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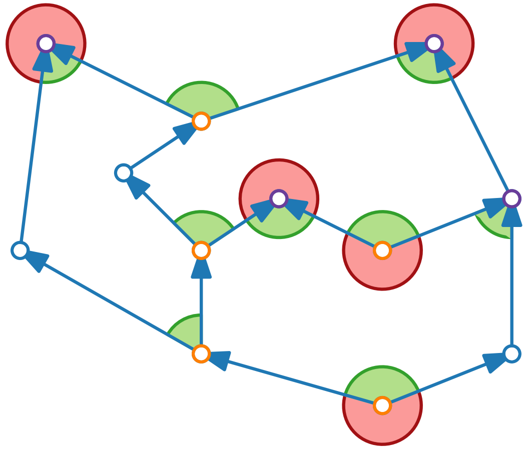
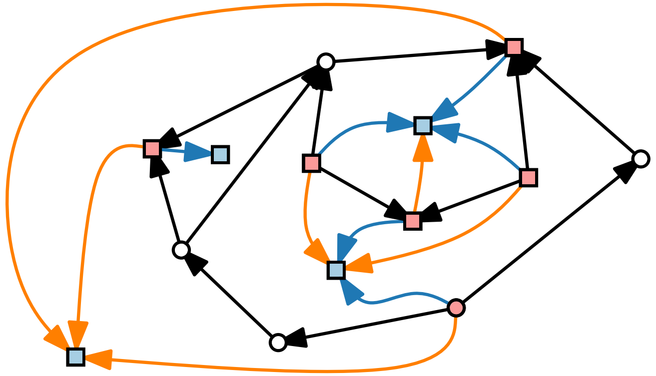
Visualization of Graphs

Lecture 6:
Upward Planar Drawings

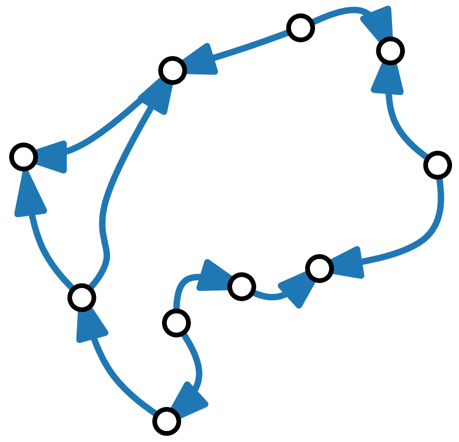


Part I:
Characterization

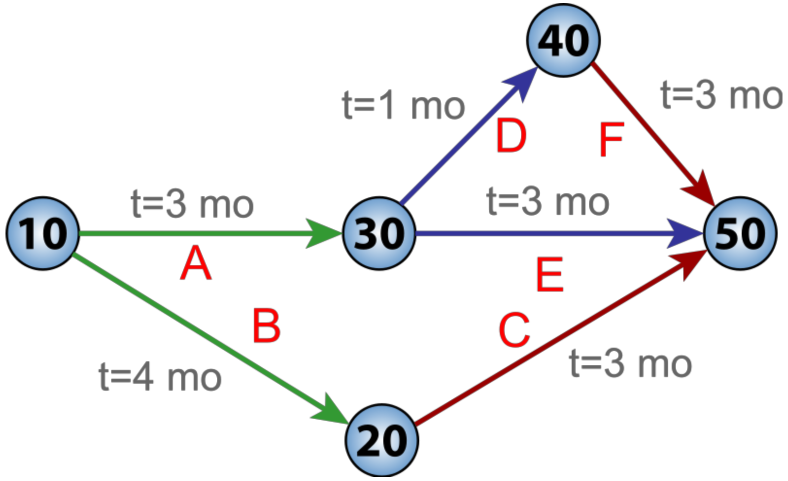
Jonathan Klawitter



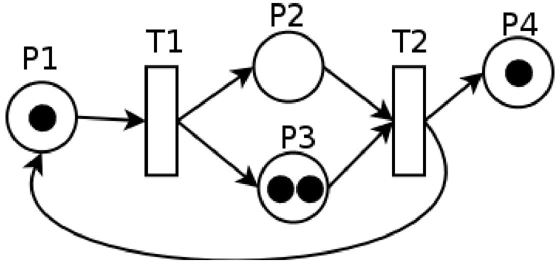
Upward Planar Drawings – Motivation



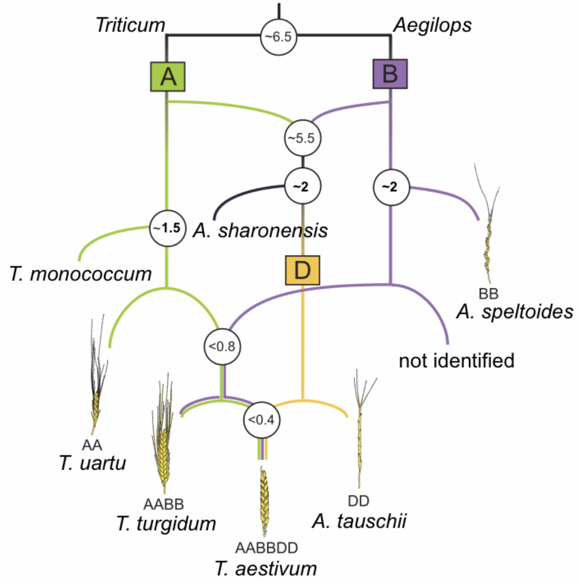
- What may the direction of edges in a digraph represent?
  - Time
  - Flow
  - Hierarchy
  - . . .
- Would be nice to have general direction preserved in drawing.



PERT diagram



Petri net

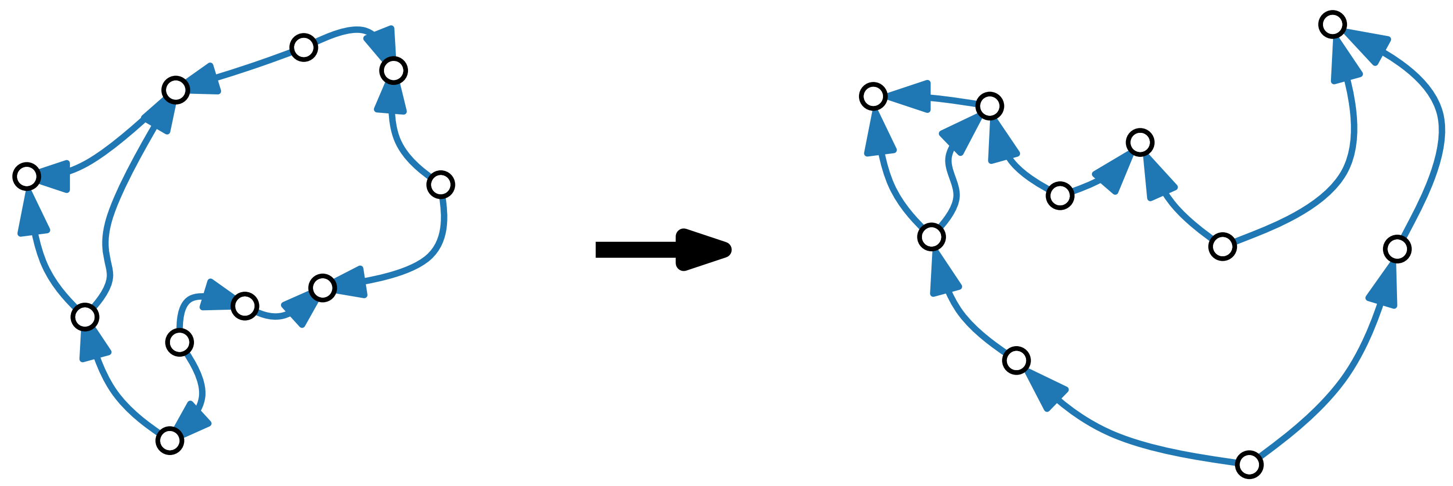


Phylogenetic network

Upward Planar Drawings – Definition

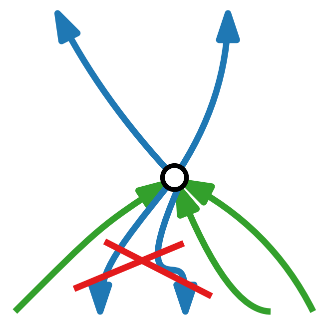
A directed graph $G = (V, E)$ is **upward planar** when it admits a drawing Γ that is

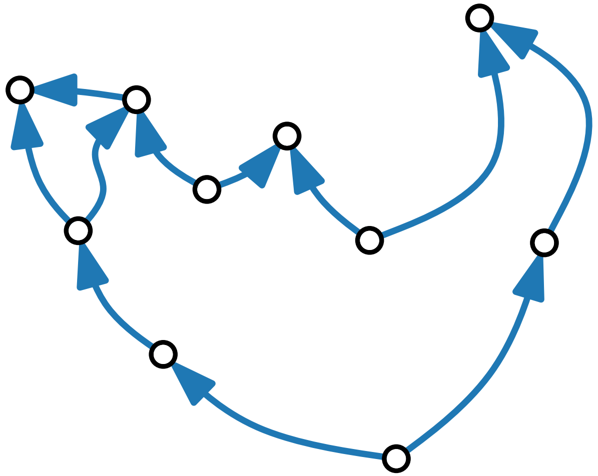
- planar and
- where each edge is drawn as an upward, y-monotone curve.

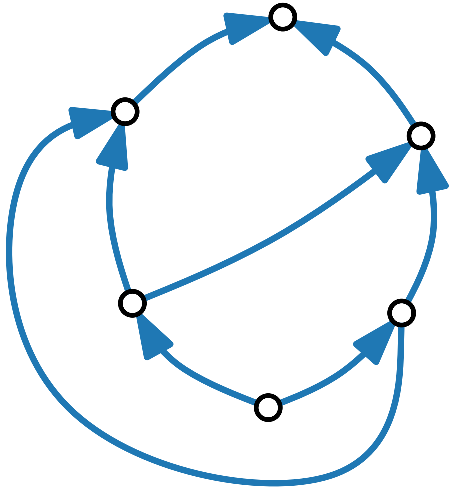


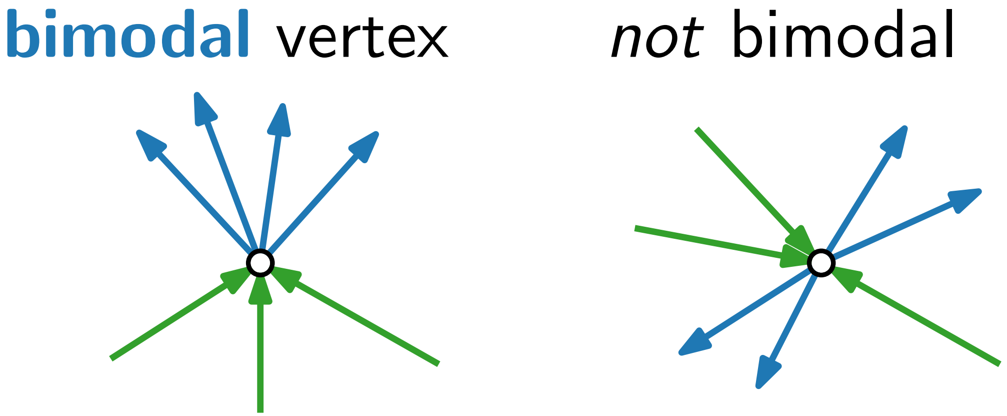
Upward Planarity – Necessary Conditions

- For a digraph G to be upward planar, it has to be:
  - planar
  - acyclic
  - bimodal
- ... but these conditions are *not sufficient*.









Upward Planarity – Characterization

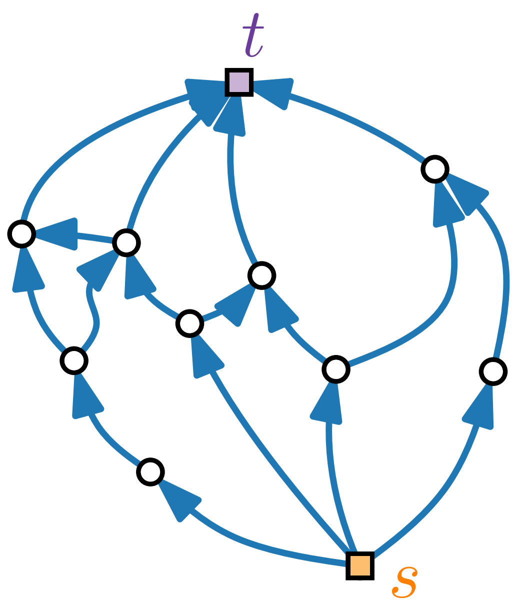
Theorem 1. [Kelly 1987, Di Battista & Tamassia 1988]
For a digraph G the following statements are equivalent:
1. G is upward planar.
2. G admits an upward planar straight-line drawing.
3. G is the spanning subgraph of a planar st-digraph.

no crossings

acyclic digraph with
a single source s and single sink t

Additionally:
Embedded such that s and t are on the outerface f_0.

or:
Edge (s, t) exists.



Upward Planarity – Characterization

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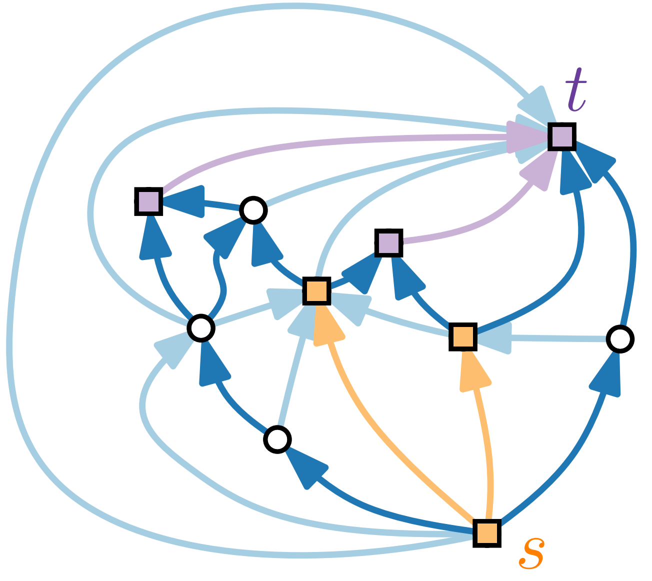


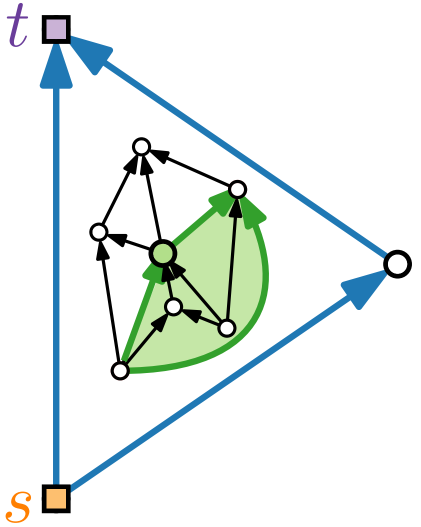
Proof.

(2) ⇒ (1) By definition. **(1) ⇔ (3)** Example:
(3) ⇒ (2) Triangulate & construct drawing:

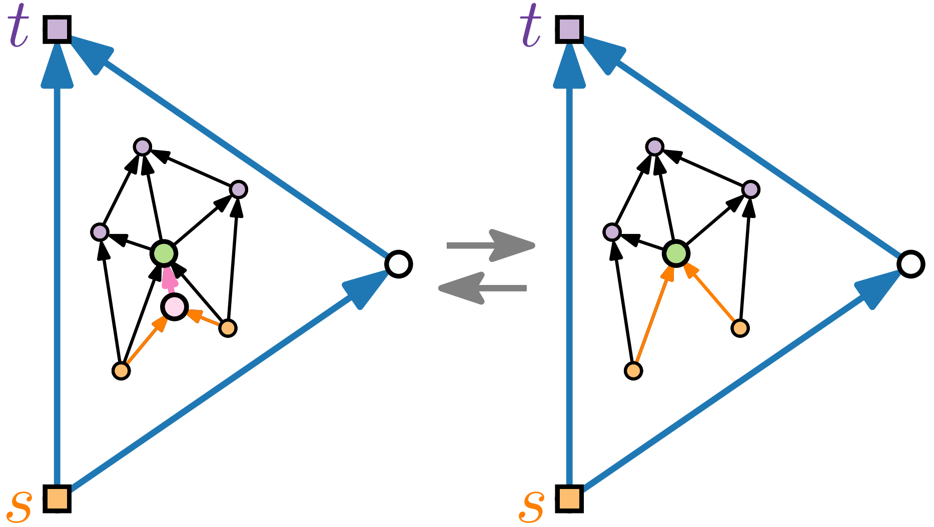
Claim.
Can draw in prespecified triangle.

Induction on n.

Case 1:
chord



Case 2:
no chord



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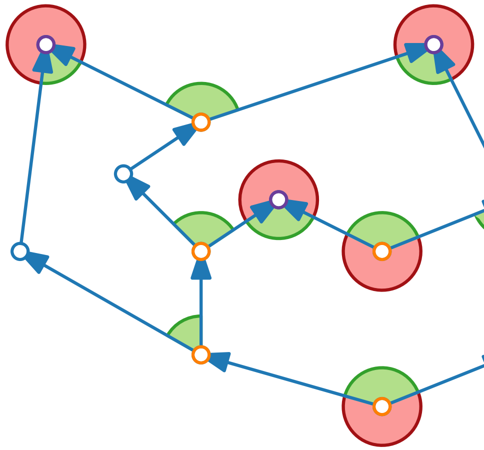
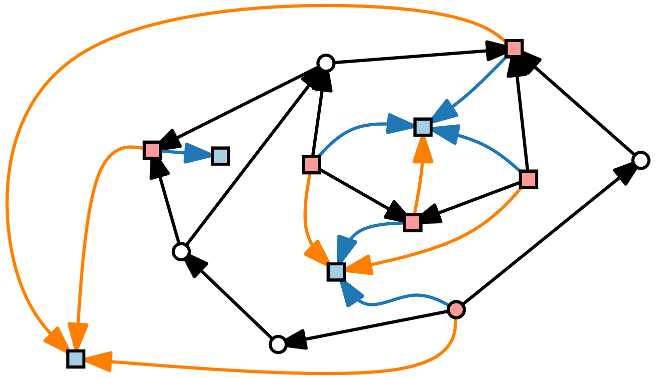
Visualization of Graphs

Lecture 6:
Upward Planar Drawings



Part II:
Assignment Problem



Jonathan Klawitter

Upward Planarity – Complexity

Theorem. [Garg, Tamassia, 1995]
For a *planar acyclic* digraph it is in general NP-hard to decide whether it is upward planar.

Theorem. [Bertolazzi et al., 1994]
For a *combinatorially embedded* planar digraph it can be tested in $\mathcal{O}(n^2)$ time whether it is upward planar.

Corollary.
For a *triconnected* planar digraph it can be tested in $\mathcal{O}(n^2)$ time whether it is upward planar.

Theorem. [Hutton, Libow, 1996]
For a *single-source* acyclic digraph it can be tested in $\mathcal{O}(n)$ time whether it is upward planar.

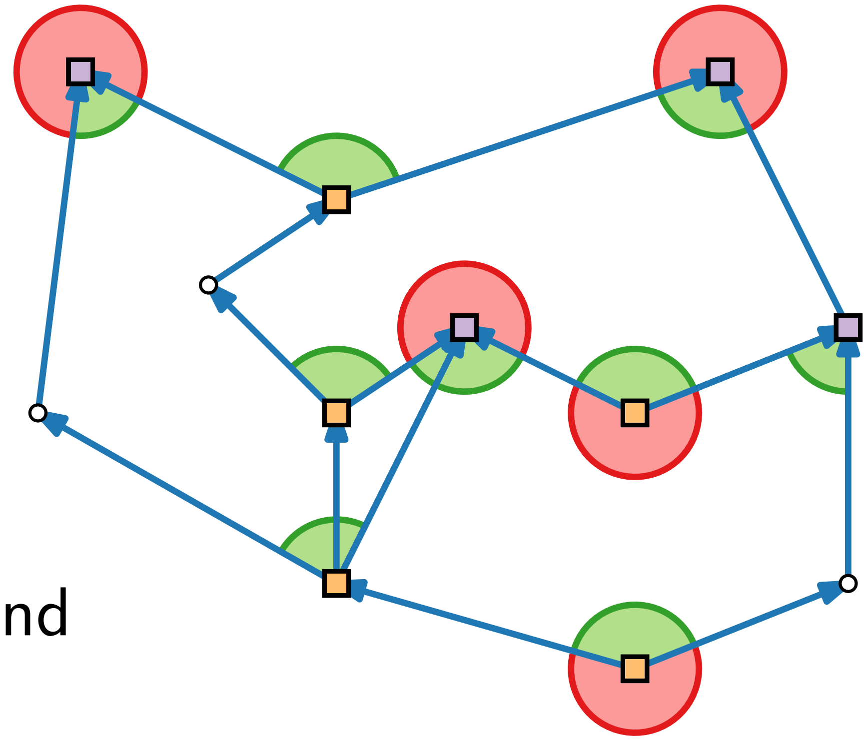
The Problem

Fixed Embedding Upward Planarity Testing.
Let $G = (V, E)$ be a plane digraph with set of faces F and outer face f_0.
Test whether G is upward planar (wrt to F, f_0).

Idea.

- Find property that any upward planar drawing of G satisfies.
- Formalise property.
- Find algorithm to test property.

Angles, Local Sources & Sinks



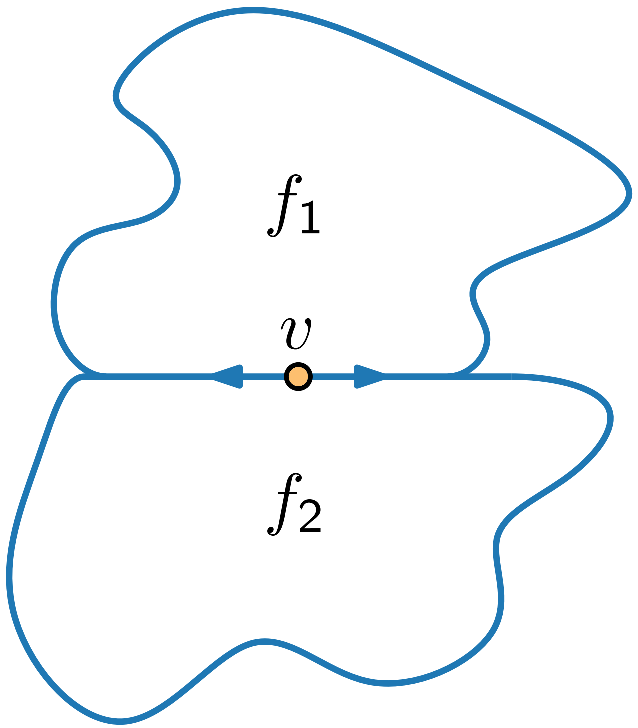
Definitions.

- A vertex v is a **local source** wrt to a face f if v has two outgoing edges on ∂f.
- A vertex v is a **local sink** wrt to a face f if v has two incoming edges on ∂f.
- An angle α at a local source / sink is **large** when $\alpha > \pi$ and **small** otherwise.
- $L(v) = \#$ large angles at v
- $L(f) = \#$ large angles in f
- $S(v)$ & $S(f)$ for $\#$ small angles
- $A(f) = \#$ local sources wrt to f
  $= \#$ local sinks wrt to f

Lemma 1.
$L(f) + S(f) = 2A(f)$

Assignment Problem

- Vertex v is a global source at faces f_1 and f_2.
- Does v have a **large** angle in f_1 or f_2?



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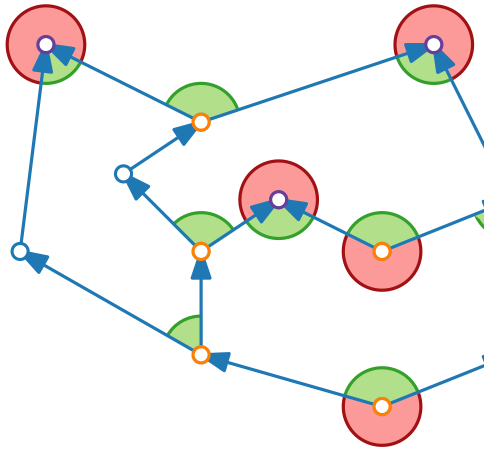
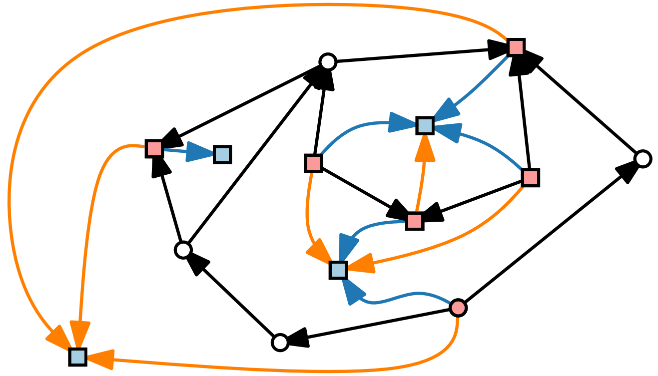


Part III:
Angle Relations



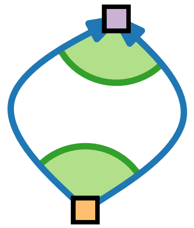
Jonathan Klawitter

Angle Relations

Lemma 2.
$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

Proof by induction.

- $L(f) = 0$

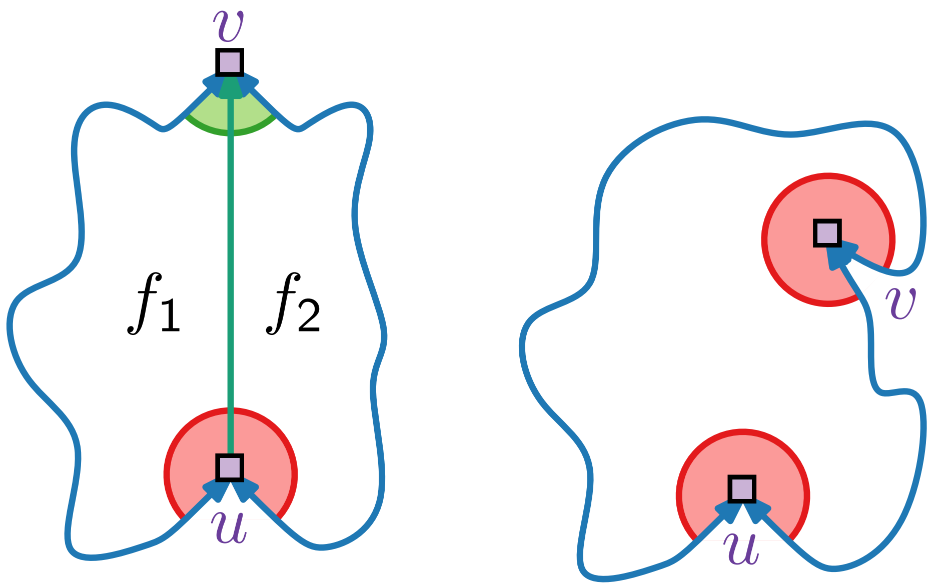


$\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- sink v with small/large angle:



$$L(f) - S(f) = \underbrace{L(f_1)}^{-2} + \underbrace{L(f_2)}^{-2} + 1 - (S(f_1) + S(f_2) - 1) = -2$$

Angle Relations

Lemma 2.

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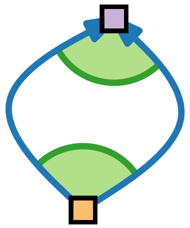

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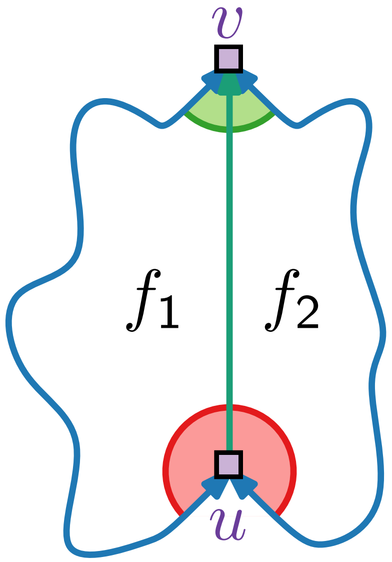

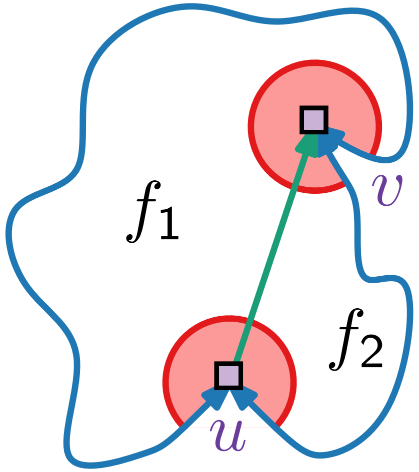

$$\begin{aligned} L(f) - S(f) &= \overset{-2}{L(f_1)} + \overset{-2}{L(f_2)} + 2 \\ &\quad - (S(f_1) + S(f_2)) \\ &= -2 \end{aligned}$$

Angle Relations

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Proof by induction.

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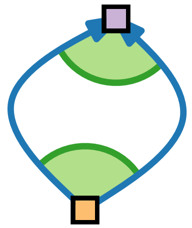
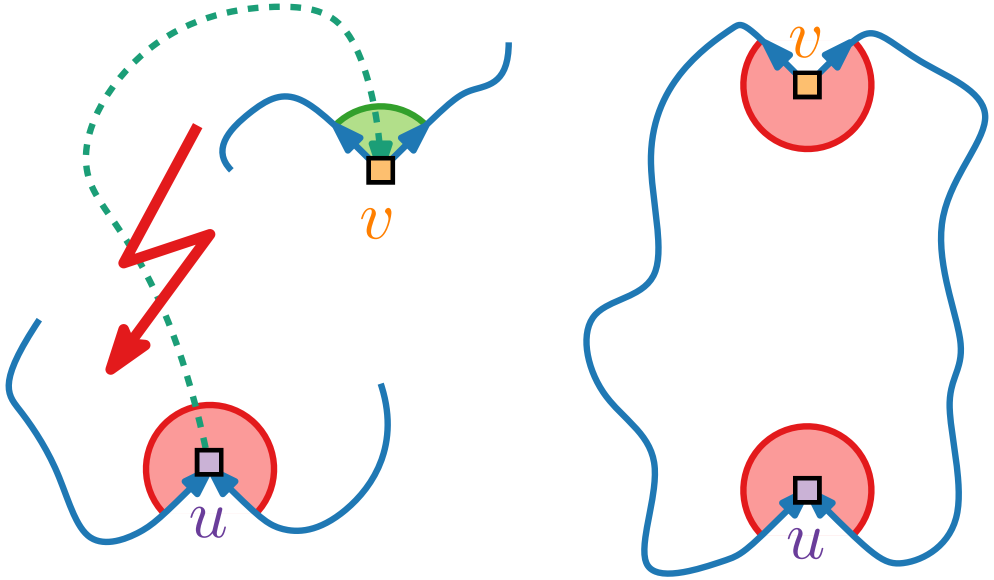


$\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- source v with ~~small~~/large angle:



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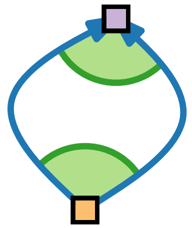
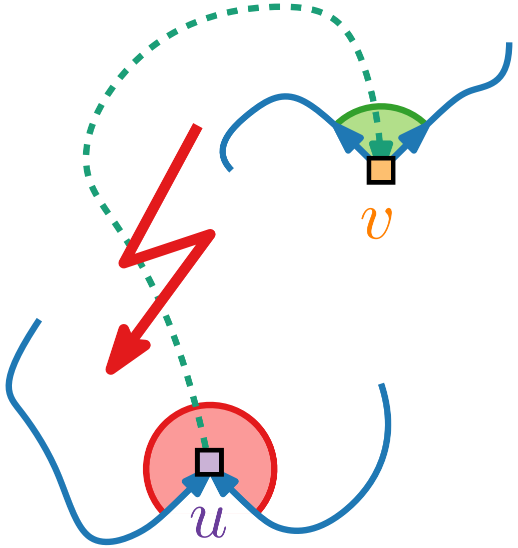


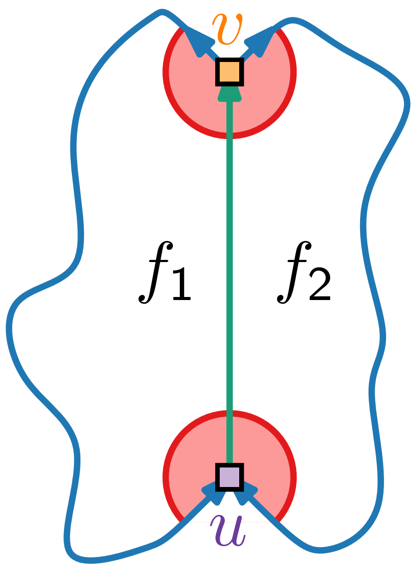
$\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- source v with ~~small~~/large angle:





$$L(f) - S(f) = \overset{-2}{L(f_1)} + \overset{-2}{L(f_2)} + 2 - (S(f_1) + S(f_2)) = -2$$

Angle Relations

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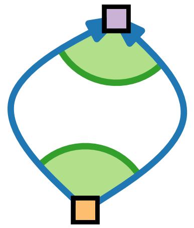
Proof by induction.

- $L(f) = 0$

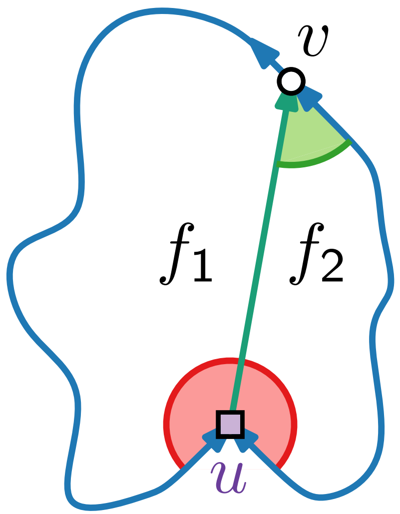


$\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

- vertex v that is neither source nor sink:



$$\begin{aligned} L(f) - S(f) &= \overbrace{L(f_1)}^{-2} + \overbrace{L(f_2)}^{-2} + 1 \\ &\quad - (S(f_1) + S(f_2) - 1) \\ &= -2 \end{aligned}$$

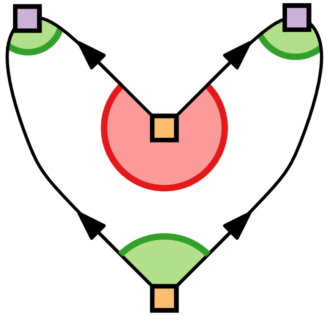
- Otherwise "high" source u exists.

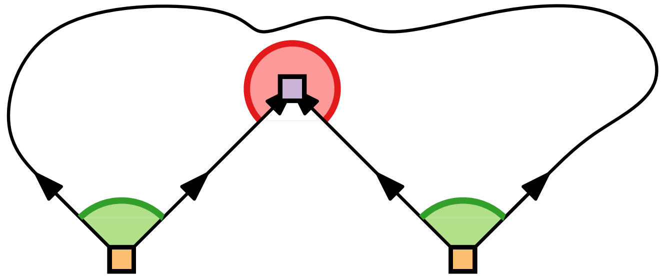
Number of Large Angles

Lemma 3.
In every upward planar drawing of G holds that

- for each vertex $v \in V$: $L(v) = \begin{cases} 0 & v \text{ inner vertex,} \\ 1 & v \text{ source / sink;} \end{cases}$
- for each face f: $L(f) = \begin{cases} A(f) - 1 & f \neq f_0, \\ A(f) + 1 & f = f_0. \end{cases}$

Proof. Lemma 1: $L(f) + S(f) = 2A(f)$
Lemma 2: $L(f) - S(f) = \pm 2.$
$\Rightarrow 2L(f) = 2A(f) \pm 2.$





Assignment of Large Angles to Faces

Let S and T be the sets of sources and sinks, respectively.

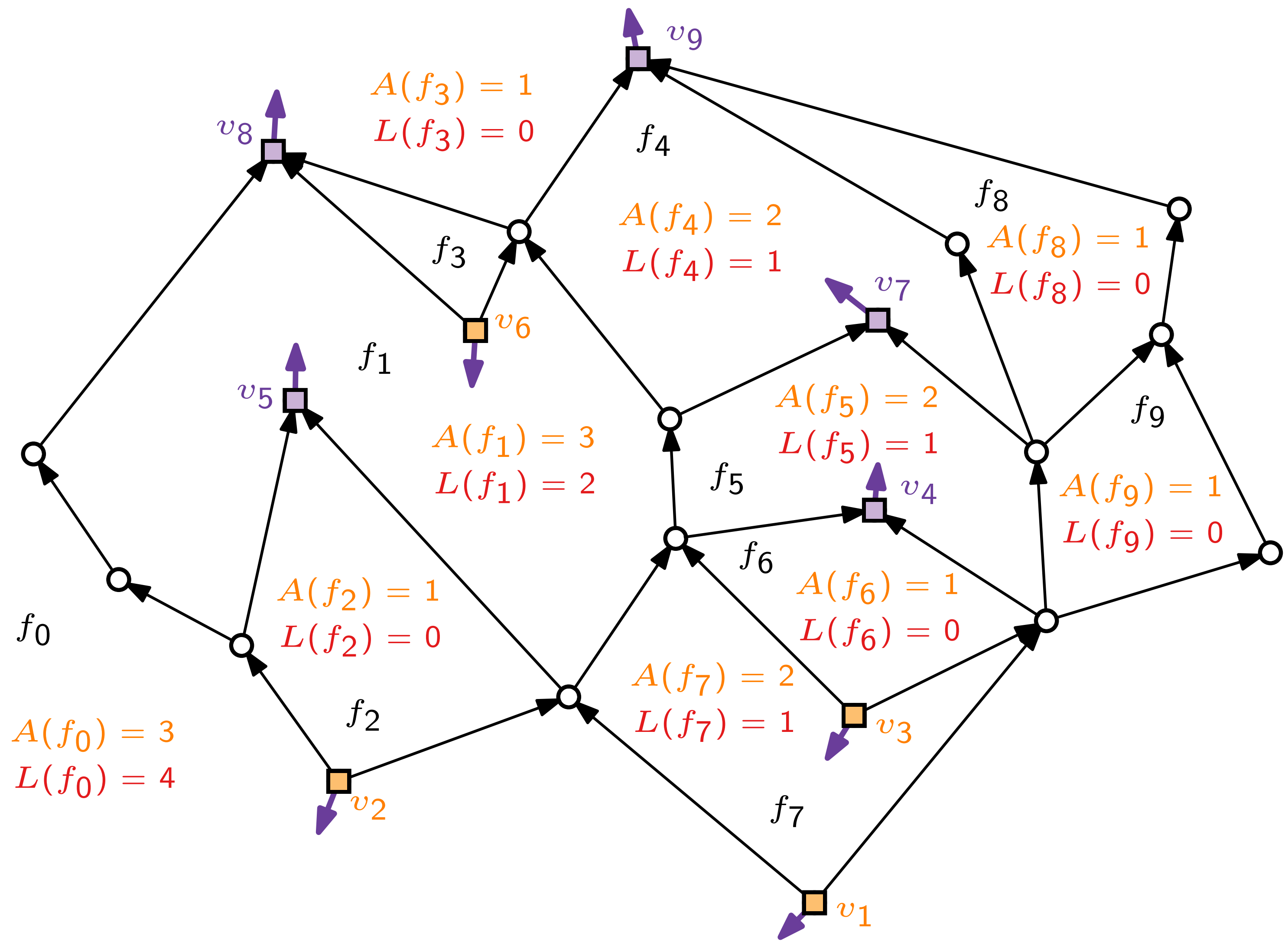
Definition.
A **consistent assignment** $\Phi \colon S \cup T \to F$ is a mapping where

$$\Phi \colon v \mapsto \text{ incident face, where } v \text{ forms large angle}$$

such that

$$|\Phi^{-1}(f)| = L(f) = \begin{cases} A(f) - 1 & \text{if } f \neq f_0, \\ A(f) + 1 & \text{if } f = f_0. \end{cases}$$

Example of Angle to Face Assignment



global sources & sinks

$A(f)$ # sources / sinks of f

$L(f)$ # large angles of f

assignment
$\Phi : S \cup T \to F$

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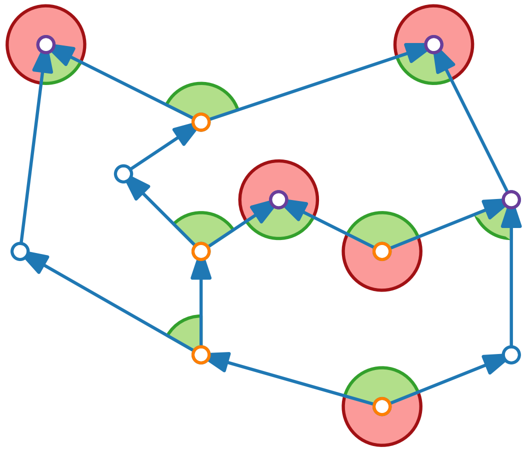
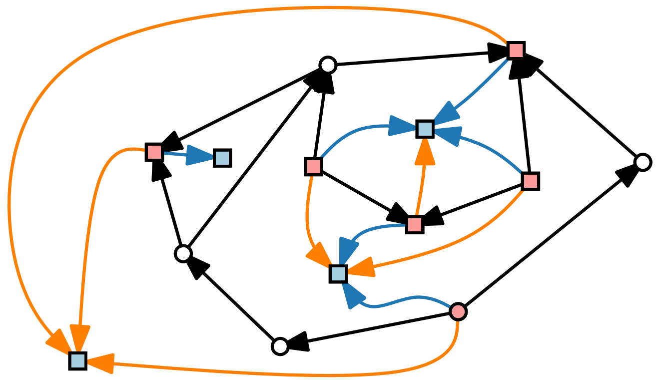


Part IV:
Refinement Algorithm



Jonathan Klawitter

Result Characterization

Theorem 3.
Let $G = (V, E)$ be an acyclic plane digraph with embedding given by F, f_0.
Then G is upward planar (respecting F, f_0) if and only if G is bimodal and there exists consistent assignment Φ.

Proof.

$\Rightarrow$: As constructed before.

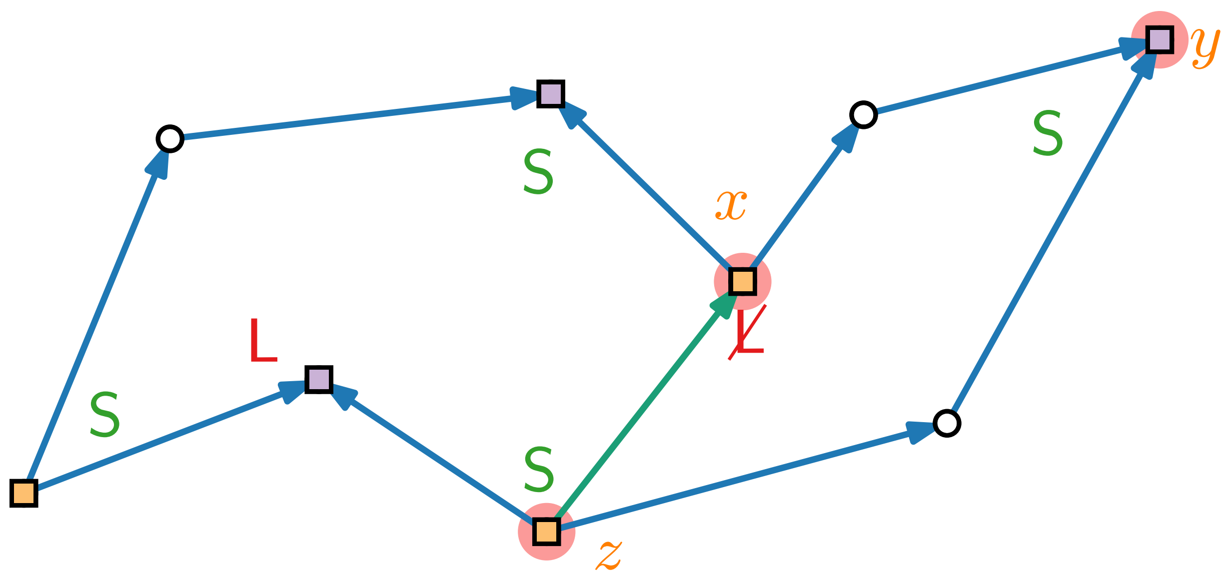
$\Leftarrow$: Idea:

- Construct planar st-digraph that is supergraph of G.
- Apply equivalence from Theorem 1.

Refinement Algorithm – $\Phi, F, f_0 \to$ st-digraph

Let f be a face. Consider the clockwise angle sequence σ_f of L/S on local sources and sinks of f.

- Goal: Add edges to break large angles (sources and sinks).
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing $\langle$L, S, S$\rangle$ at vertices x, y, z:
- x source $\Rightarrow$ insert edge (z, x)



Refinement Algorithm – $\Phi, F, f_0 \rightarrow$ st-digraph

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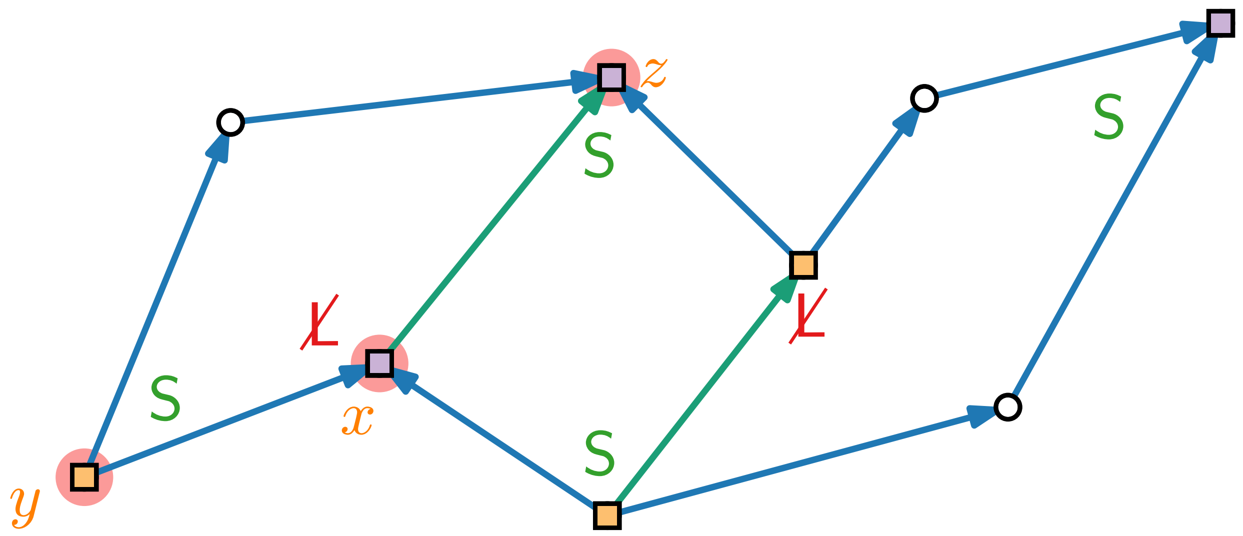
- x sink $\Rightarrow$ insert edge (x, z).



Refinement Algorithm – $\Phi, F, f_0 \to$ st-digraph

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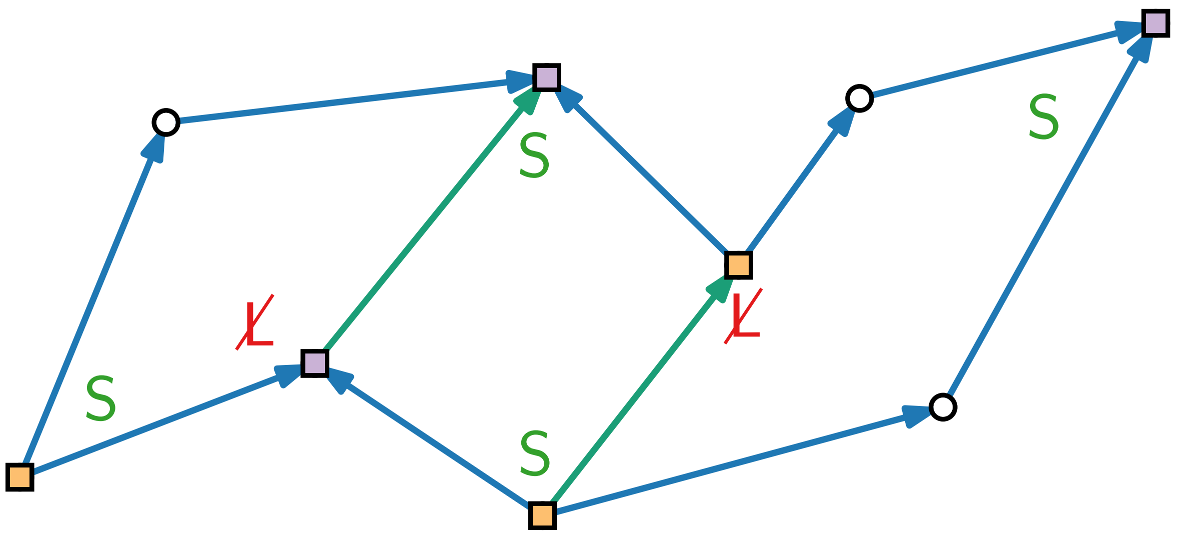
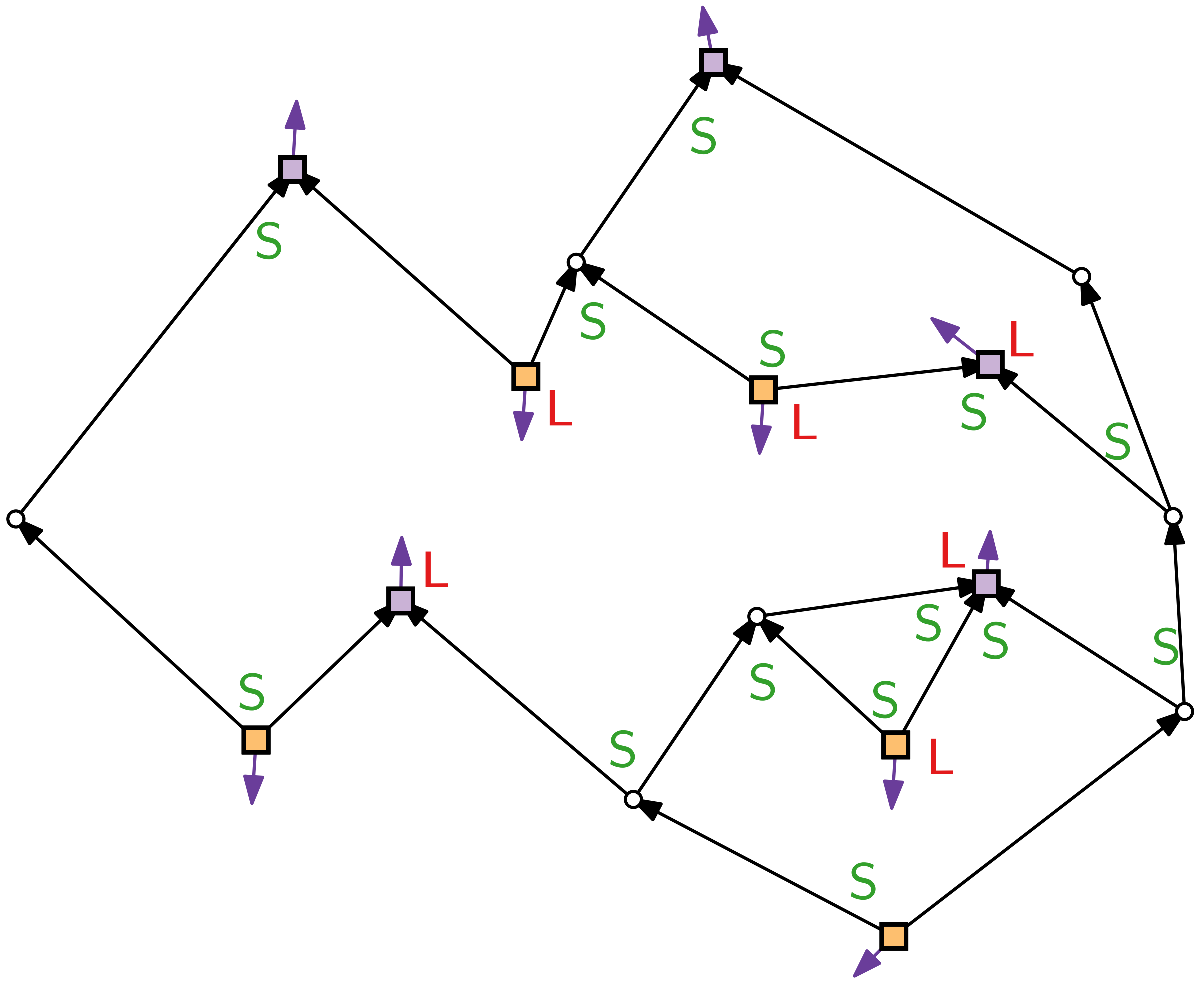
- For $f \neq f_0$ with $|\sigma_f| \geq 2$ containing ⟨L, S, S⟩ at vertices x, y, z:
- x source $\Rightarrow$ insert edge (z, x)
- x sink $\Rightarrow$ insert edge (x, z).
- Refine outer face f_0.

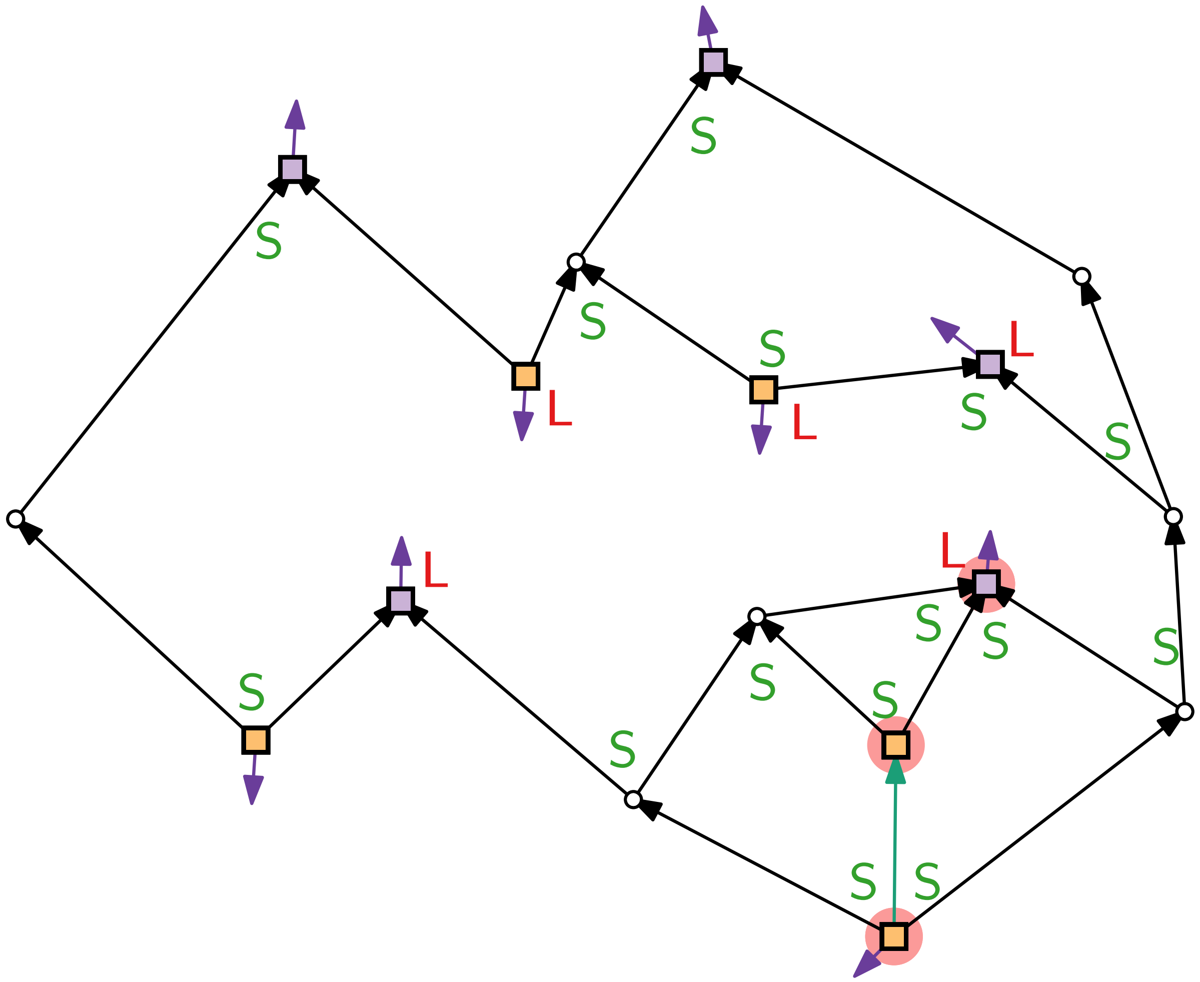


- Refine all faces. $\Rightarrow$ G is contained in a planar st-digraph.
- Planarity, acyclicity, bimodality are invariants under construction.

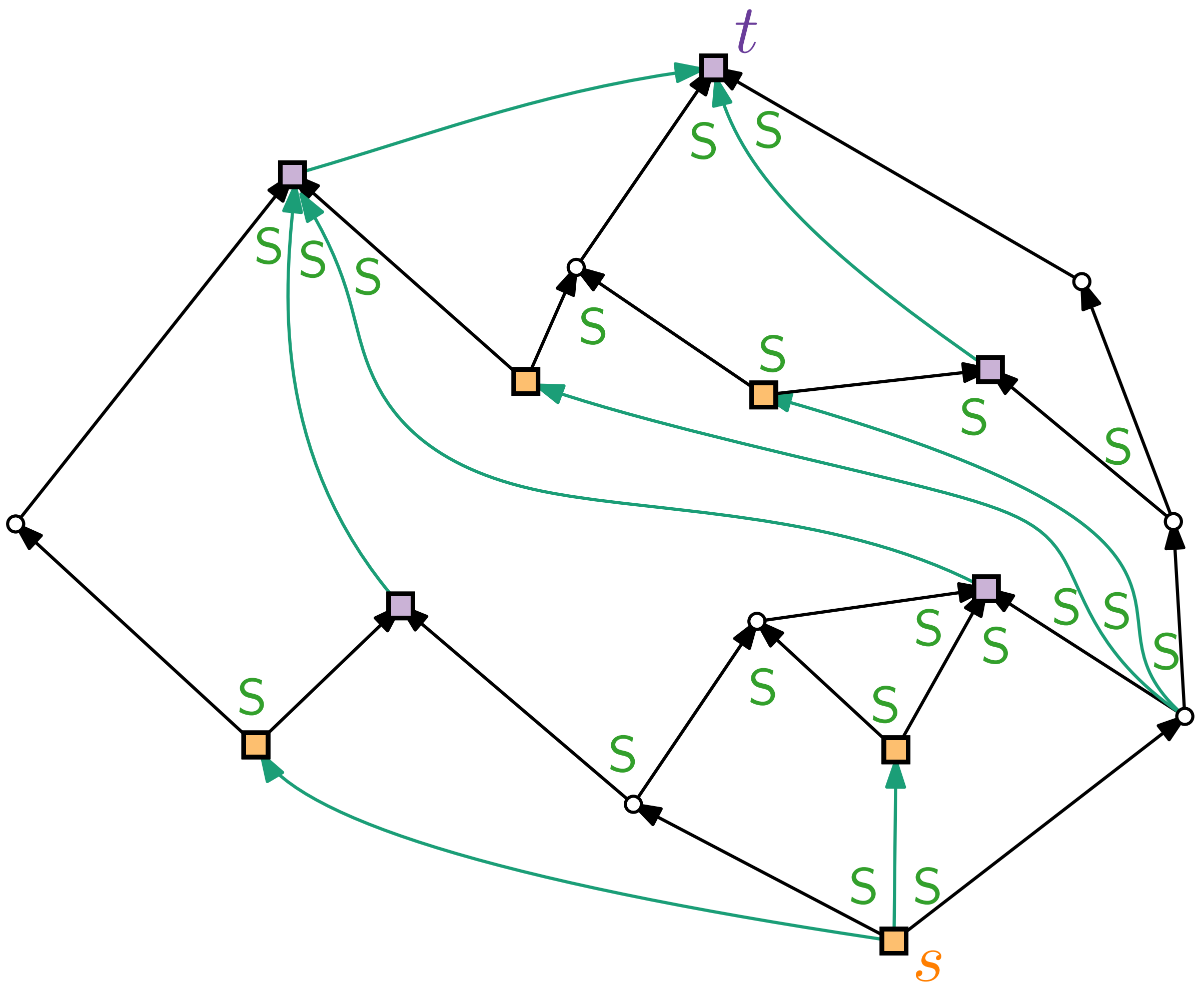
Refinement Example



Refinement Example



Refinement Example



Result Upward Planarity Test

Theorem 2. [Bertolazzi et al., 1994]
For a *combinatorially embedded* planar digraph G it can be tested in $\mathcal{O}(n^2)$ time whether it is upward planar.

Proof.

- Test for bimodality.
- Test for a consistent assignment Φ (via flow network).
- If G bimodal and Φ exists, refine G to plane st-digraph H.
- Draw H upward planar.
- Deleted edges added in refinement step.

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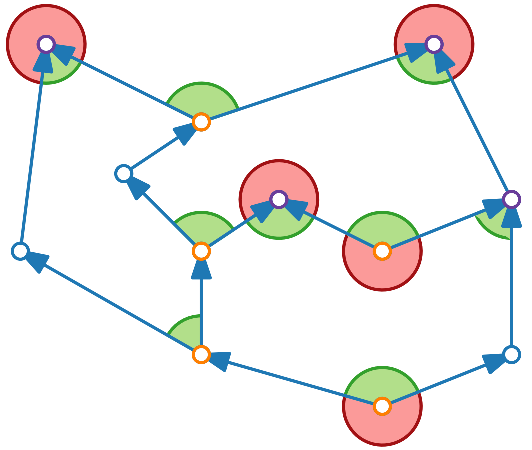
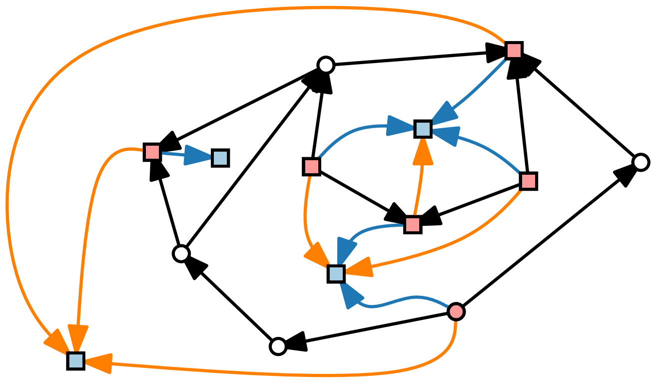


Part V:
Finding a Consistent Assignment



Jonathan Klawitter

Finding a Consistent Assignment

Idea.

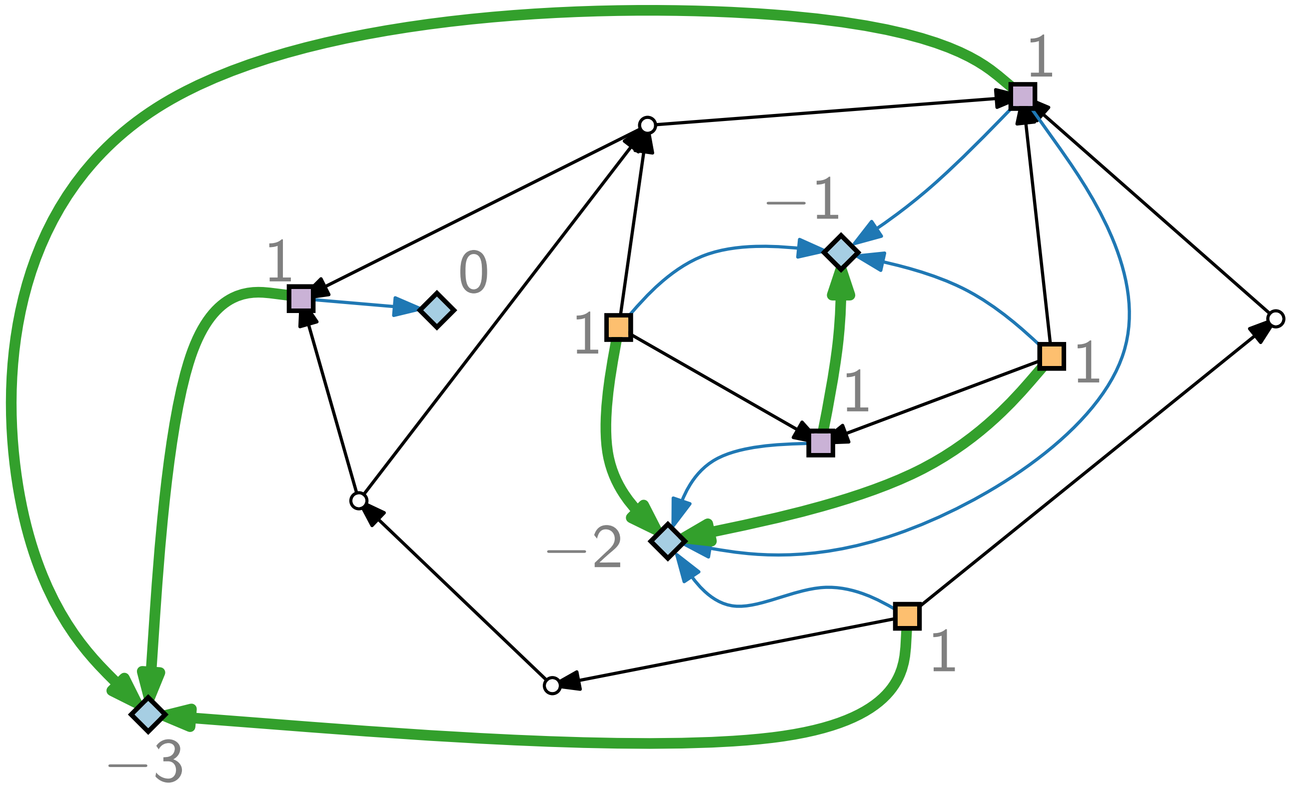
Flow $(v, f) = 1$ from global source / sink v to the incident face f its large angle gets assigned to.

Flow network.

$N_{F,f_0}(G) = ((W, E'); b; \ell; u)$

- $W = \{v \in V \mid v$ source or sink$\} \cup F$
- $E' = \{(v, f) \mid v$ incident to $f\}$
- $\ell(e) = 0 \ \forall e \in E'$
- $u(e) = 1 \ \forall e \in E'$
- $b(w) = \begin{cases} 1 & \forall w \in W \cap V \\ -(A(w) - 1) & \forall w \in F \setminus \{f_0\} \\ -(A(w) + 1) & w = f_0 \end{cases}$

Example.



Discussion

- There exist fixed-parameter tractable algorithms to test upward planarity of general digraphs with the parameter being the number of triconnected components.
  [Healy, Lynch 2005, Didimo et al. 2009]

- Finding assignment in Theorem 2 can be sped up to $\mathcal{O}(n + r^{1.5})$ where $r = \#$ sources / sinks.
  [Abbasi, Healy, Rextin 2010]

- Many related concepts have been studied: quasi-planarity, upward drawings of mixed graphs, upward planarity on cyclinder/torus, ...

Literature

- [GD Ch. 6] for detailed explanation

Orginal papers referenced:

- [Kelly '87] Fundamentals of Planar Ordered Sets
- [Di Battista, Tamassia '88] Algorithms for Plane Representations of Acyclic Digraphs
- [Garg, Tamassia '95] On the Computational Complexity of Upward and Rectilinear Planarity Testing
- [Hutton, Lubiw '96] Upward Planar Drawing of Single-Source Acyclic Digraphs
- [Bertolazzi, Di Battista, Mannino, Tamassia '94] Upward Drawings of Triconnected Digraphs
- [Healy, Lynch '05] Building Blocks of Upward Planar Digraphs
- [Didimo, Giardano, Liotta '09] Upward Spirality and Upward Planarity Testing
- [Abbasi, Healy, Rextin '10] Improving the running time of embedded upward planarity testing