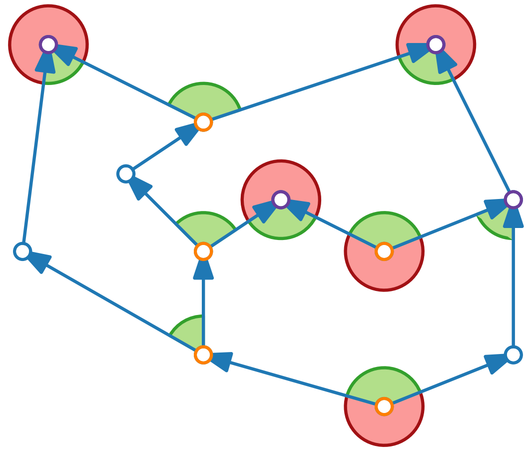
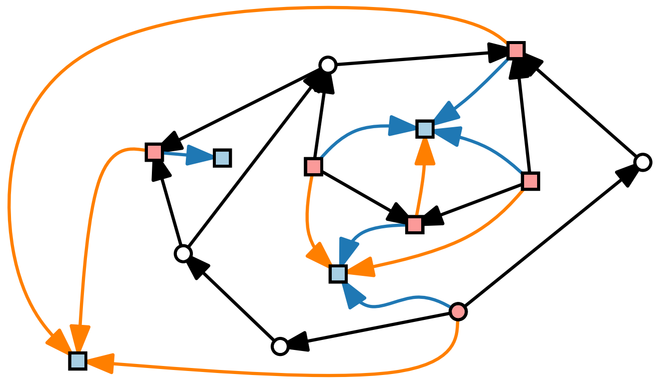
Visualization of Graphs

Lecture 6:
Upward Planar Drawings

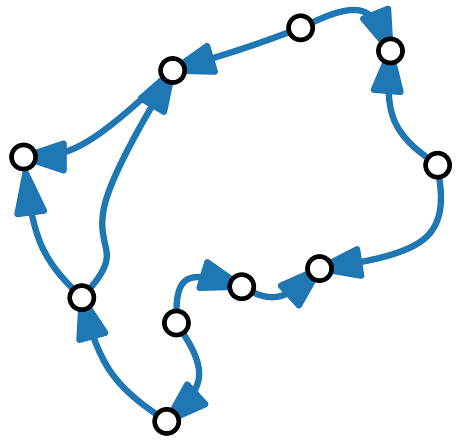


Part I:
Characterization



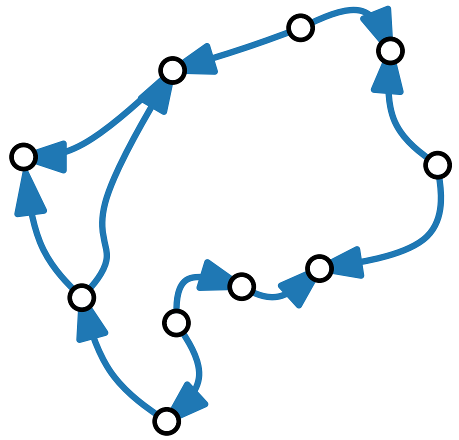
Jonathan Klawitter

Upward Planar Drawings – Motivation



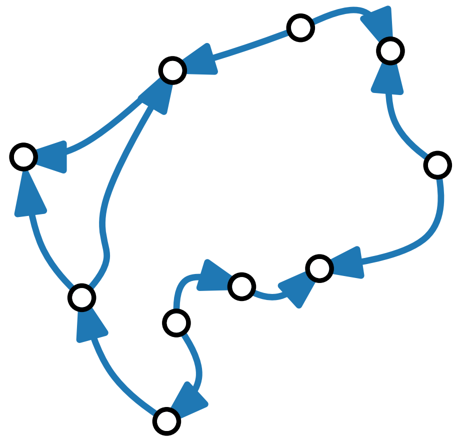
Upward Planar Drawings – Motivation

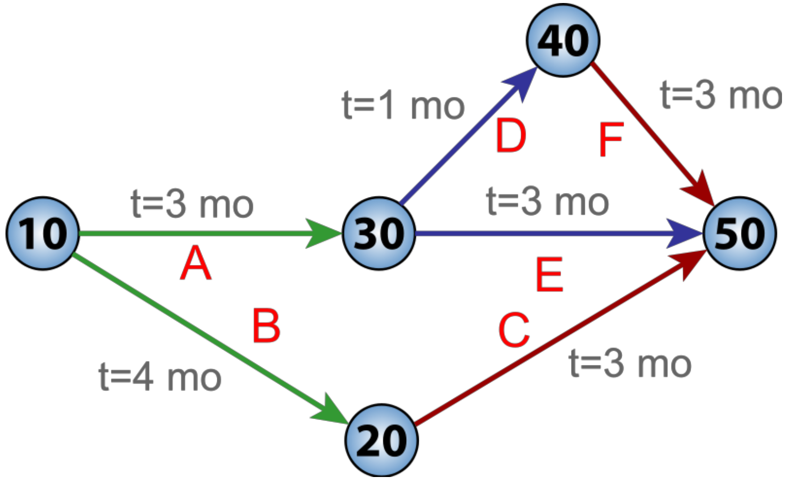
- What may the direction of edges in a digraph represent?



Upward Planar Drawings – Motivation

- What may the direction of edges in a digraph represent?
  - Time

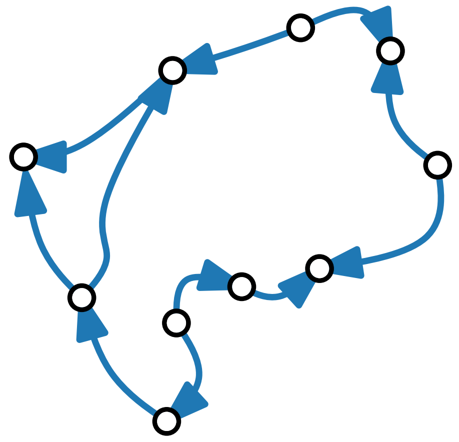


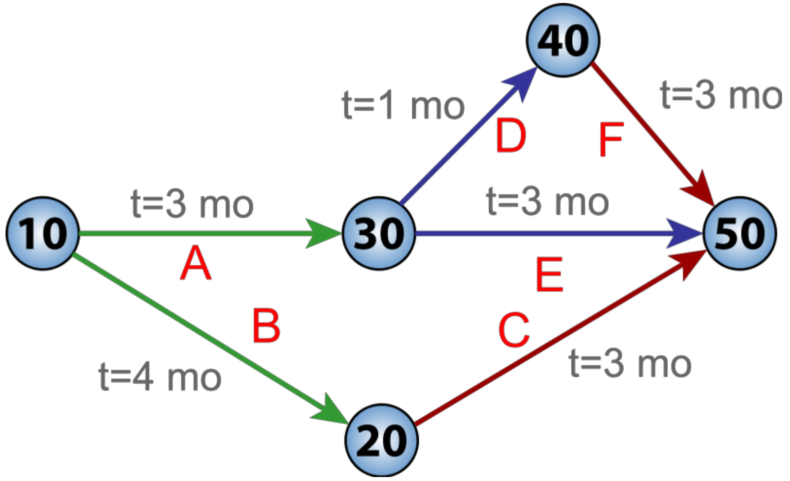


PERT diagram

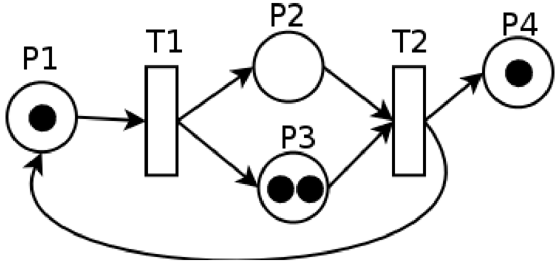
Upward Planar Drawings – Motivation

- What may the direction of edges in a digraph represent?
  - Time
  - Flow



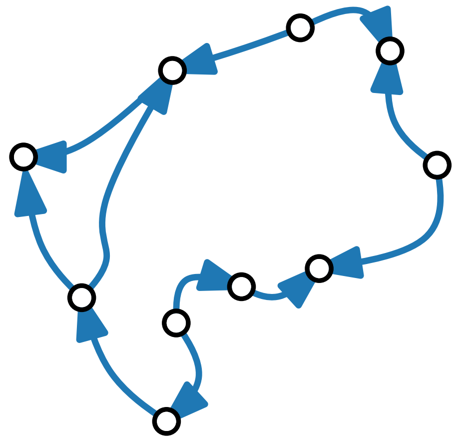


PERT diagram

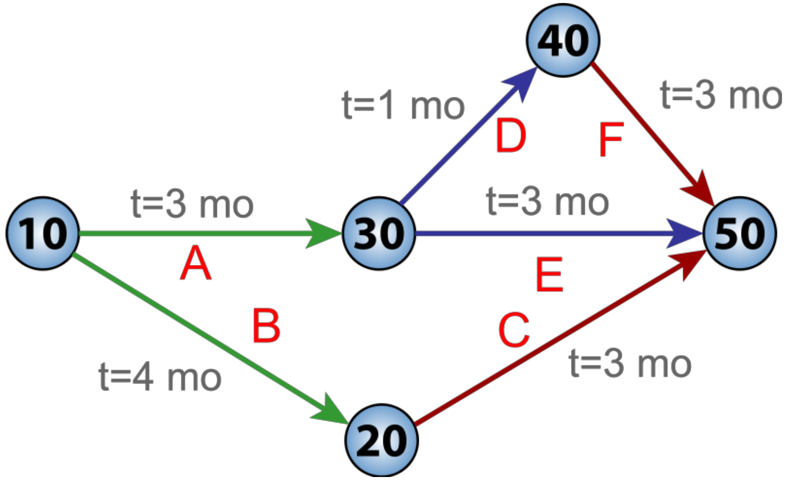


Petri net

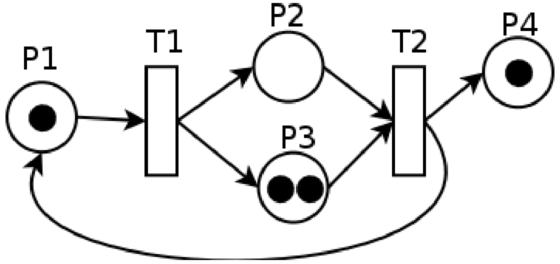
Upward Planar Drawings – Motivation



- What may the direction of edges in a digraph represent?
  - Time
  - Flow
  - Hierarchy



PERT diagram



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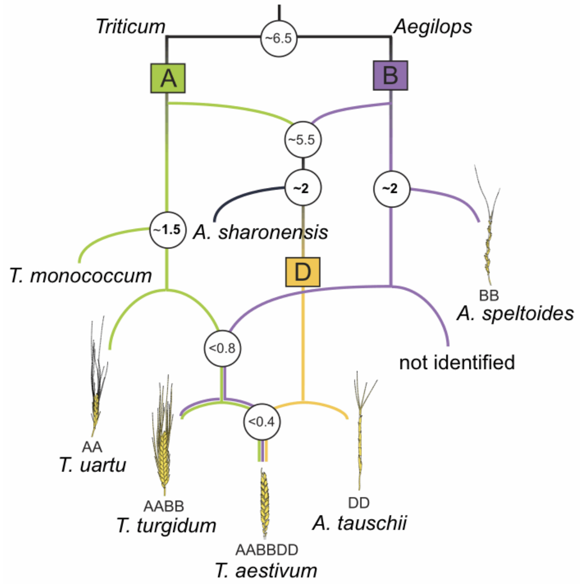
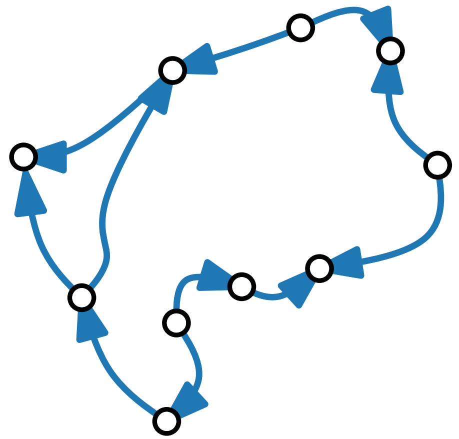
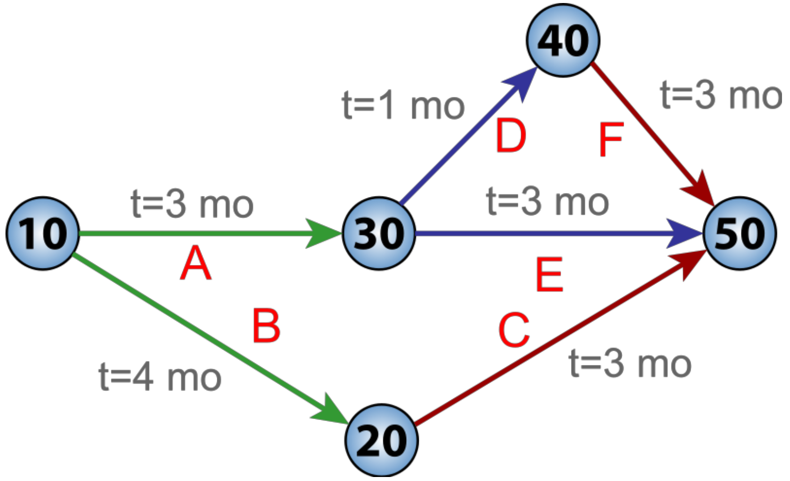


Phylogenetic network

Upward Planar Drawings – Motivation

- What may the direction of edges in a digraph represent?
  - Time
  - Flow
  - Hierarchy
  - ...





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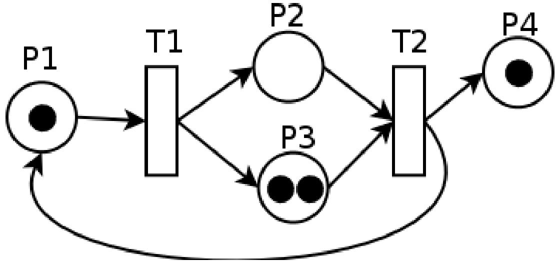

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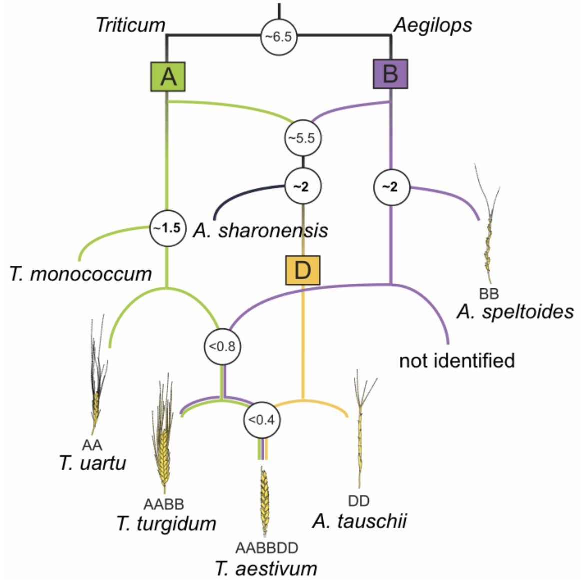
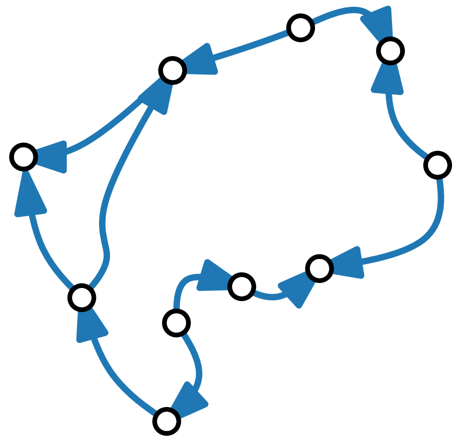
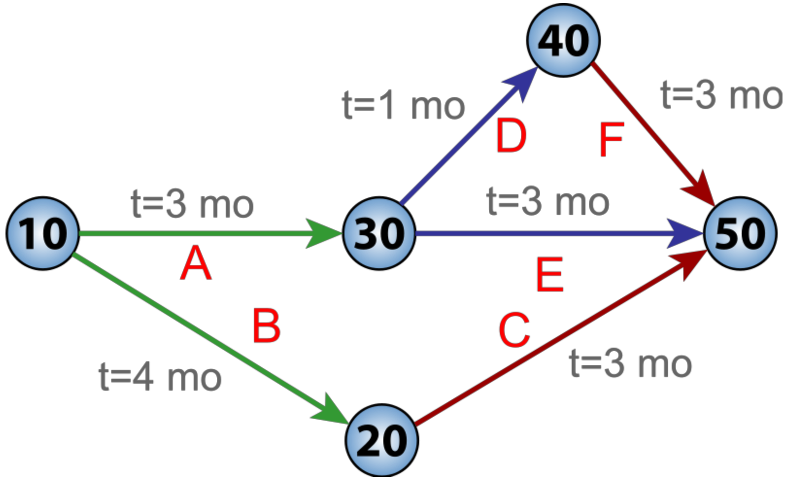


Phylogenetic network

Upward Planar Drawings – Motivation



- What may the direction of edges in a digraph represent?
  - Time
  - Flow
  - Hierarchy
  - . . .
- Would be nice to have general direction preserved in drawing.



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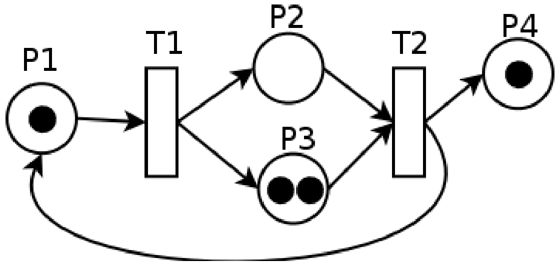
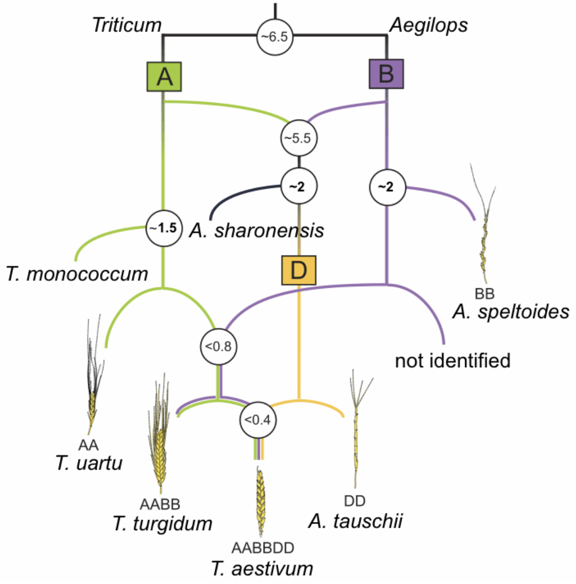


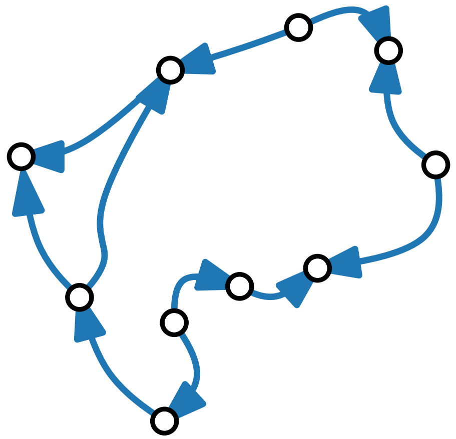
Petri net



Phylogenetic network

Upward Planar Drawings – Definition

A directed graph $G = (V, E)$ is **upward planar** when it admits a drawing Γ that is



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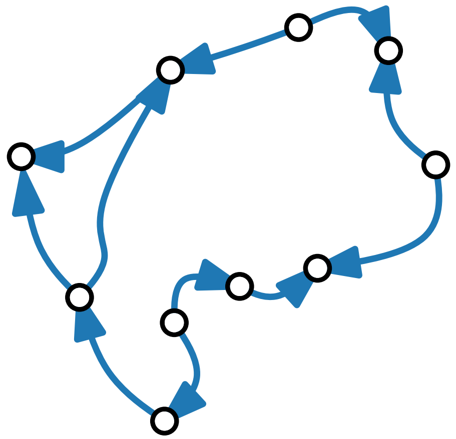

Upward Planar Drawings – Definition

A directed graph $G = (V, E)$ is **upward planar** when it admits a drawing Γ that is

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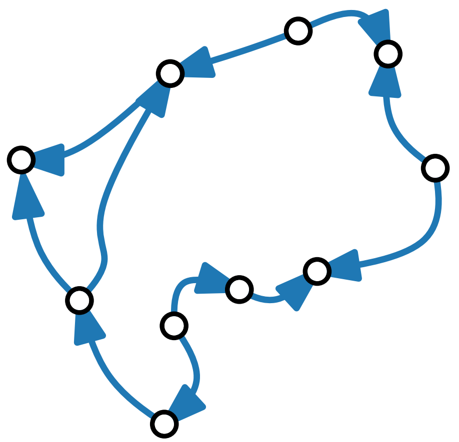

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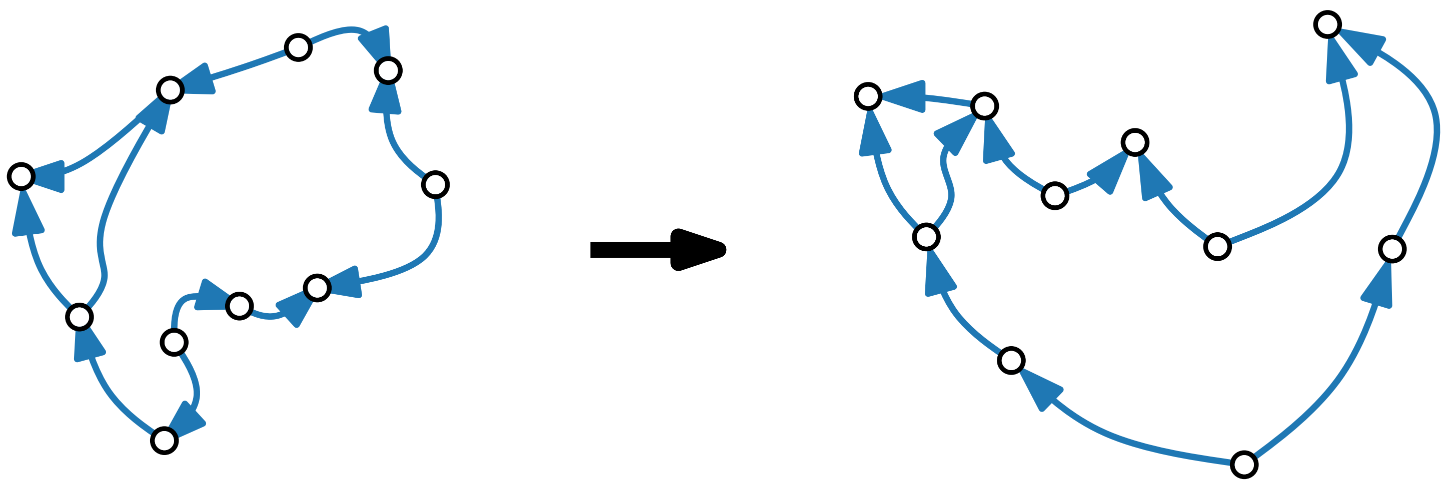
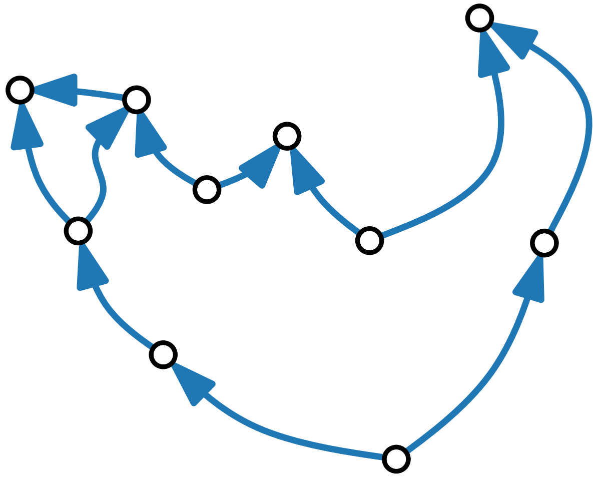


Upward Planarity – Necessary Conditions

- For a digraph G to be upward planar, it has to be:
  - planar



Upward Planarity – Necessary Conditions

- For a digraph G to be upward planar, it has to be:
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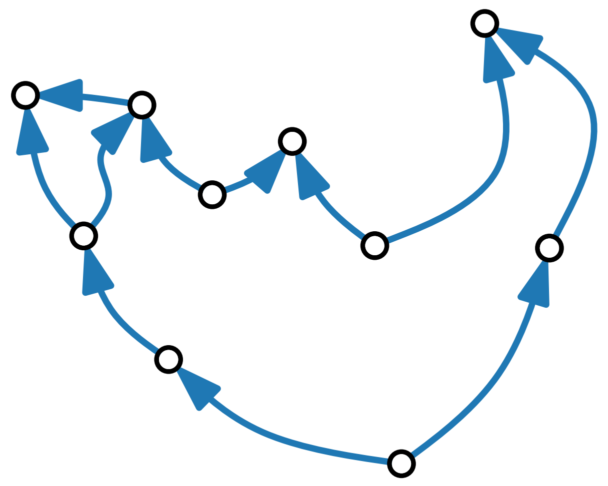

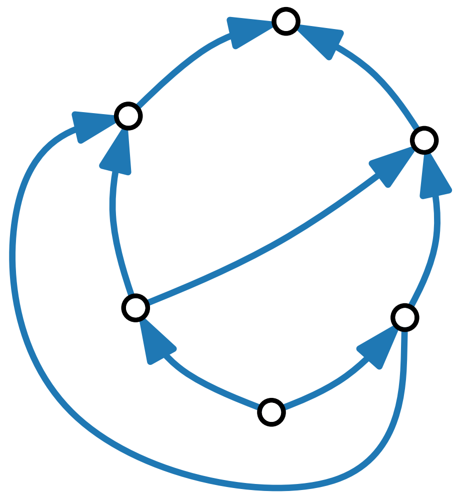

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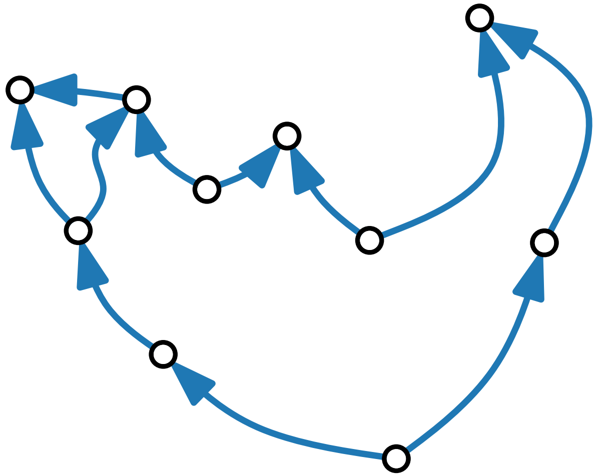

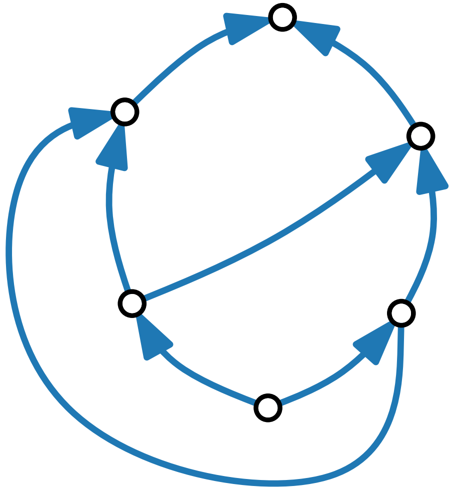

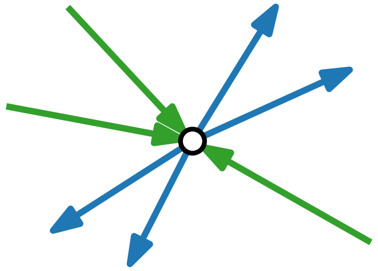

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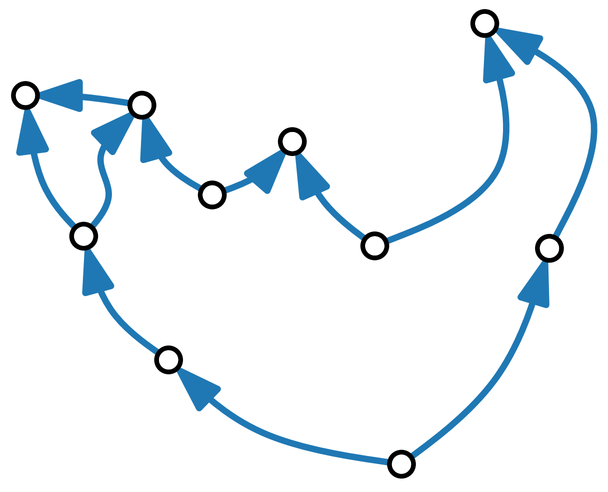

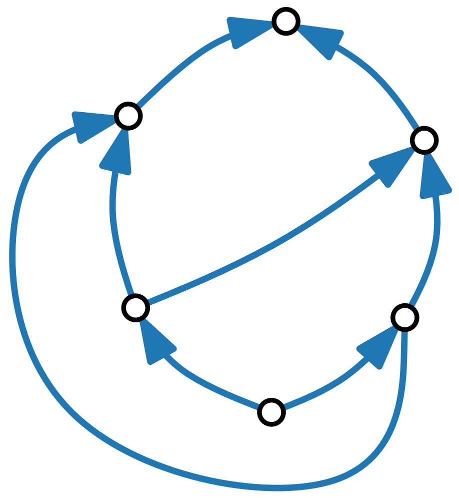

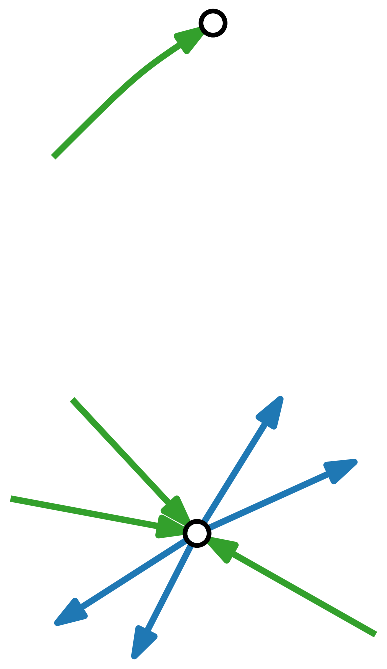

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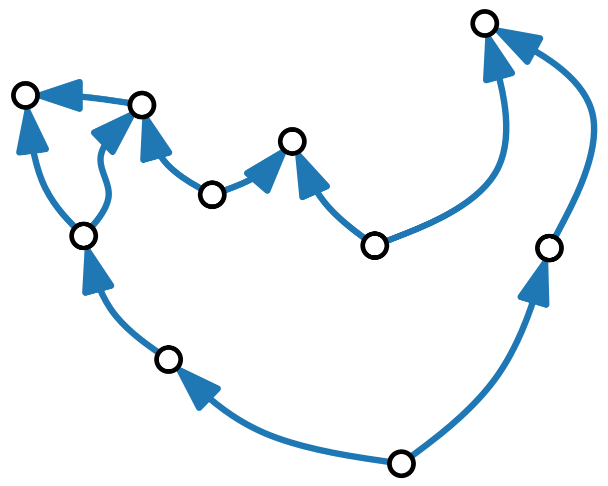

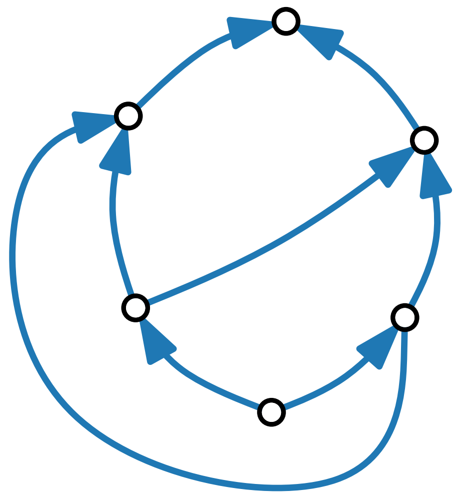

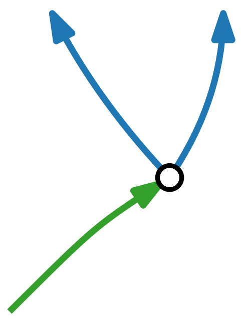

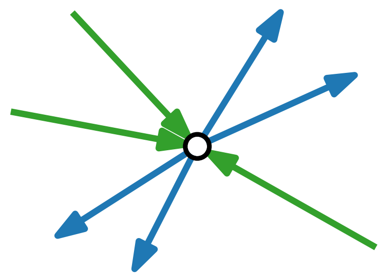

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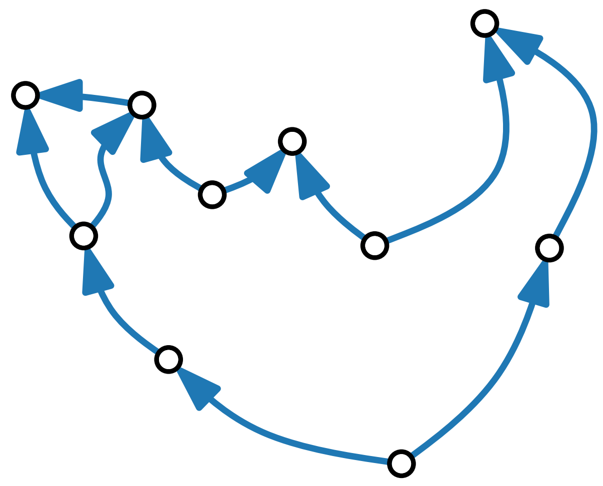

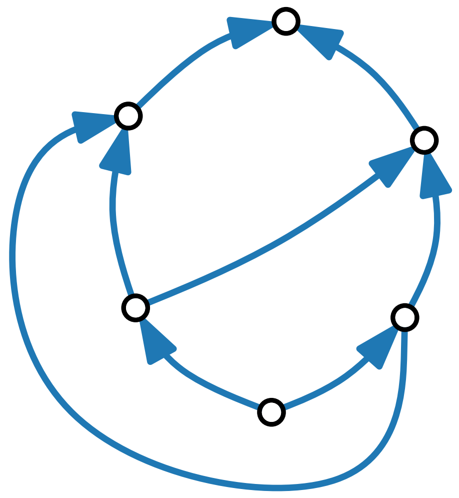

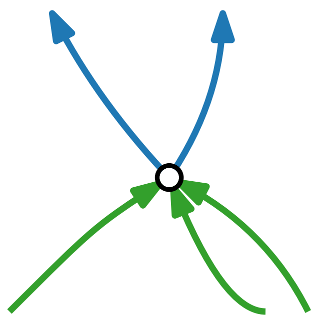

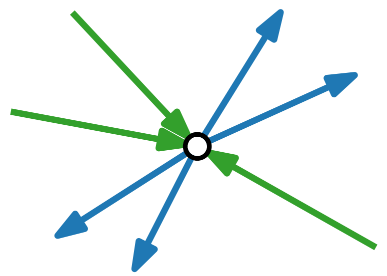

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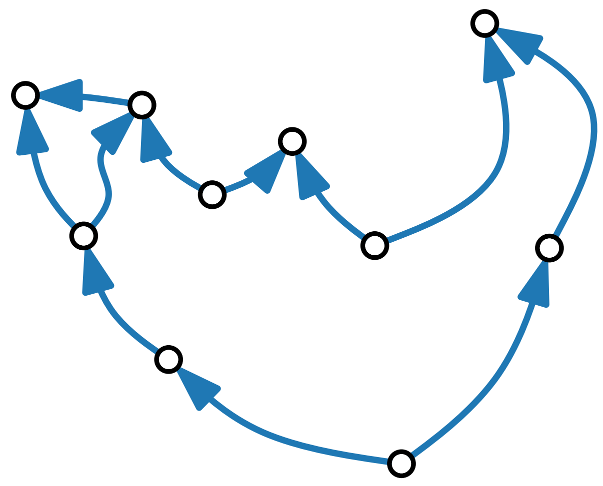

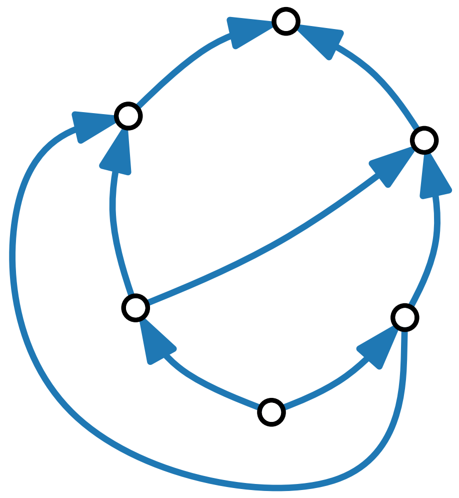

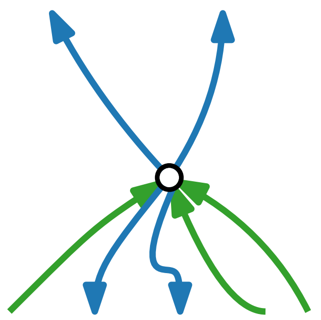

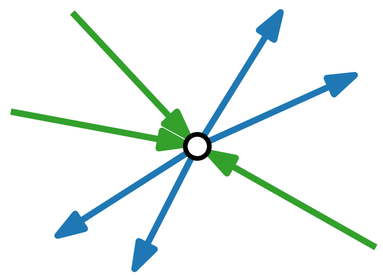

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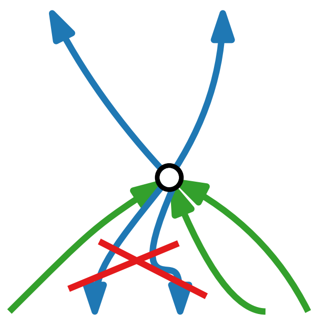

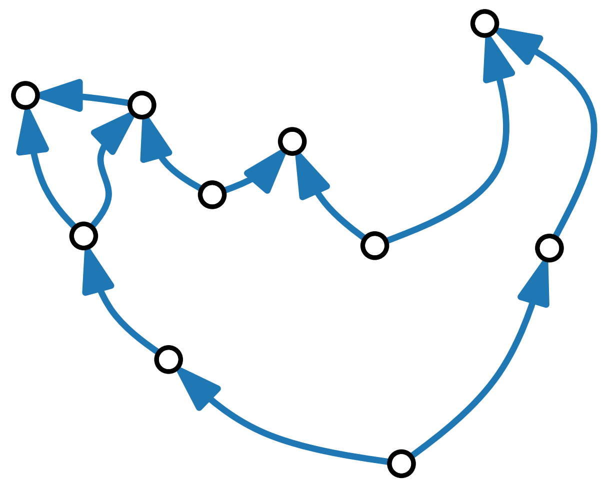

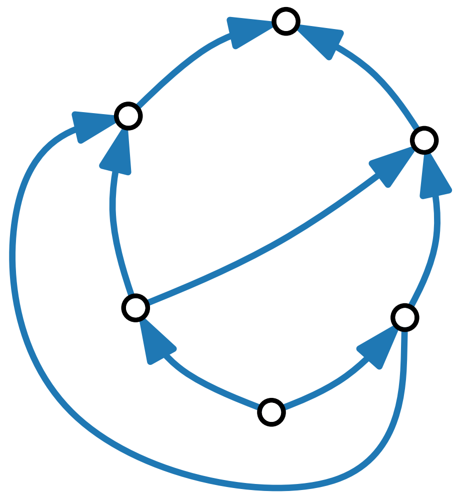

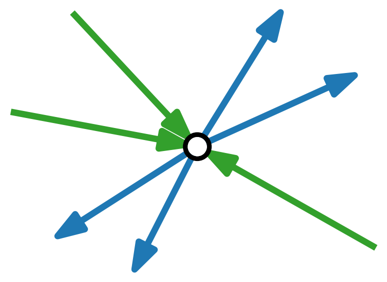

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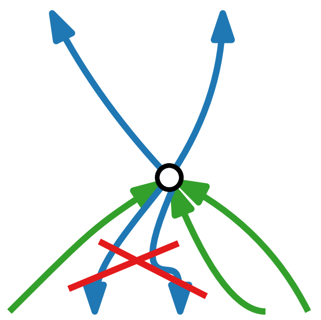

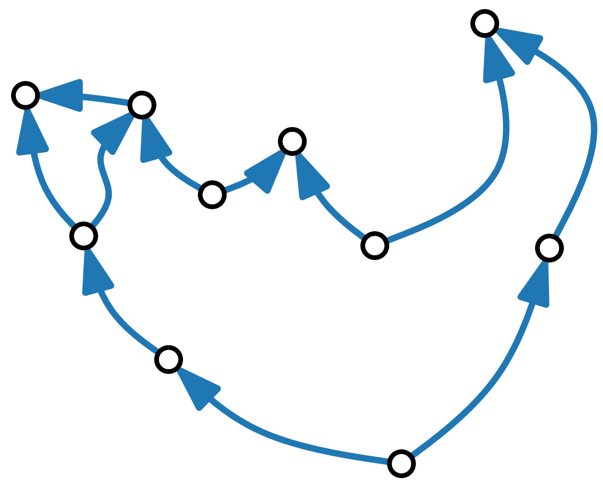

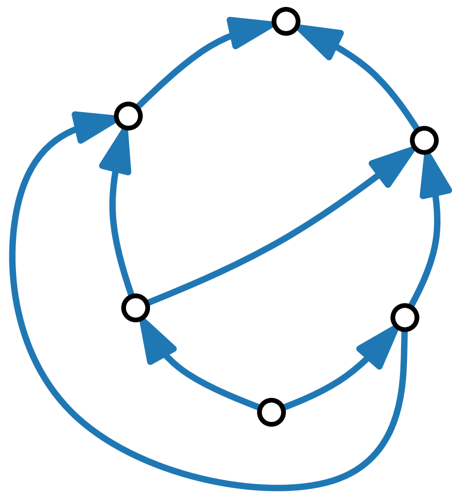

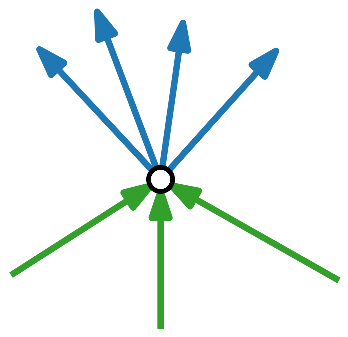

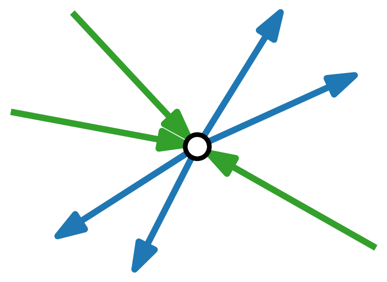

Upward Planarity – Necessary Conditions

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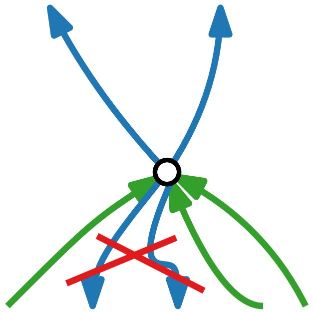
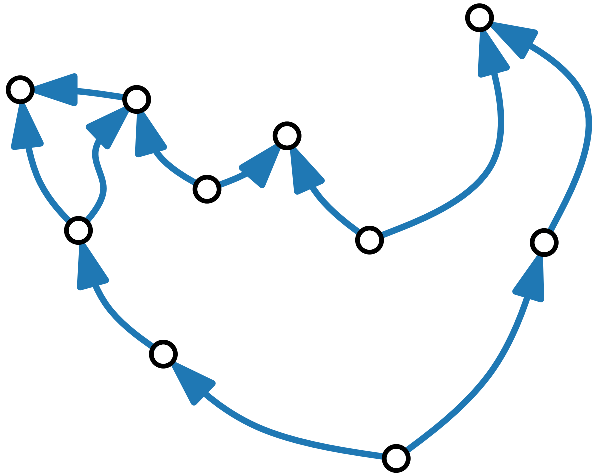
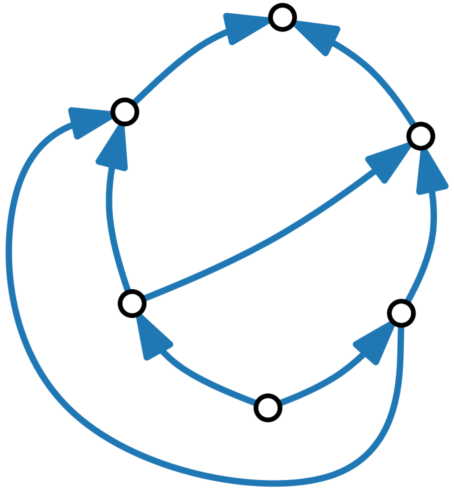
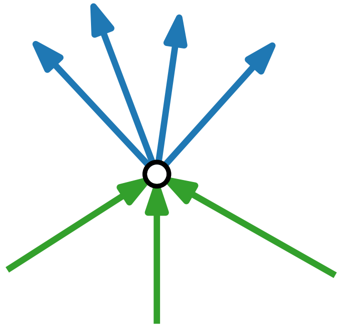


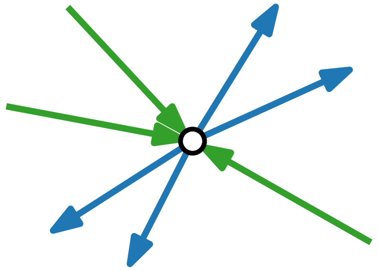




bimodal vertex



not bimodal



Upward Planarity – Necessary Conditions

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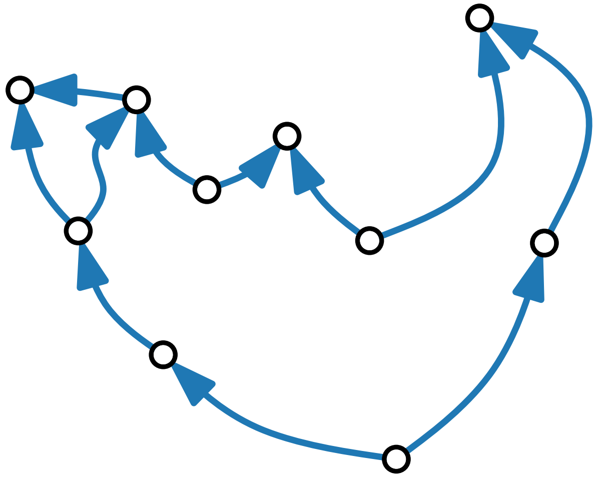
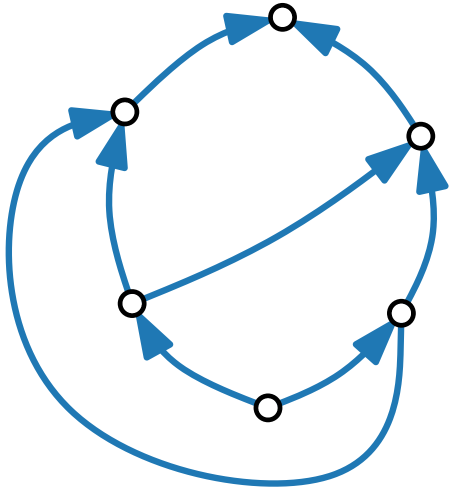
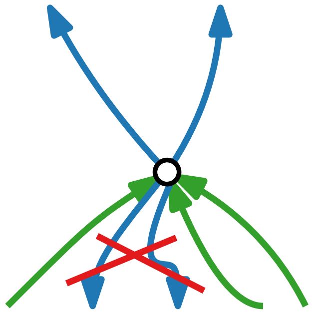
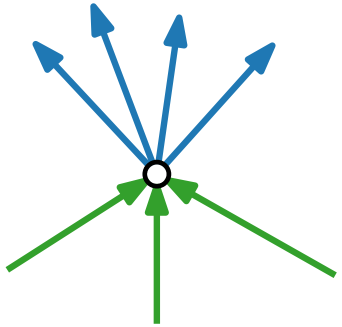


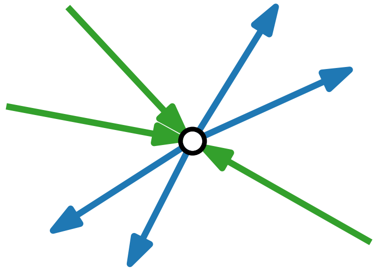




bimodal vertex

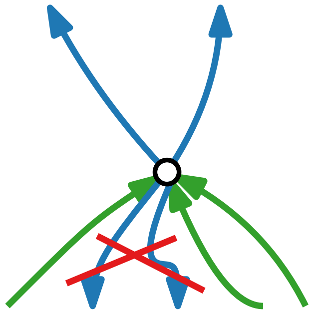


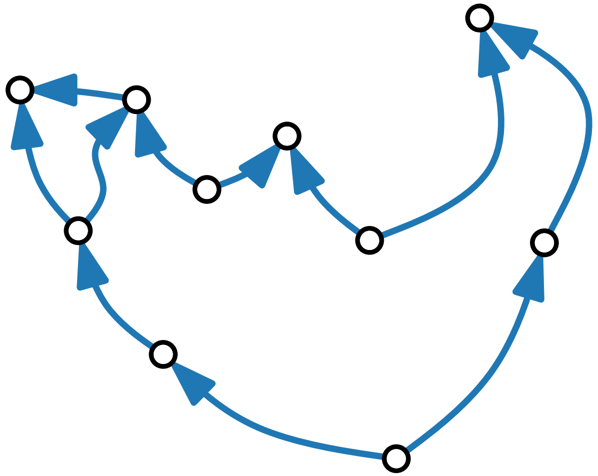
not bimodal

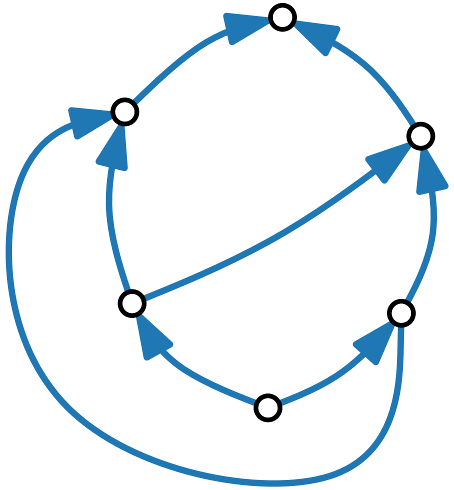


Upward Planarity – Necessary Conditions

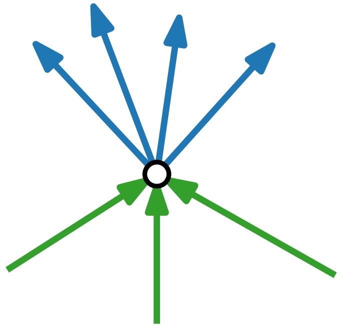
- For a digraph G to be upward planar, it has to be:
  - planar
  - acyclic
  - bimodal
- ... but these conditions are *not sufficient*.



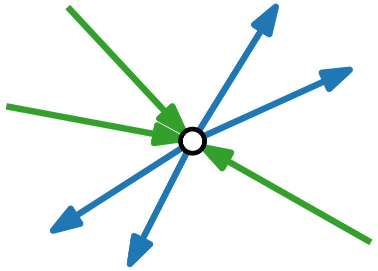




bimodal vertex



not bimodal



Upward Planarity – Characterization

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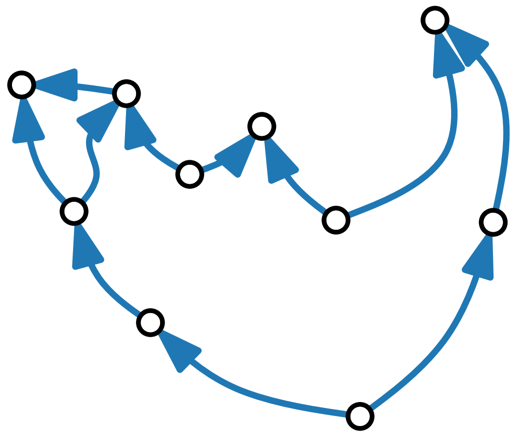

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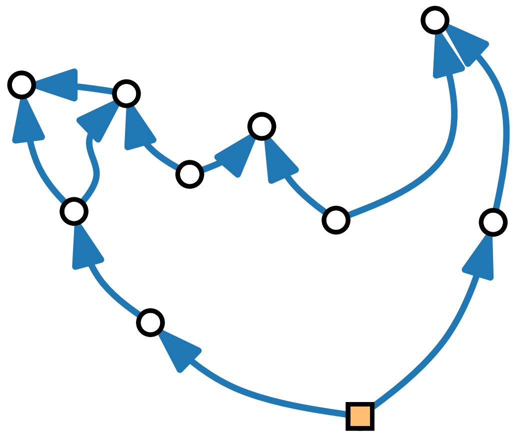

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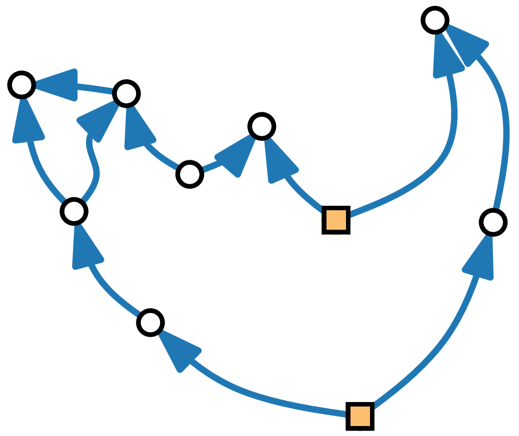

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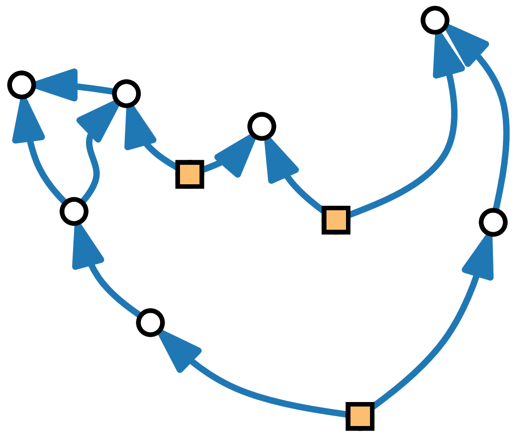

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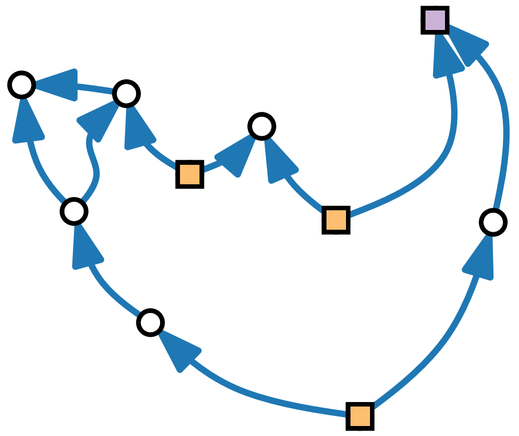

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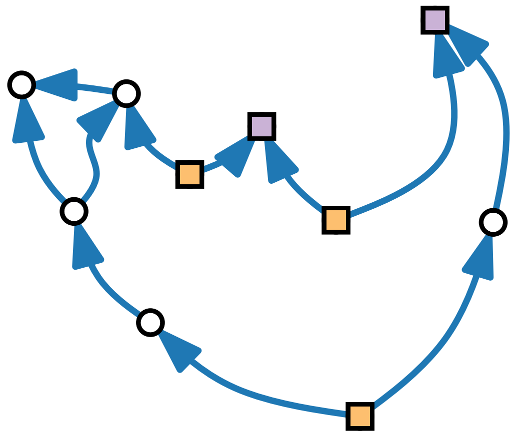

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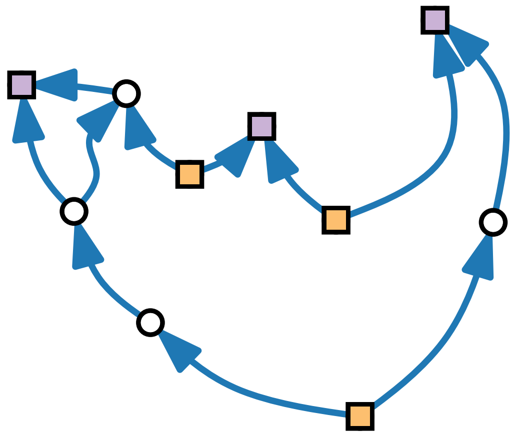

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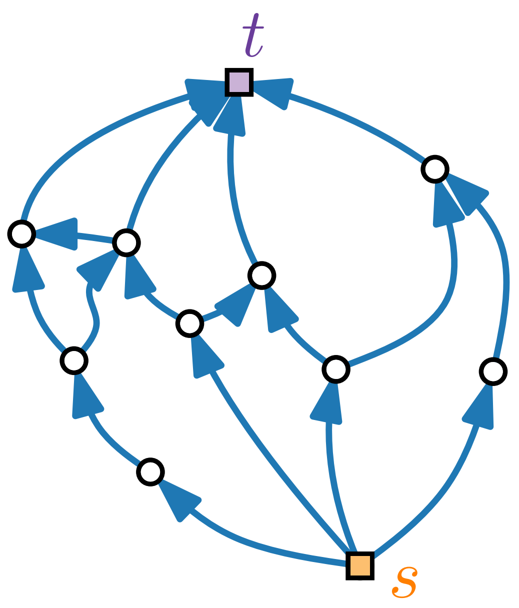

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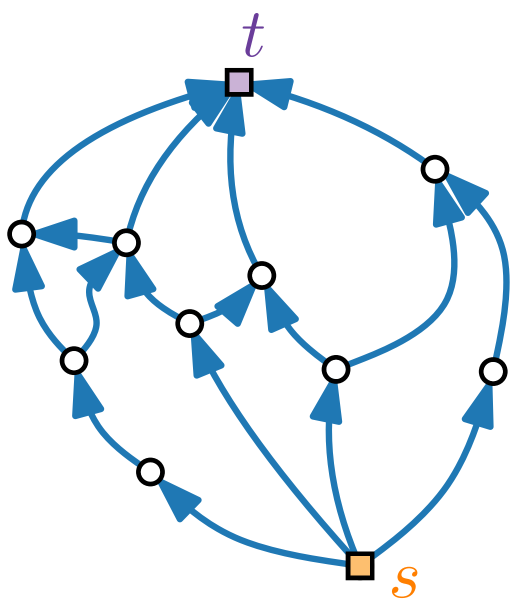

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Additionally:
Embedded such that s and t are on the outerface f_0.

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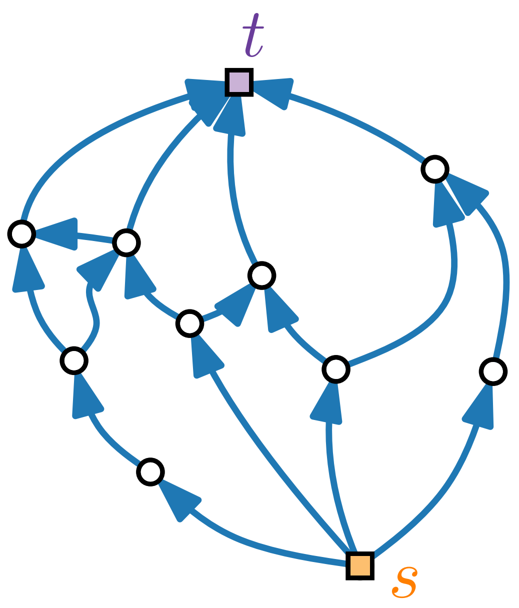

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Additionally:
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or:
Edge (s, t) exists.

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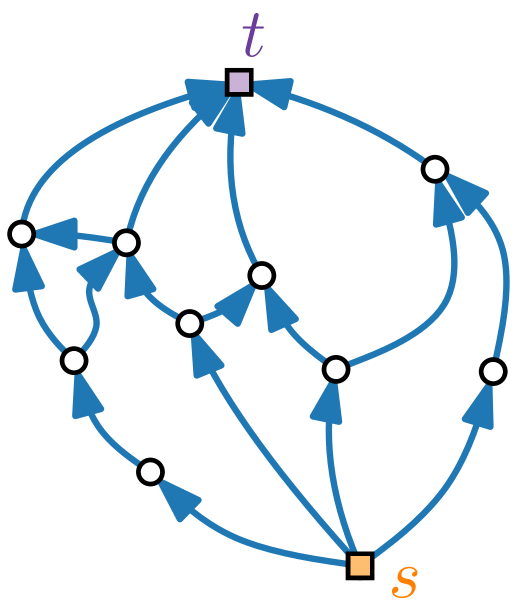

Proof.

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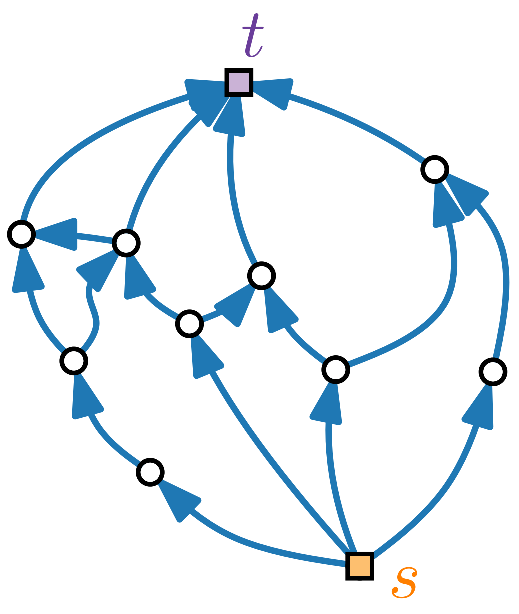

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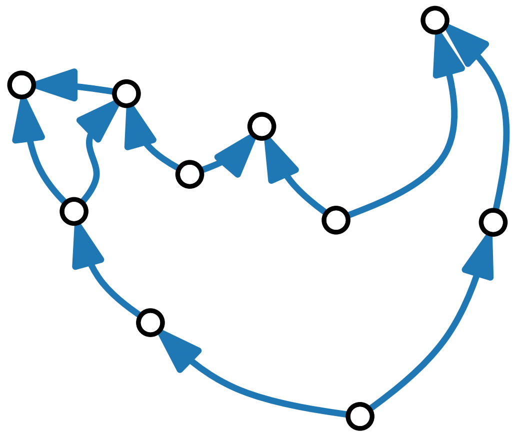

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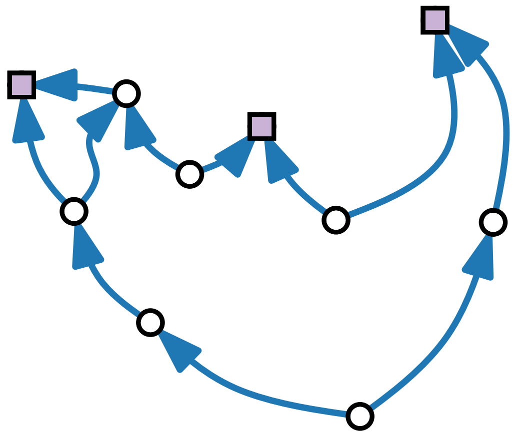

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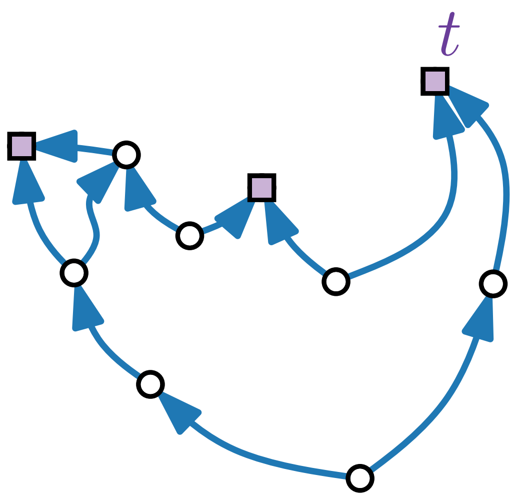

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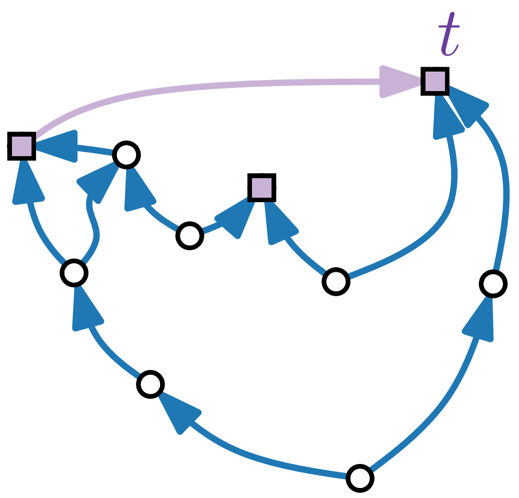

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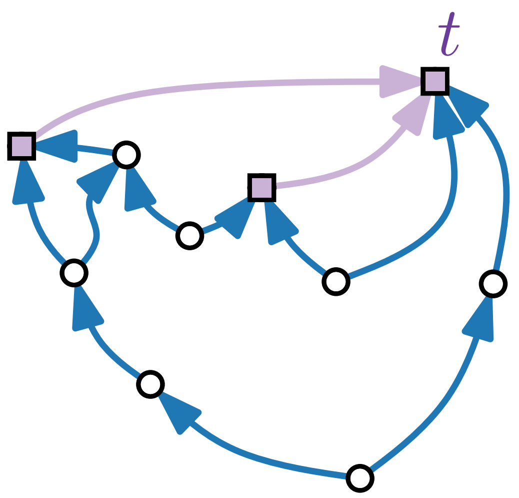

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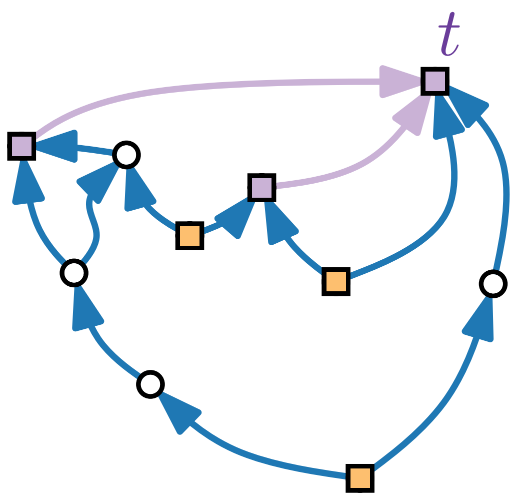

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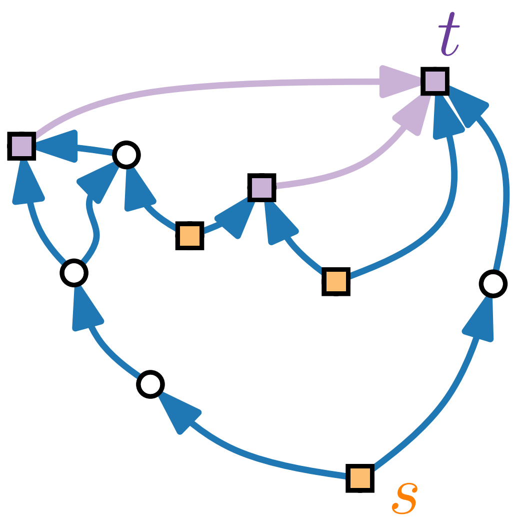

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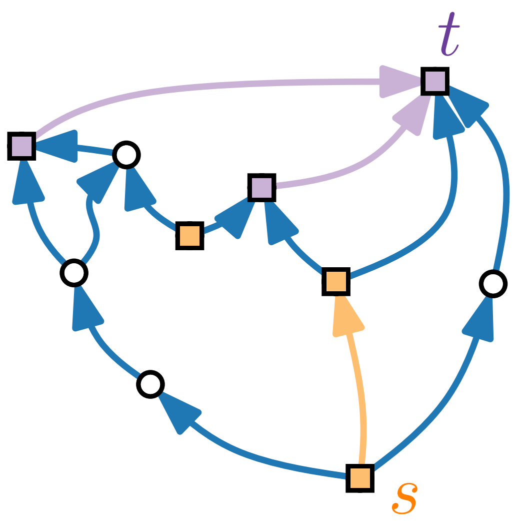

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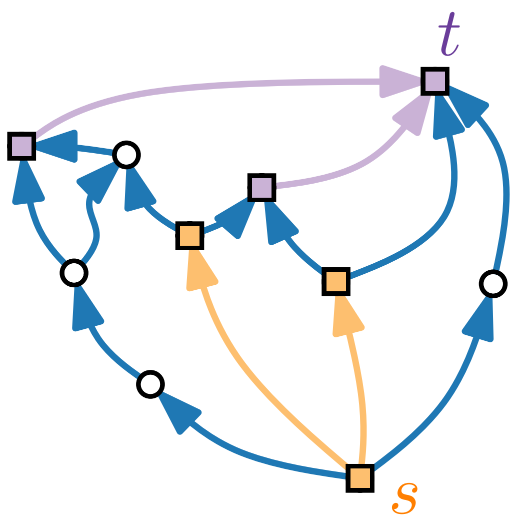

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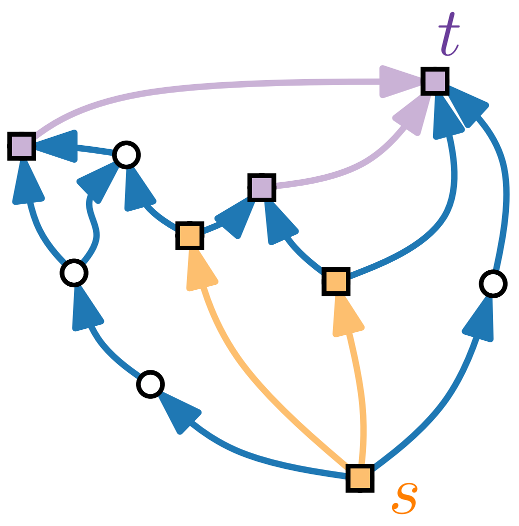

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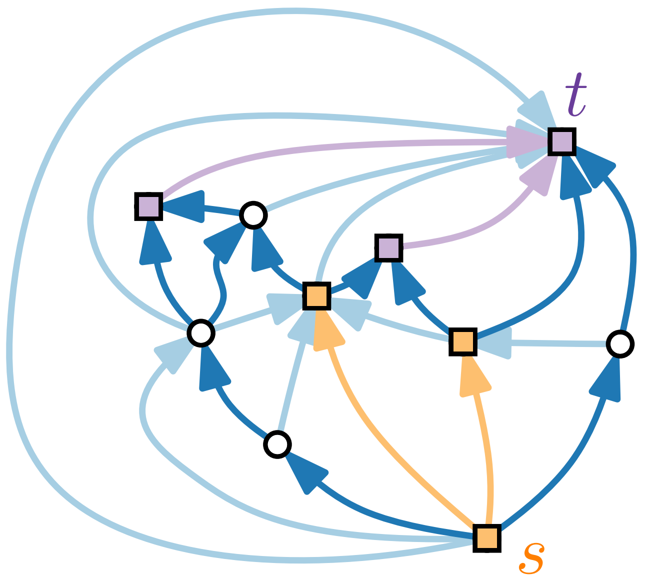

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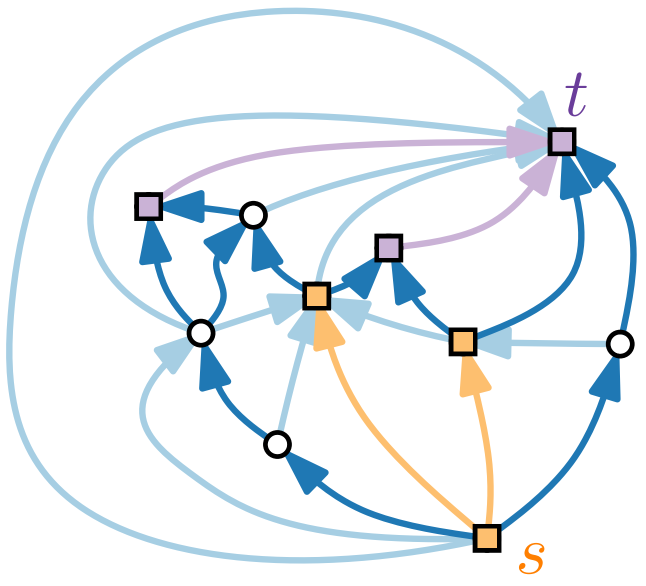

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Claim.
Can draw in prespecified triangle.

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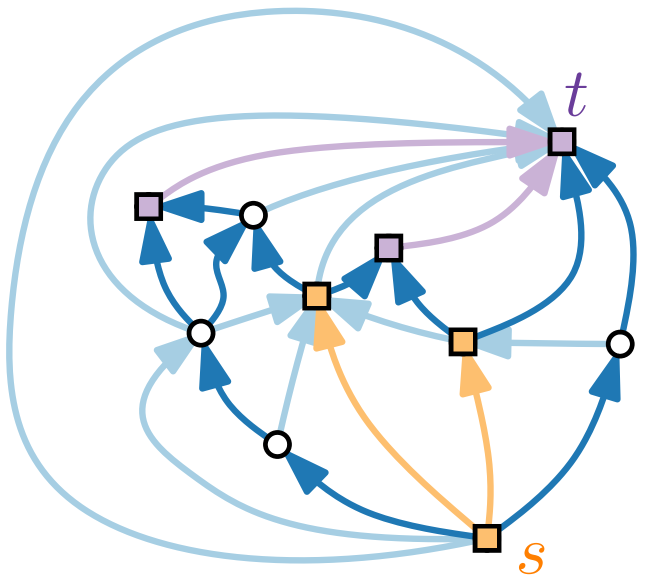

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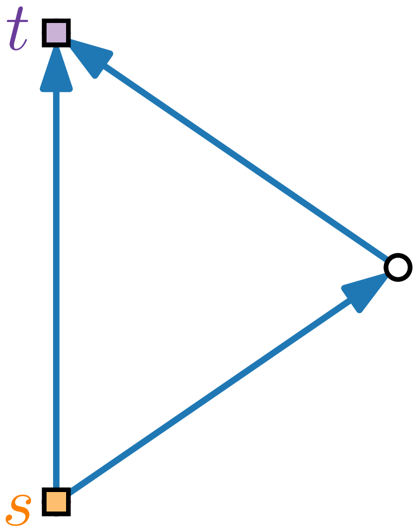

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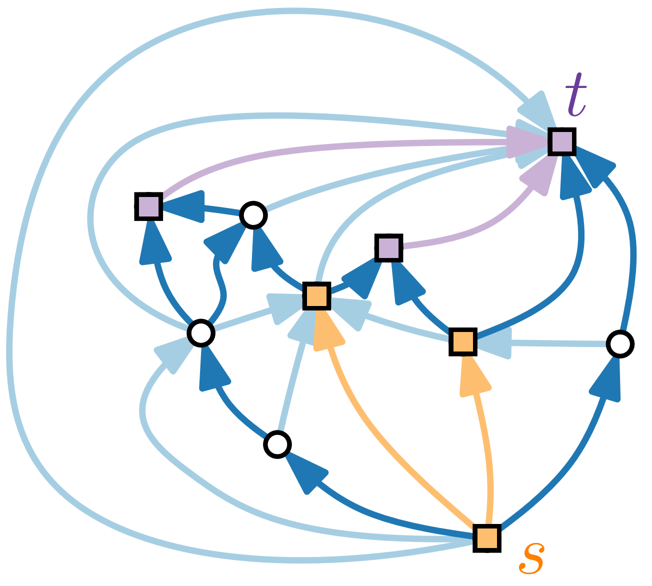

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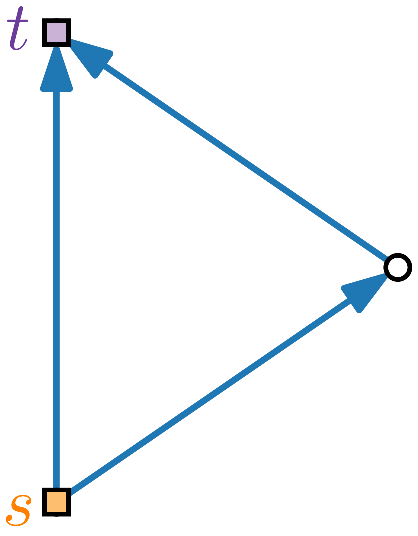
Induction on n.



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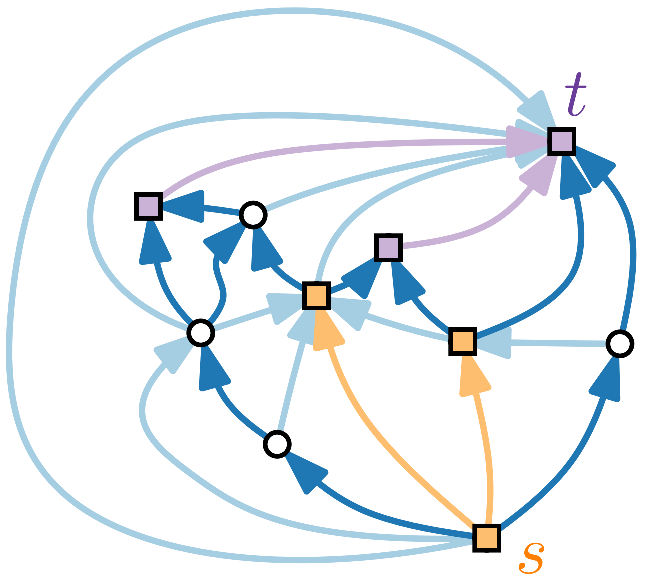

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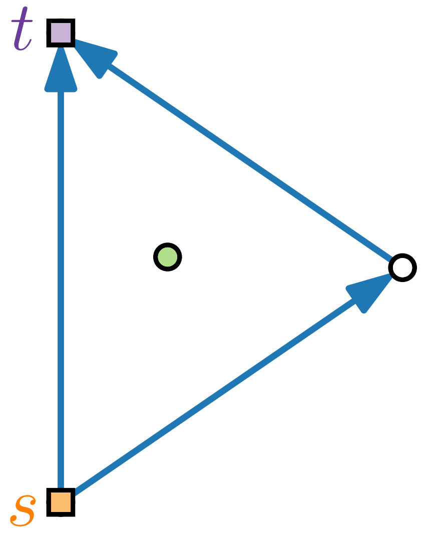

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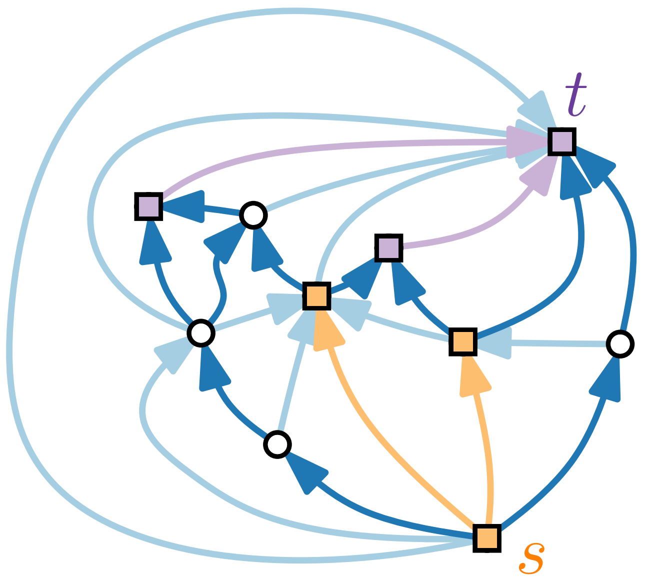

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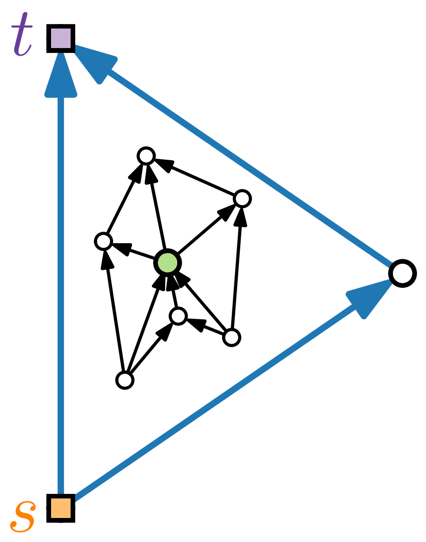

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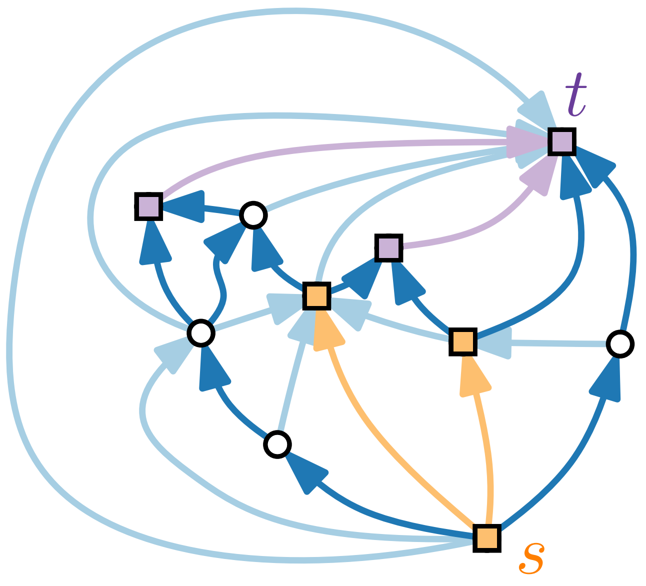

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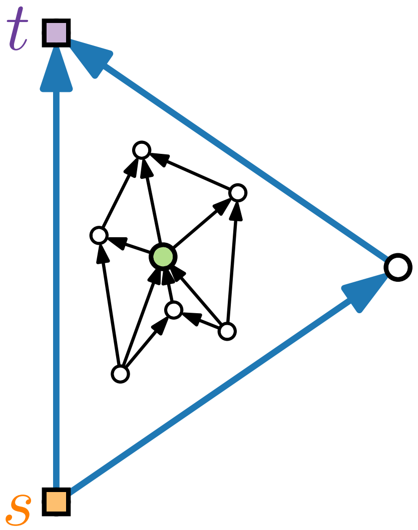
Case 1: chord



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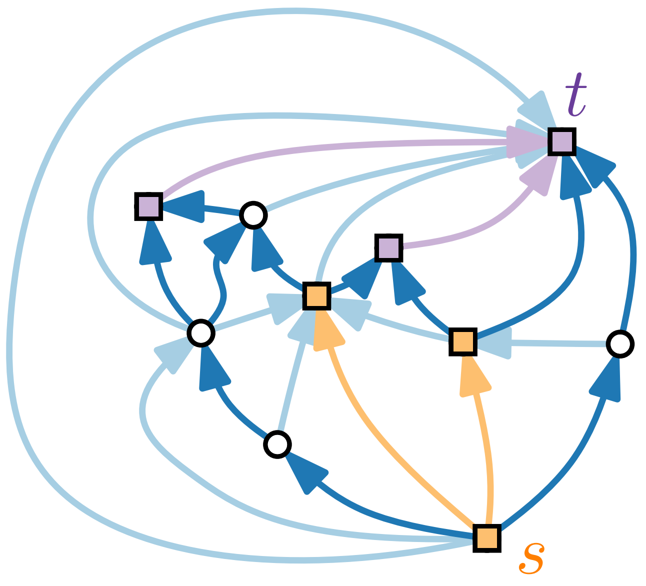

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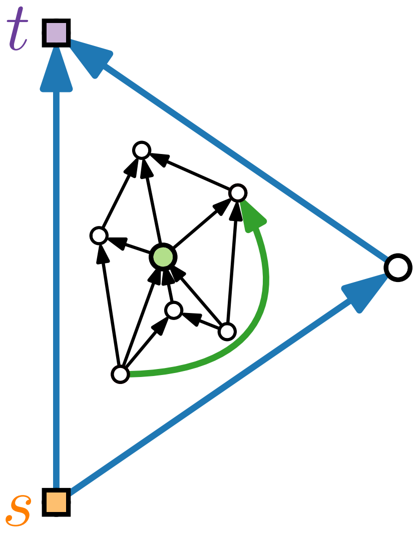

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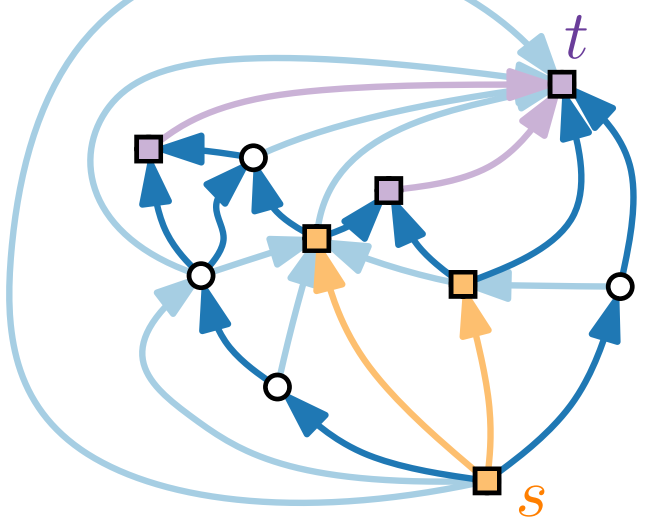

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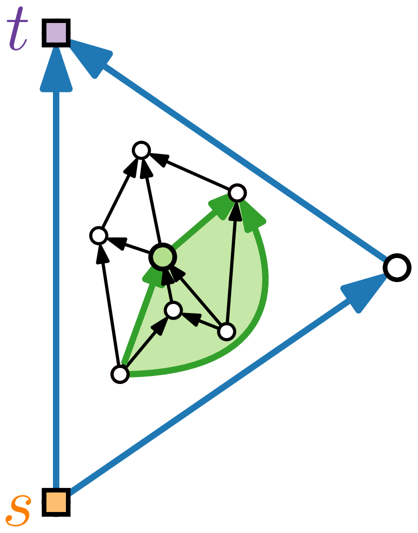

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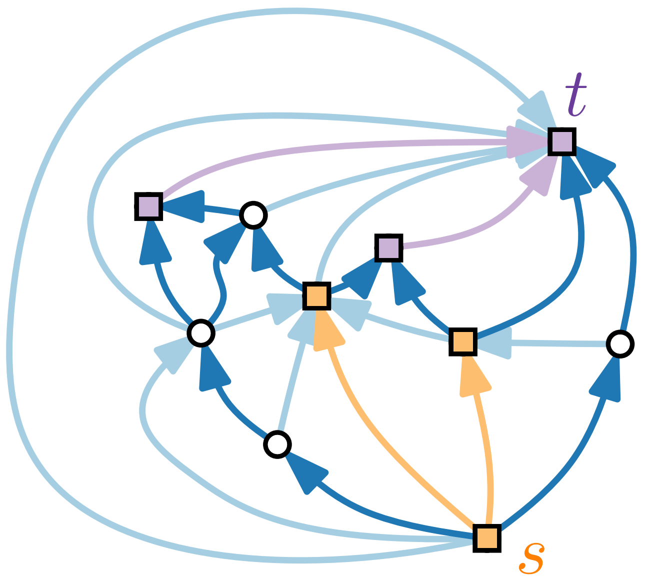

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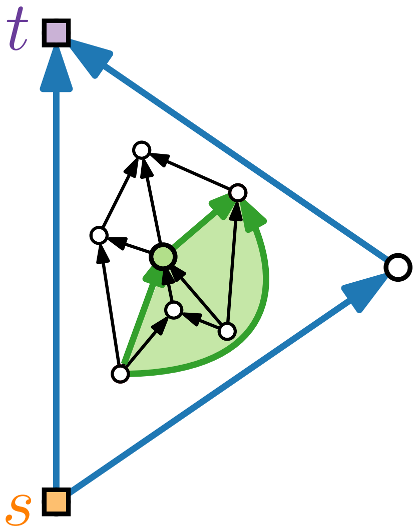

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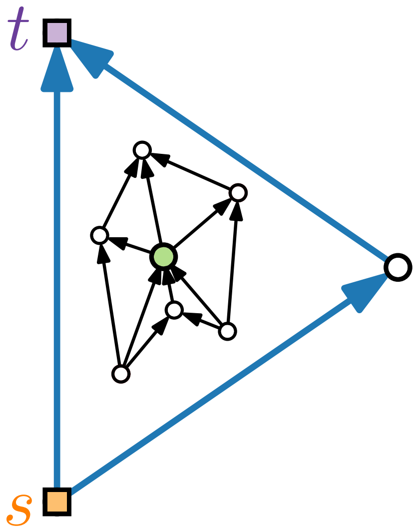

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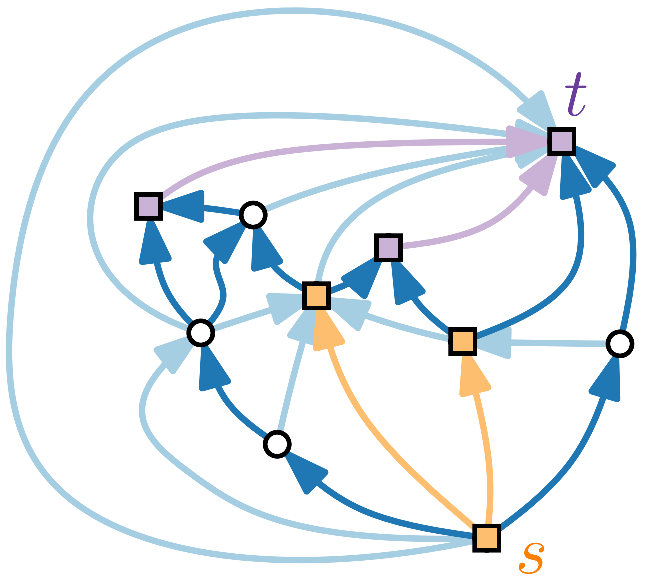

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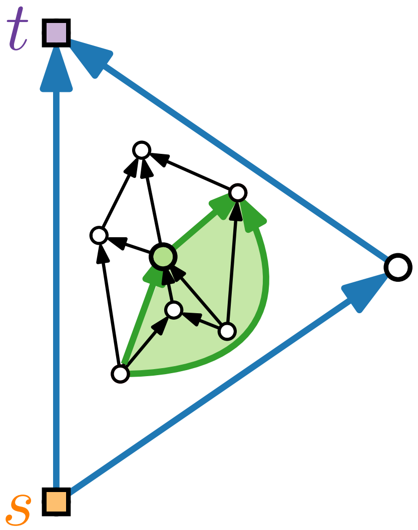

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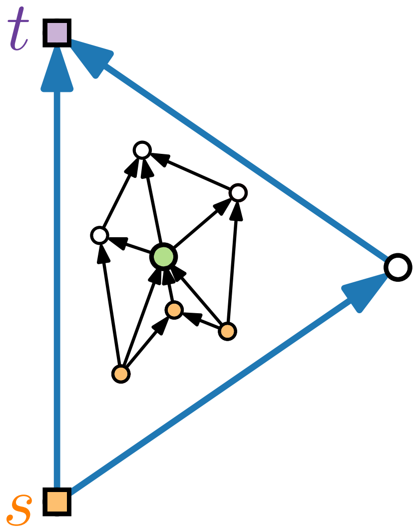

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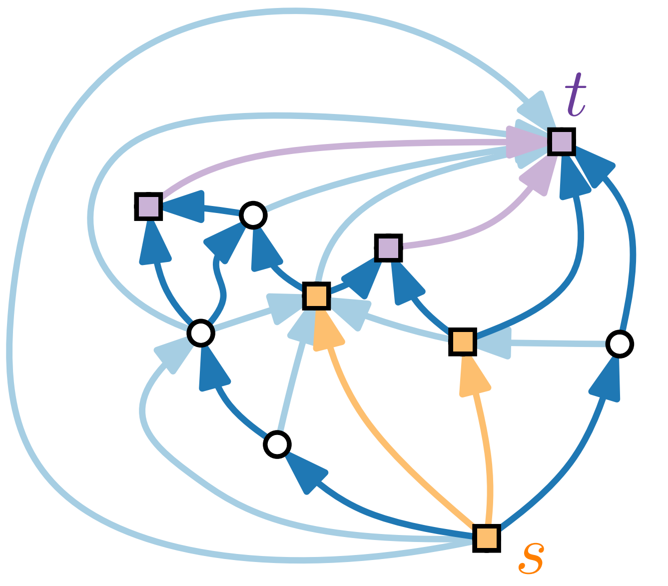

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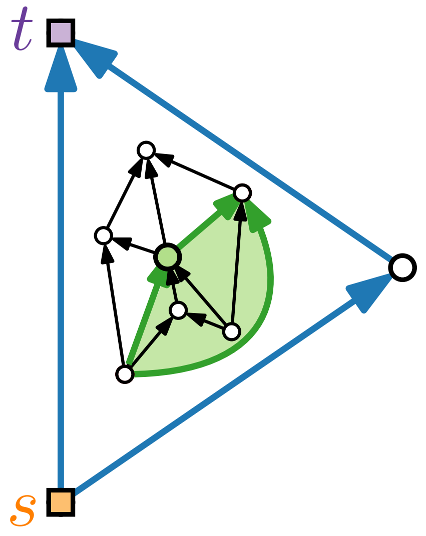

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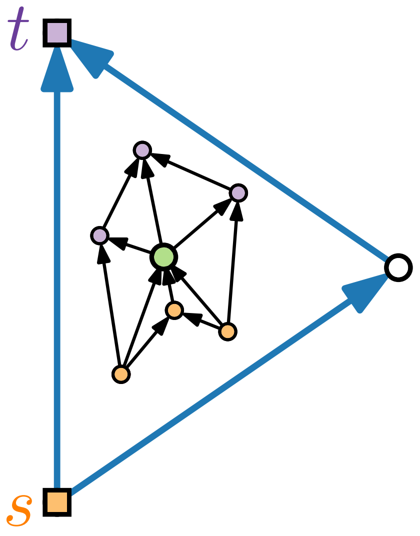

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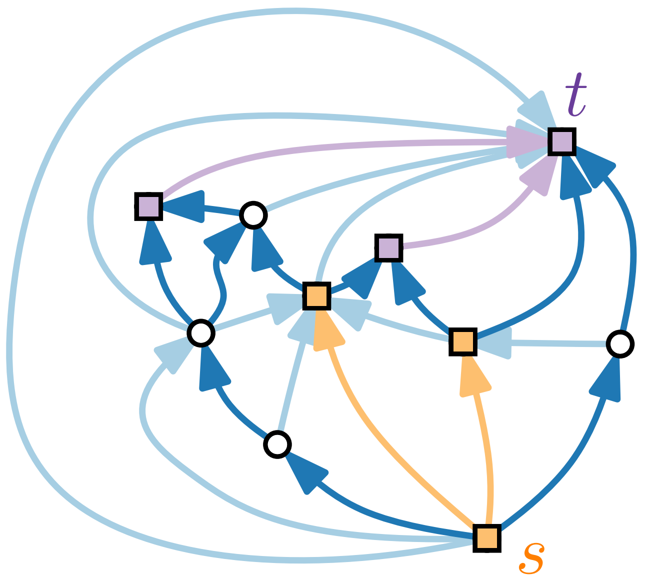

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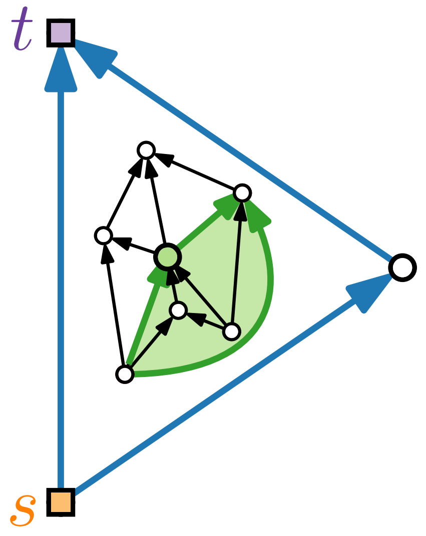

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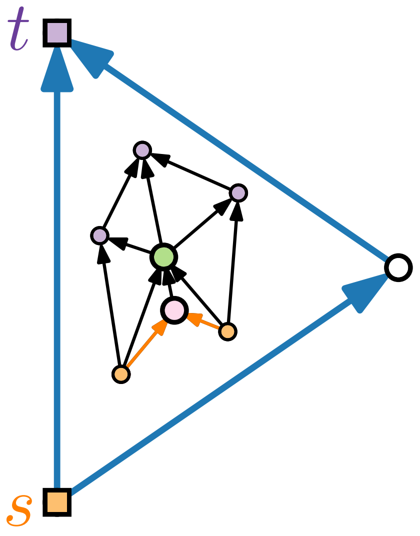

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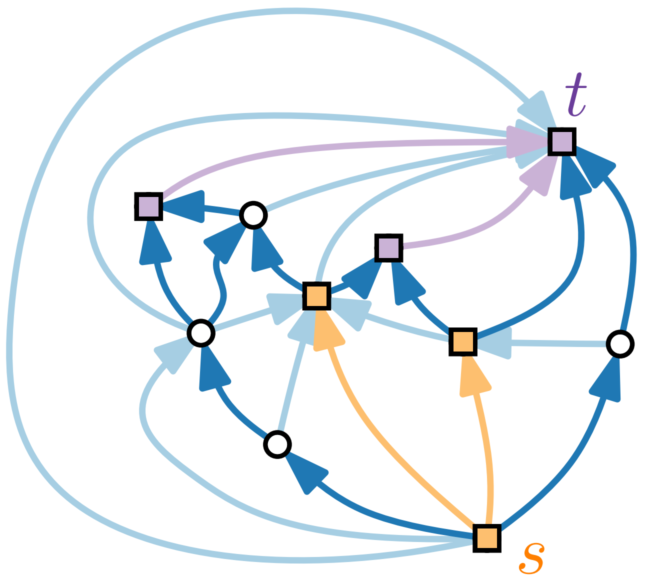

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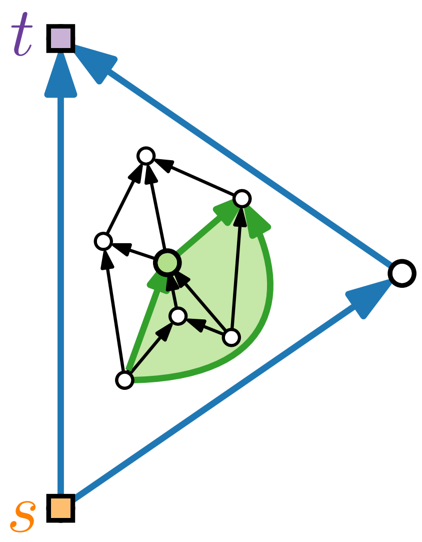

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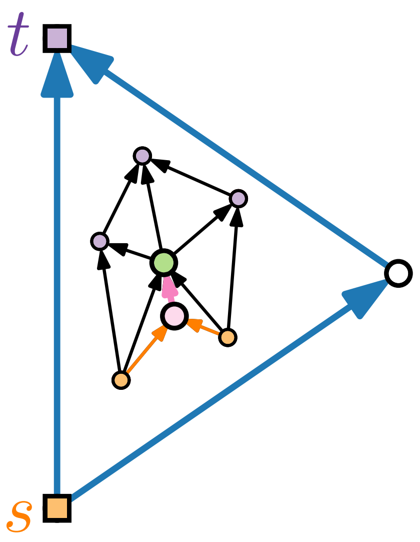

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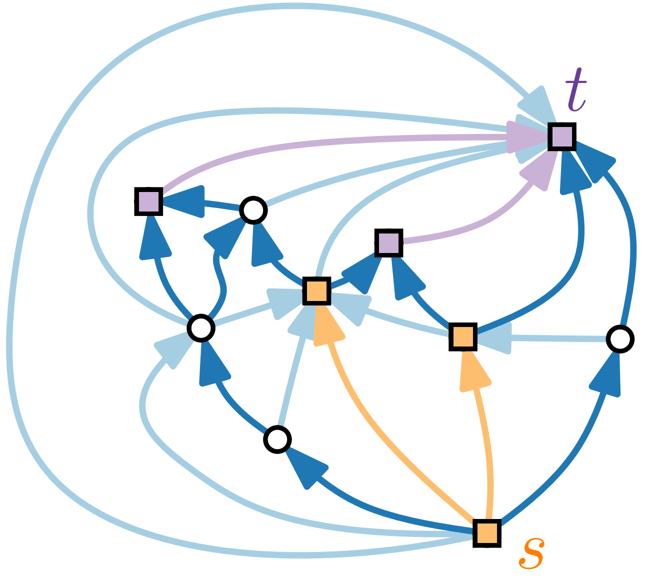

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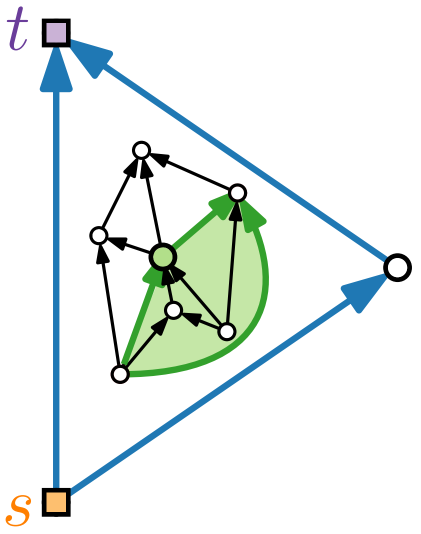

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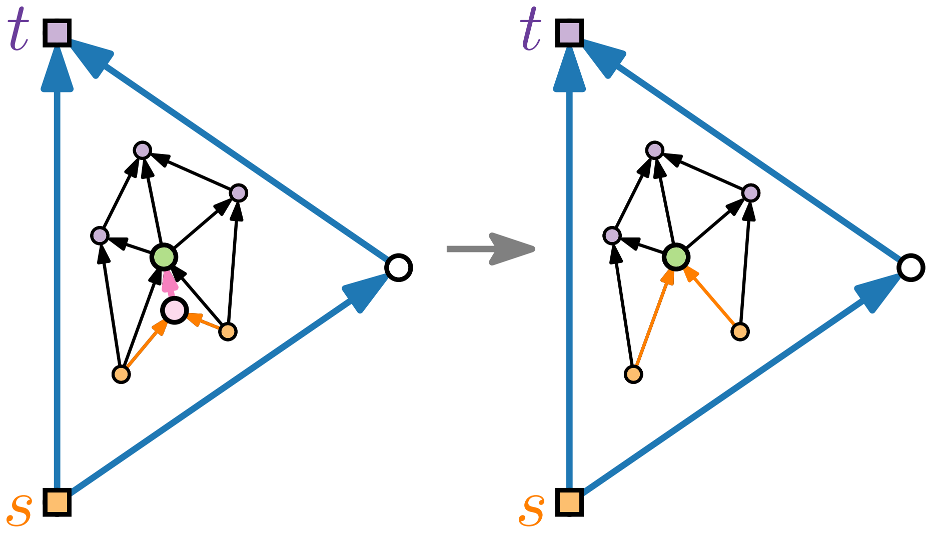

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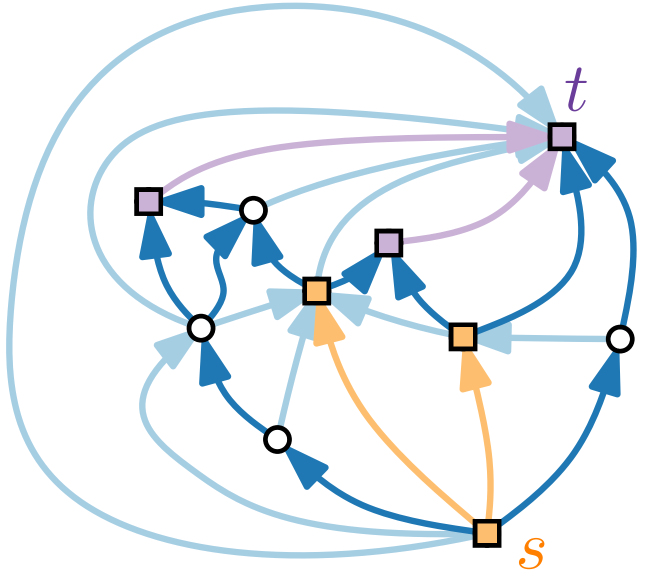

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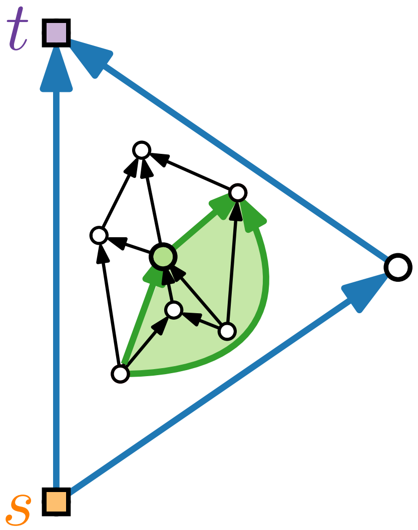

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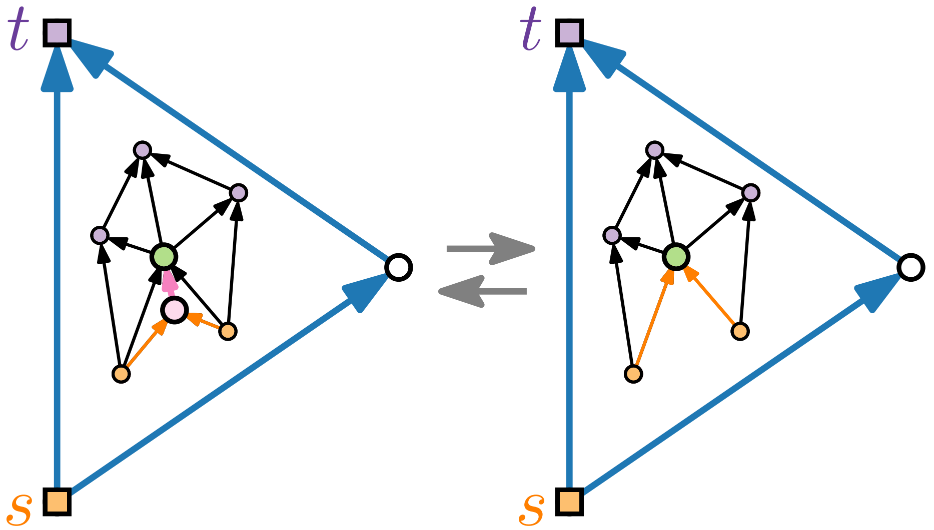

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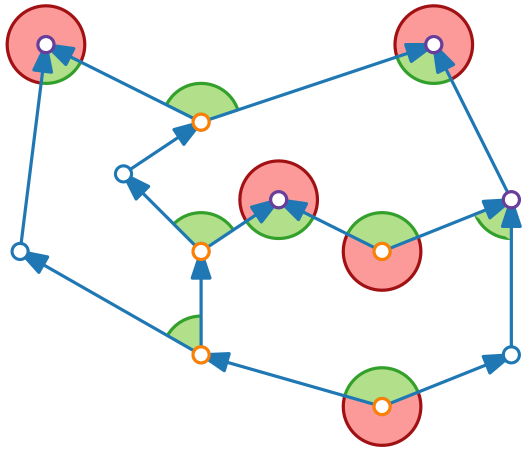
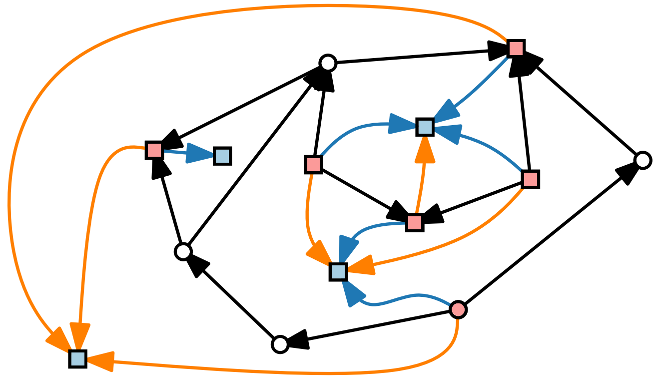
Visualization of Graphs

Lecture 6:
Upward Planar Drawings



Part II:
Assignment Problem

Jonathan Klawitter



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Fixed Embedding Upward Planarity Testing.
Let $G = (V, E)$ be a plane digraph with set of faces F and outer face f_0.
Test whether G is upward planar (wrt to F, f_0).

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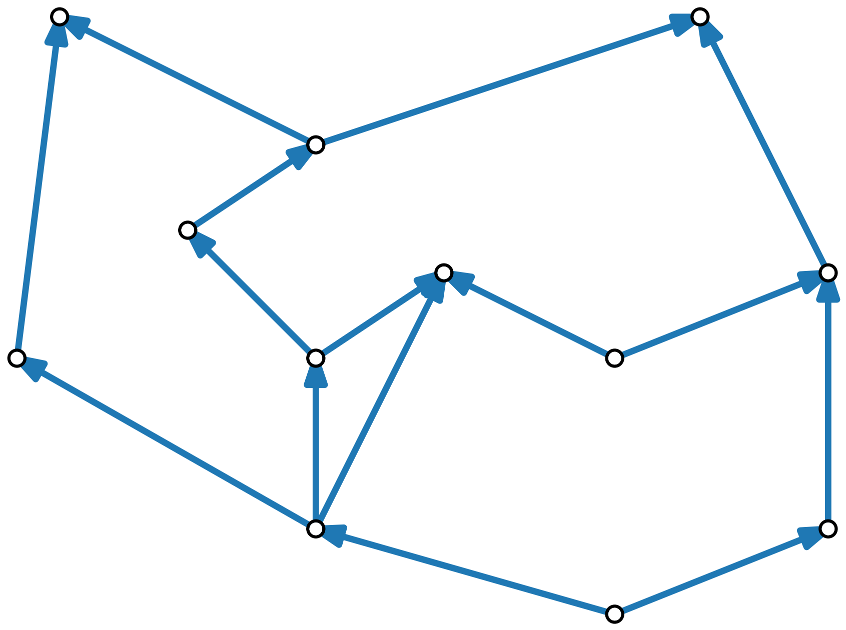
Idea.

- Find property that any upward planar drawing of G satisfies.
- Formalise property.
- Find algorithm to test property.

Angles, Local Sources & Sinks



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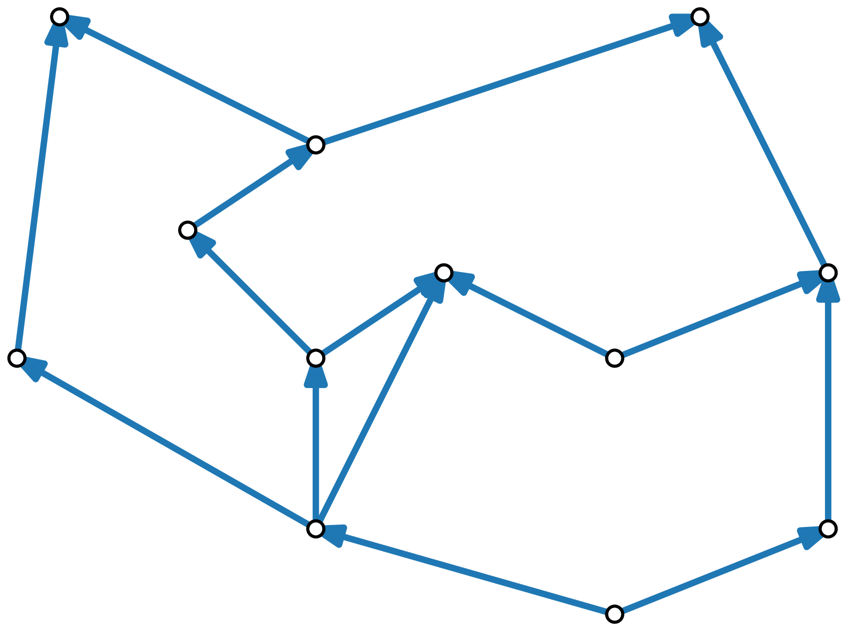

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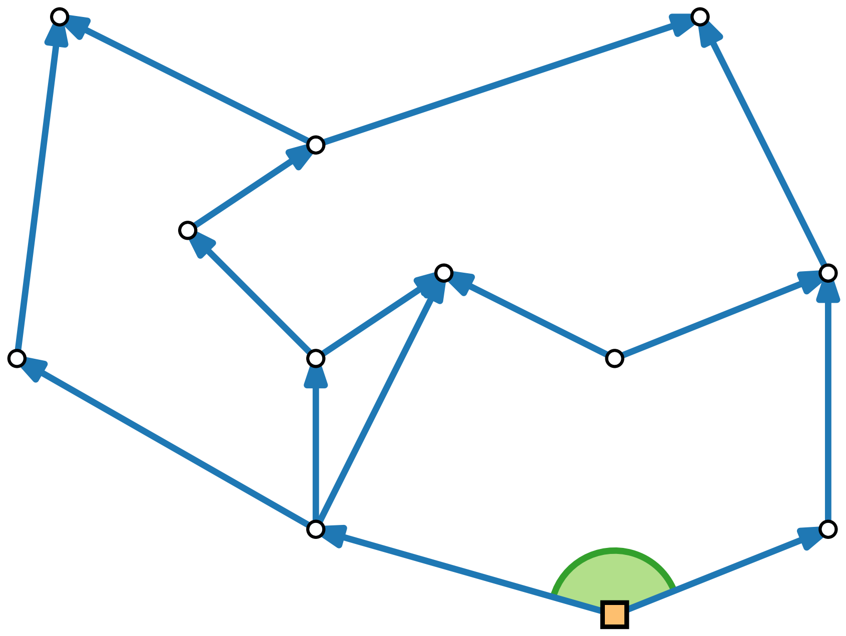

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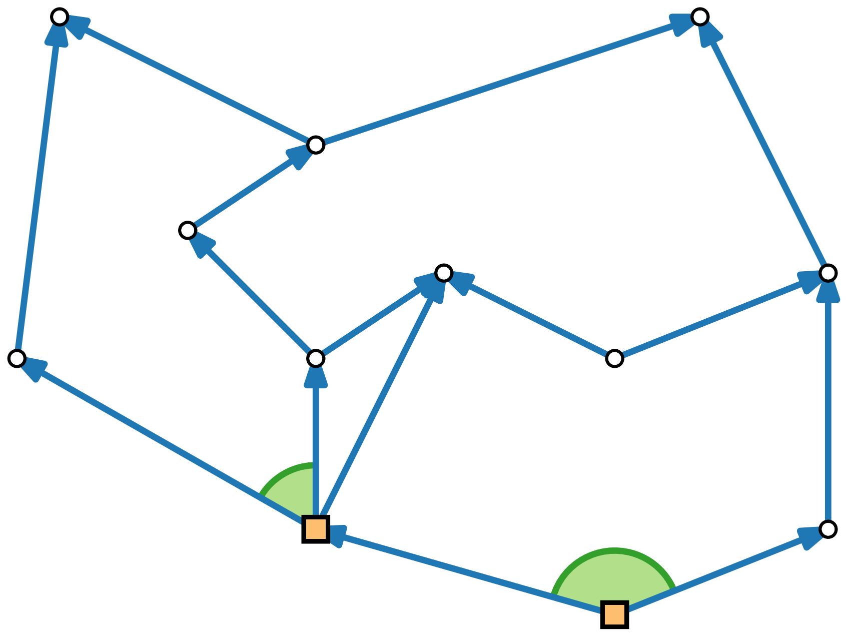

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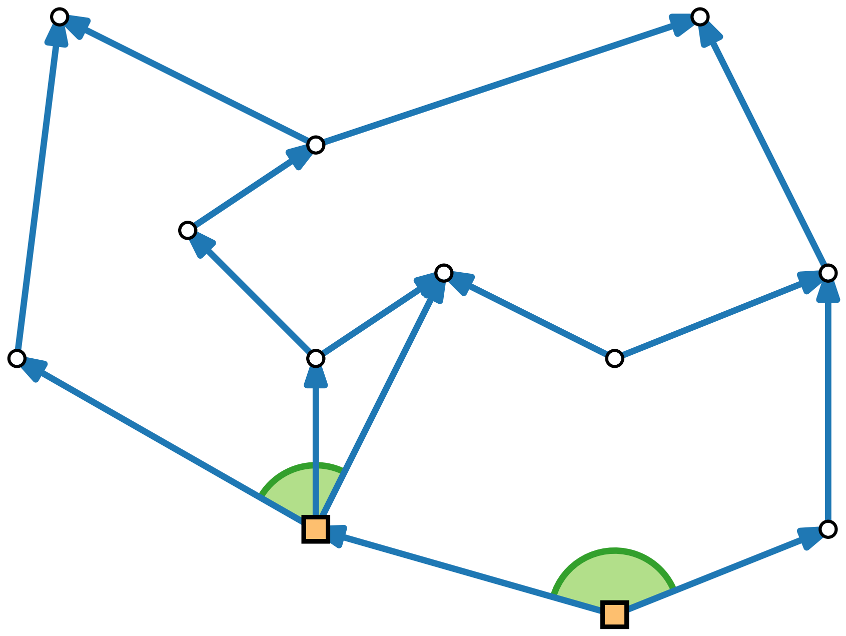

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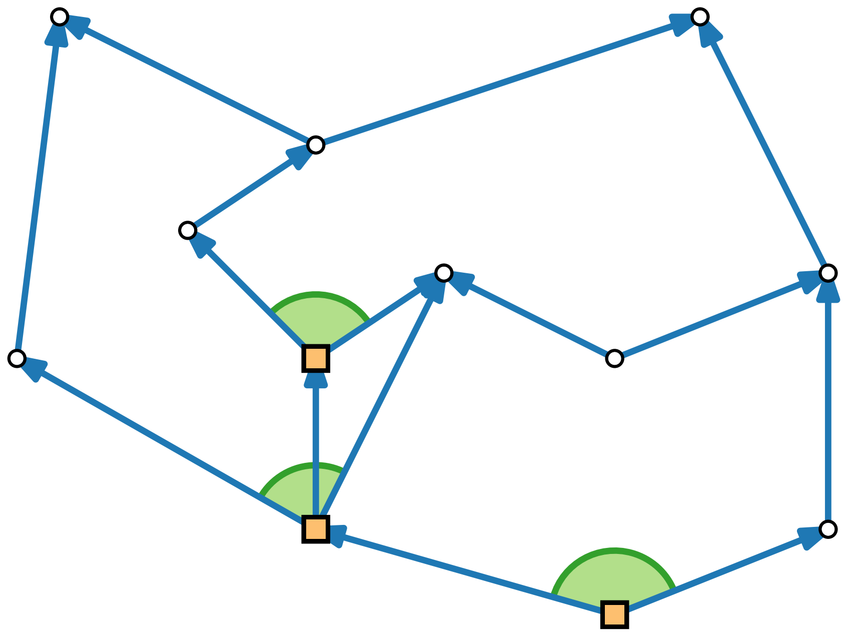

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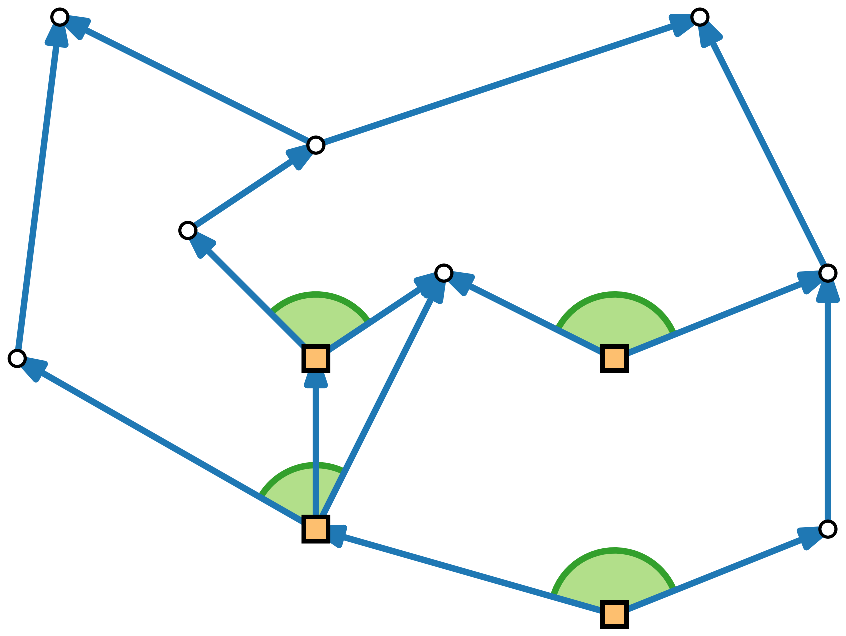

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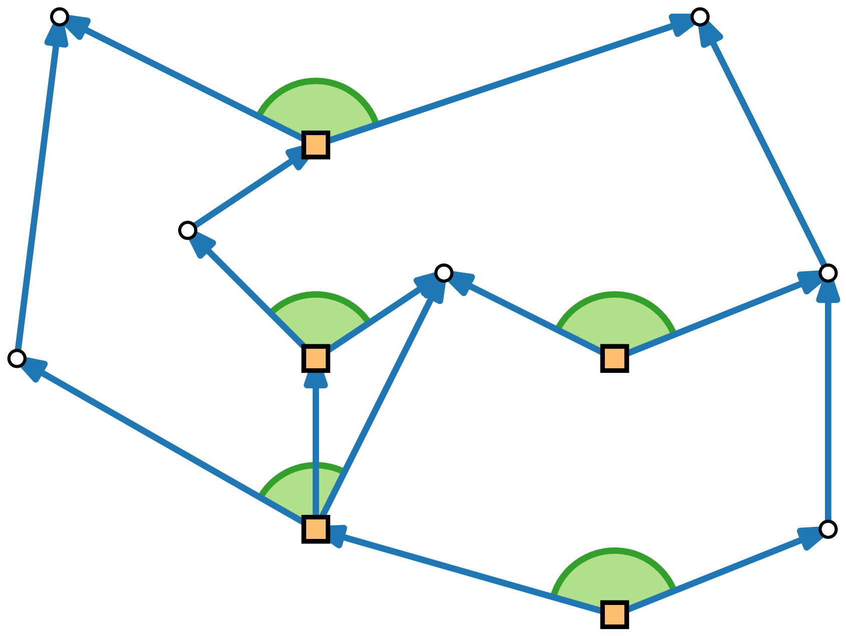

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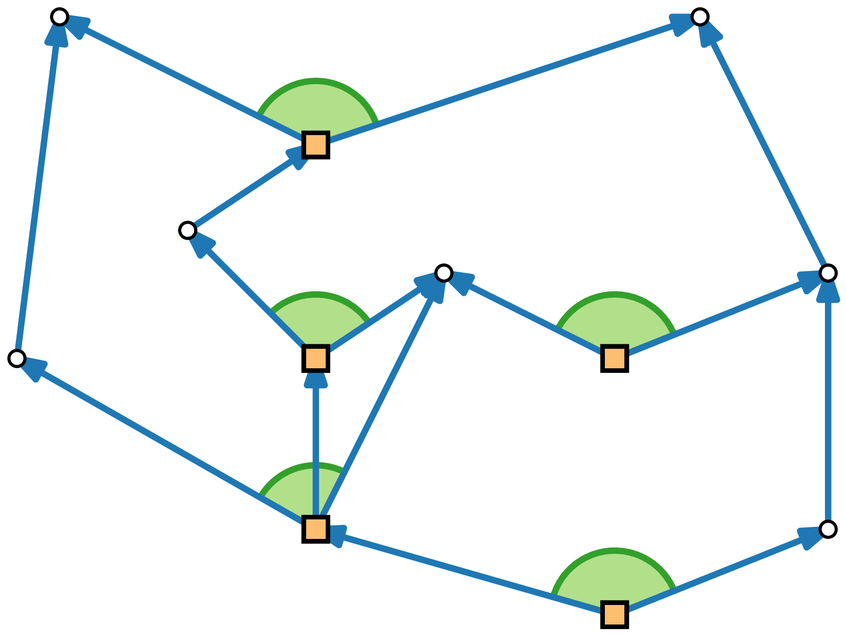

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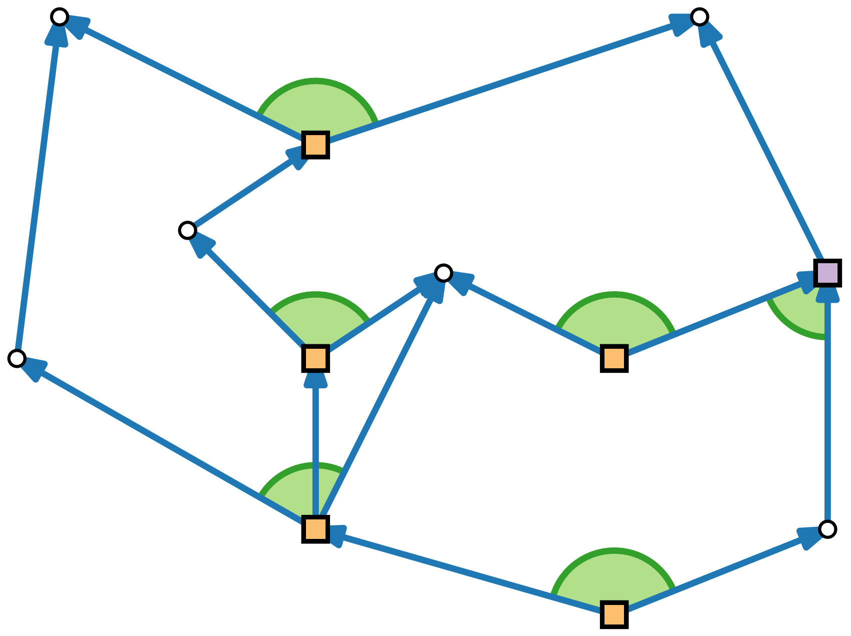

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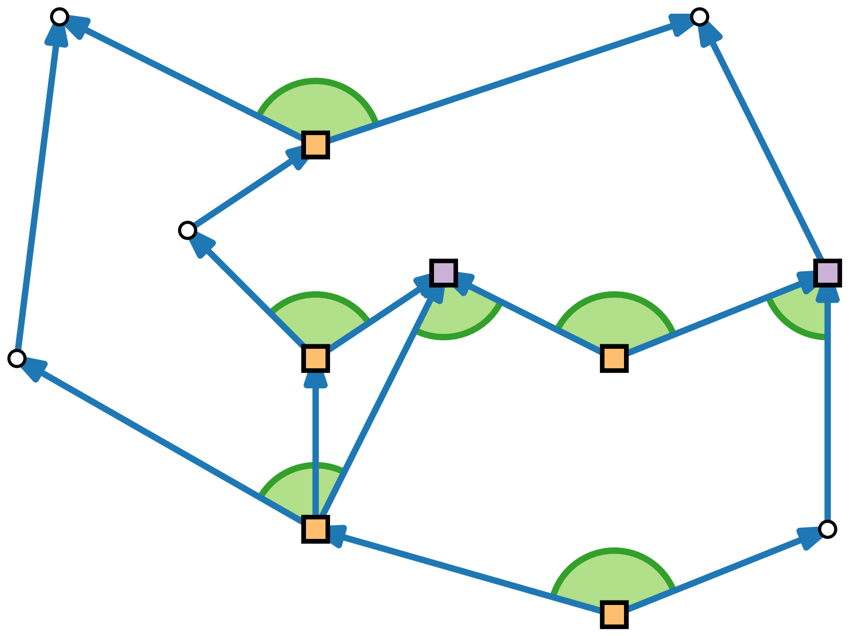

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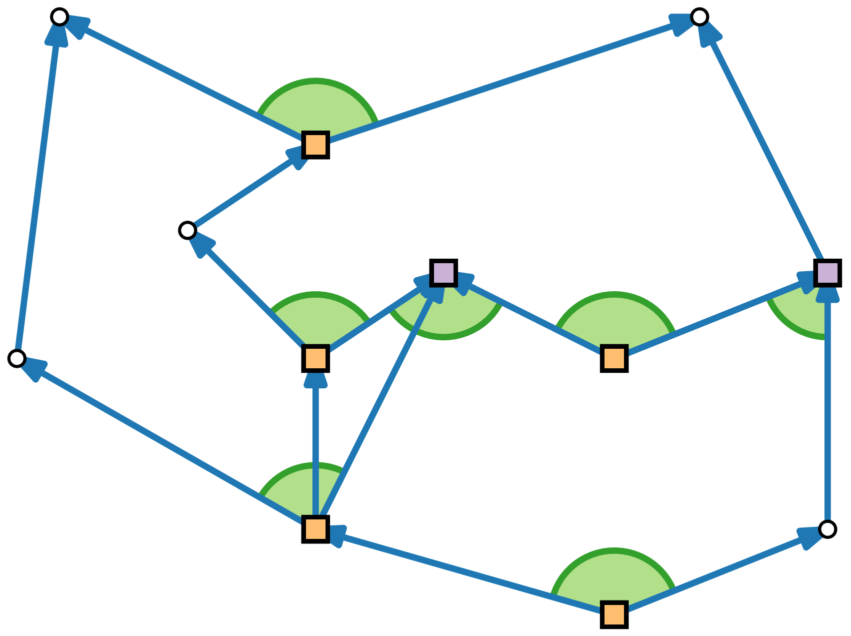

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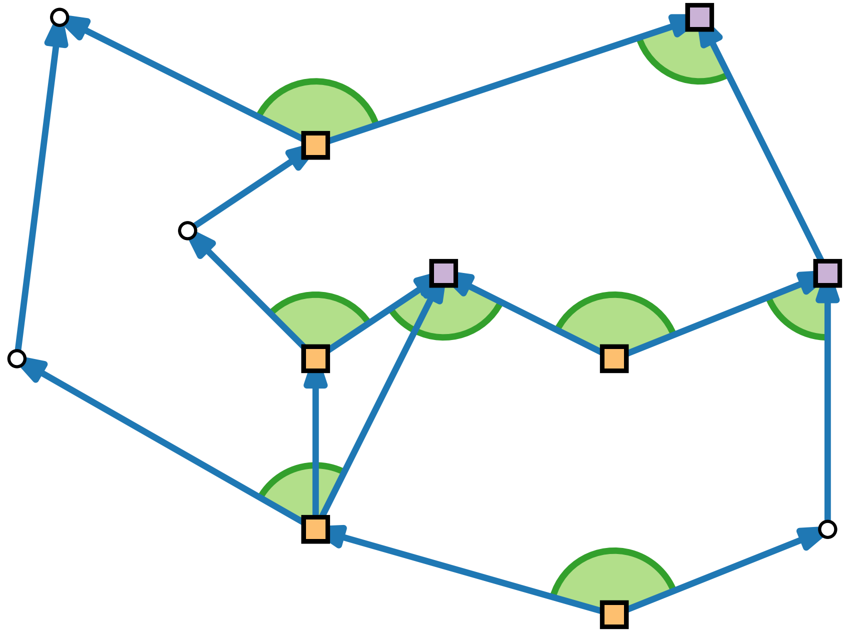

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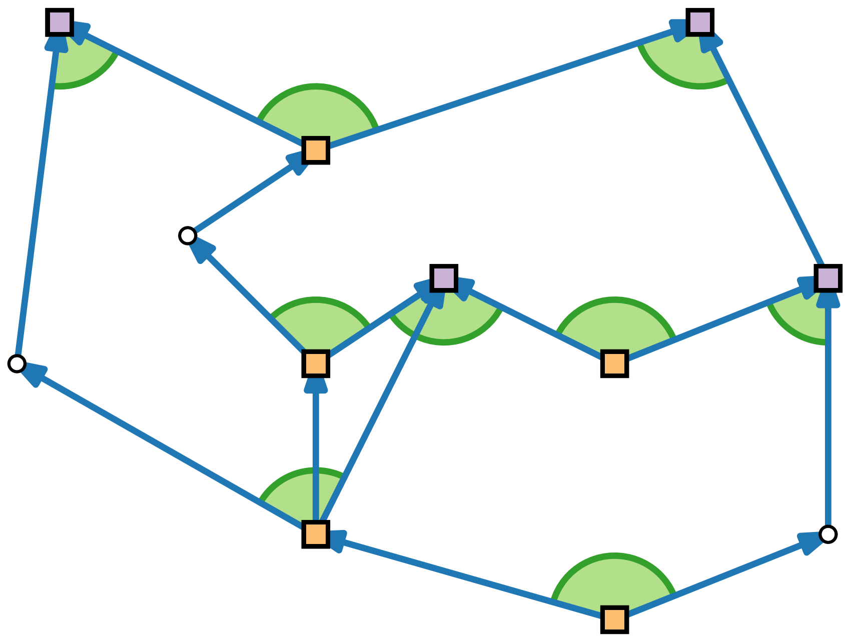

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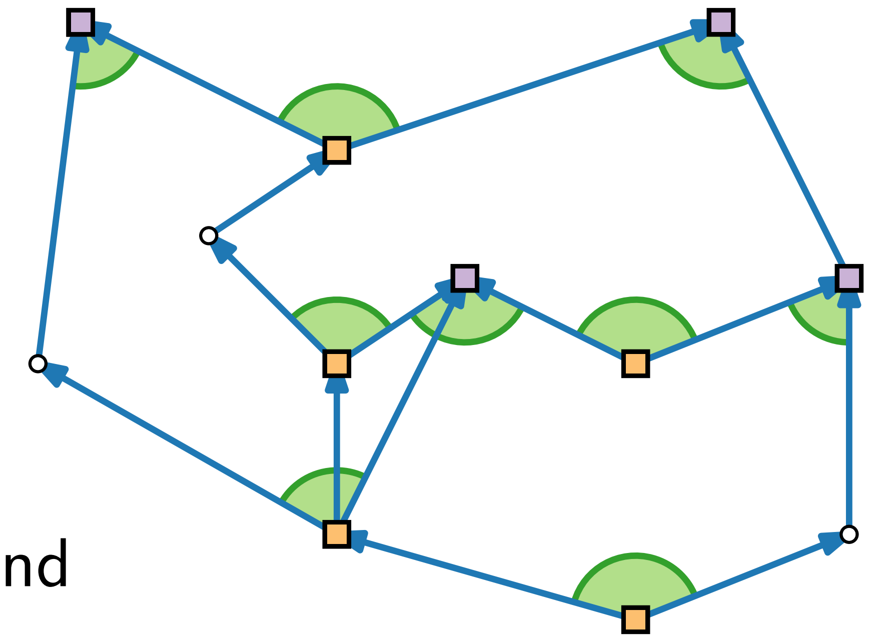

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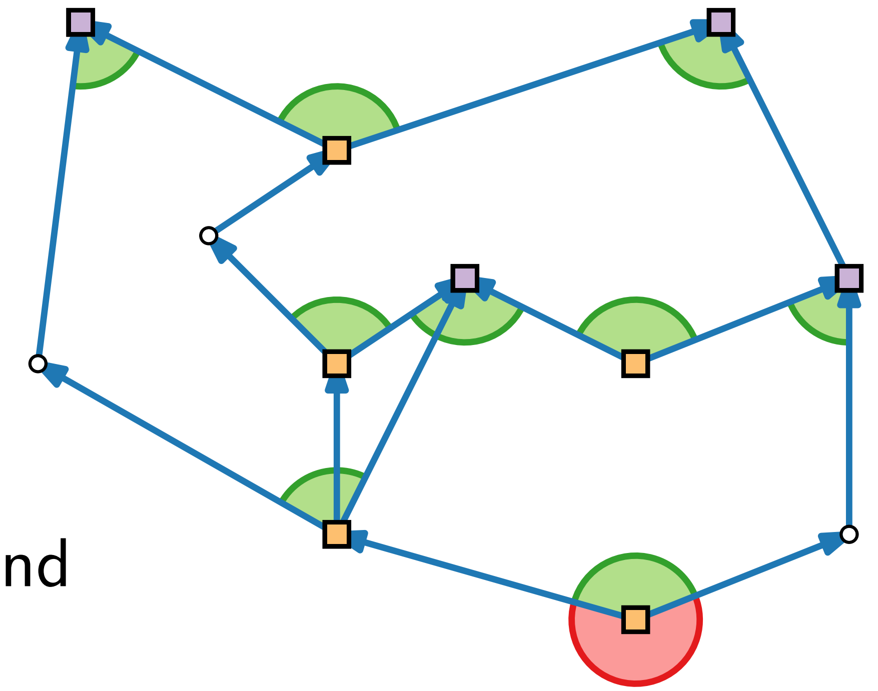

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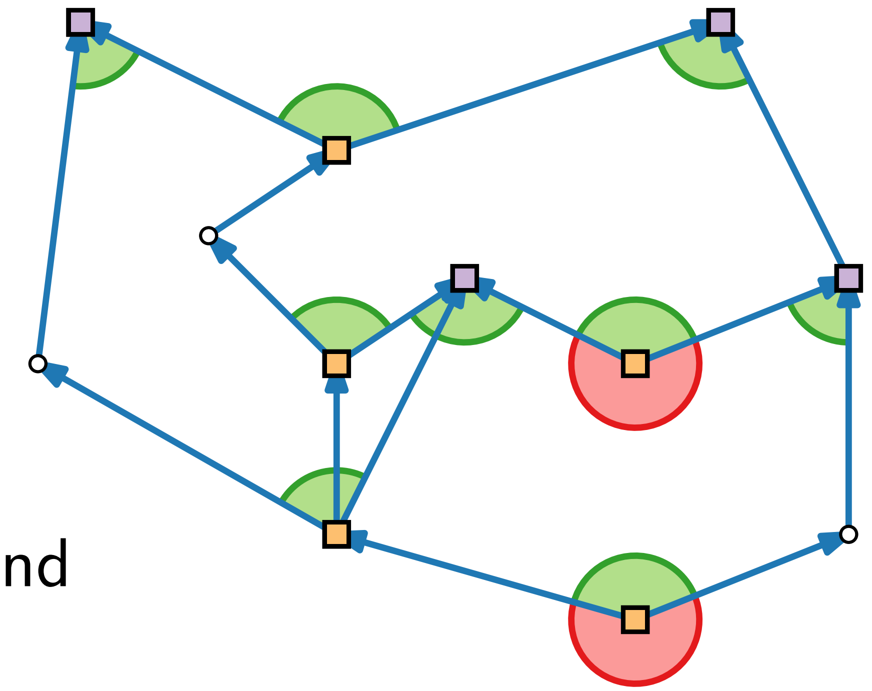

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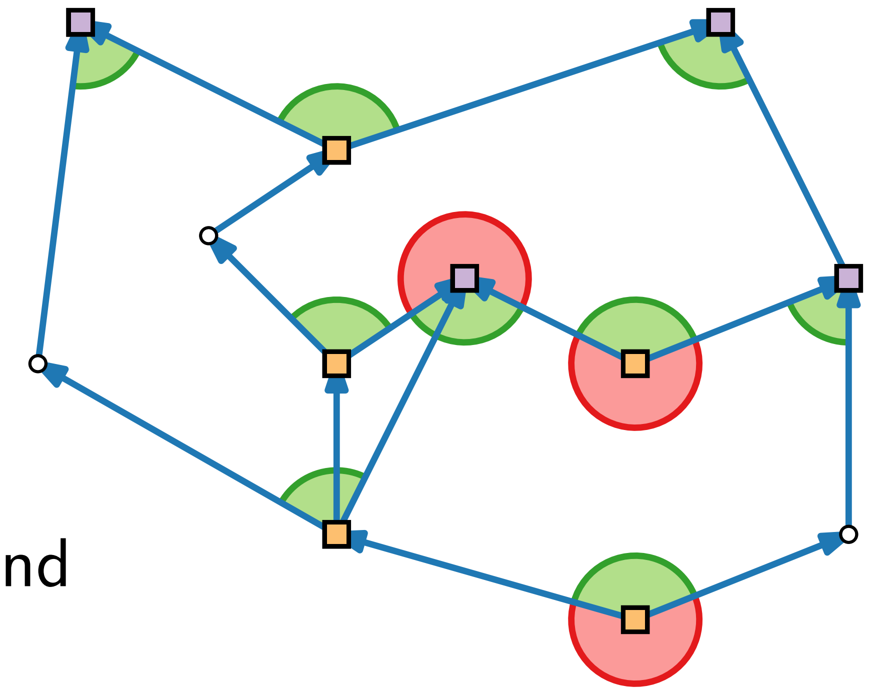

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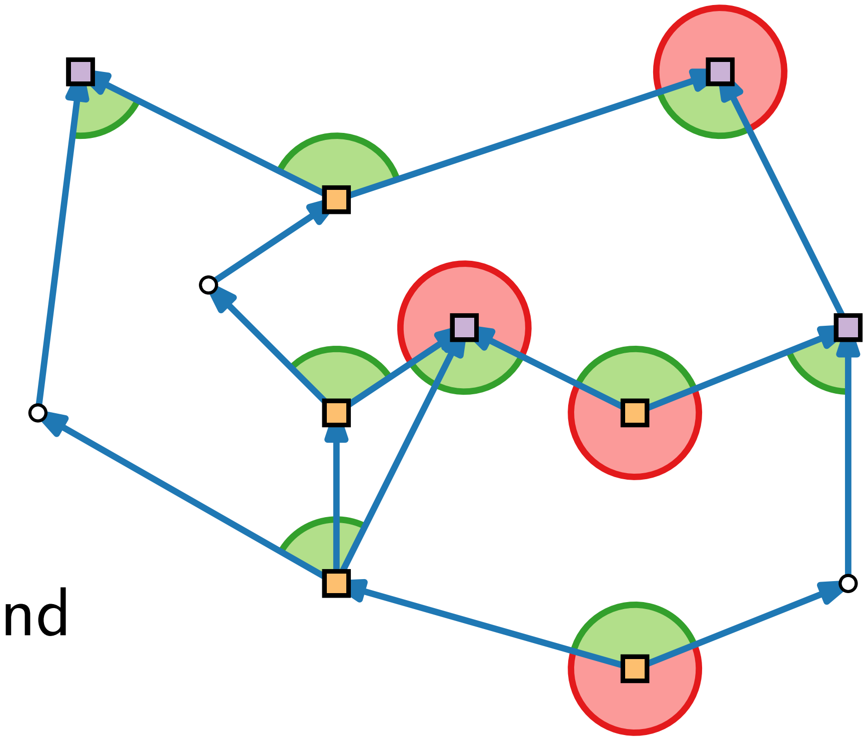

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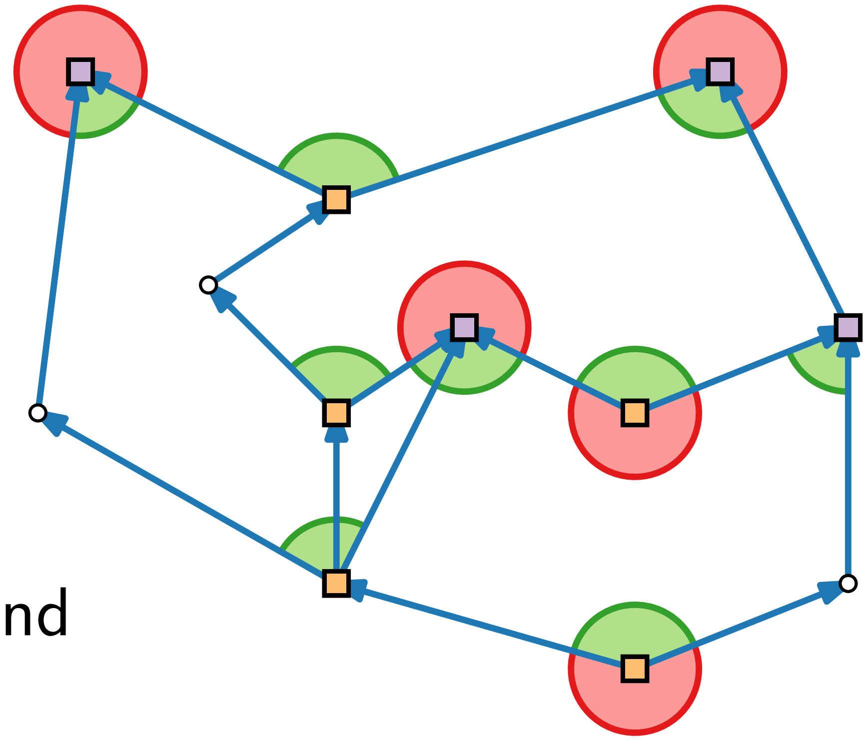

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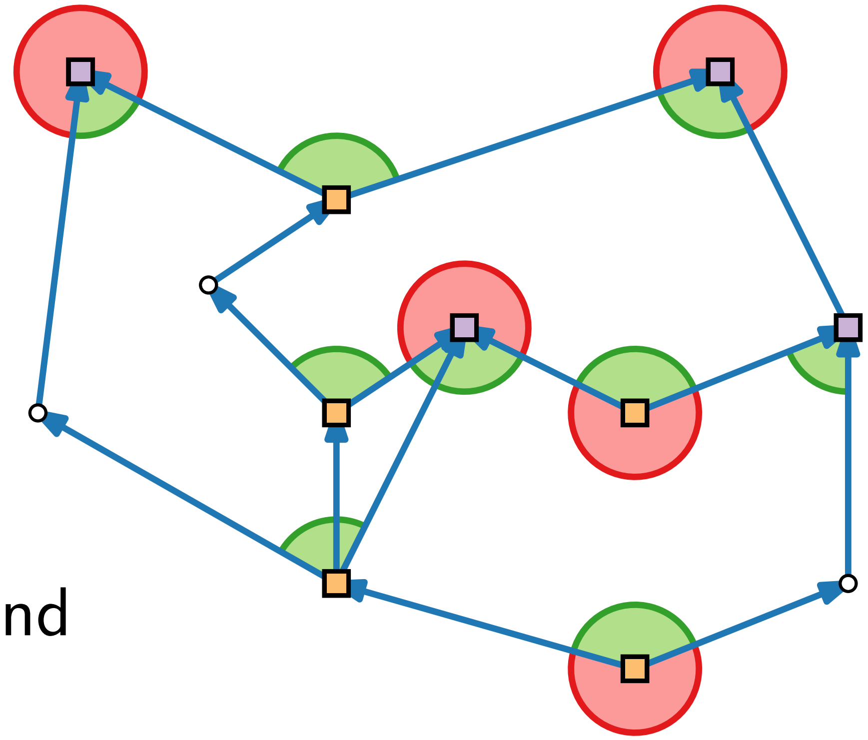

Definitions.

- A vertex v is a **local source** wrt to a face f if v has two outgoing edges on ∂f.
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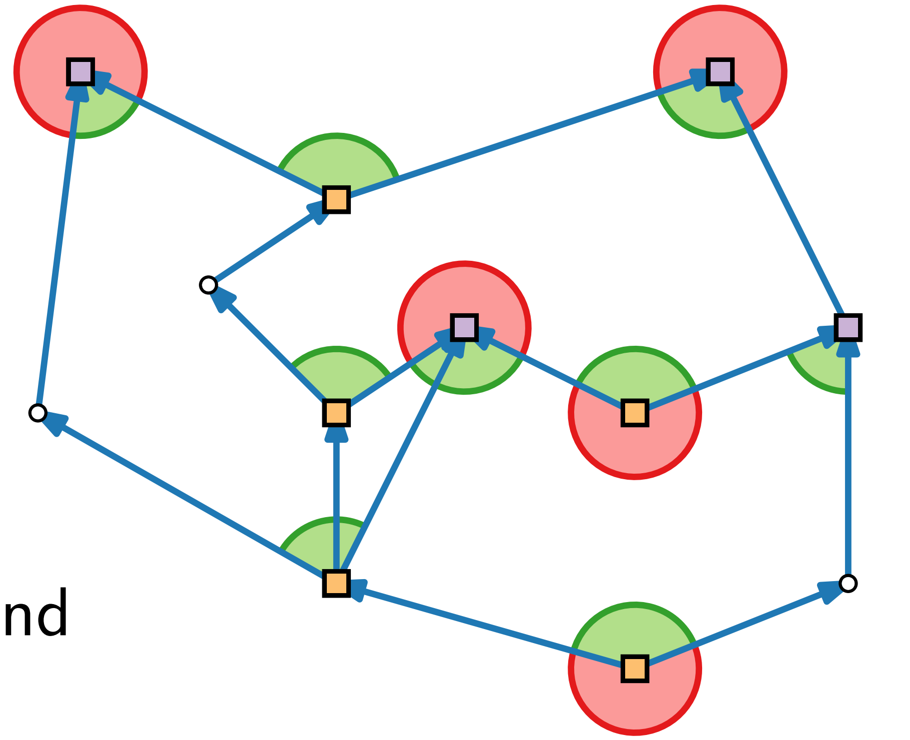

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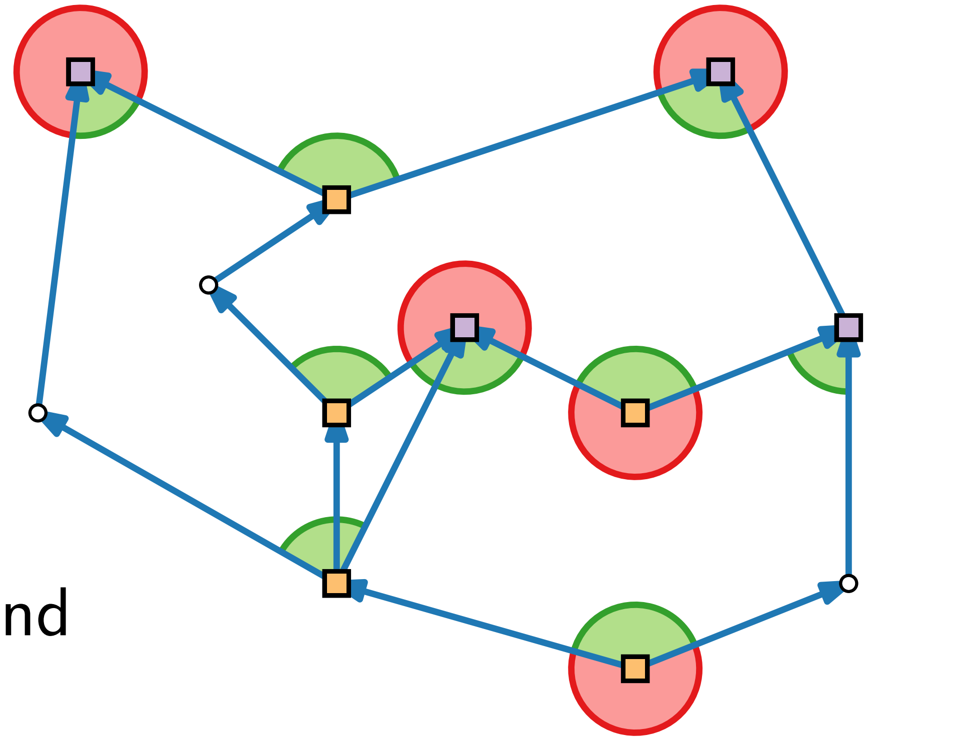

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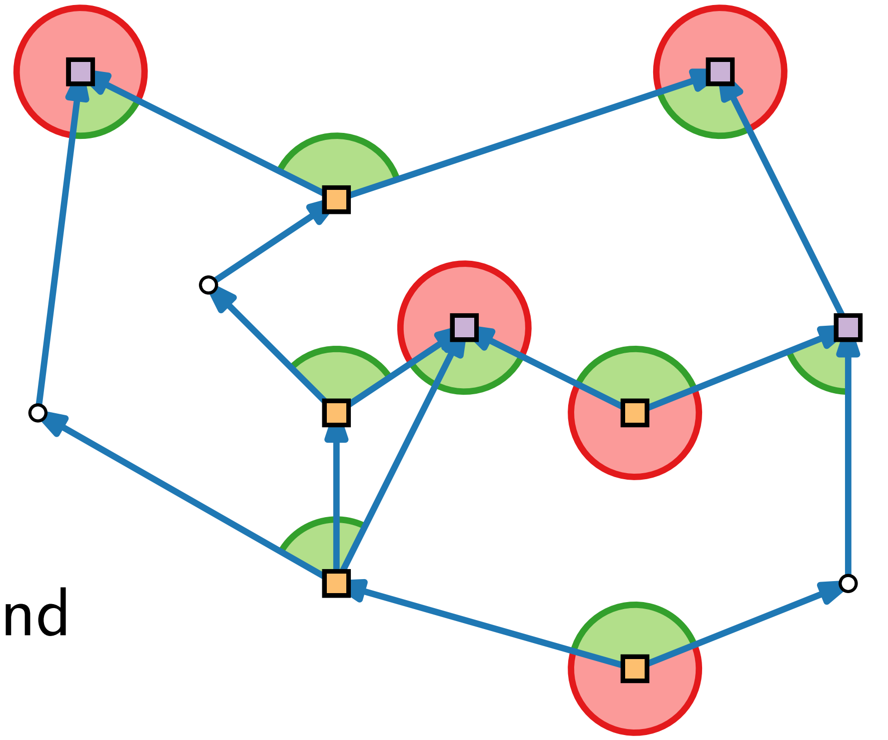

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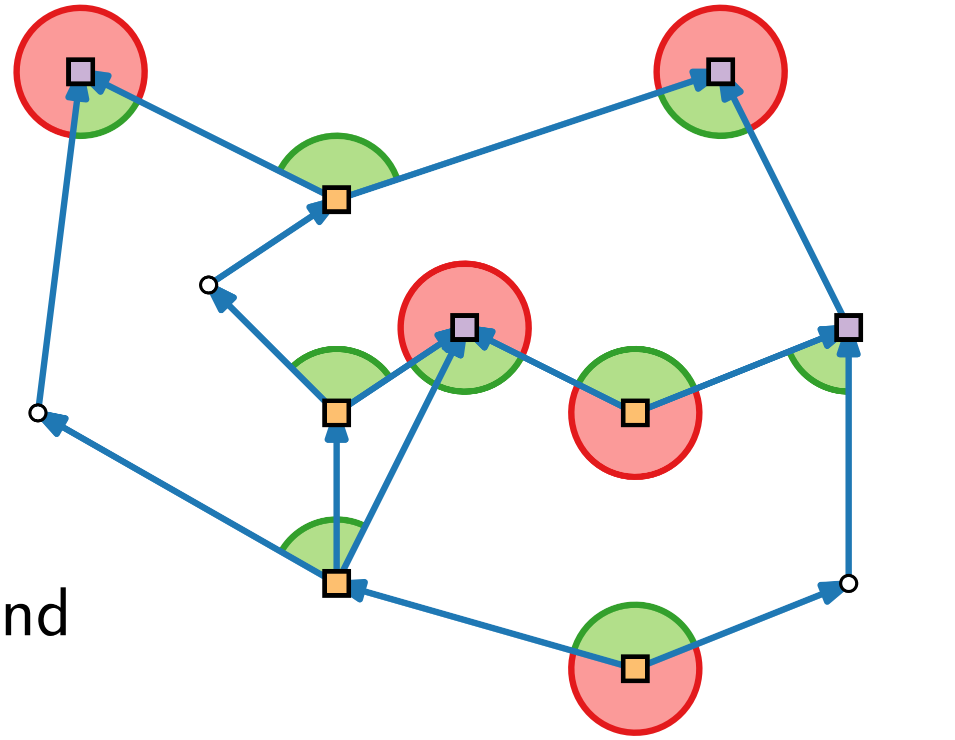
Angles, Local Sources & Sinks



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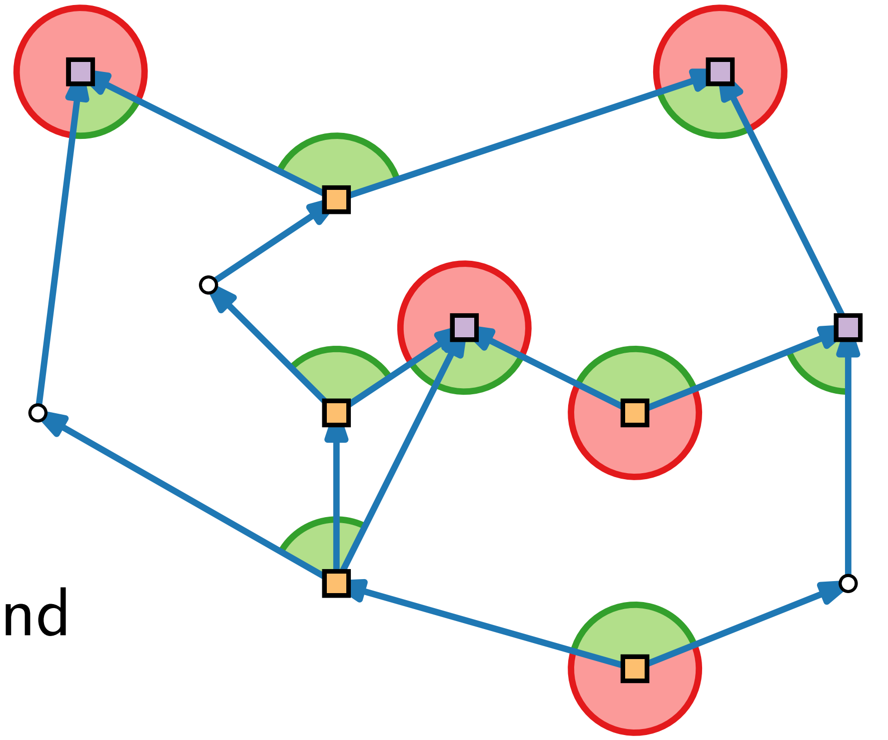
Angles, Local Sources & Sinks



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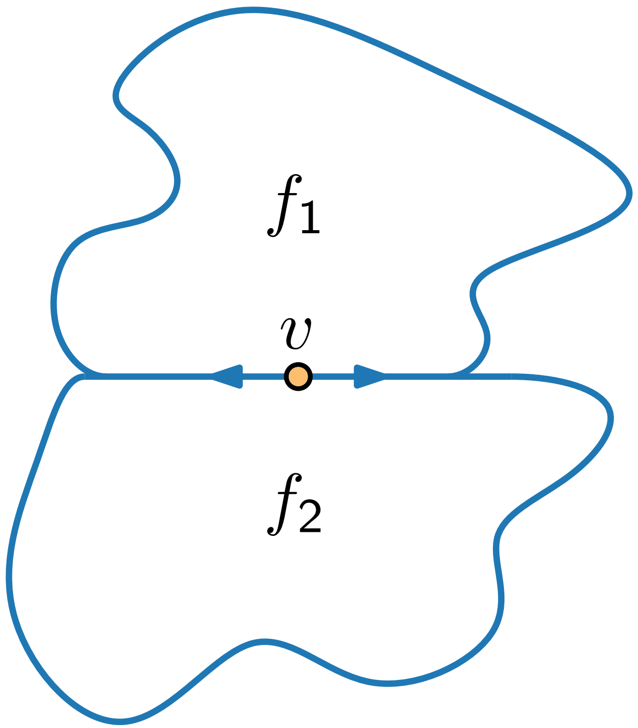
Lemma 1.
$L(f) + S(f) = 2A(f)$

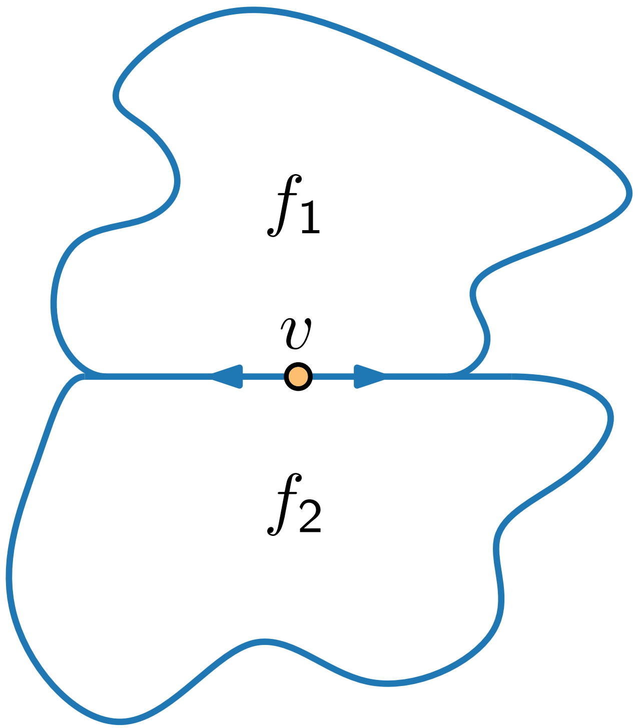
Assignment Problem

- Vertex v is a global source at faces f_1 and f_2.



Assignment Problem

- Vertex v is a global source at faces f_1 and f_2.
- Does v have a **large** angle in f_1 or f_2?



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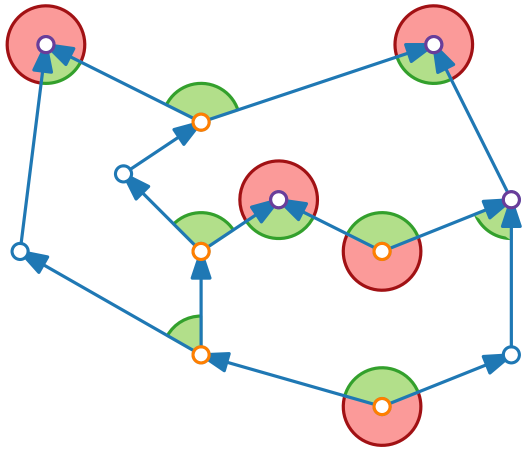
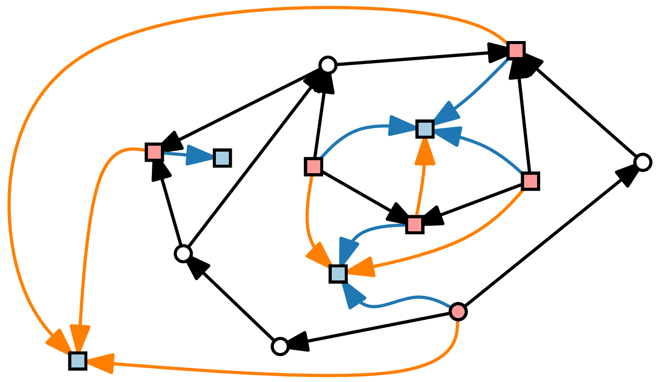
Visualization of Graphs

Lecture 6:
Upward Planar Drawings



Part III:
Angle Relations



Jonathan Klawitter

Angle Relations

Lemma 2.

$$L(f) - S(f) = \begin{cases} -2, & f \neq f_0 \\ +2, & f = f_0 \end{cases}$$

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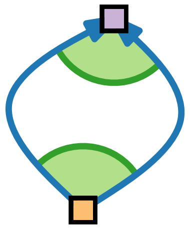

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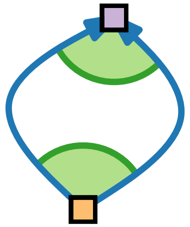

$\Rightarrow S(f) = 2$

Angle Relations

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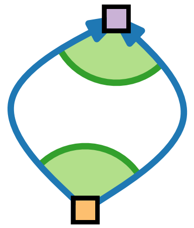

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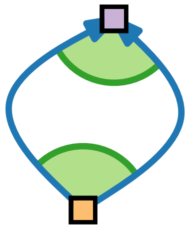

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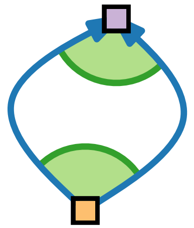

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Split f with edge from a large angle at a "low" sink u to

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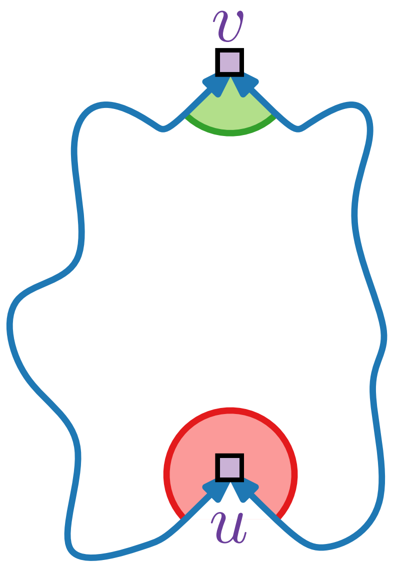

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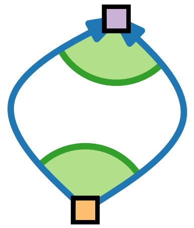
- $L(f) = 0$  $\Rightarrow S(f) = 2$
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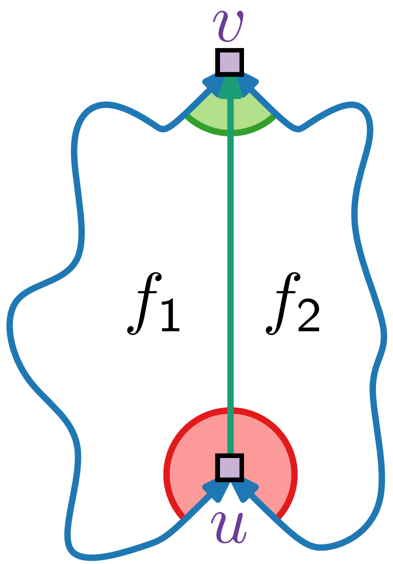

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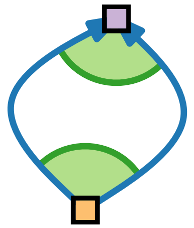

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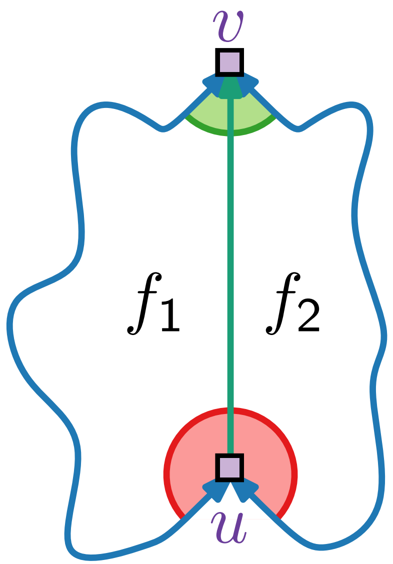

$L(f) - S(f)$

Angle Relations

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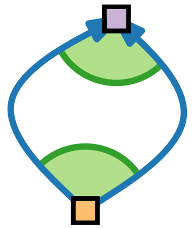

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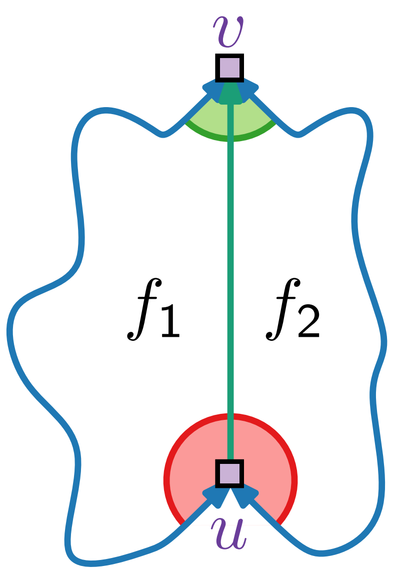

$$L(f) - S(f) = L(f_1) + L(f_2) + 1$$

Angle Relations

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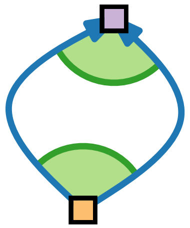

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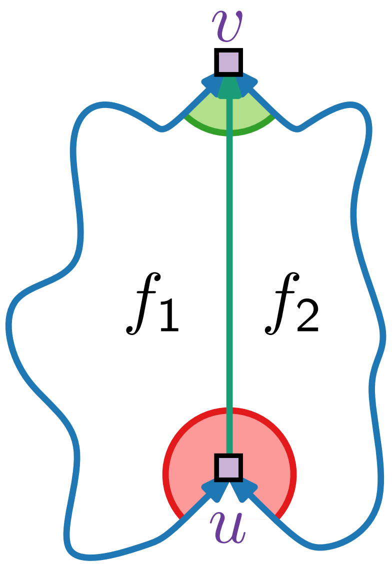

$$L(f) - S(f) = L(f_1) + L(f_2) + 1 - (S(f_1) + S(f_2) - 1)$$

Angle Relations

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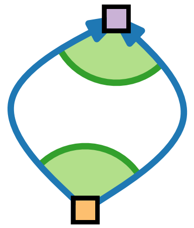

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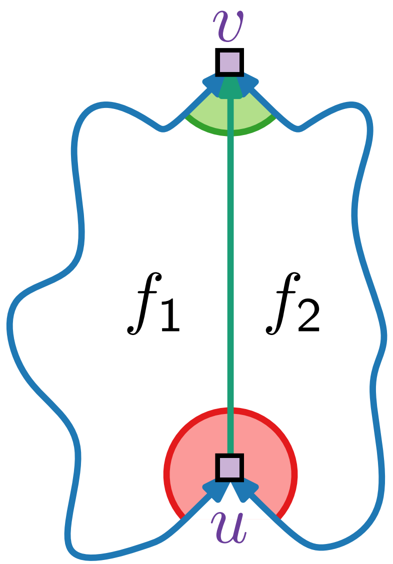

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-2 -2

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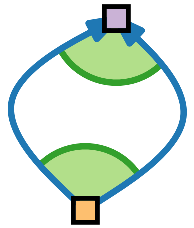

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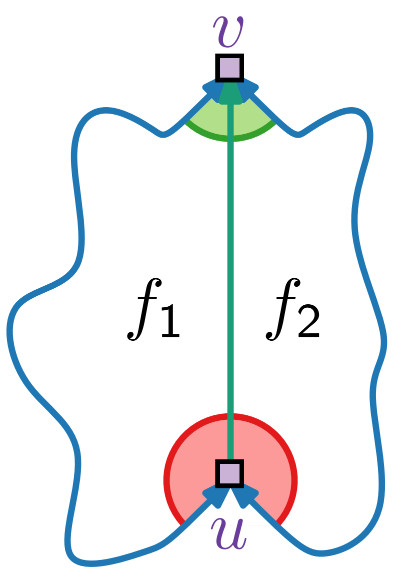

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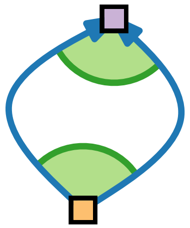

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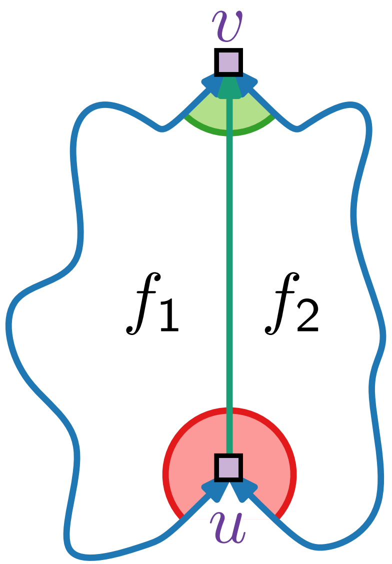

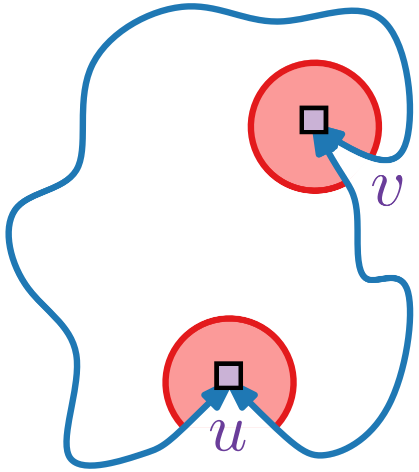

-2 -2

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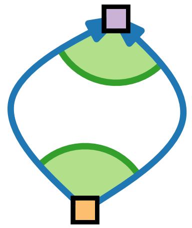

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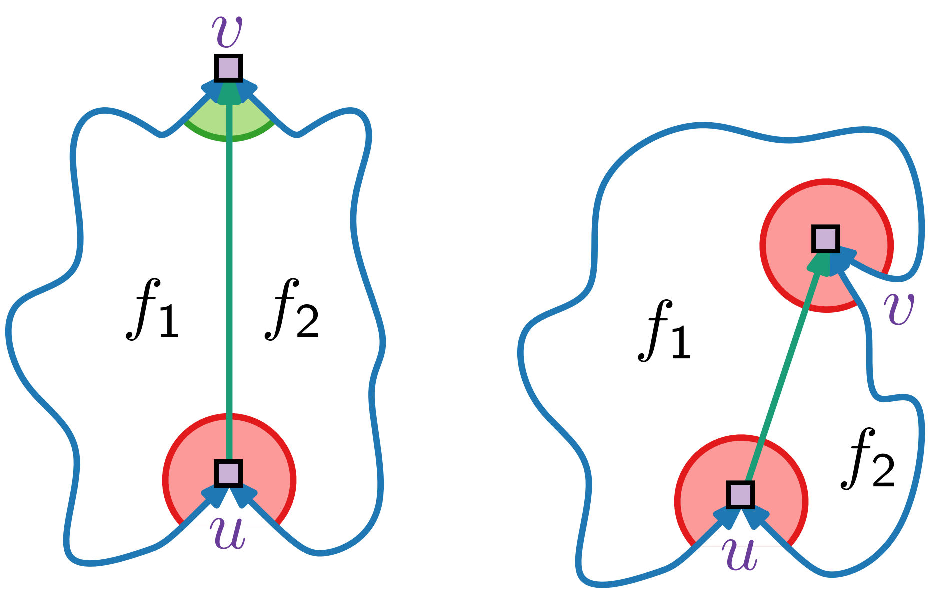

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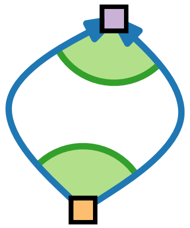

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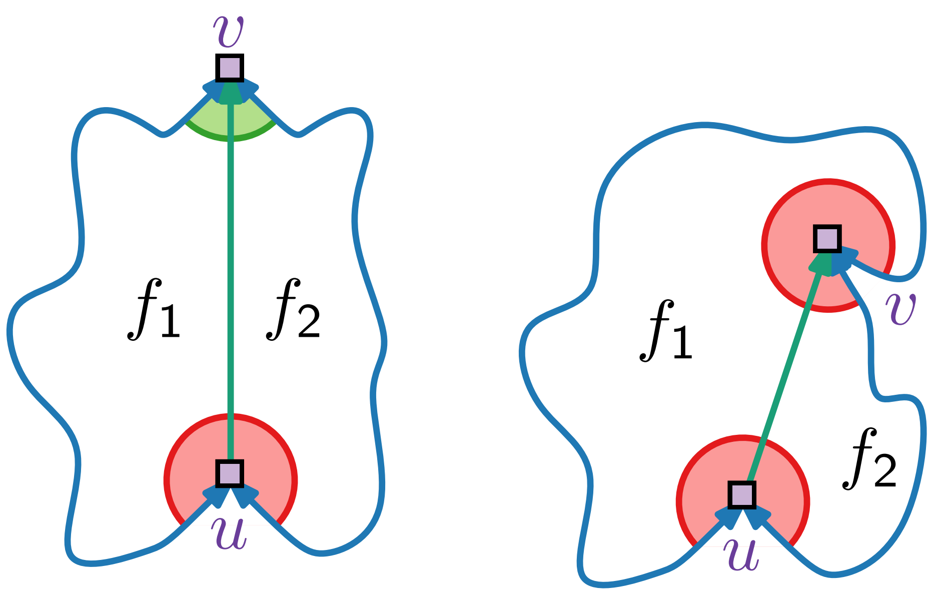

$$\begin{aligned} L(f) - S(f) &= \overbrace{L(f_1)}^{-2} + \overbrace{L(f_2)}^{-2} + 2 \\ &\quad - (S(f_1) + S(f_2)) \\ &= -2 \end{aligned}$$

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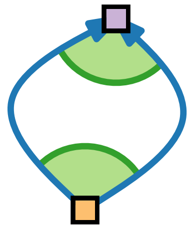
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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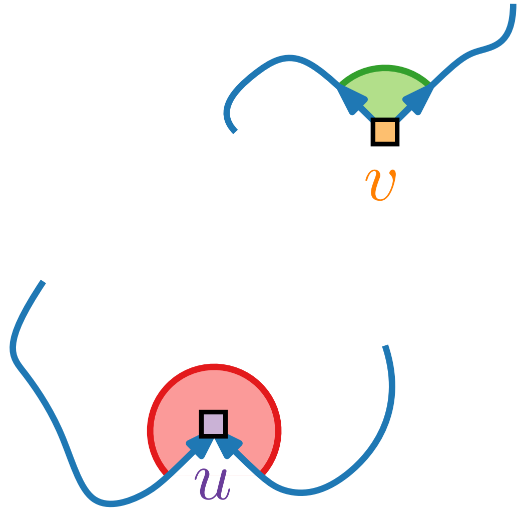

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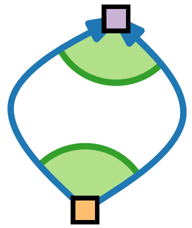
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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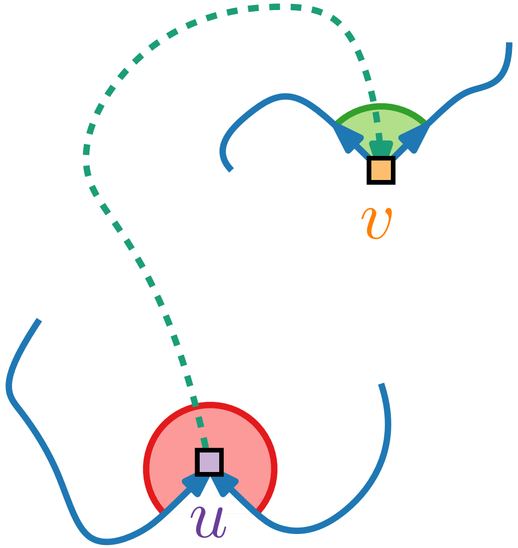

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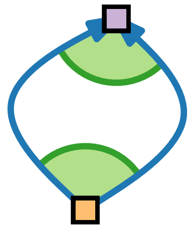
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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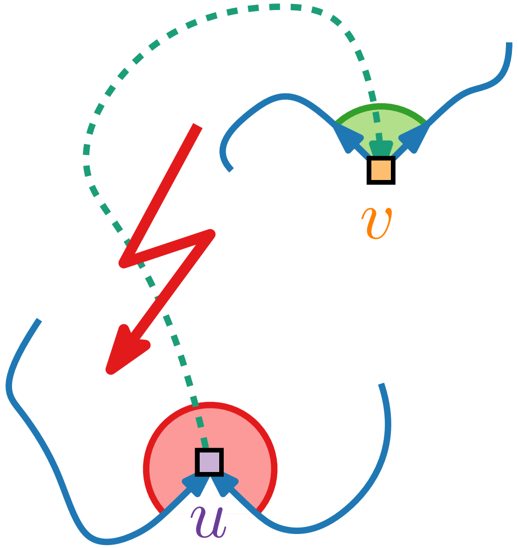

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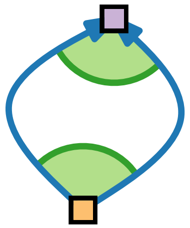

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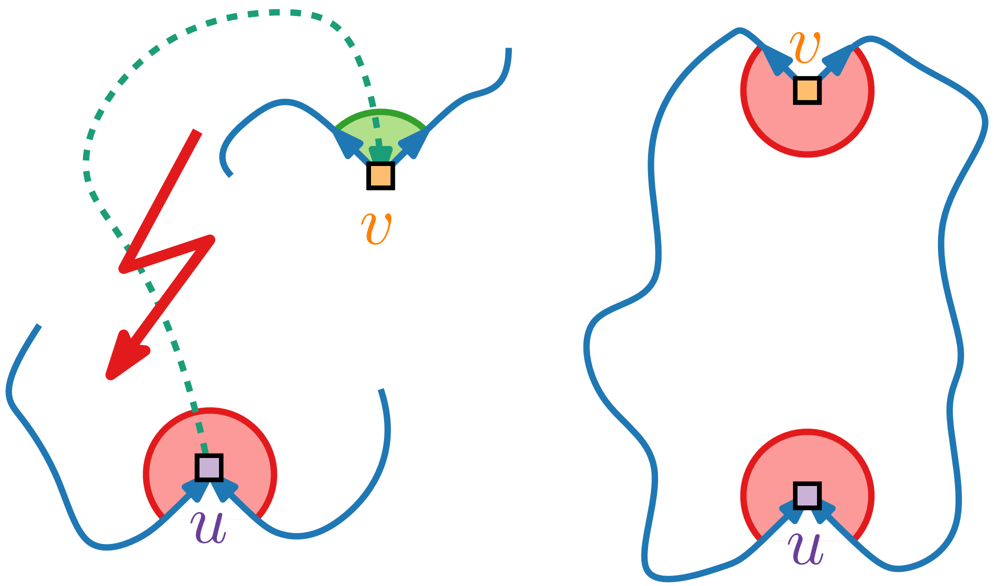

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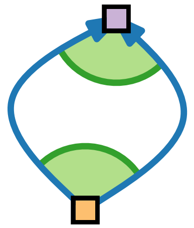
- $L(f) = 0$  $\Rightarrow S(f) = 2$

- $L(f) \geq 1$

Split f with edge from a large angle at a "low" sink u to

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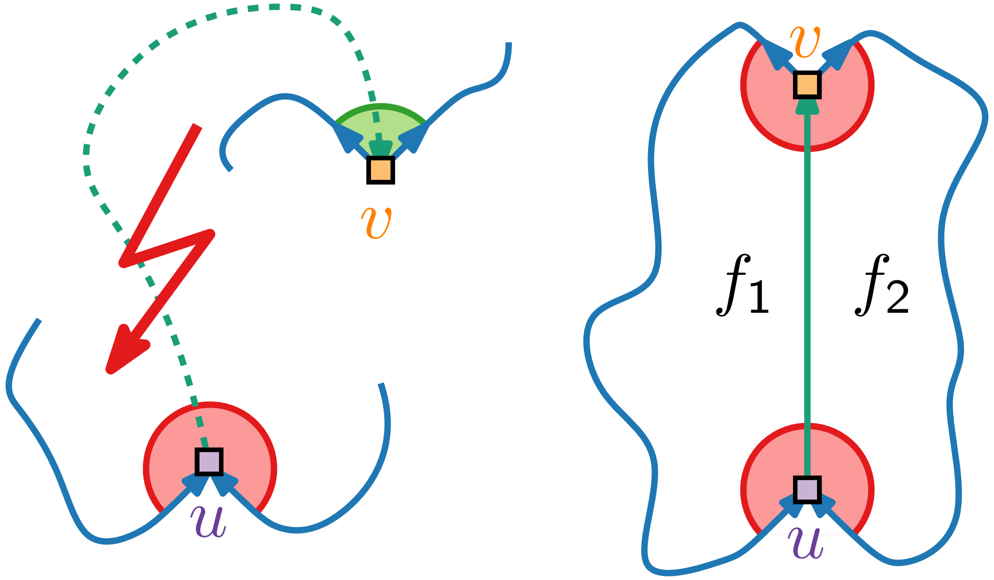

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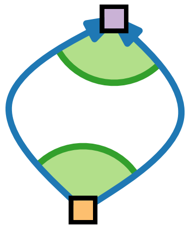

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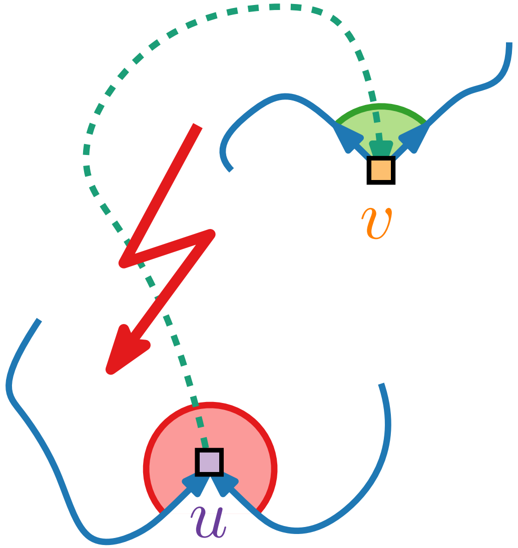

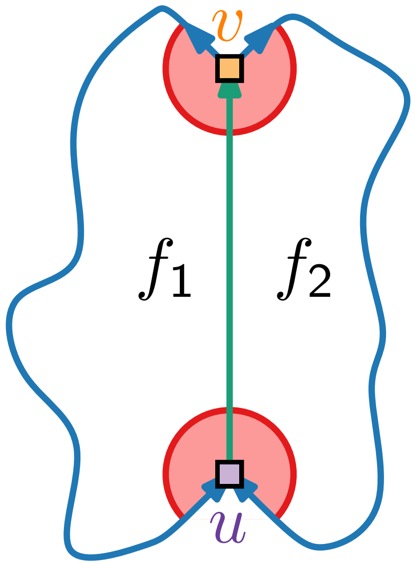

$L(f) - S(f)$

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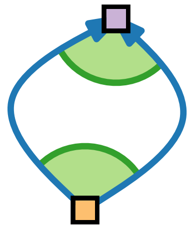

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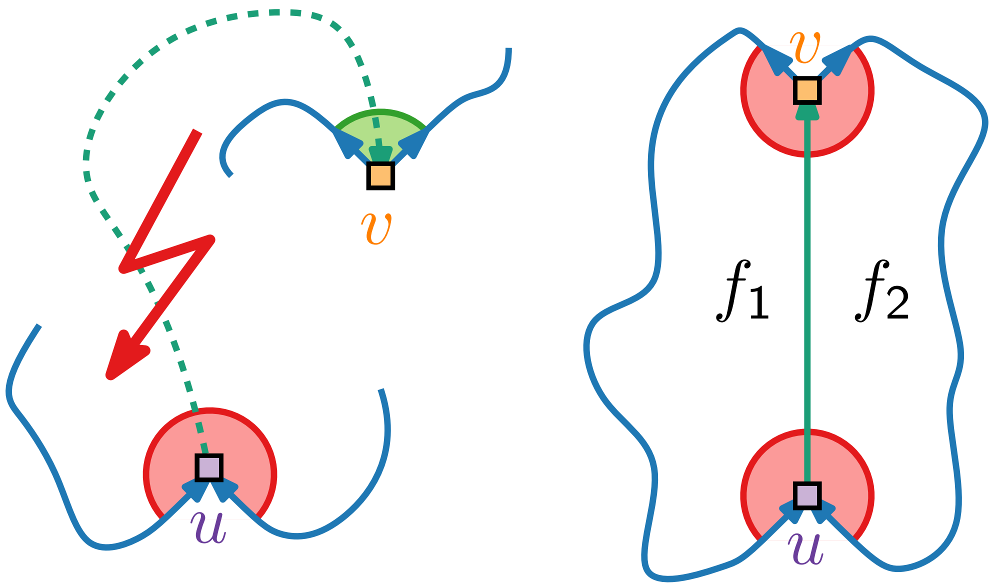

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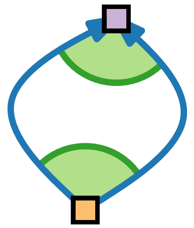

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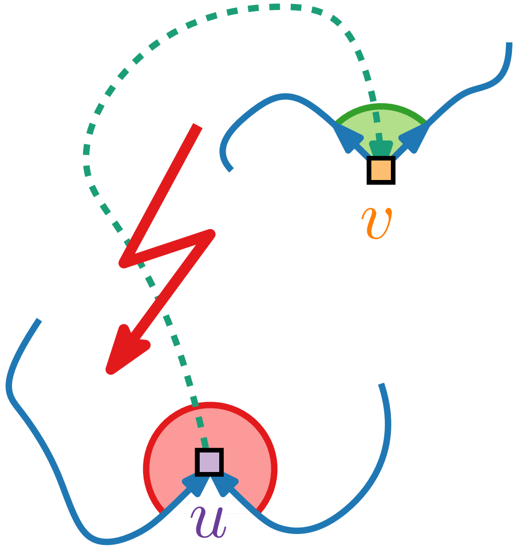

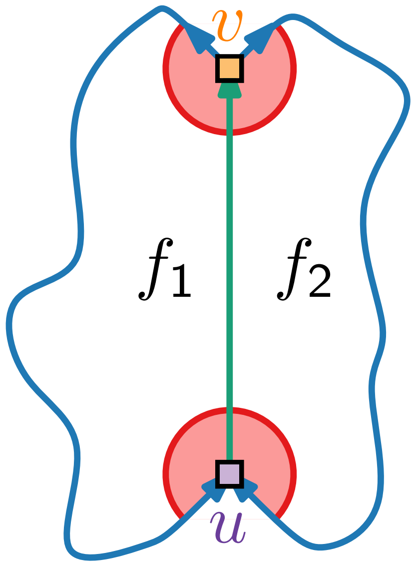

$$L(f) - S(f) = L(f_1) + L(f_2) + 2 - (S(f_1) + S(f_2))$$

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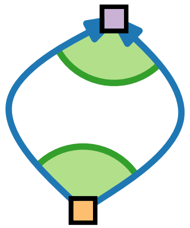

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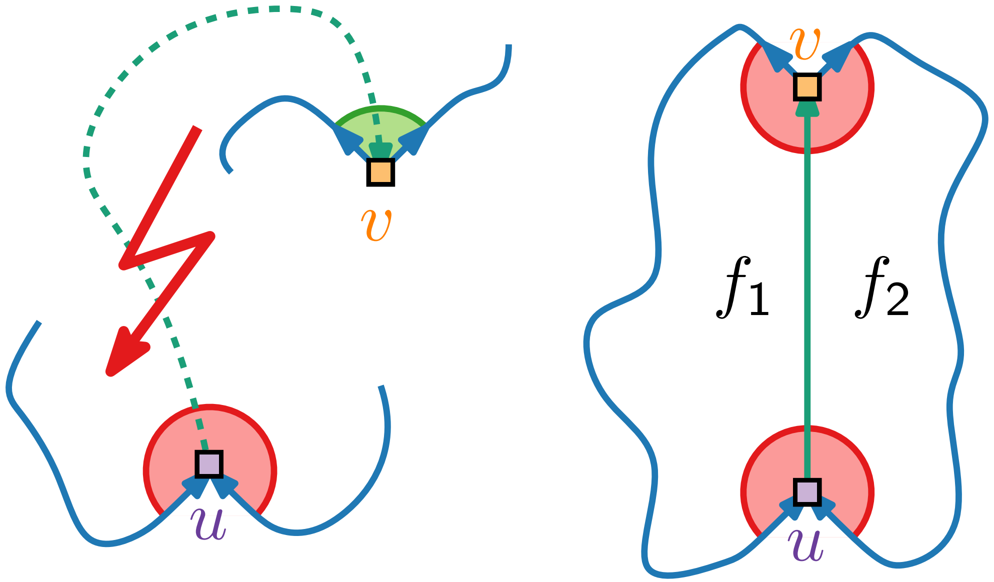

-2 -2

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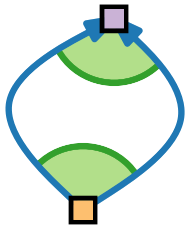
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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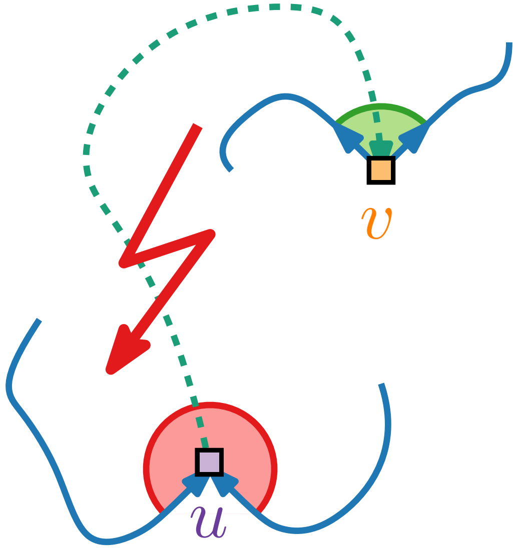

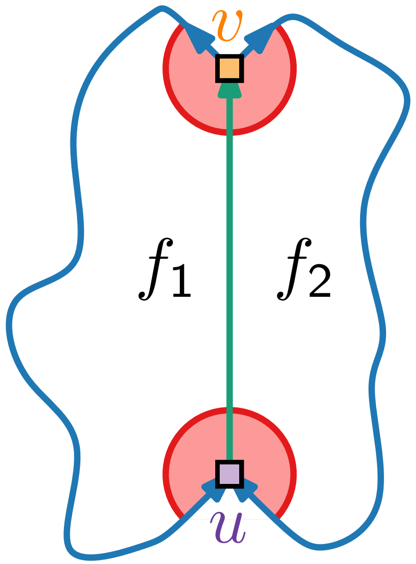

$$\begin{aligned} L(f) - S(f) &= \underbrace{L(f_1)}_{-2} + \underbrace{L(f_2)}_{-2} + 2 \\ &\quad - (S(f_1) + S(f_2)) \\ &= -2 \end{aligned}$$

(-2 is written above $L(f_1)$ … $S(f_1)$ and above $L(f_2)$ … $S(f_2)$)

Angle Relations

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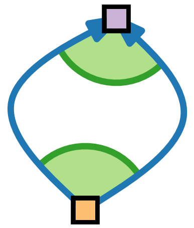

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Split f with edge from a large angle at a “low” sink u to

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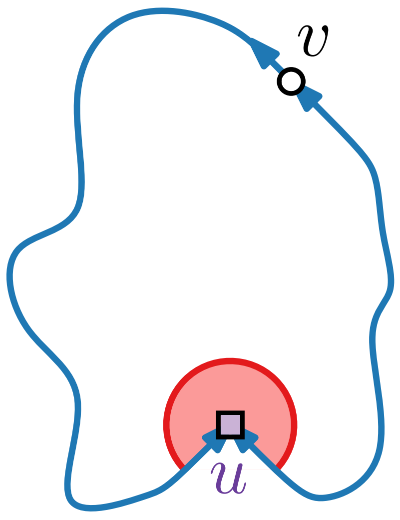

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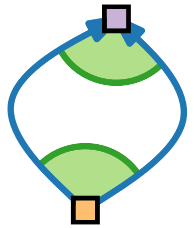
- $L(f) = 0$  $\Rightarrow S(f) = 2$
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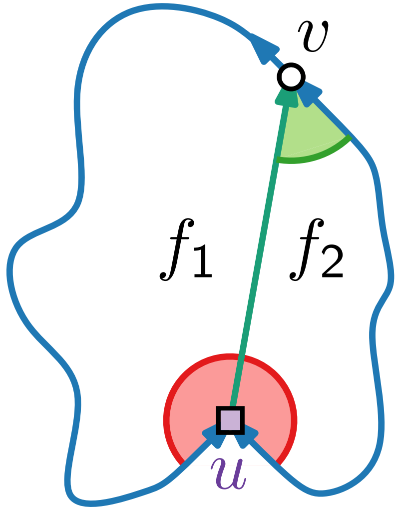

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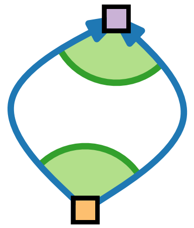

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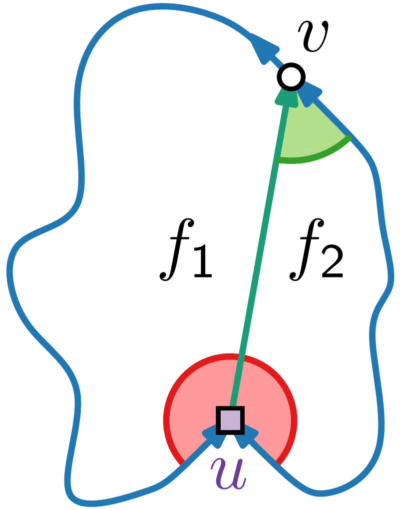

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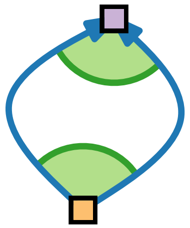
- $L(f) = 0$  $\Rightarrow S(f) = 2$

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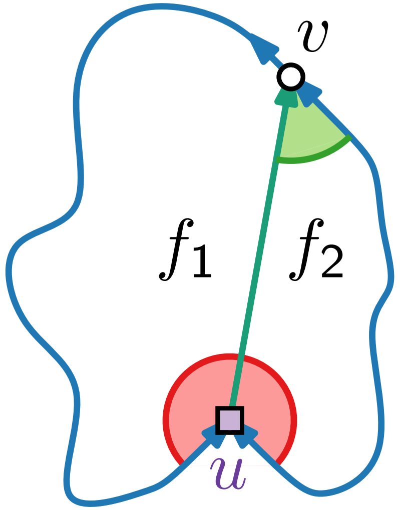

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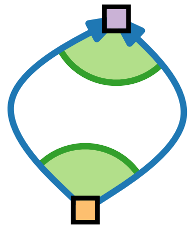

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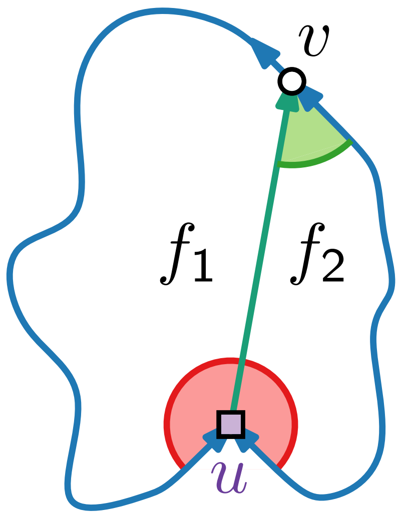

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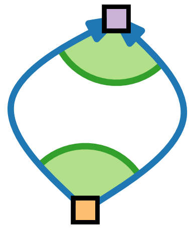

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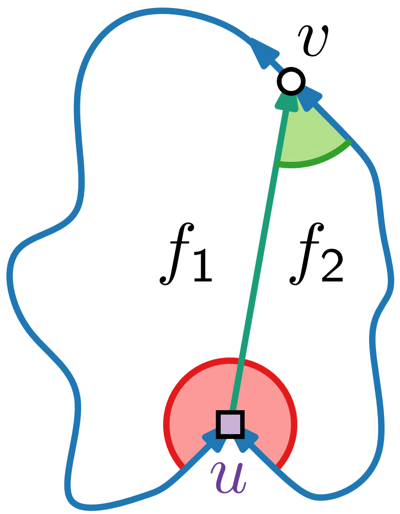

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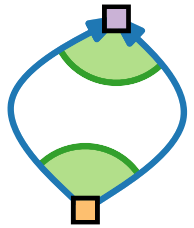

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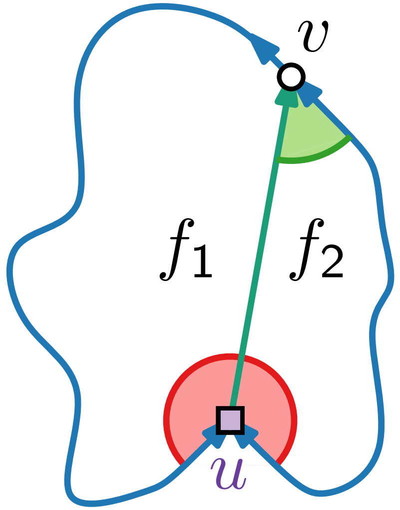

$$L(f) - S(f) = L(f_1) + L(f_2) + 1 - (S(f_1) + S(f_2) - 1) = -2$$

(annotations: -2 above $L(f_1) - S(f_1)$, -2 above $L(f_2) - S(f_2)$)

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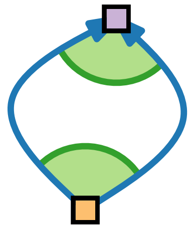

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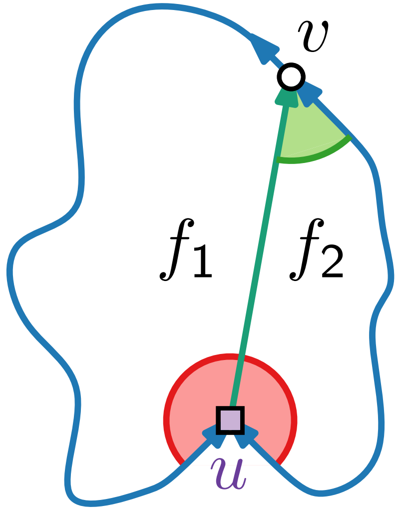

$$\begin{aligned} L(f) - S(f) &= \overbrace{L(f_1)}^{-2} + \overbrace{L(f_2)}^{-2} + 1 \\ &\quad - (S(f_1) + S(f_2) - 1) \\ &= -2 \end{aligned}$$

- Otherwise "high" source u exists.

Number of Large Angles

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In every upward planar drawing of G holds that

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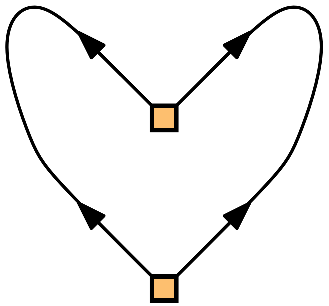

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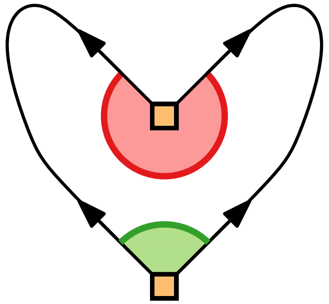

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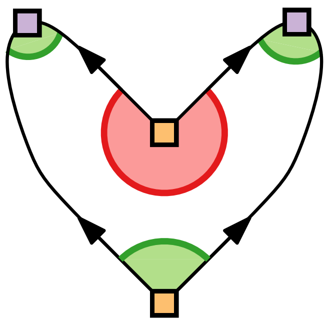

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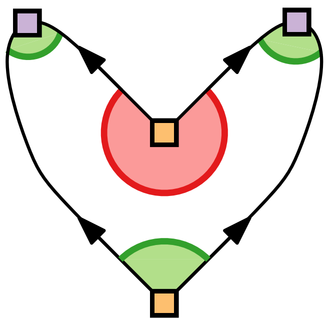

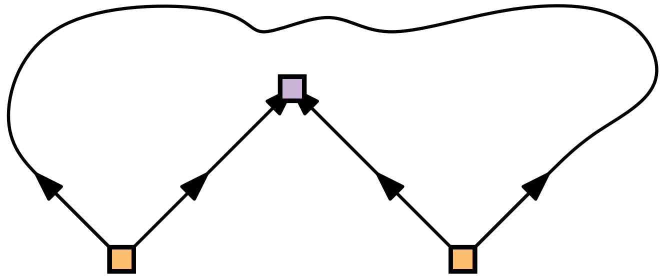

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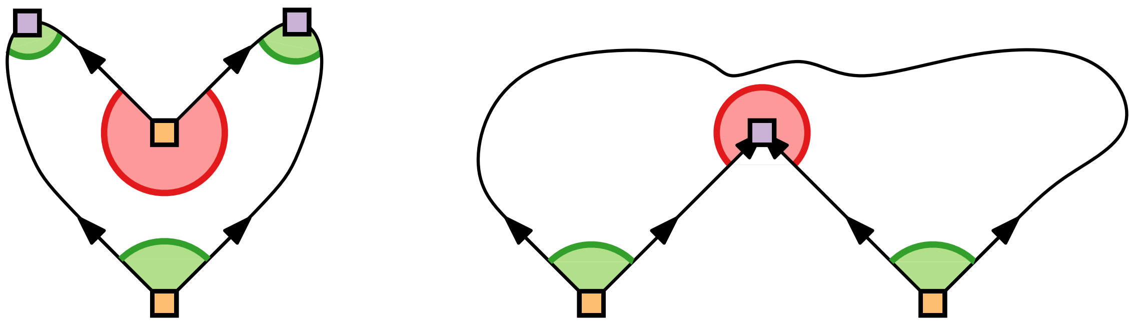

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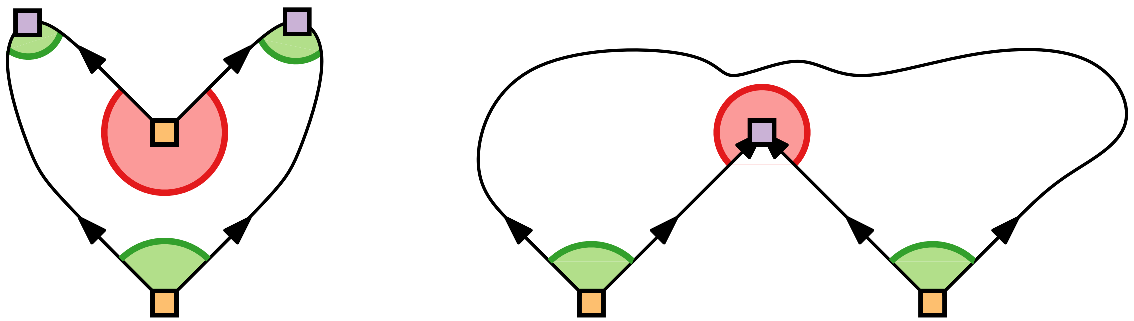
Proof.



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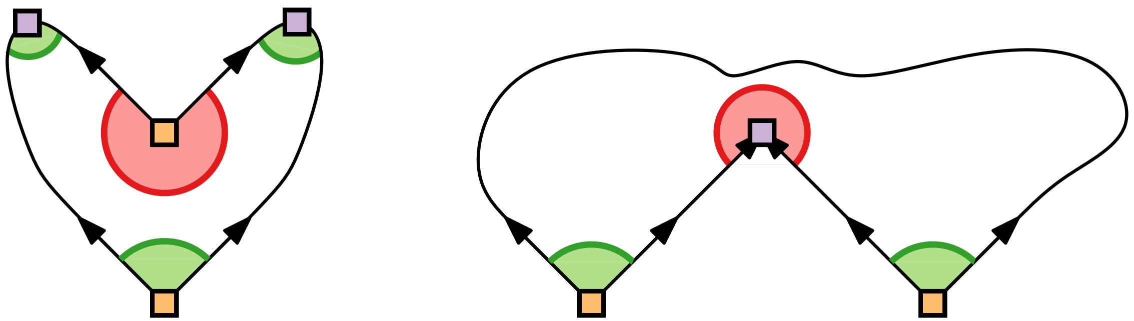
Proof. Lemma 1: $L(f) + S(f) = 2A(f)$



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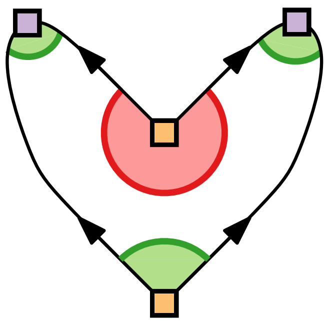
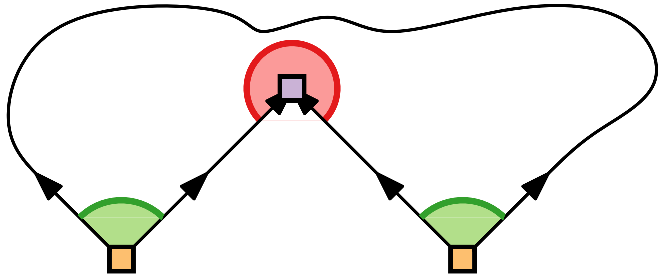
Proof. Lemma 1: $L(f) + S(f) = 2A(f)$
Lemma 2: $L(f) - S(f) = \pm 2$.





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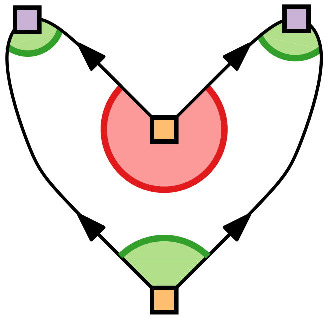
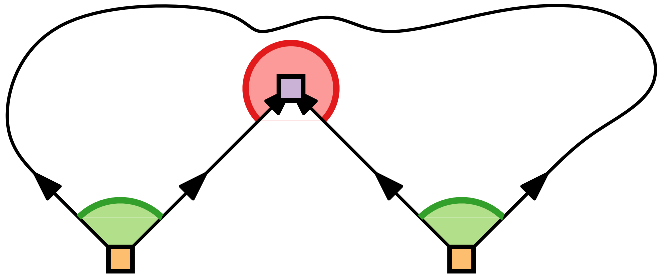
Proof. Lemma 1: $L(f) + S(f) = 2A(f)$
Lemma 2: $L(f) - S(f) = \pm 2$.
$\Rightarrow 2L(f) = 2A(f) \pm 2$.





Assignment of Large Angles to Faces

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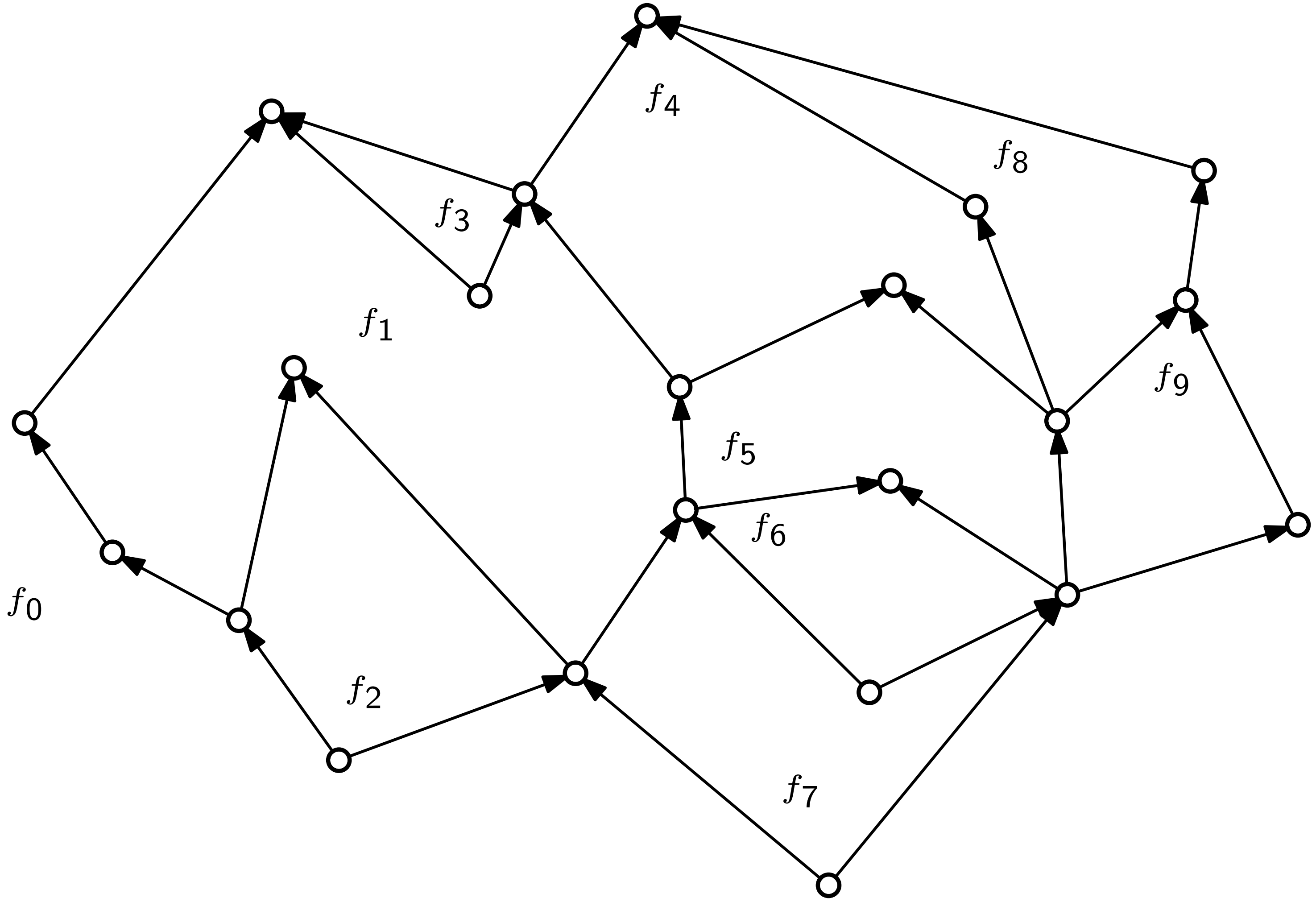

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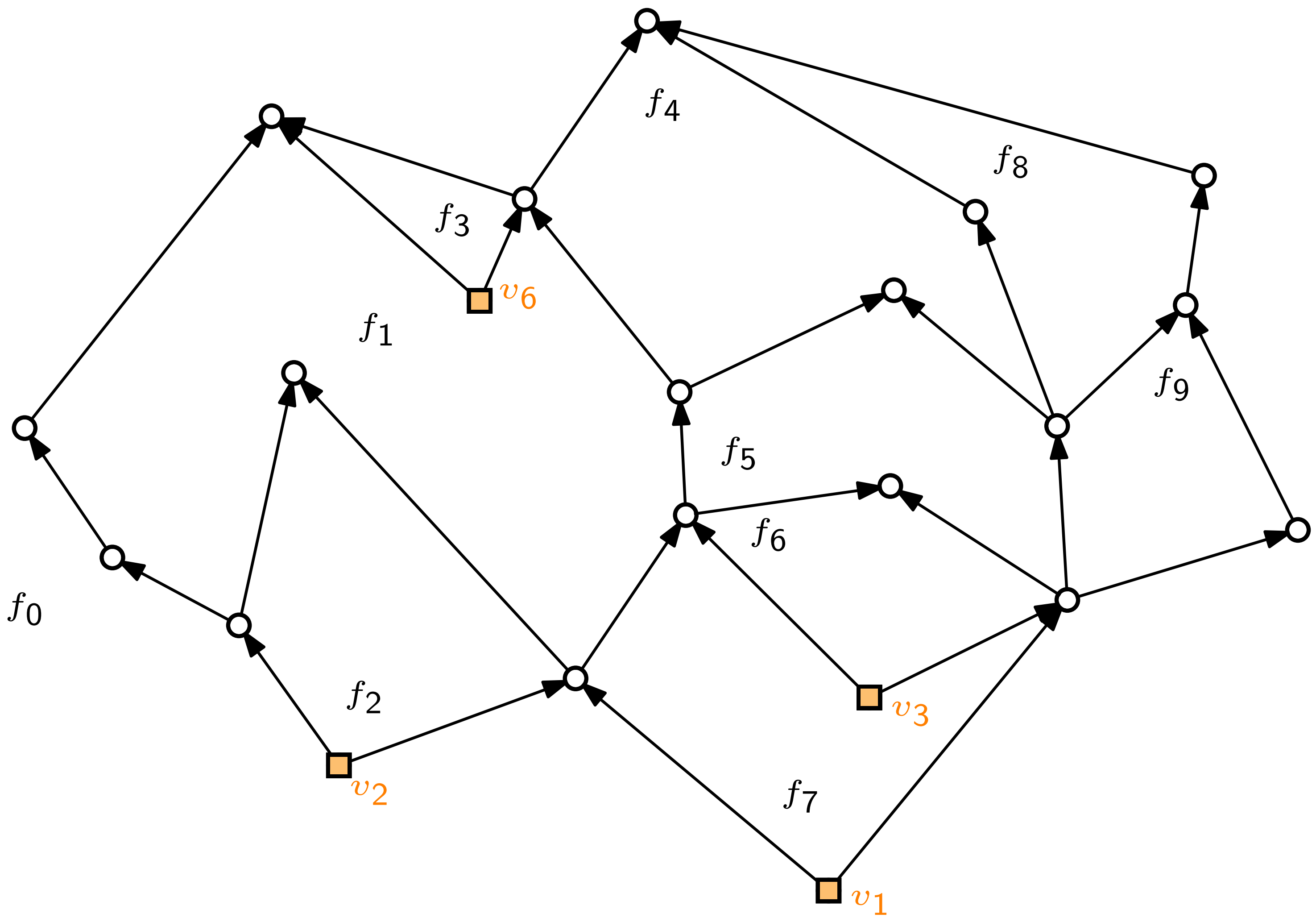

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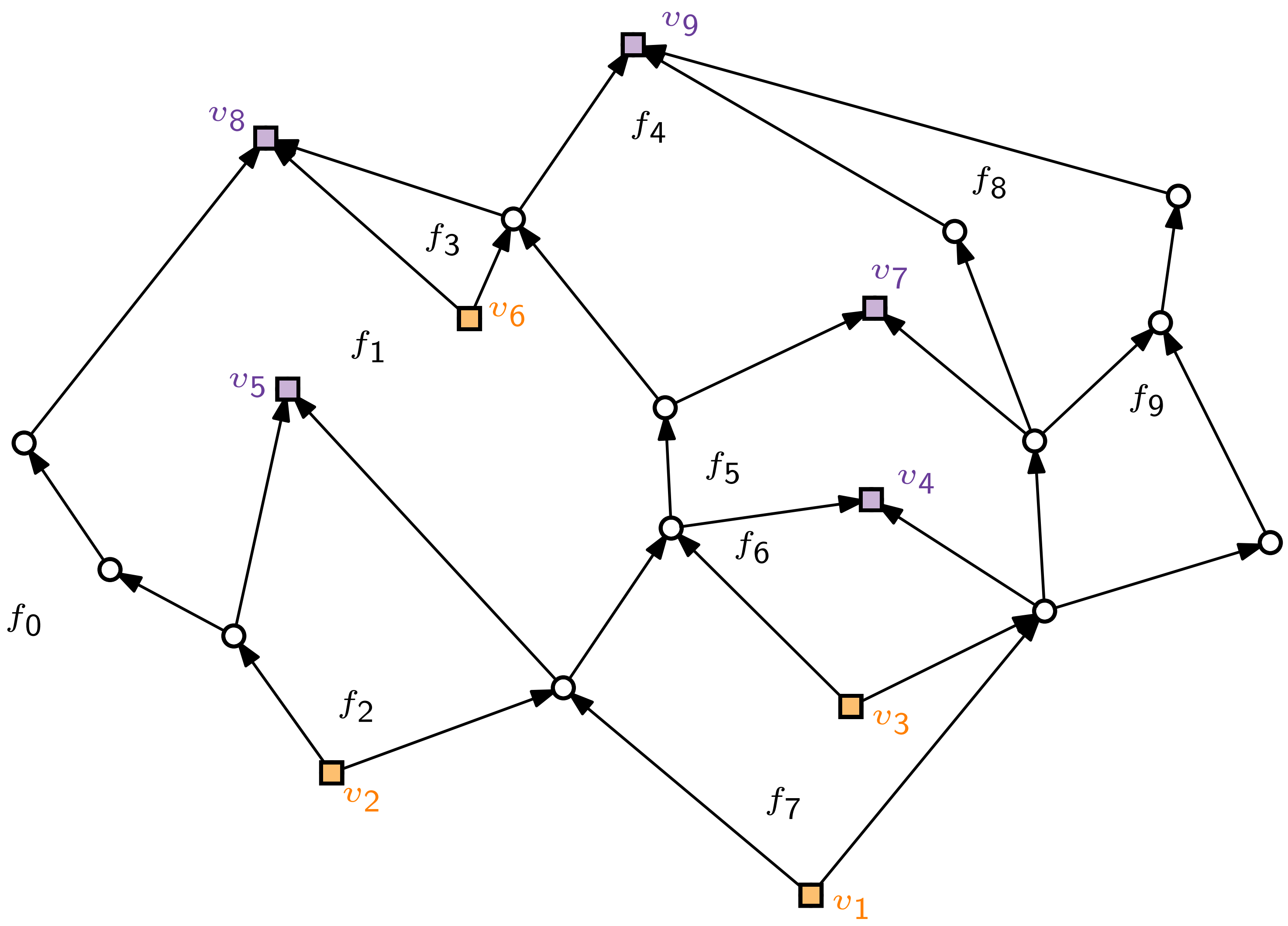

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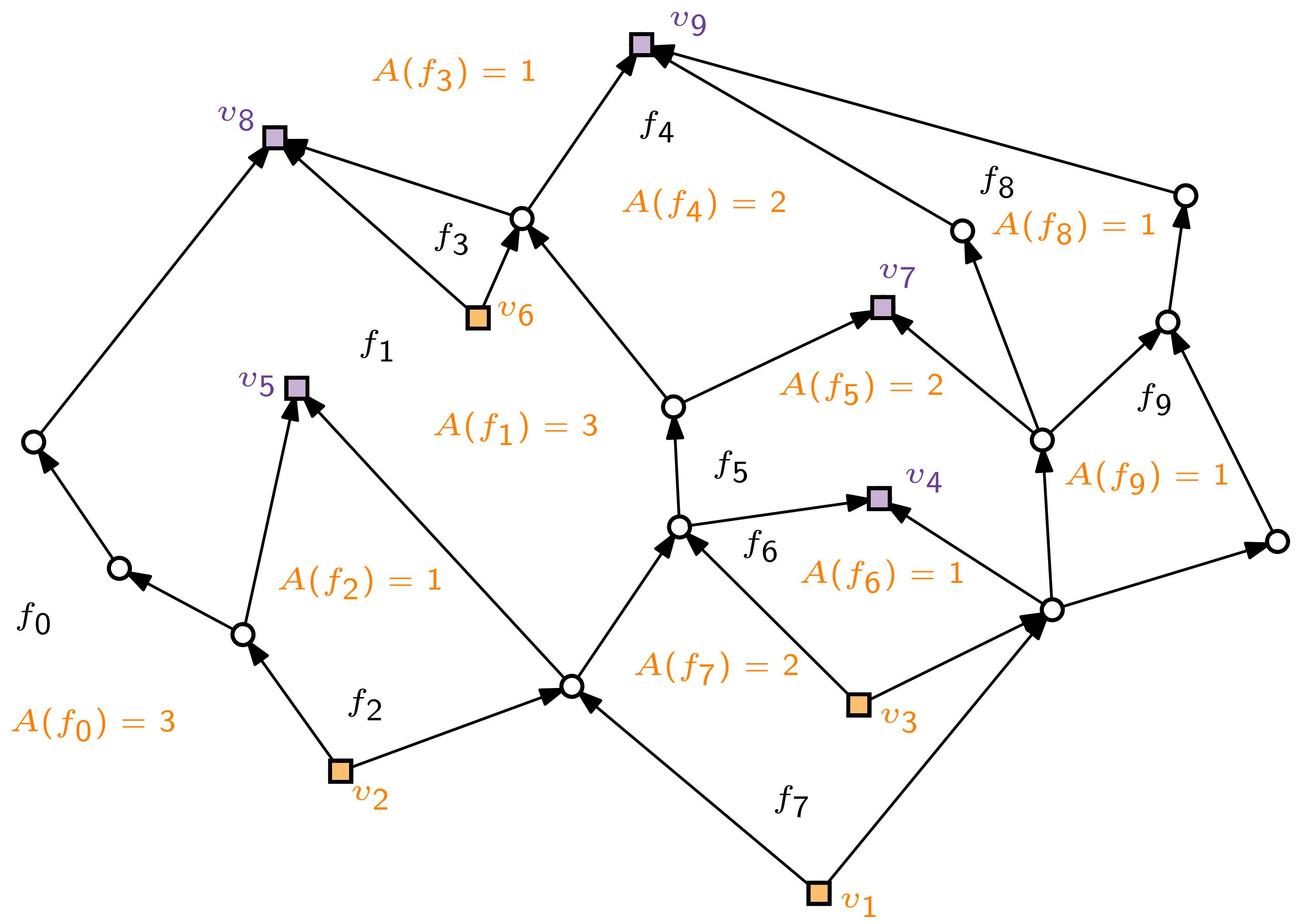

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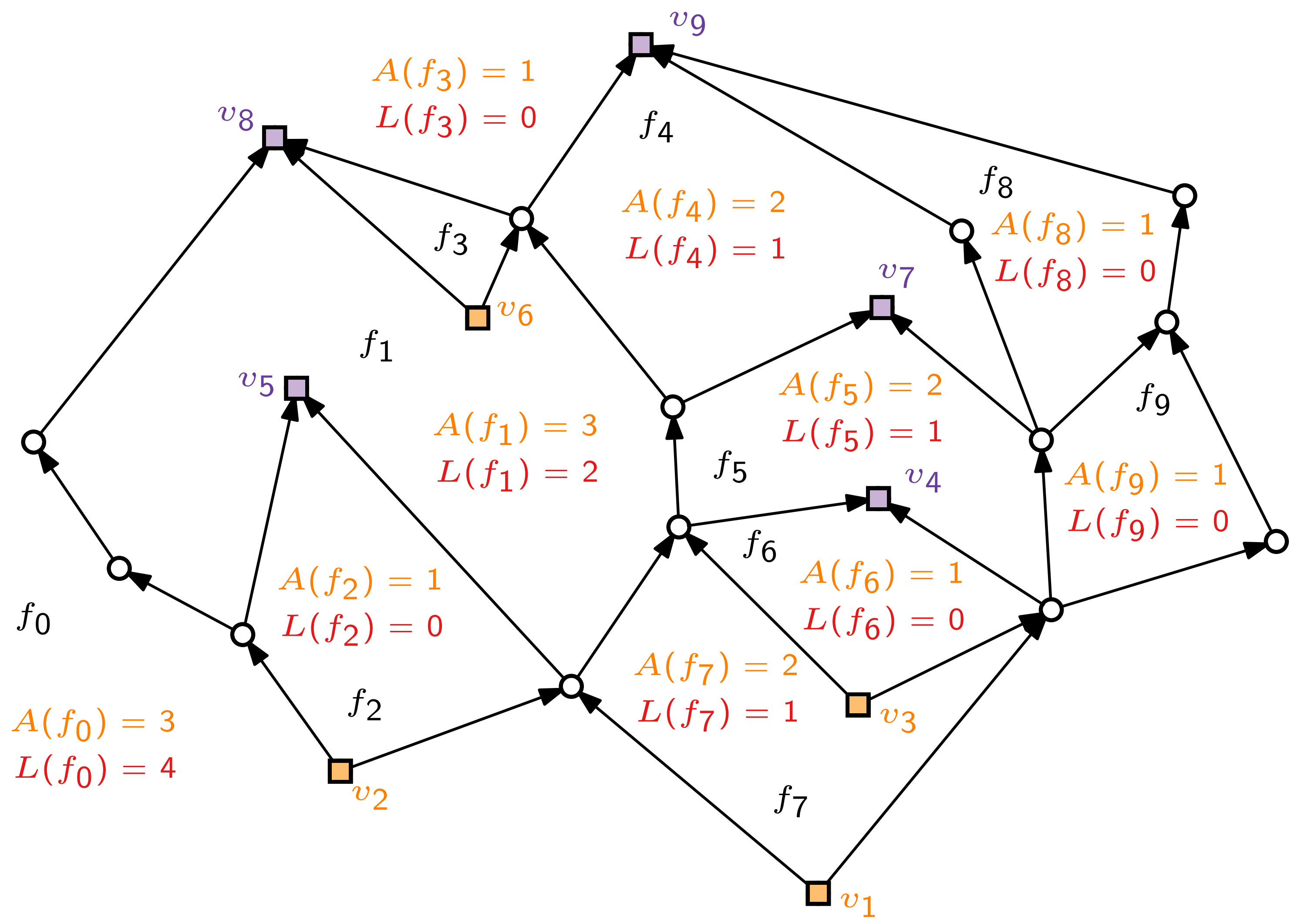

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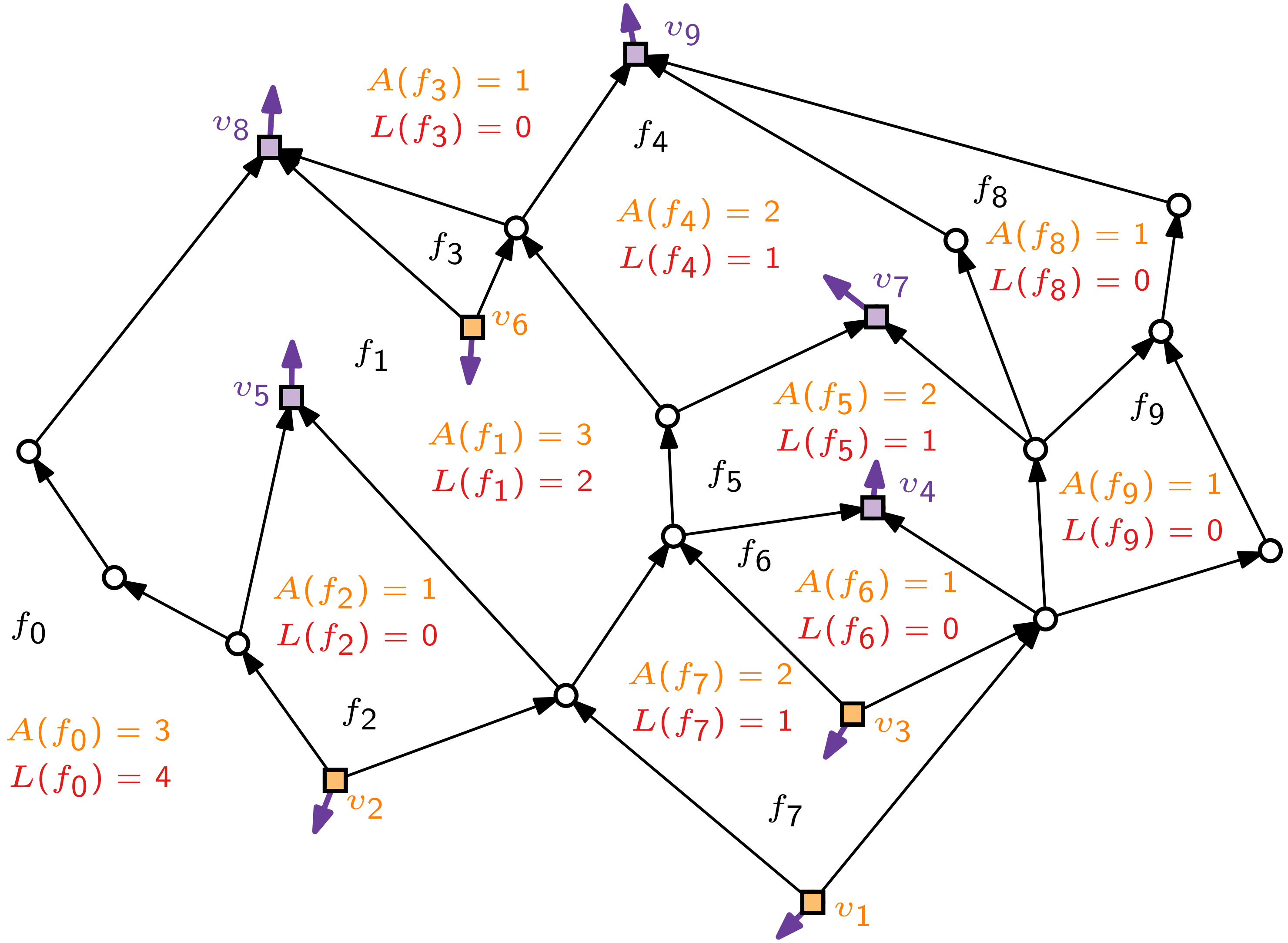

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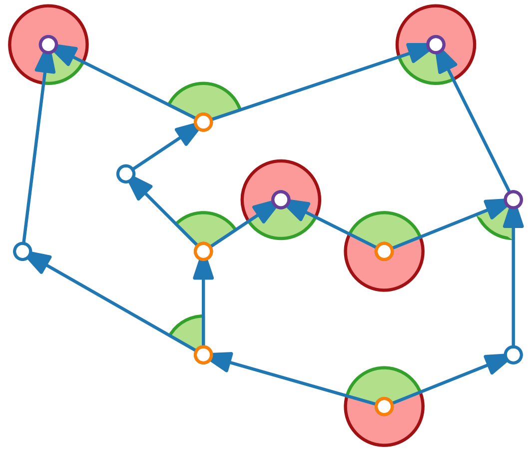
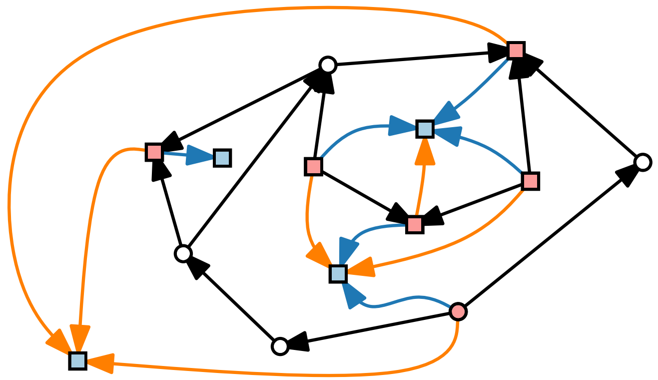
Visualization of Graphs

Lecture 6:
Upward Planar Drawings



Part IV:
Refinement Algorithm



Jonathan Klawitter

Result Characterization

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Let $G = (V, E)$ be an acyclic plane digraph with embedding given by F, f_0.

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$\Leftarrow$: Idea:

- Construct planar st-digraph that is supergraph of G.
- Apply equivalence from Theorem 1.

Refinement Algorithm – $\Phi, F, f_0 \rightarrow$ st-digraph

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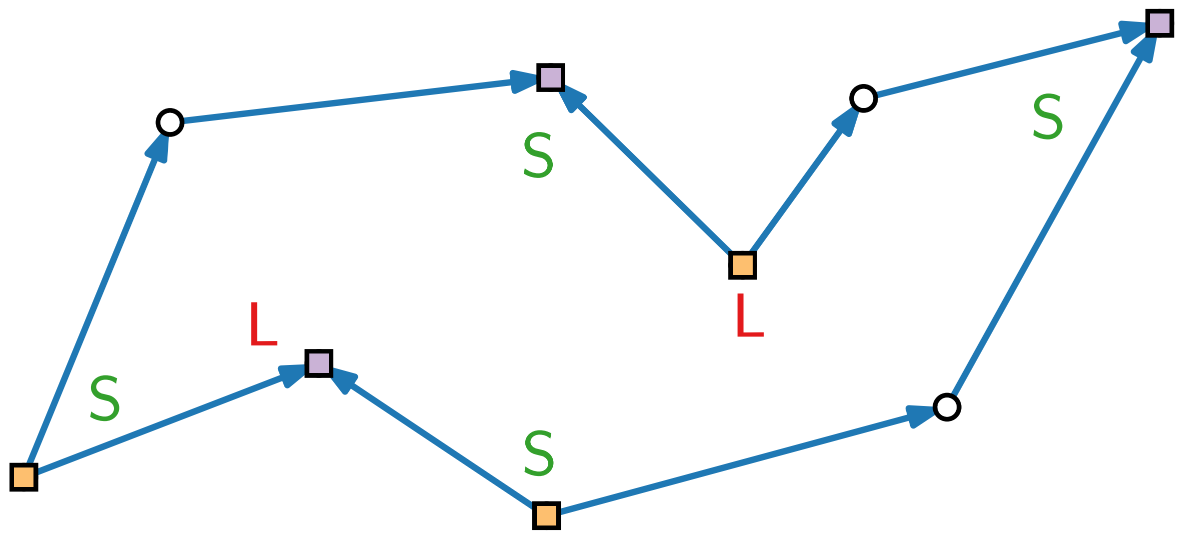

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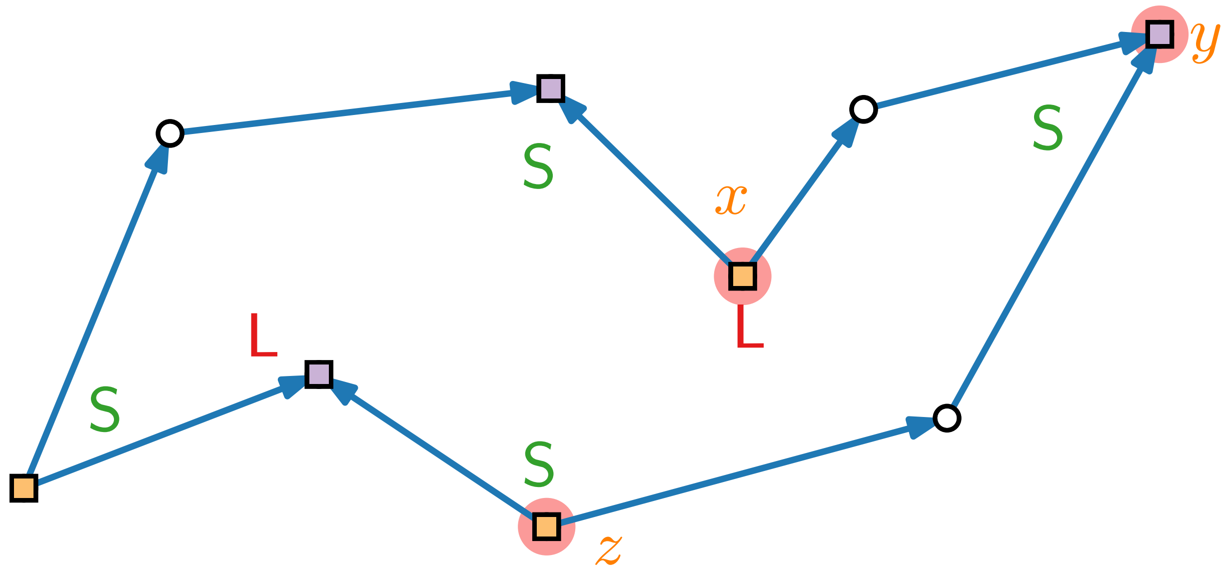

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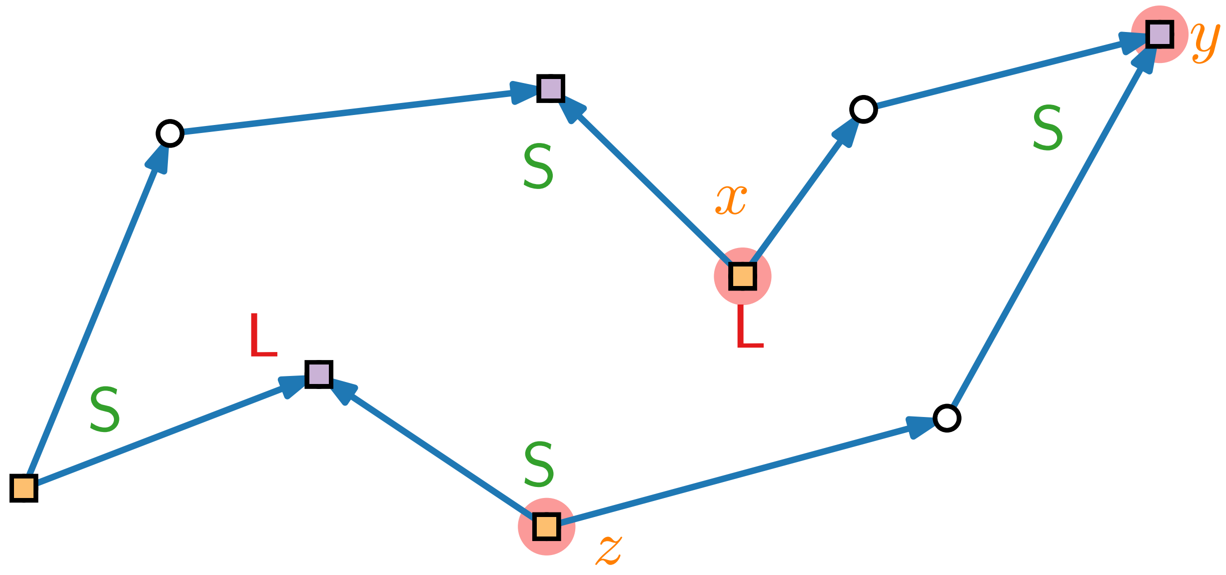

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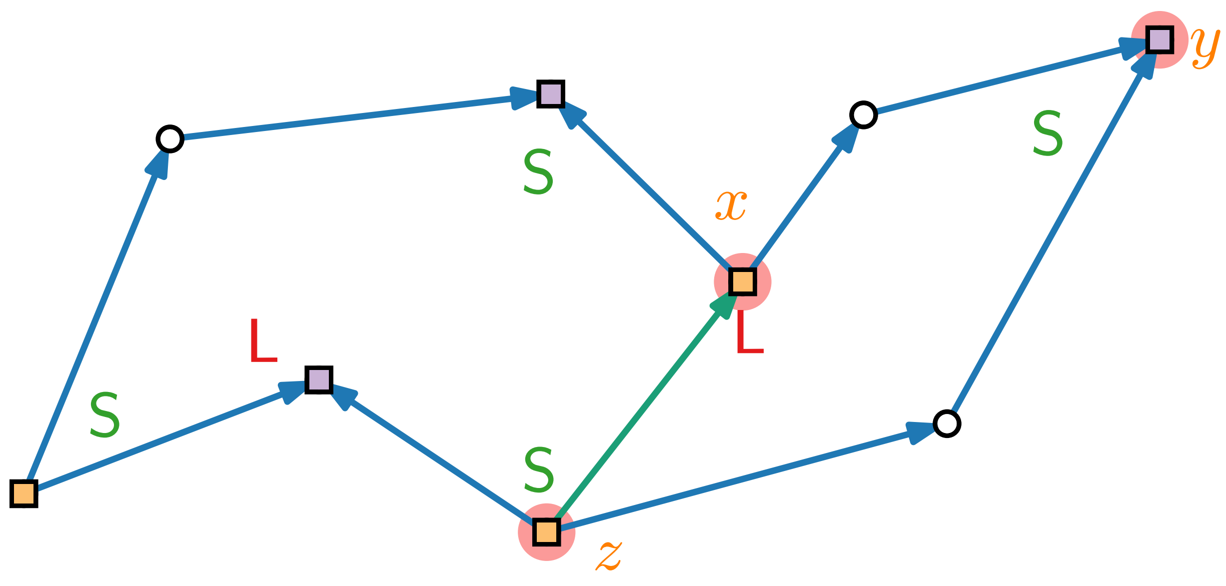

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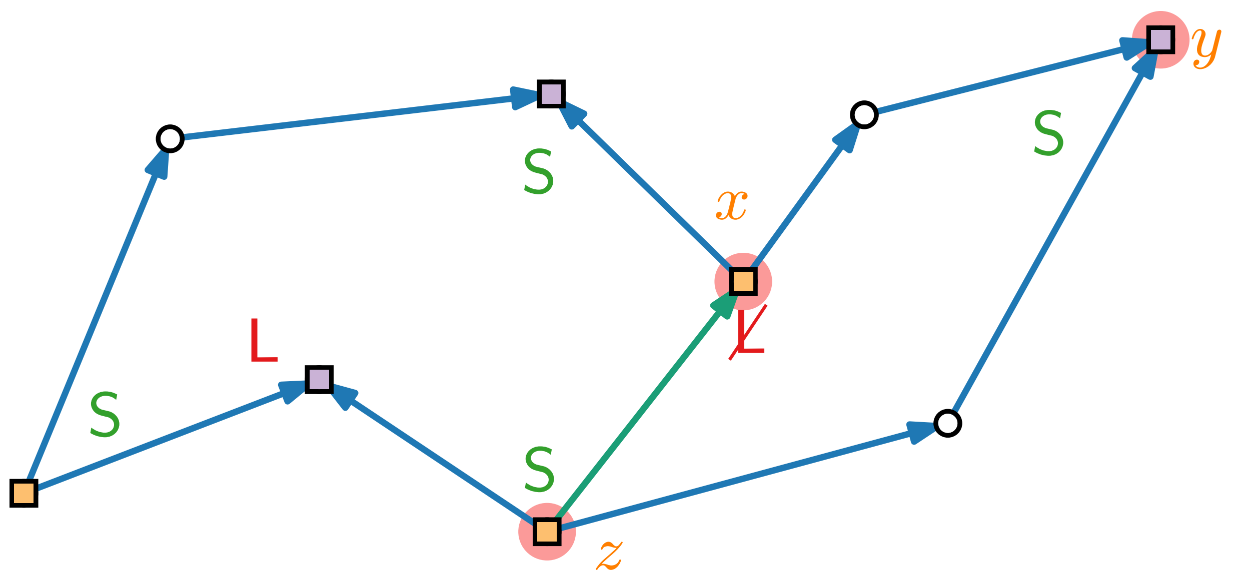

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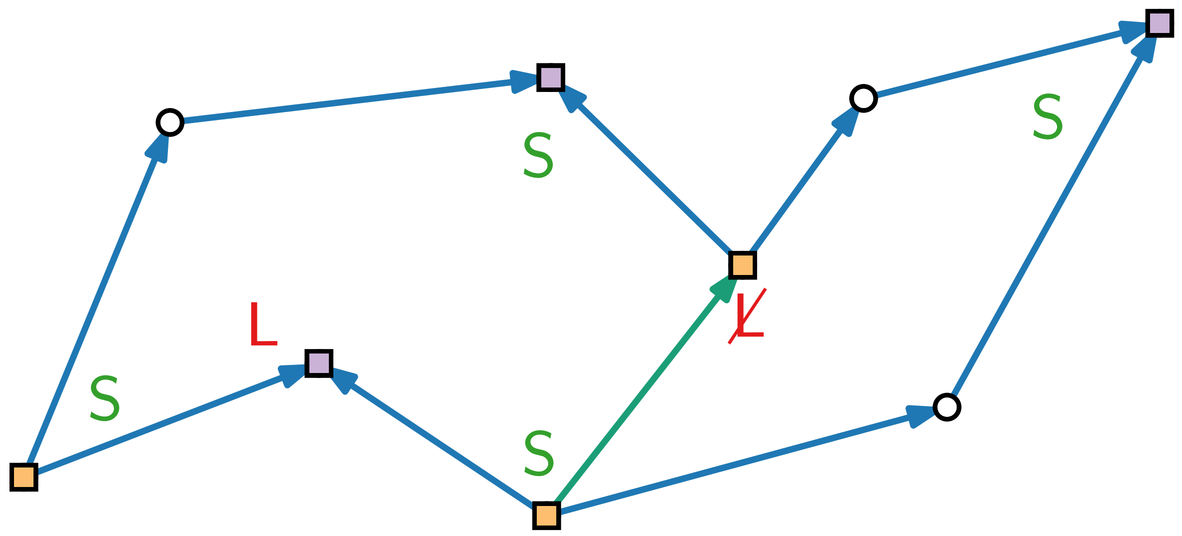

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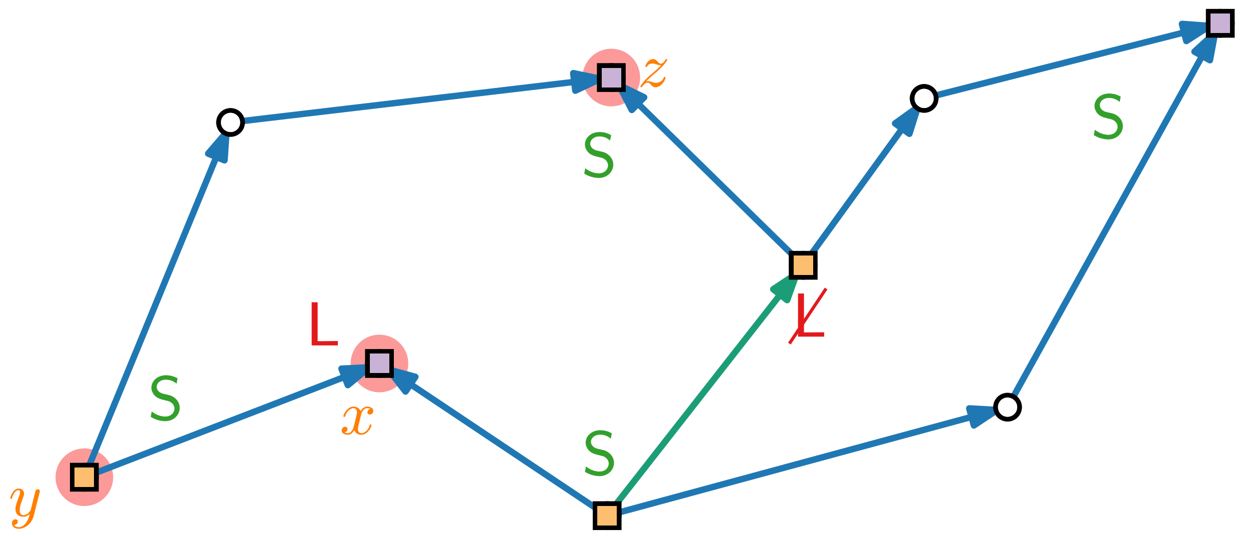

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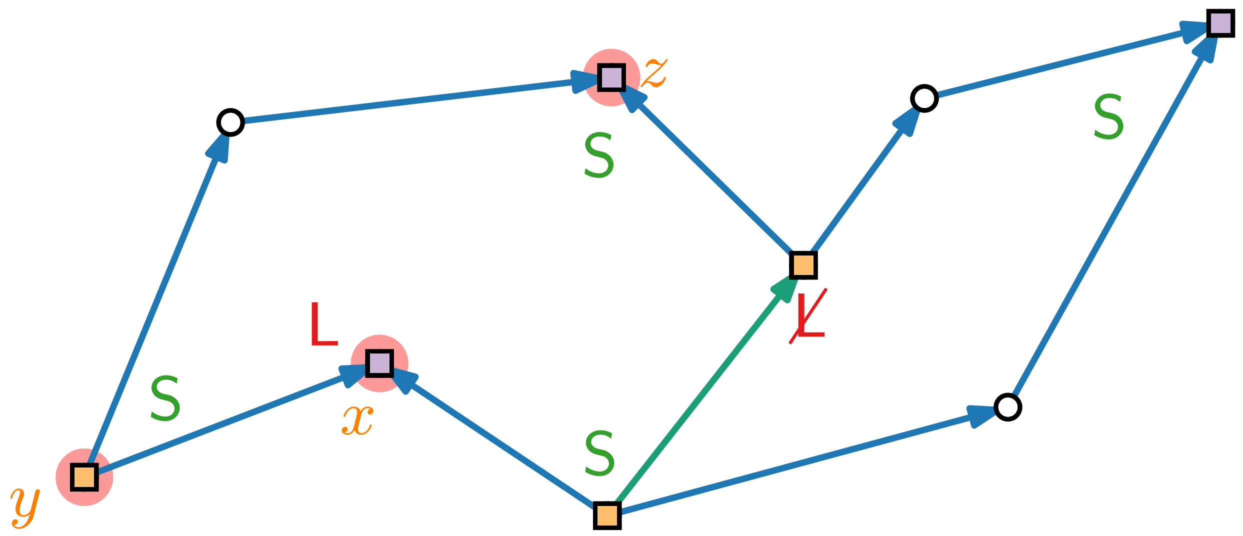

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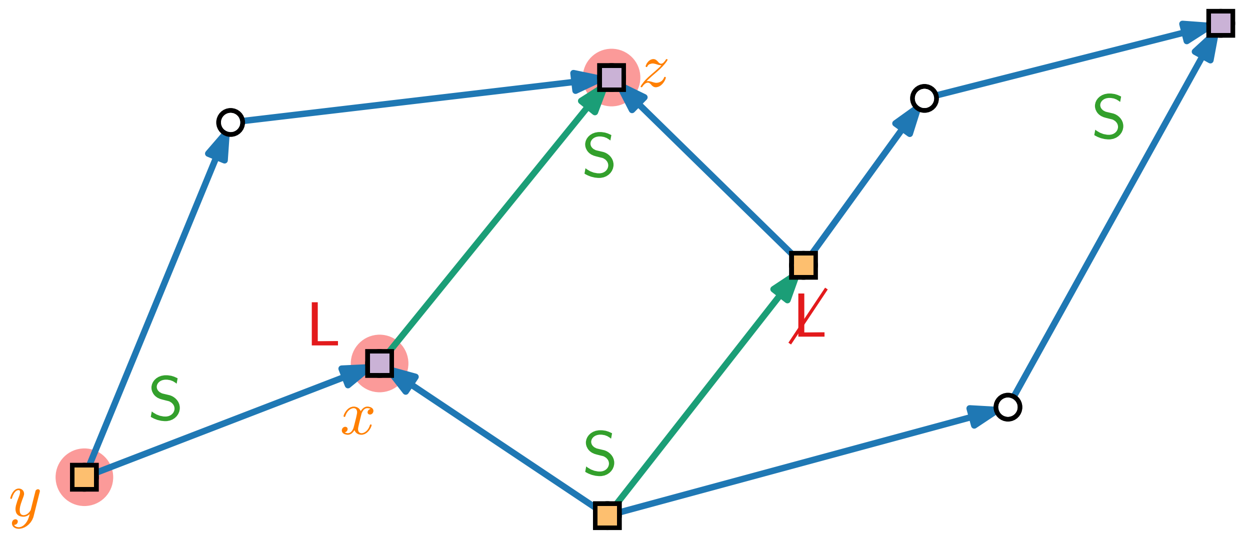

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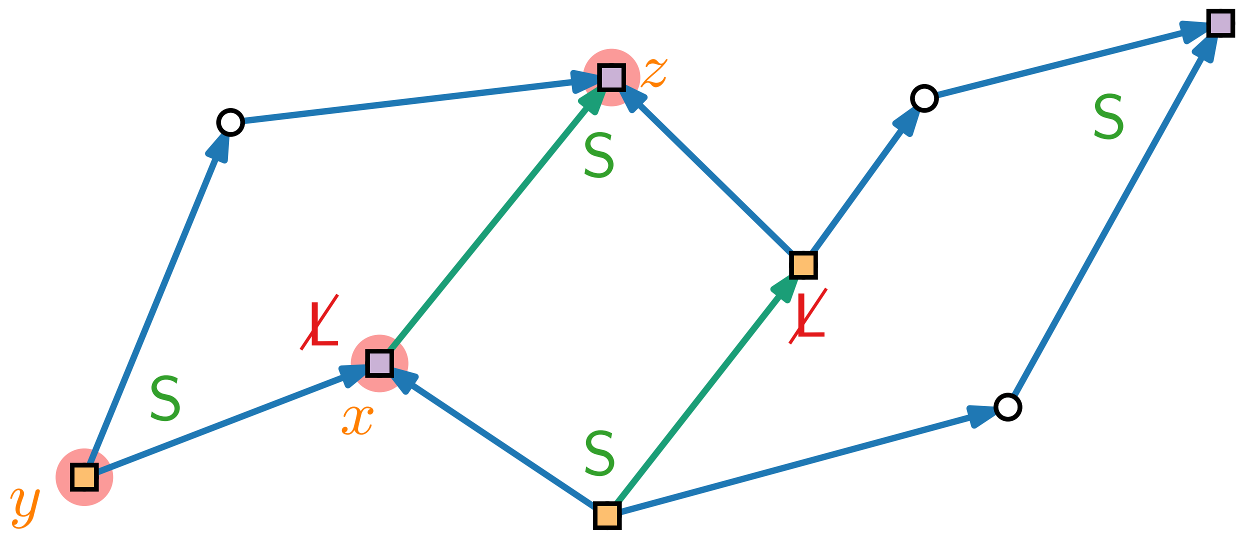

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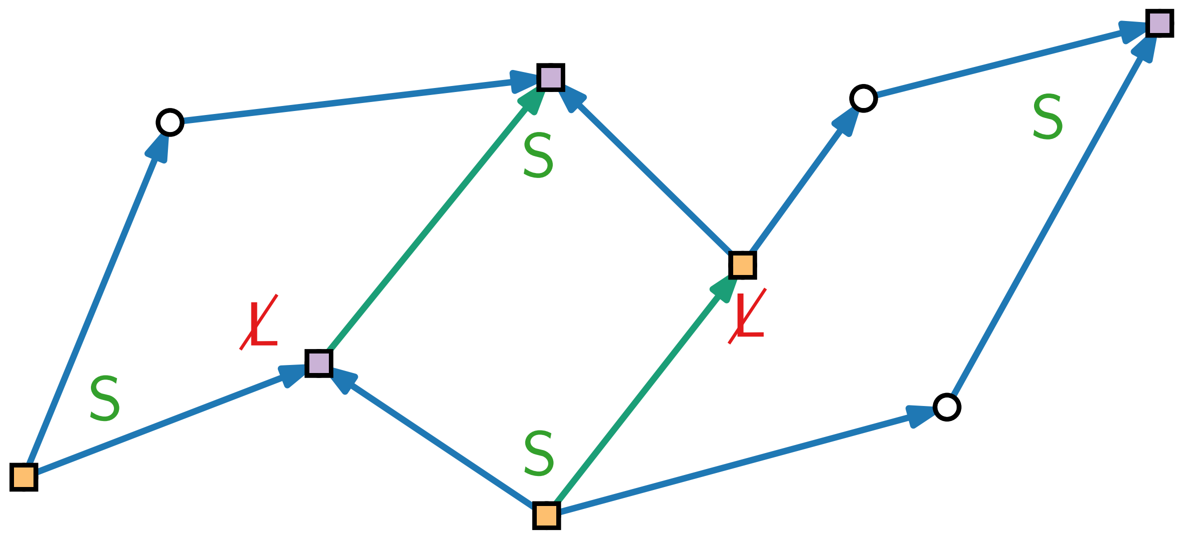

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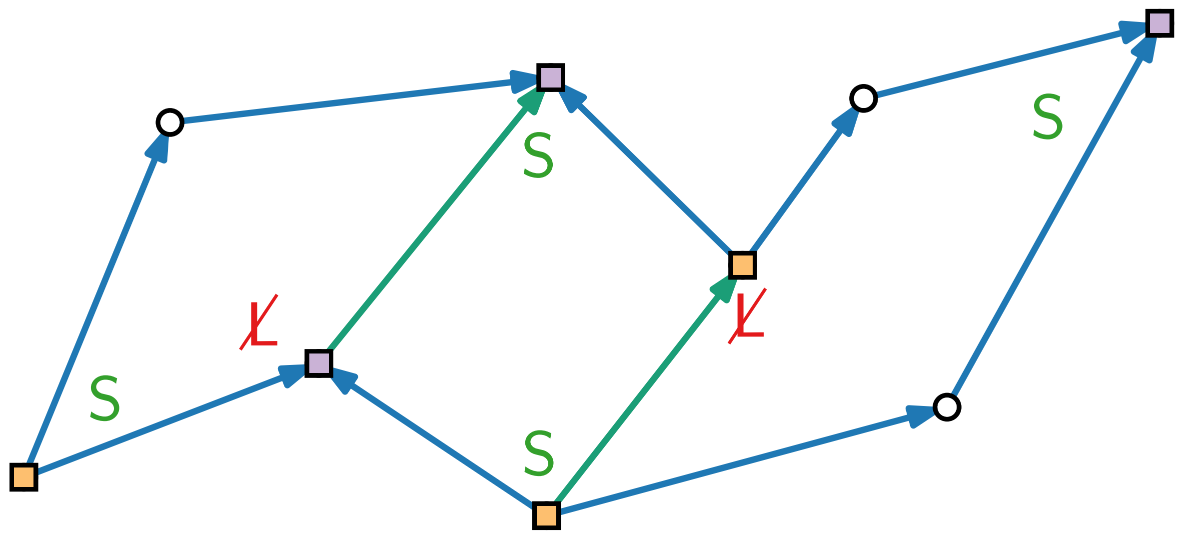

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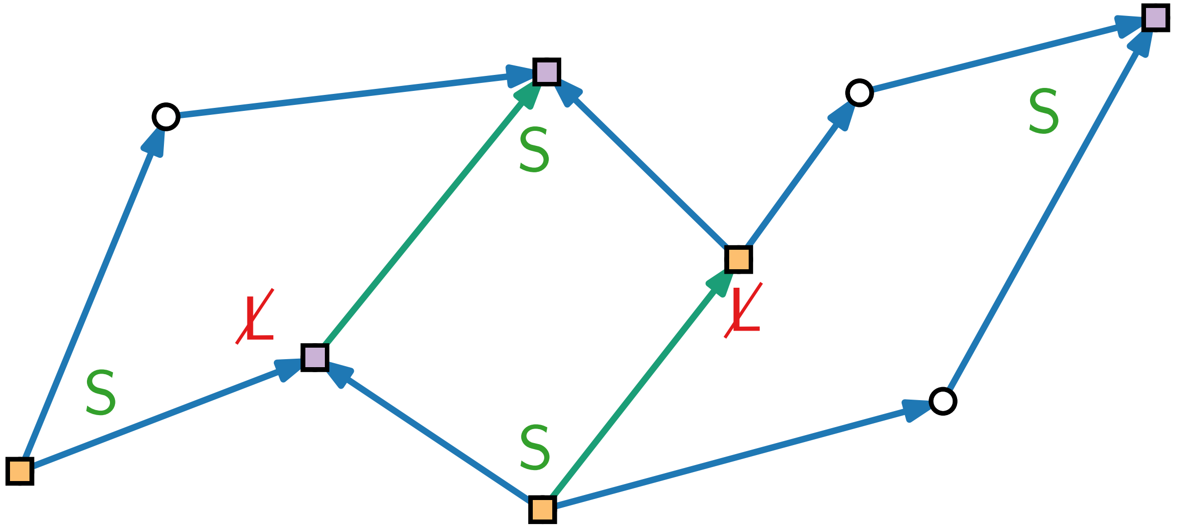

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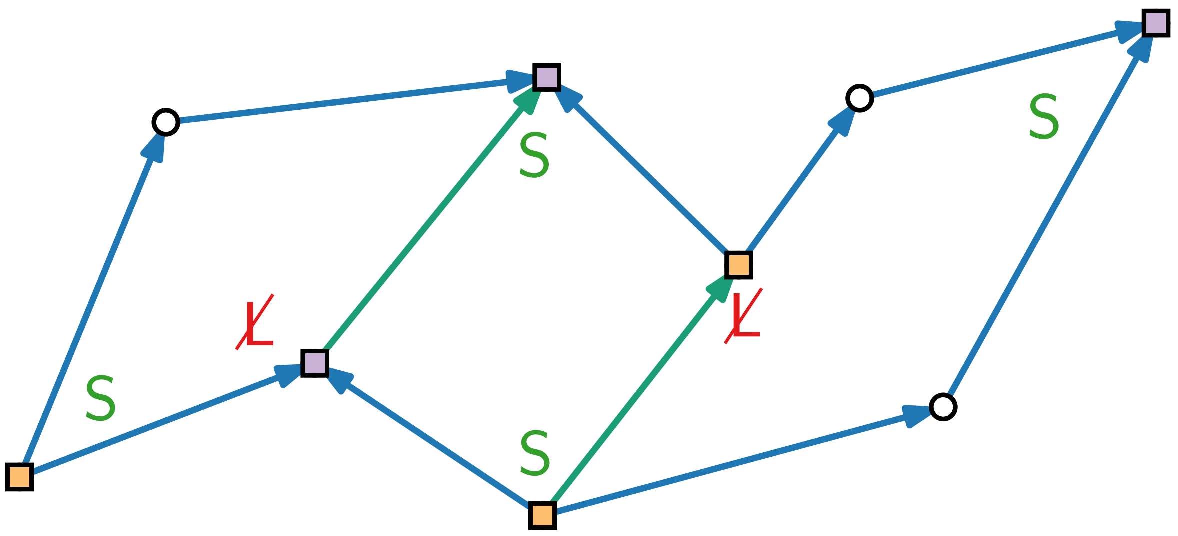

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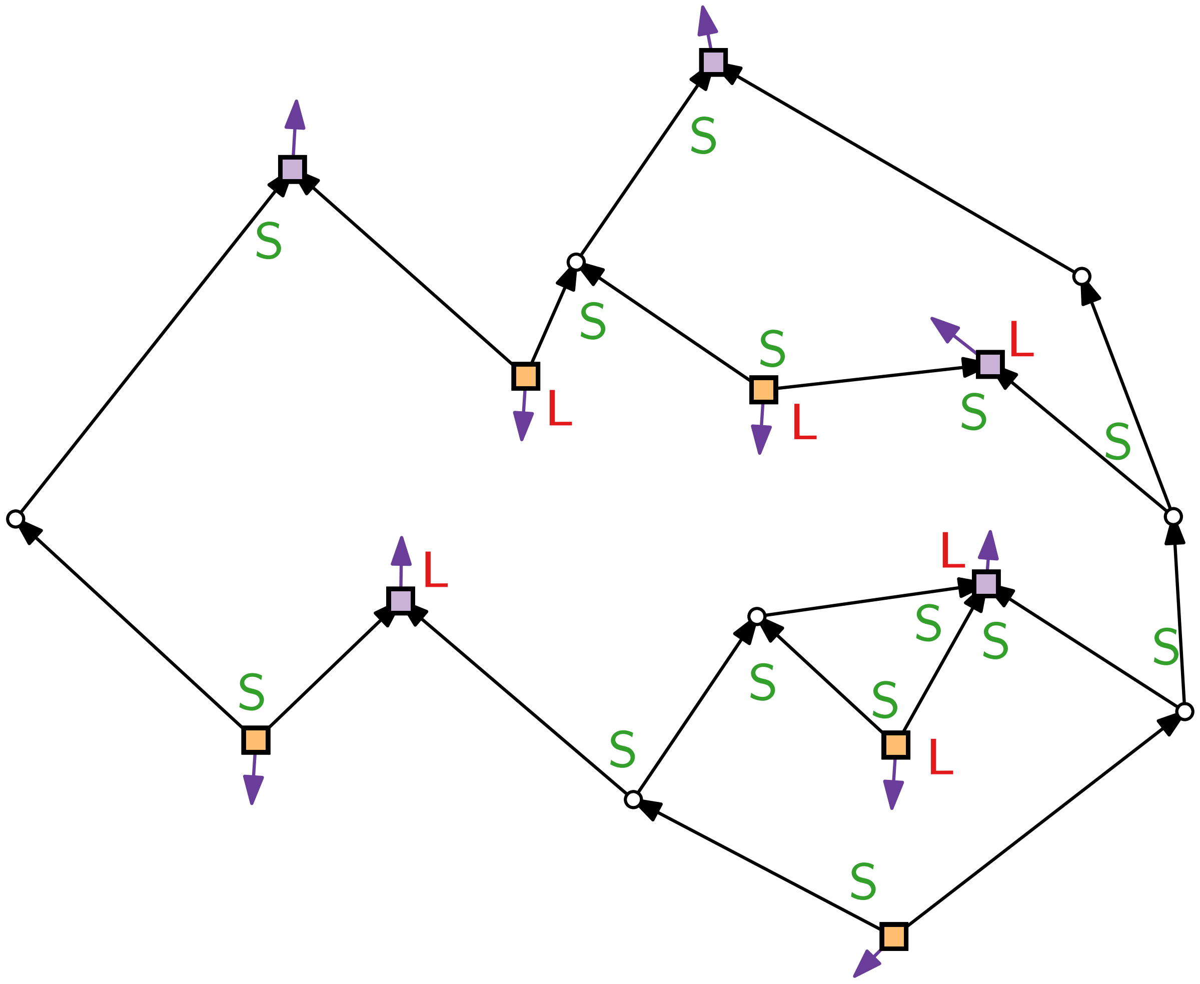
- Planarity, acyclicity, bimodality are invariants under construction.

Refinement Example



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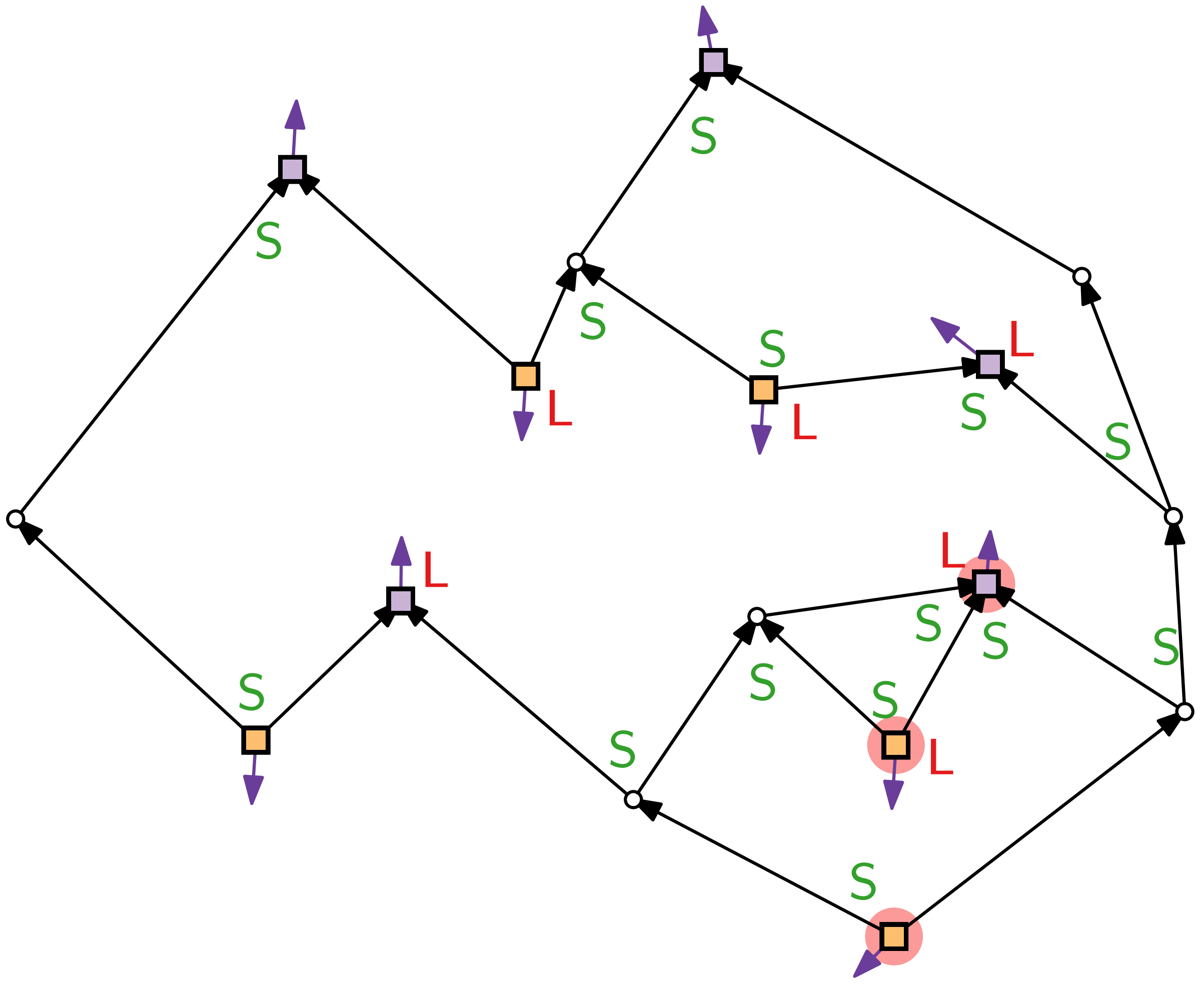

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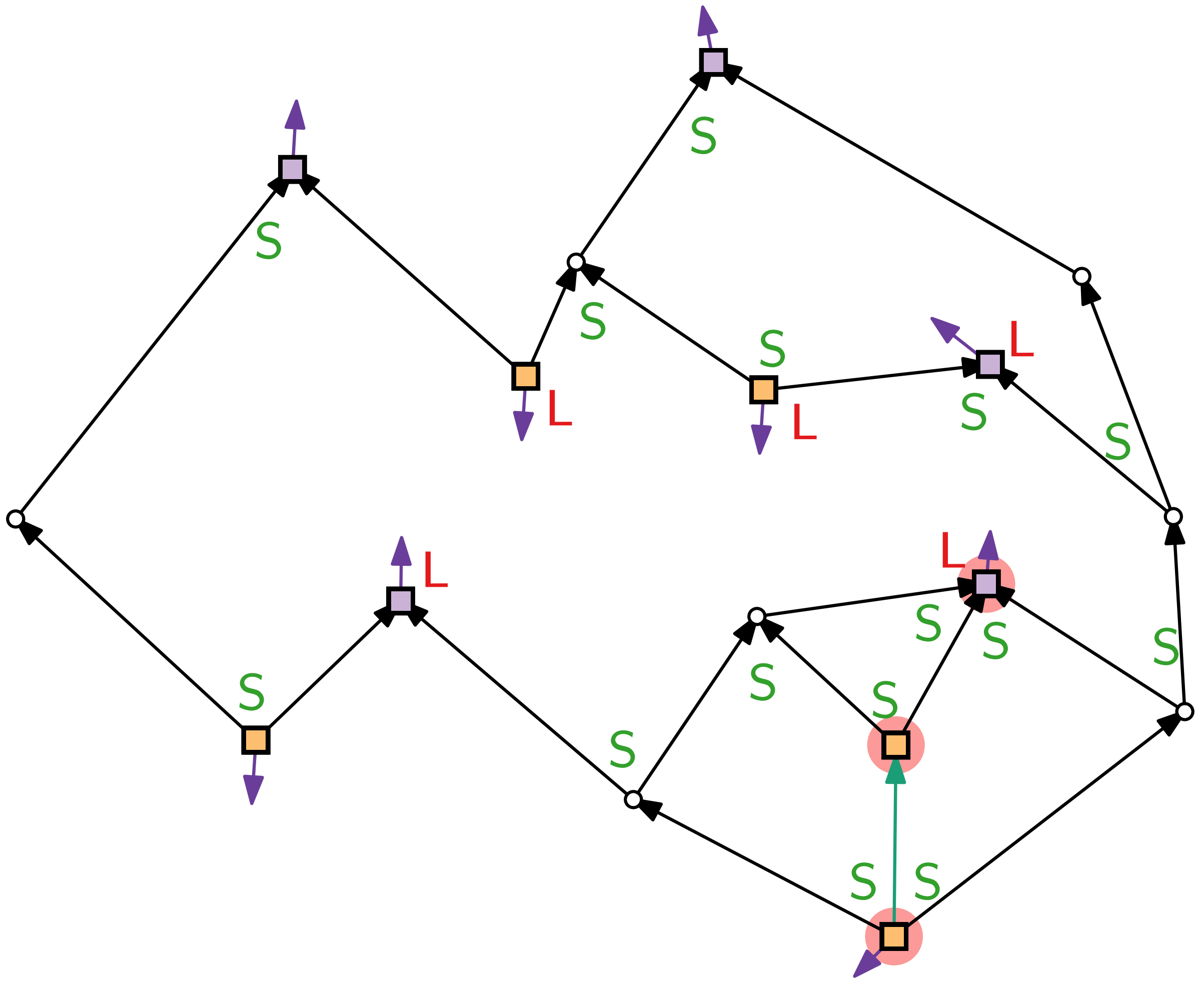

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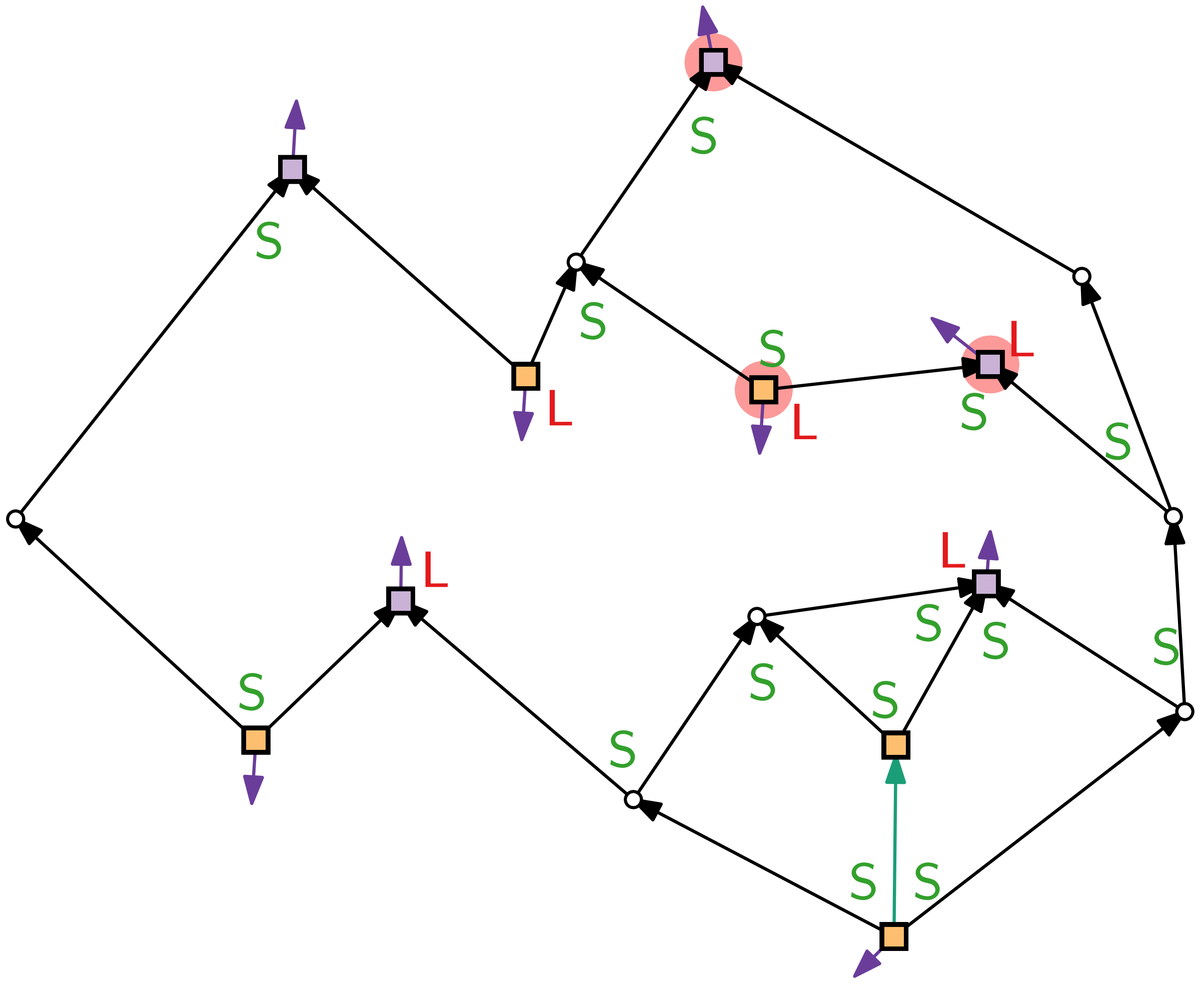

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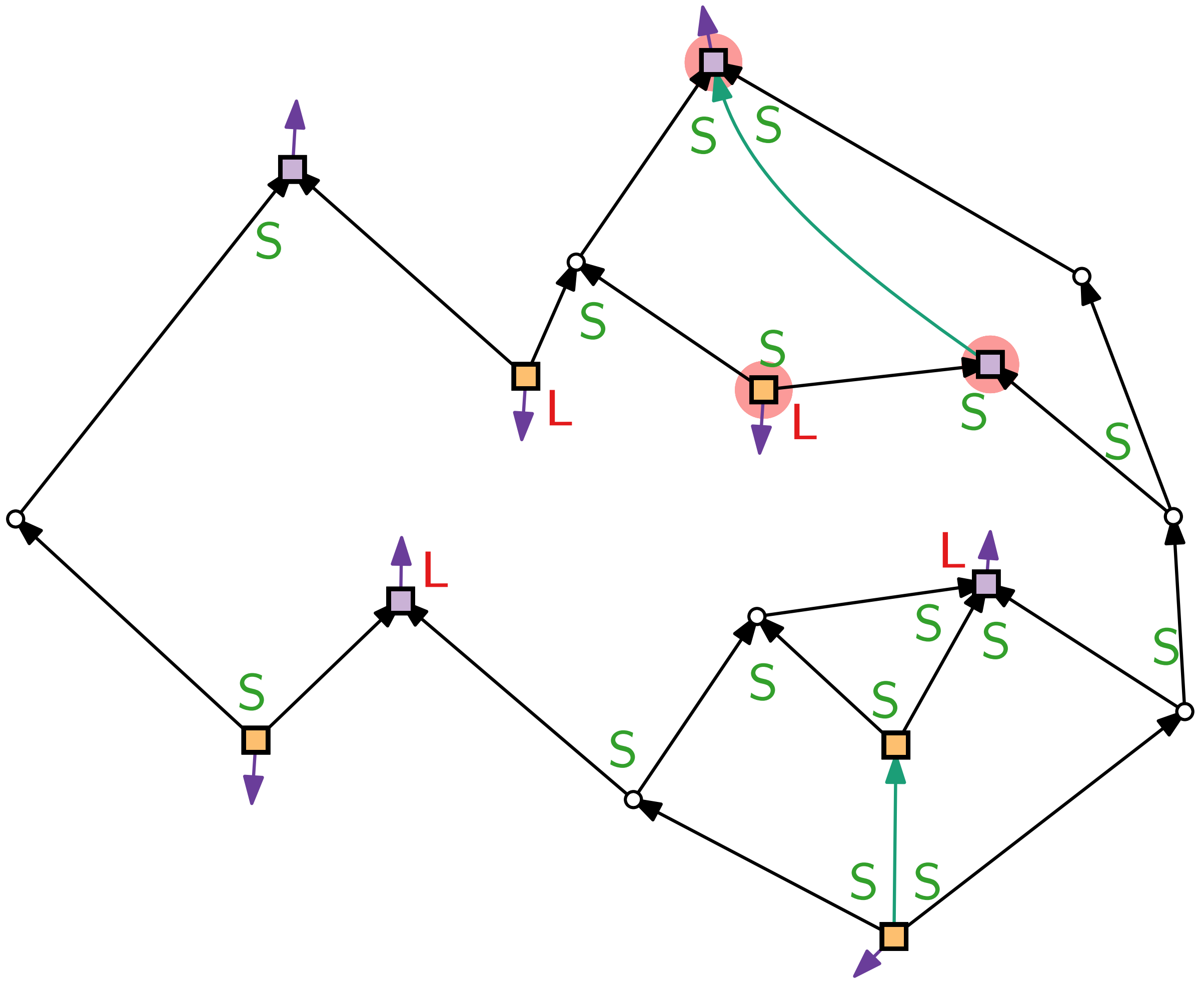

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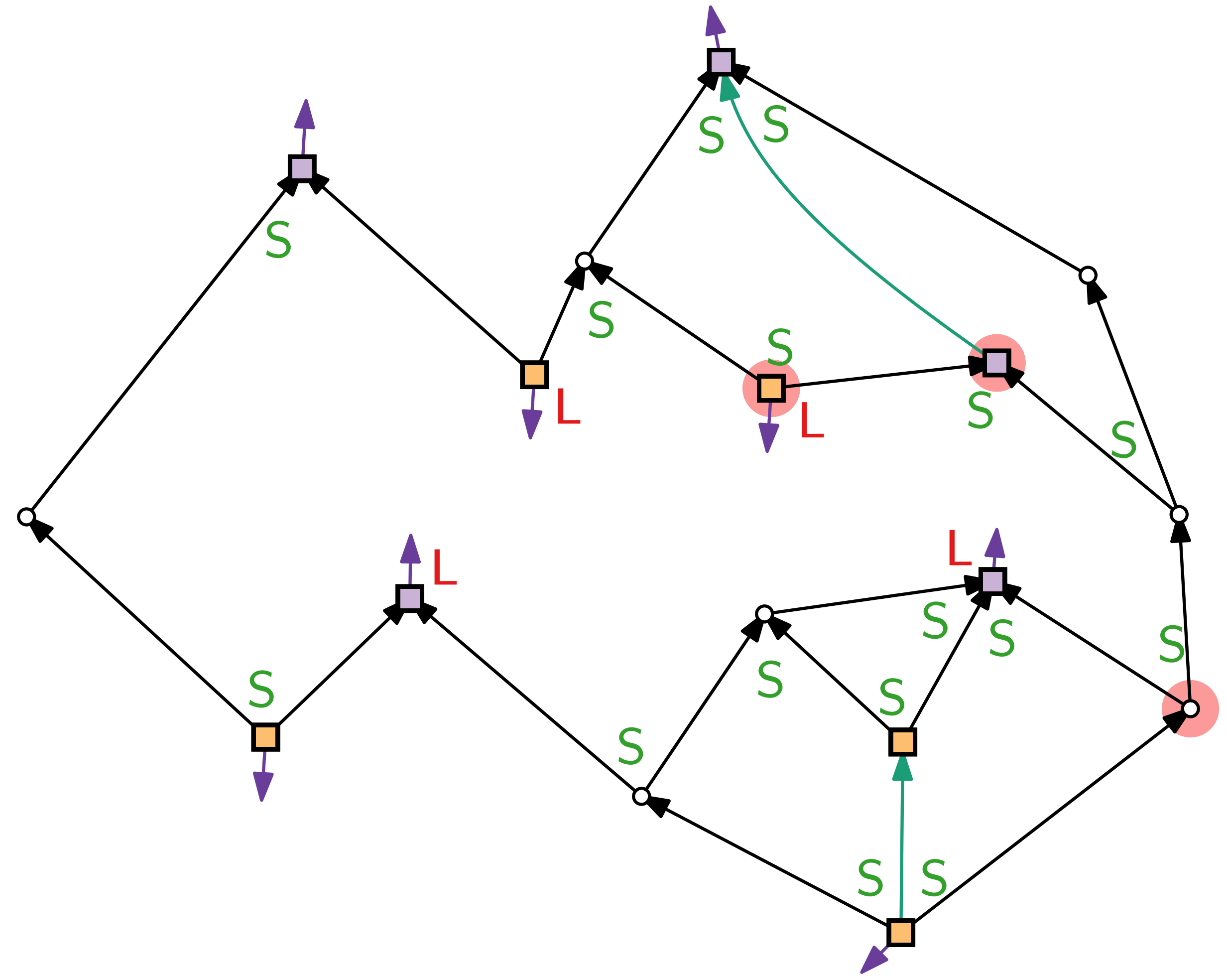

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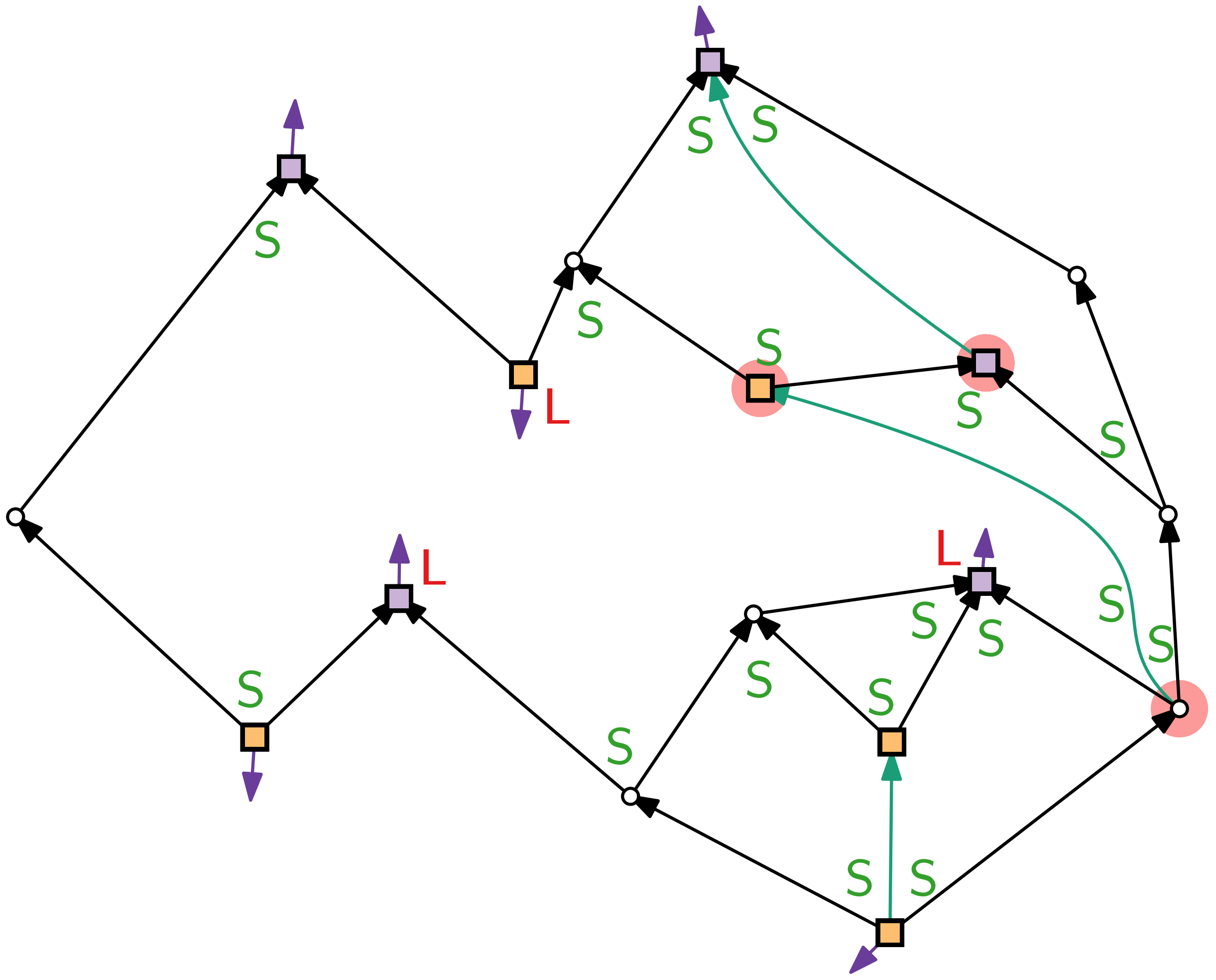

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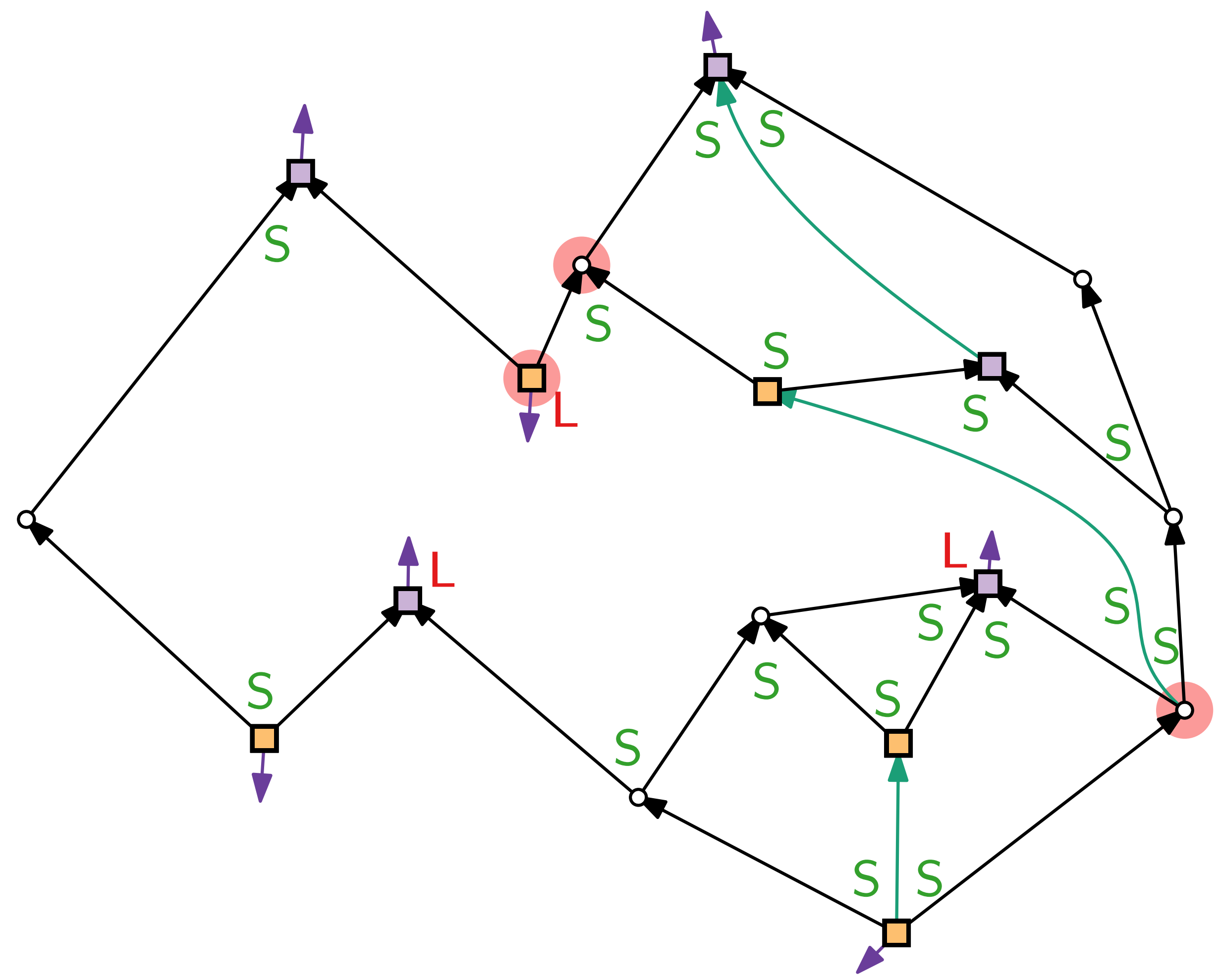

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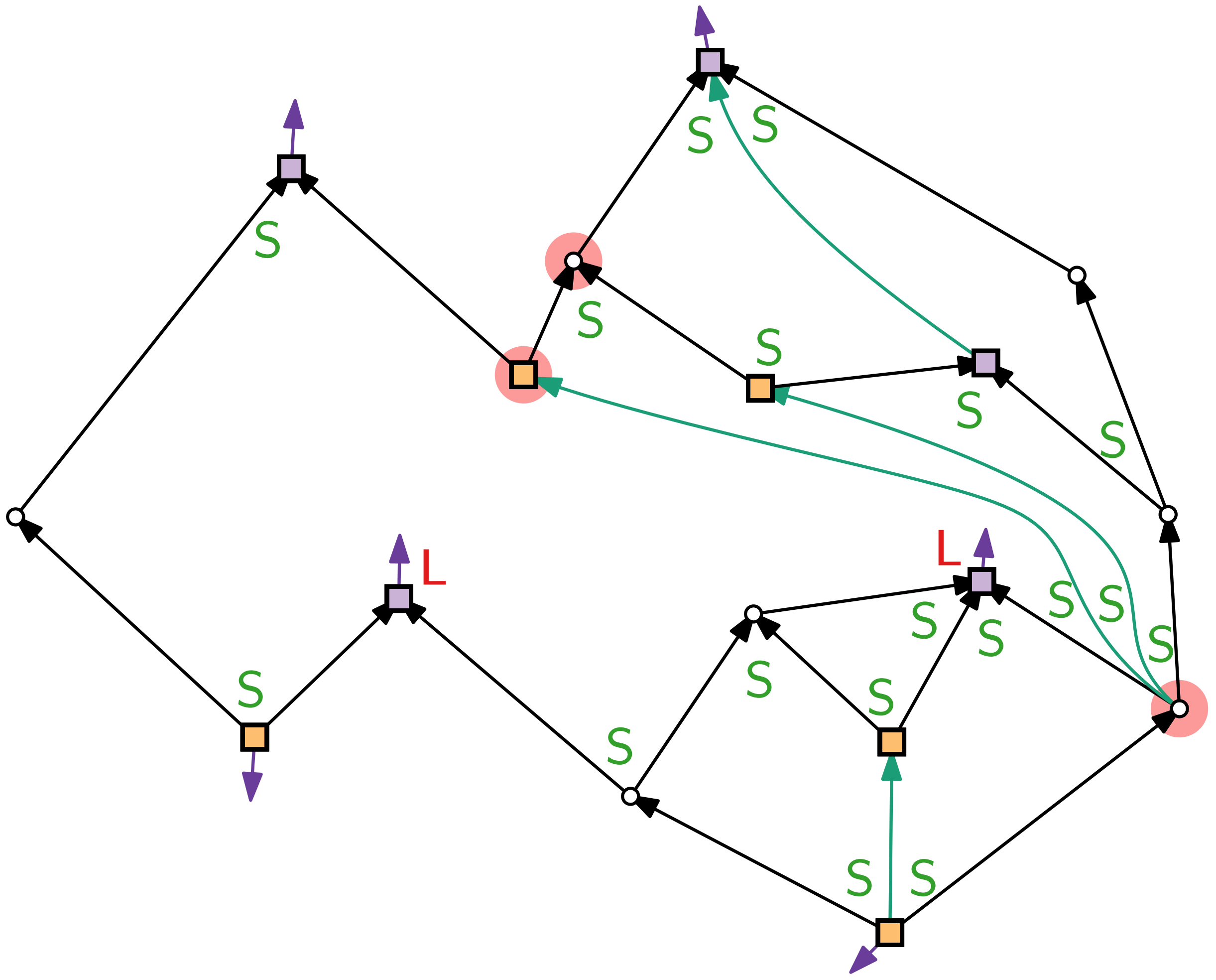

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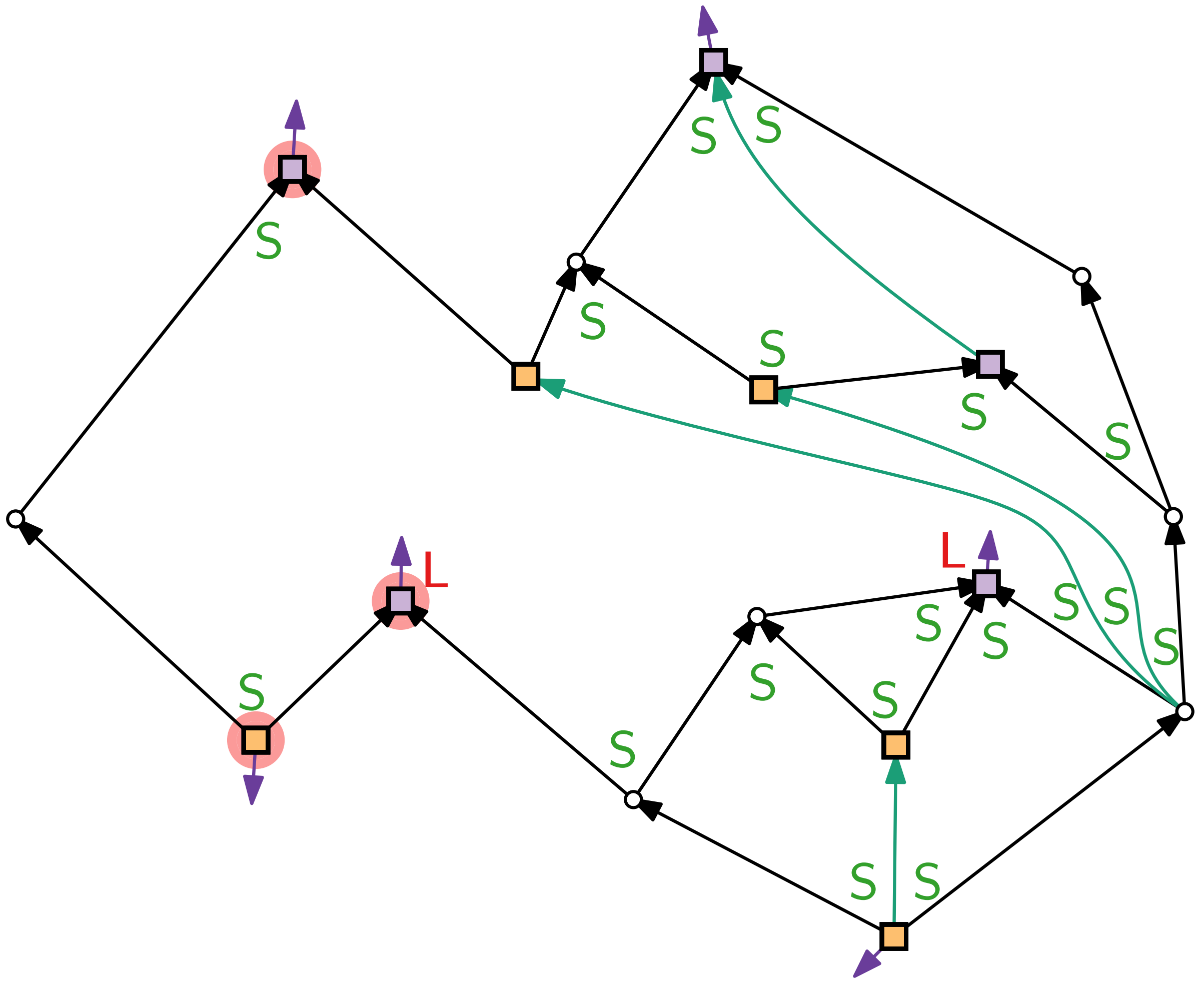

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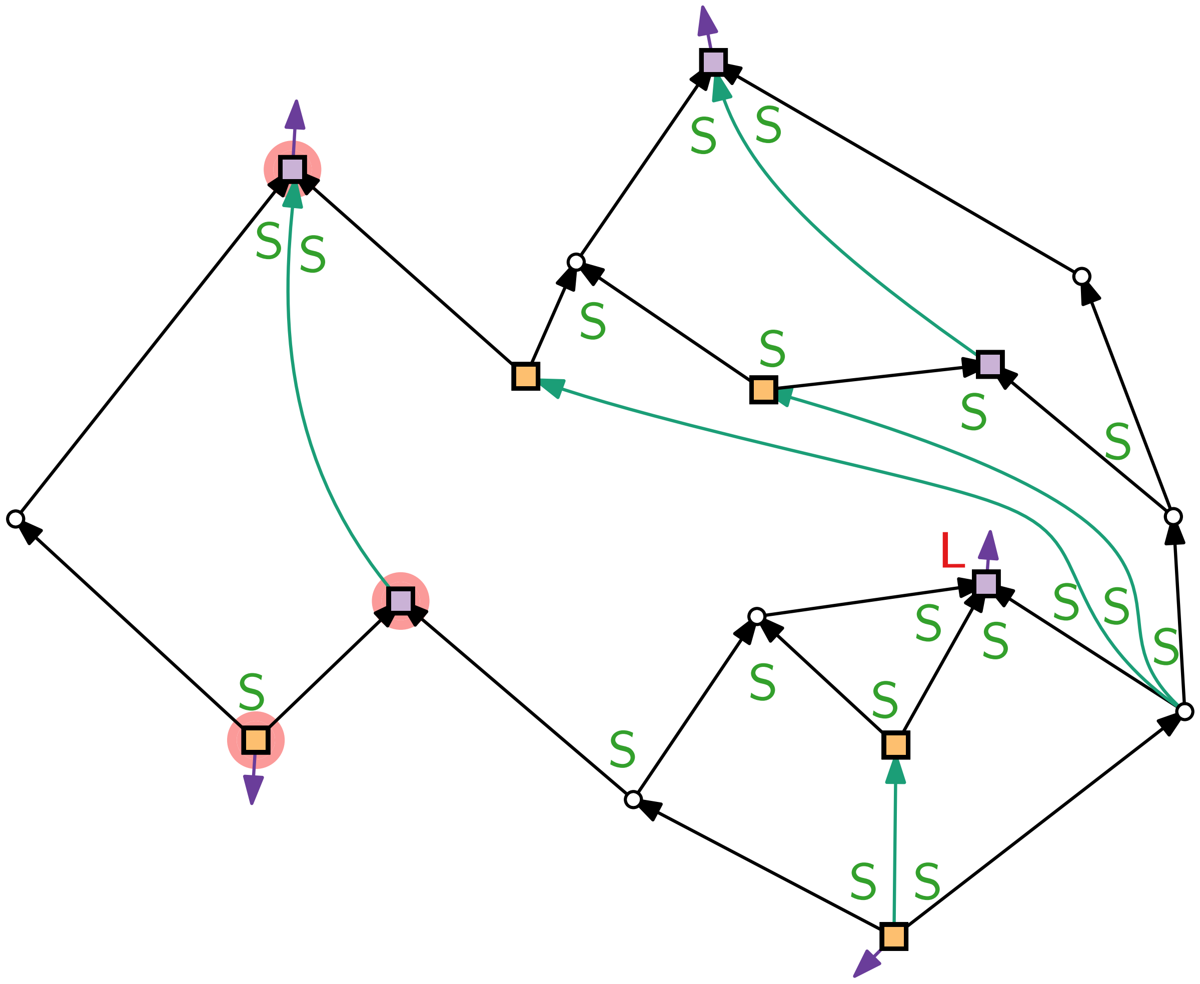

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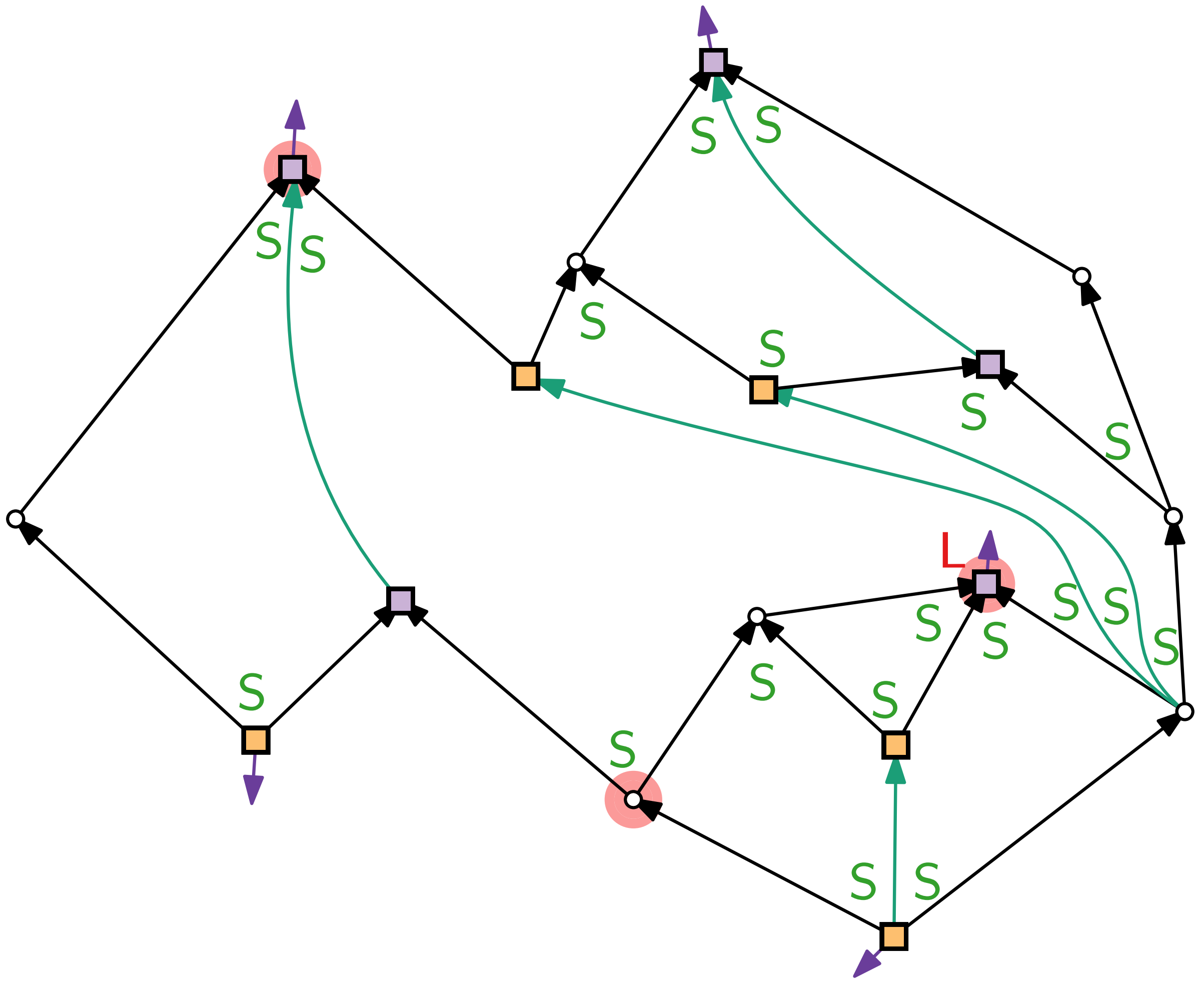

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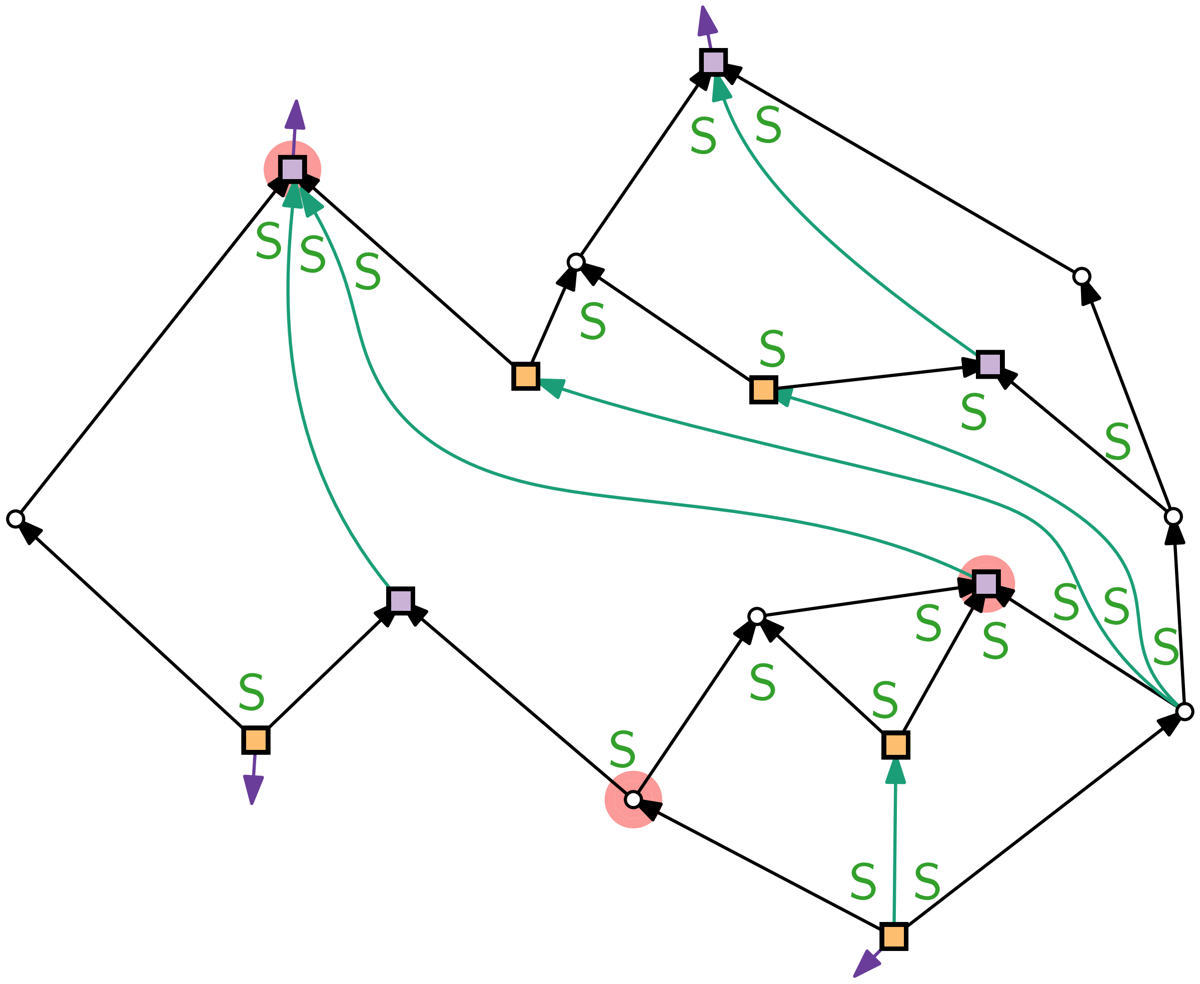

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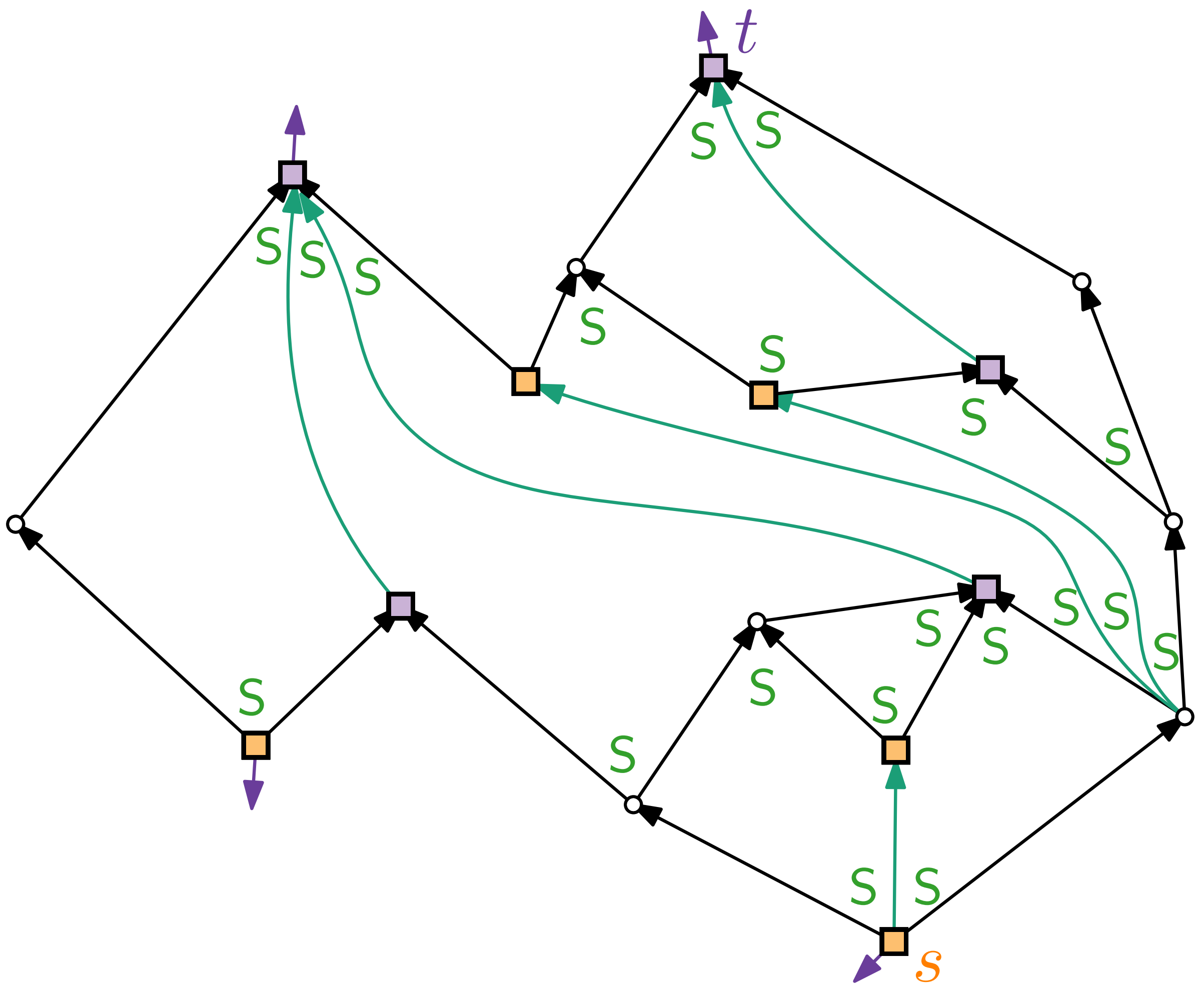

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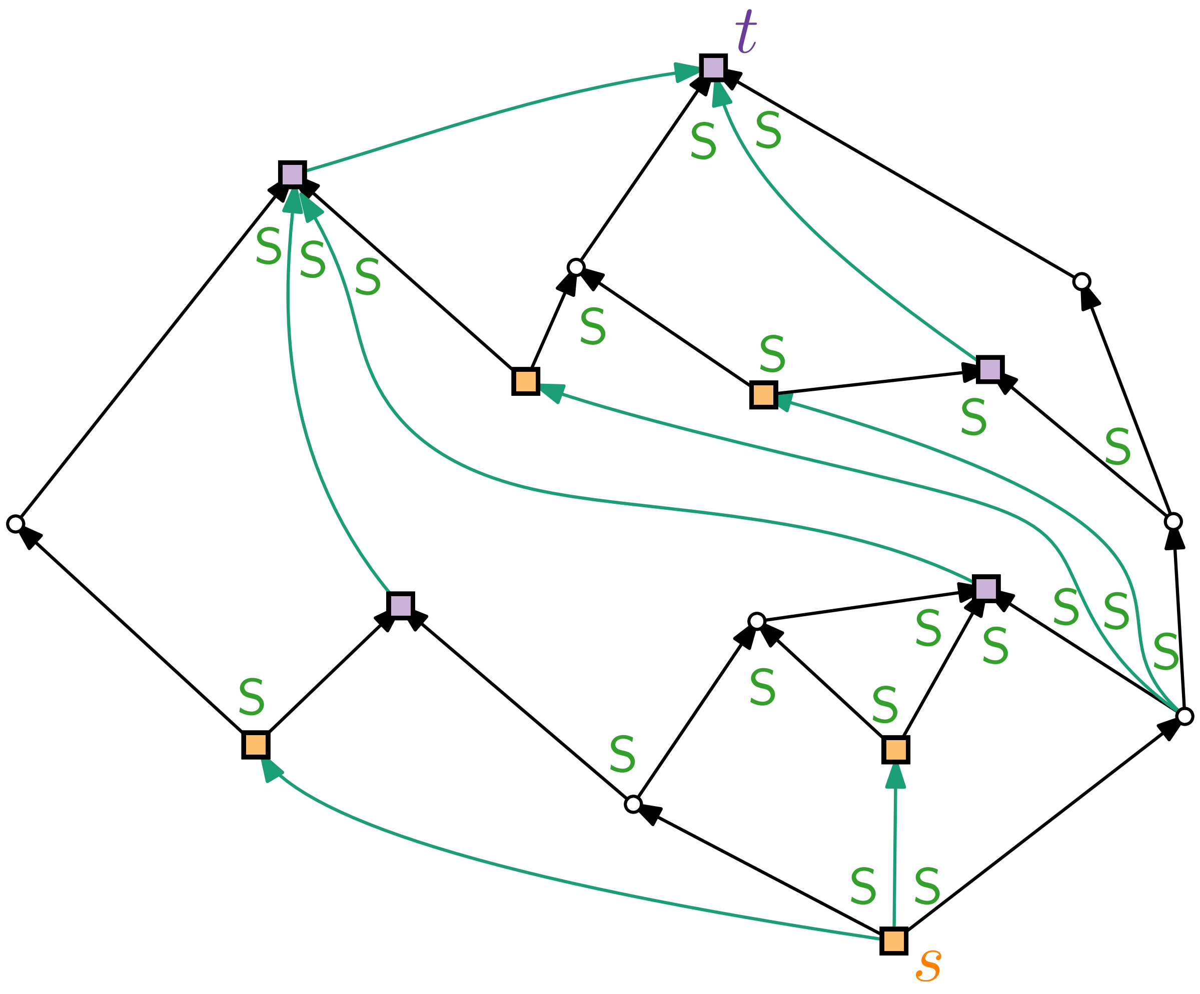

Result Upward Planarity Test

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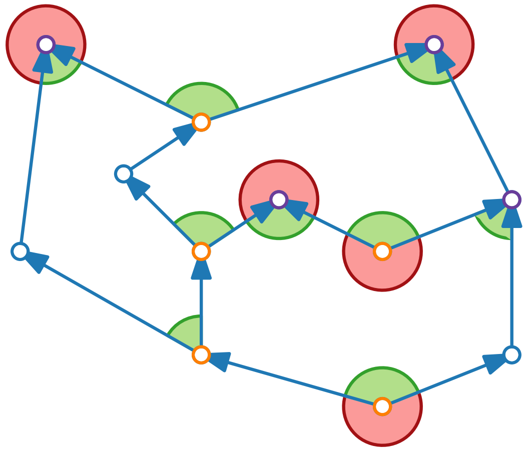
- Deleted edges added in refinement step.

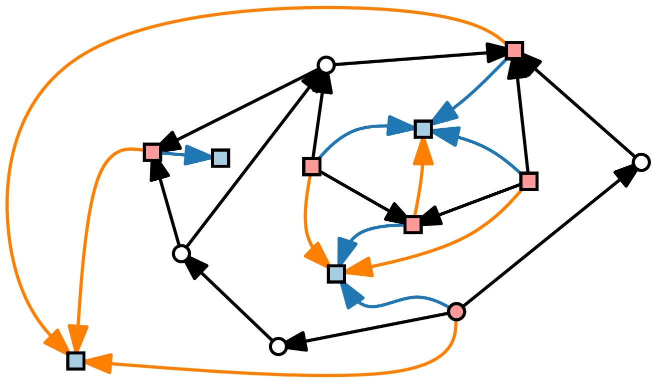
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Visualization of Graphs

Lecture 6:
Upward Planar Drawings



Part V:
Finding a Consistent Assignment



Jonathan Klawitter

Finding a Consistent Assignment

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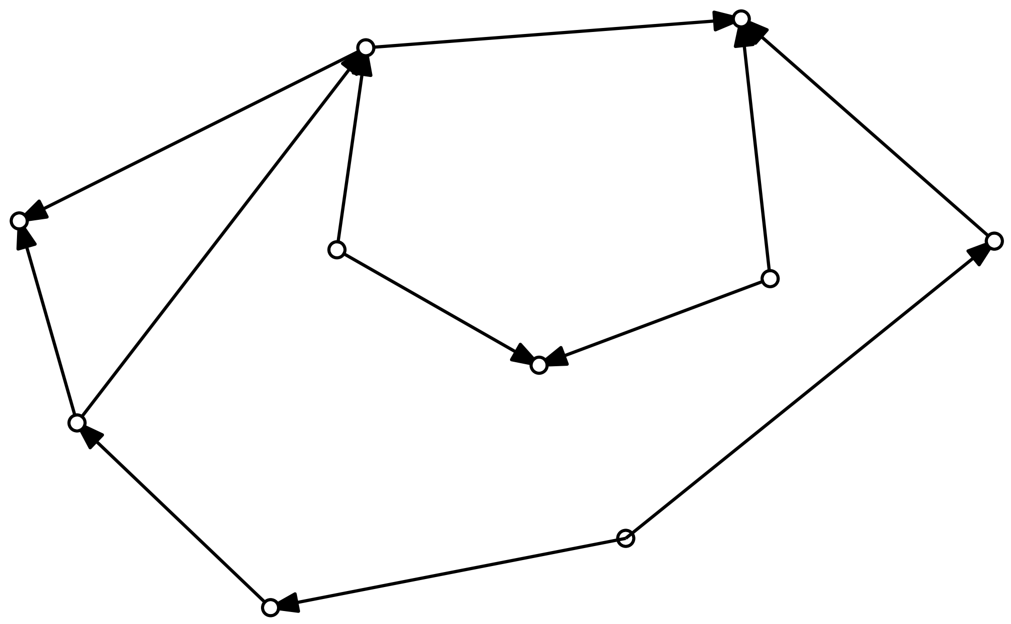

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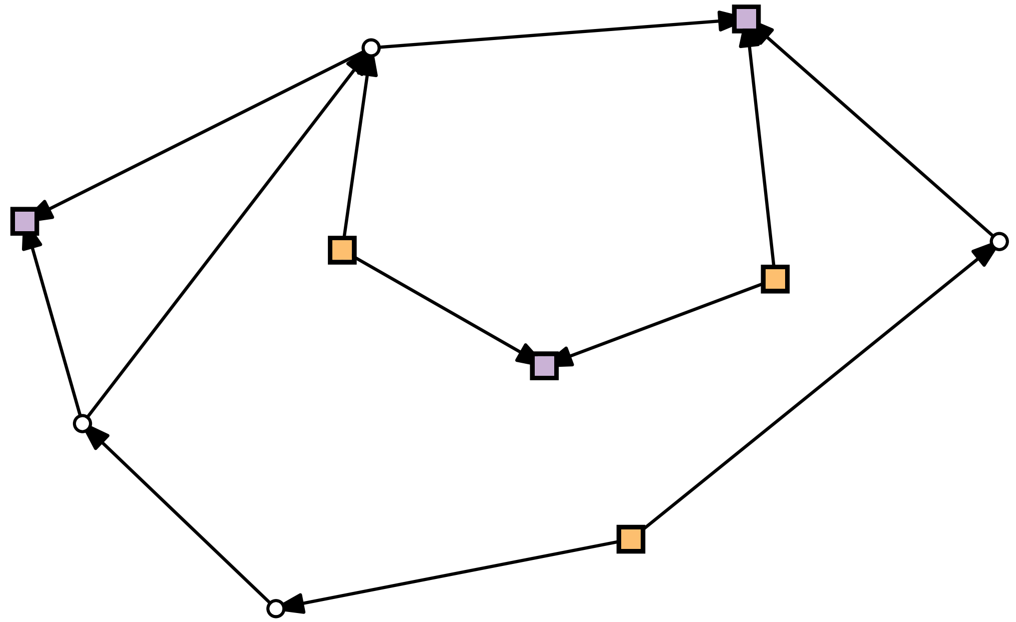

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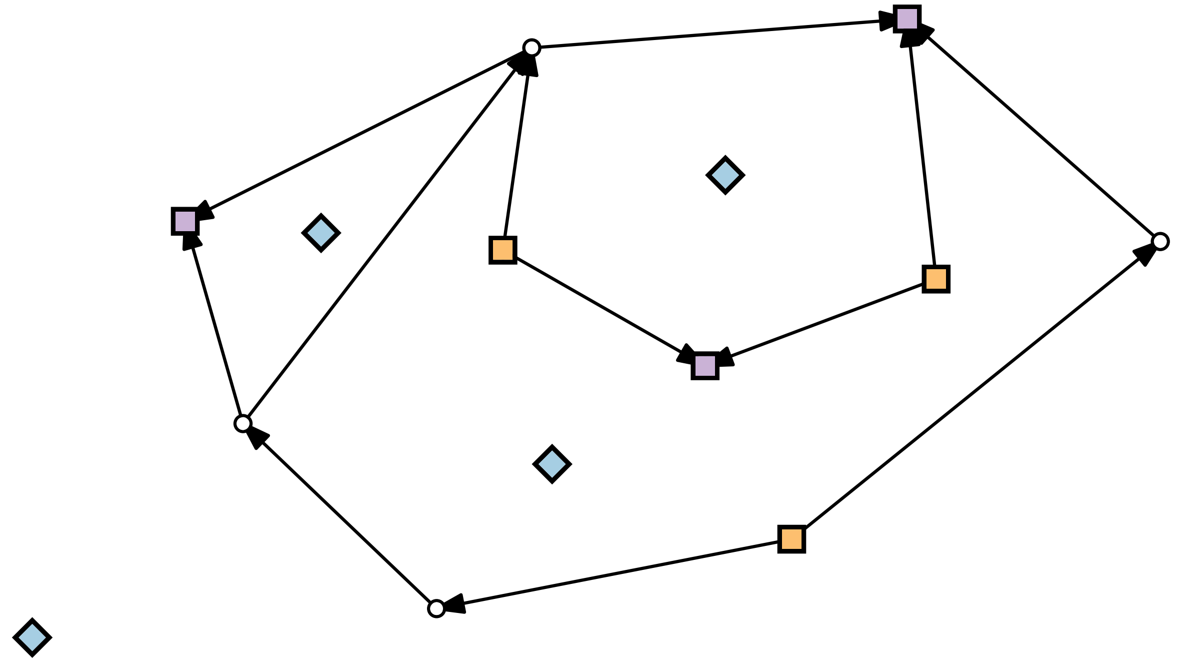

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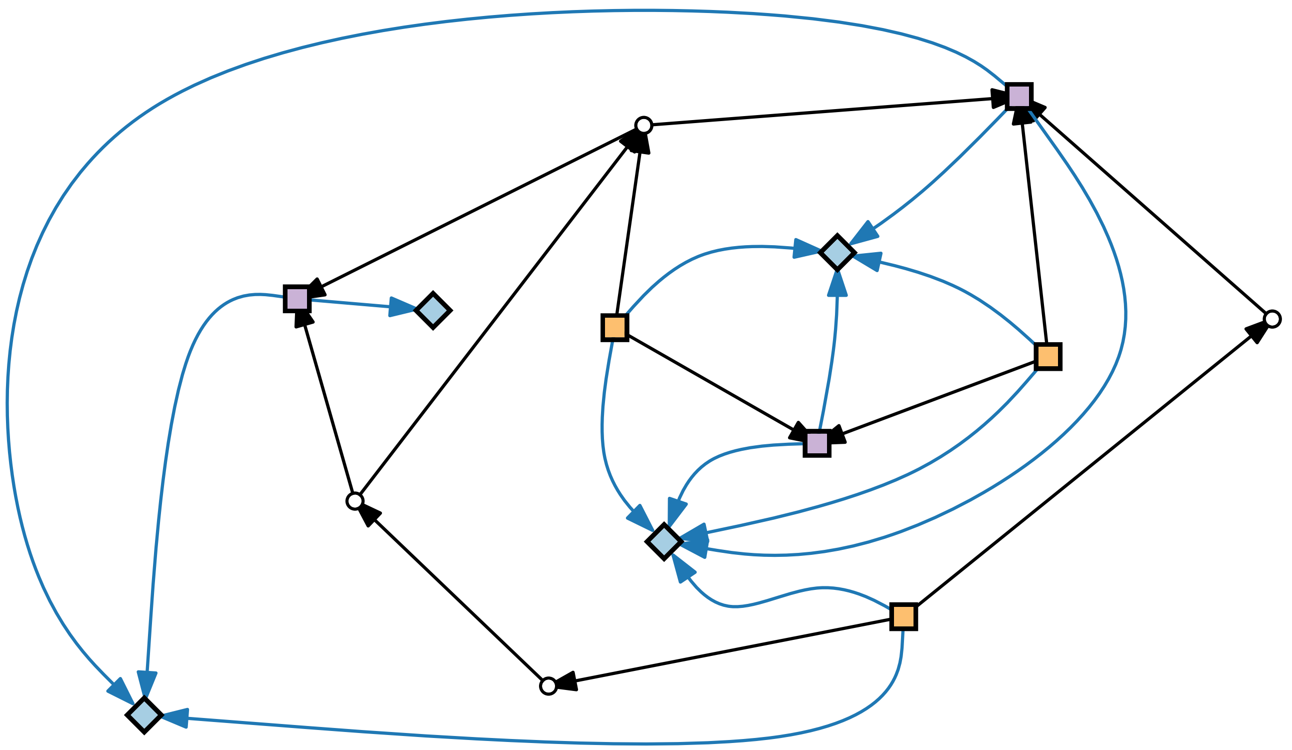

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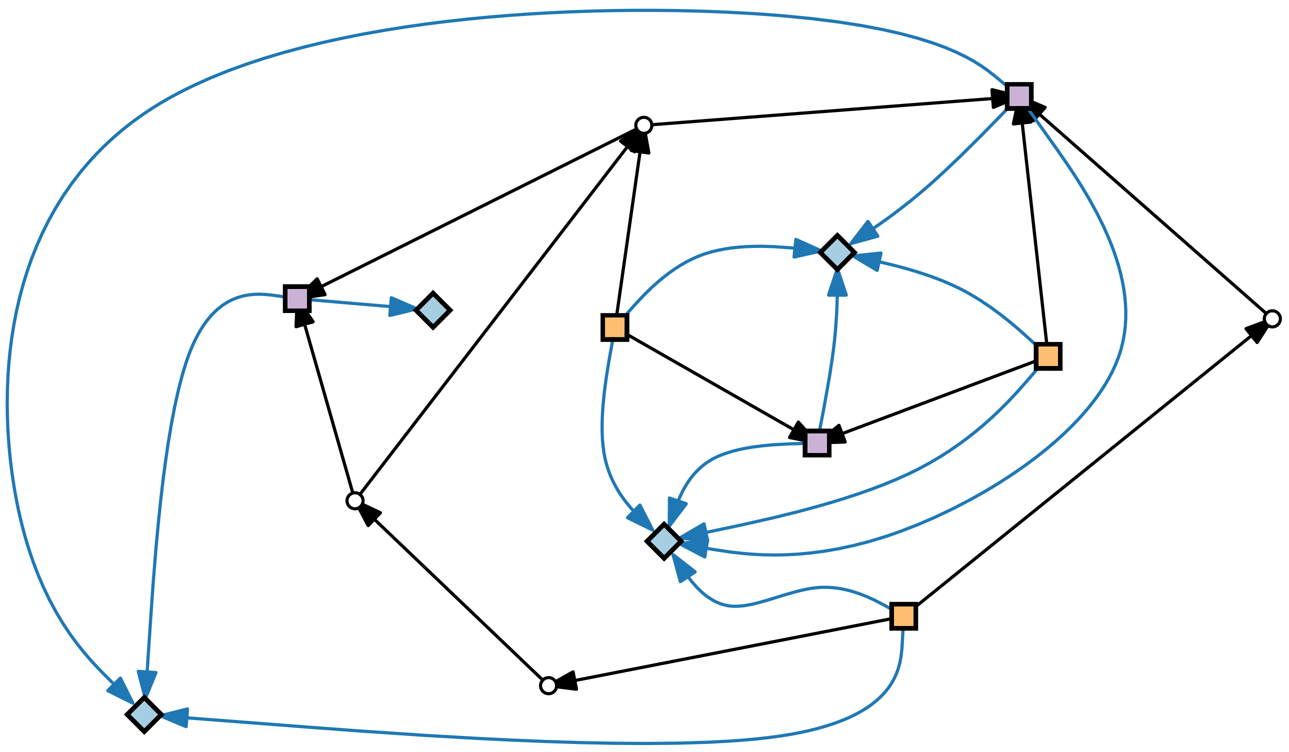

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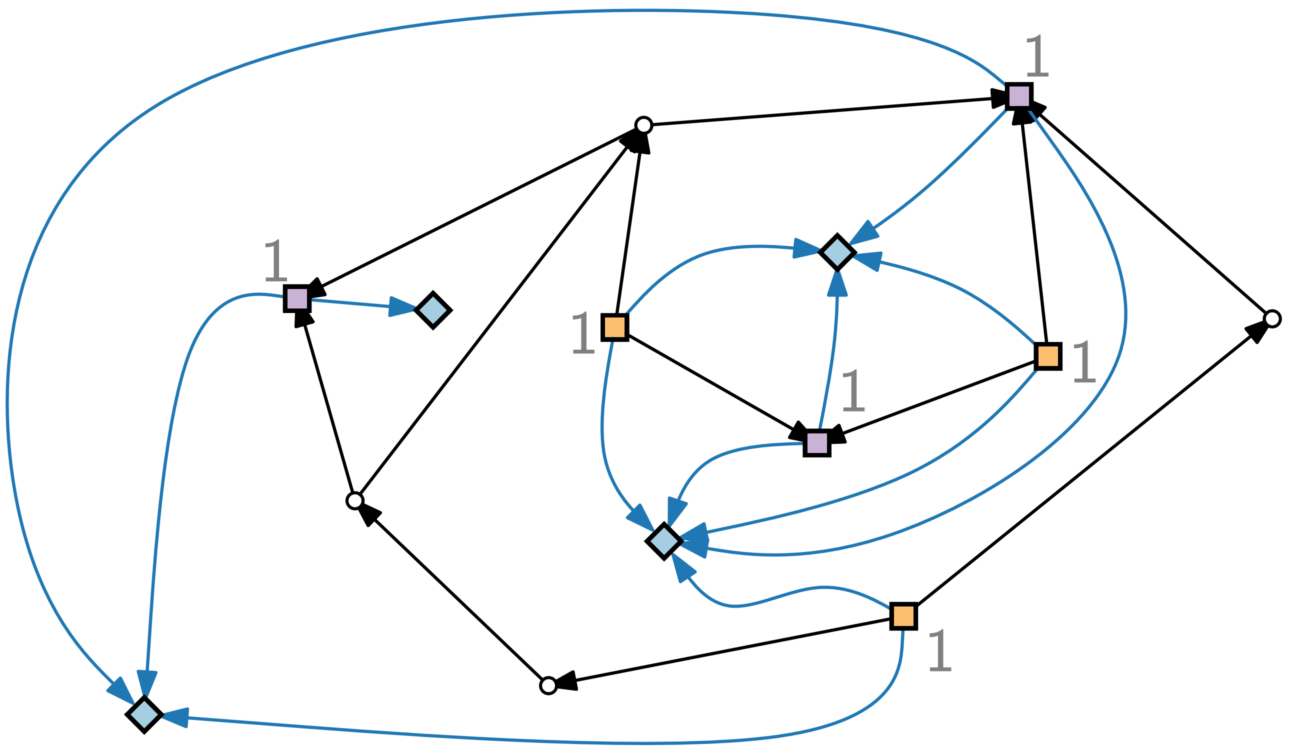

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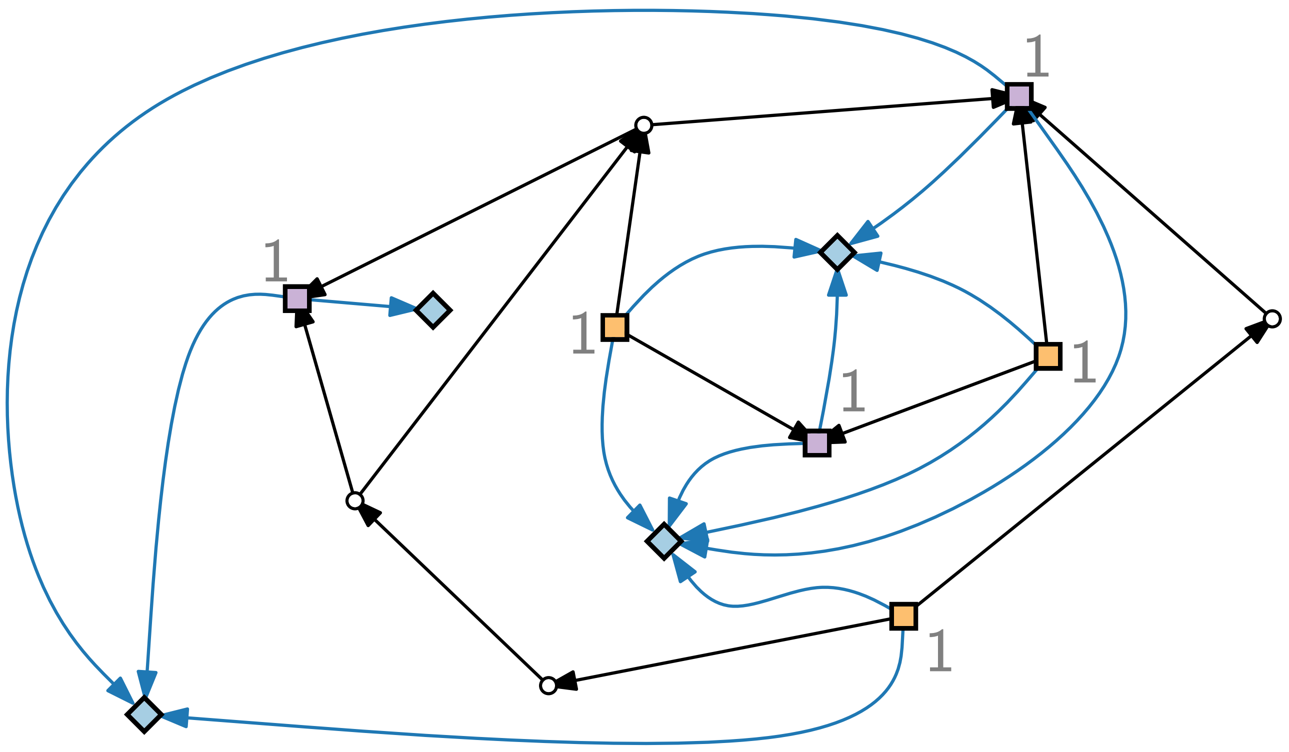

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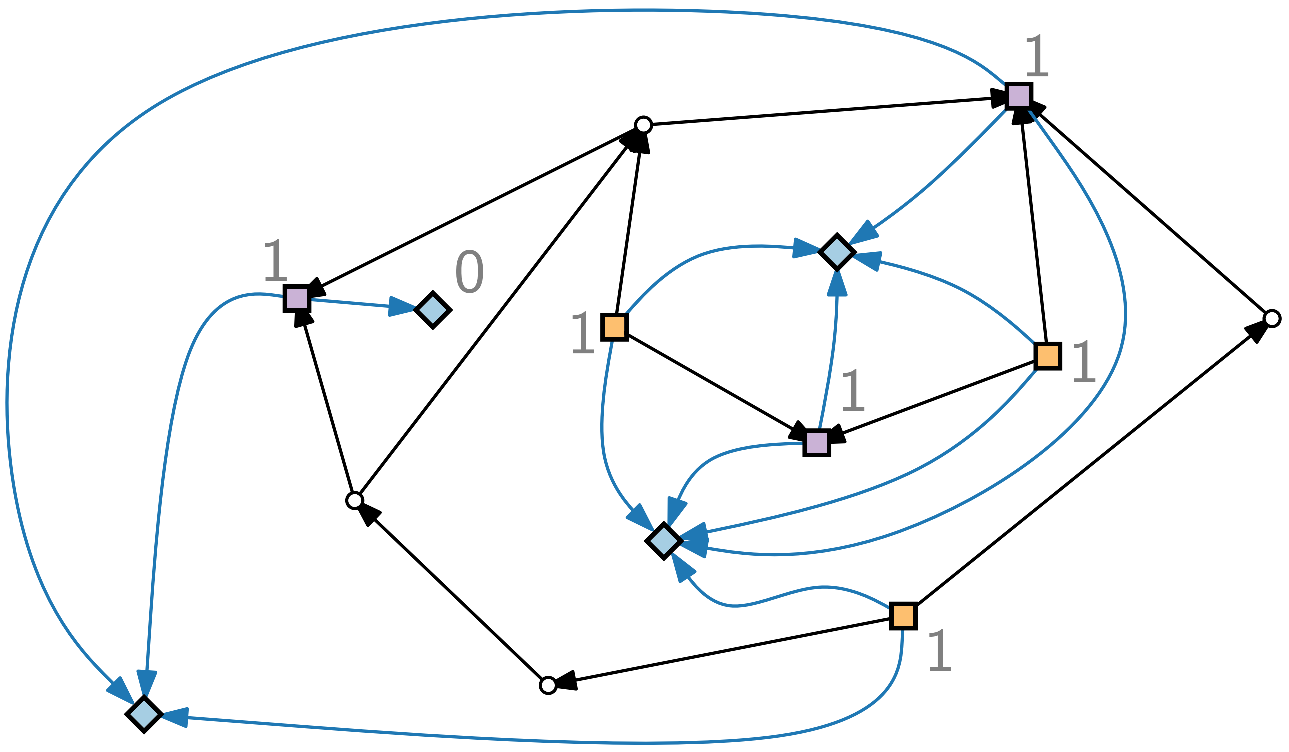

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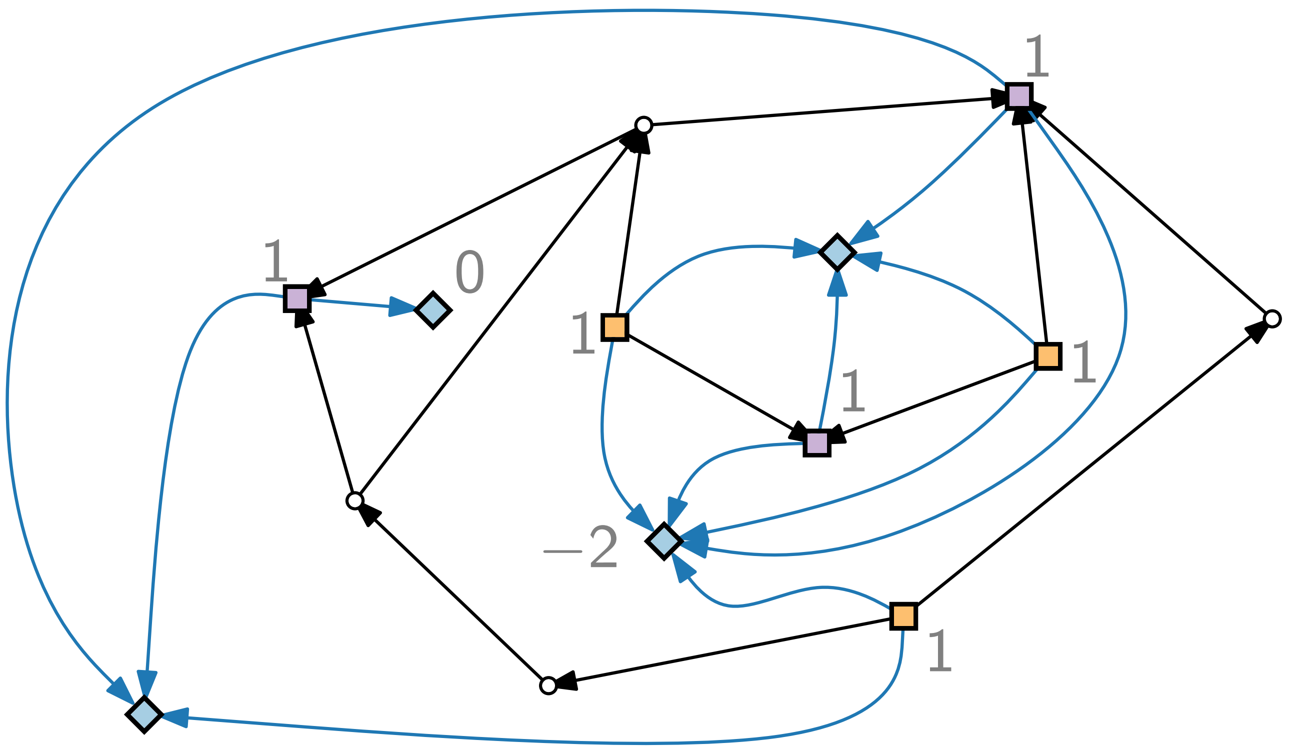

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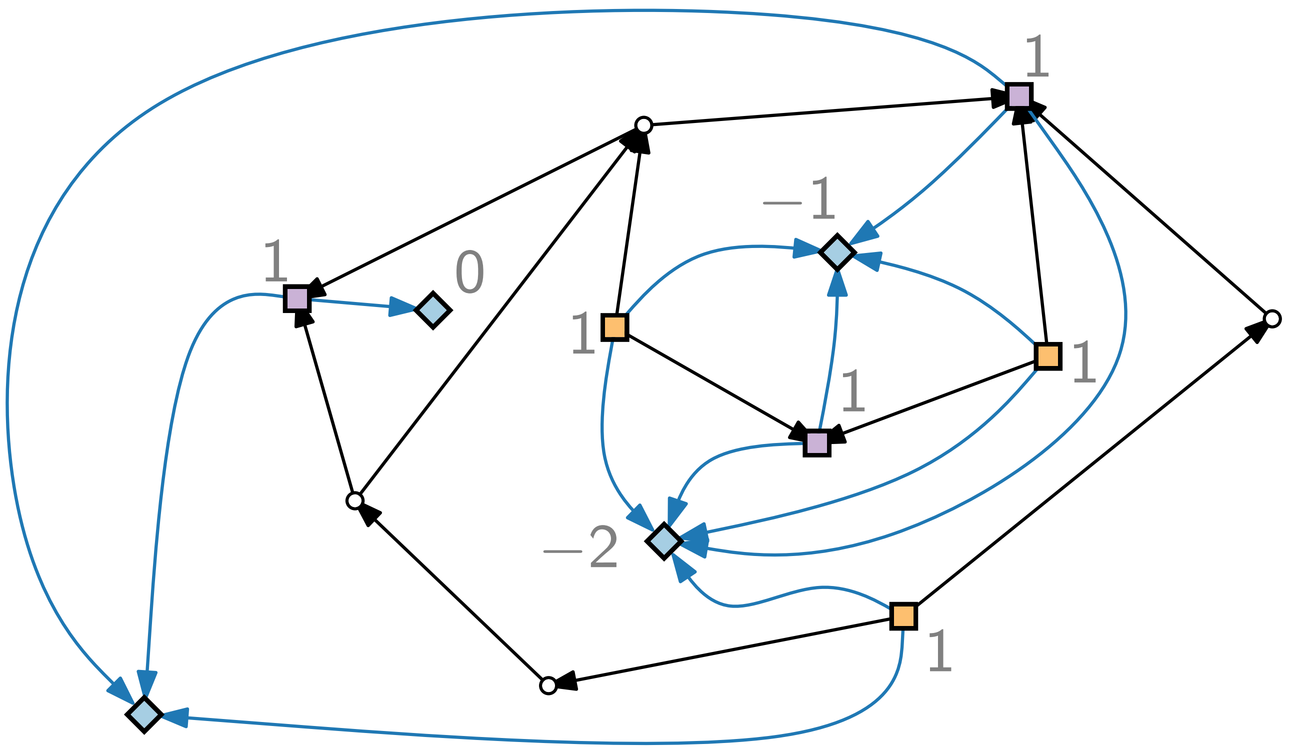

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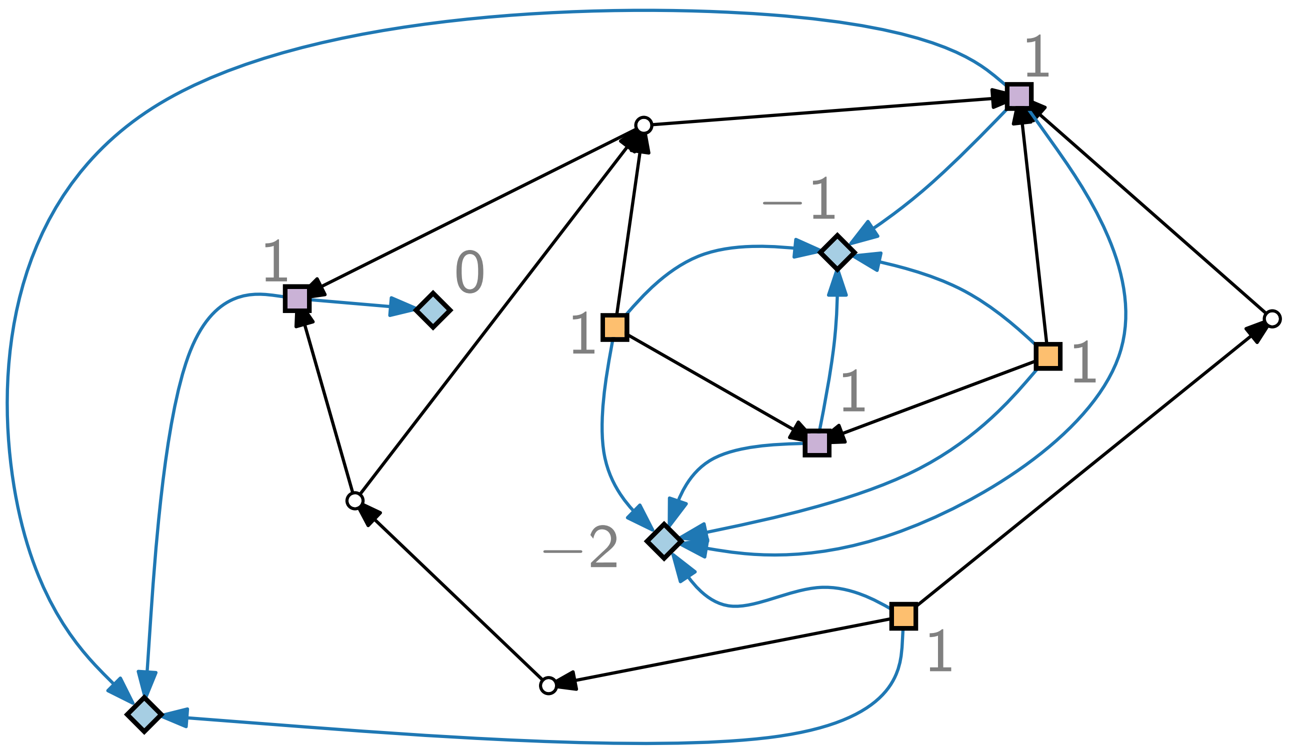

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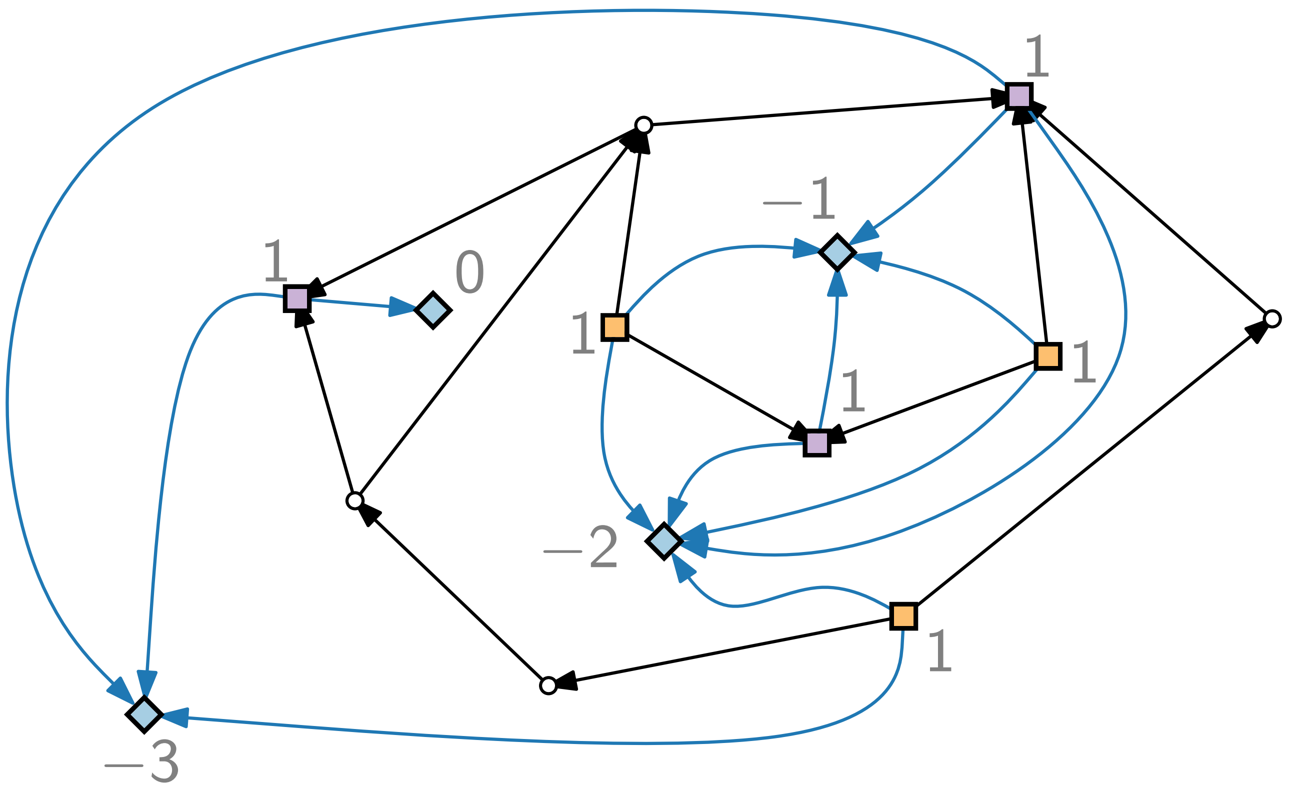

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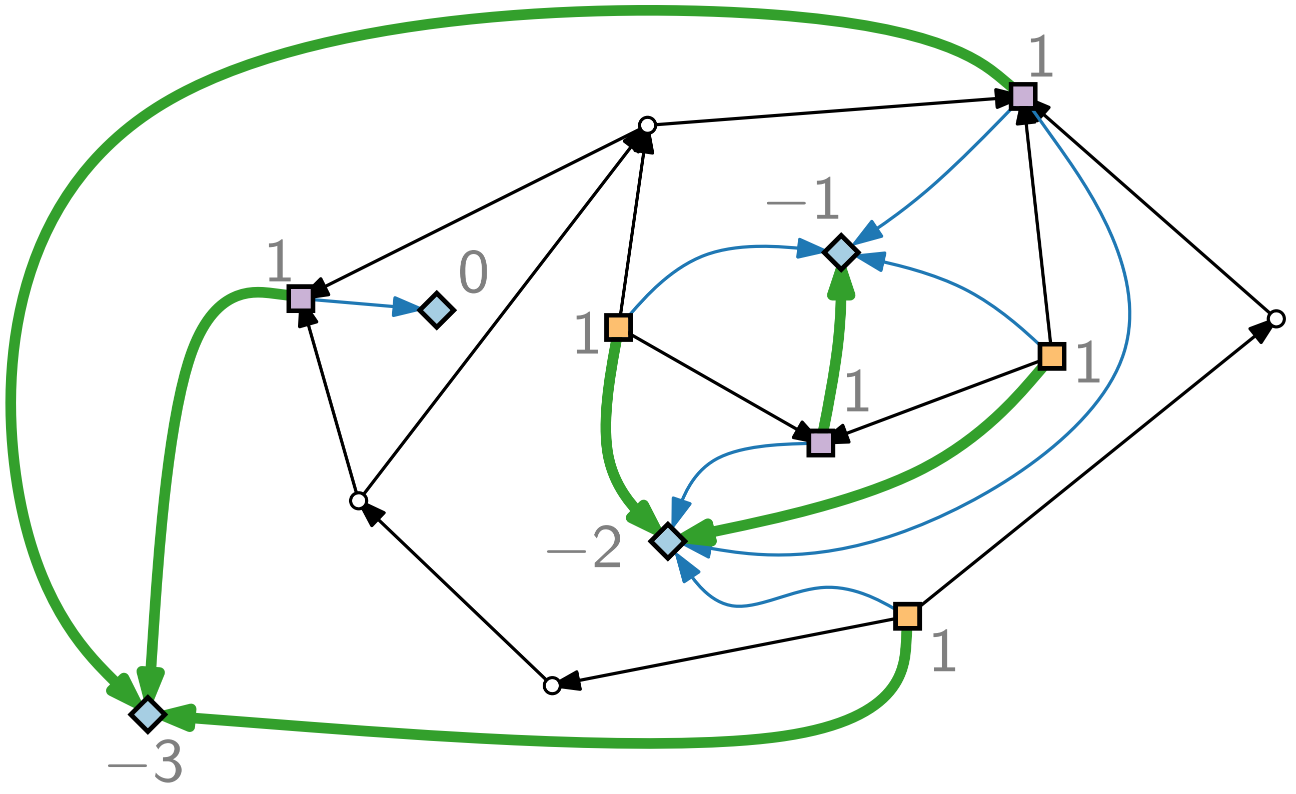

Discussion

- There exist fixed-parameter tractable algorithms to test upward planarity of general digraphs with the parameter being the number of triconnected components.
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- Many related concepts have been studied: quasi-planarity, upward drawings of mixed graphs, upward planarity on cyclinder/torus, ...

Literature

- [GD Ch. 6] for detailed explanation

Orginal papers referenced:

- [Kelly '87] Fundamentals of Planar Ordered Sets
- [Di Battista, Tamassia '88] Algorithms for Plane Representations of Acyclic Digraphs
- [Garg, Tamassia '95] On the Computational Complexity of Upward and Rectilinear Planarity Testing
- [Hutton, Lubiw '96] Upward Planar Drawing of Single-Source Acyclic Digraphs
- [Bertolazzi, Di Battista, Mannino, Tamassia '94] Upward Drawings of Triconnected Digraphs
- [Healy, Lynch '05] Building Blocks of Upward Planar Digraphs
- [Didimo, Giardano, Liotta '09] Upward Spirality and Upward Planarity Testing
- [Abbasi, Healy, Rextin '10] Improving the running time of embedded upward planarity testing