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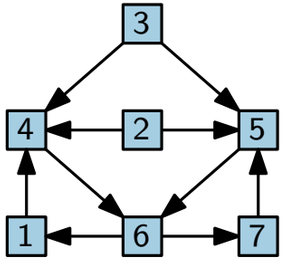
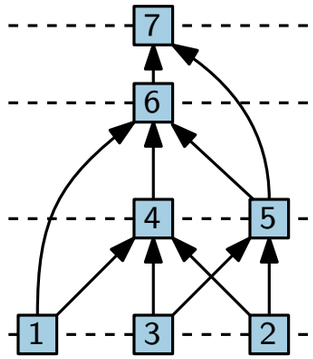
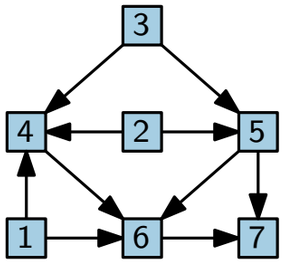
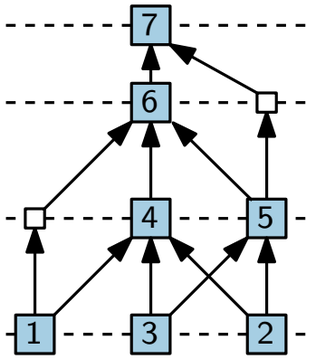


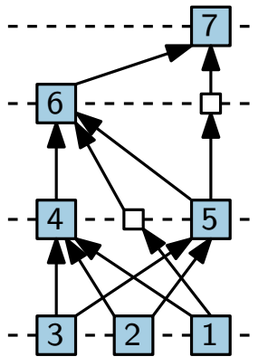
Visualization of Graphs



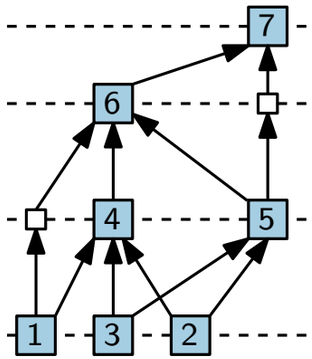
Lecture 7: Hierarchical Layouts: Sugiyama Framework





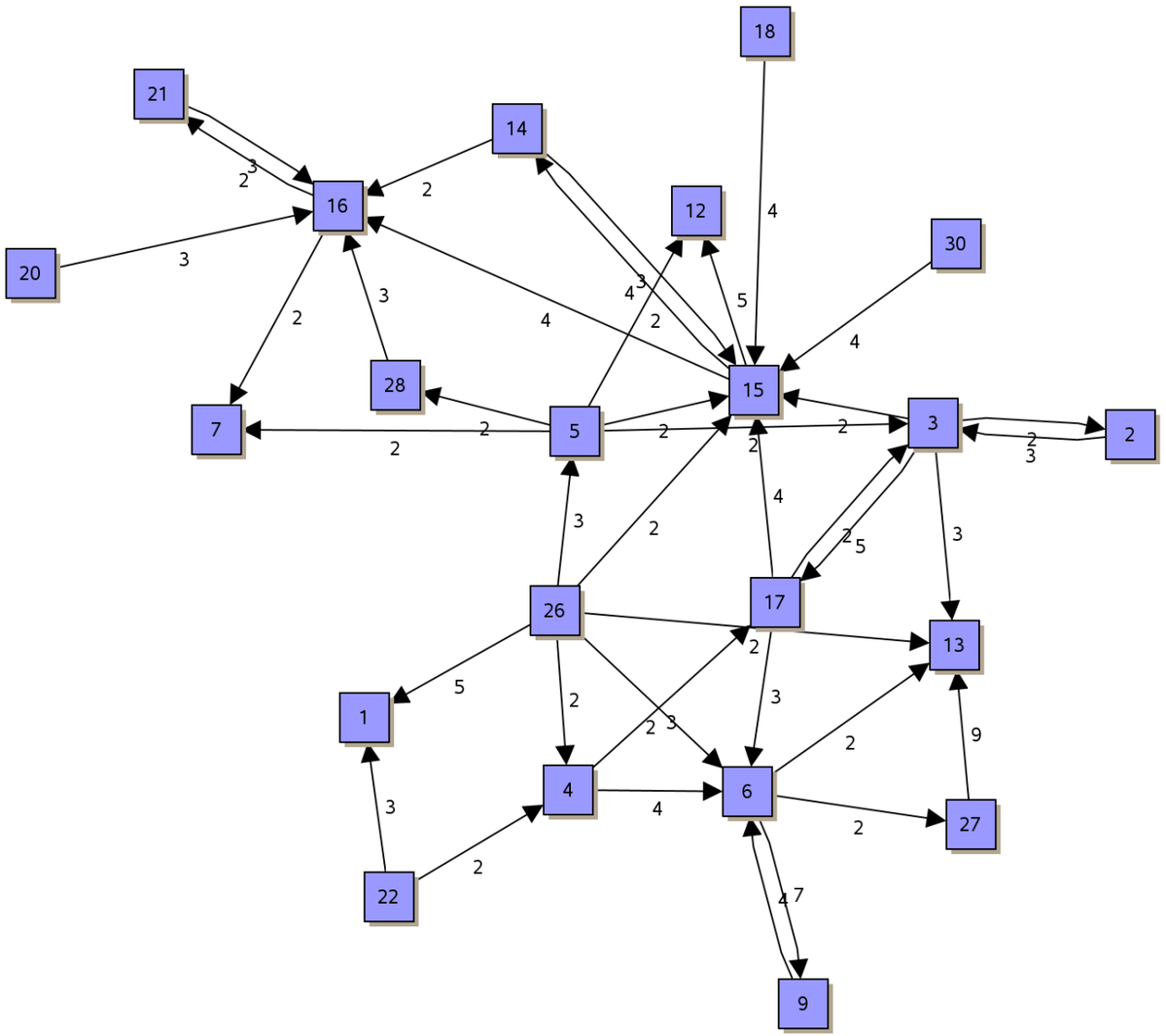


Part I:
The Framework

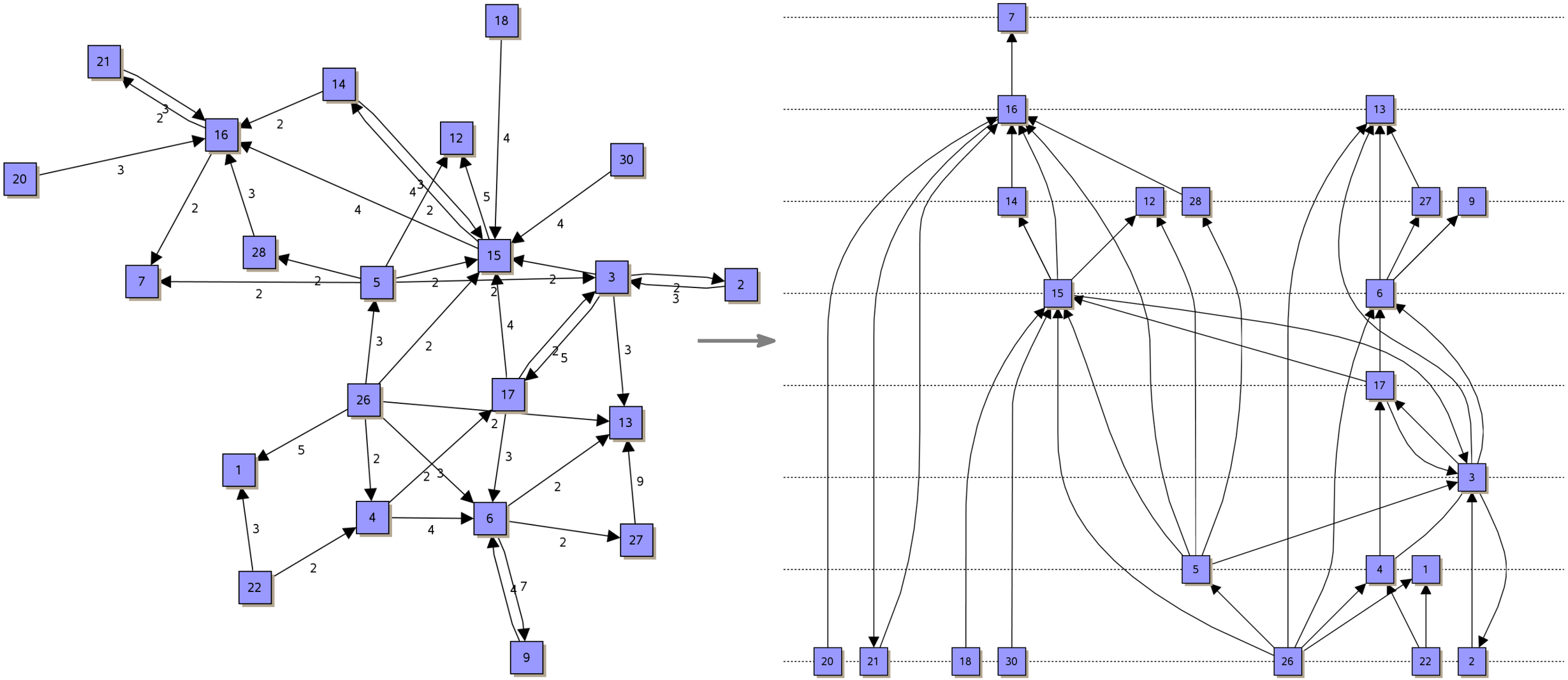


Jonathan Klawitter

Hierarchical Drawings – Motivation



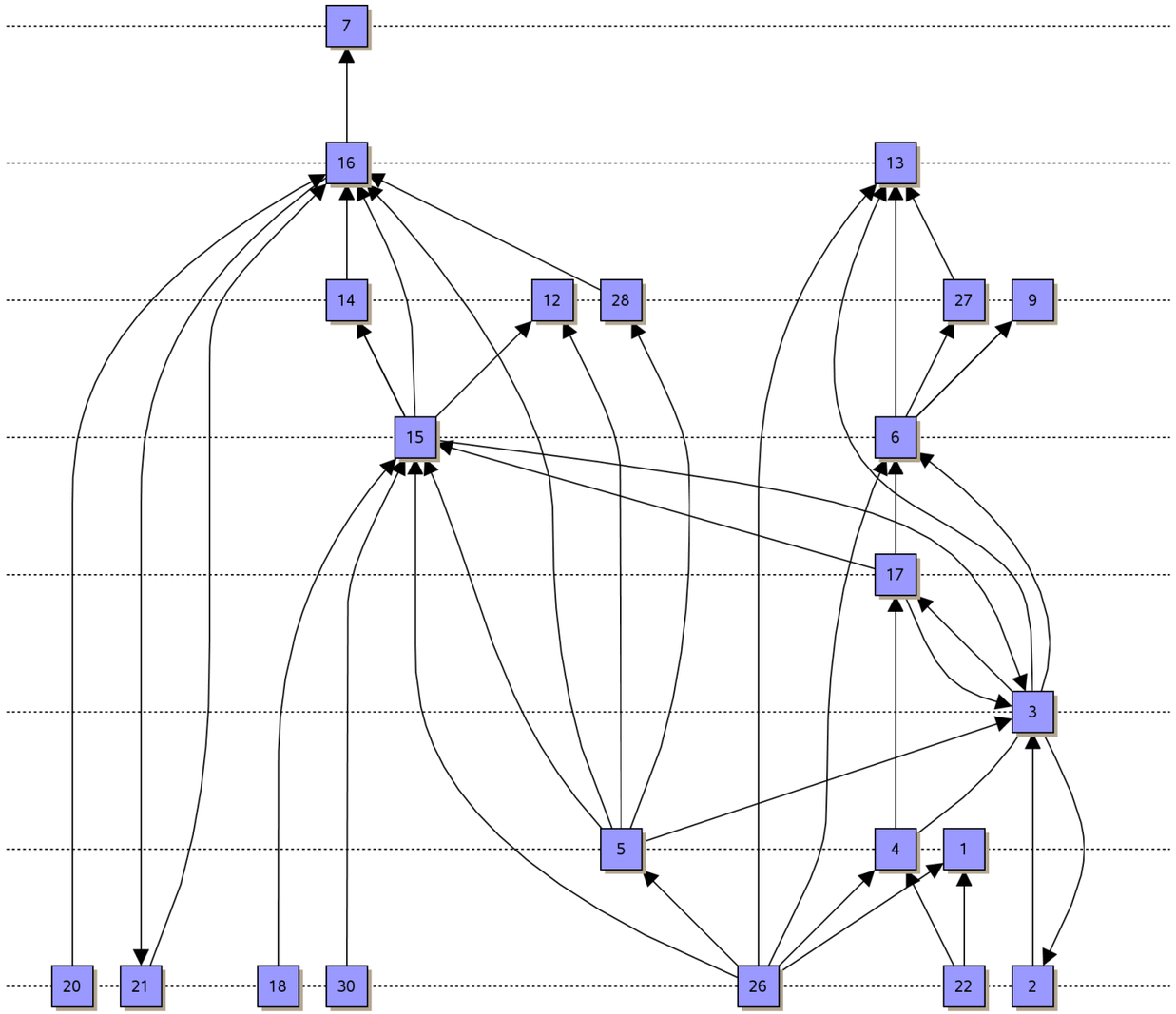
Hierarchical Drawings – Motivation



Hierarchical Drawing

Problem Statement.

- Input: digraph $G = (V, E)$
- Output: drawing of G that "closely" reproduces the hierarchical properties of G



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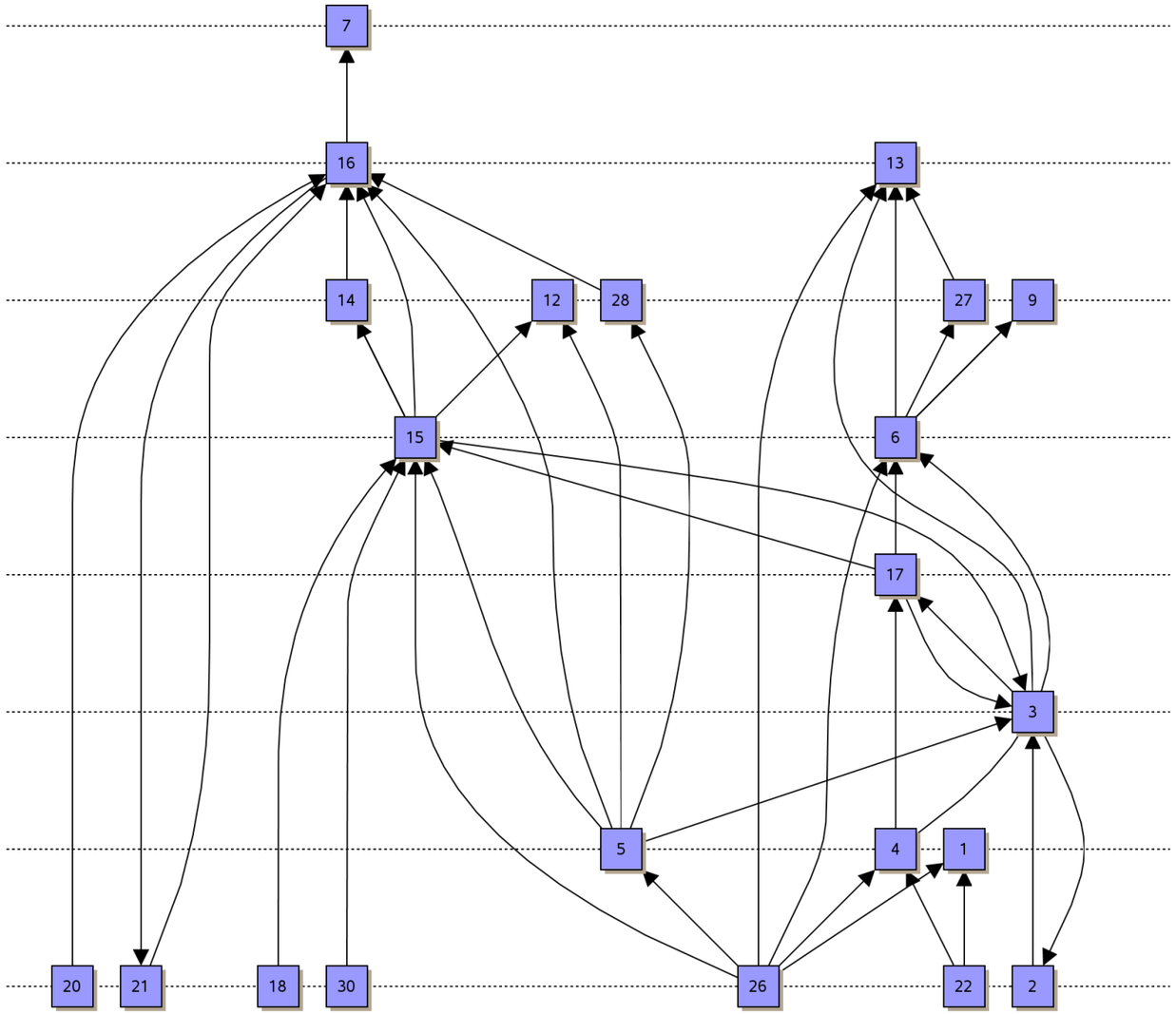
Desirable Properties.



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- Input: digraph $G = (V, E)$
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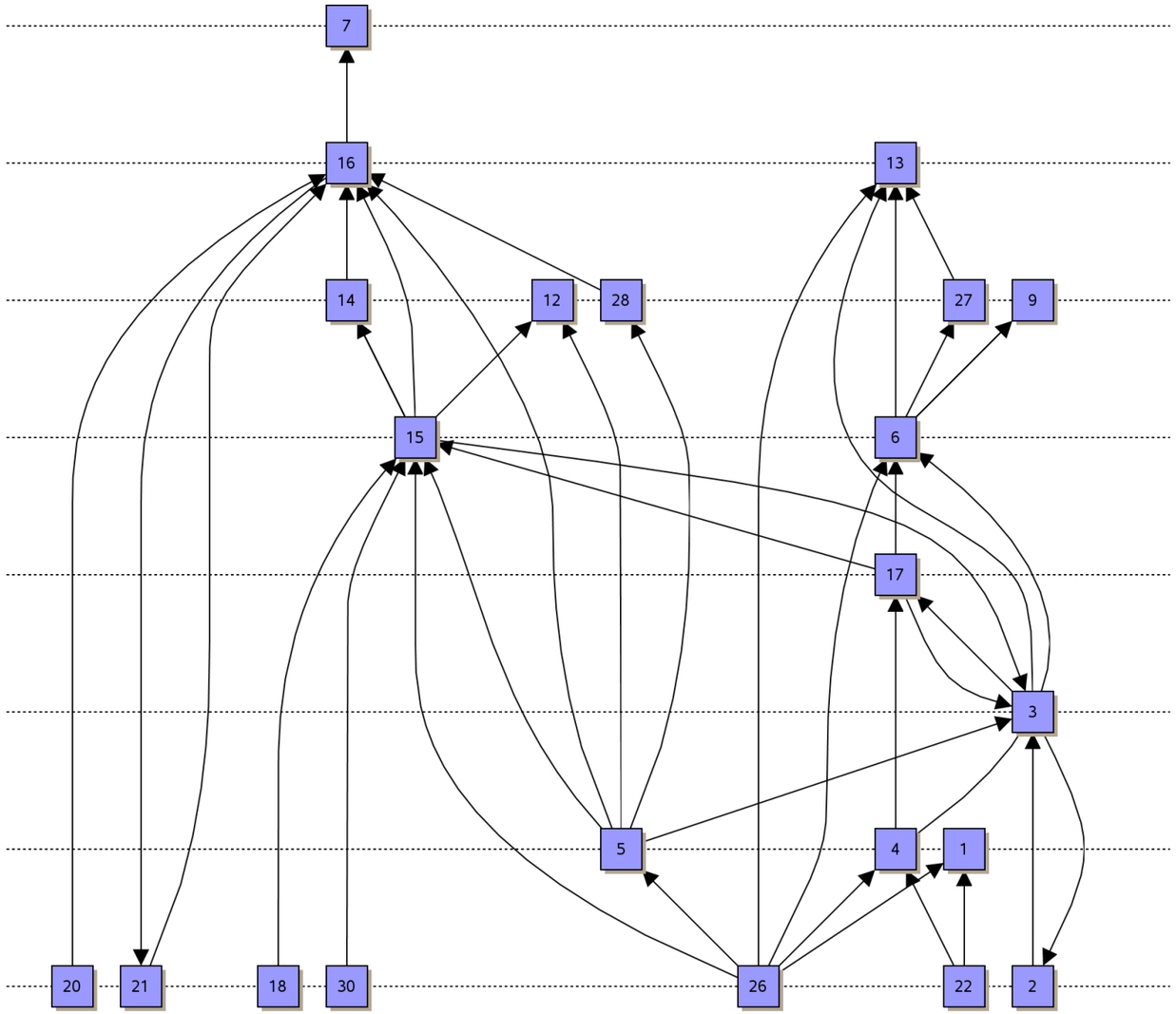
Desirable Properties.

- vertices occur on (few) horizontal lines



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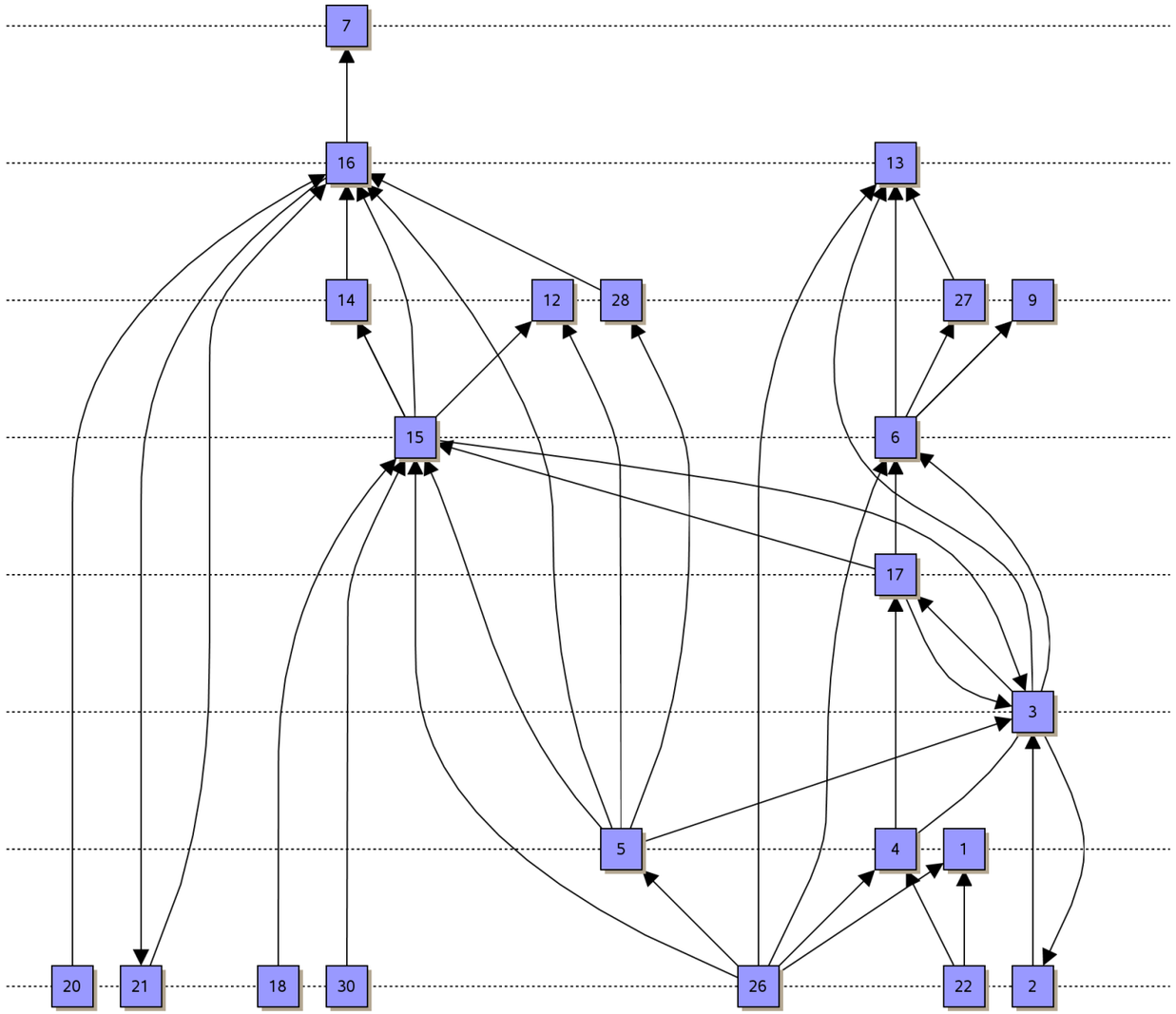

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- Input: digraph $G = (V, E)$
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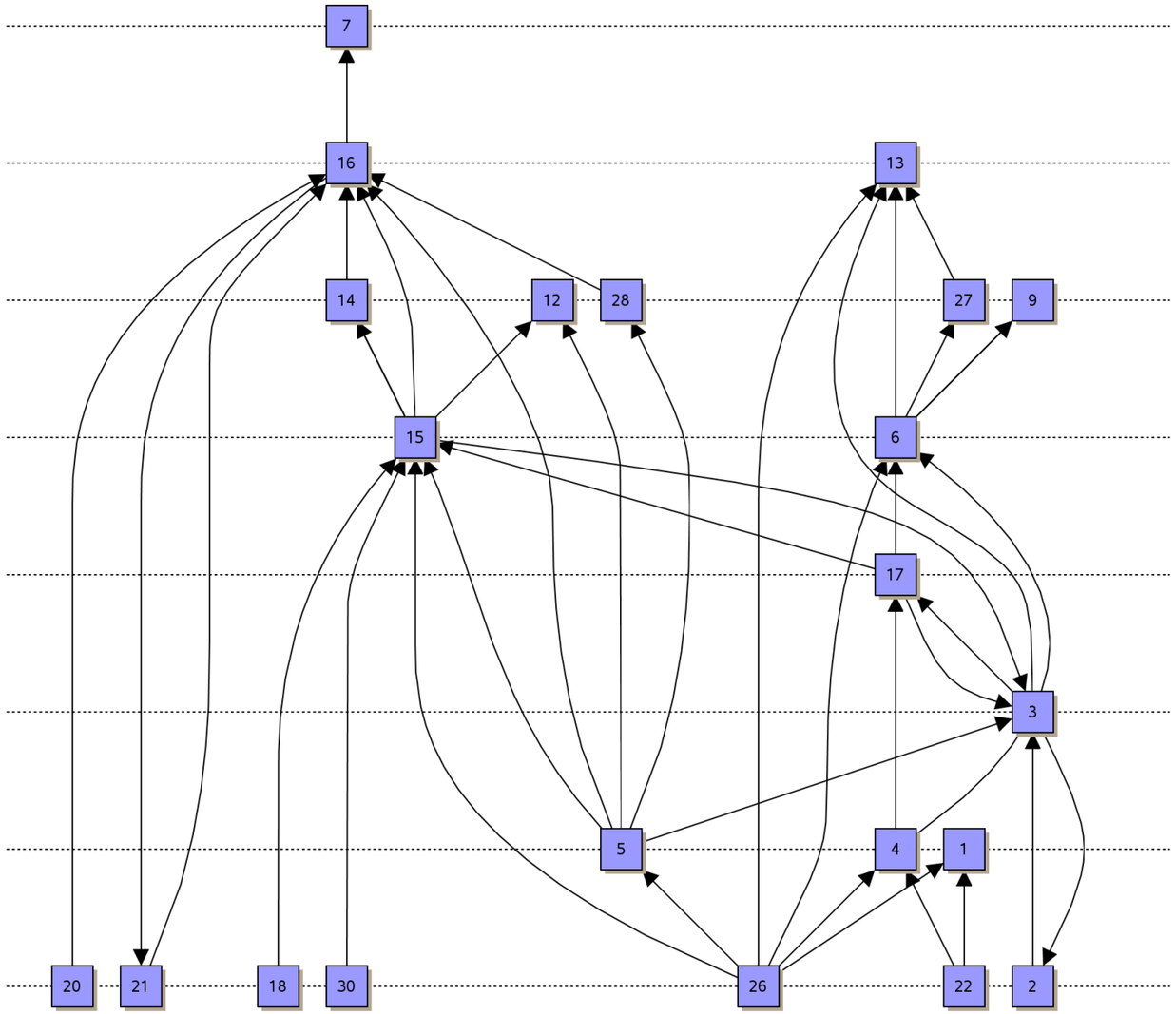
Desirable Properties.

- vertices occur on (few) horizontal lines
- edges directed upwards
- edge crossings minimized



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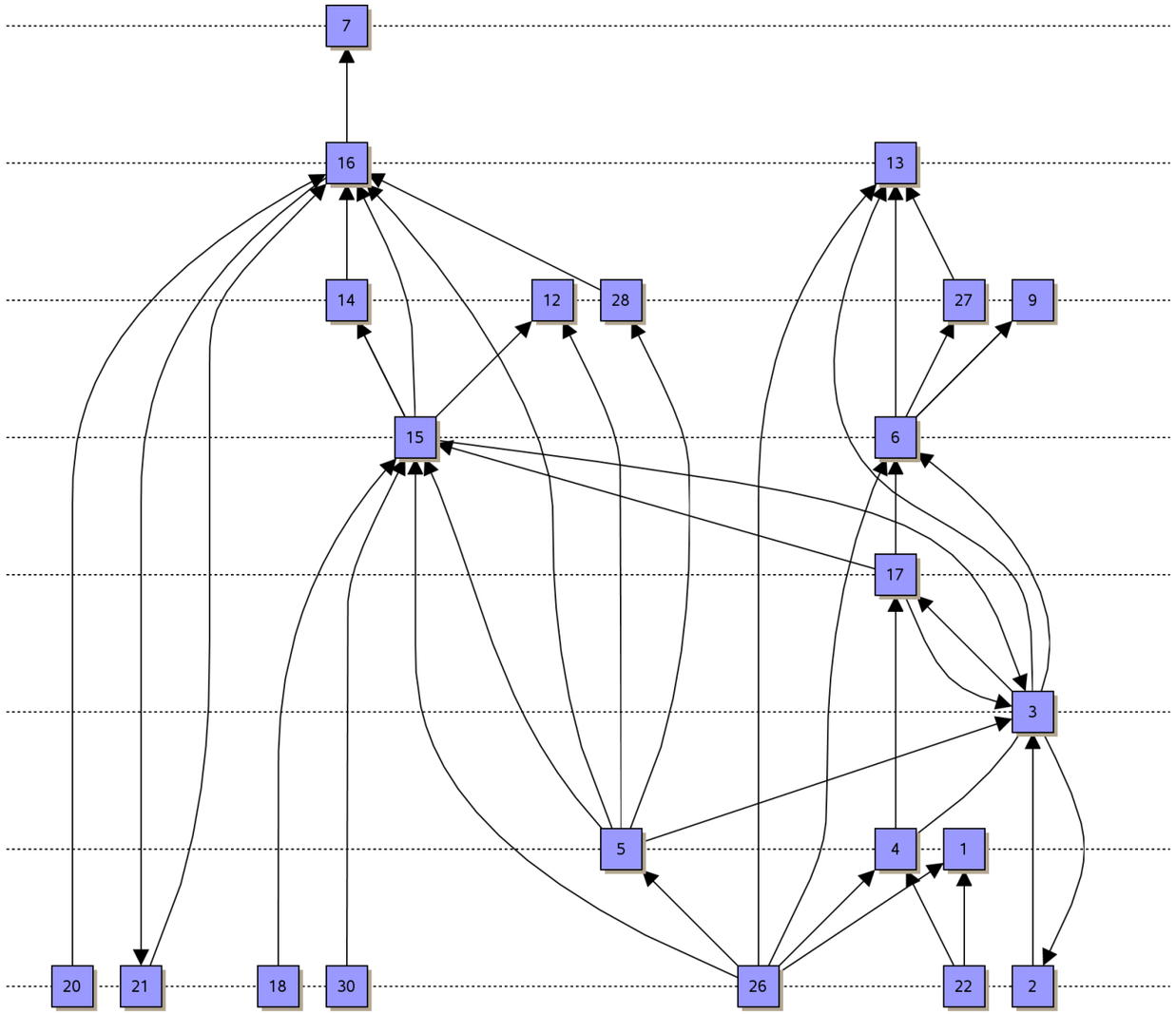

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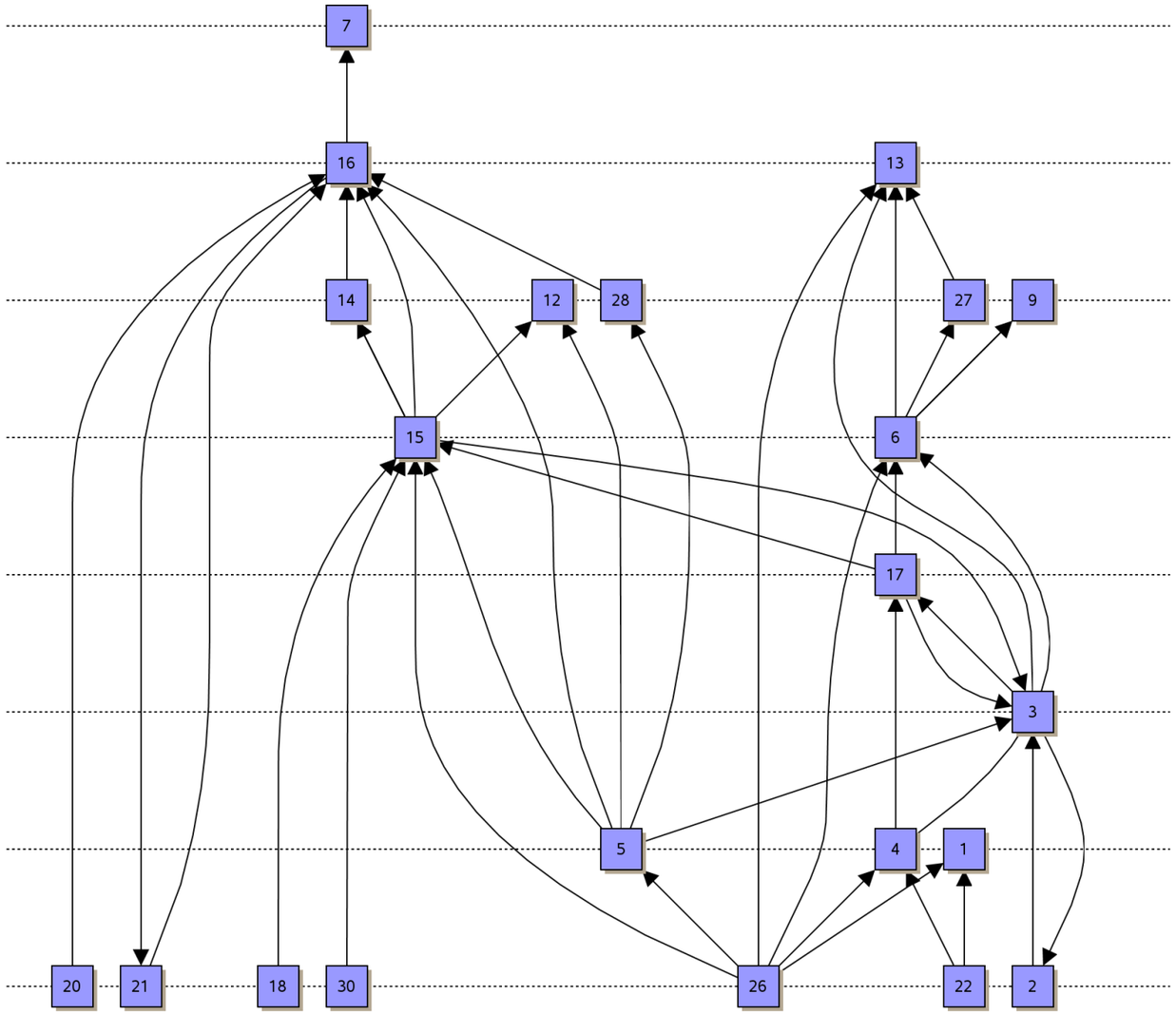

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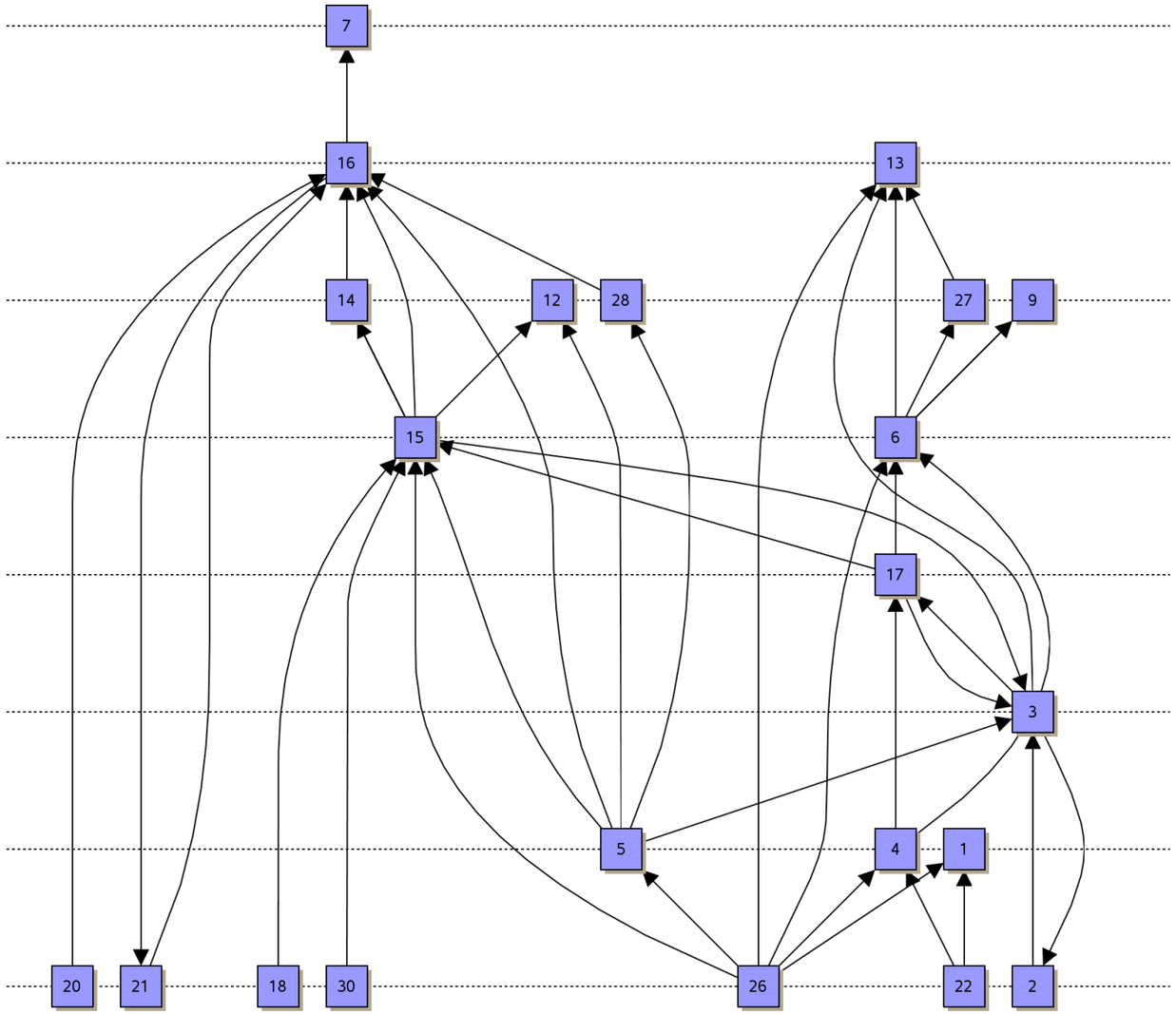
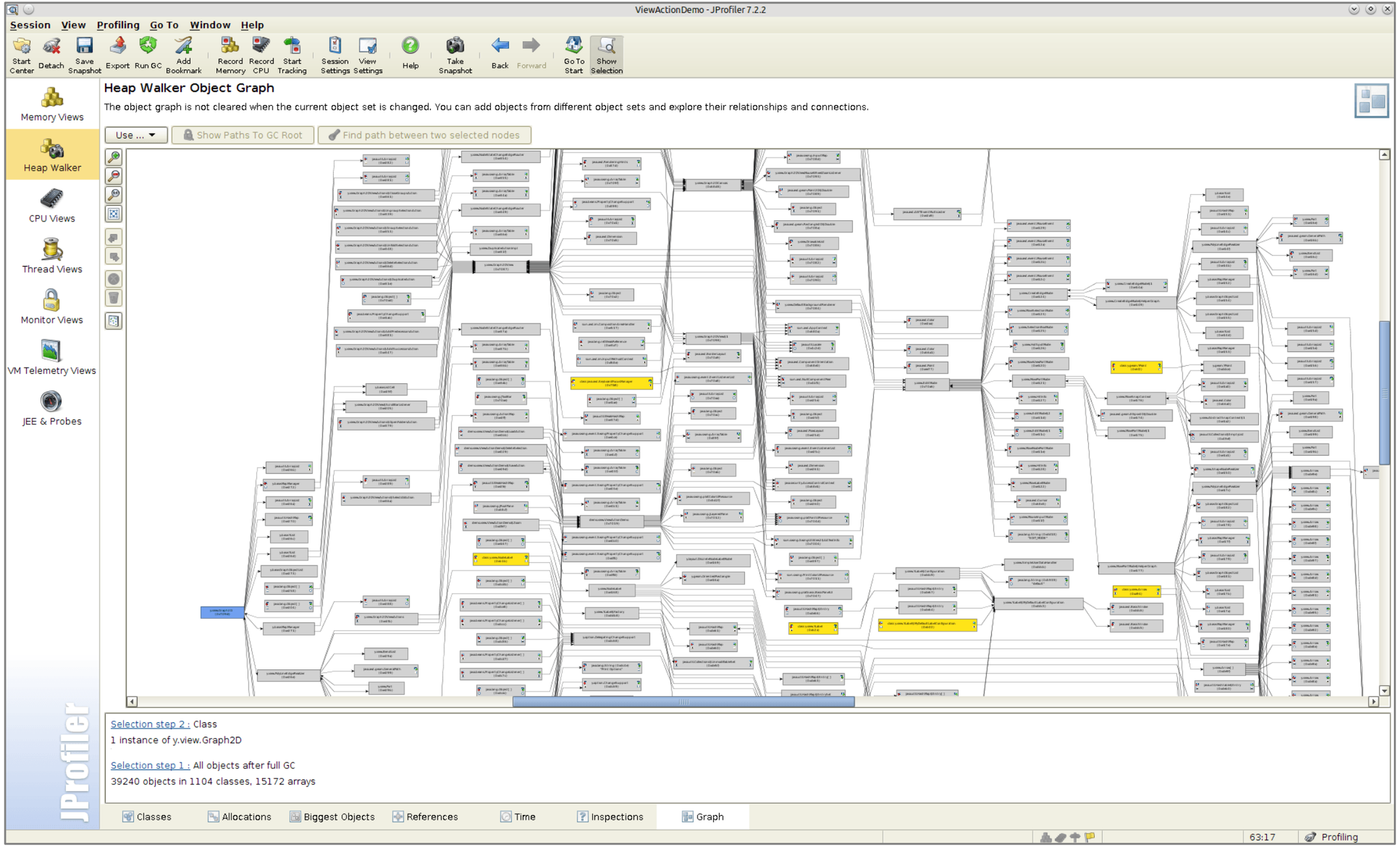
Criteria can be contradictory!



Hierarchical Drawing – Applications

yEd Gallery: Java profiler JProfiler using yFiles



Hierarchical Drawing – Applications

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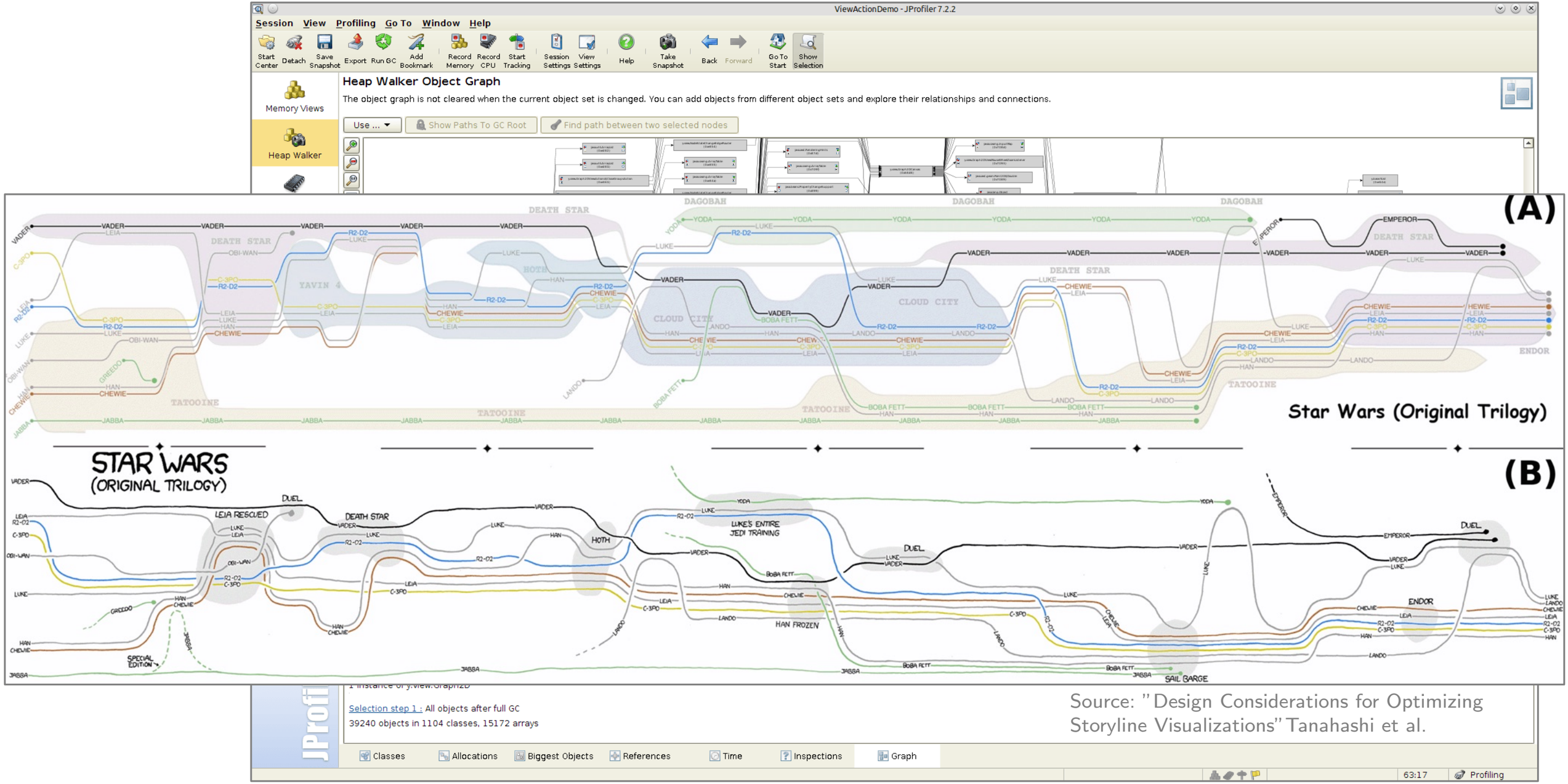

Source: "Design Considerations for Optimizing Storyline Visualizations" Tanahashi et al.

Hierarchical Drawing – Applications

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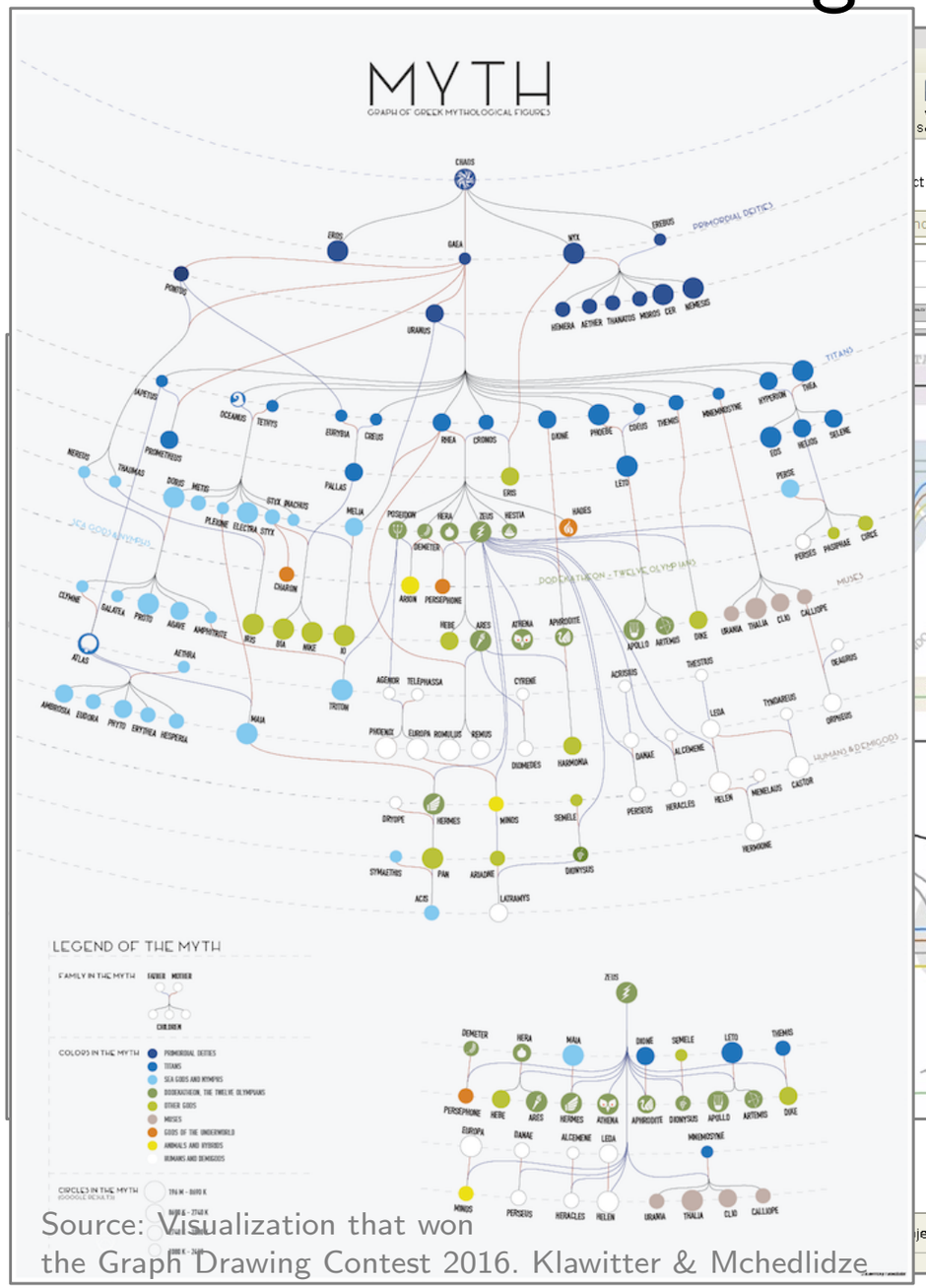
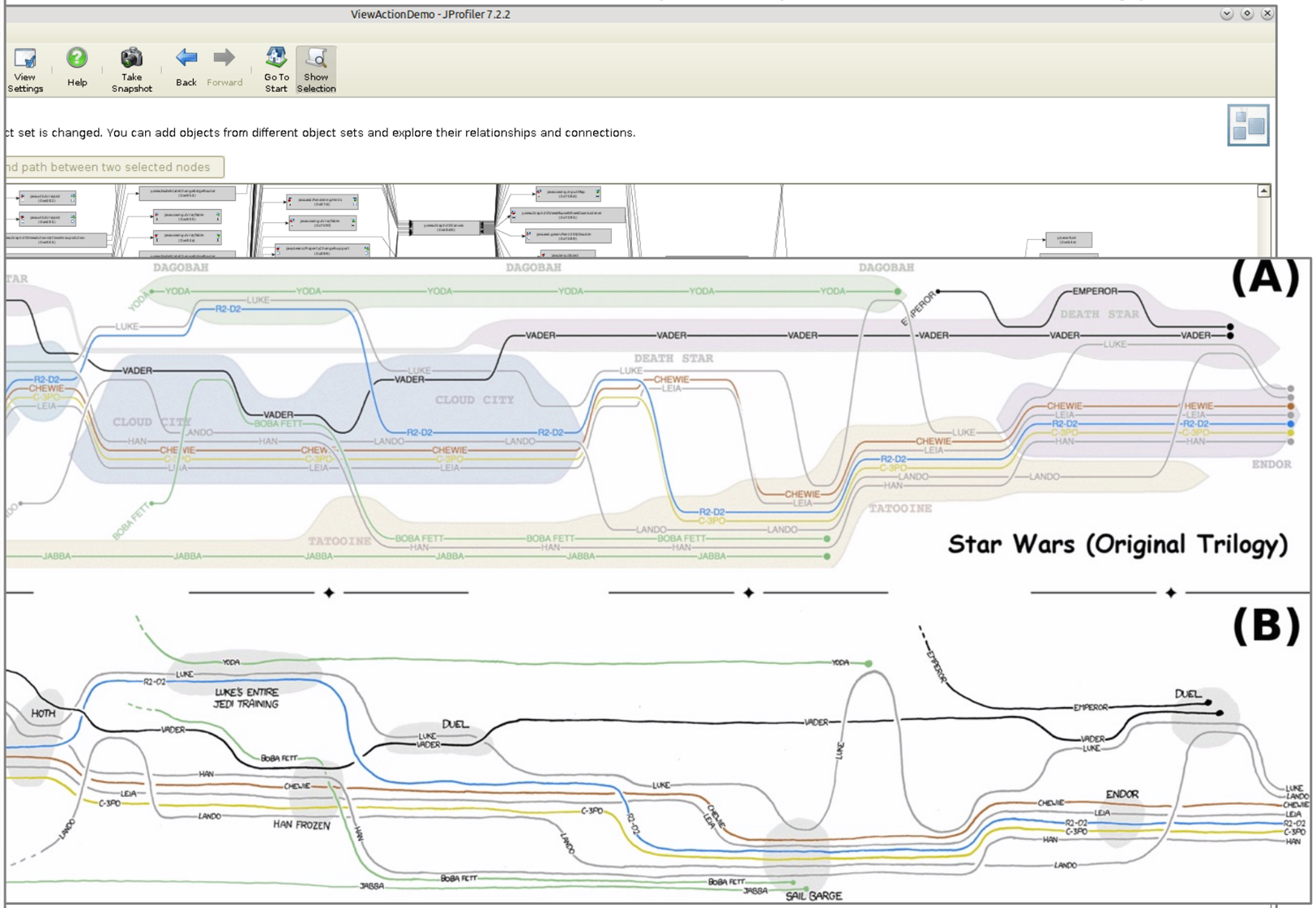


Source: Visualization that won the Graph Drawing Contest 2016. Klawitter & Mchedlidze



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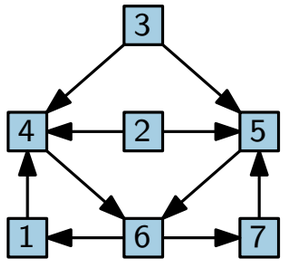
Classical Approach – Sugiyama Framework

[Sugiyama, Tagawa, Toda '81]

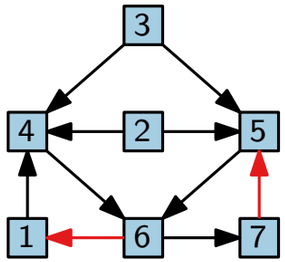
Input



Classical Approach – Sugiyama Framework

[Sugiyama, Tagawa, Toda '81]

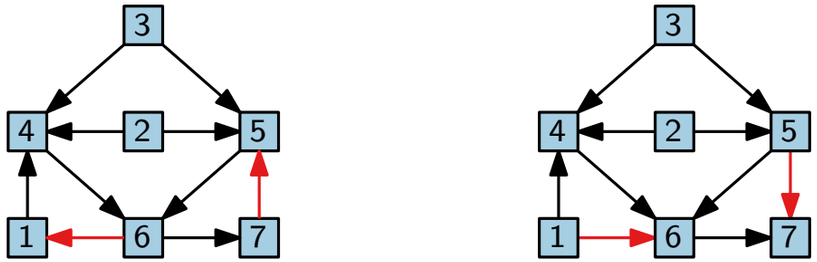
Input



Classical Approach – Sugiyama Framework

[Sugiyama, Tagawa, Toda '81]

Input → Cycle Breaking



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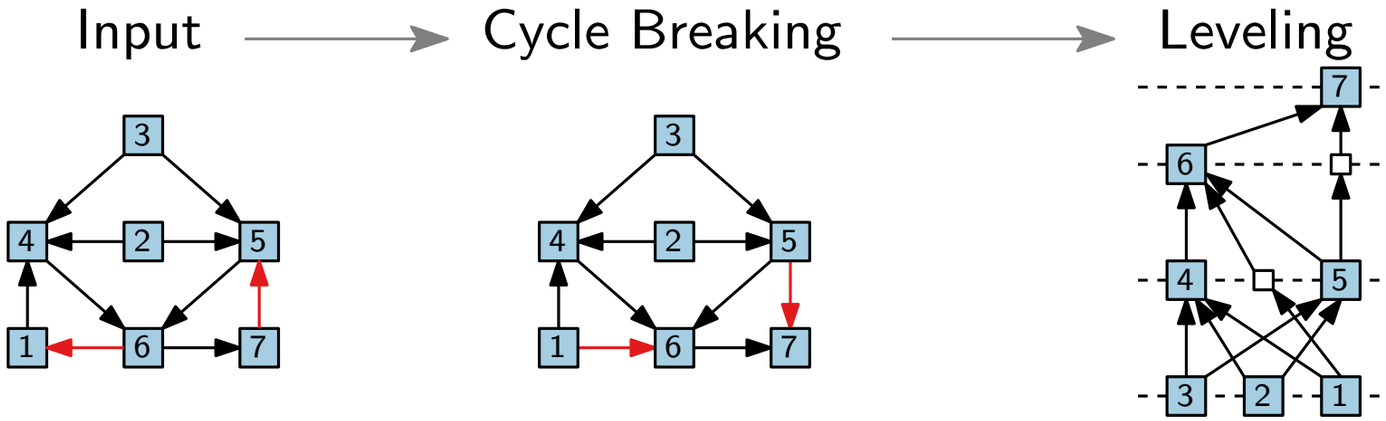

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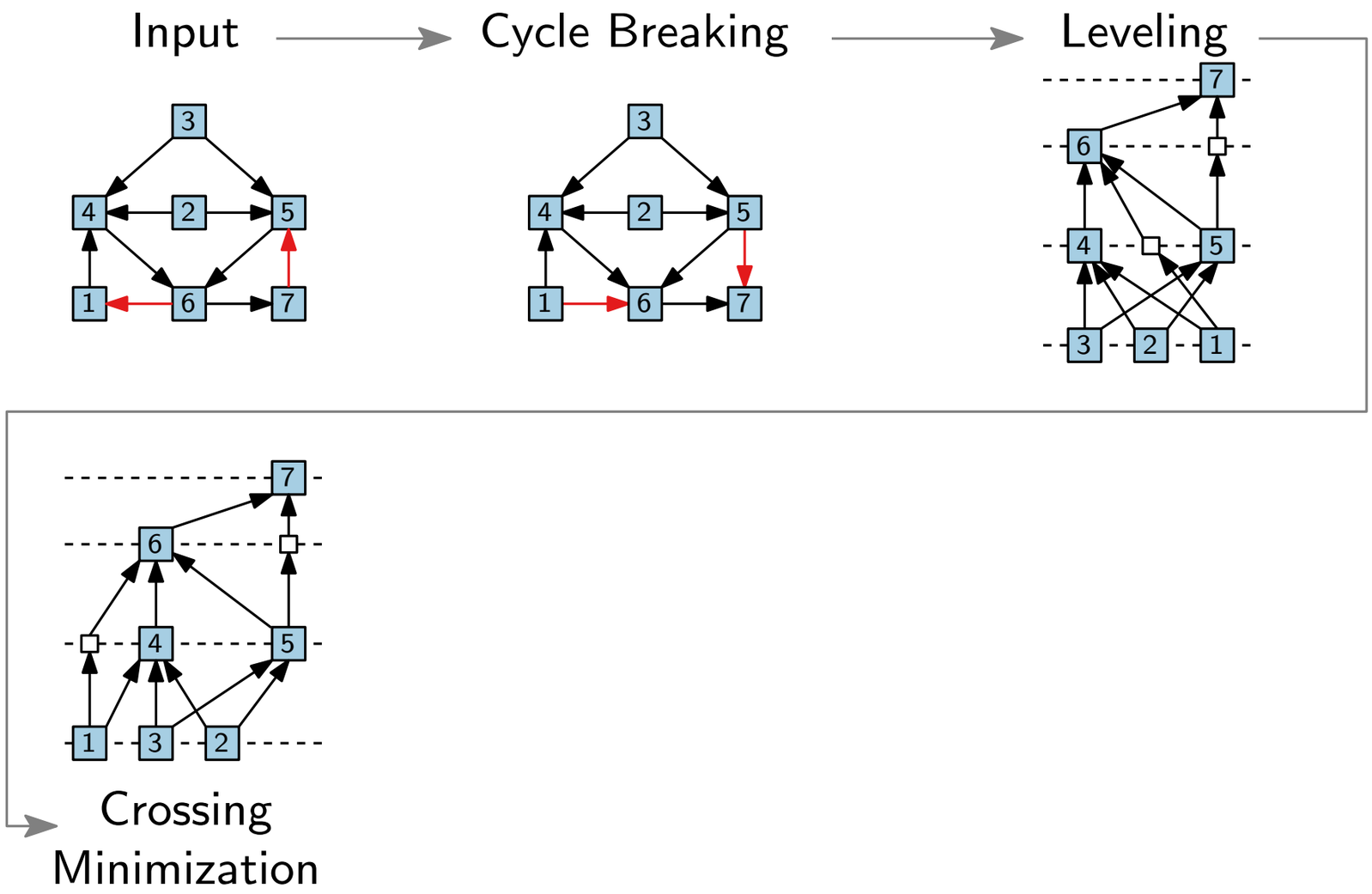

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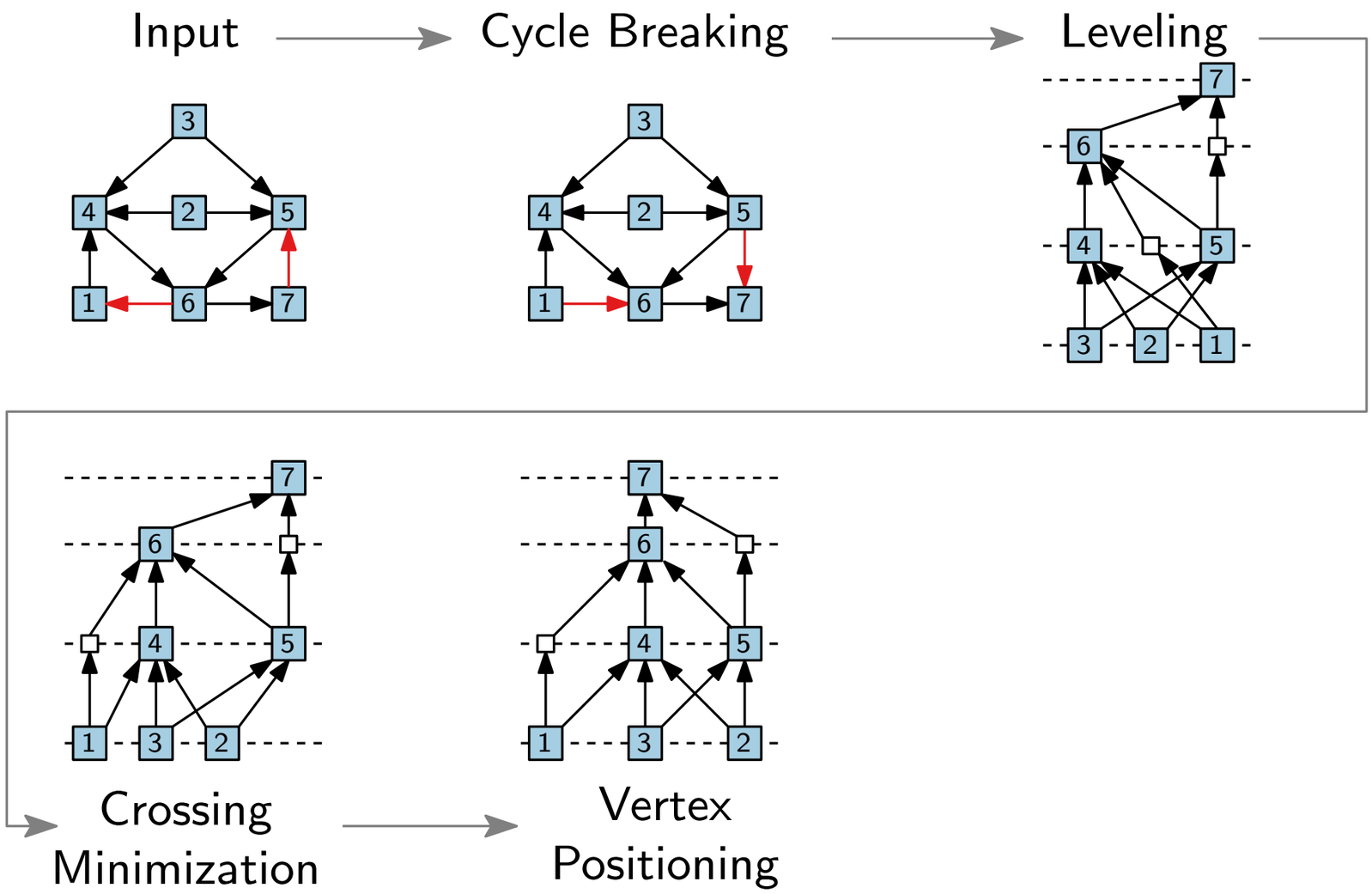

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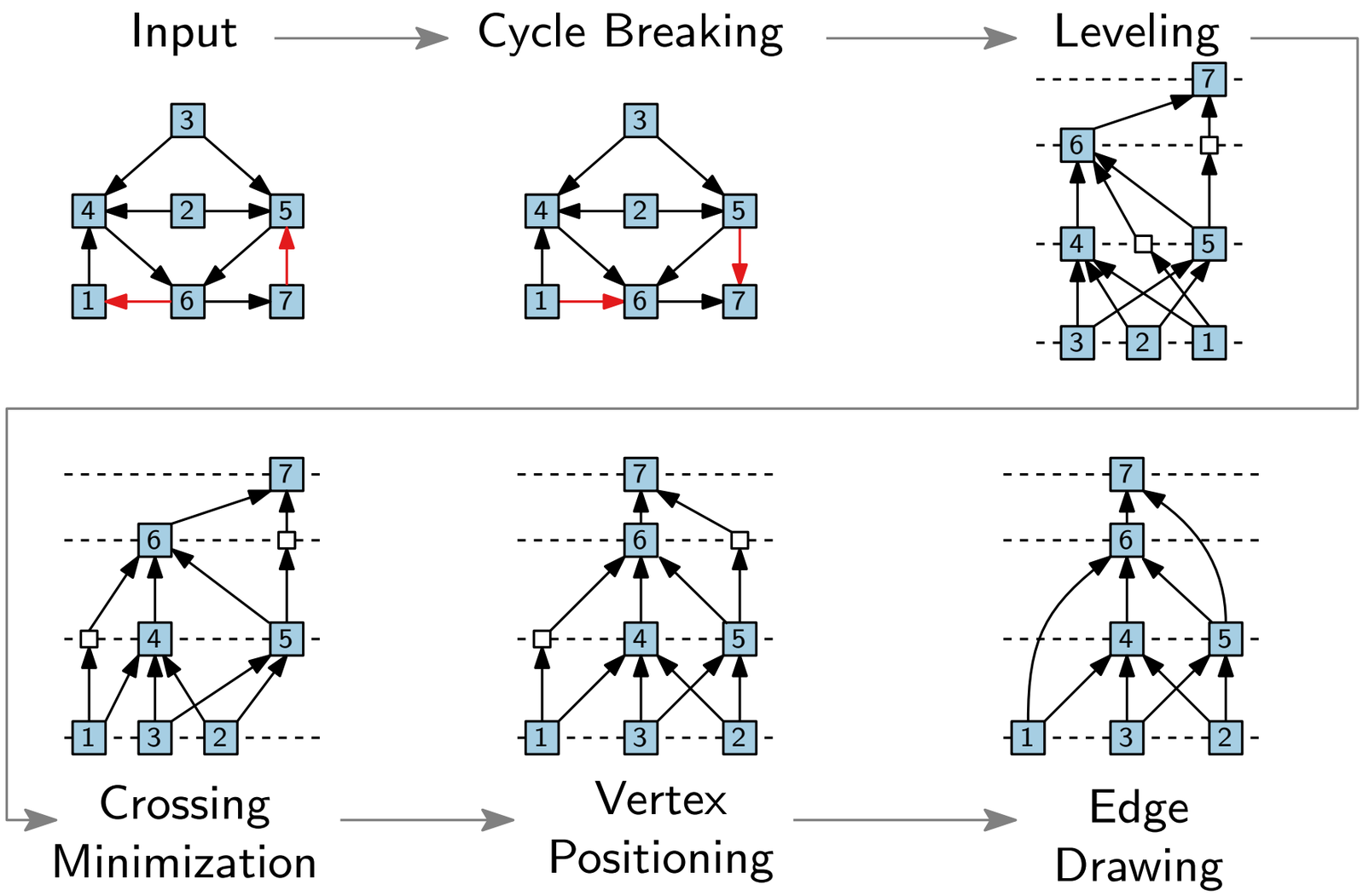

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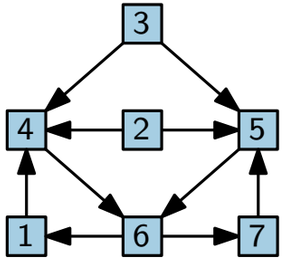
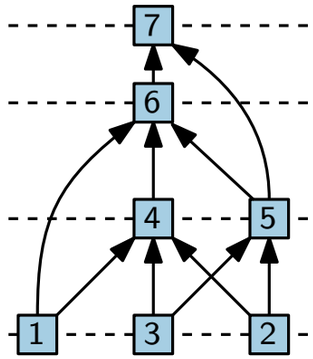
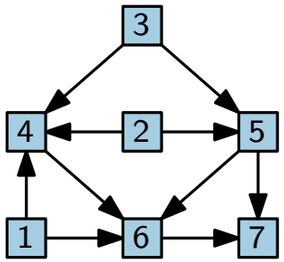
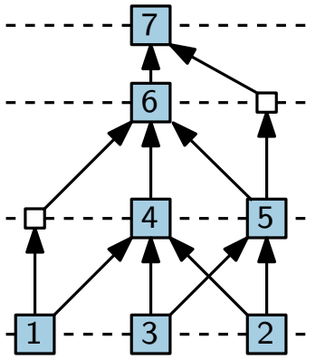


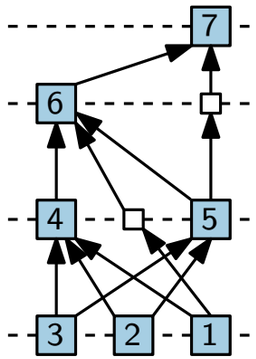
Visualization of Graphs



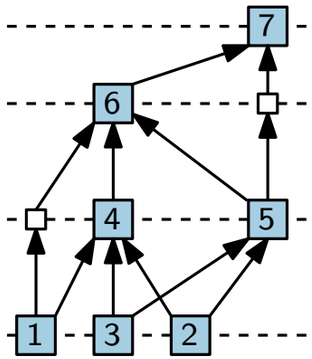
Lecture 7: Hierarchical Layouts: Sugiyama Framework





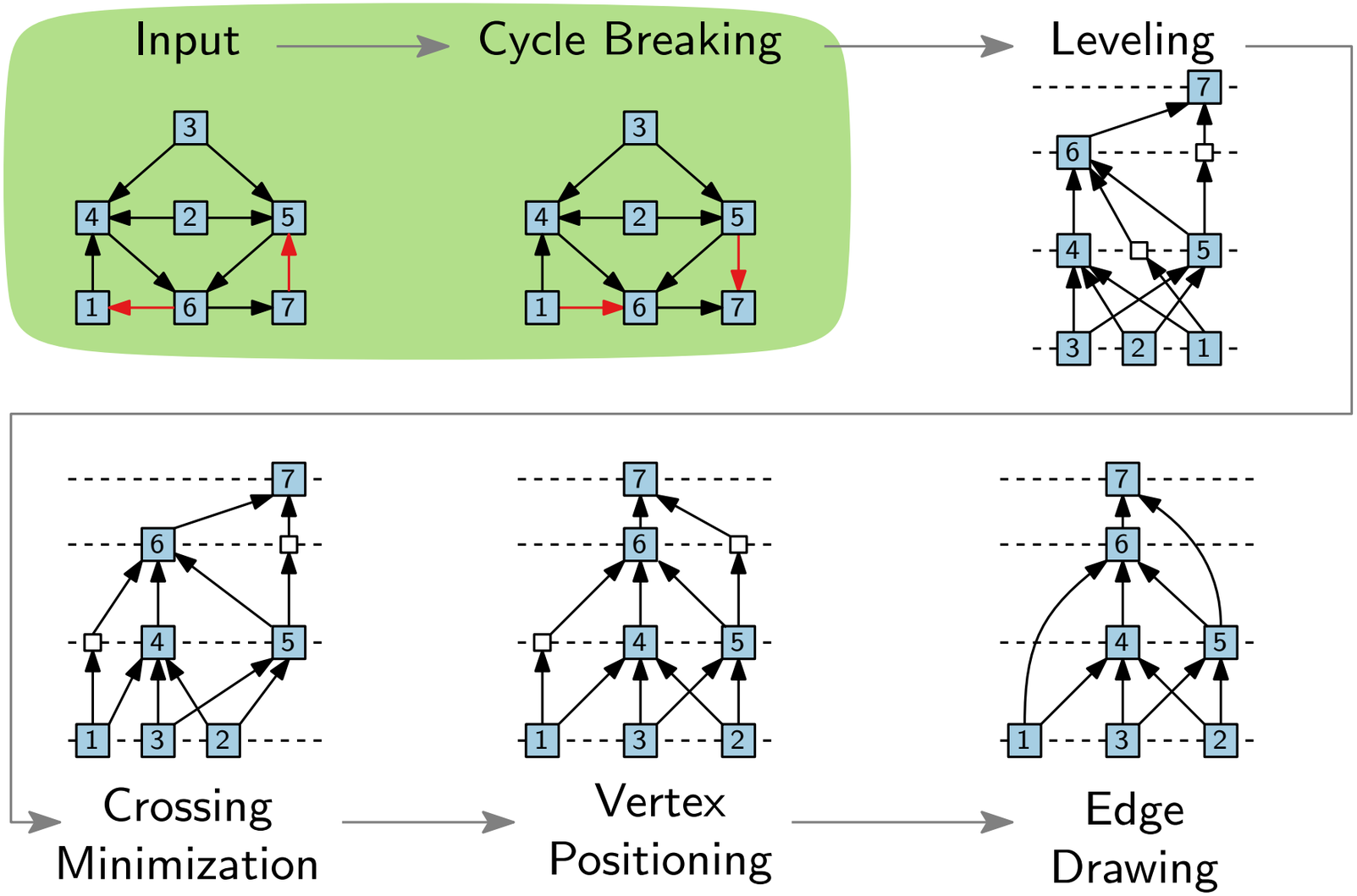


Part II: Cycle Breaking



Jonathan Klawitter

Step 1: Cycle breaking



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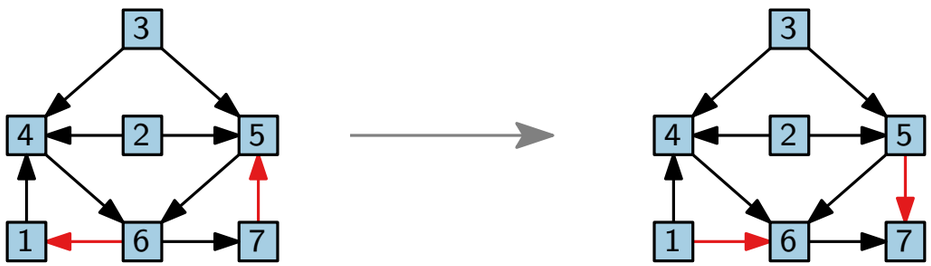
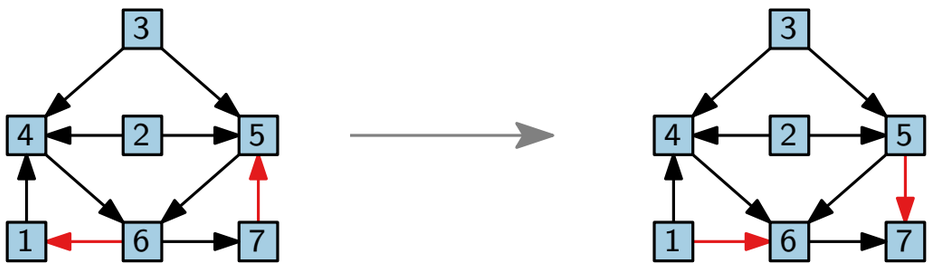


Approach.

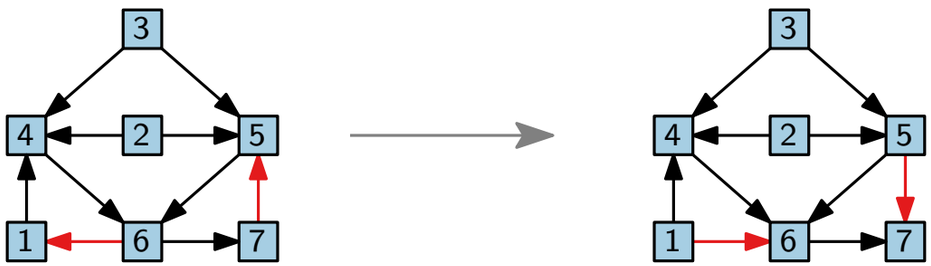
Step 1: Cycle breaking



Approach.

- Find minimum set $E^\star$ of edges which are not upwards.

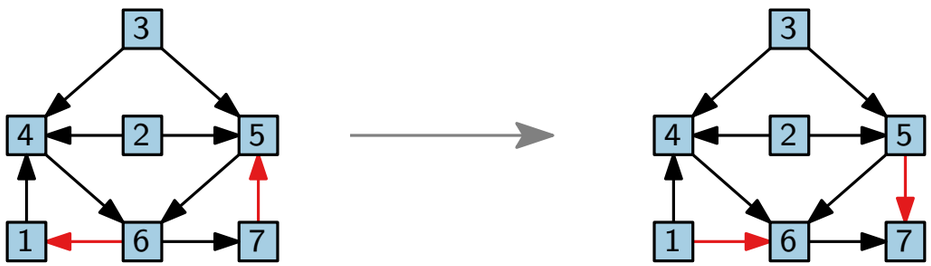
Step 1: Cycle breaking



Approach.

- Find minimum set $E^\star$ of edges which are not upwards.
- Remove $E^\star$ and insert reversed edges.

Step 1: Cycle breaking



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Problem MINIMUM FEEDBACK ARC SET **(FAS).**

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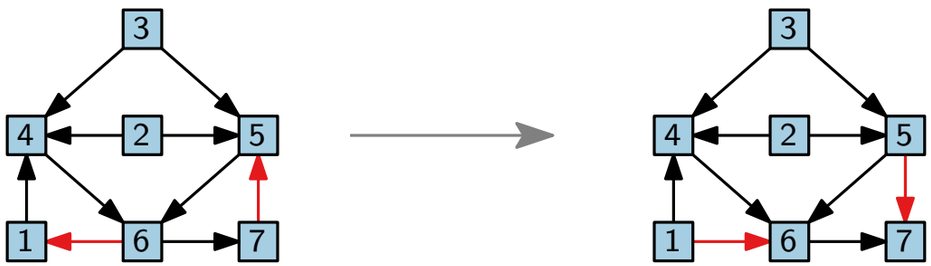

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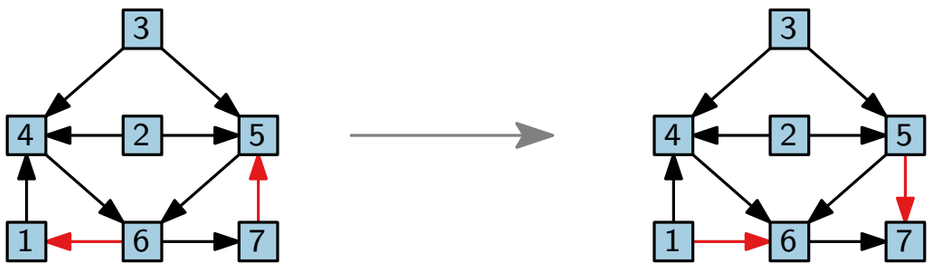
Problem MINIMUM FEEDBACK ARC SET **(FAS).**

- Input: directed graph $G = (V, E)$
- Output:

Step 1: Cycle breaking



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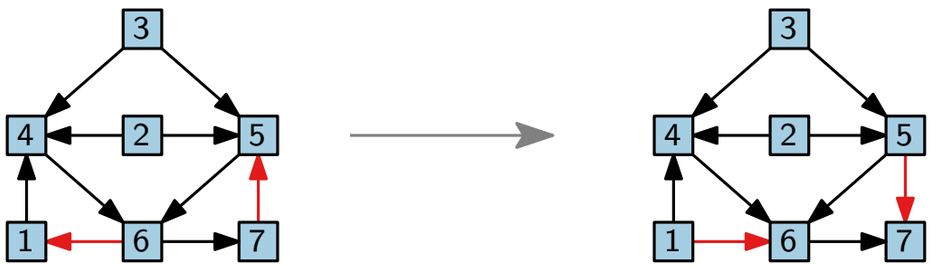
Problem MINIMUM FEEDBACK ARC SET **(FAS).**

- Input: directed graph $G = (V, E)$
- Output: min. set $E^\star \subseteq E$, so that $G - E^\star$ acyclic

Step 1: Cycle breaking



Approach.

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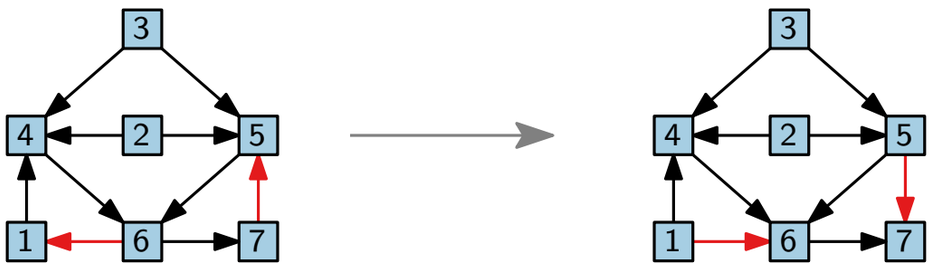
Problem MINIMUM FEEDBACK ~~ARC~~ SET **(F~~A~~S).**

- Input: directed graph $G = (V, E)$
- Output: min. set $E^\star \subseteq E$, so that $\cancel{G - E^\star}$ acyclic

$$G - E^\star + E^\star_r$$

Step 1: Cycle breaking



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  $G - E^\star + E^\star_r$

. . . NP-hard ☹

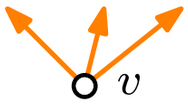
Heuristic 1

[Berger, Shor '90]

v

Heuristic 1

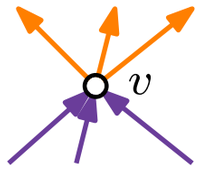
[Berger, Shor '90]



$$N^{\rightarrow}(v) \quad := \quad \{(v,u) | (v,u) \in E\}$$

Heuristic 1

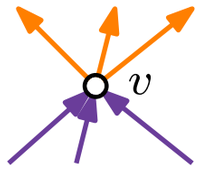
[Berger, Shor '90]



$$N^{\rightarrow}(v) := \{(v,u) | (v,u) \in E\}$$
$$N^{\leftarrow}(v) := \{(u,v) | (u,v) \in E\}$$

Heuristic 1

[Berger, Shor '90]



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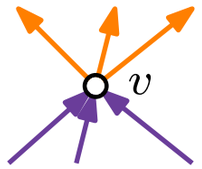
GreedyMakeAcyclic(Digraph $G = (V, E)$)



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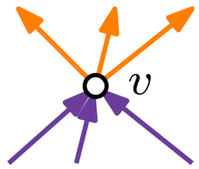

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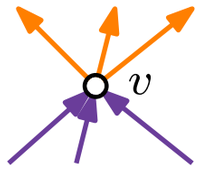

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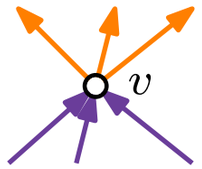

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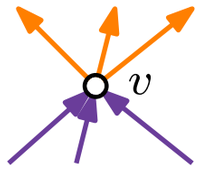

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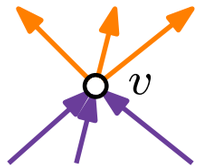

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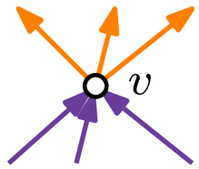

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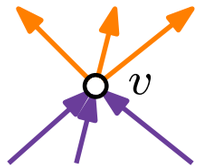

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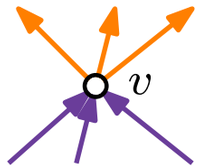

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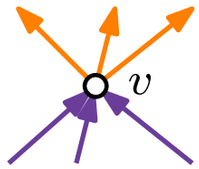

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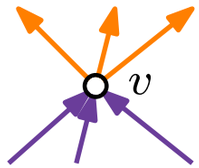

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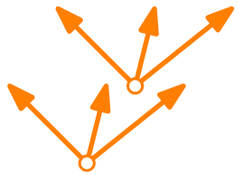

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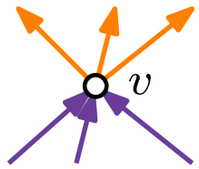

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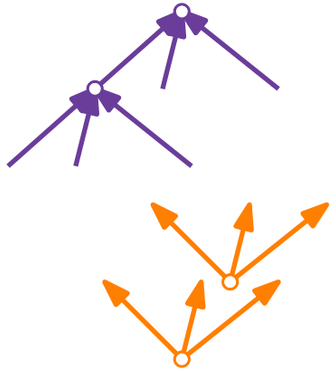

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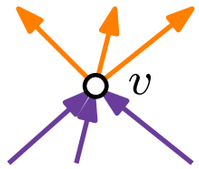

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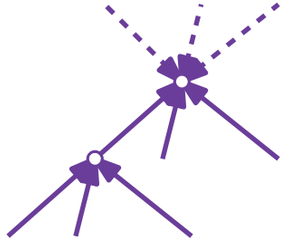

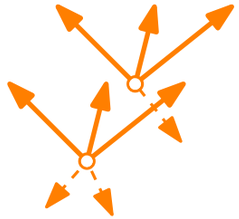

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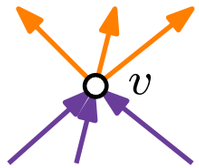

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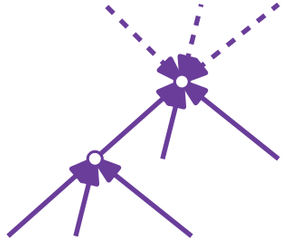

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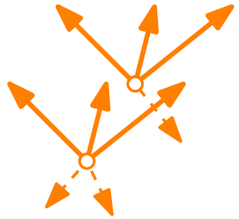

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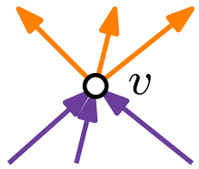

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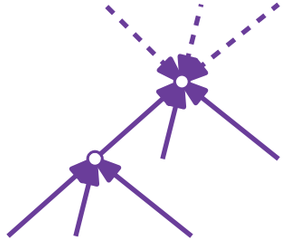

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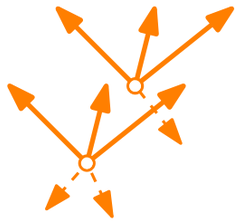

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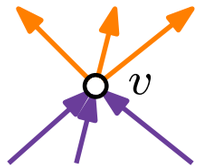

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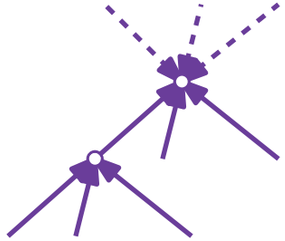

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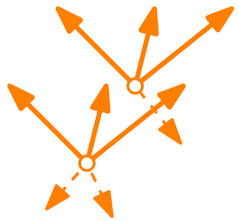

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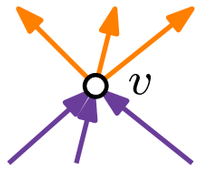

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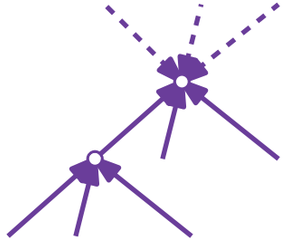

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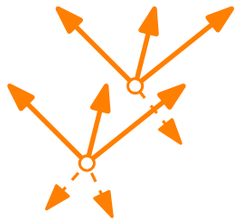

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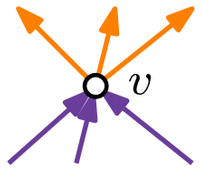

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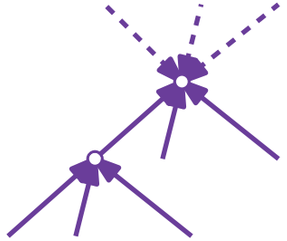

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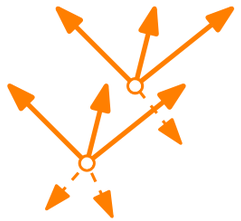

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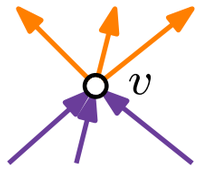

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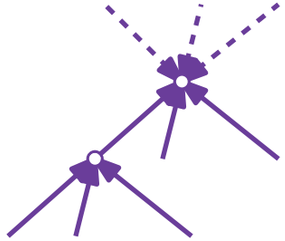

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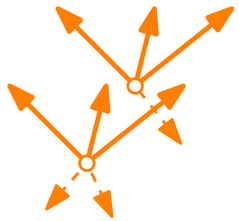

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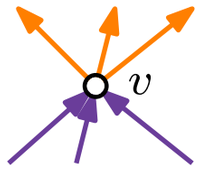

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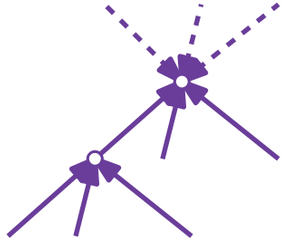

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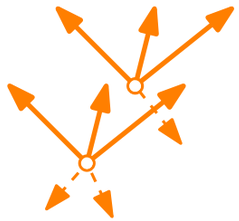

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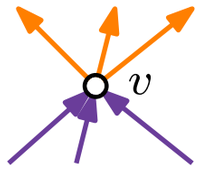

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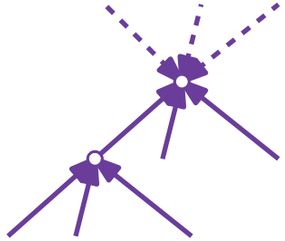

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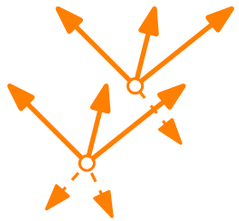

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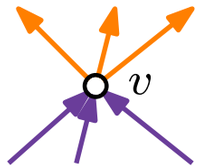

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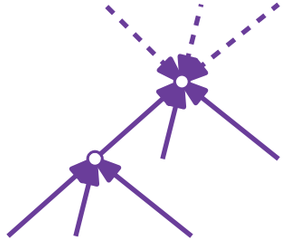

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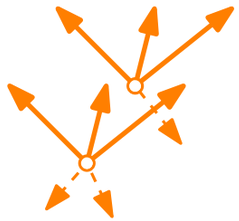

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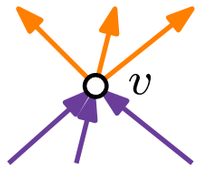

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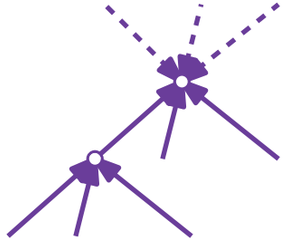

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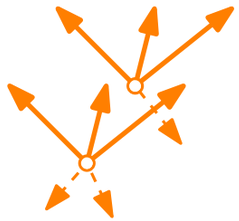

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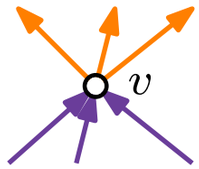

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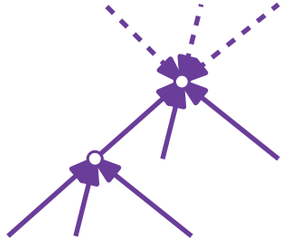
- Time: $\mathcal{O}(n + m)$



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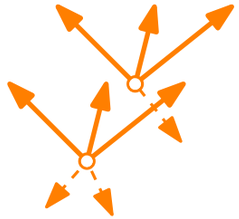

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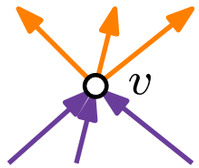

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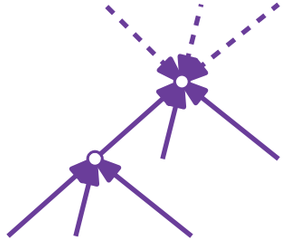
- Time: $\mathcal{O}(n + m)$
- Quality guarantee: $|E'| \geq$



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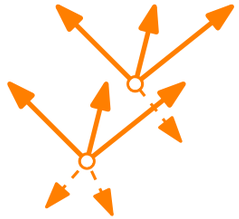

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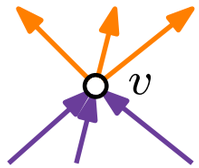

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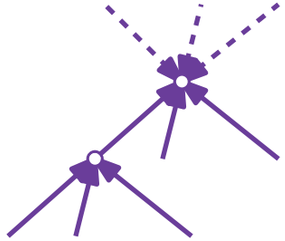
- Quality guarantee: $|E'| \geq |E|/2$



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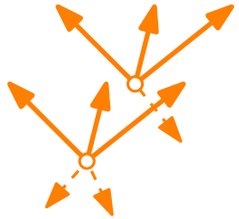

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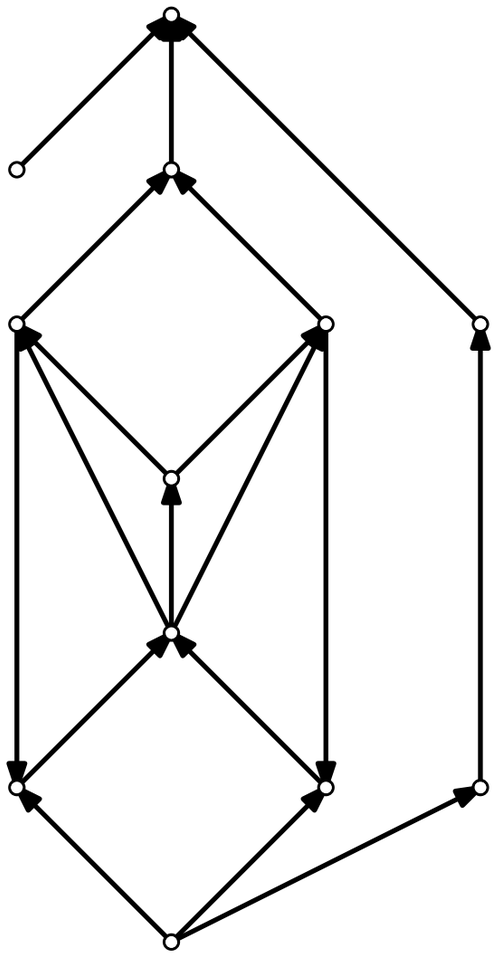
[Eades, Lin, Smyth '93]



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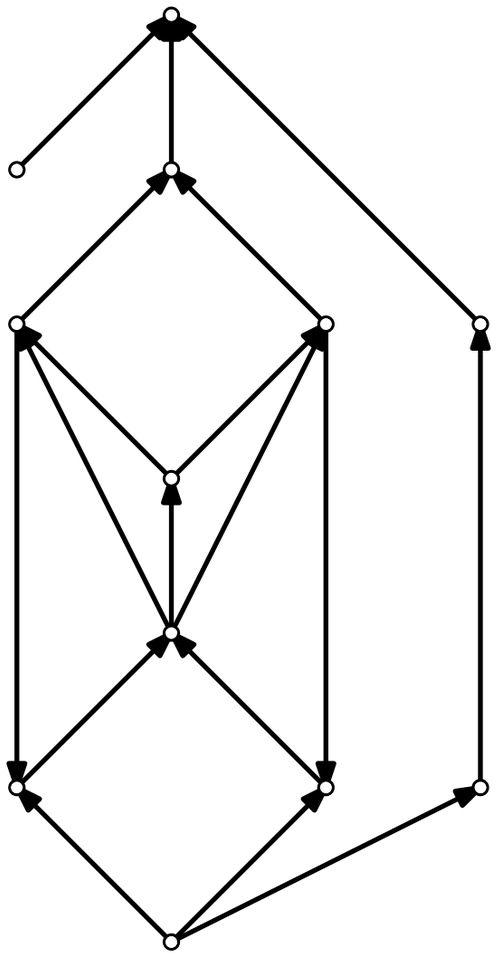

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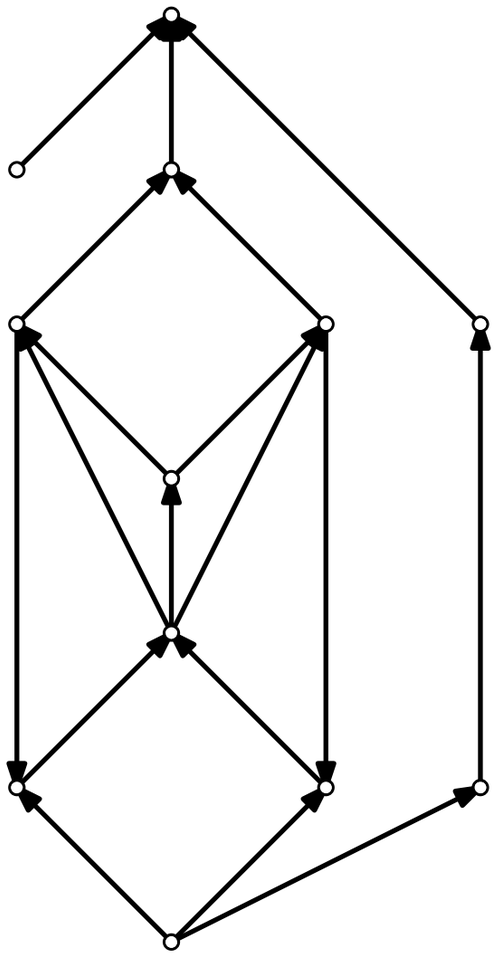

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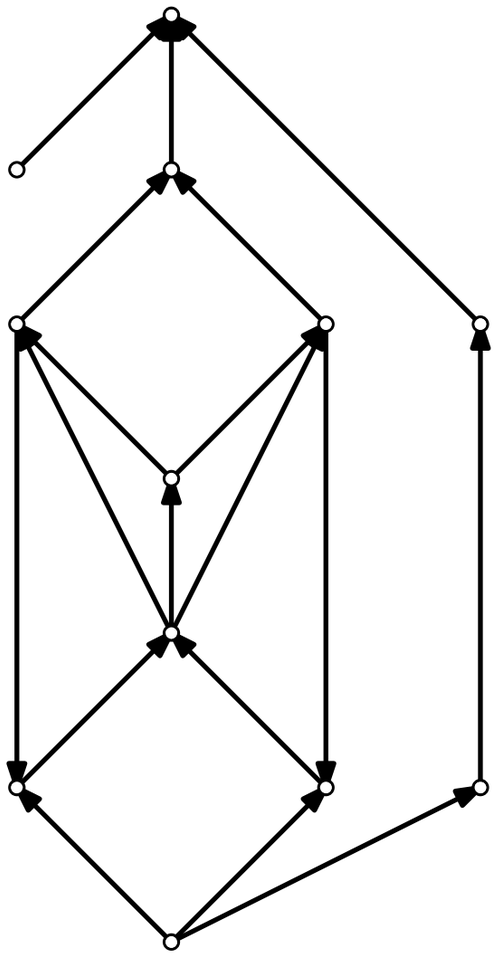

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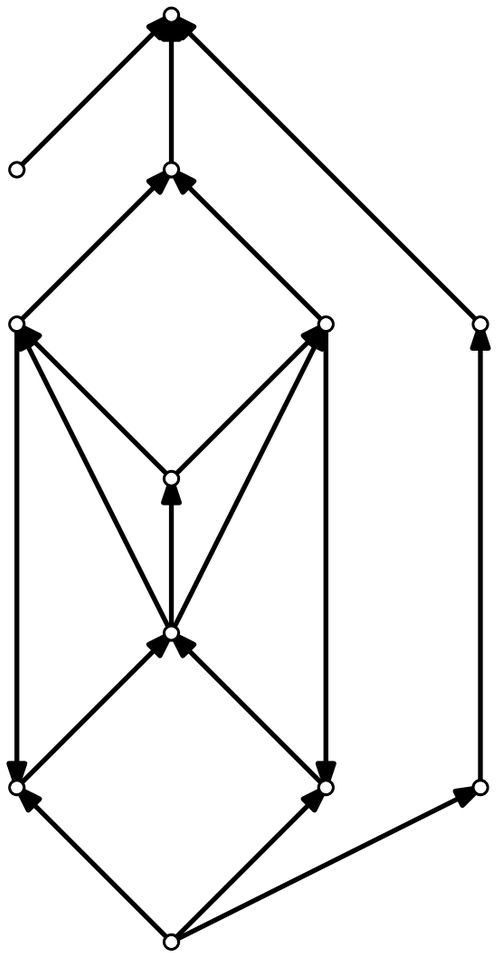

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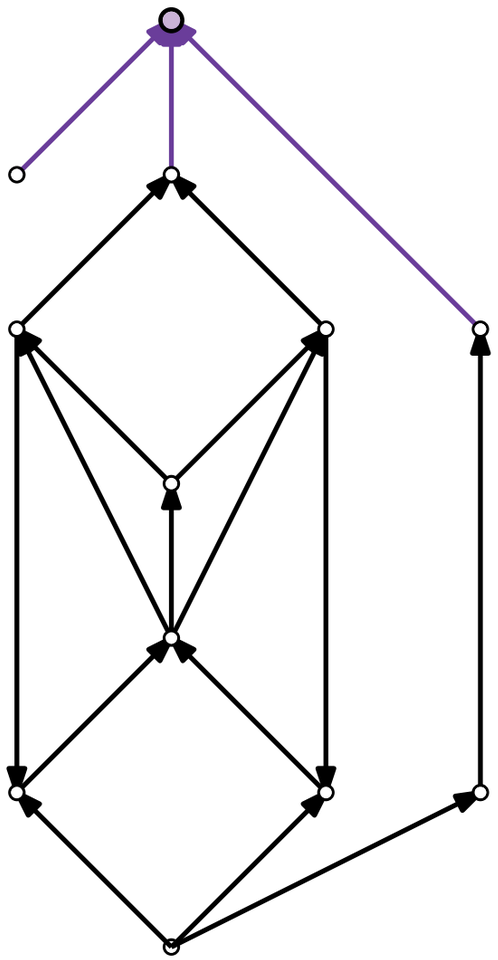

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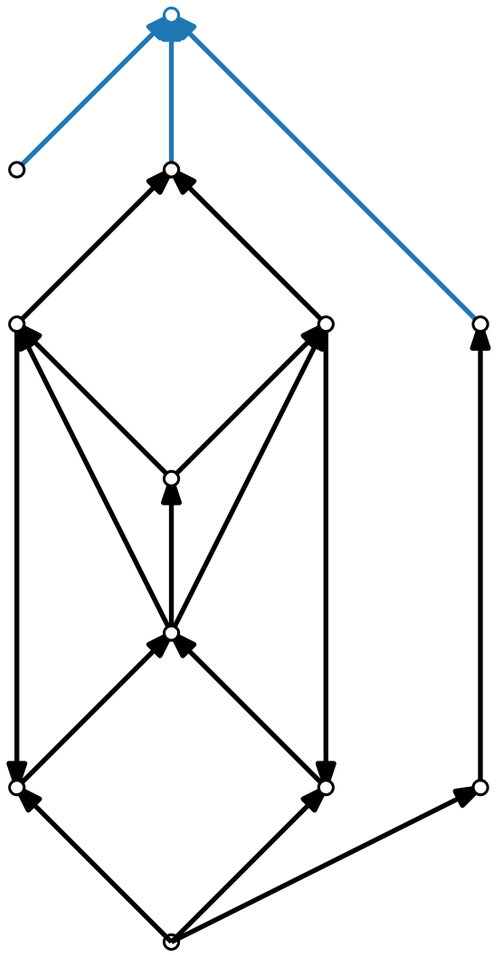

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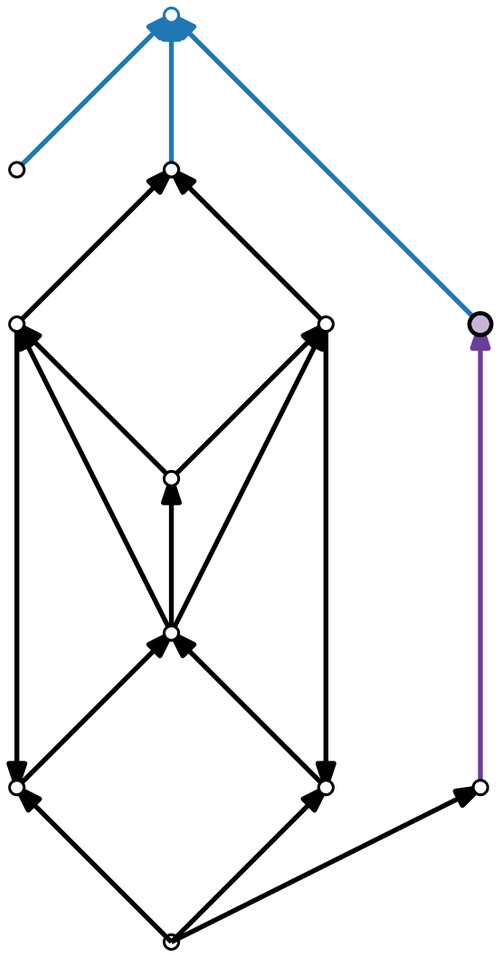

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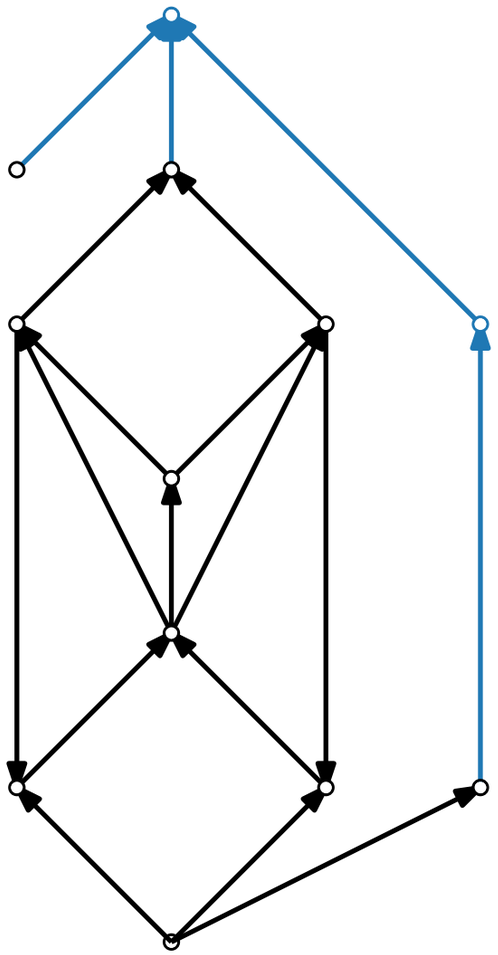

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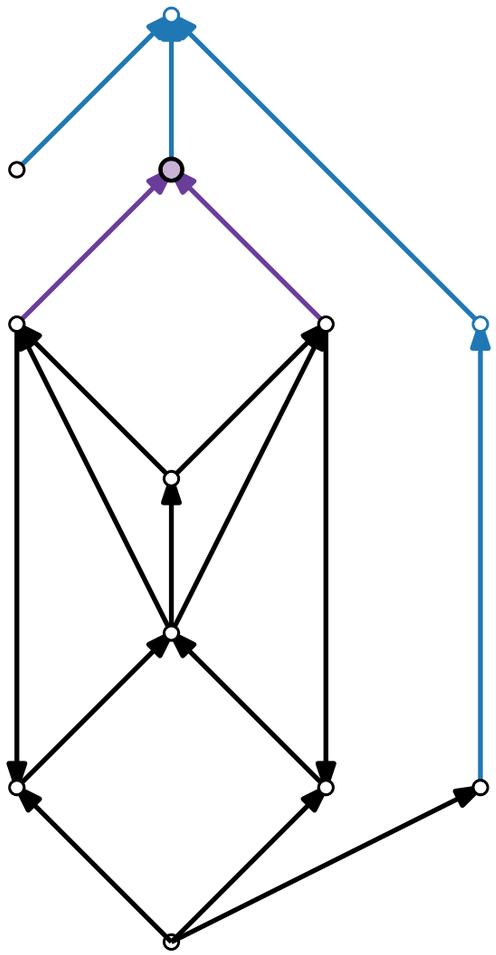

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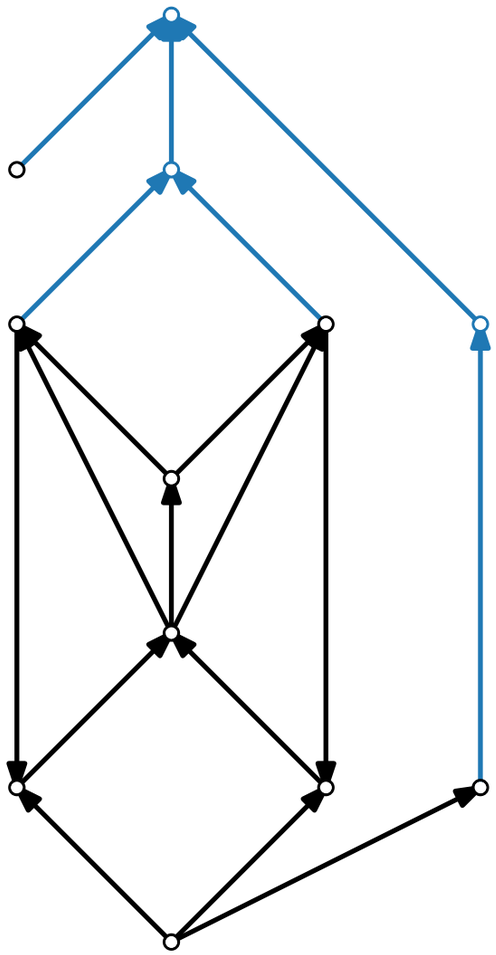

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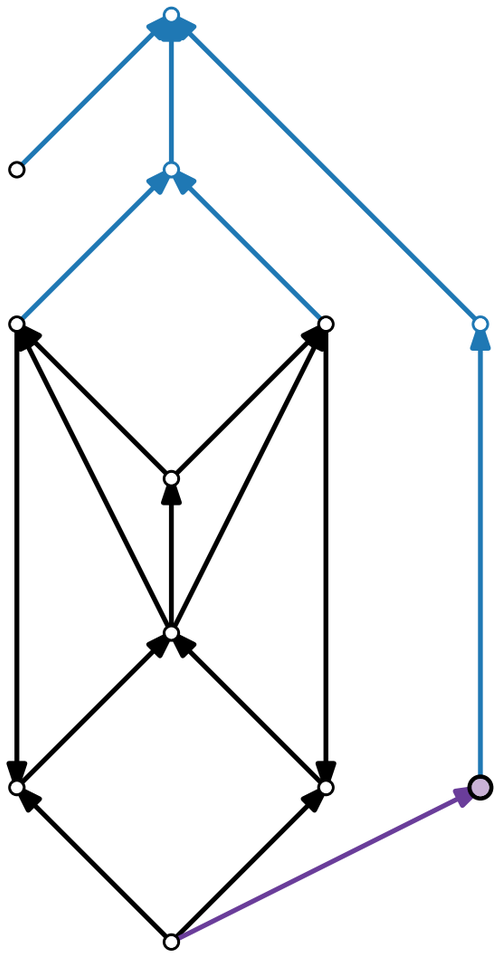

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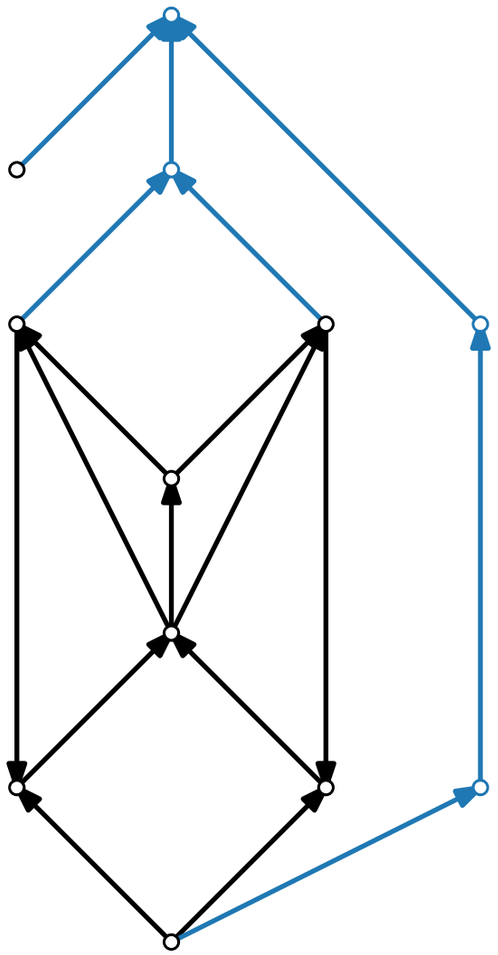

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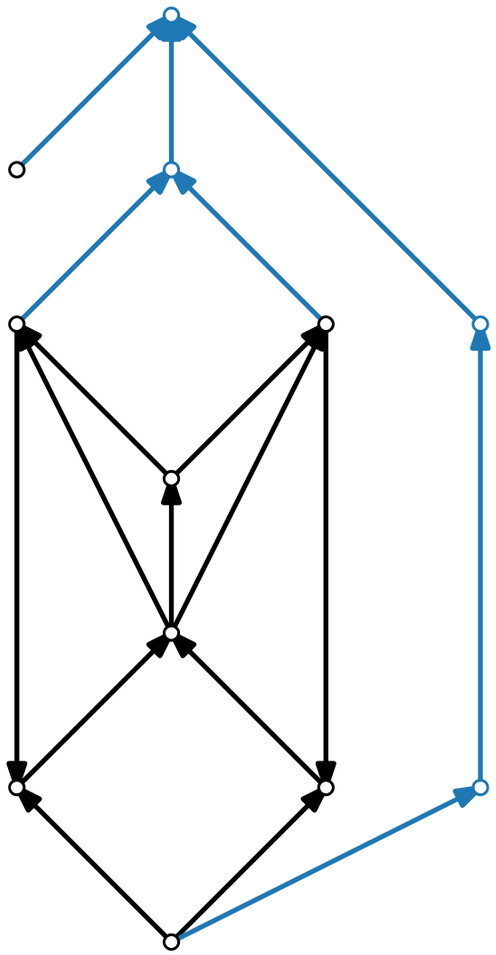

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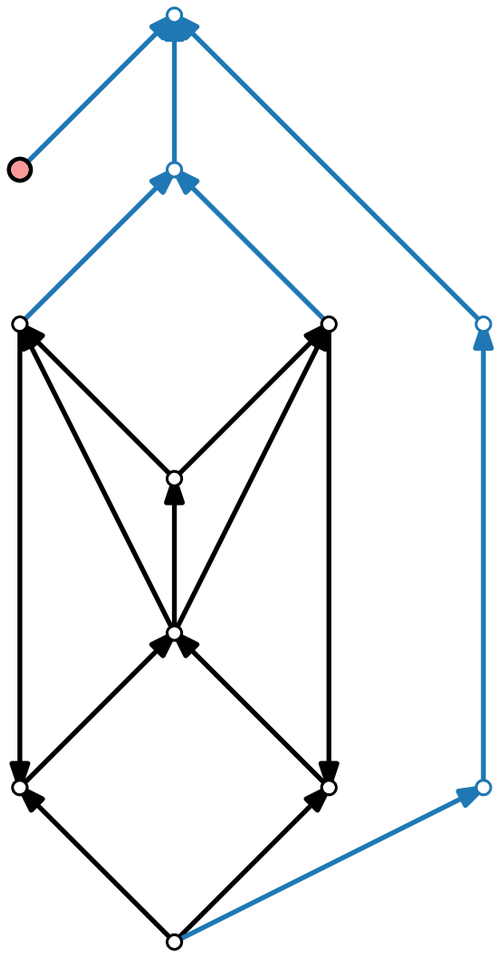

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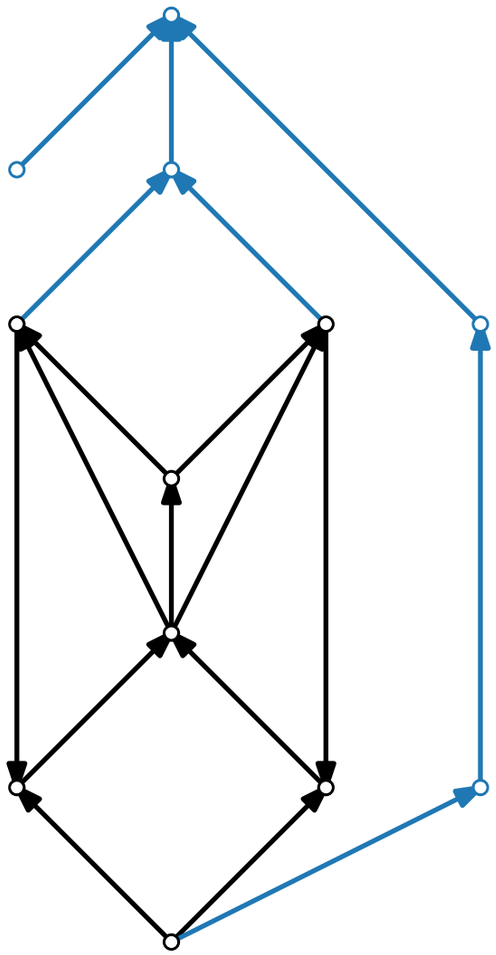

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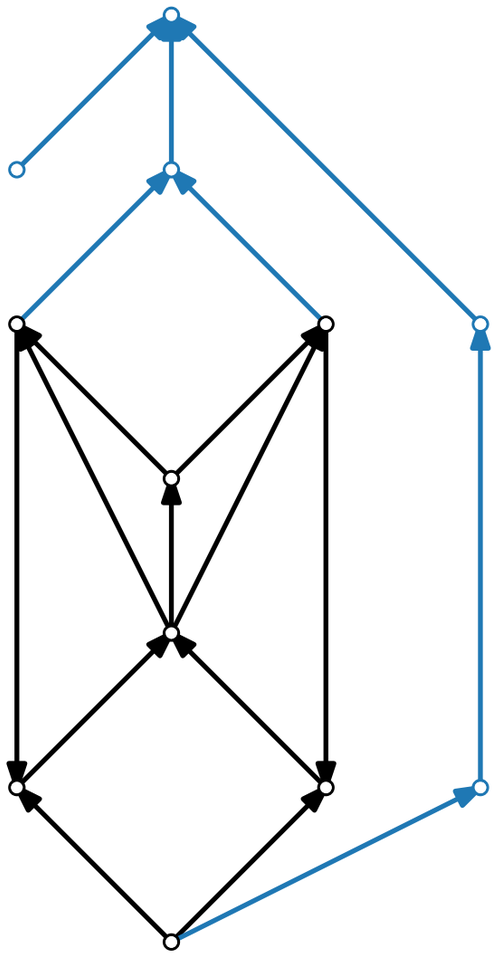

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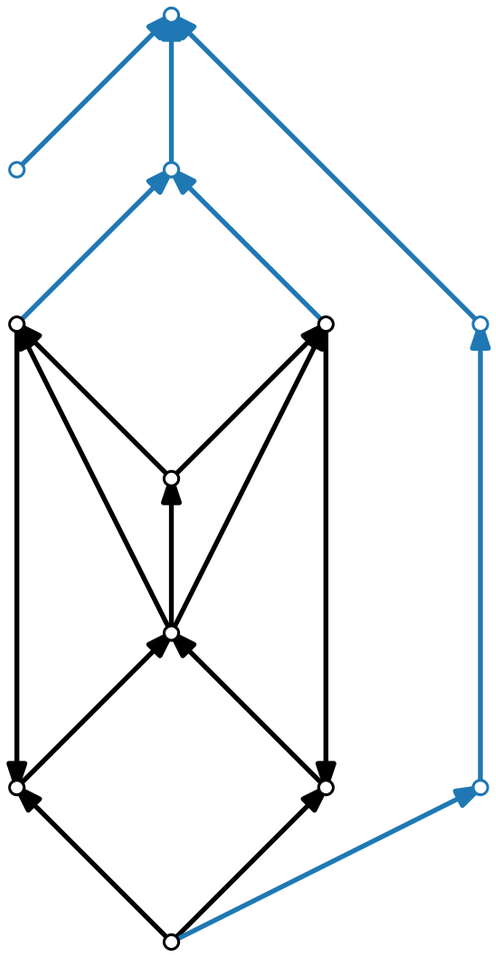

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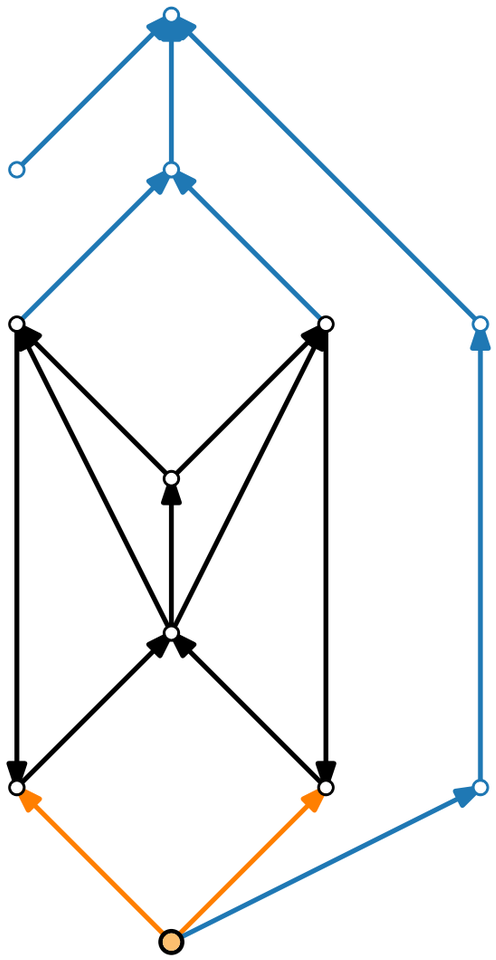

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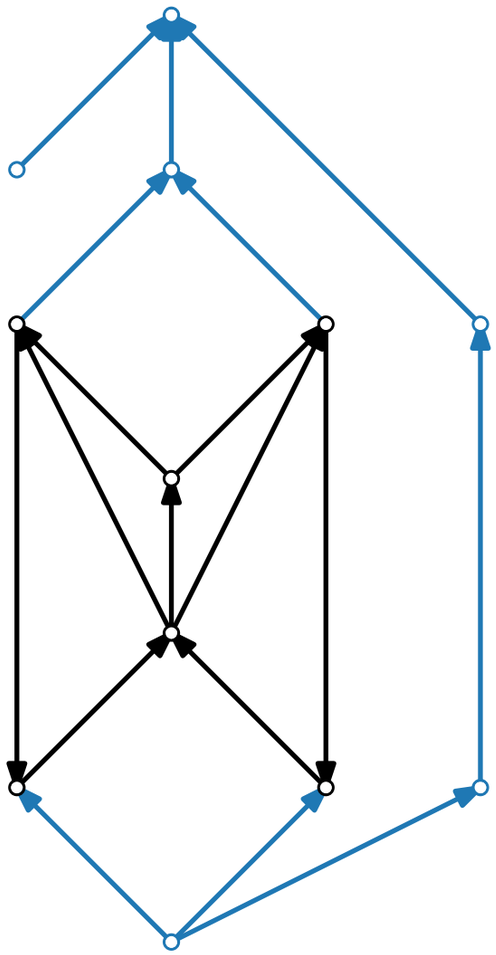

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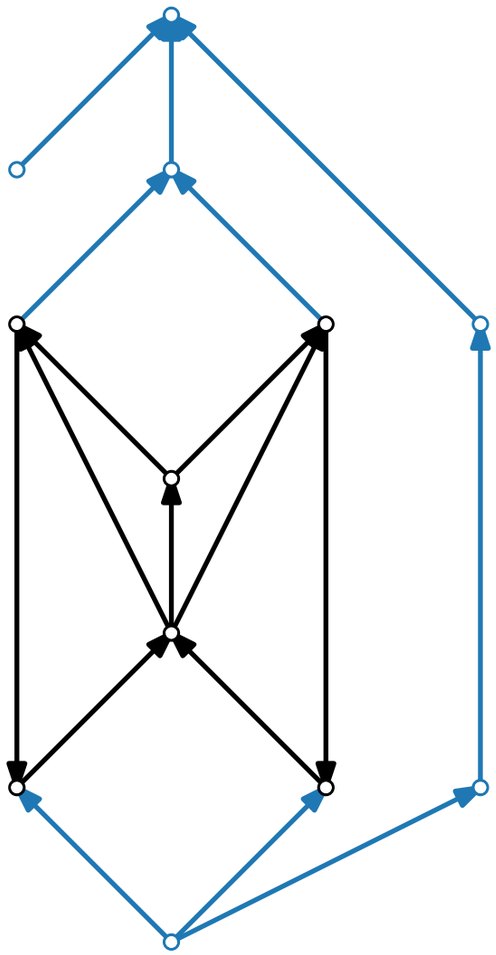

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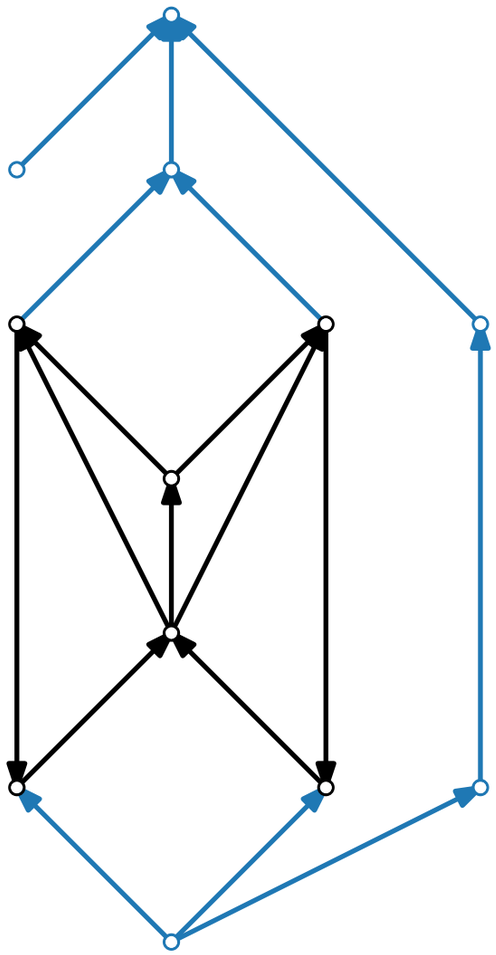

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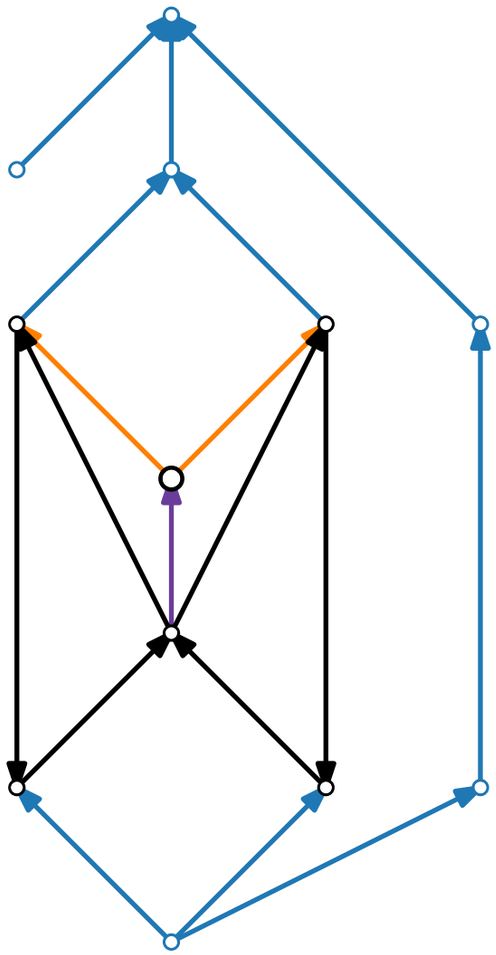

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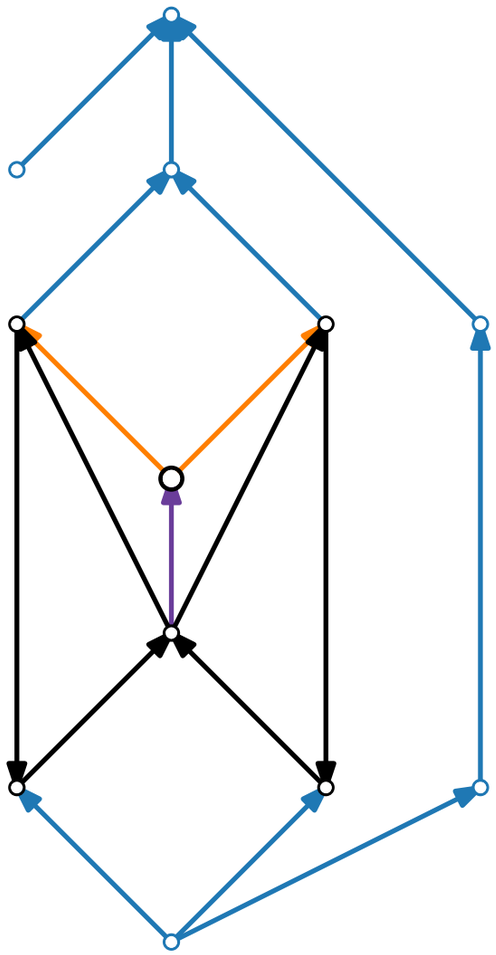

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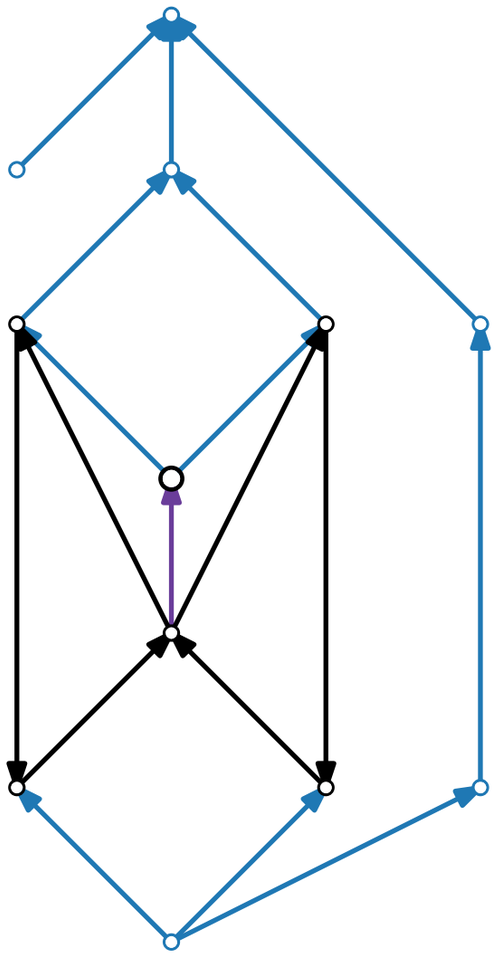

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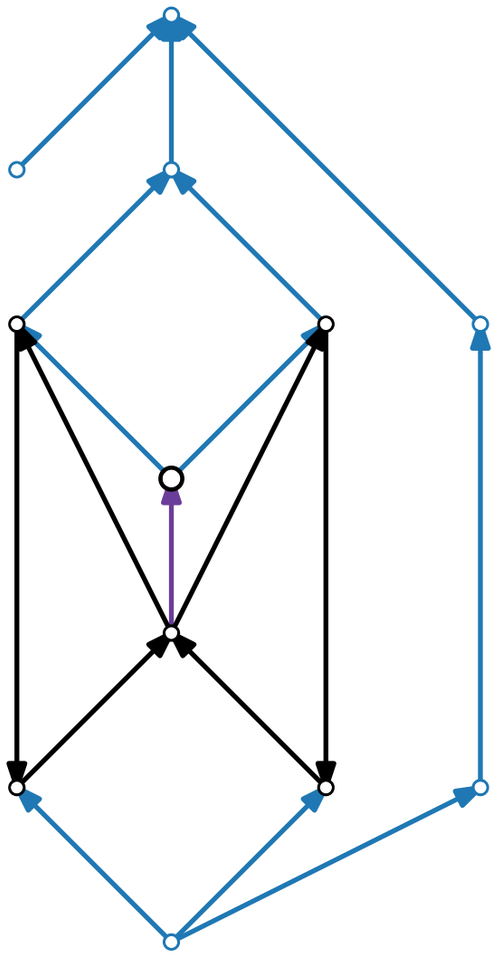

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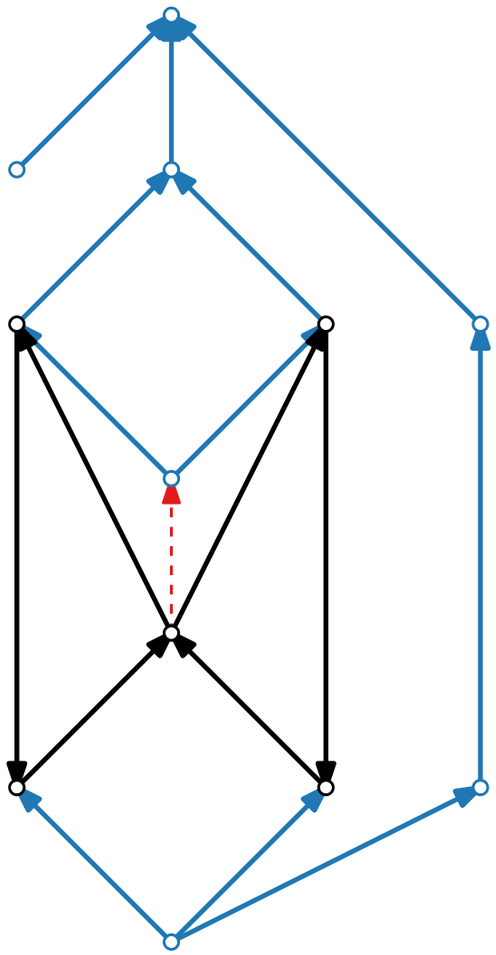

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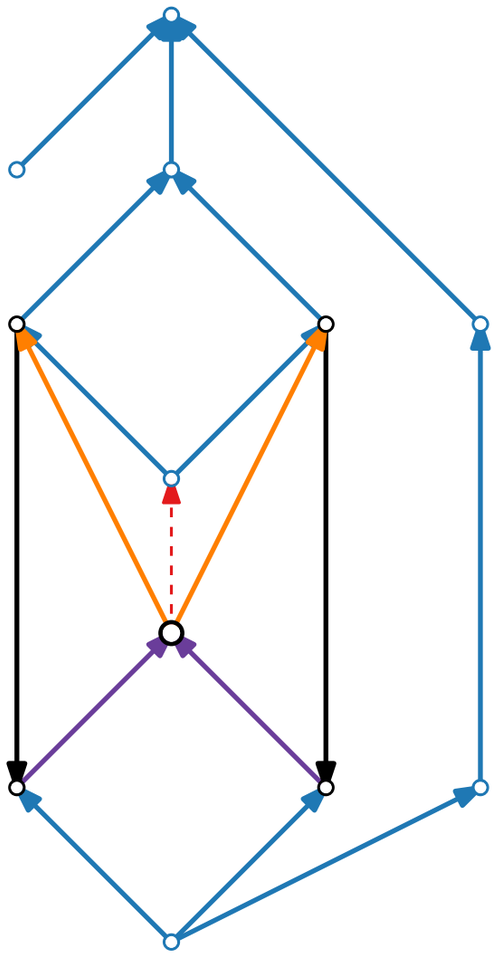

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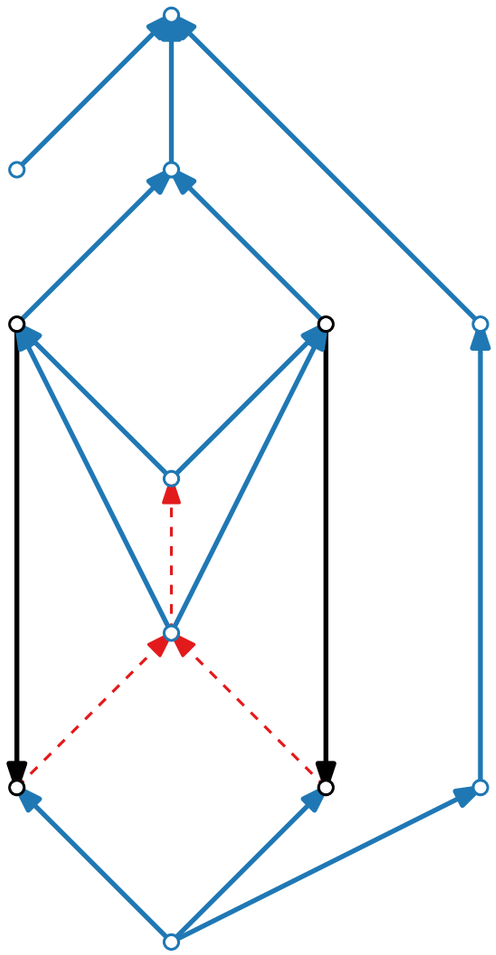

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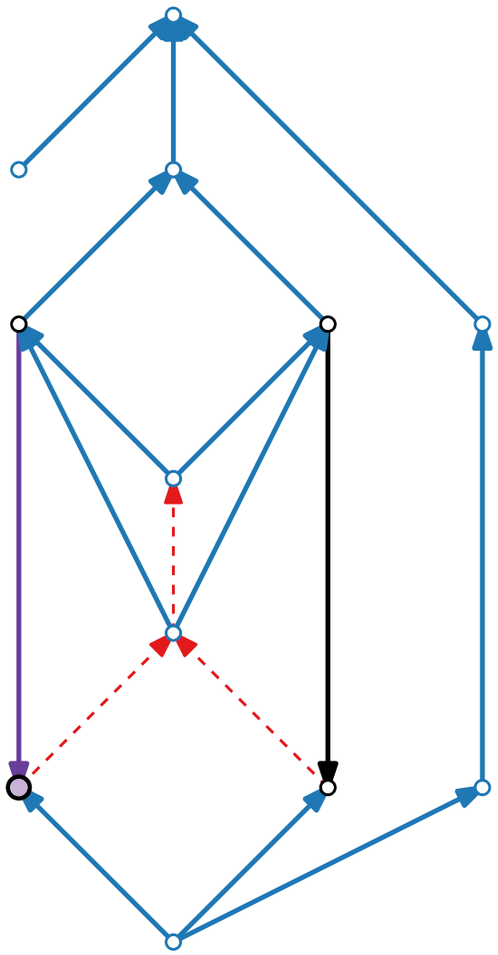

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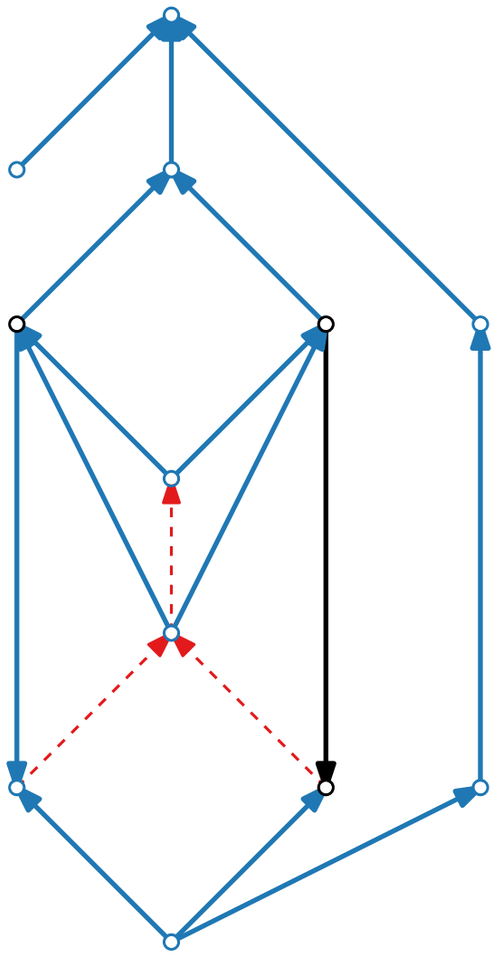

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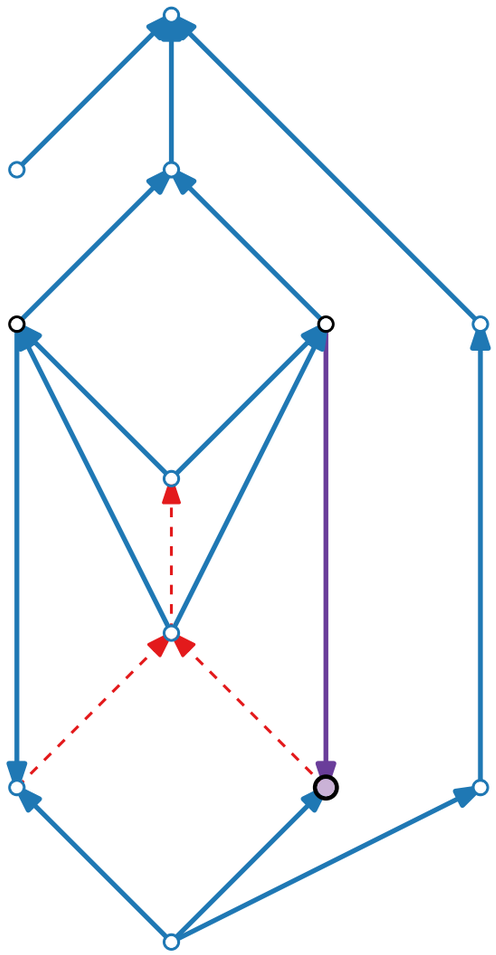

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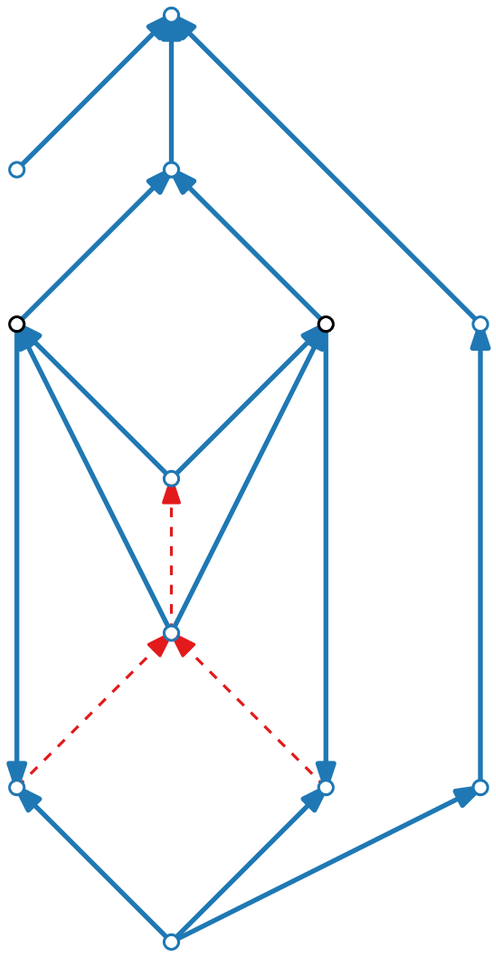

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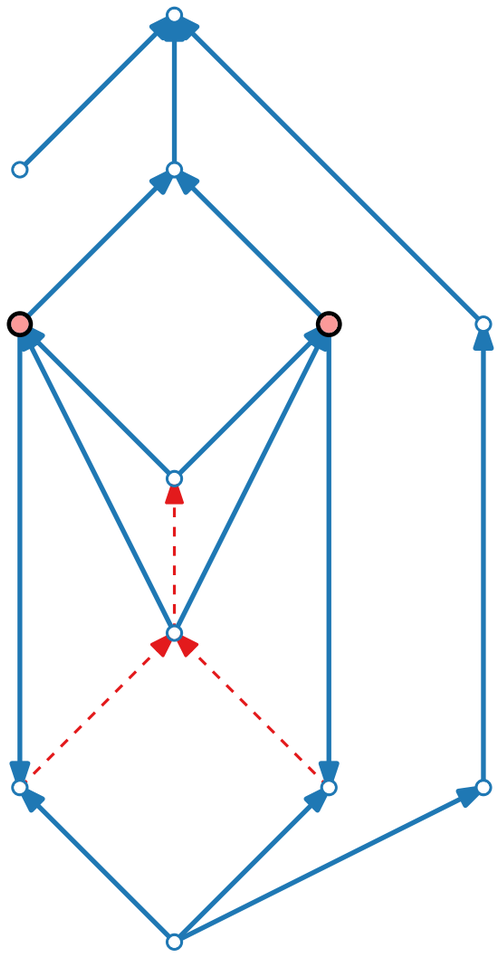

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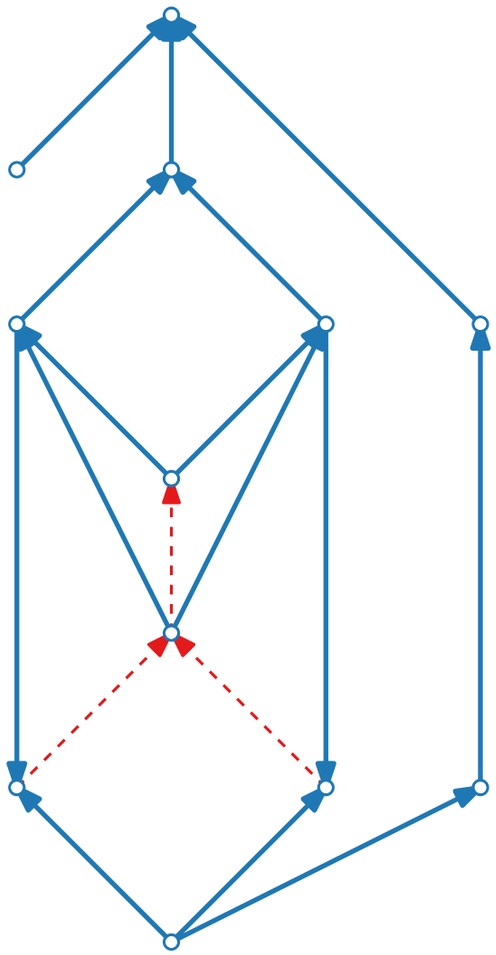

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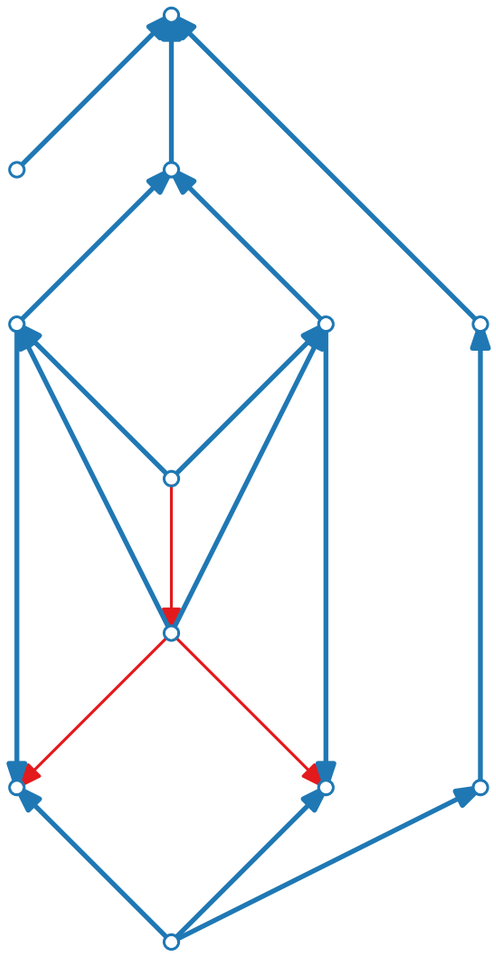

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[Eades, Lin, Smyth '93]

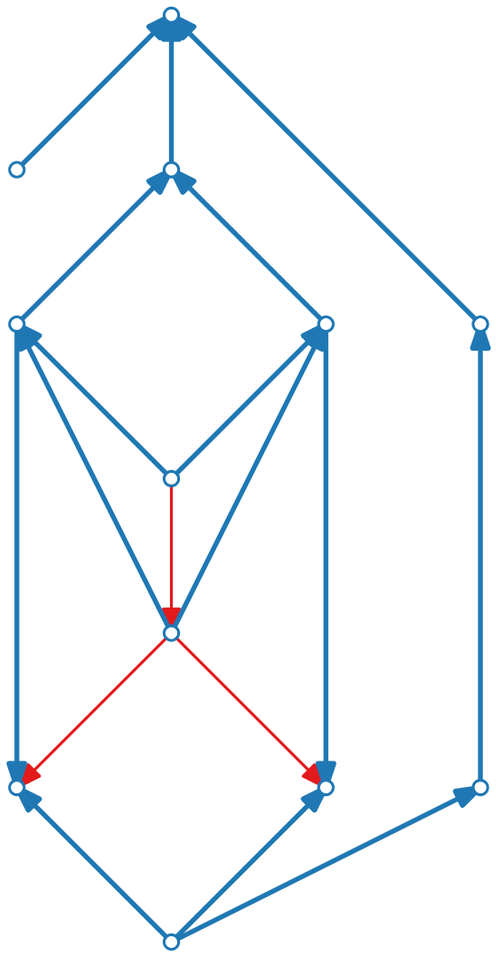
$E' \leftarrow \emptyset$
while $V \neq \emptyset$ **do**
  while in V exists a sink v **do**
    $E' \leftarrow E' \cup N^{\leftarrow}(v)$
    remove v and $N^{\leftarrow}(v)$

  Remove all isolated vertices from V

  while in V exists a source v **do**
    $E' \leftarrow E' \cup N^{\rightarrow}(v)$
    remove v and $N^{\rightarrow}(v)$

  if $V \neq \emptyset$ **then**
    let $v \in V$ such that $|N^{\rightarrow}(v)| - |N^{\leftarrow}(v)|$ maximal
    $E' \leftarrow E' \cup N^{\rightarrow}(v)$
    remove v and $N(v)$

- Time: $\mathcal{O}(n + m)$



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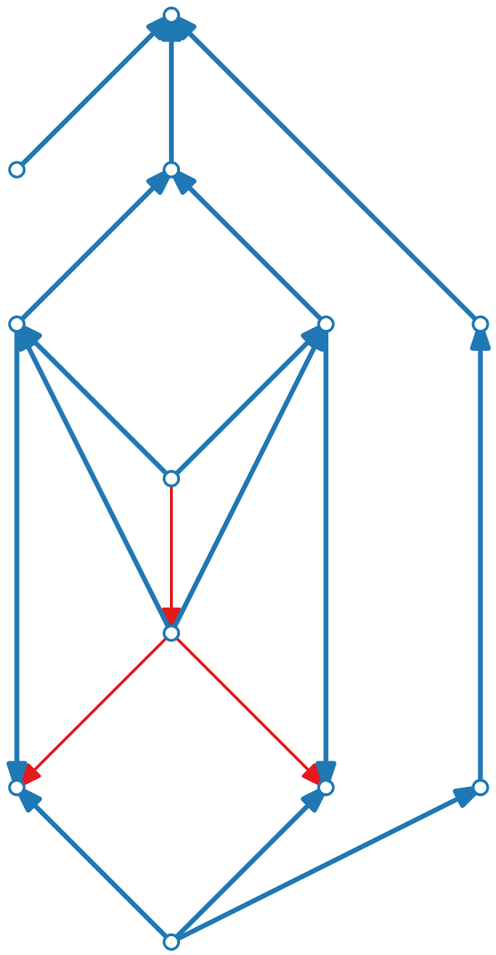
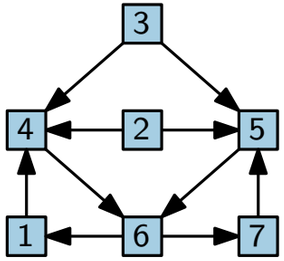
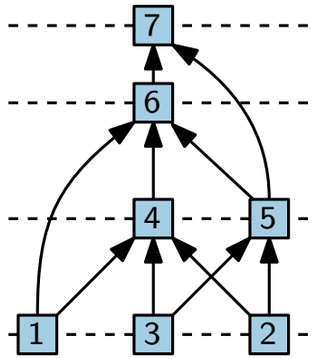
- Time: $\mathcal{O}(n+m)$
- Quality guarantee: $|E'| \geq |E|/2 + |V|/6$



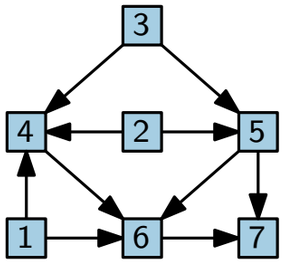
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

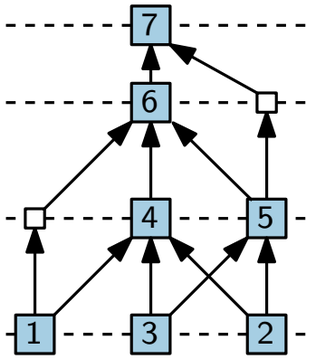


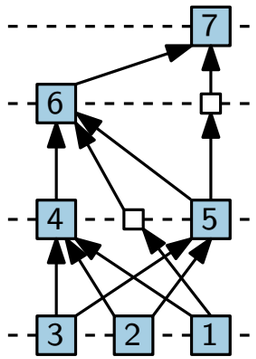
Visualization of Graphs



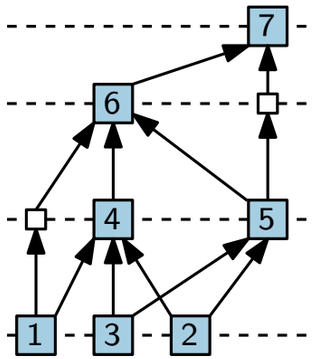
Lecture 7: Hierarchical Layouts: Sugiyama Framework





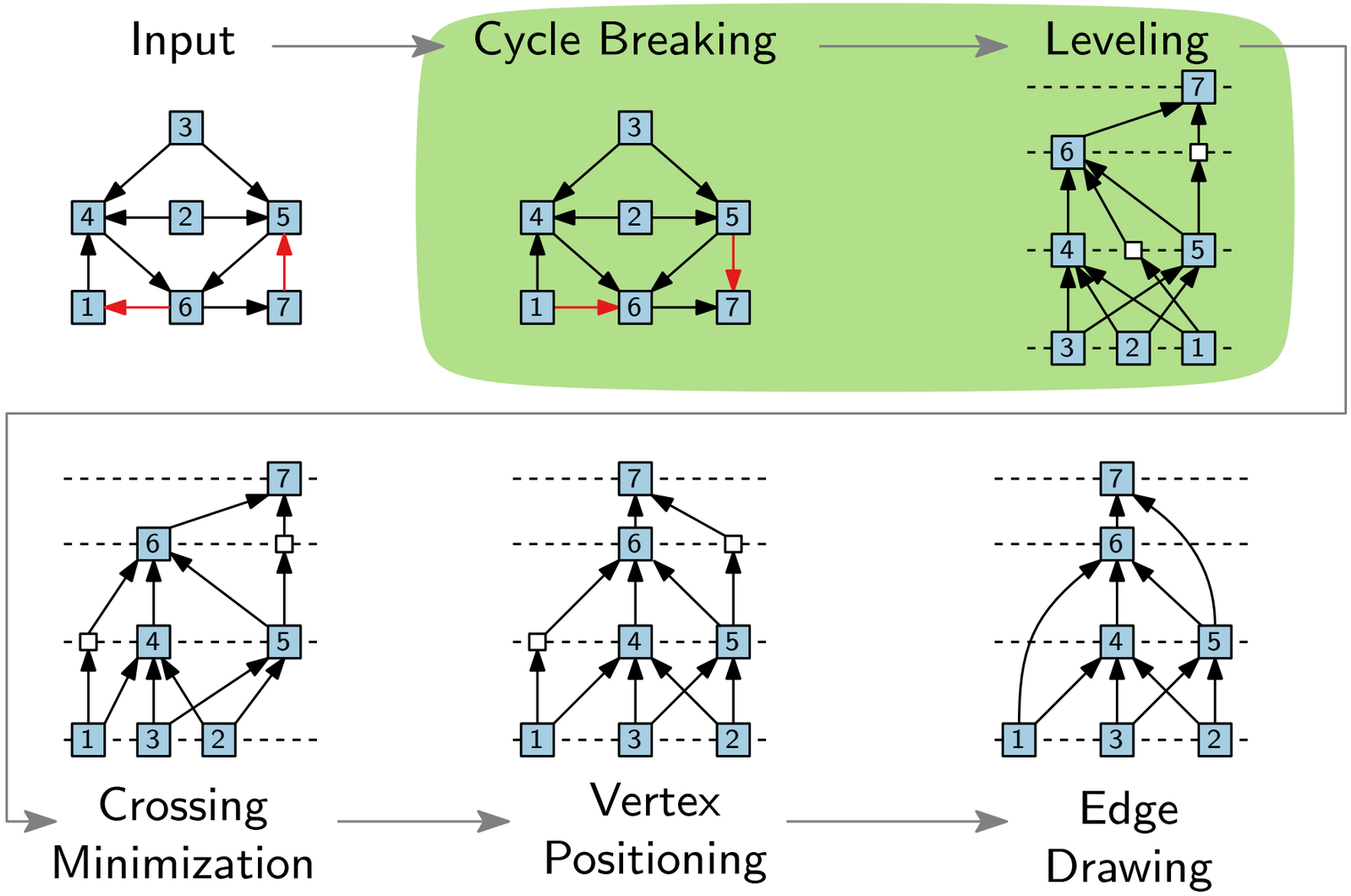


Part III:
Leveling

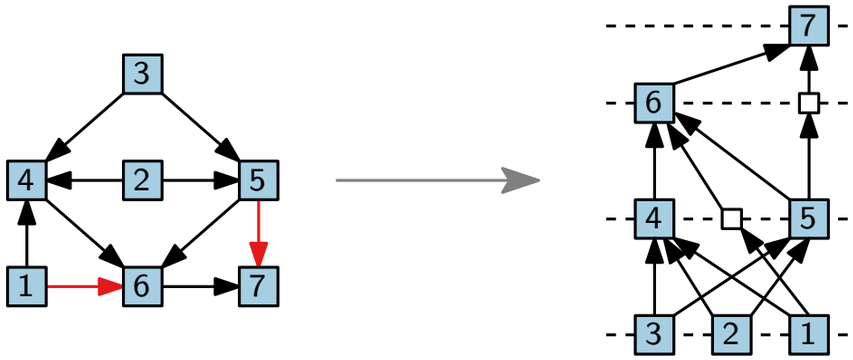


Jonathan Klawitter

Step 2: Leveling

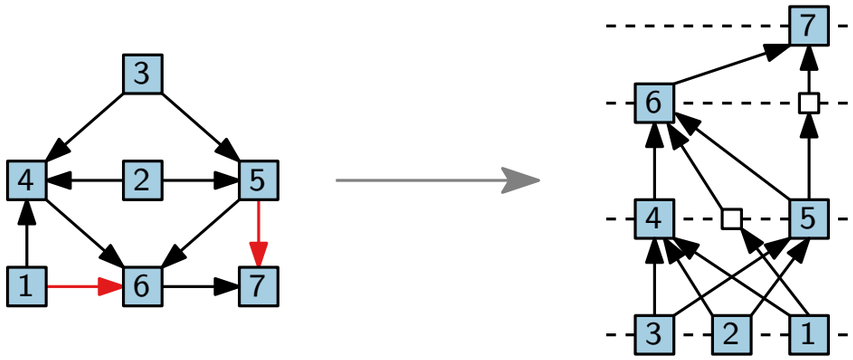


Step 2: Leveling



Problem.

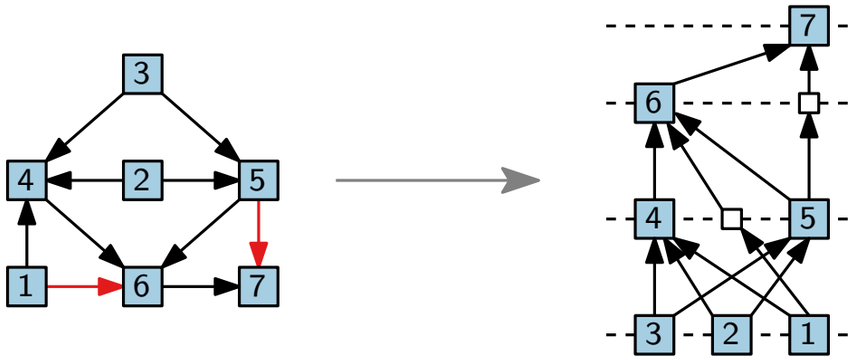
Step 2: Leveling



Problem.

- Input: acyclic digraph $G = (V, E)$

Step 2: Leveling



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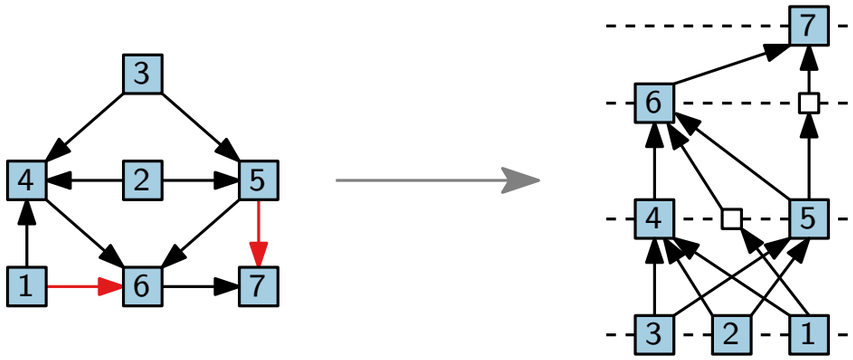
- Input: acyclic digraph $G = (V, E)$
- Output: Mapping $y\colon V \to \{1, \ldots n\}$,
  so that for every $uv \in E$, $y(u) < y(v)$.

Step 2: Leveling



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Objective is to *minimize* . . .

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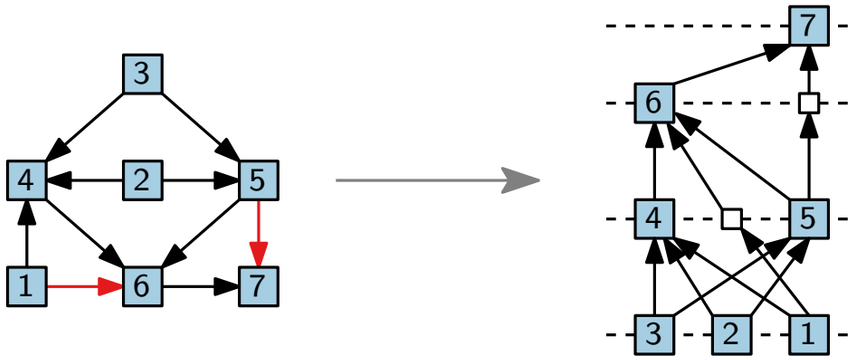

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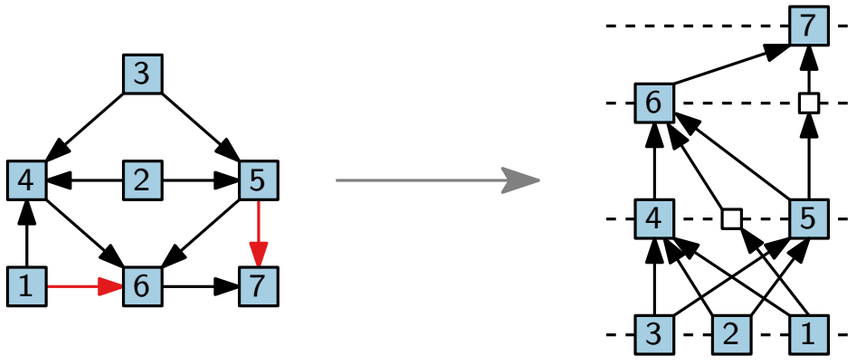

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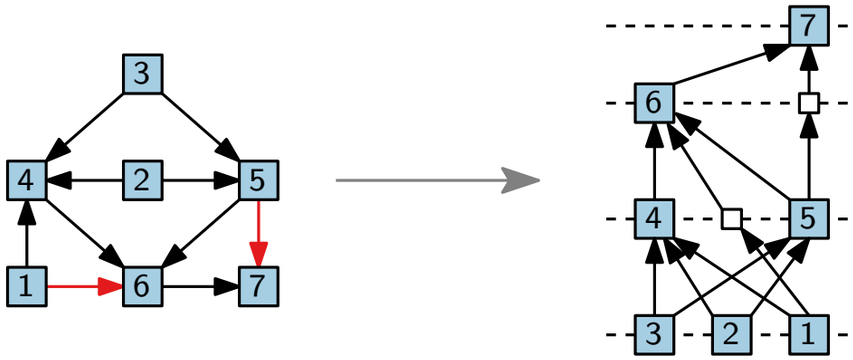
Objective is to *minimize ...*

- number of layers, i.e. $|y(V)|$

Step 2: Leveling



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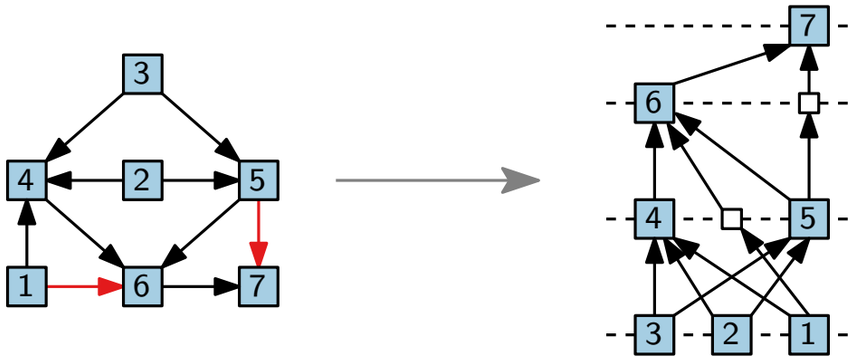
Objective is to *minimize* . . .

- number of layers, i.e. $|y(V)|$
- length of the longest edge,

Step 2: Leveling



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Objective is to *minimize* . . .

- number of layers, i.e. $|y(V)|$
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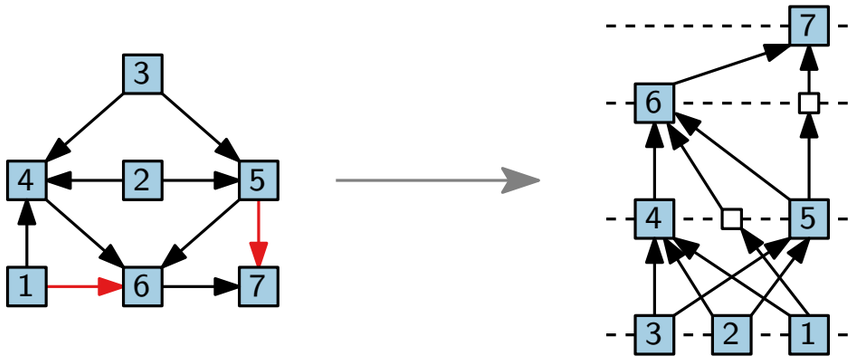
Step 2: Leveling



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- number of layers, i.e. $|y(V)|$
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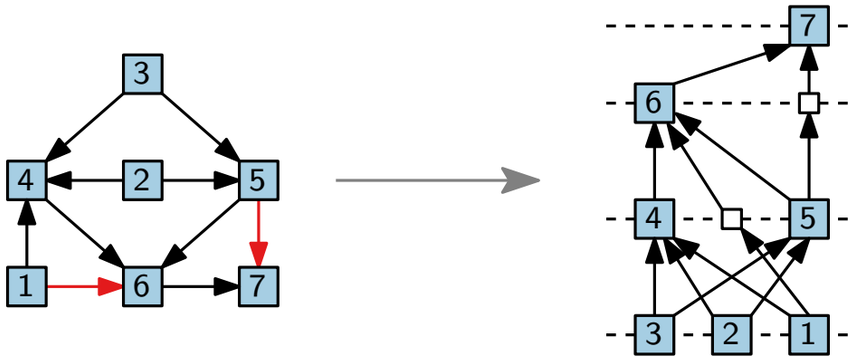
- width,

Step 2: Leveling



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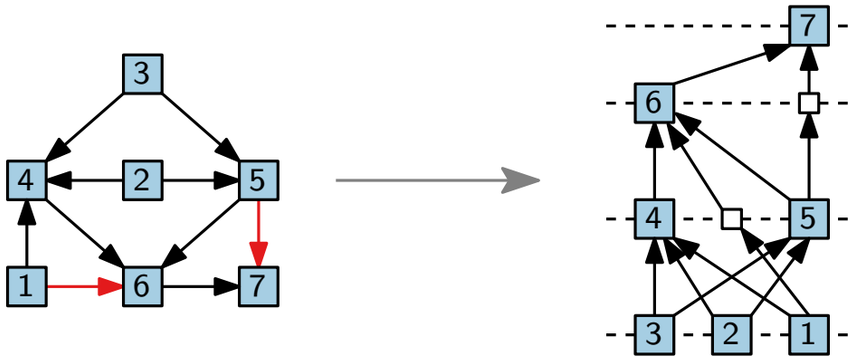
Step 2: Leveling



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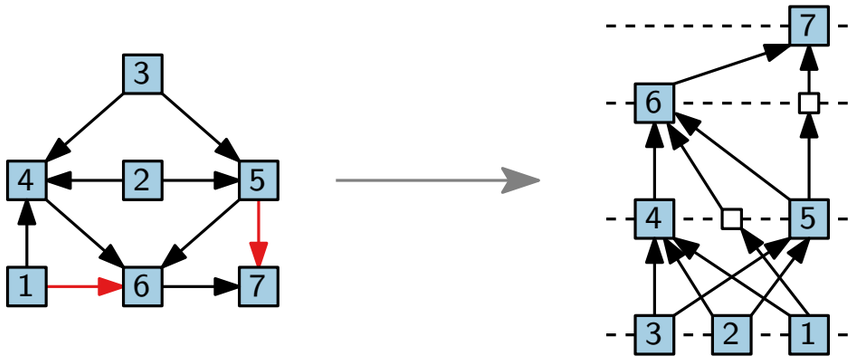
- total edge length,

Step 2: Leveling



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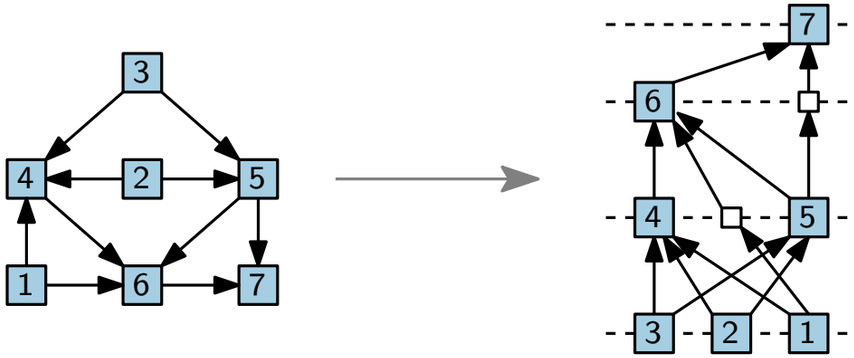
Objective is to *minimize* ...

- number of layers, i.e. $|y(V)|$
- length of the longest edge, i.e. $\max_{uv \in E} y(v) - y(u)$
- width, i.e. $\max\{|L_i| \mid 1 \leq i \leq h\}$
- total edge length, i.e. number of dummy vertices

Min Number of Layers

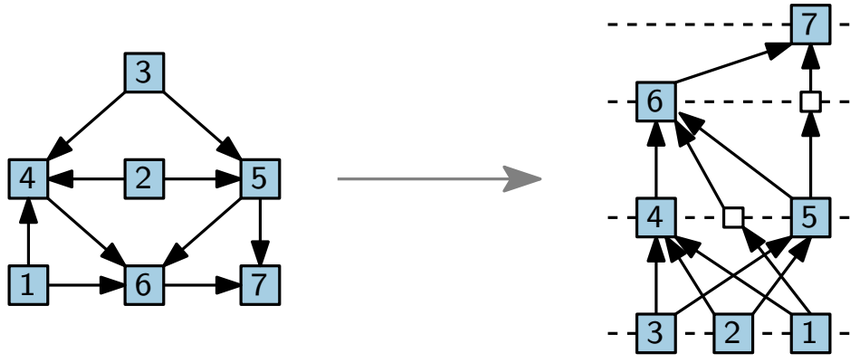
Algorithm.



Min Number of Layers

Algorithm.

- for each source q
  set $y(q) := 1$



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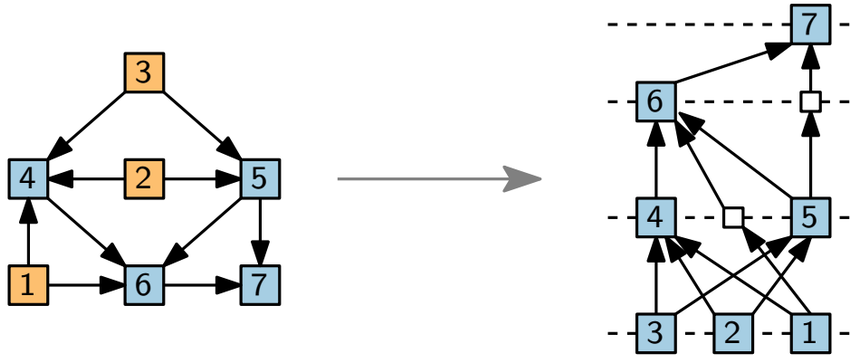

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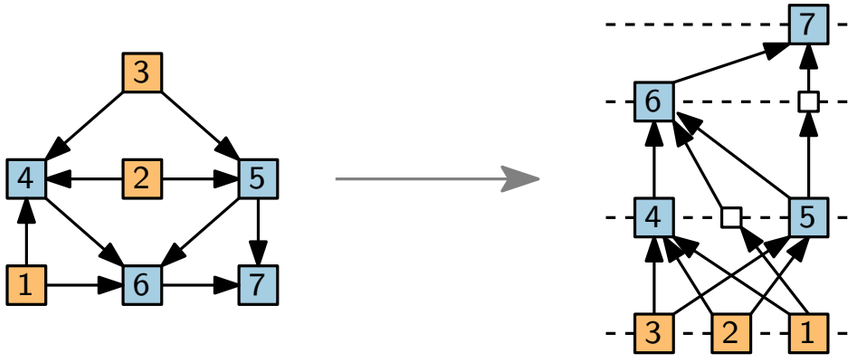
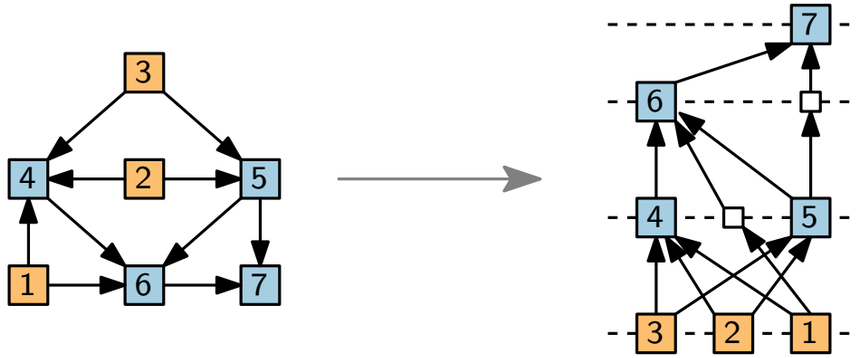


Min Number of Layers

Algorithm.

- for each source q
  set $y(q) := 1$
- for each non-source v
  set $y(v) := \max\{y(u) \mid uv \in E\} + 1$



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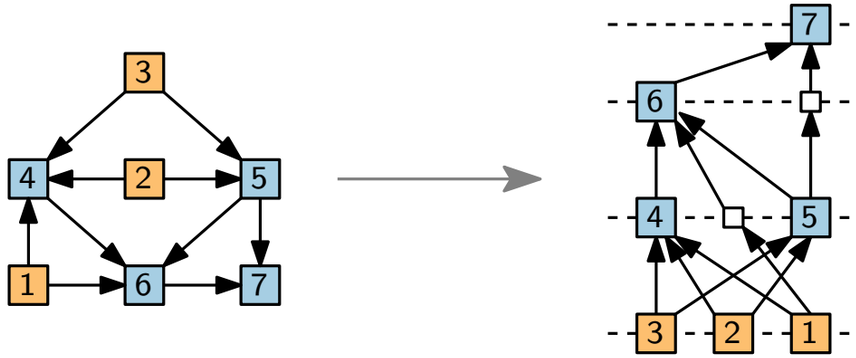

Observation.

- $y(v)$

Min Number of Layers

Algorithm.

- for each source q
  set $y(q) := 1$
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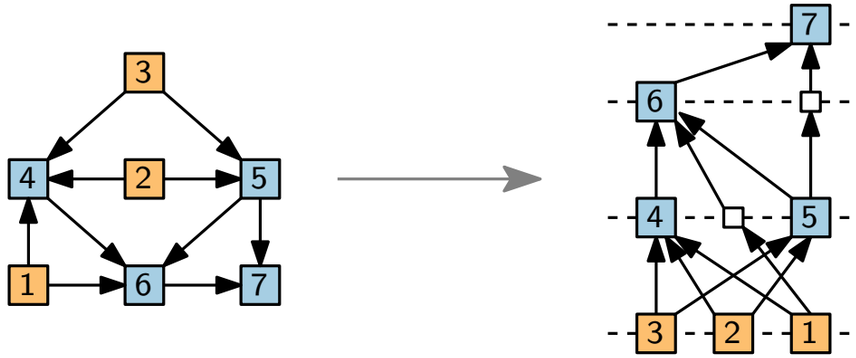

Observation.

- $y(v)$ is length of the longest path from a source to v plus 1.

Min Number of Layers

Algorithm.

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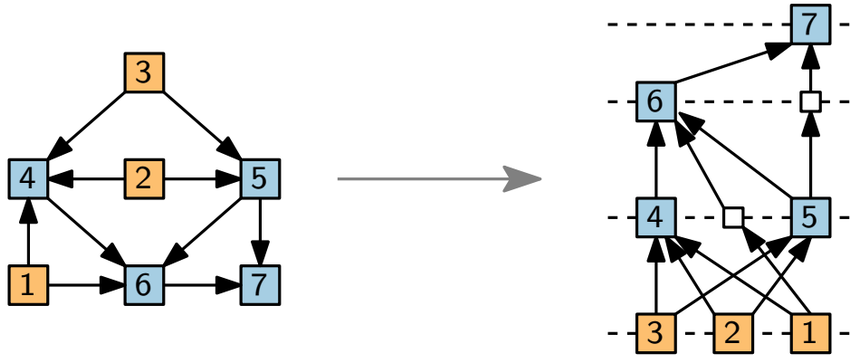
  set $y(v) := \max\{y(u) \mid uv \in E\} + 1$



Observation.

- $y(v)$ is length of the longest path from a source to v plus 1.

  ...which is optimal!

Min Number of Layers

Algorithm.

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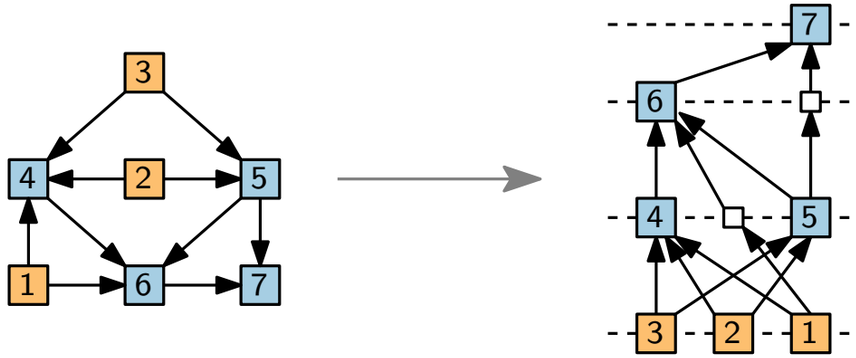
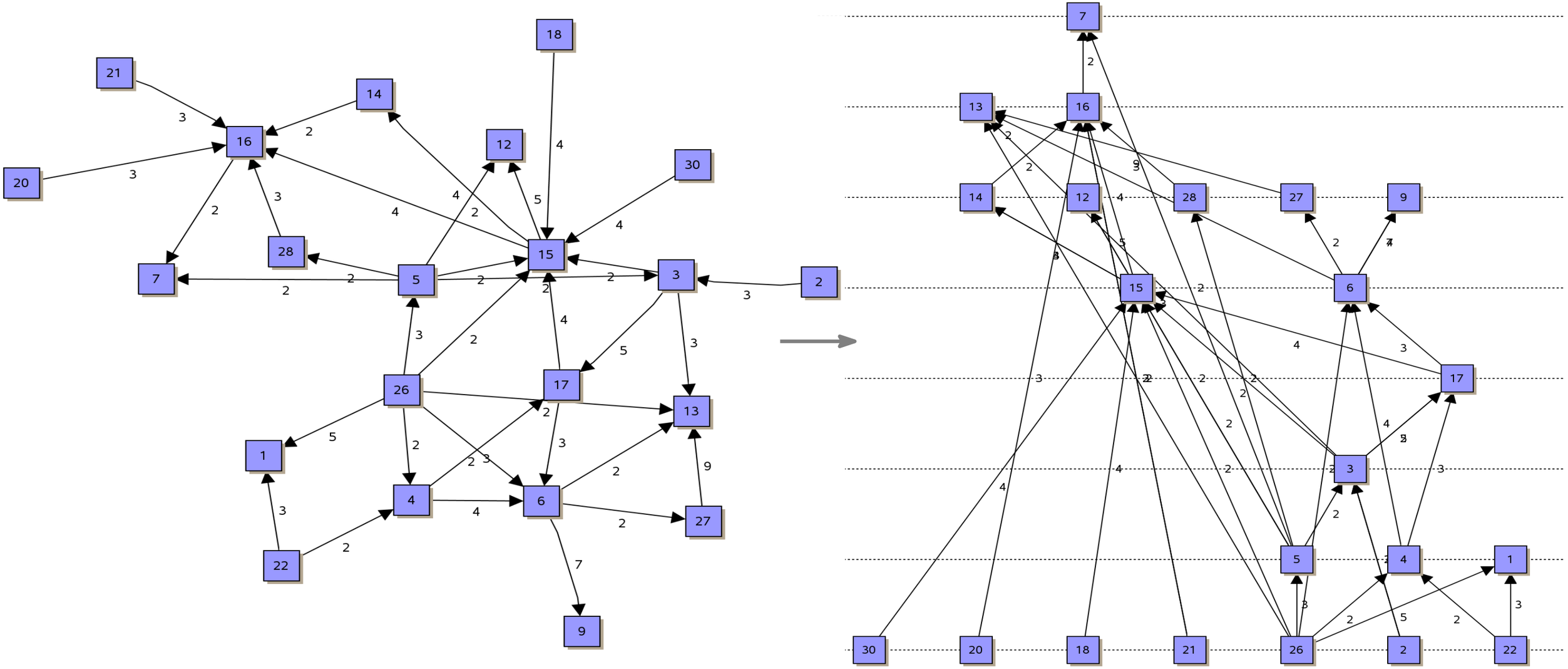


Observation.

- $y(v)$ is length of the longest path from a source to v plus 1.

  ...which is optimal!
- Can be implemented in linear time with recursive algorithm.

Example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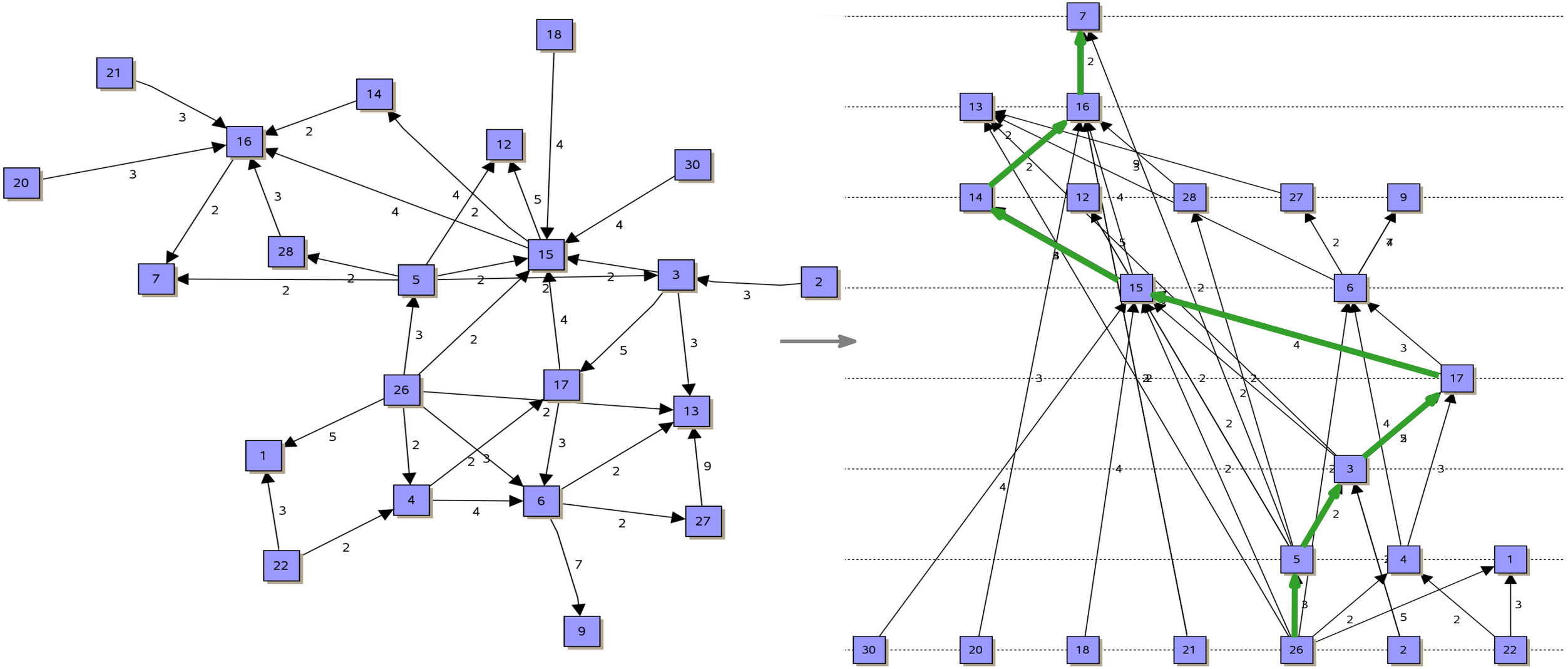
Example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Total Edge Length – ILP

Can be formulated as an integer linear program:

Total Edge Length – ILP

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$$\text{min} \quad \sum_{(u,v)\in E}(y(v) - y(u))$$

Total Edge Length – ILP

Can be formulated as an integer linear program:

$$\begin{aligned} \text{min} \quad & \textstyle\sum_{(u,v)\in E}(y(v) - y(u)) \\ \text{subject to} \quad & \end{aligned}$$

Total Edge Length – ILP

Can be formulated as an integer linear program:

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Total Edge Length – ILP

Can be formulated as an integer linear program:

$$
\begin{array}{rll}
\text{min} & \sum_{(u,v)\in E}(y(v)-y(u)) & \\
\text{subject to} & y(v)-y(u)\geq 1 & \forall (u,v)\in E \\
 & y(v)\geq 1 & \forall v\in V
\end{array}
$$

Total Edge Length – ILP

Can be formulated as an integer linear program:

$$
\begin{array}{rll}
\text{min} & \sum_{(u,v)\in E}(y(v)-y(u)) & \\
\text{subject to} & y(v)-y(u)\geq 1 & \forall (u,v)\in E \\
 & y(v)\geq 1 & \forall v\in V \\
 & y(v)\in \mathbb{Z} & \forall v\in V
\end{array}
$$

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\end{array}
$$

One can show that:

Total Edge Length – ILP

Can be formulated as an integer linear program:

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\end{array}
$$

One can show that:

- Constraint-matrix is **totally unimodular**

Total Edge Length – ILP

Can be formulated as an integer linear program:

$$\begin{array}{rll} \text{min} & \sum_{(u,v)\in E}(y(v)-y(u)) & \\ \text{subject to} & y(v)-y(u)\geq 1 & \forall (u,v)\in E \\ & y(v)\geq 1 & \forall v\in V \\ & y(v)\in\mathbb{Z} & \forall v\in V \end{array}$$

One can show that:

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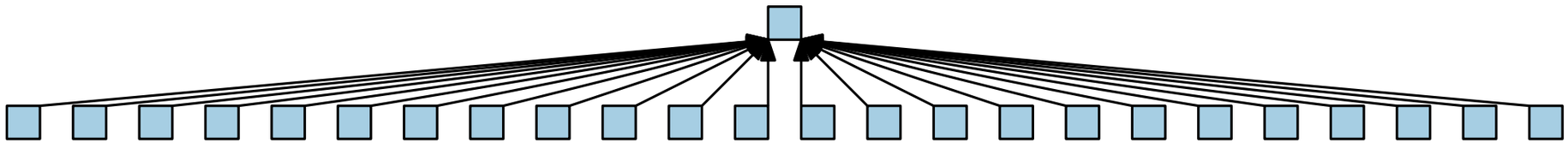
- The total edge length can be minimized in polynomial time

Width



Drawings can be very wide.

Narrower Layer Assignment

Problem: Leveling With a Given Width.

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Approximating PCMPS

- jobs stored in a list L
  (in any order, e.g., topologically sorted)

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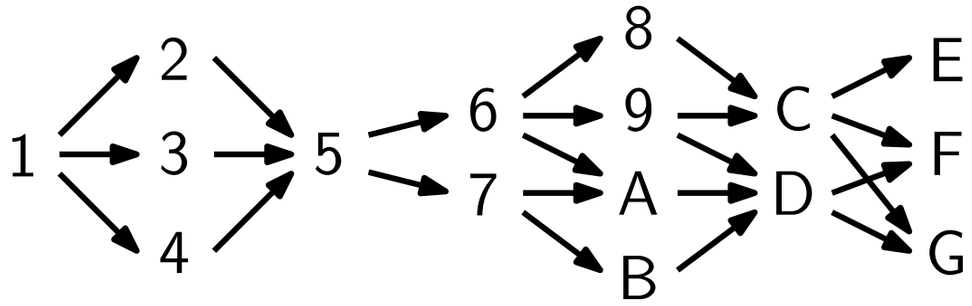
- a job in L is *available* when all its predecessors have been scheduled
- as long as there are free machines and available jobs, take the first available job and assign it to a free machine

Approximating PCMPS

Input: Precedence graph (divided into layers of arbitrary width)



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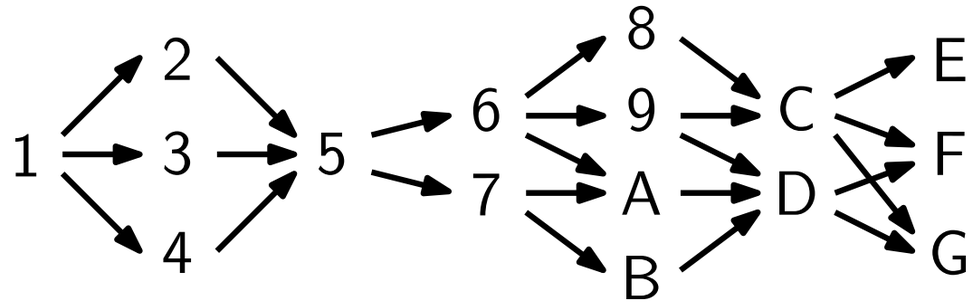

Number of Machines is $W = 2$.

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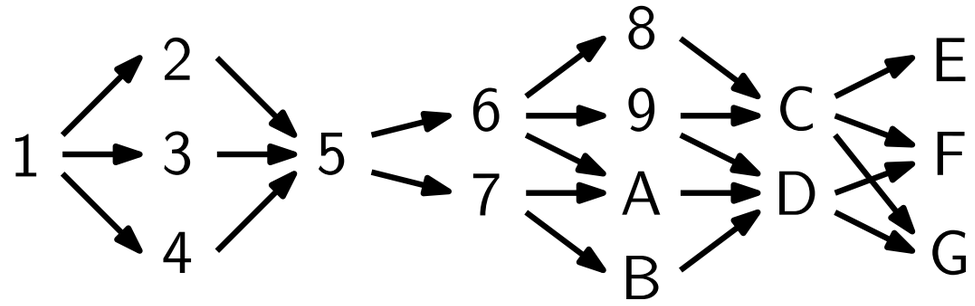
Input: Precedence graph (divided into layers of arbitrary width)



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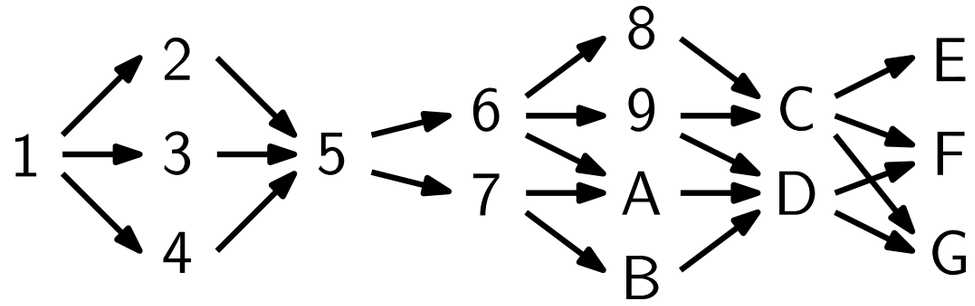

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Output: Schedule

M_1										
M_2										
t	1	2	3	4	5	6	7	8	9	10

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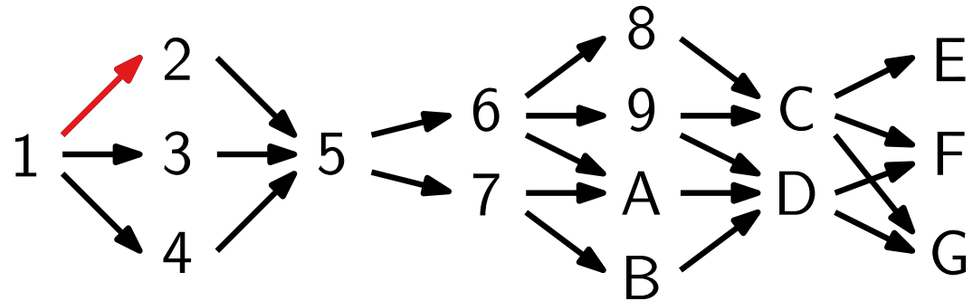

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Output: Schedule

M_1	1									
M_2	–									
t	1	2	3	4	5	6	7	8	9	10

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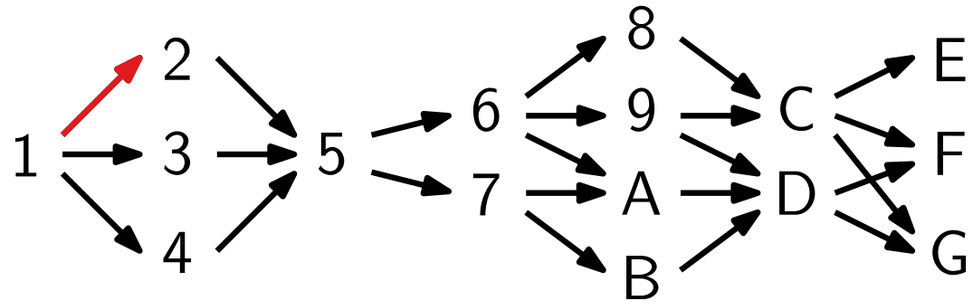

Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2								
M_2	–	3								
t	1	2	3	4	5	6	7	8	9	10

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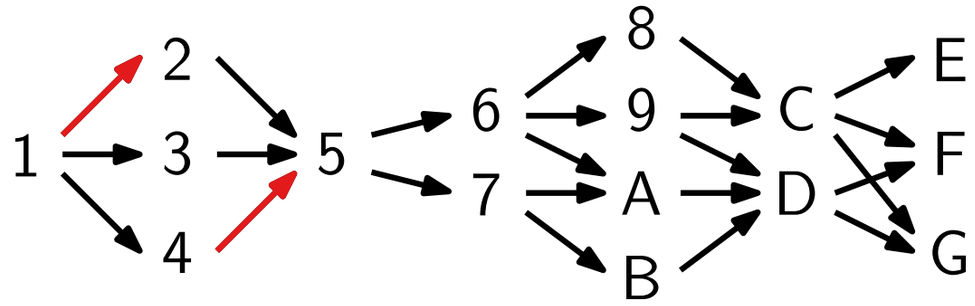

Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4							
M_2	–	3	–							
t	1	2	3	4	5	6	7	8	9	10

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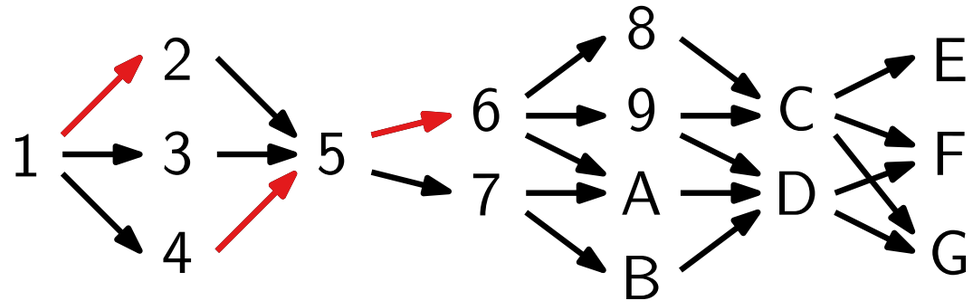

Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5						
M_2	–	3	–	–						
t	1	2	3	4	5	6	7	8	9	10

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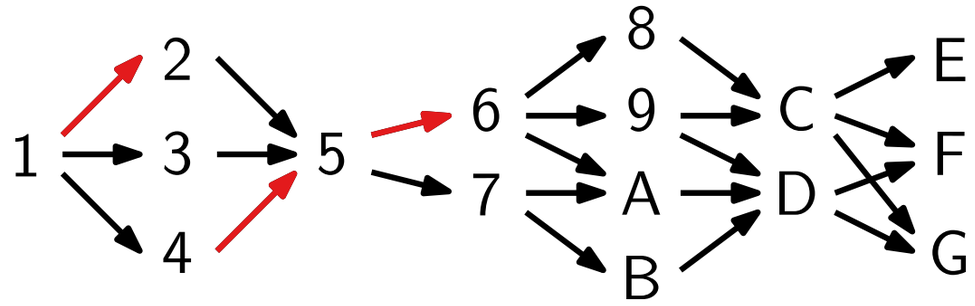
Input: Precedence graph (divided into layers of arbitrary width)



Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6					
M_2	–	3	–	–	7					
t	1	2	3	4	5	6	7	8	9	10

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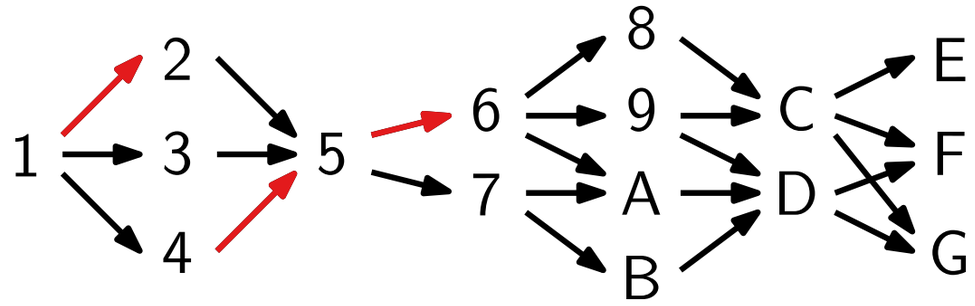

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Output: Schedule

M_1	1	2	4	5	6	8				
M_2	–	3	–	–	7	9				
t	1	2	3	4	5	6	7	8	9	10

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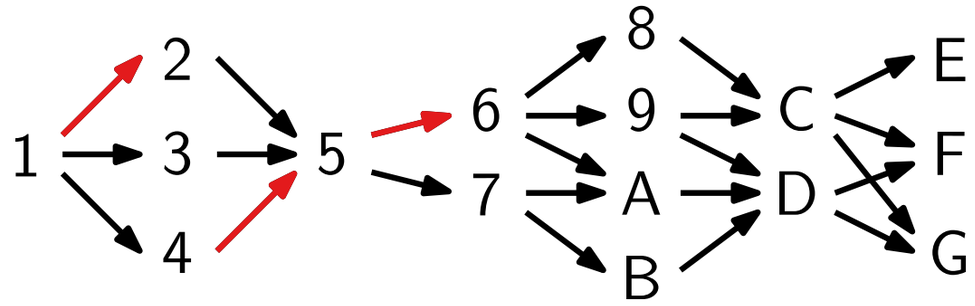

Number of Machines is $W = 2$.

Output: Schedule

M_1	1	2	4	5	6	8	A			
M_2	–	3	–	–	7	9	B			
t	1	2	3	4	5	6	7	8	9	10

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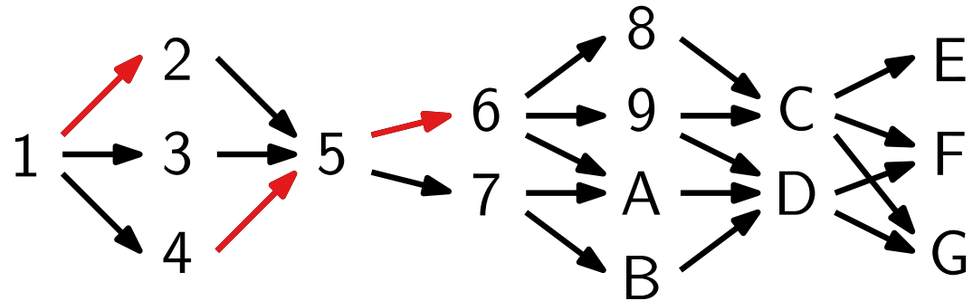

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Output: Schedule

M_1	1	2	4	5	6	8	A	C		
M_2	–	3	–	–	7	9	B	D		
t	1	2	3	4	5	6	7	8	9	10

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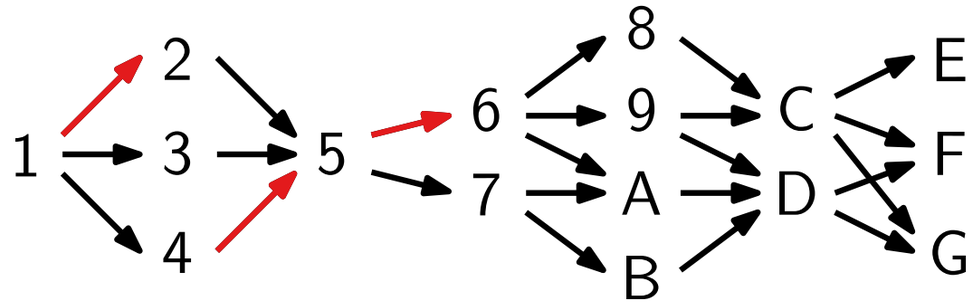

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M_1	1	2	4	5	6	8	A	C	E	
M_2	–	3	–	–	7	9	B	D	F	
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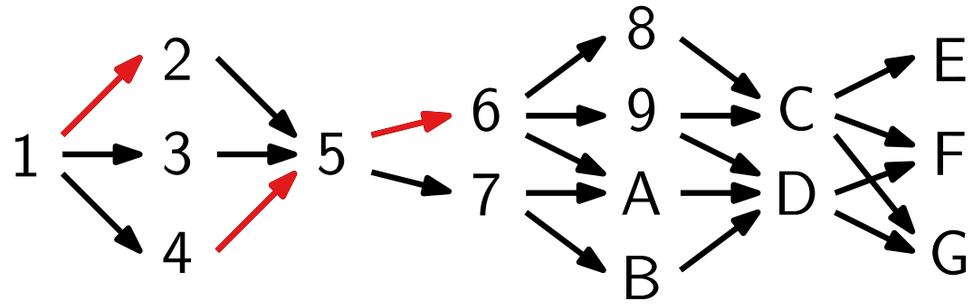

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Output: Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
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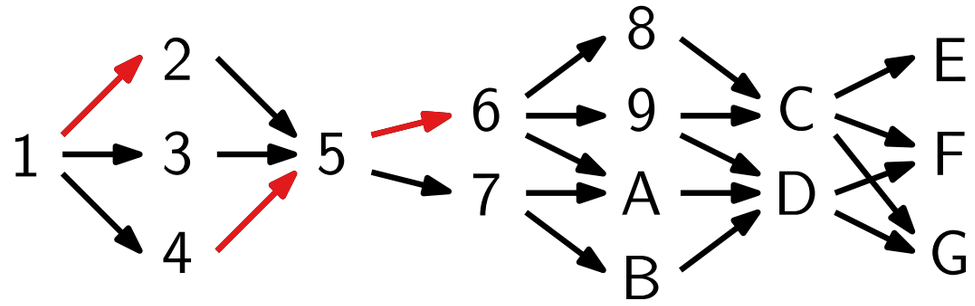

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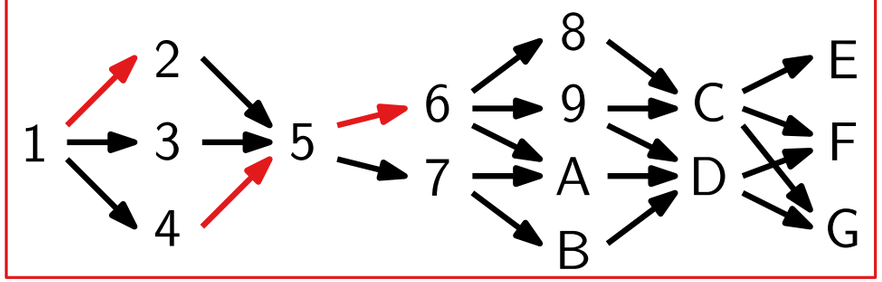
Question: Good approximation factor?

Approximating PCMPS - Analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

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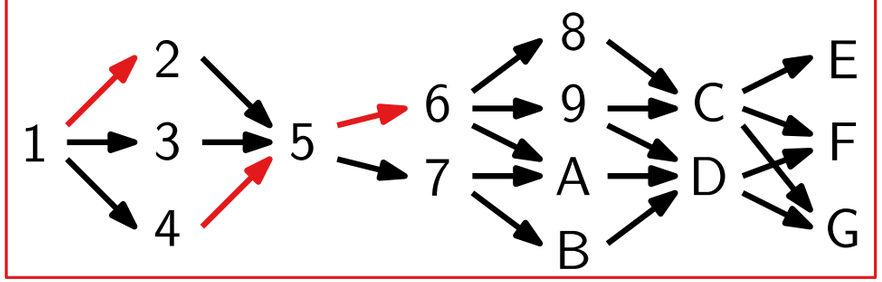

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OPT $\geq$

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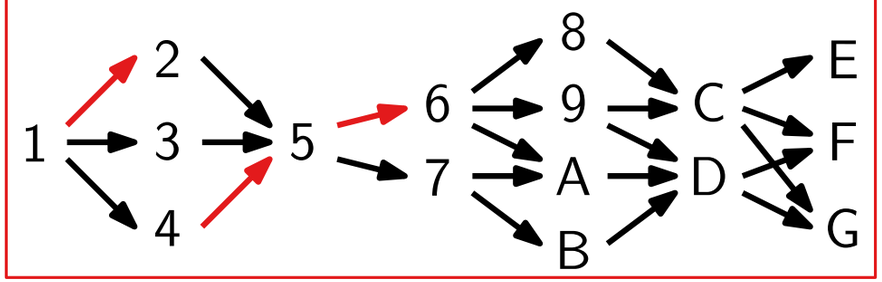

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„The art of the lower bound“

OPT $\geq \lceil n/2 \rceil$

Approximating PCMPS - Analysis for $W = 2$

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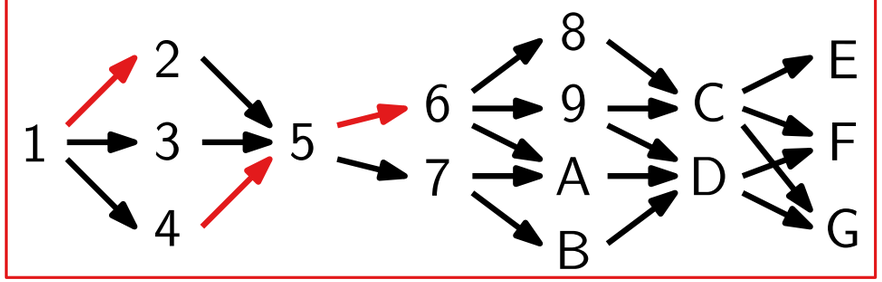

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OPT $\geq \lceil n/2 \rceil$ and OPT $\geq$

Approximating PCMPS - Analysis for $W = 2$

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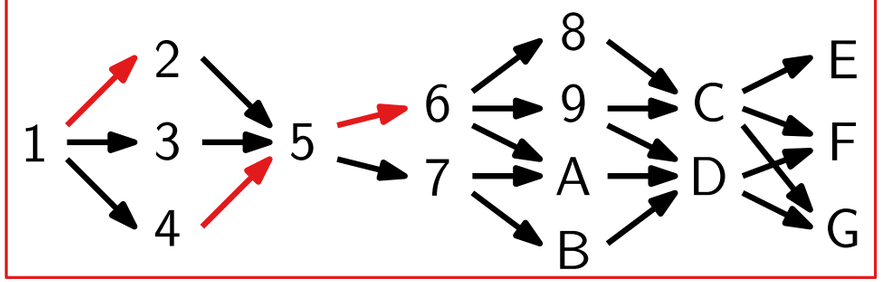

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„The art of the lower bound“

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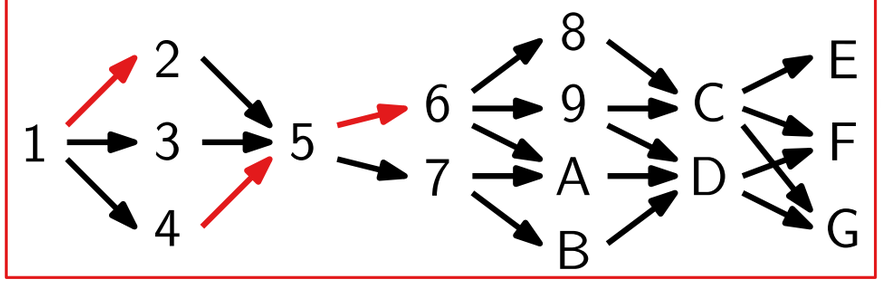
Approximating PCMPS - Analysis for $W = 2$

Precedence graph $G_<$



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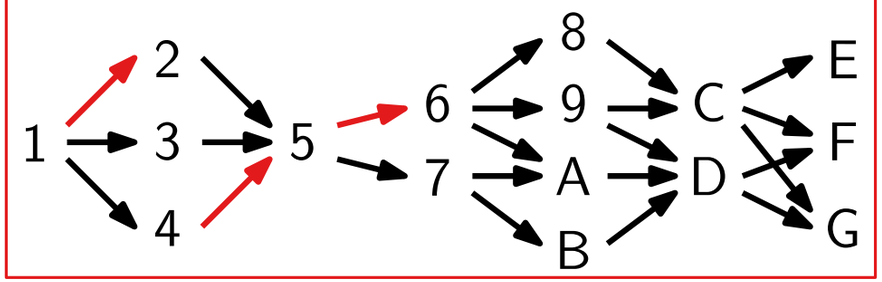
Goal: measure the quality of our algorithm using the lower bounds

Approximating PCMPS - Analysis for $W = 2$

Precedence graph $G_<$



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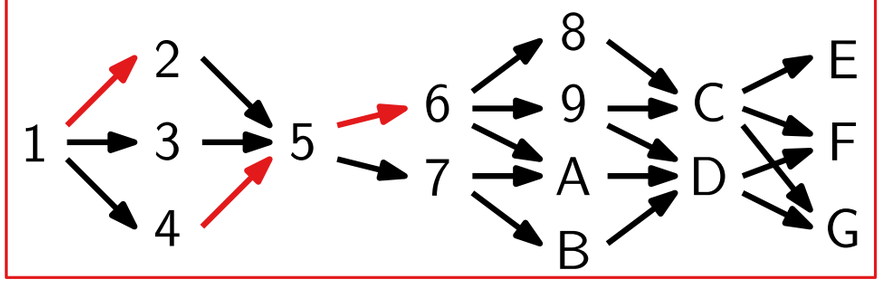
Bound. ALG $\leq$

Approximating PCMPS - Analysis for $W = 2$

Precedence graph $G_<$



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Approximating PCMPS - Analysis for $W = 2$

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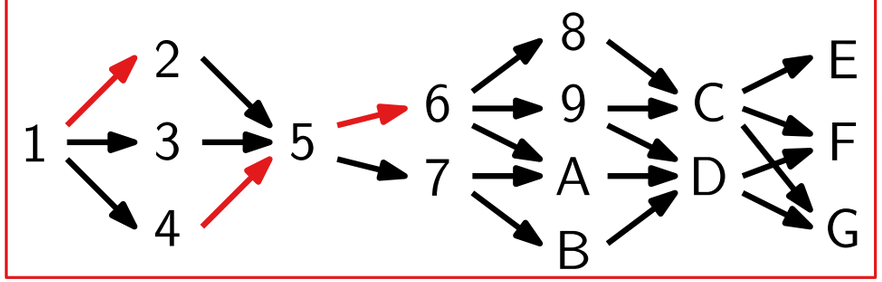

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M_2	–	3	–	–	7	9	B	D	F	–
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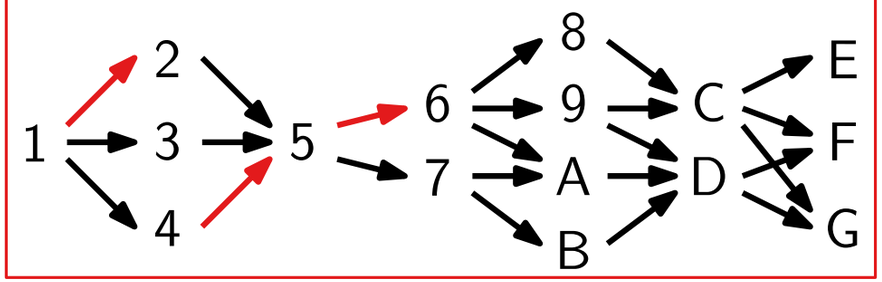
Bound. ALG $\leq$

insertion of pauses (–) in the schedule (except the last) maps to layers of $G_<$

Approximating PCMPS - Analysis for $W = 2$

Precedence graph $G_<$



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OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

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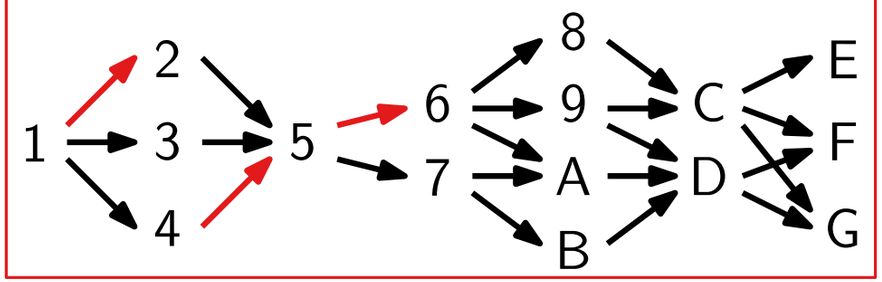
Bound. ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil$

insertion of pauses (–) in the schedule (except the last) maps to layers of $G_<$

Approximating PCMPS - Analysis for $W = 2$

Precedence graph $G_<$



Schedule

M_1	1	2	4	5	6	8	A	C	E	G
M_2	–	3	–	–	7	9	B	D	F	–
t	1	2	3	4	5	6	7	8	9	10

„The art of the lower bound“

OPT $\geq \lceil n/2 \rceil$ and OPT $\geq \ell :=$ Number of layers of $G_<$

Goal: measure the quality of our algorithm using the lower bounds

Bound. ALG $\leq \left\lceil \frac{n+\ell}{2} \right\rceil \approx$

insertion of pauses (–) in the schedule (except the last) maps to layers of $G_<$

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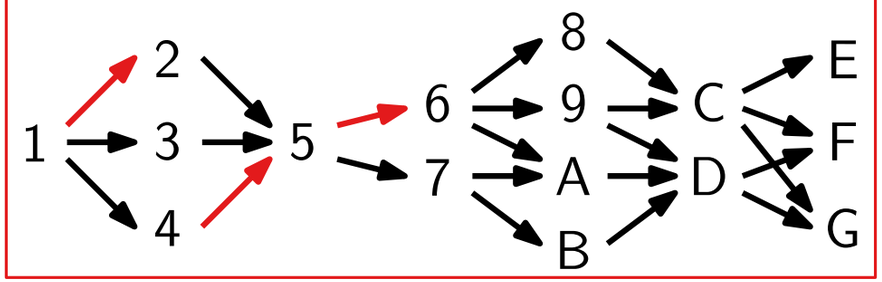

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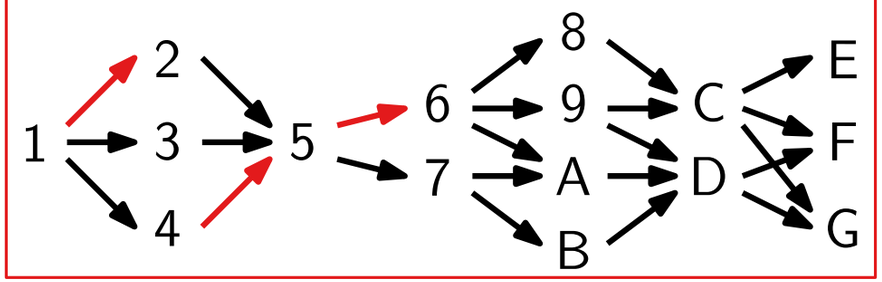

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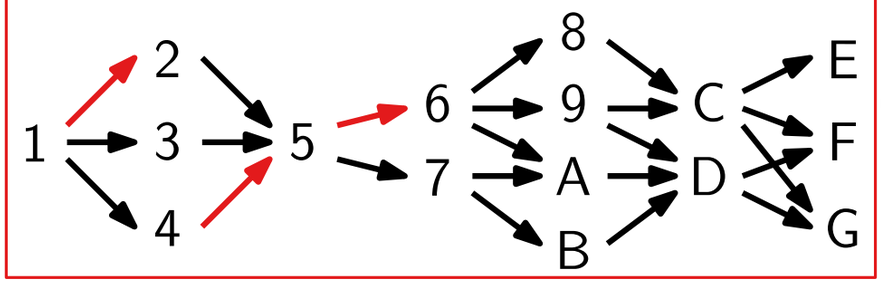

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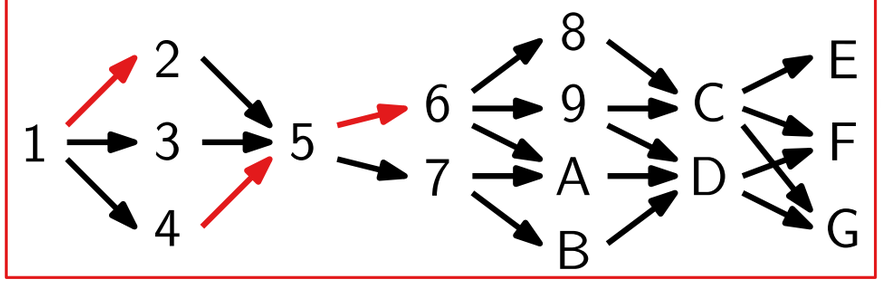

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$\leq (2 - 1/W) \cdot$ OPT in general case

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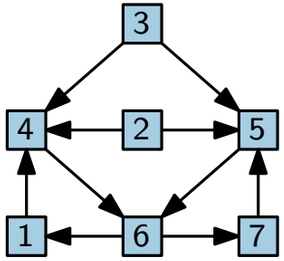
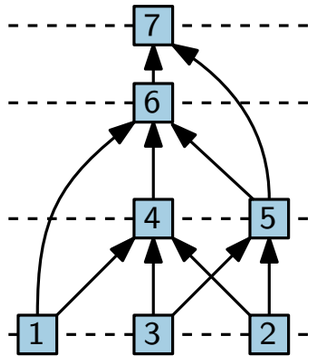
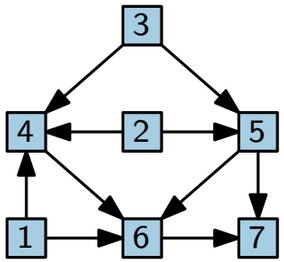
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

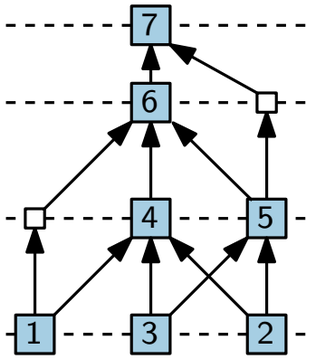


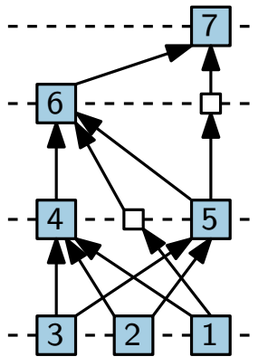
Visualization of Graphs



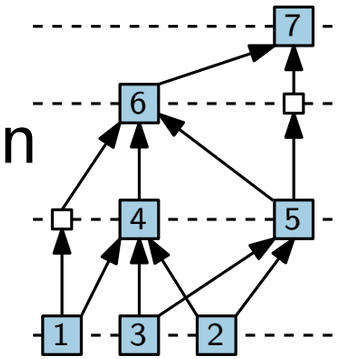
Lecture 7: Hierarchical Layouts: Sugiyama Framework





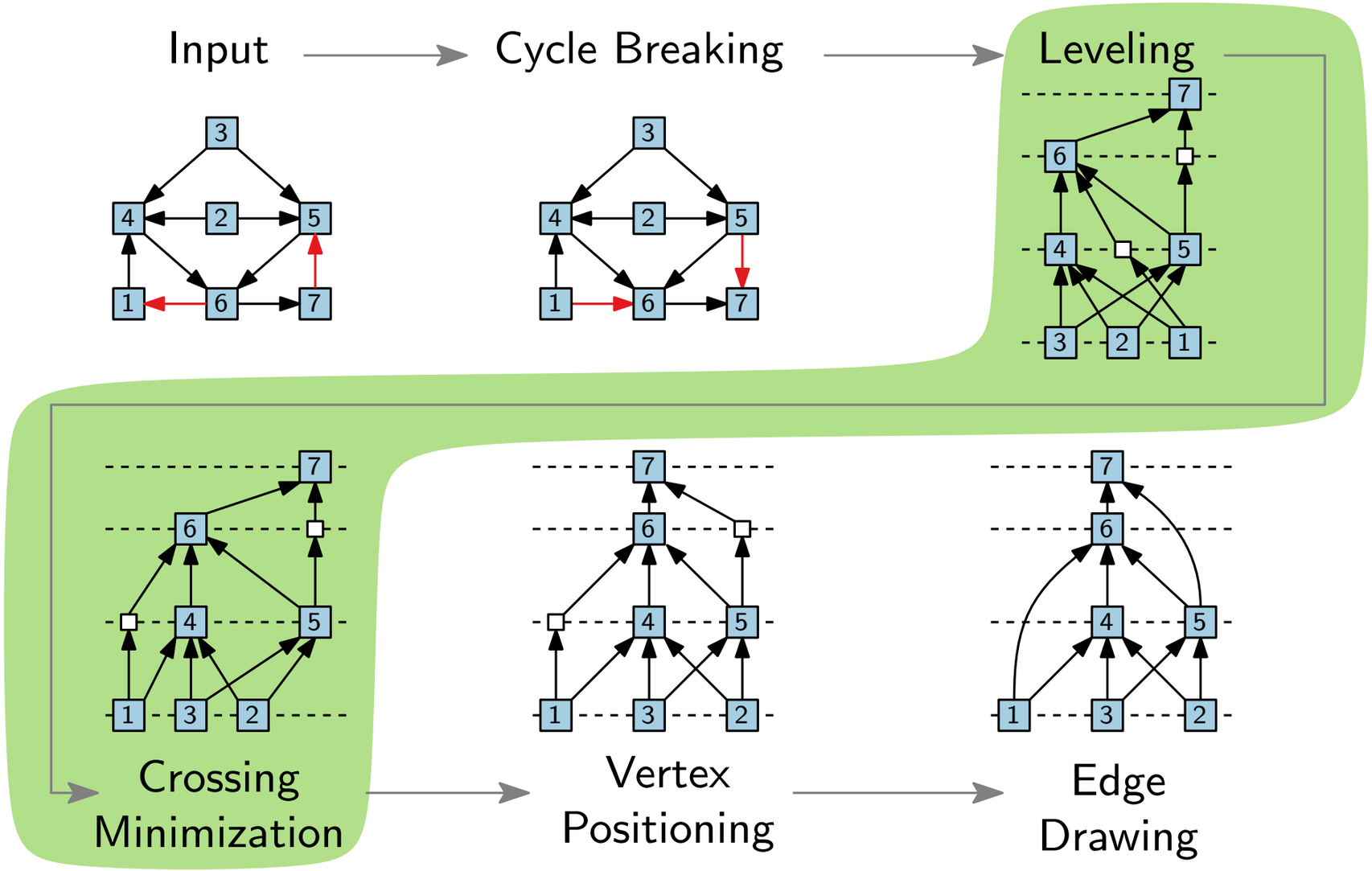


Part IV:
Crossing Minimisation



Jonathan Klawitter

Step 3: Crossing Minimization



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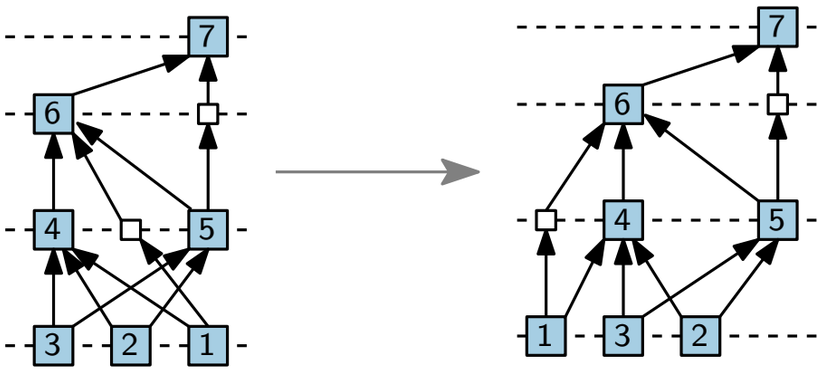
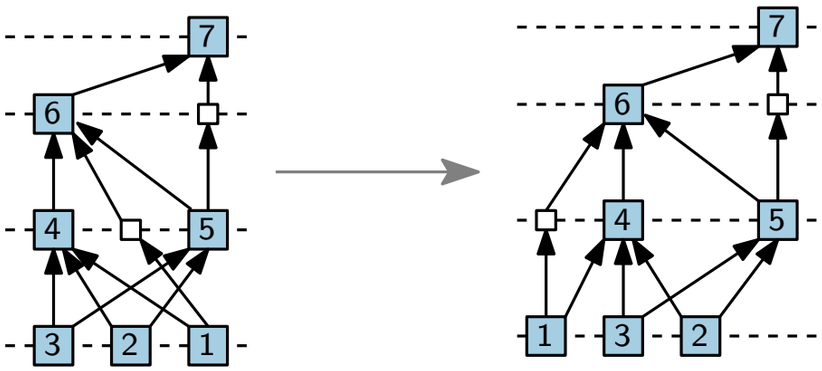


Problem.

Step 3: Crossing Minimization



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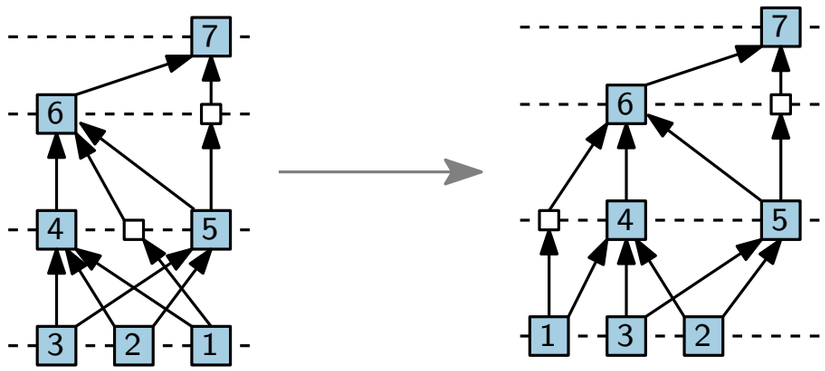

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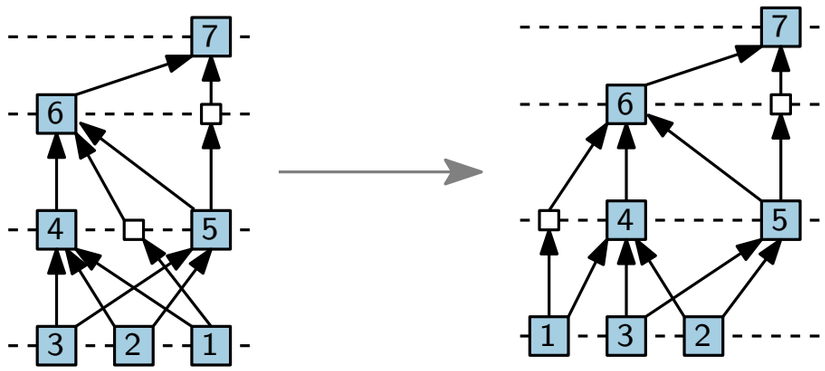

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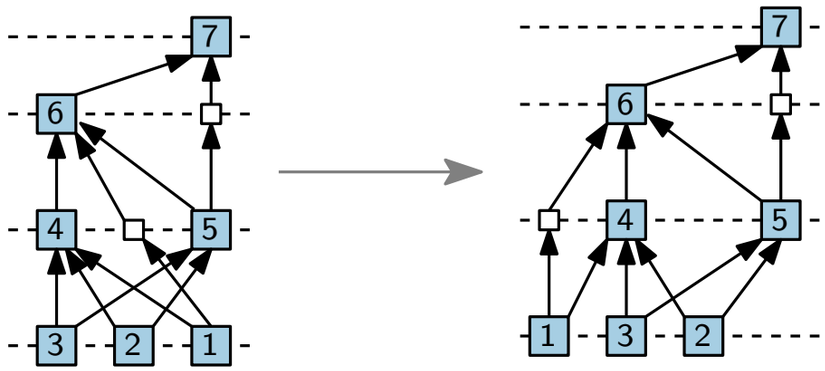

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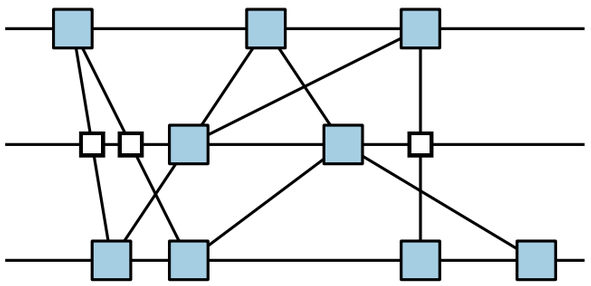
- hardly any approaches optimize over multiple layers :(

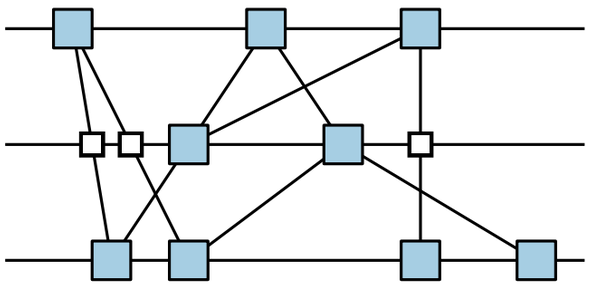
Iterative Crossing Reduction – Idea



Iterative Crossing Reduction – Idea

Observation.

The number of crossings only depends on permutations of adjacent layers.



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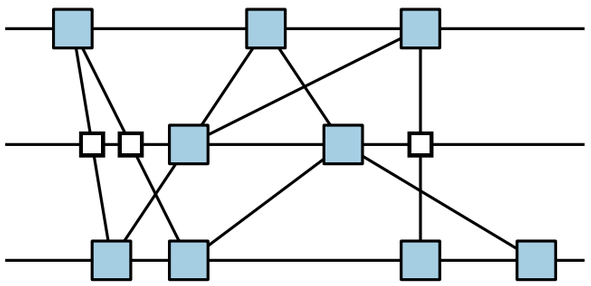

- Add dummy-vertices for edges connecting “far” layers.

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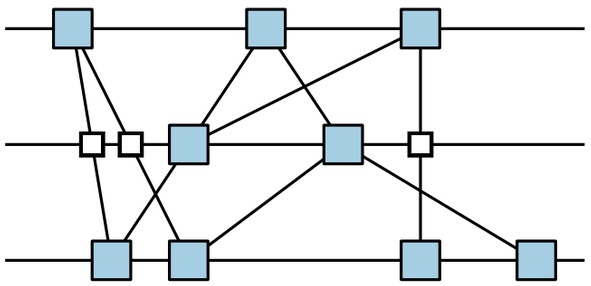

- Add dummy-vertices for edges connecting "far" layers.
- Consider adjacent layers $(L_1, L_2), (L_2, L_3), \ldots$ bottom-to-top.

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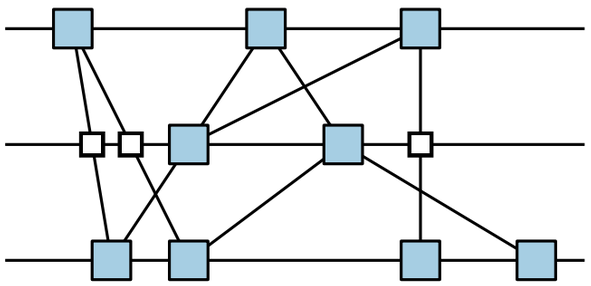

- Add dummy-vertices for edges connecting "far" layers.
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- Minimize crossings by permuting L_{i+1} while keeping L_i fixed.

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(1) choose a random permutation of L_1

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one-sided crossing minimization

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One-Sided Crossing Minimization

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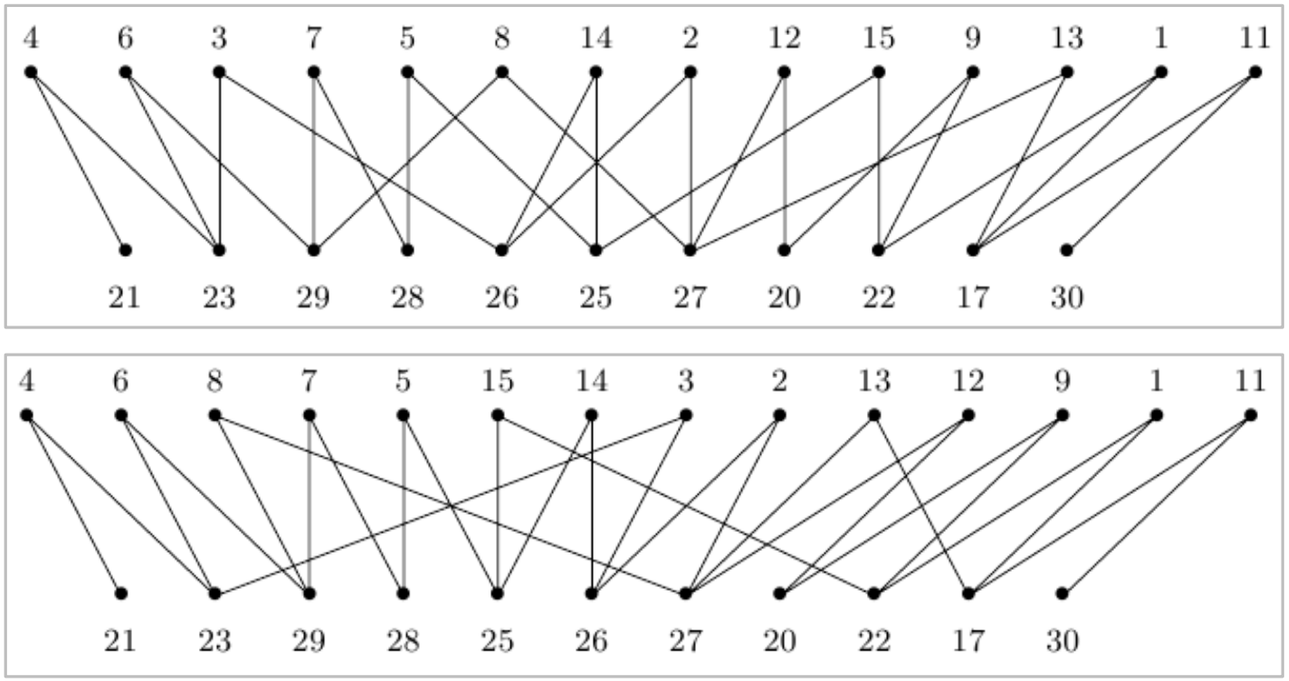

Abb. aus [Kaufmann und Wagner: Drawing Graphs]
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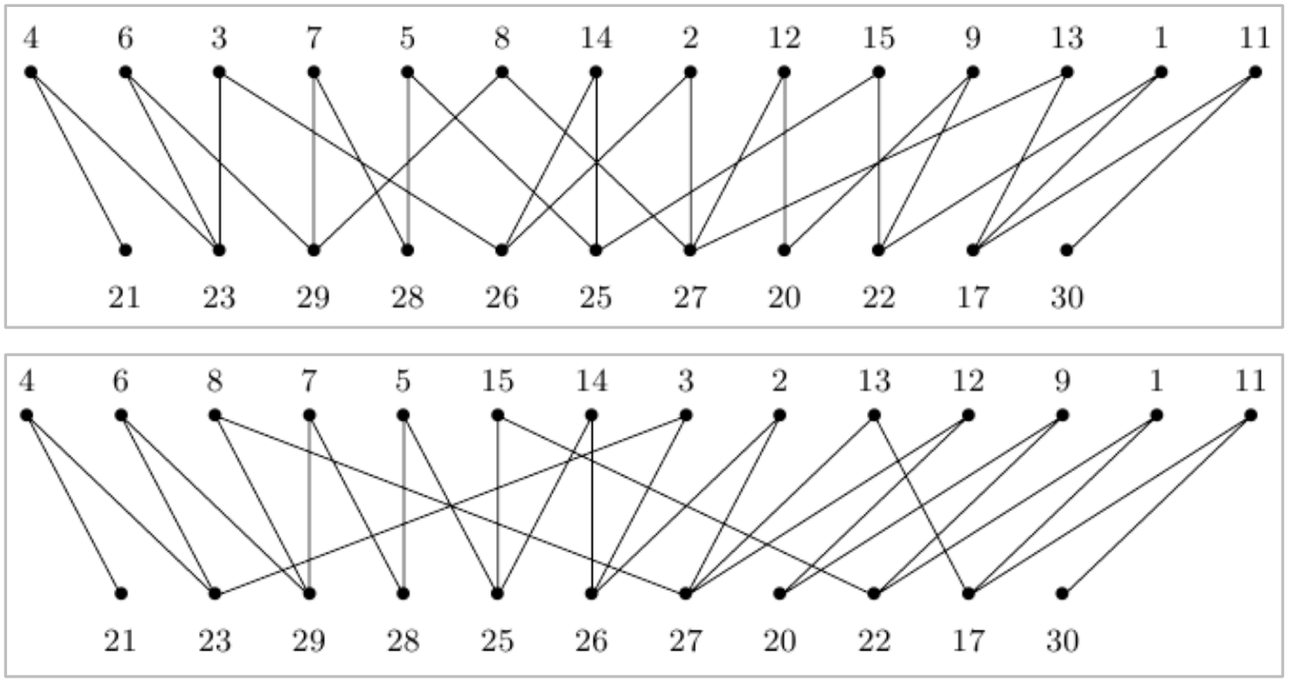

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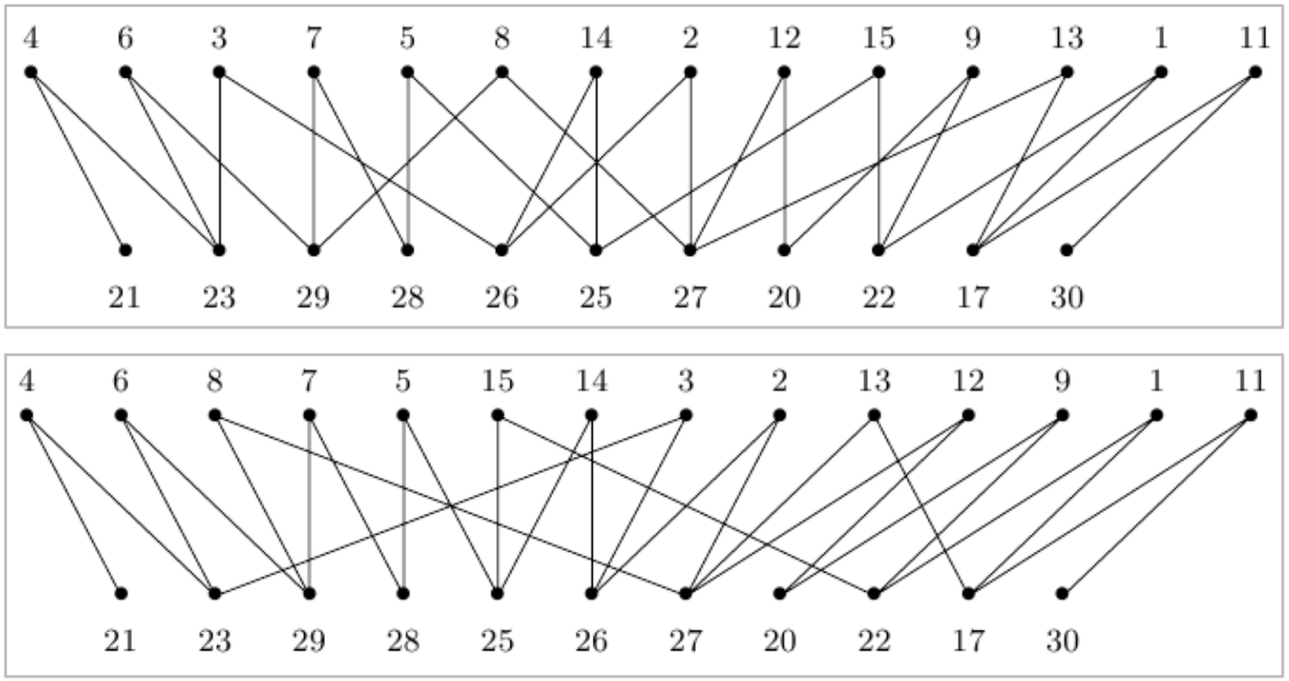

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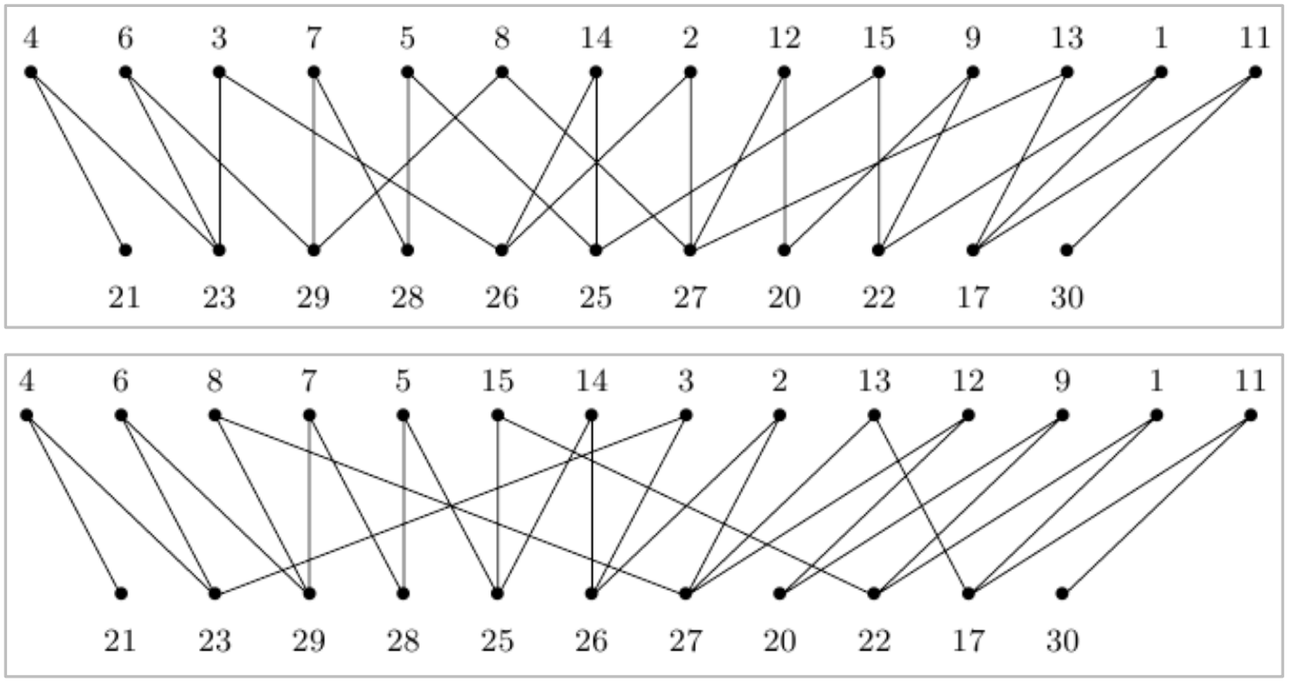

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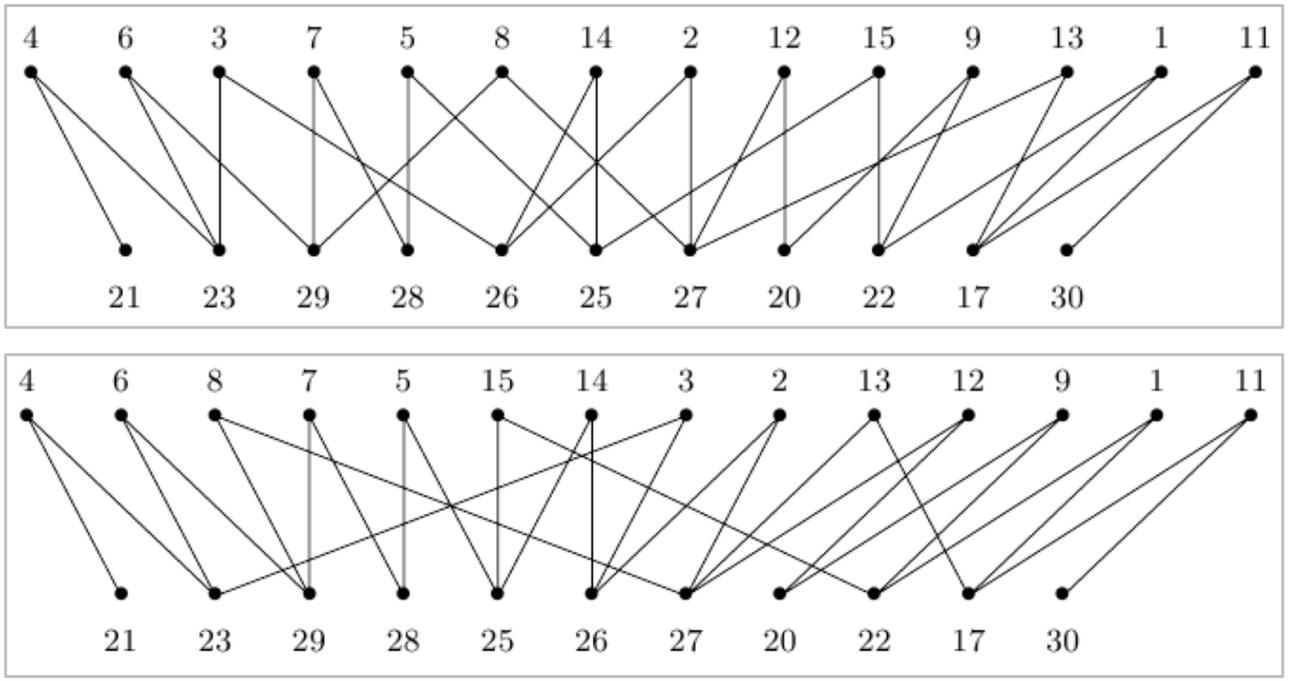

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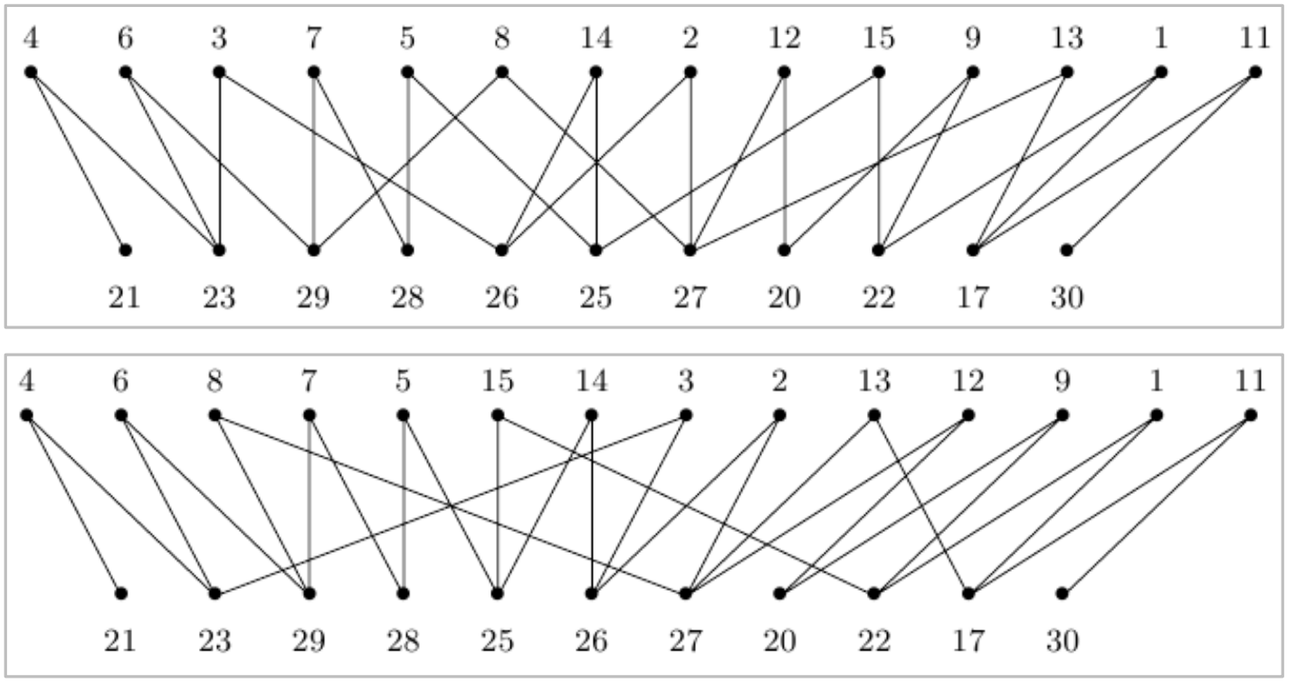

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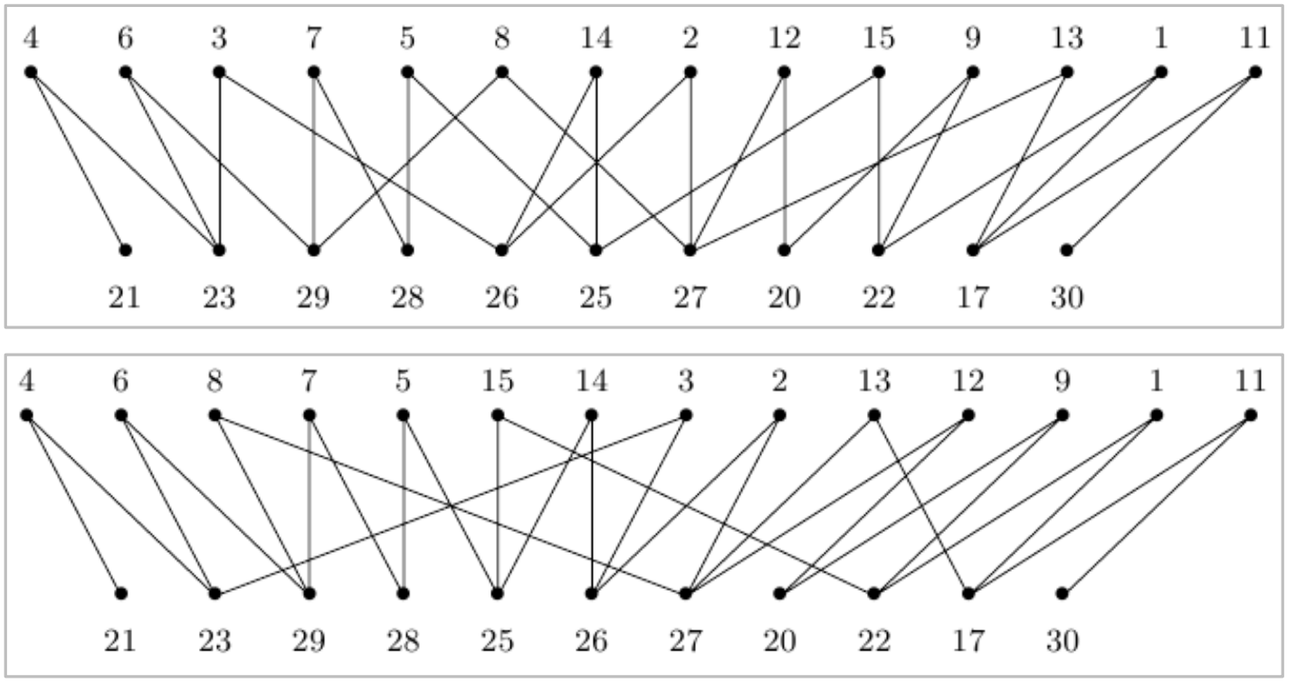

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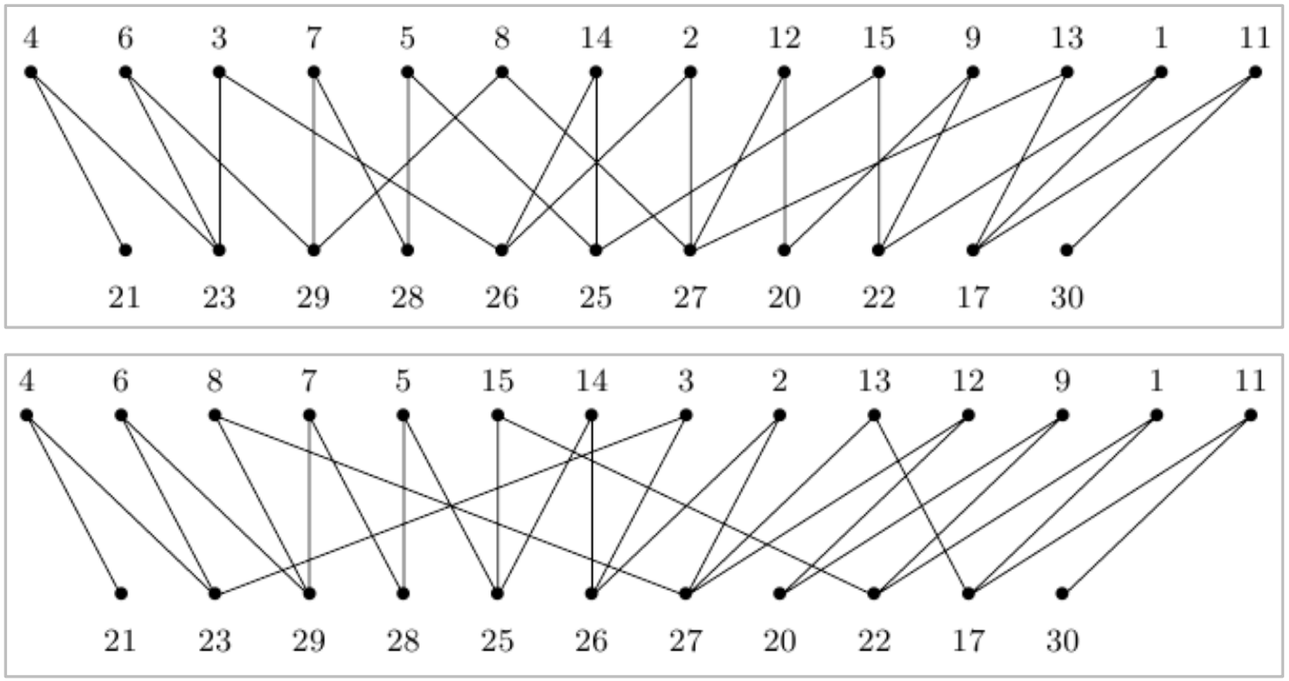

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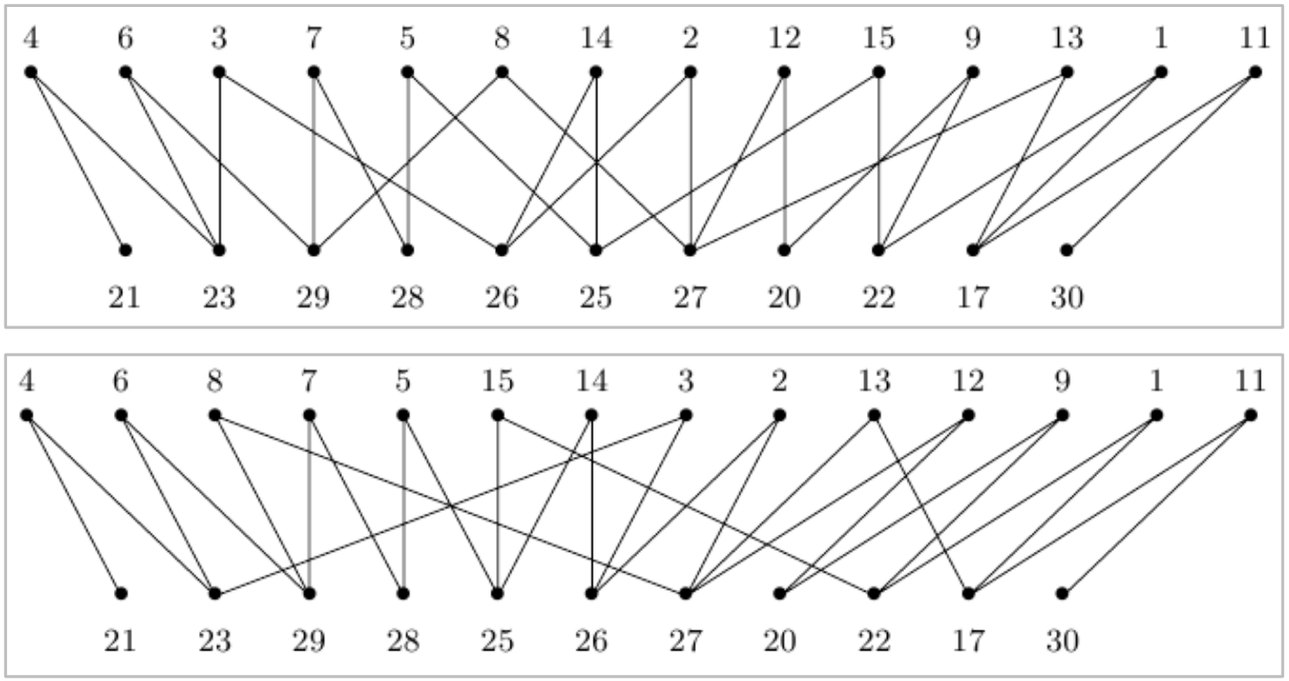

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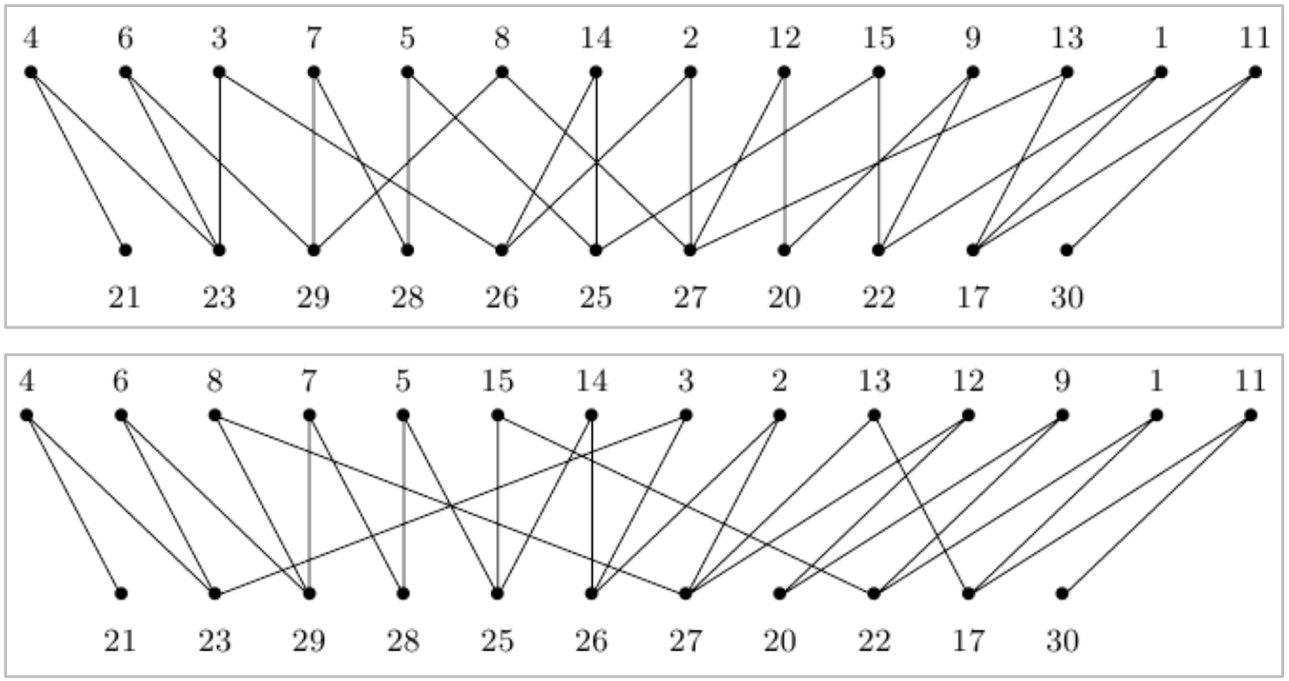

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Barycenter Heuristic

[Sugiyama et al. '81]

- Intuition: few intersections occur when vertices are close to their neighbors

Barycenter Heuristic

[Sugiyama et al. '81]

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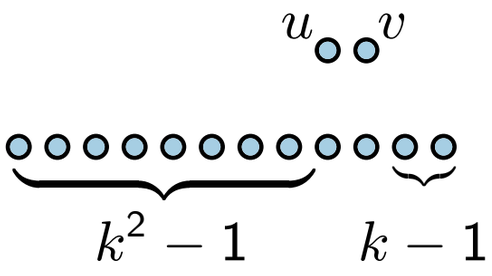

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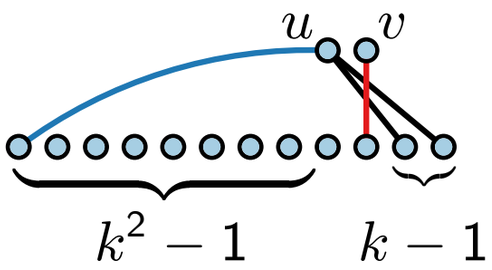

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Median Heuristic

[Eades & Wormald '94]

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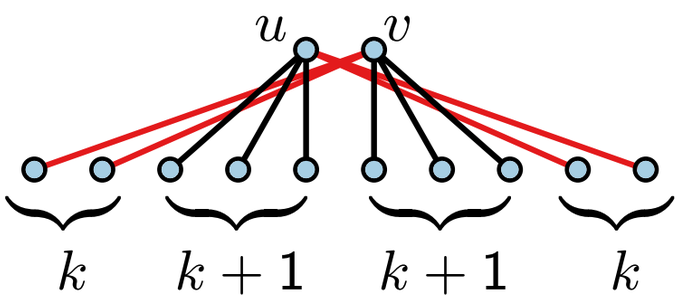

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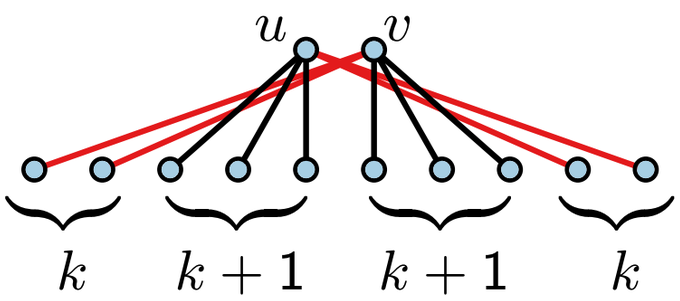

$2k(k+1) + k^2$ vs. $(k+1)^2$

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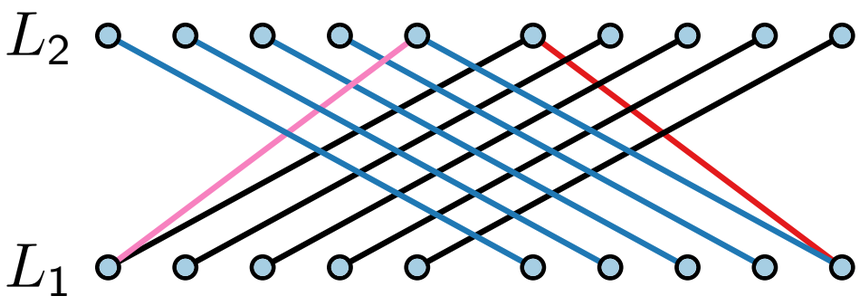

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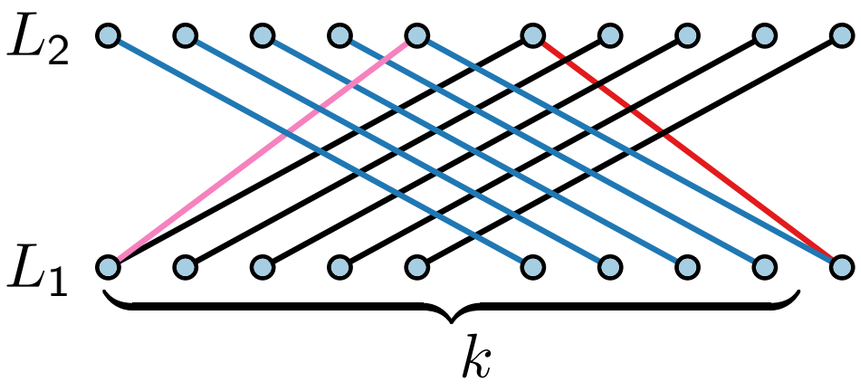

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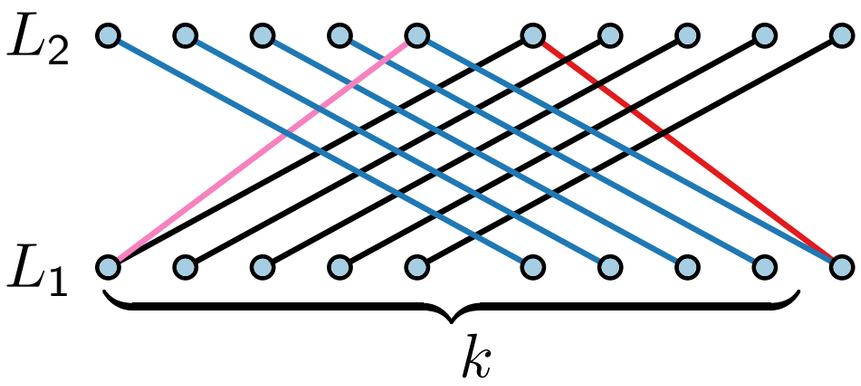

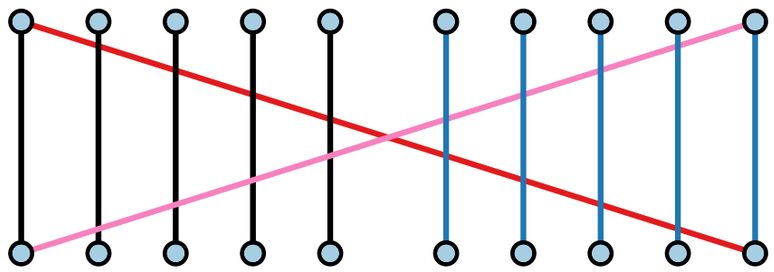

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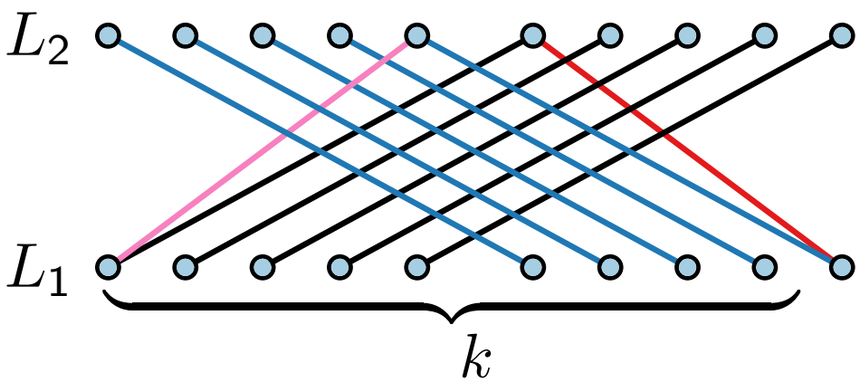
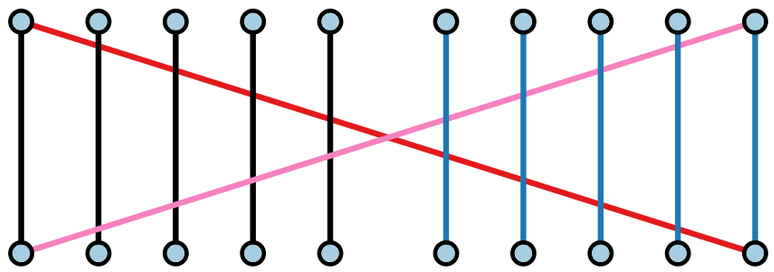


$\approx k^2/4$



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Integer Linear Program

[Jünger & Mutzel, '97]

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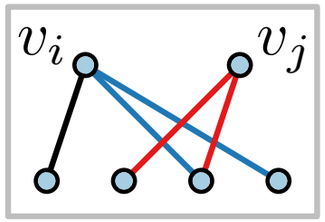

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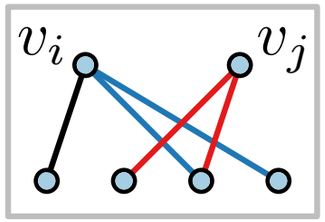

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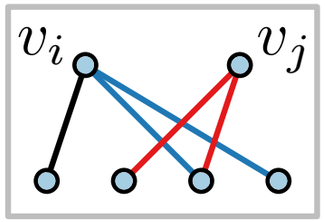

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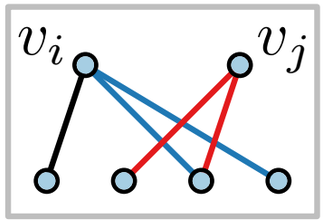

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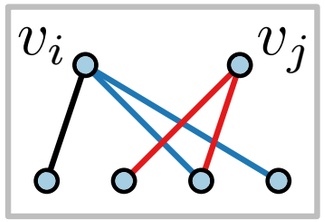

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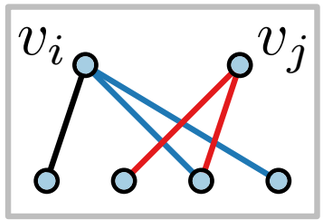

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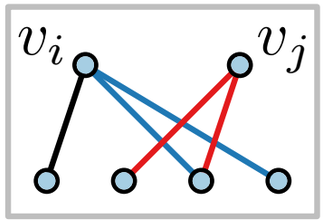

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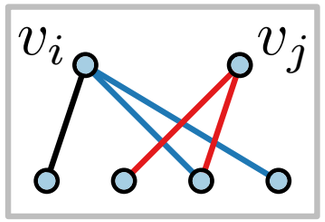

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Integer Linear Program

- Minimize the number of crossings:

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i.e., if $x_{ij} = 1$ and $x_{jk} = 1$, then $x_{ik} = 1$

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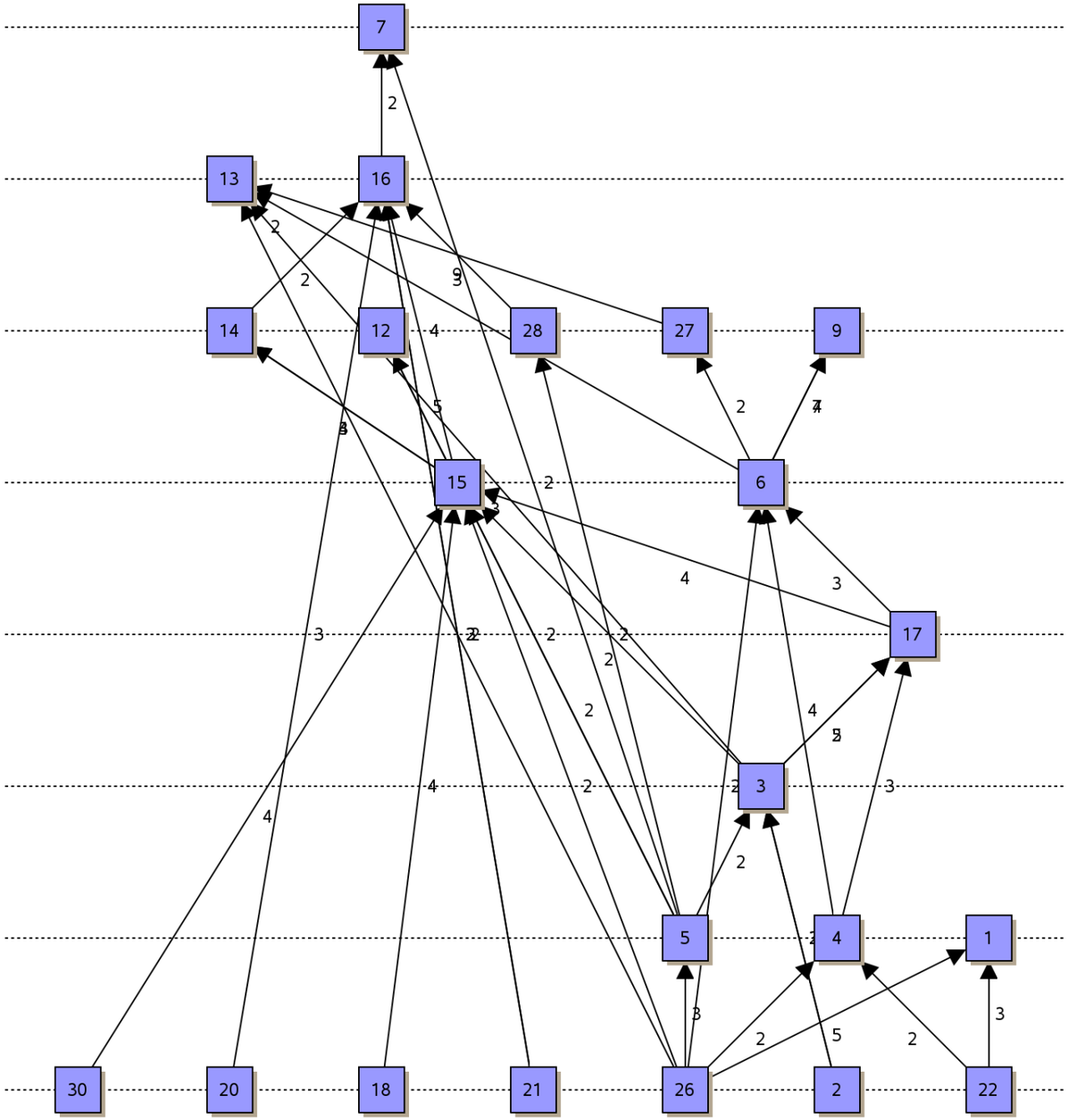
Properties.

- Branch-and-cut technique for DAGs of limited size
- Useful for graphs of small to medium size
- Finds optimal solution
- Solution in polynomial time is not guaranteed

Iterations on Example



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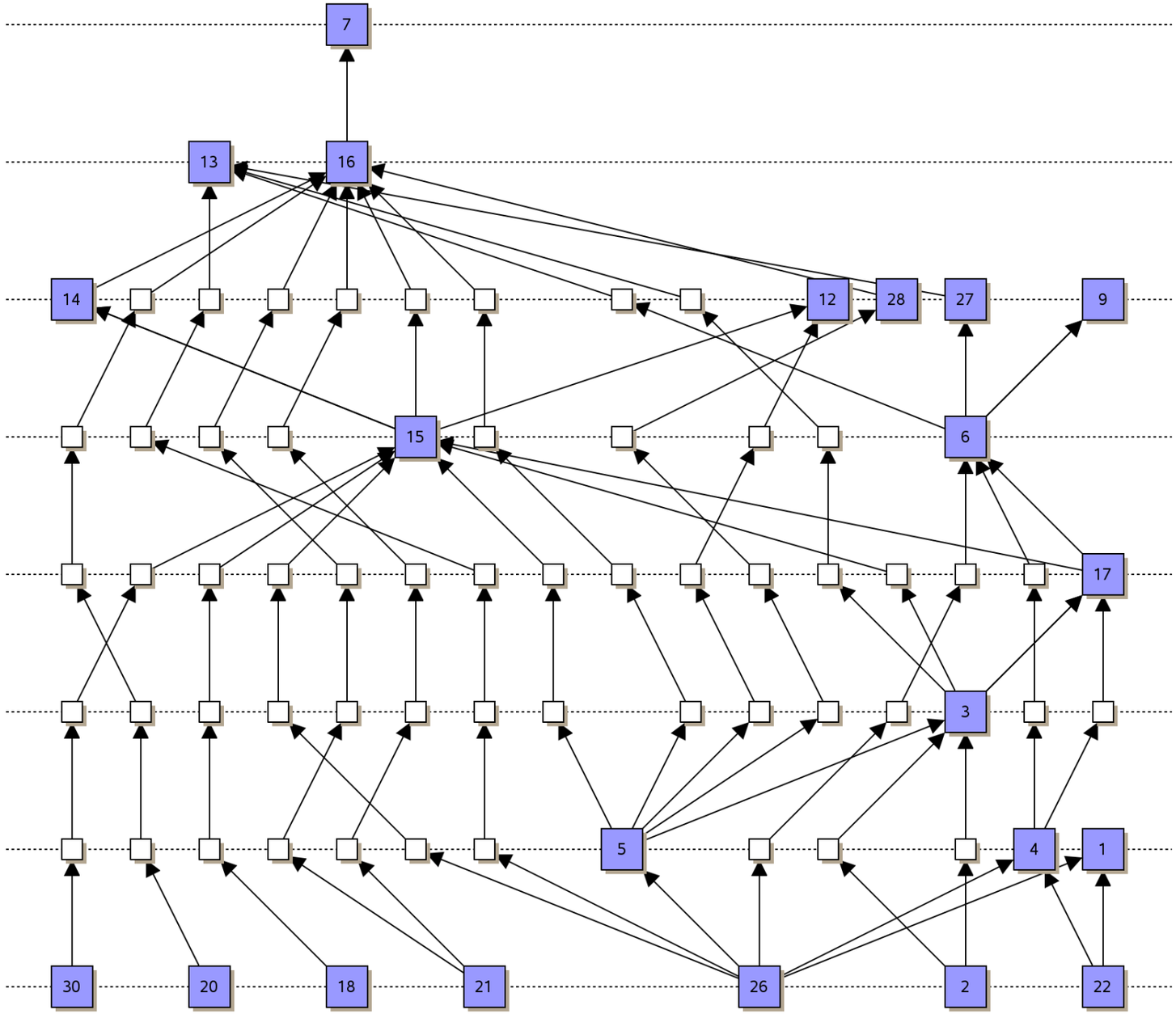

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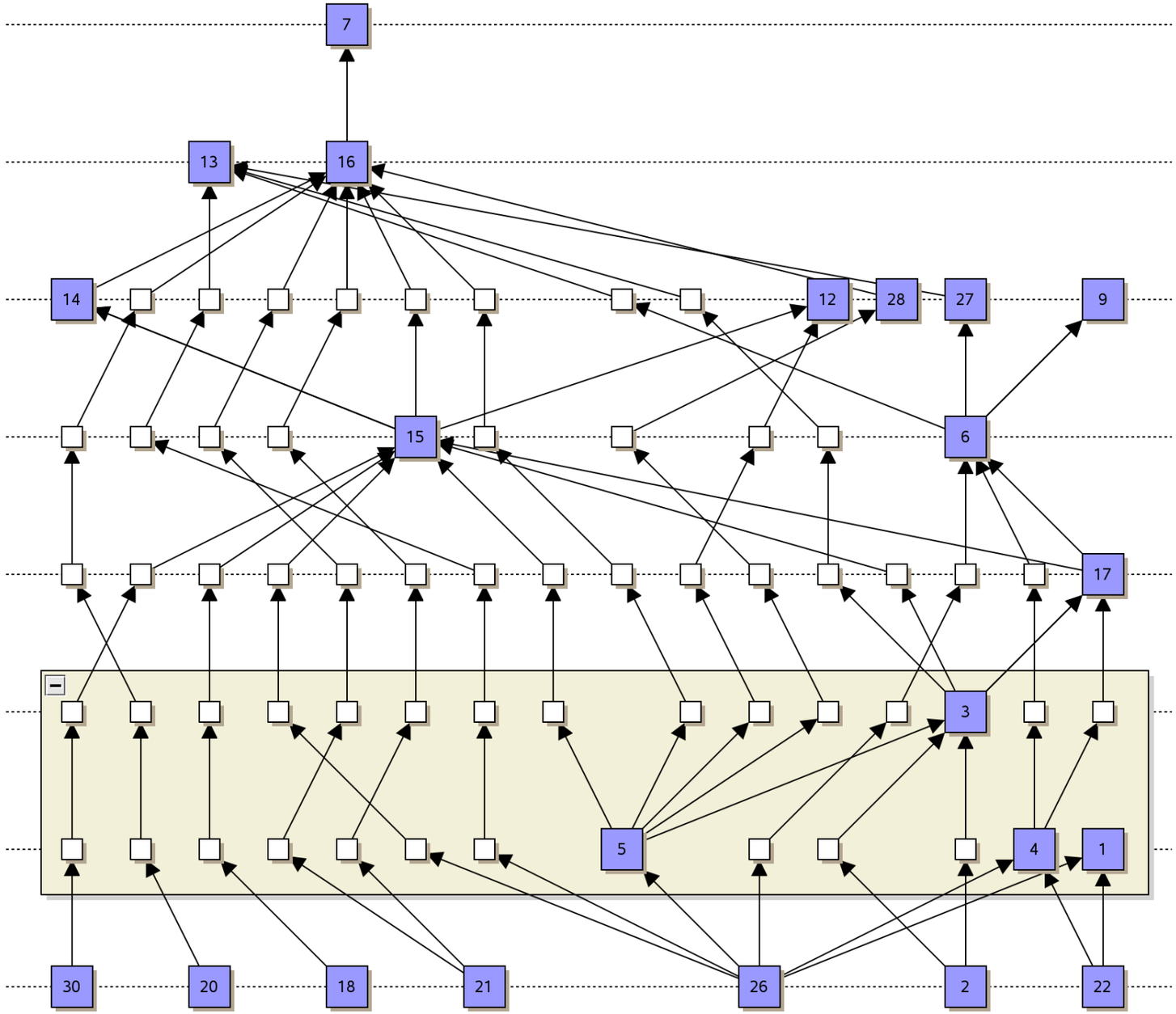

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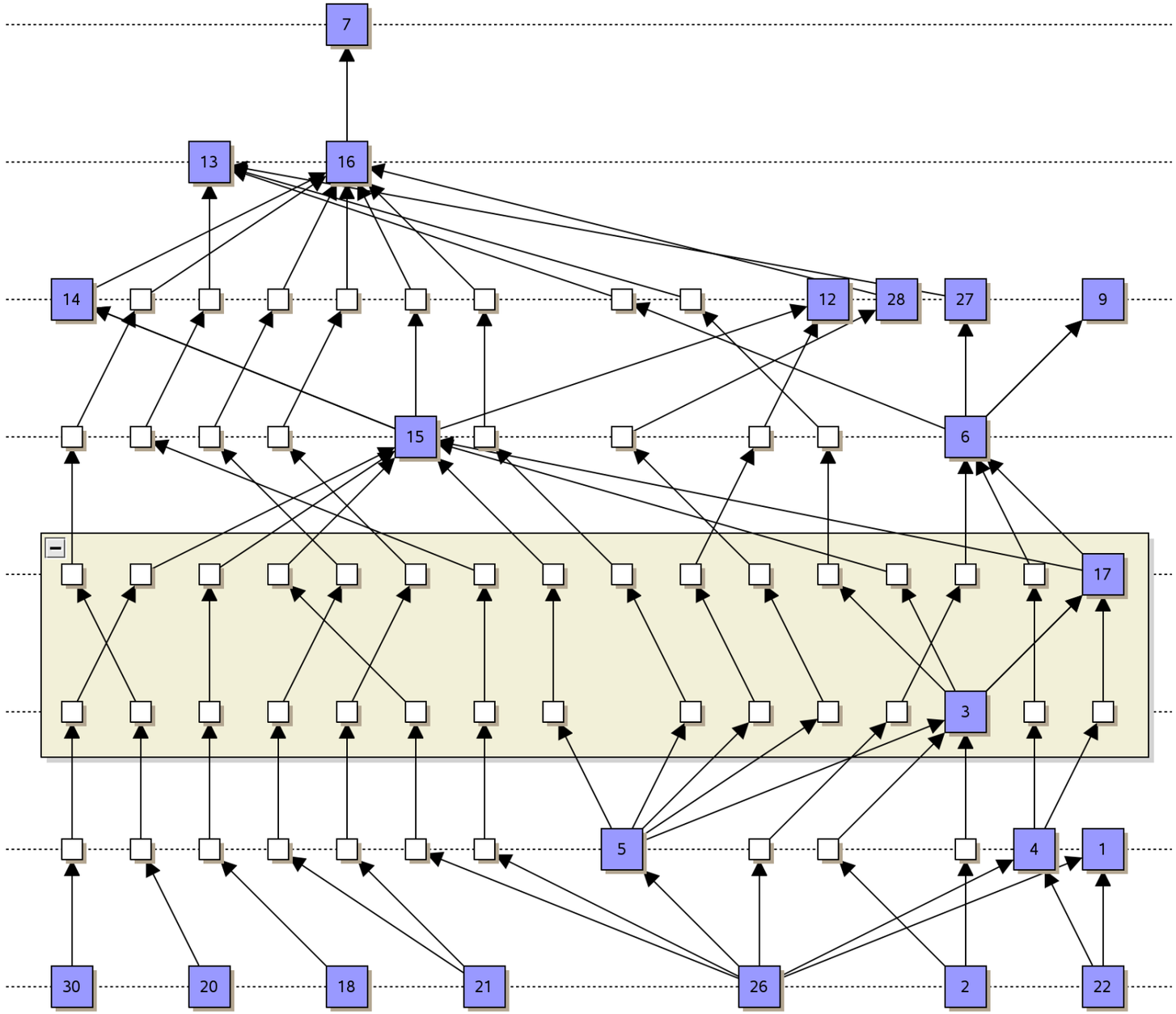

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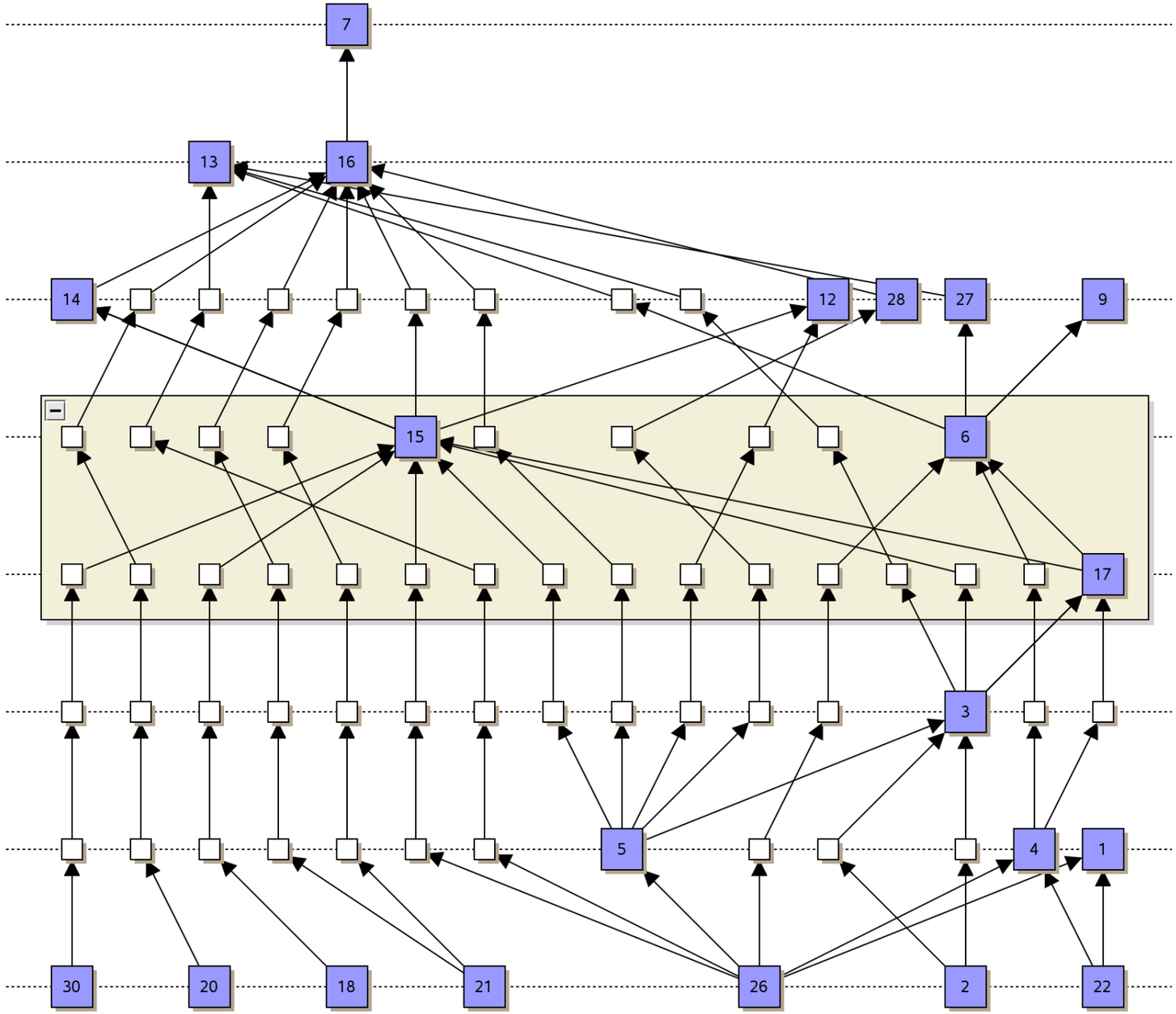

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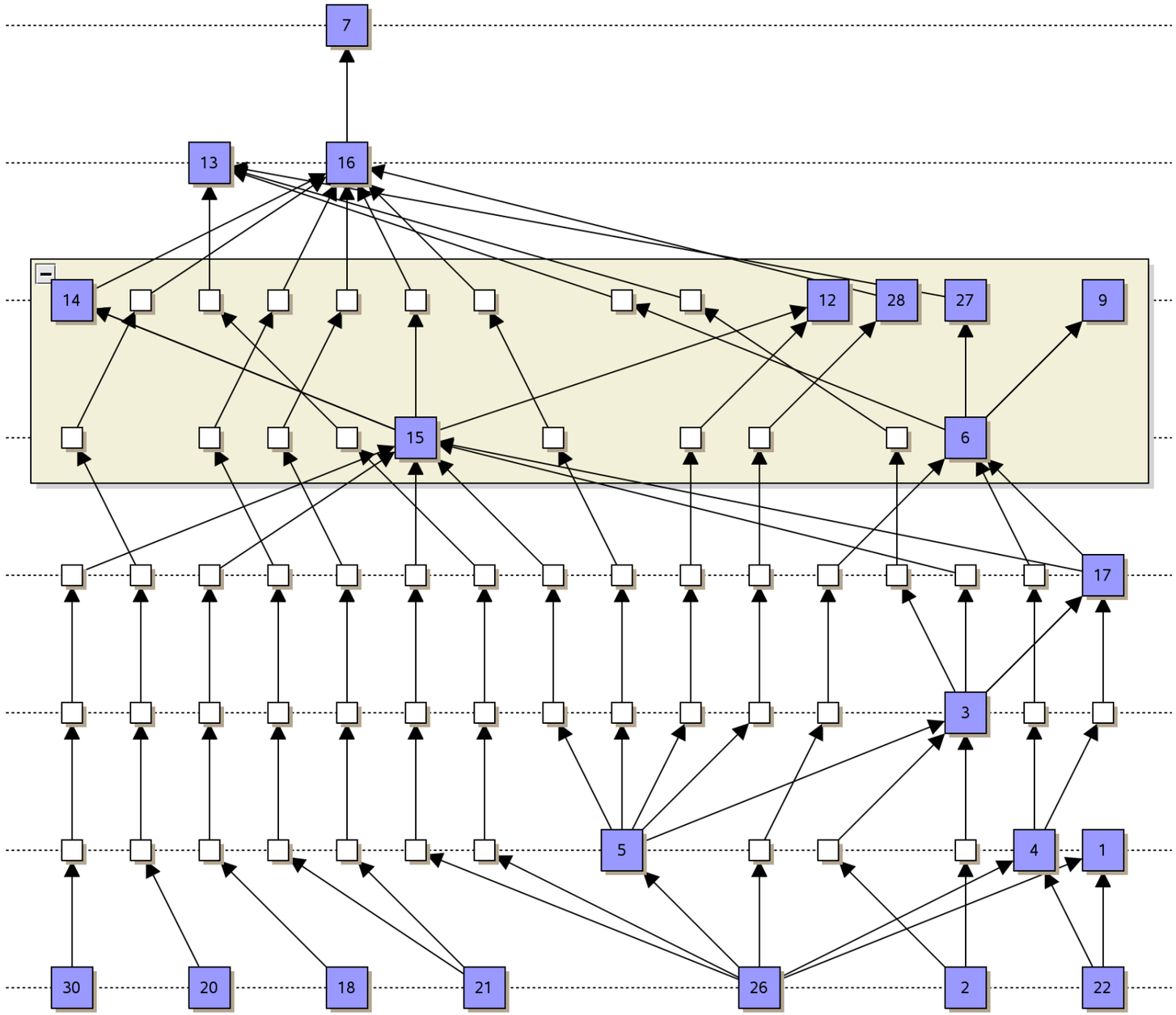

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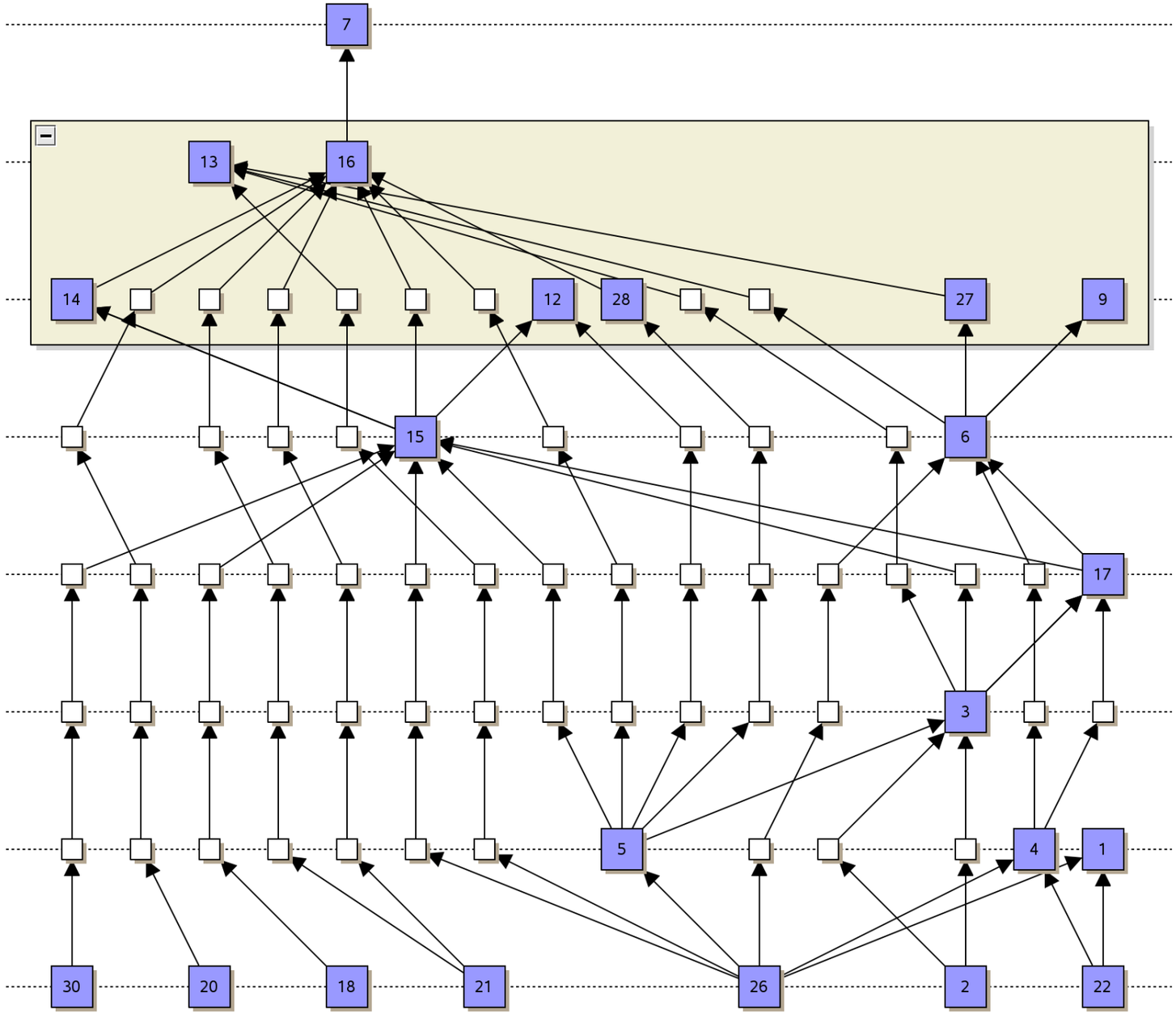

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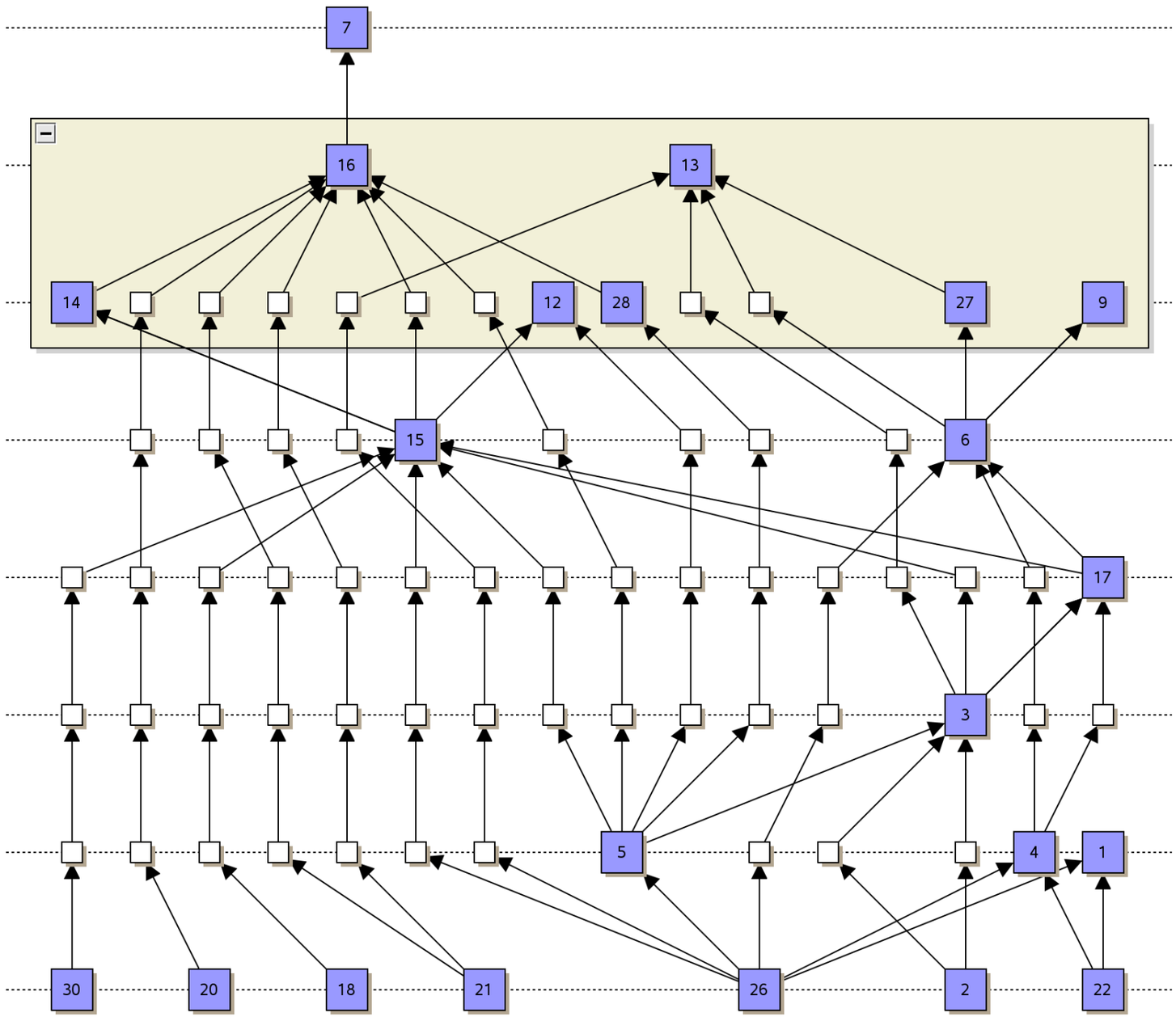

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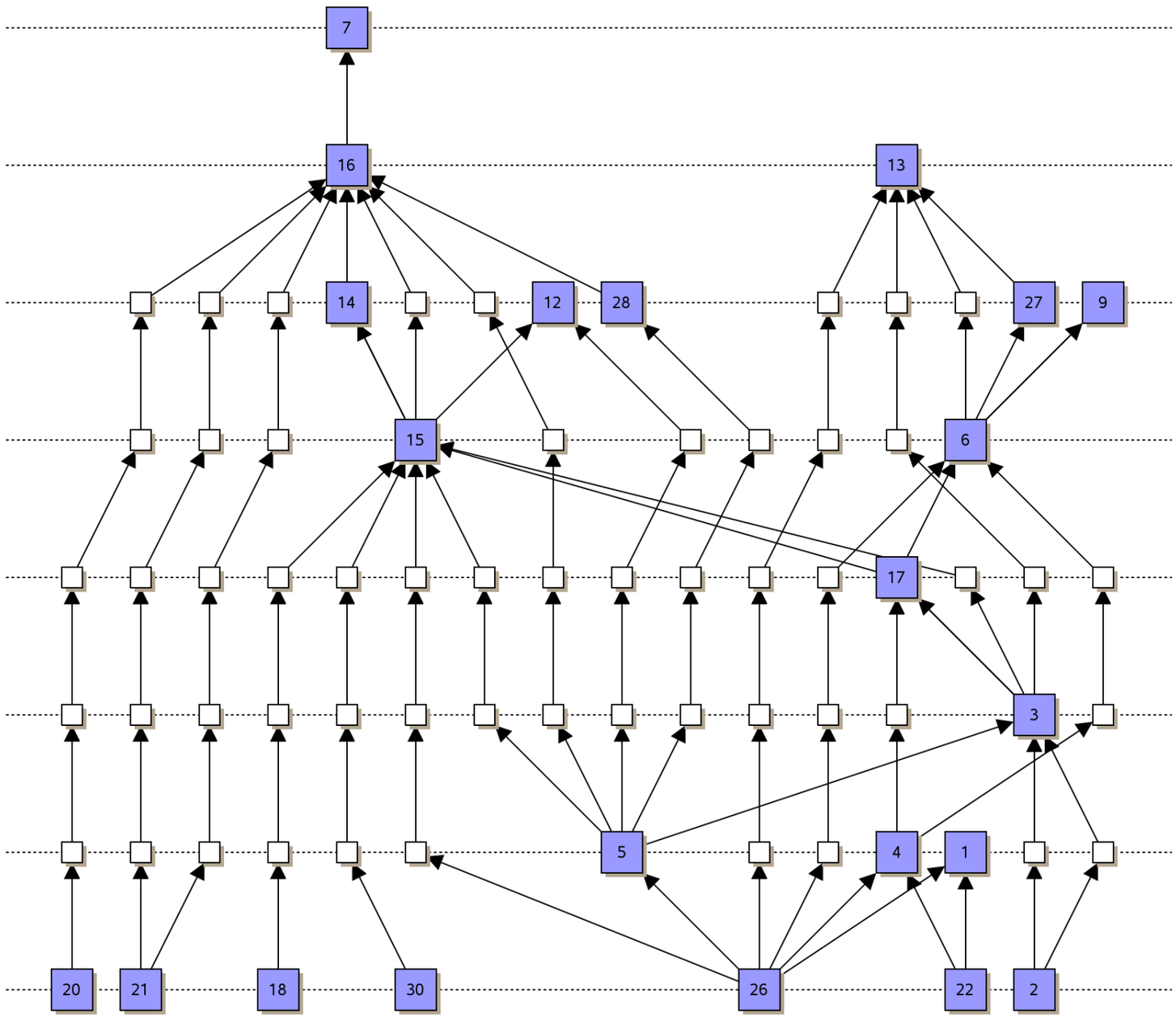

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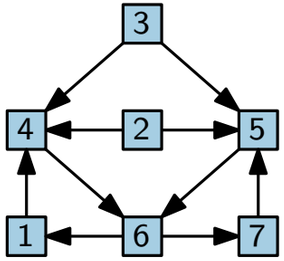
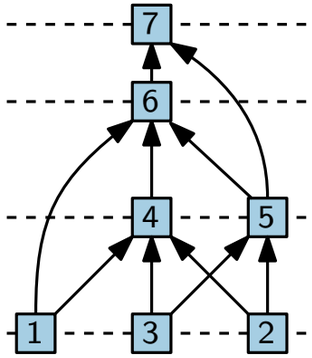
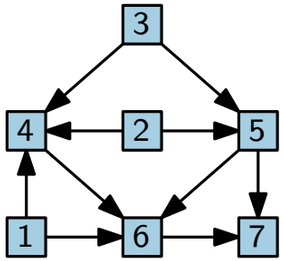
WÜRZBURG

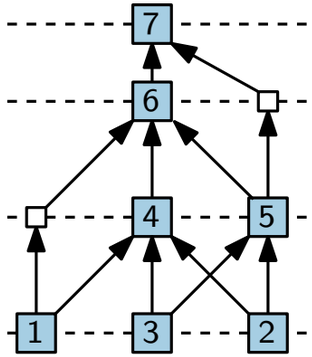


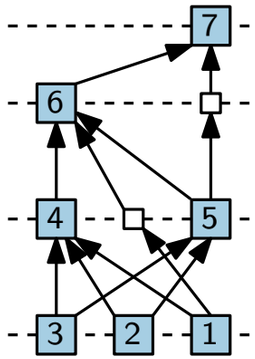


Visualization of Graphs

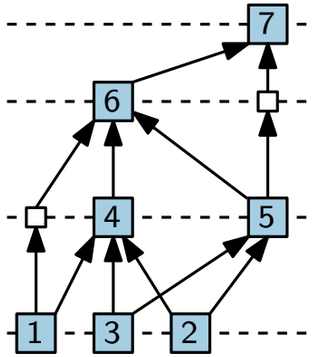
Lecture 7: Hierarchical Layouts: Sugiyama Framework





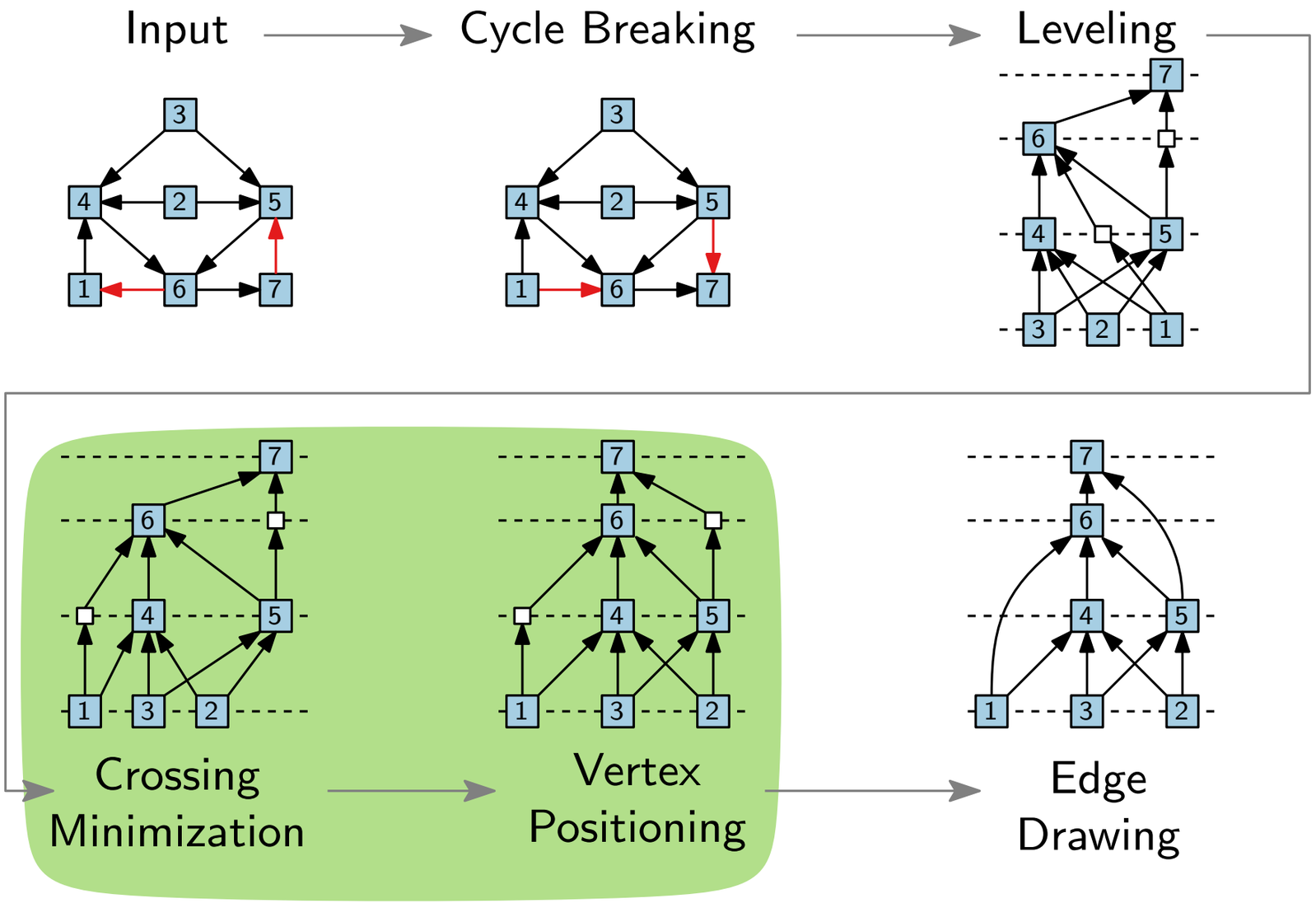


Part V: Vertex Positioning & Drawing Edges

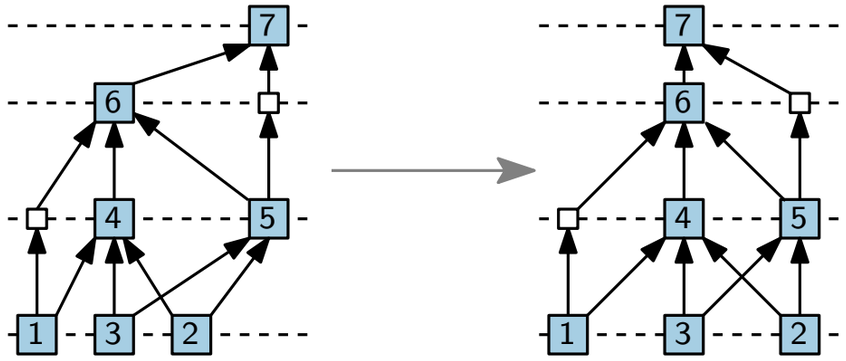


Jonathan Klawitter

Step 4: Vertex Positioning



Step 4: Vertex Positioning



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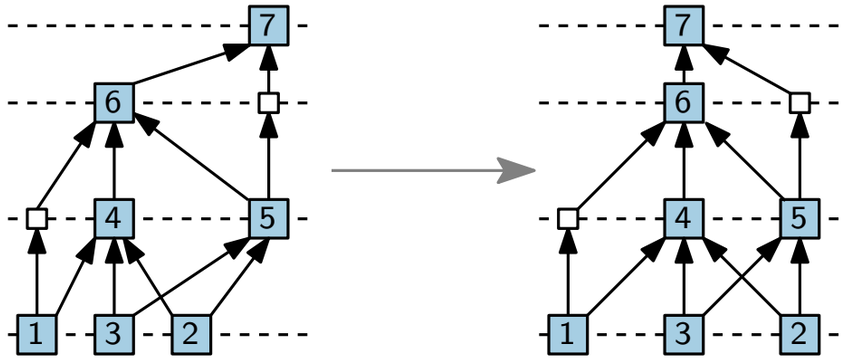

Goal.

Paths should be close to straight, vertices evenly spaced

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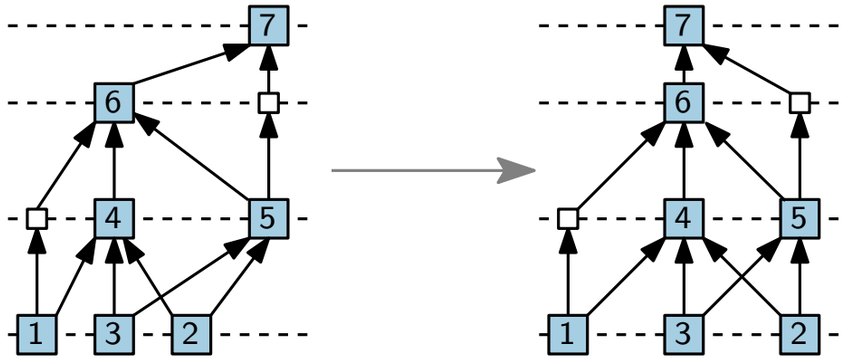

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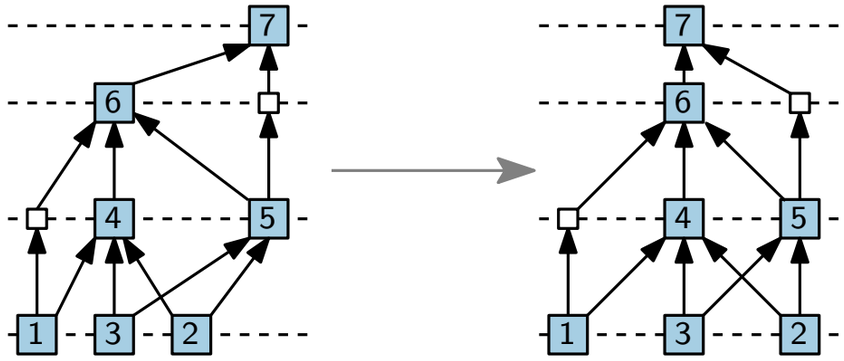

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- **Exact:** Quadratic Program (QP)
- **Heuristic:** Iterative approach

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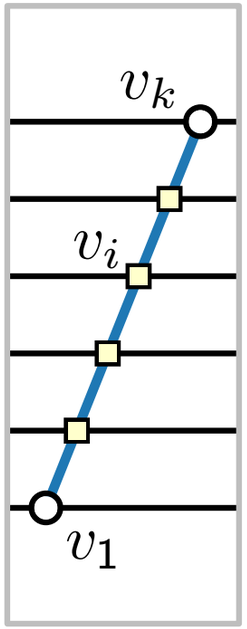
- Consider the path $p_e = (v_1, \ldots, v_k)$ of an edge $e = v_1 v_k$ with dummy vertices: $v_2, \ldots, v_{k-1}$



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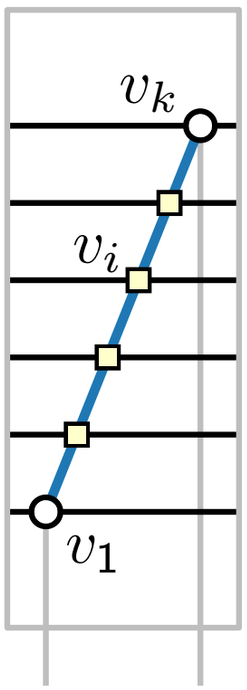

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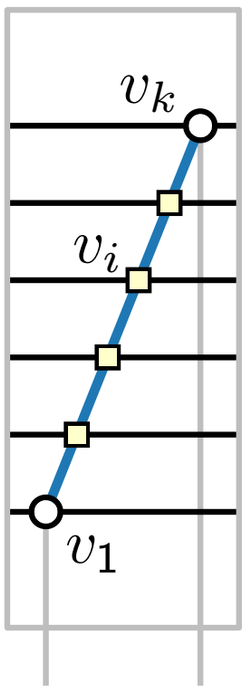

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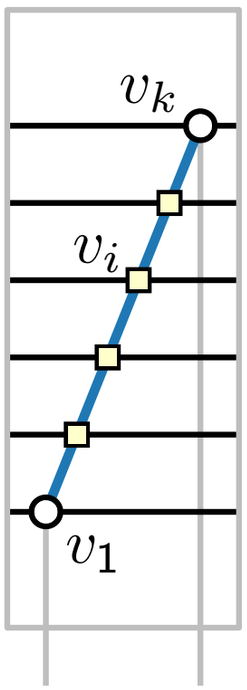

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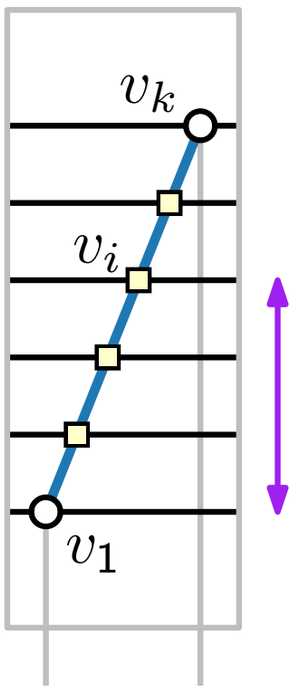

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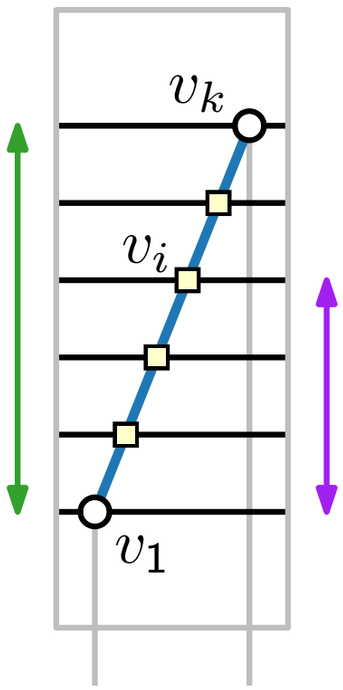

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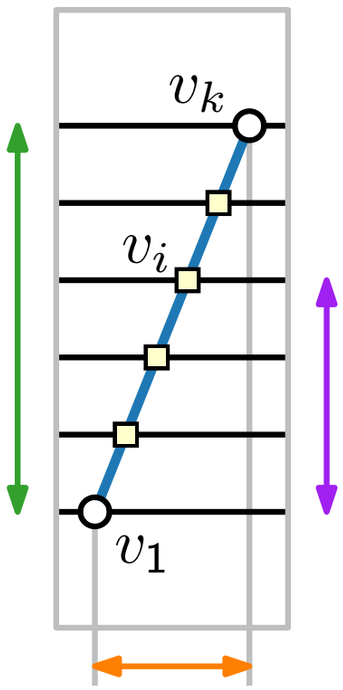

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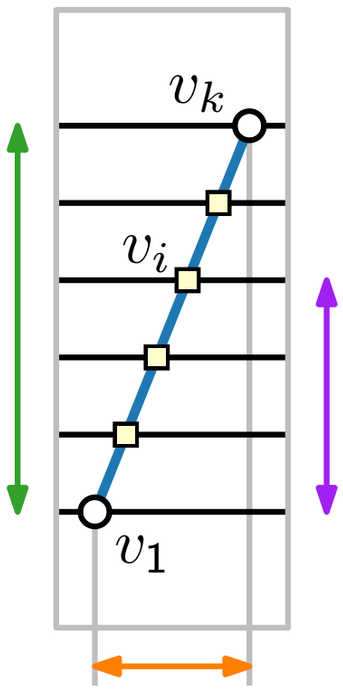
- Define the deviation from the line



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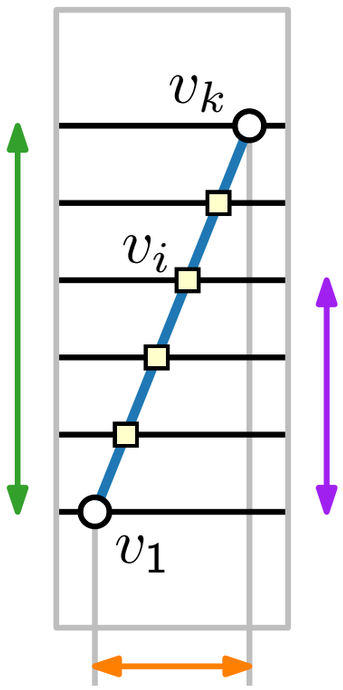

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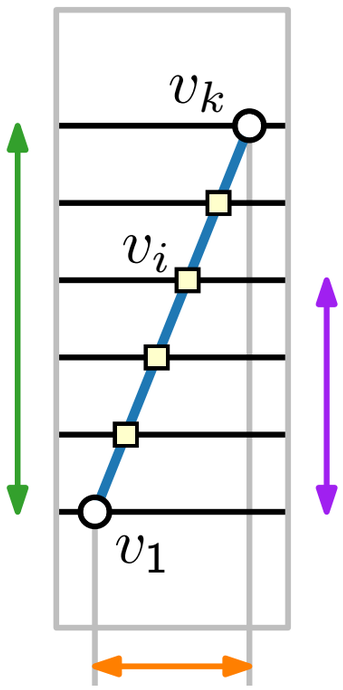
$$\mathrm{dev}(p_e) := \sum_{i=2}^{k-1}$$



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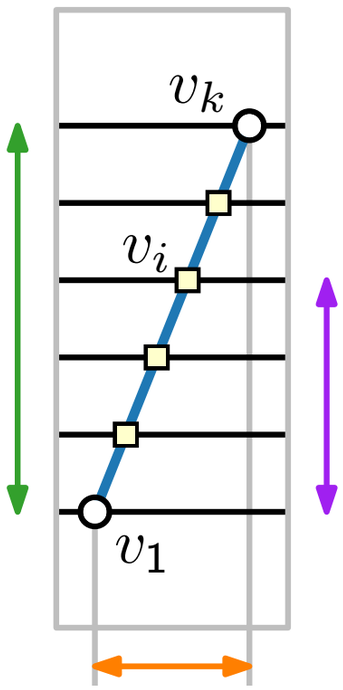

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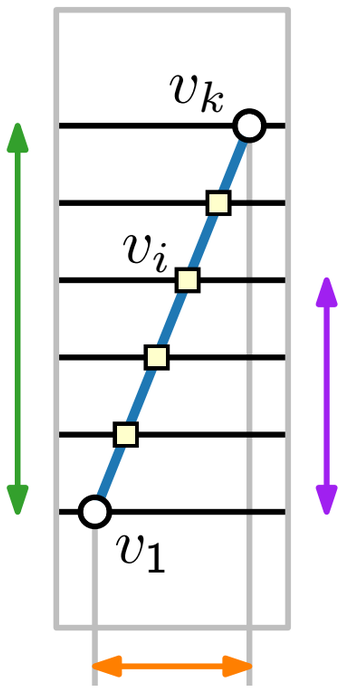

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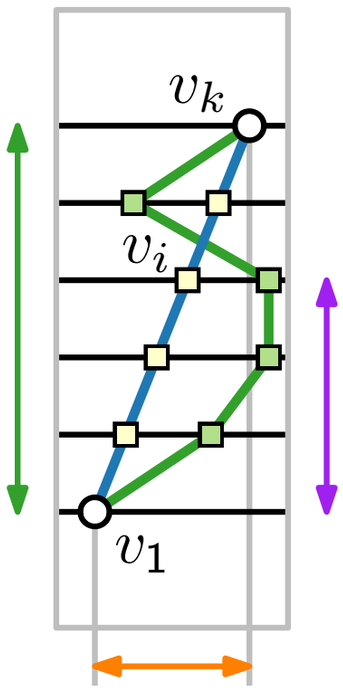

Quadratic Program

- Consider the path $p_e = (v_1, \ldots, v_k)$ of an edge $e = v_1 v_k$ with dummy vertices: $v_2, \ldots, v_{k-1}$
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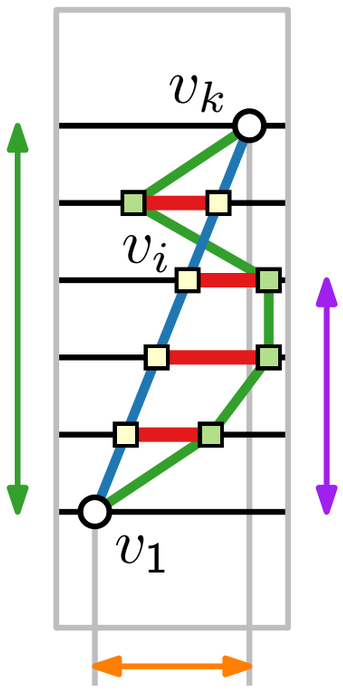
$$\mathrm{dev}(p_e) := \sum_{i=2}^{k-1} \Big(x(v_i) - \overline{x(v_i)}\Big)^2$$



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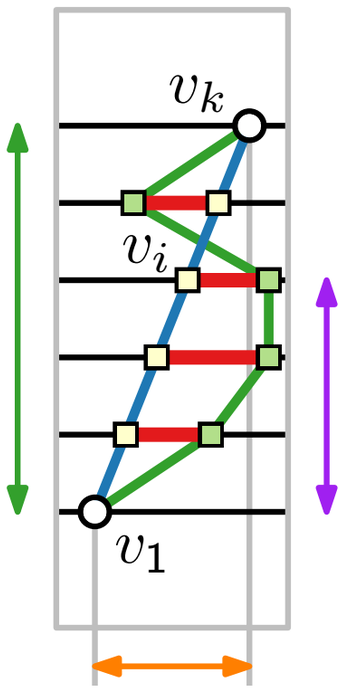

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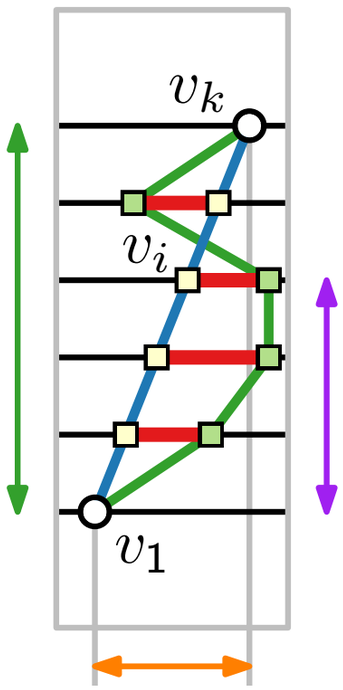

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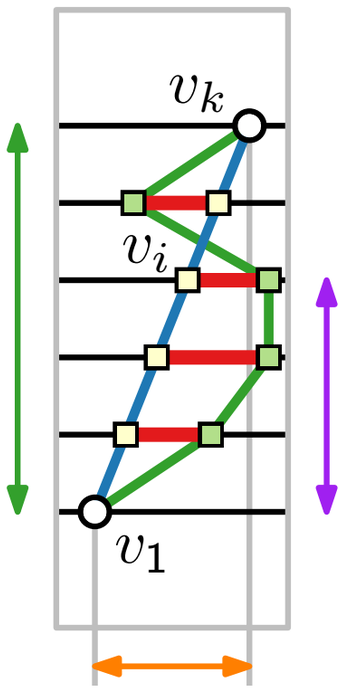

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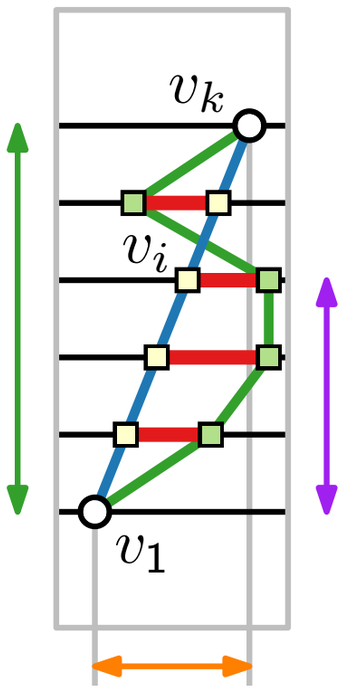

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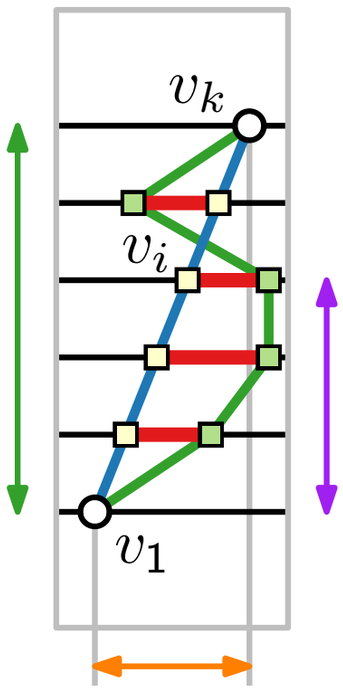
min. horizontal distance



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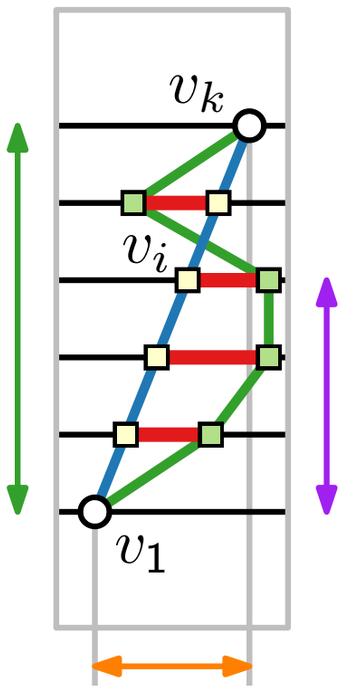

- QP is time-expensive

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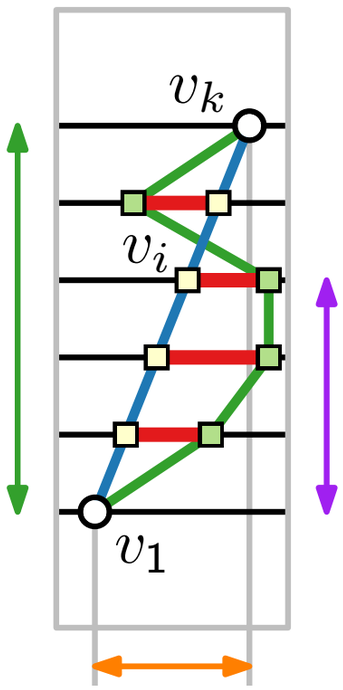

- QP is time-expensive
- width can be exponential

Iterative Heuristic

- Compute an initial layout

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- Apply the following steps as long as improvements can be made:

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Iterative Heuristic

- Compute an initial layout
- Apply the following steps as long as improvements can be made:

  1. Vertex positioning,
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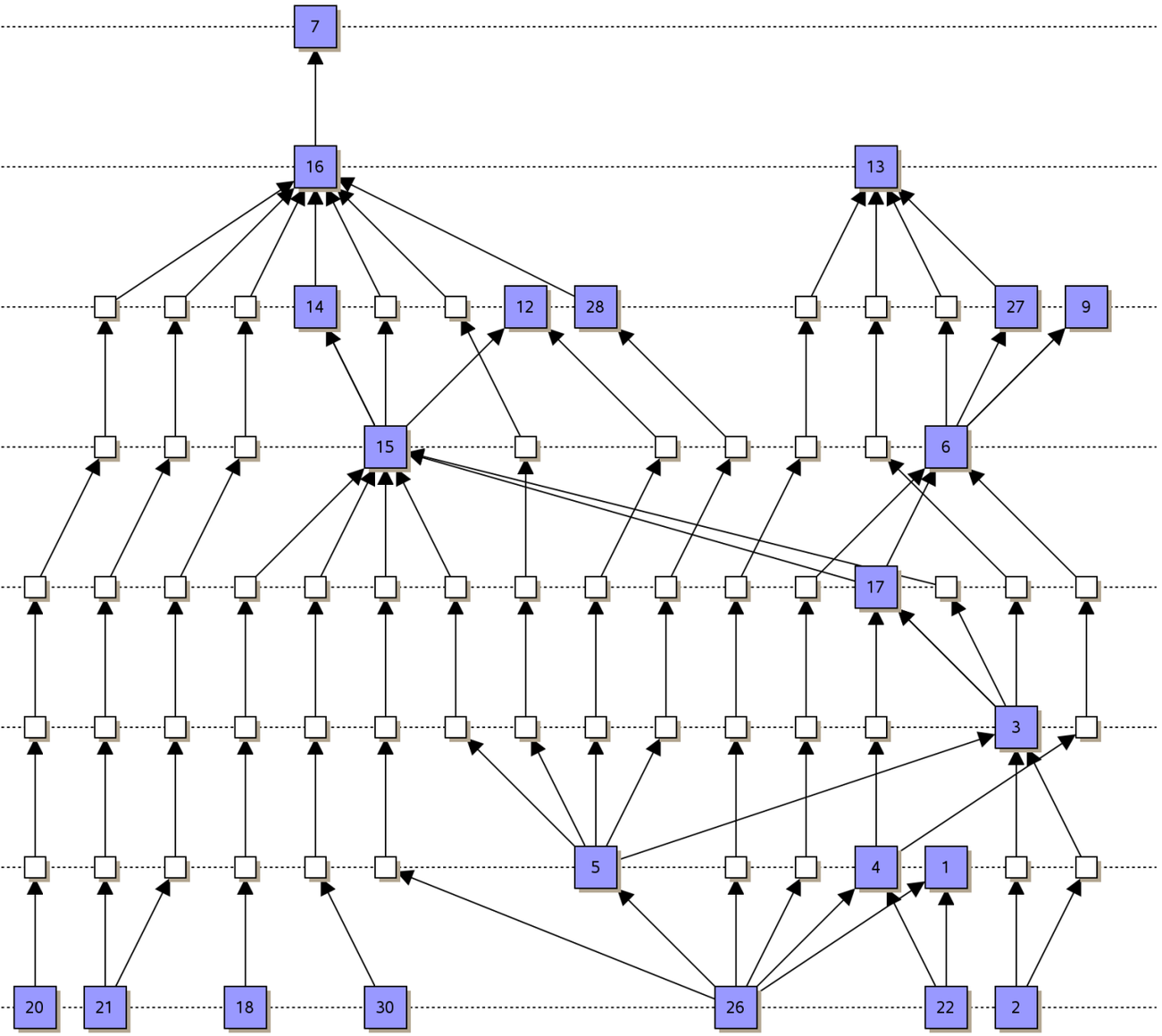
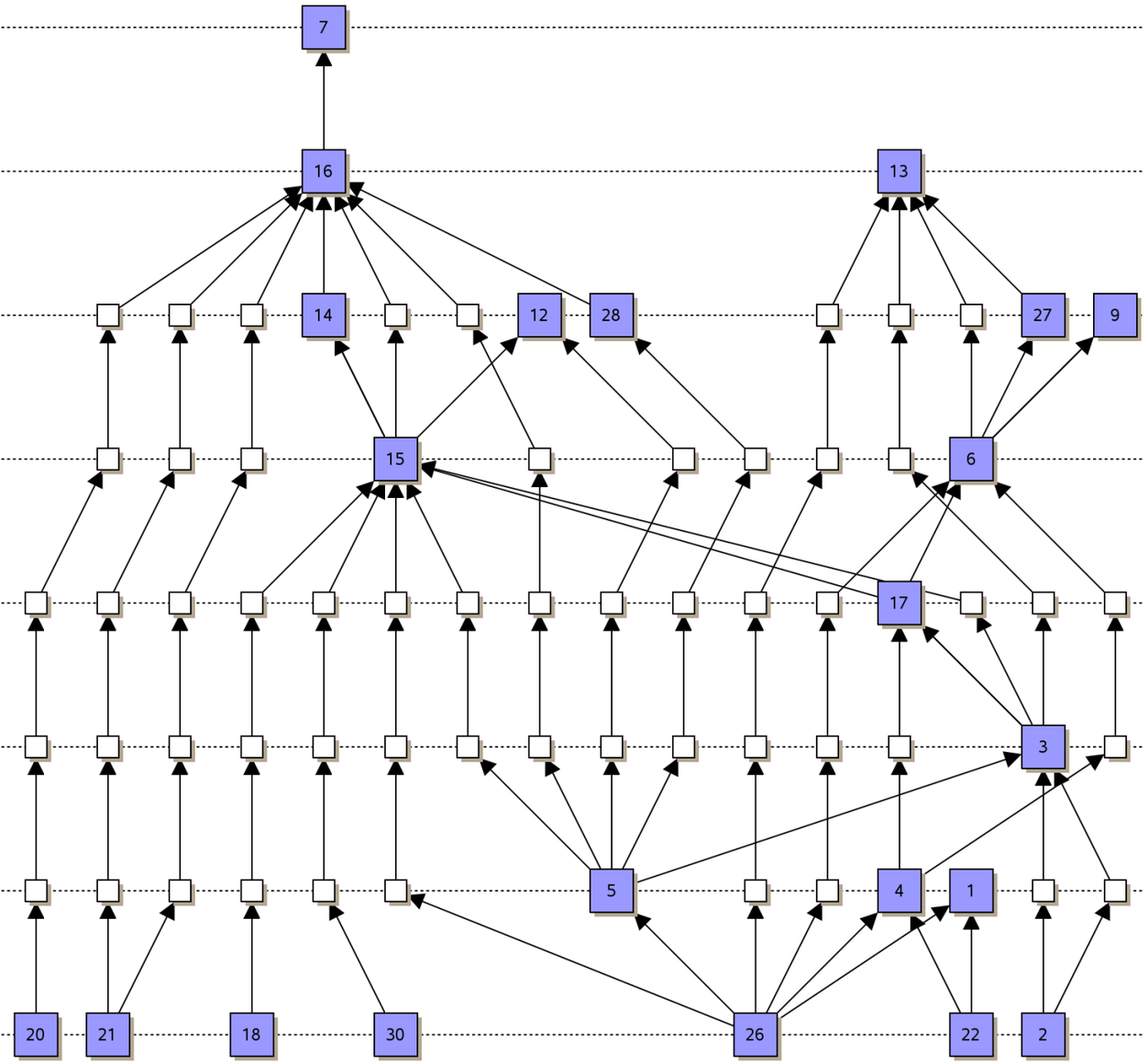
  3. Compactifying the layout width.

Example



Example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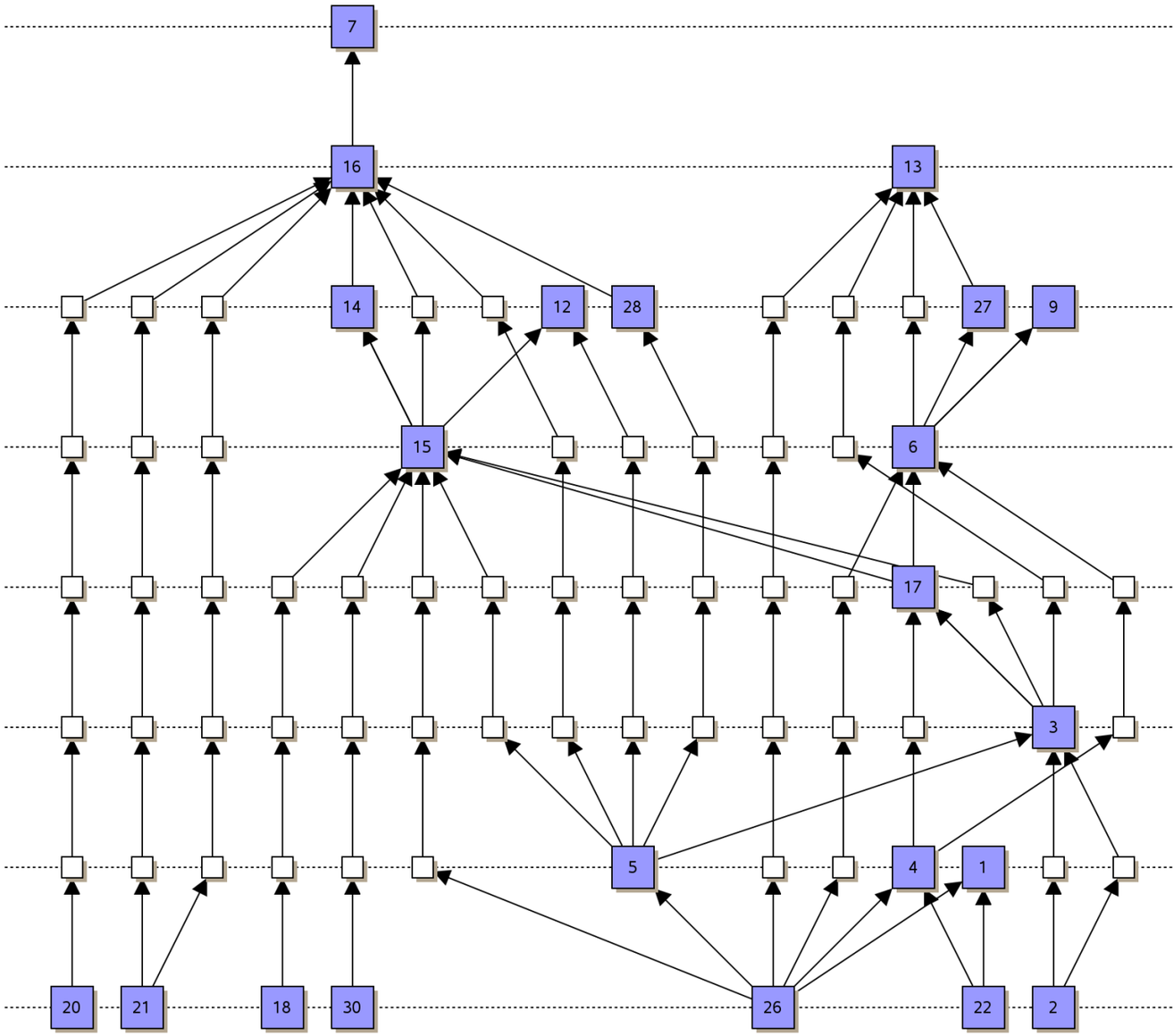
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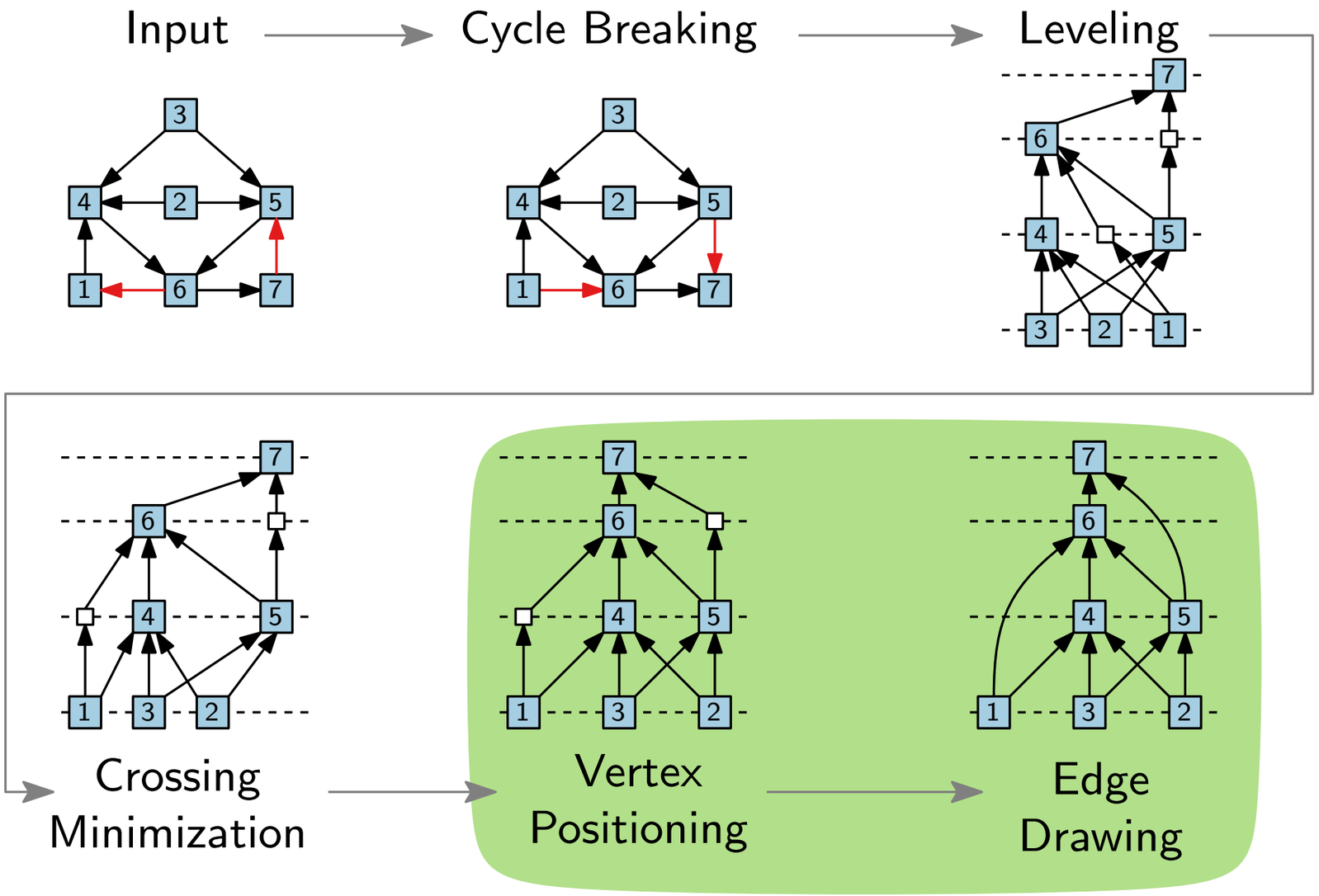
Step 5: Drawing Edges



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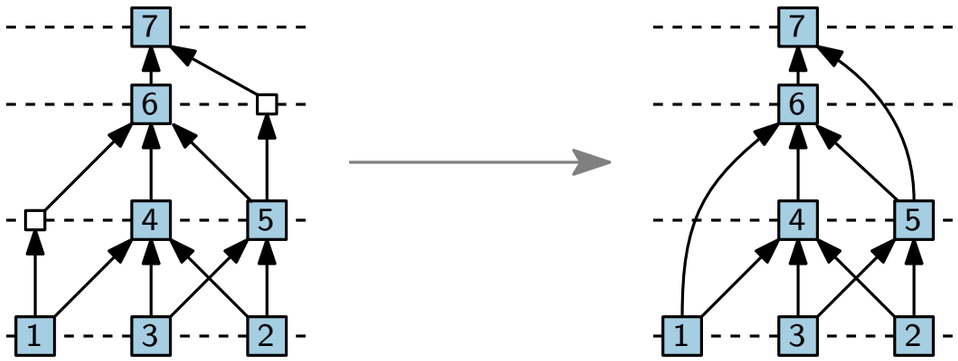
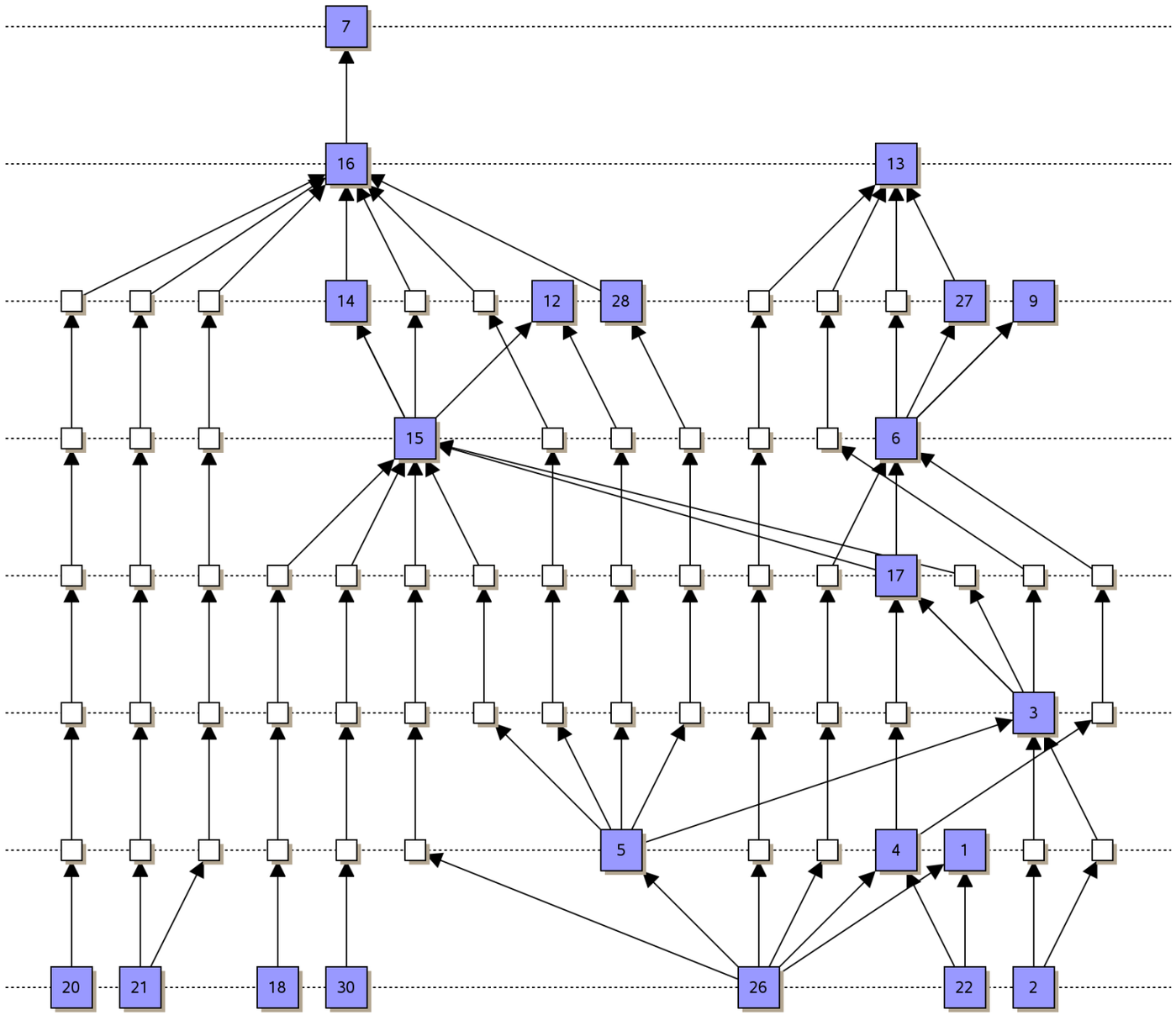


Possibility.
Substitute polylines by Bézier curves

Example



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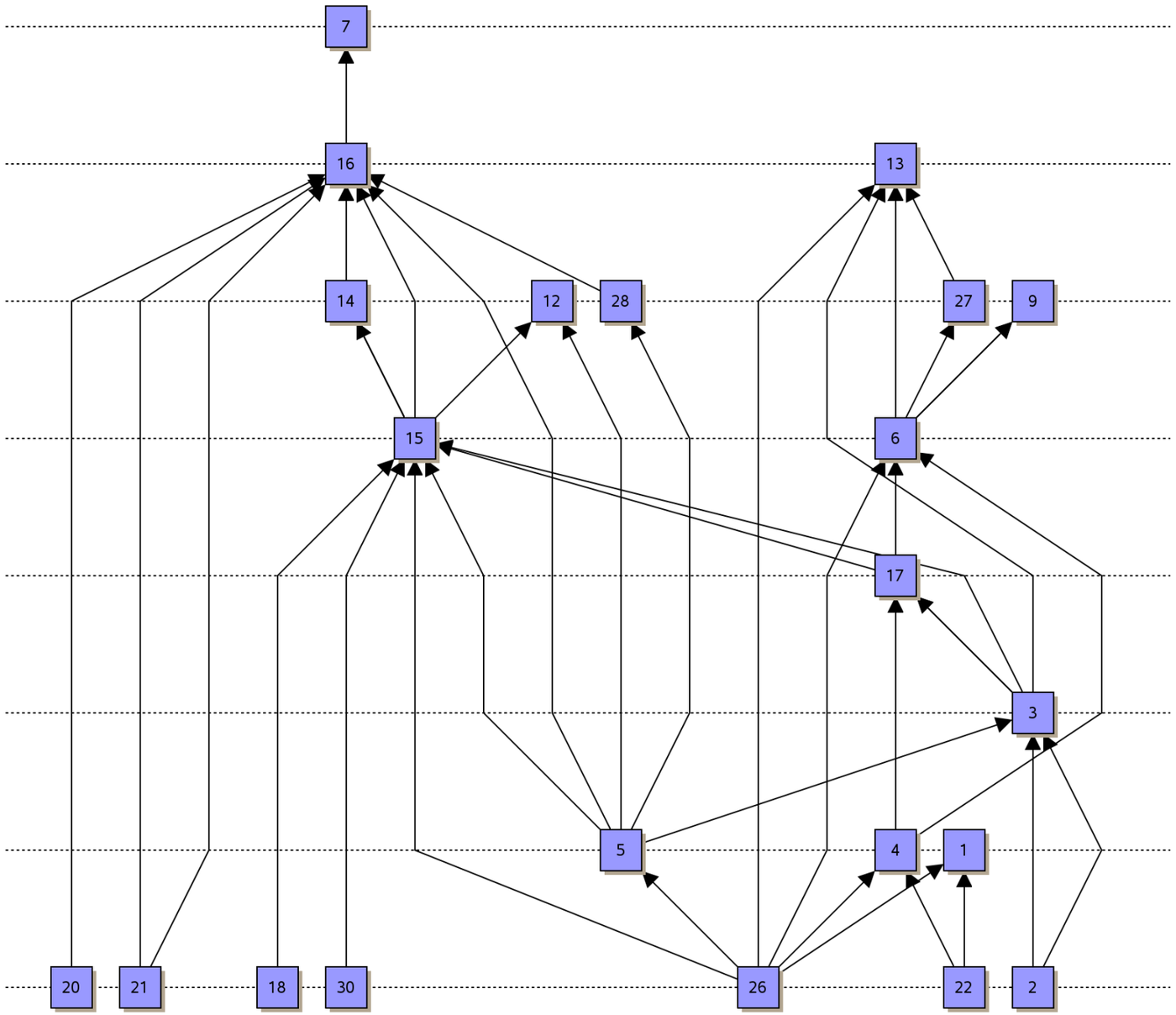

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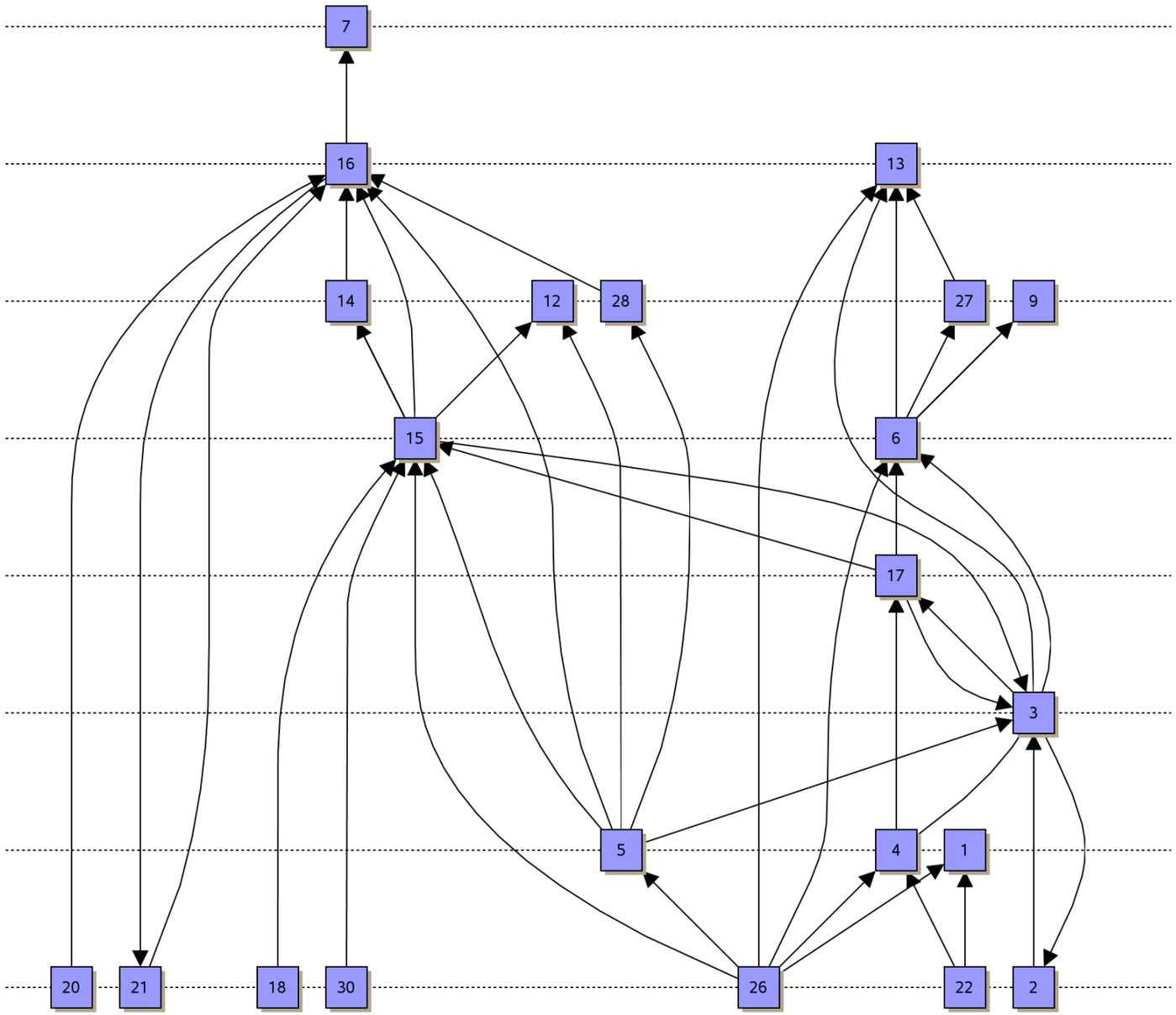
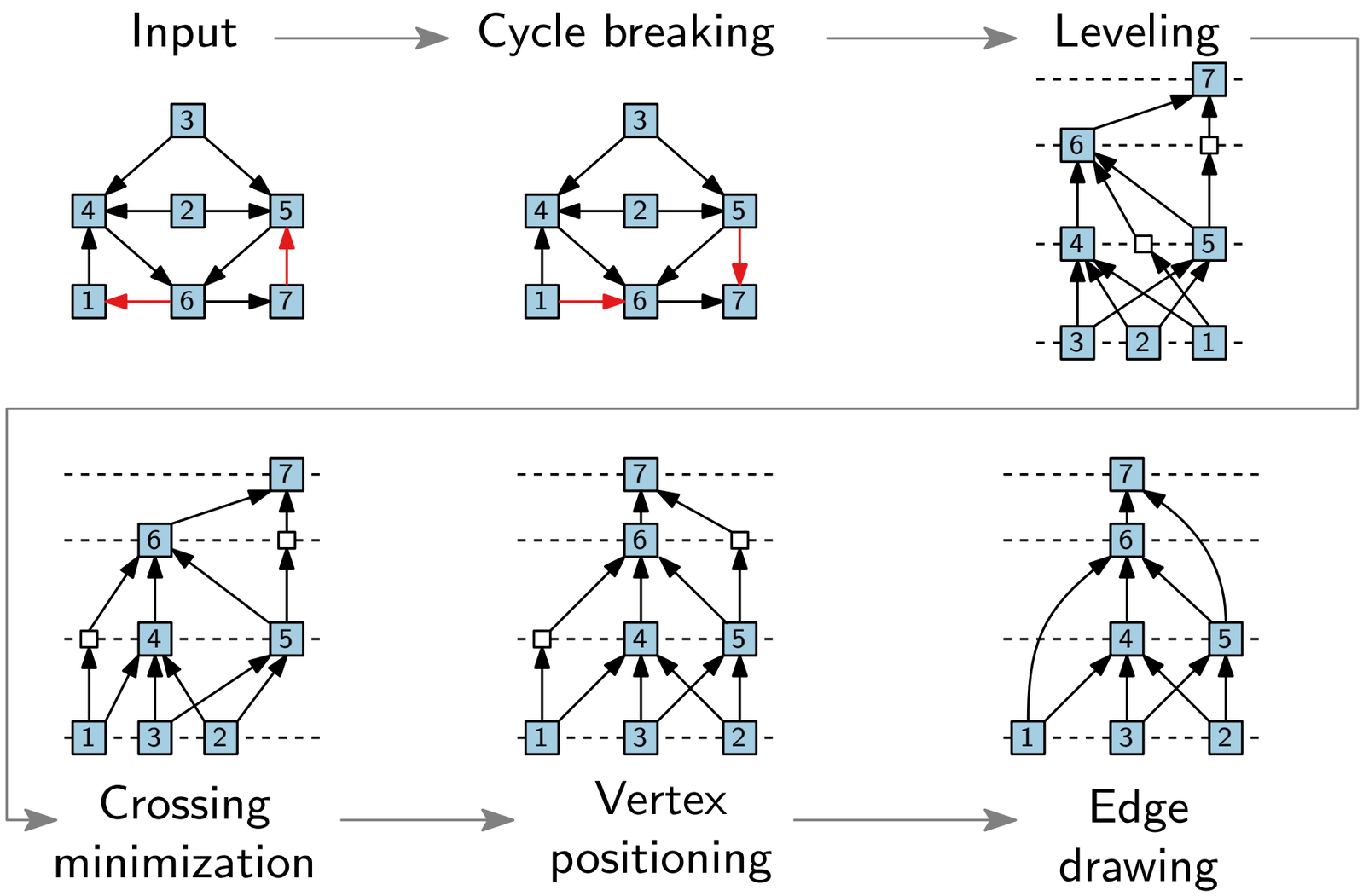


Classical Approach – Sugiyama Framework

[Sugiyama, Tagawa, Toda '81]



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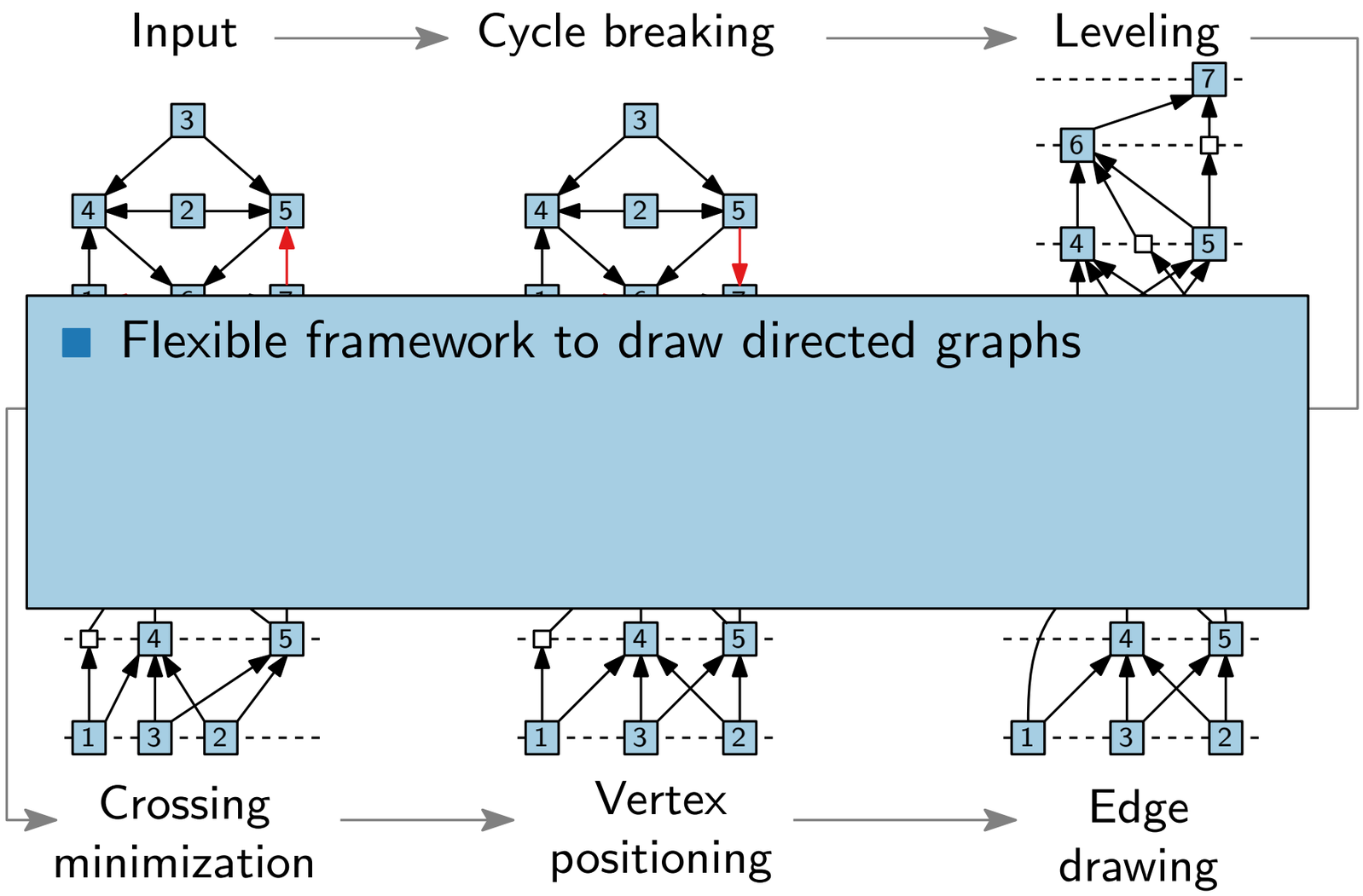

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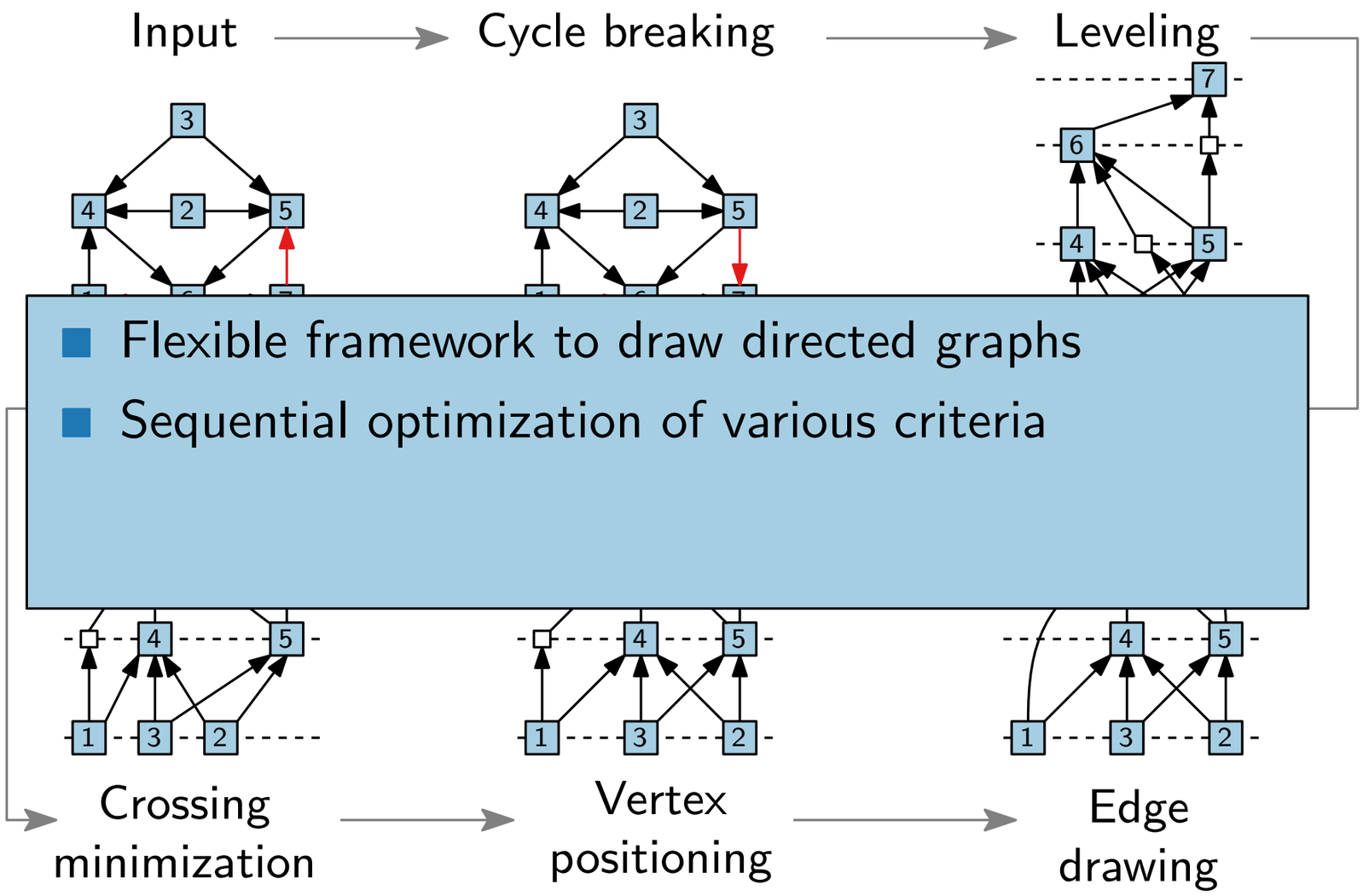

- Flexible framework to draw directed graphs
- Sequential optimization of various criteria

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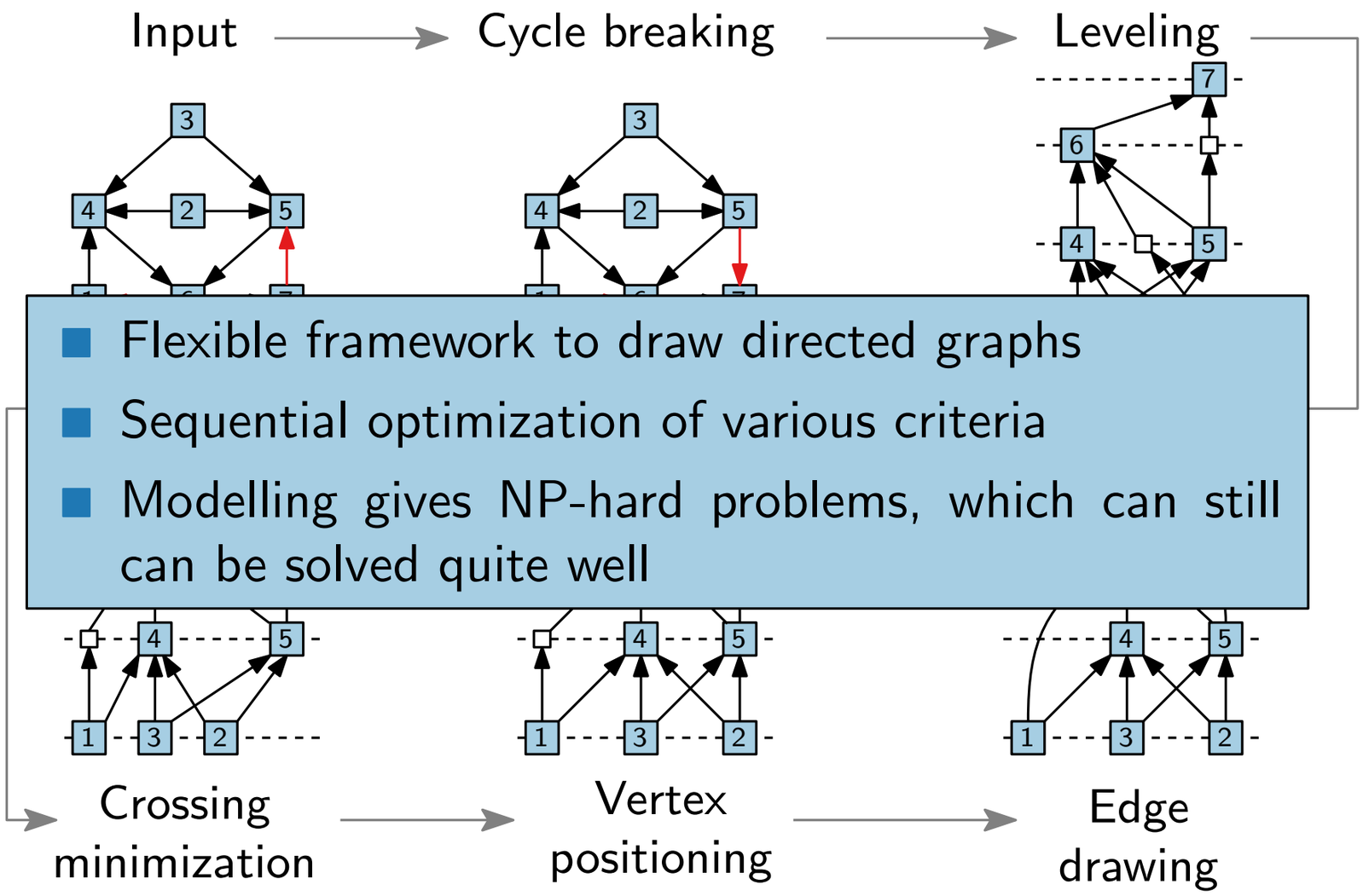

- Flexible framework to draw directed graphs
- Sequential optimization of various criteria
- Modelling gives NP-hard problems, which can still can be solved quite well

Literature

Detailed explanations of steps and proofs in

- [GD Ch. 11] and [DG Ch. 5]

based on

- [Sugiyama, Tagawa, Toda '81] Methods for visual understanding of hierarchical system structures

and refined with results from

- [Berger, Shor '90] Approximation alogorithms for the maximum acyclic subgraph problem
- [Eades, Lin, Smith '93] A fast and effective heuristic for the feedback arc set problem
- [Garey, Johnson '83] Crossing number is NP-complete
- [Eades, Whiteside '94] Drawing graphs in two layers
- [Eades, Wormland '94] Edge crossings in drawings of bipartite graphs
- [Jünger, Mutzel '97] 2-Layer Straightline Crossing Minimization: Performance of Exact and Heuristic Algorithms