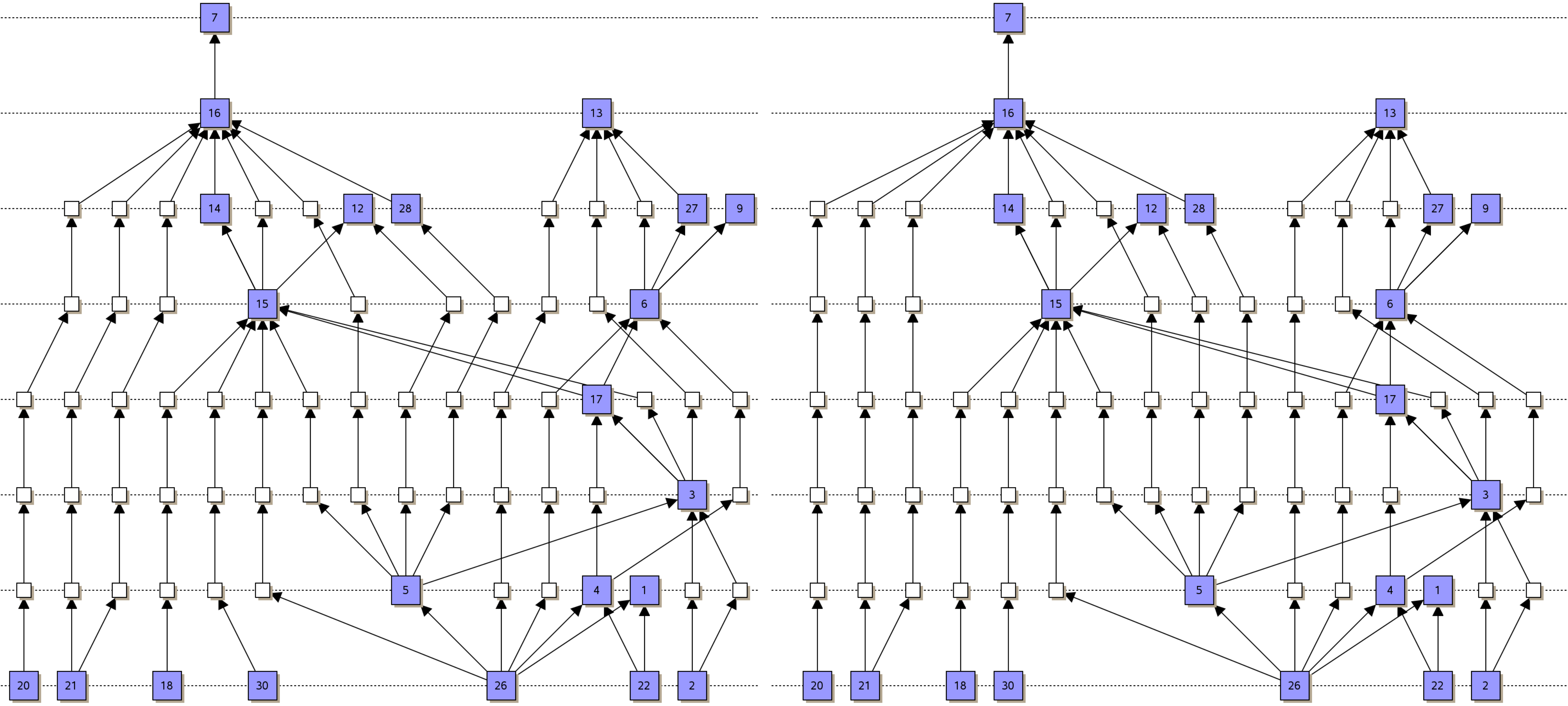
- QP is time-expensive
- width can be exponential

Iterative Heuristic

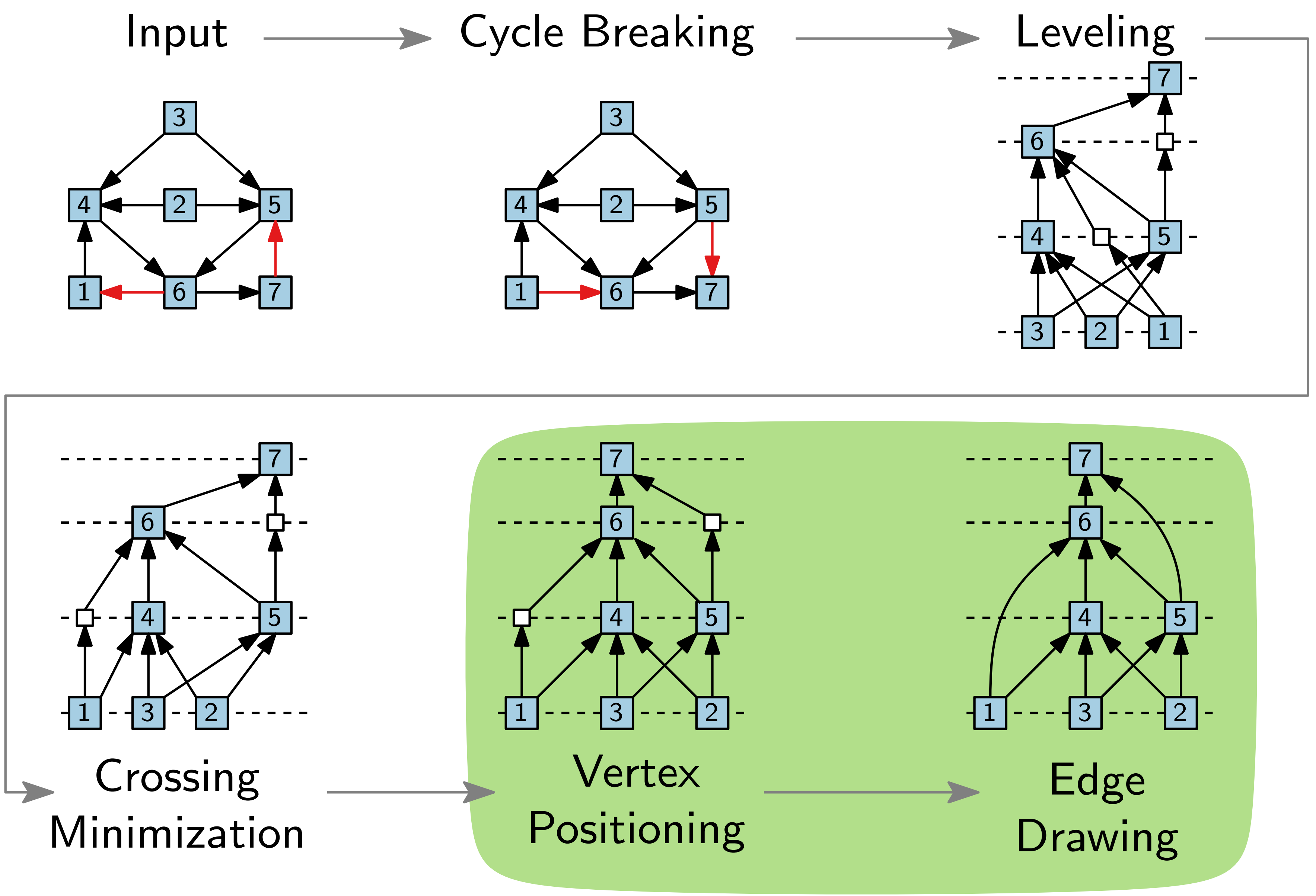
- Compute an initial layout
- Apply the following steps as long as improvements can be made:

  1. Vertex positioning,
  2. edge straightening,
  3. Compactifying the layout width.

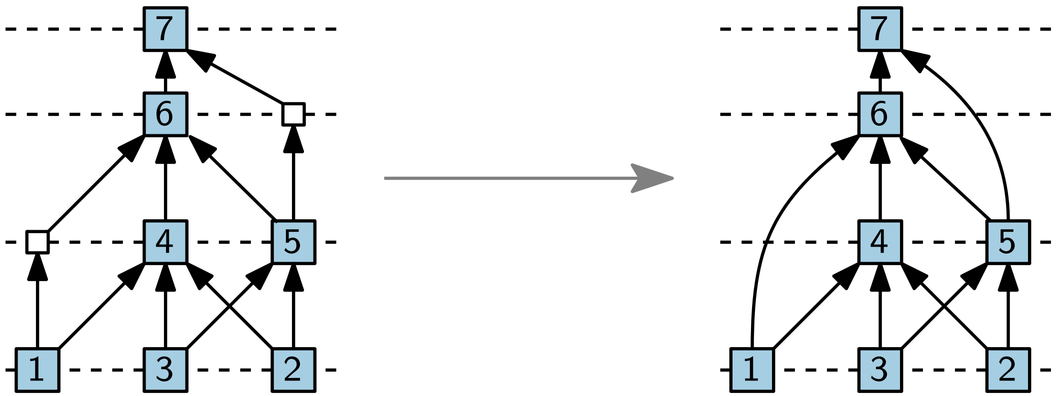
Example


7
16
13
14
12
28
27
9
15
6
17
3
5
4
1
20
21
18
30
26
22
2
7
16
13
14
12
28
27
9
15
6
17
3
5
4
1
20
21
18
30
26
22
2

Step 5: Drawing Edges

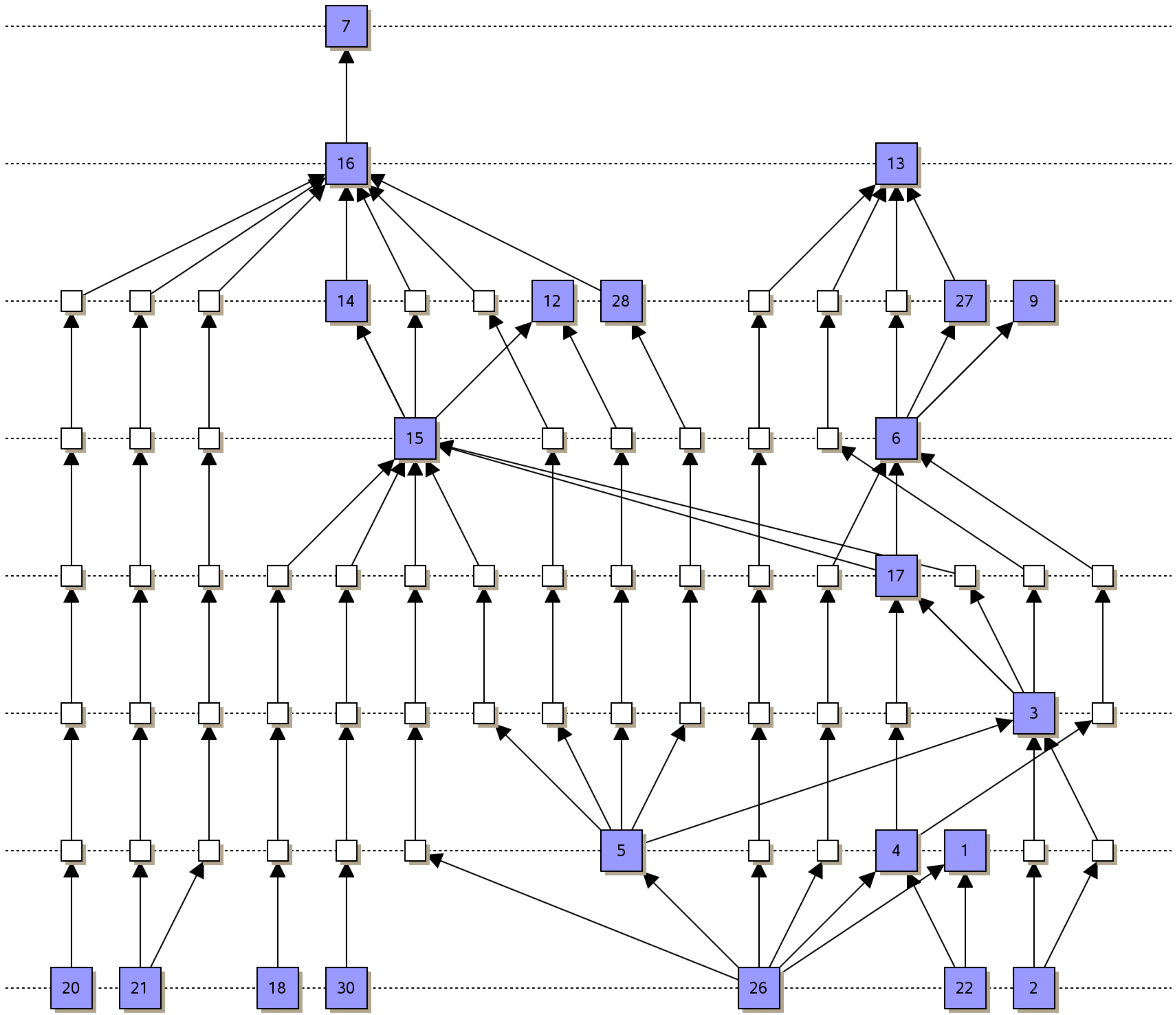


Step 5: Drawing Edges

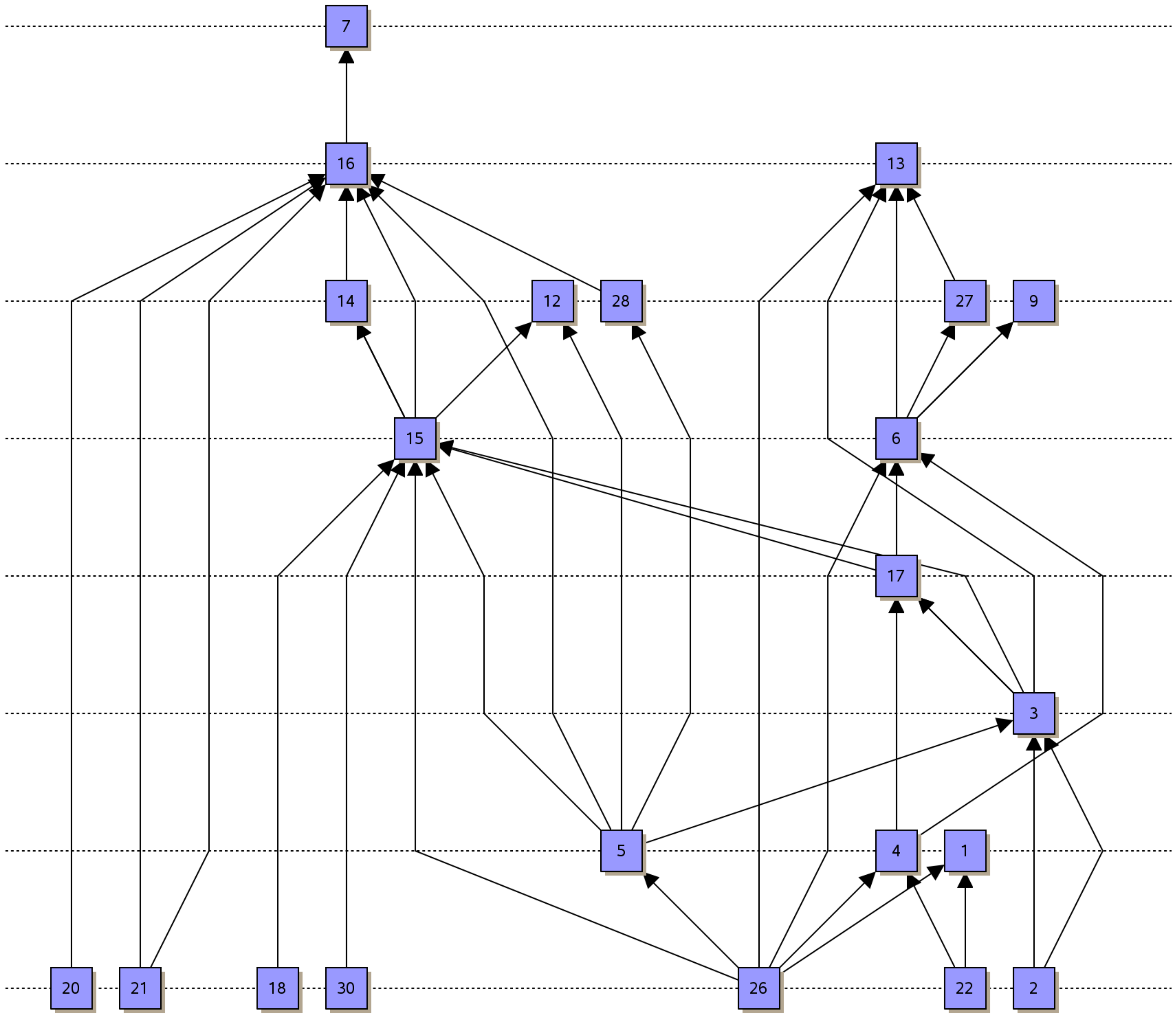


Possibility.
Substitute polylines by Bézier curves

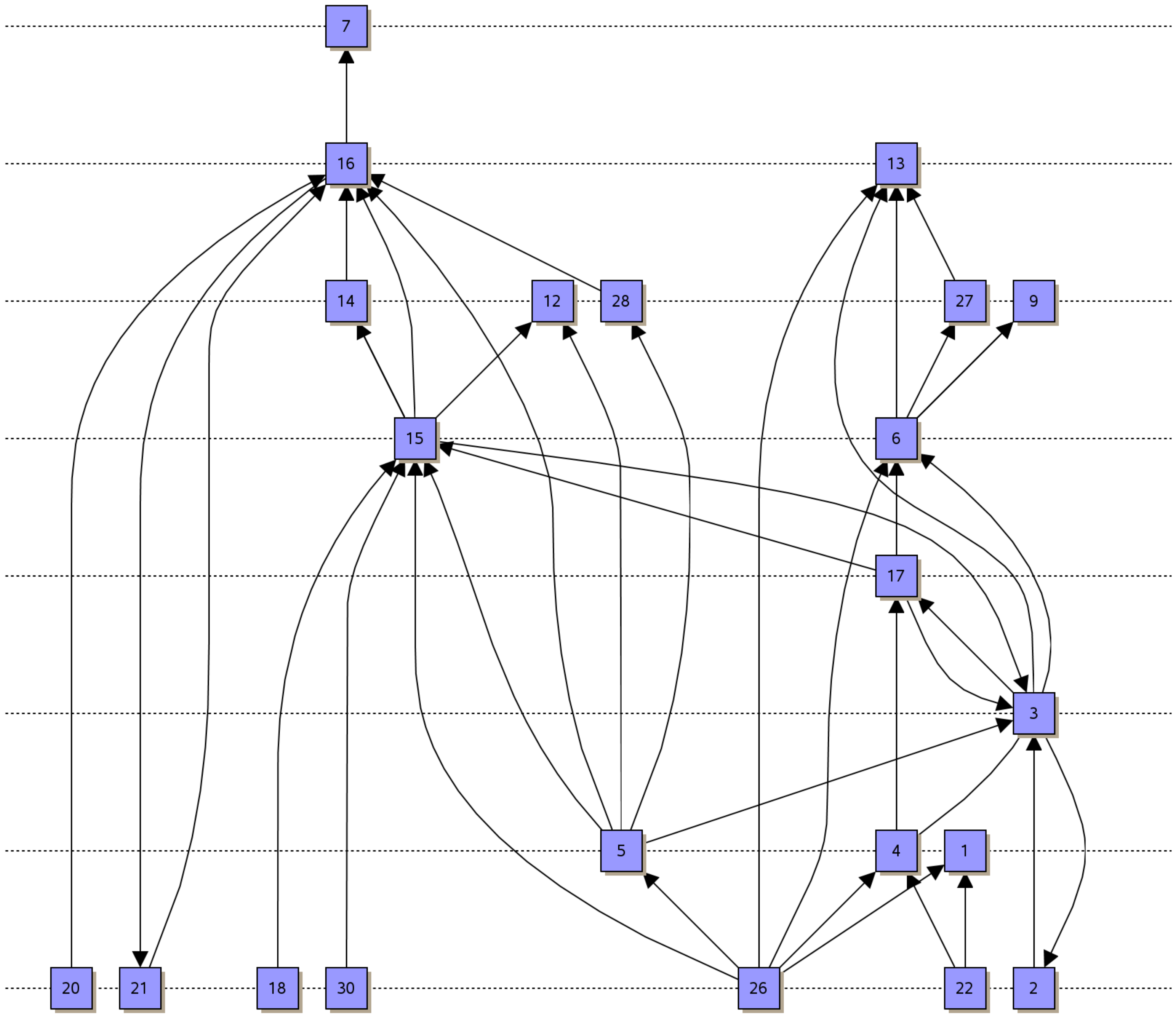
Example



Example

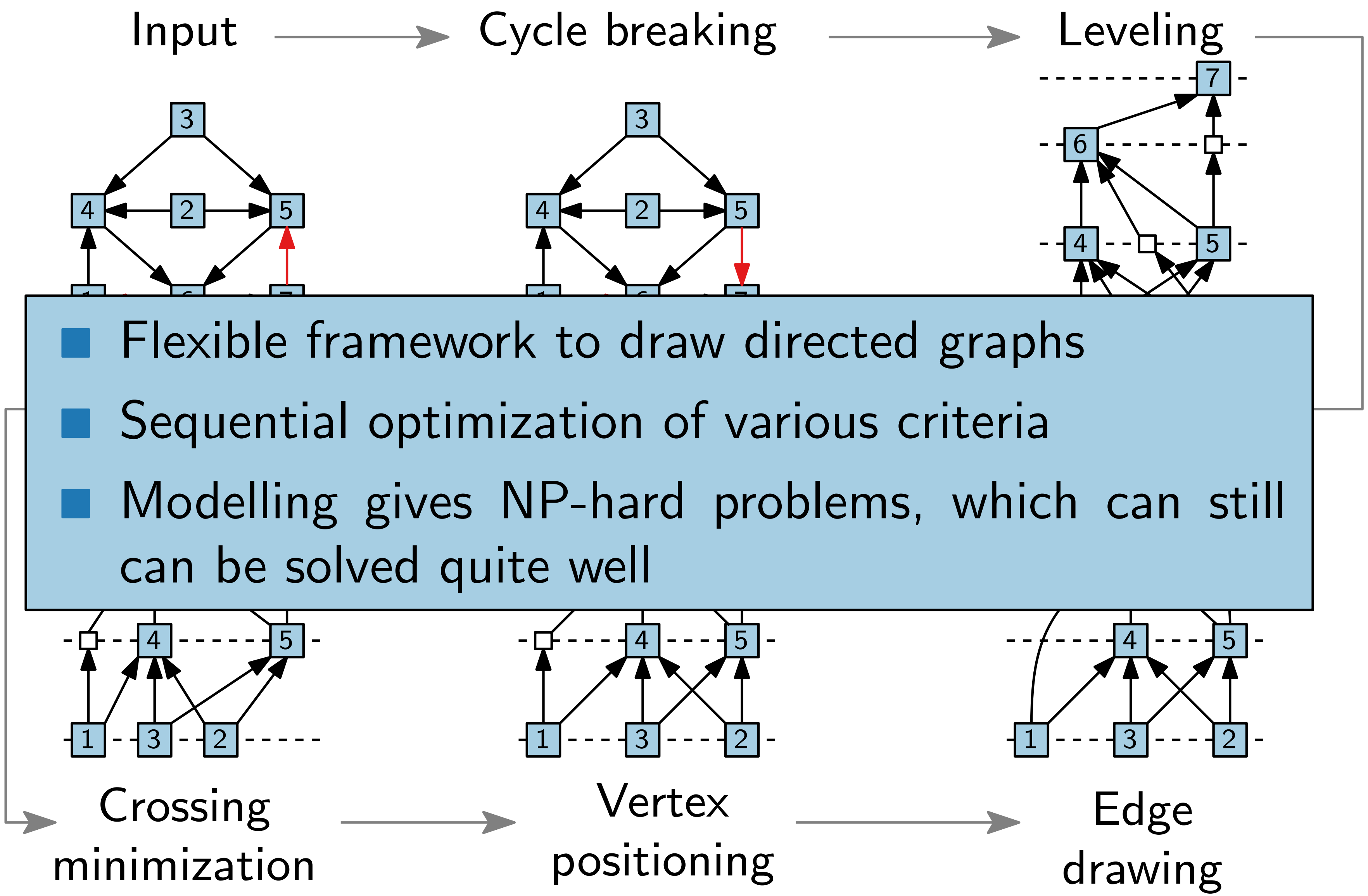


Example



Classical Approach – Sugiyama Framework

[Sugiyama, Tagawa, Toda ’81]



- Flexible framework to draw directed graphs
- Sequential optimization of various criteria
- Modelling gives NP-hard problems, which can still can be solved quite well

Literature

Detailed explanations of steps and proofs in

- [GD Ch. 11] and [DG Ch. 5]

based on

- [Sugiyama, Tagawa, Toda '81] Methods for visual understanding of hierarchical system structures

and refined with results from

- [Berger, Shor '90] Approximation alogorithms for the maximum acyclic subgraph problem
- [Eades, Lin, Smith '93] A fast and effective heuristic for the feedback arc set problem
- [Garey, Johnson '83] Crossing number is NP-complete
- [Eades, Whiteside '94] Drawing graphs in two layers
- [Eades, Wormland '94] Edge crossings in drawings of bipartite graphs
- [Jünger, Mutzel '97] 2-Layer Straightline Crossing Minimization: Performance of Exact and Heuristic Algorithms