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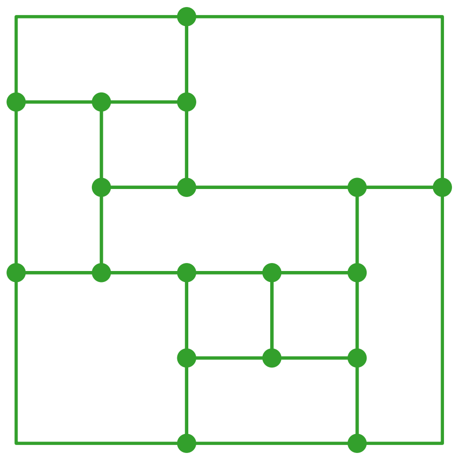
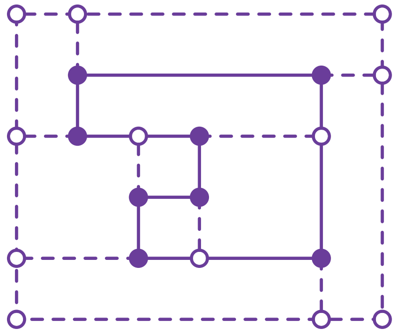
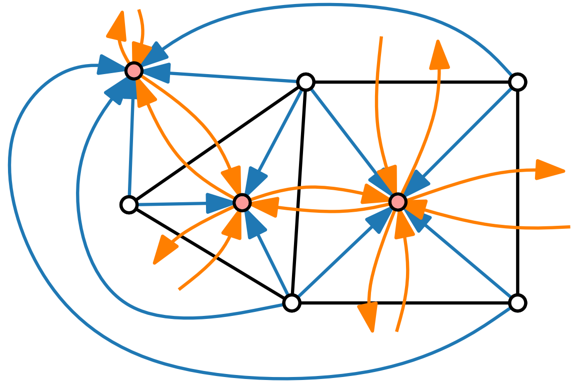
# Visualization of Graphs



## Lecture 5:
## Orthogonal Layouts

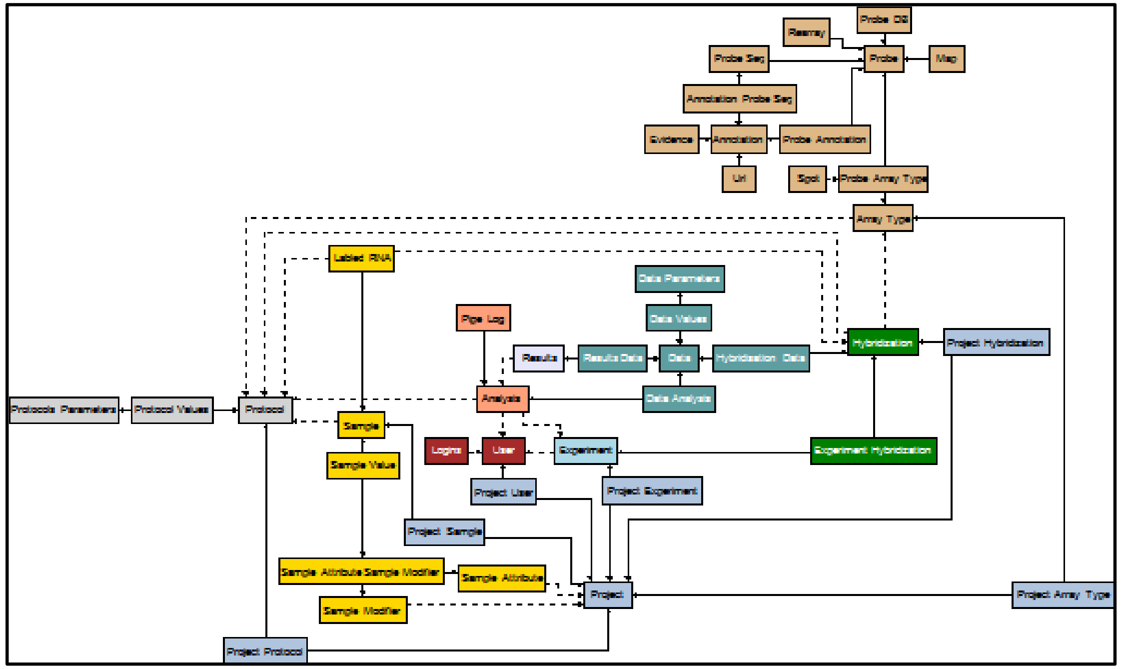


### Part I:
### Topolgy – Shape – Metric



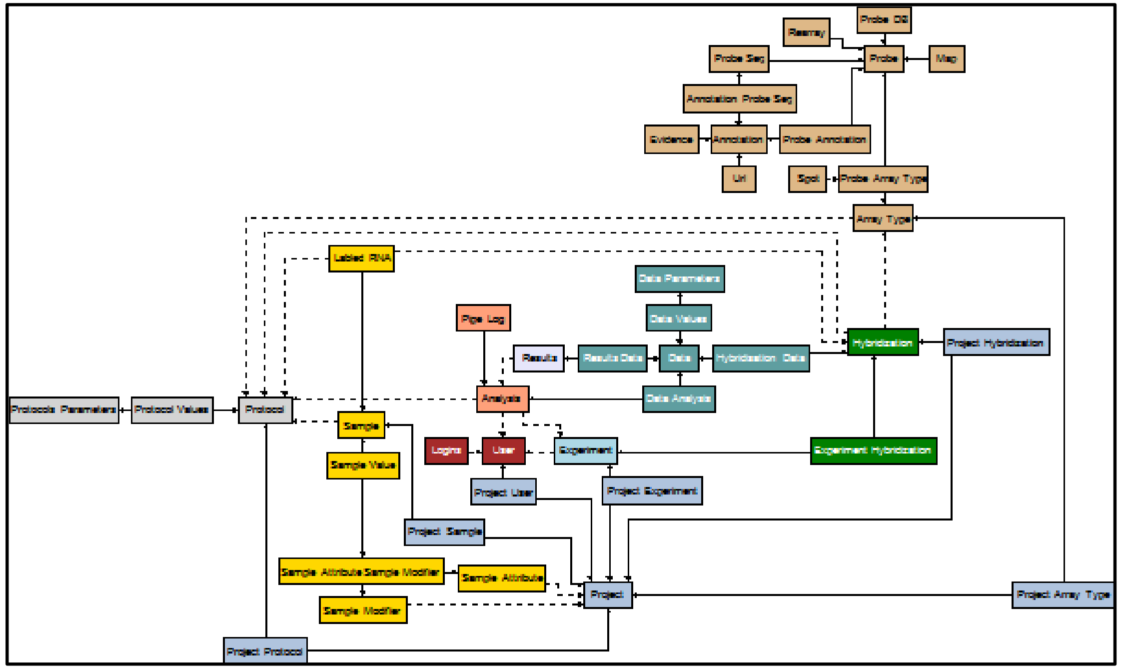
Jonathan Klawitter

# Orthogonal Layout – Applications




ER diagram in OGDF

# Orthogonal Layout – Applications



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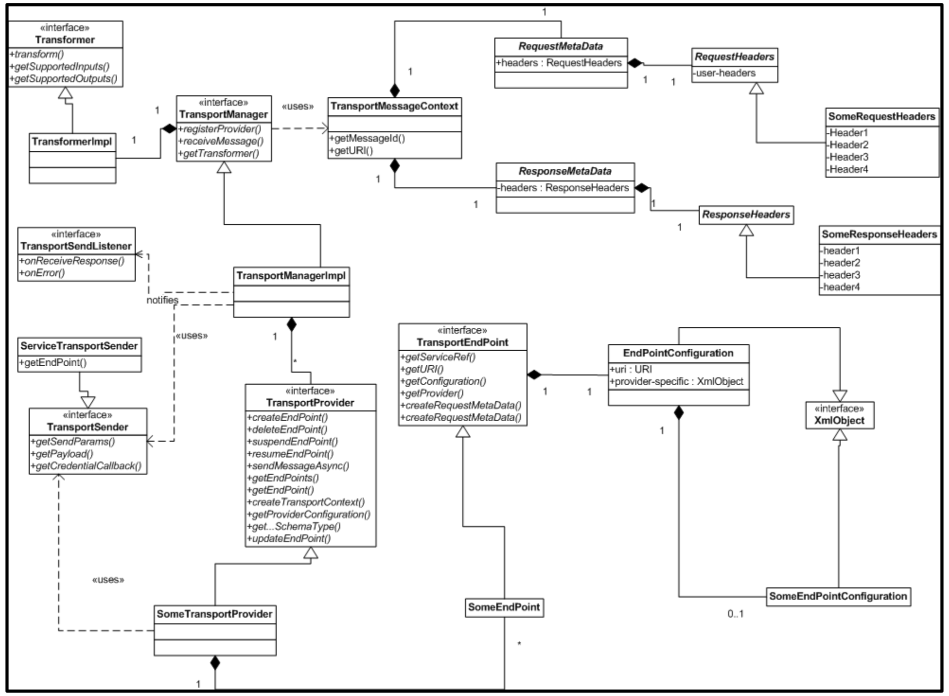
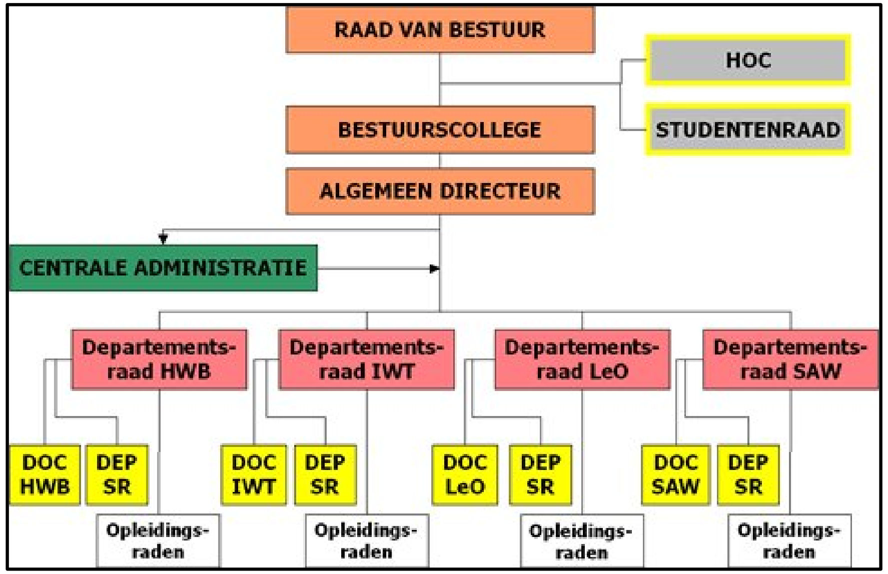
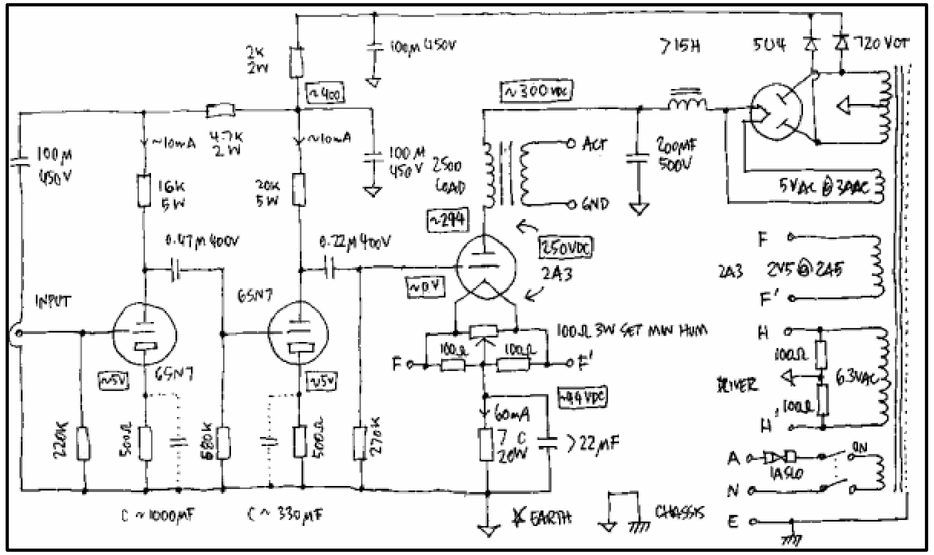


UML diagram by Oracle

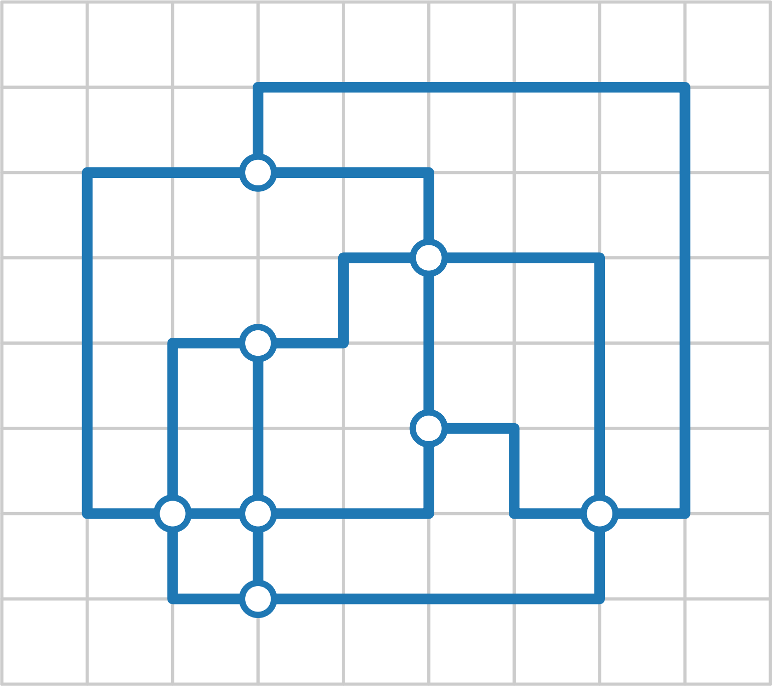



Organigram of HS Limburg



Circuit diagram by Jeff Atwood

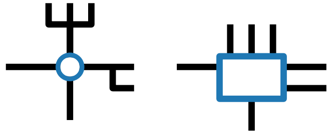
# Orthogonal Layout – Definition



**Definition.**
A drawing $\Gamma$ of a graph $G = (V, E)$ is called **orthogonal** if
- vertices are drawn as points on a grid,
- each edge is represented as a sequence of alternating horizontal and vertical segments, and
- pairs of edges are disjoint or cross orthogonally.

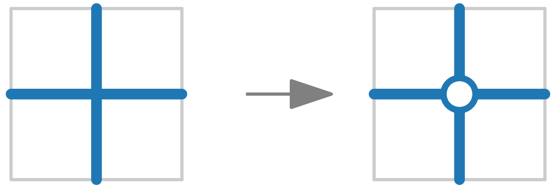
## Observations.

- Edges lie on grid $\Rightarrow$ **bends** lie on grid points
- Max degree of each vertex is at most 4
- Otherwise



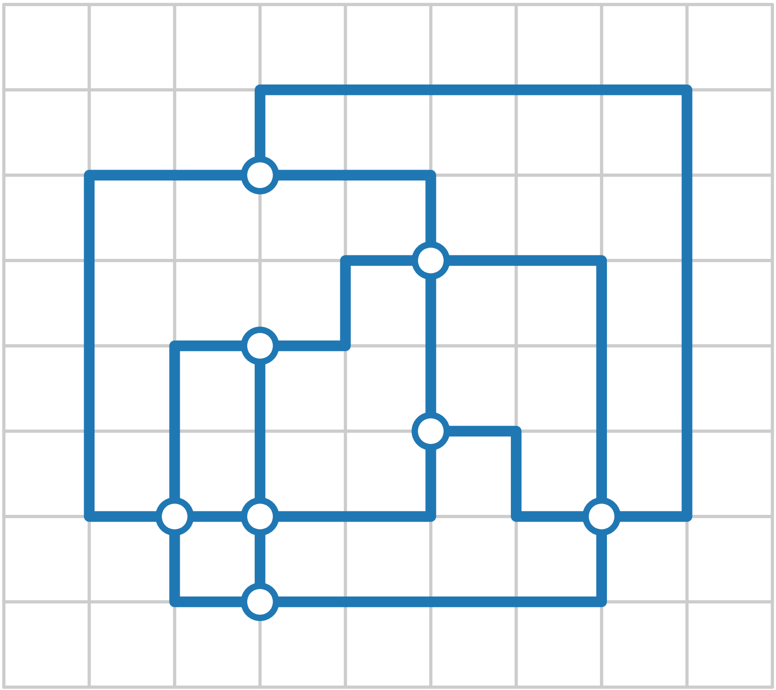
## Planarization.

- Fix embedding
- Crossings become vertices



## Aesthetic criteria.

# Orthogonal Layout – Definition



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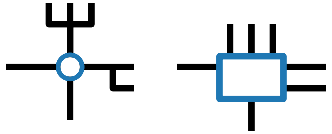
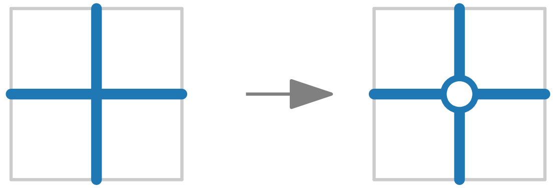


## Planarization.
- Fix embedding
- Crossings become vertices



## Aesthetic criteria.
- Number of bends
- Length of edges
- Width, height, area
- Monotonicity of edges
- ...

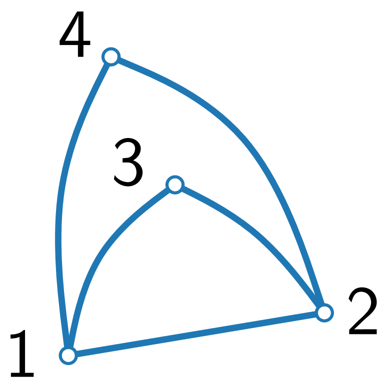
# Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

$$V = \{v_1, v_2, v_3, v_4\}$$
$$E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$$

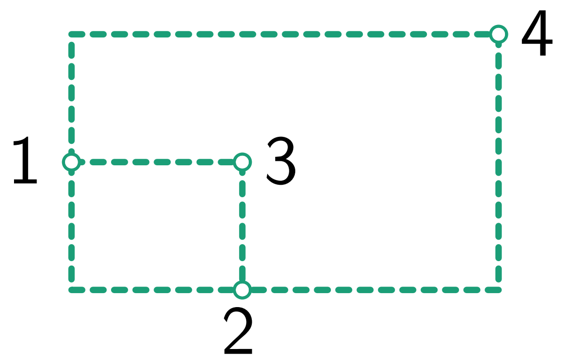
combinatorial embedding/ planarization

reduce crossings



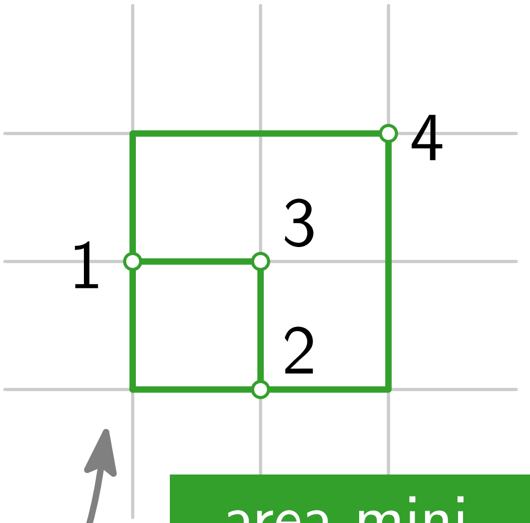

bend minimization

orthogonal representation




planar orthogonal drawing

area minimization




TOPOLOGY — SHAPE — METRICS

# Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

$V = \{v_1, v_2, v_3, v_4\}$
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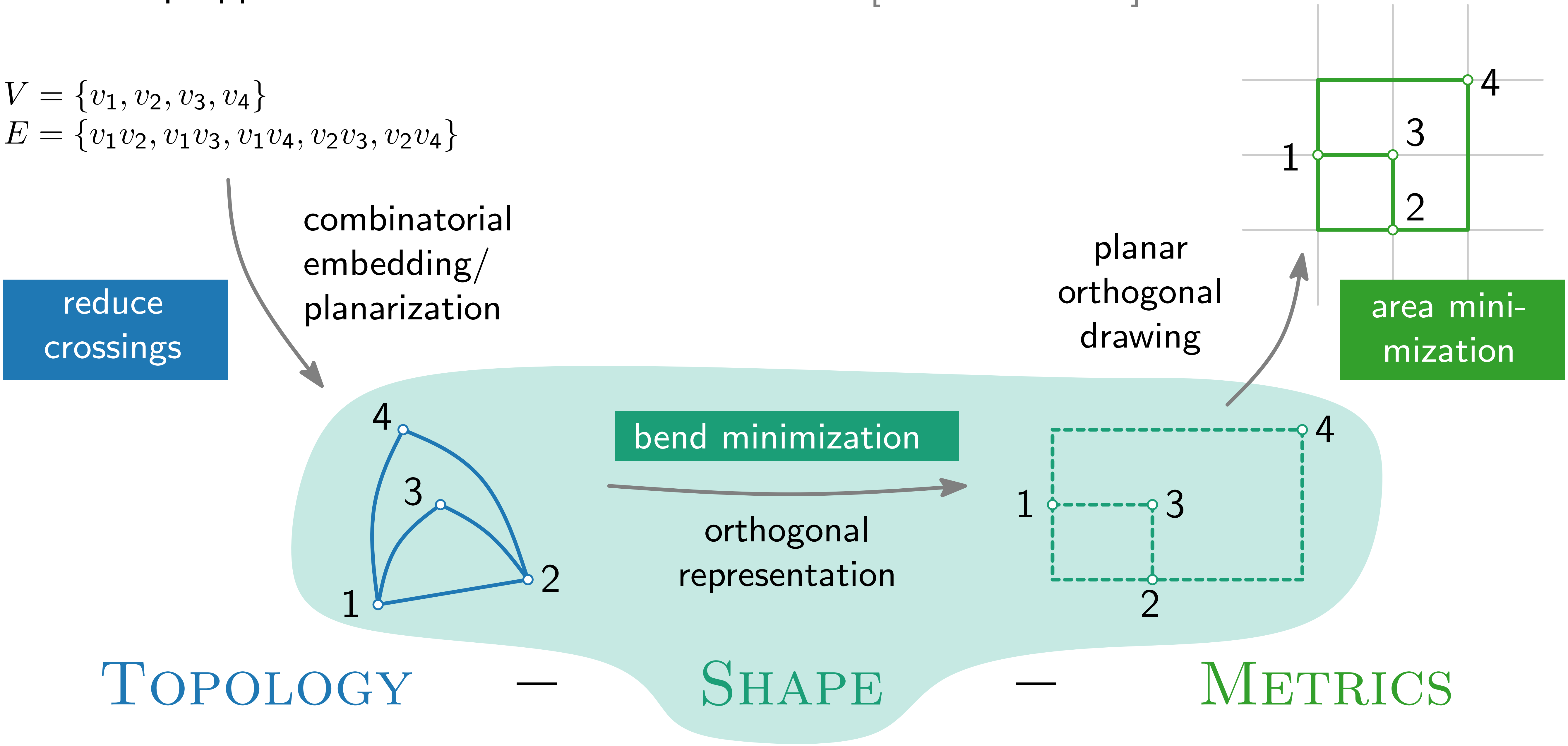

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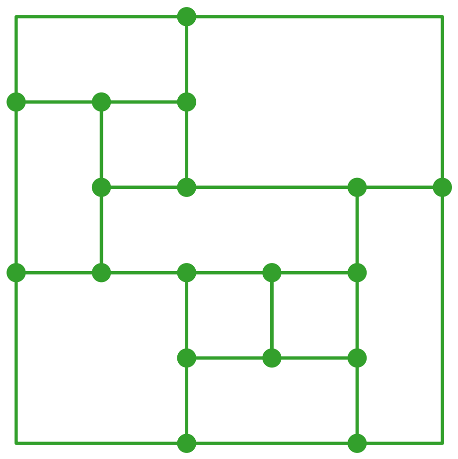
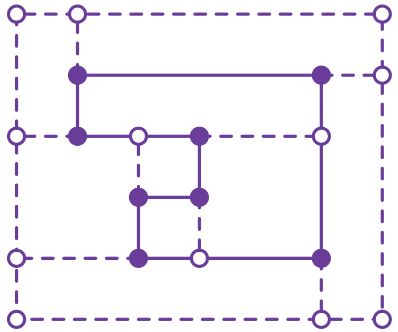
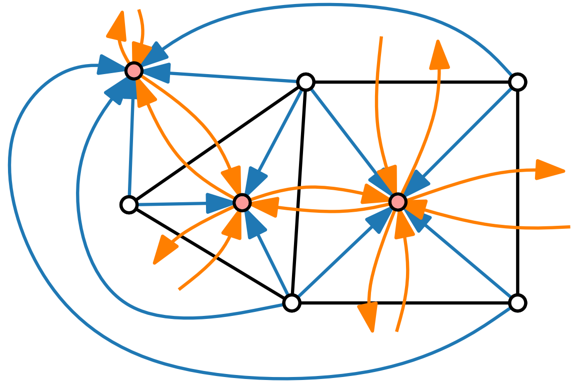
# Visualization of Graphs



## Lecture 5:
## Orthogonal Layouts



### Part II:
### Orthogonal Representation



Jonathan Klawitter

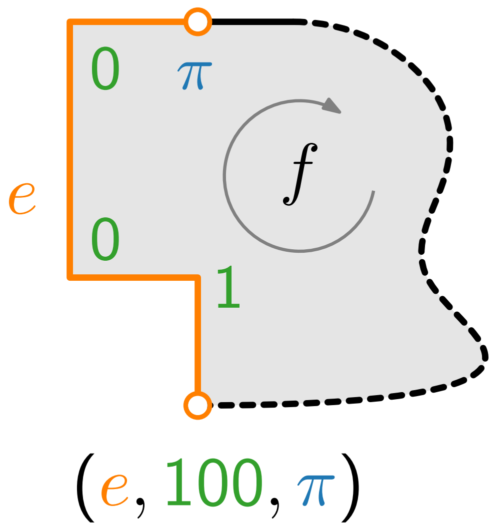
# Orthogonal Representation

**Idea.**
Describe orthogonal drawing combinatorically.

**Definitions.**
Let $G = (V, E)$ be a plane graph with faces $F$ and outer face $f_0$.

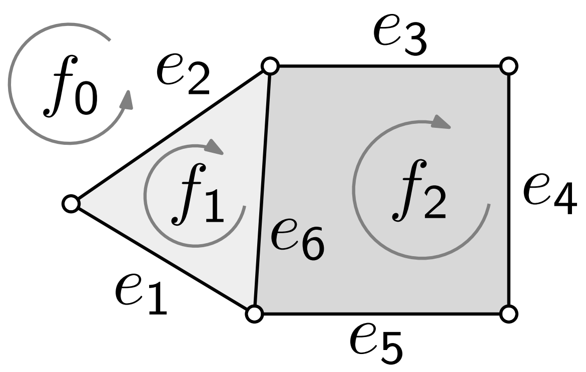
- Let $e$ be an edge with the face $f$ to the right.
  An **edge description** of $e$ wrt $f$ is a triple $(e, \delta, \alpha)$ where
  - $\delta$ is a sequence of $\{0, 1\}^*$ ($0 =$ right bend, $1 =$ left bend)
  - $\alpha$ is angle $\in \{\frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi\}$ between $e$ and next edge $e'$
- A **face representation** $H(f)$ of $f$ is a clockwise ordered sequence of edge descriptions $(e, \delta, \alpha)$.
- An **orthogonal representation** $H(G)$ of $G$ is defined as

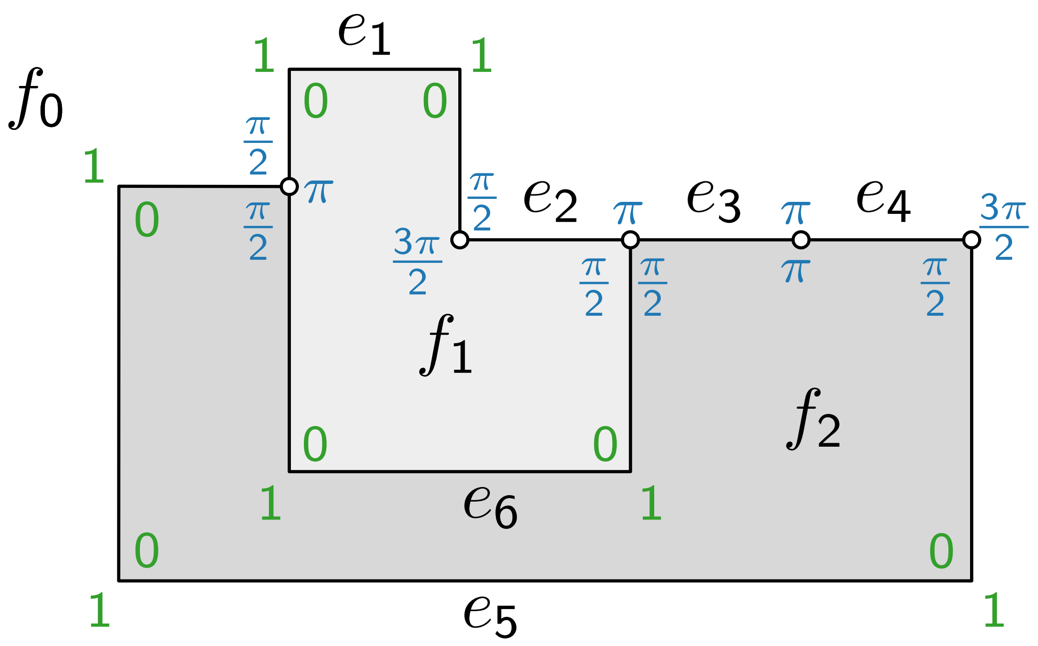
$$H(G) = \{H(f) \mid f \in F\}.$$



# Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$




Concrete coordinates are not fixed yet!

# Correctness of an Orthogonal Representation

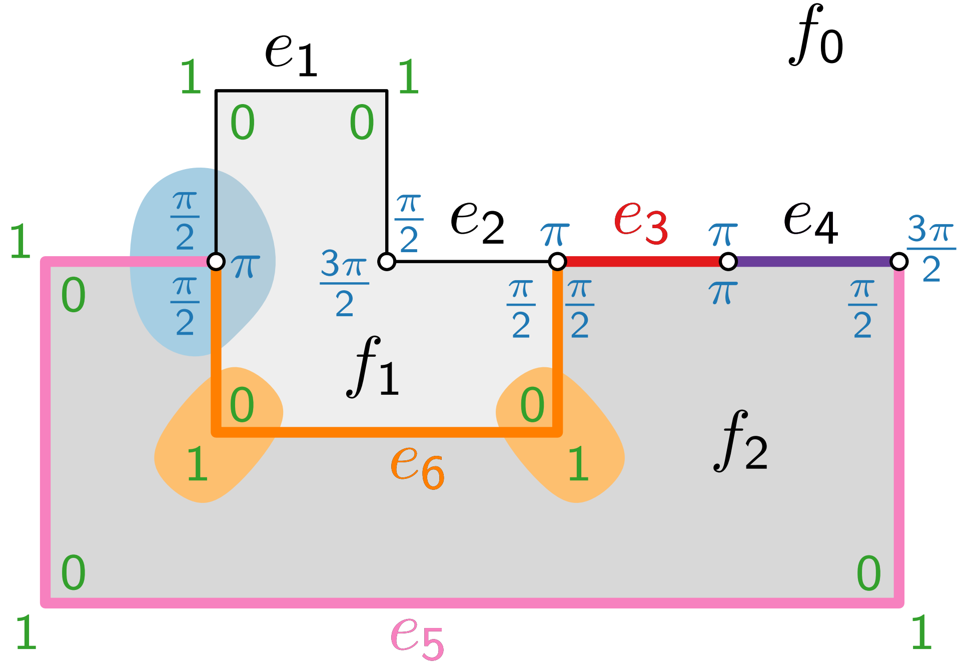
(H1) $H(G)$ corresponds to $F$, $f_0$.

(H2) For each **edge** $\{u, v\}$ shared by faces $f$ and $g$ with $((u,v), \delta_1, \alpha_1) \in H(f)$ and $((v,u), \delta_2, \alpha_2) \in H(g)$ sequence $\delta_1$ is reversed and inverted $\delta_2$.

(H3) Let $|\delta|_0$ (resp. $|\delta|_1$) be the number of zeros (resp. ones) in $\delta$ and $r = (e, \delta, \alpha)$.
Let $C(r) := |\delta|_0 - |\delta|_1 + 2 - \alpha \cdot 2/\pi$.
For each **face** $f$ it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

(H4) For each **vertex** $v$ the sum of incident angles is $2\pi$.




$$C(e_3) = 0 - 0 + 2 - \pi \cdot \frac{2}{\pi} = 0$$

$$C(e_4) = 0 - 0 + 2 - \frac{\pi}{2} \cdot \frac{2}{\pi} = 1$$

$$C(e_5) = 3 - 0 + 2 - \frac{\pi}{2} \cdot \frac{2}{\pi} = 4$$

$$C(e_6) = 0 - 2 + 2 - \frac{\pi}{2} \cdot \frac{2}{\pi} = -1$$

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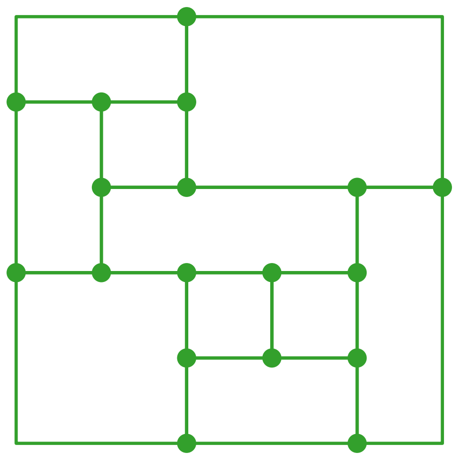
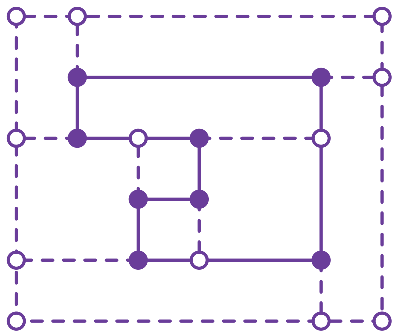
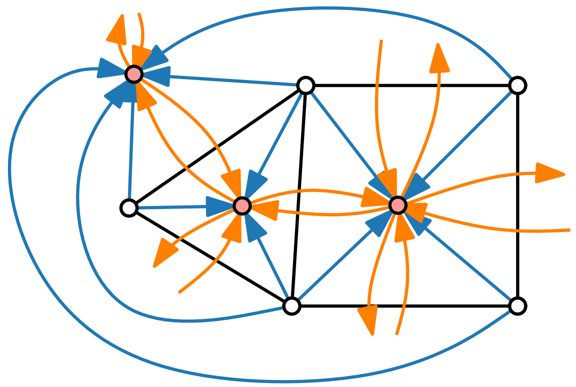
# Visualization of Graphs



## Lecture 5:
## Orthogonal Layouts

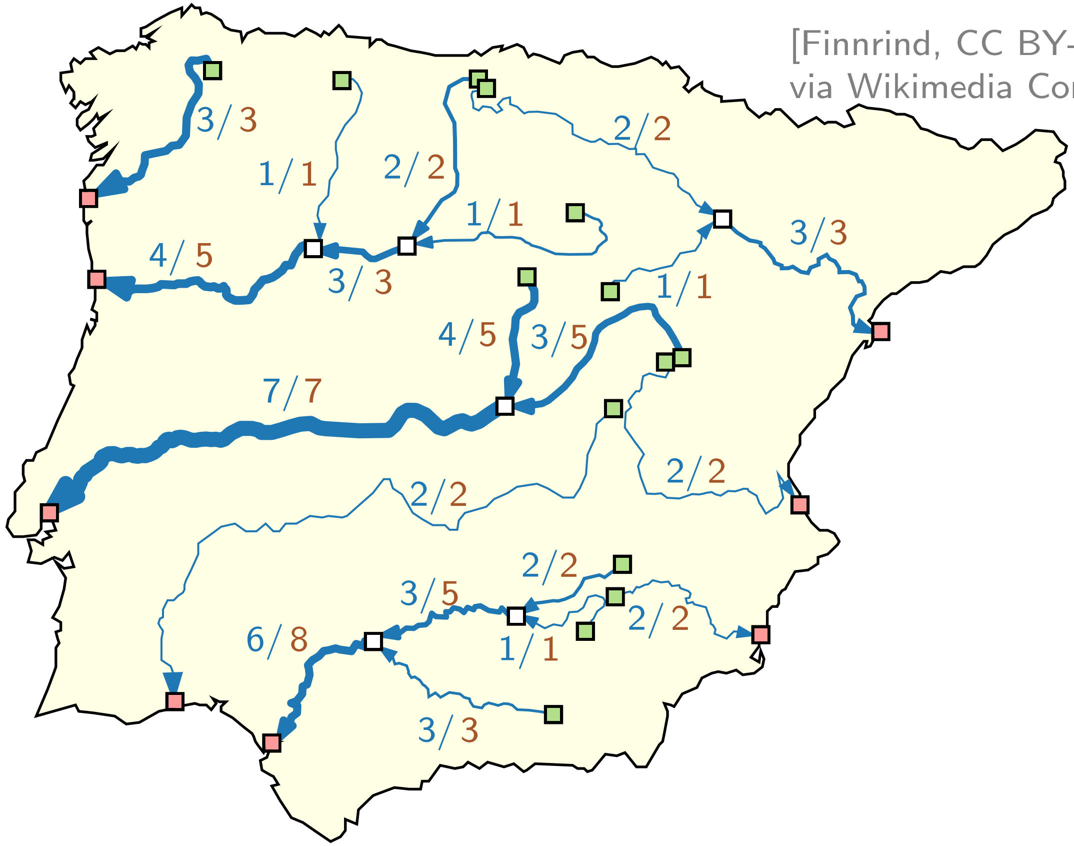


### Part III:
### Bend Minimization



Jonathan Klawitter

# Reminder: $s$-$t$-Flow Networks






**Flow network** $(G = (V, E); S, T; u)$ with

- directed graph $G = (V, E)$
- *sources* $S \subseteq V$, *sinks* $T \subseteq V$
- edge *capacity* $u \colon E \to \mathbb{R}_0^+$

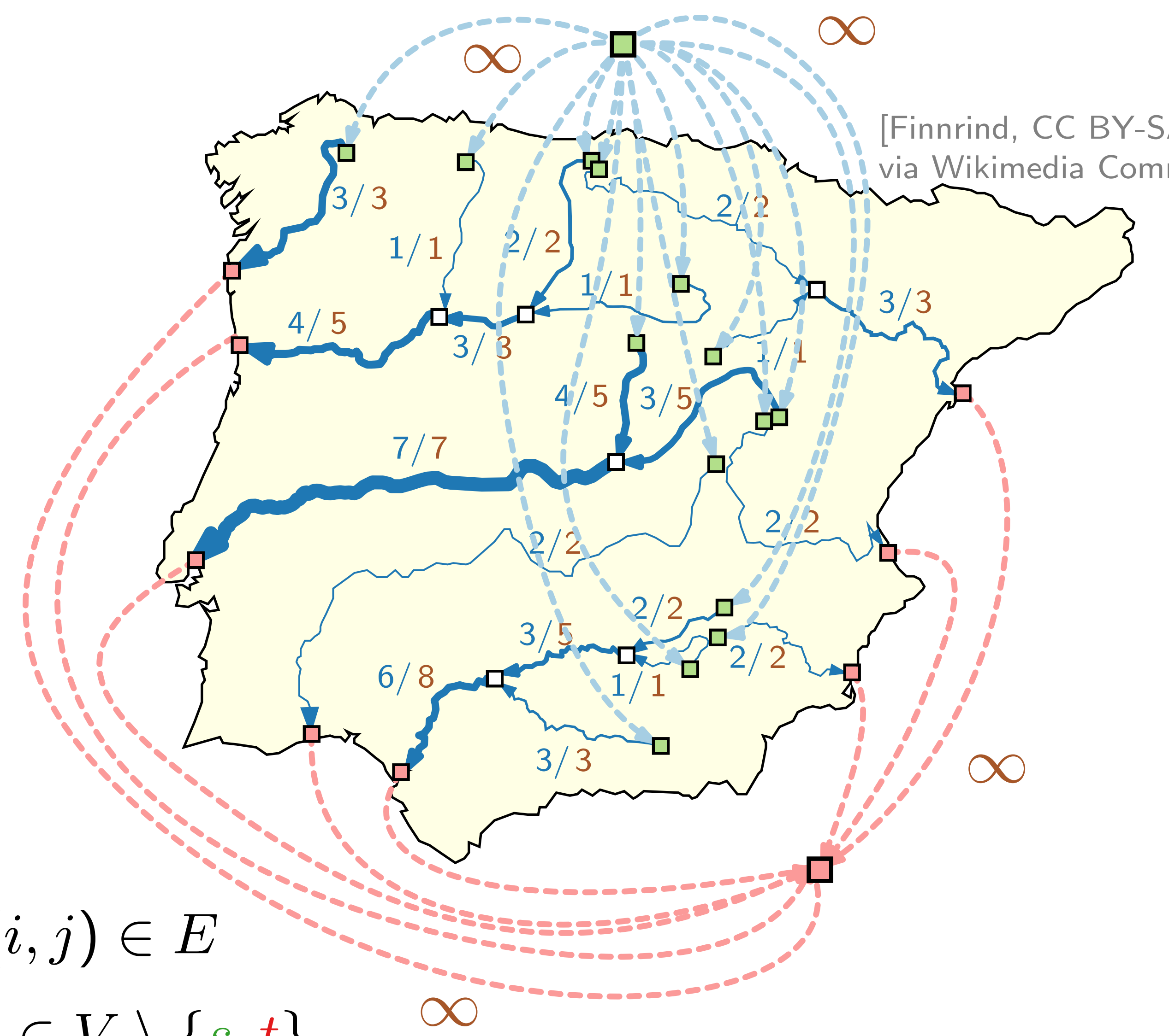
A function $X \colon E \to \mathbb{R}_0^+$ is called $S$-$T$-**flow**, if:

$$0 \leq X(i,j) \leq u(i,j) \qquad \forall (i,j) \in E$$

$$\sum_{(i,j) \in E} X(i,j) - \sum_{(j,i) \in E} X(j,i) = 0 \qquad \forall i \in V \setminus (S \cup T)$$

A **maximum** $S$-$T$-flow is an $S$-$T$-flow where $\sum_{(i,j) \in E, i \in S} X(i,j)$ is maximized.

# Reminder: $s$-$t$-Flow Networks




**Flow network** $(G = (V, E); s, t; u)$ with

- directed graph $G = (V, E)$
- *source* $s \in V$, *sink* $t \in V$
- edge *capacity* $u \colon E \to \mathbb{R}_0^+$

A function $X \colon E \to \mathbb{R}_0^+$ is called $s$-$t$-**flow**, if:

$$0 \leq X(i,j) \leq u(i,j) \qquad \forall (i,j) \in E$$

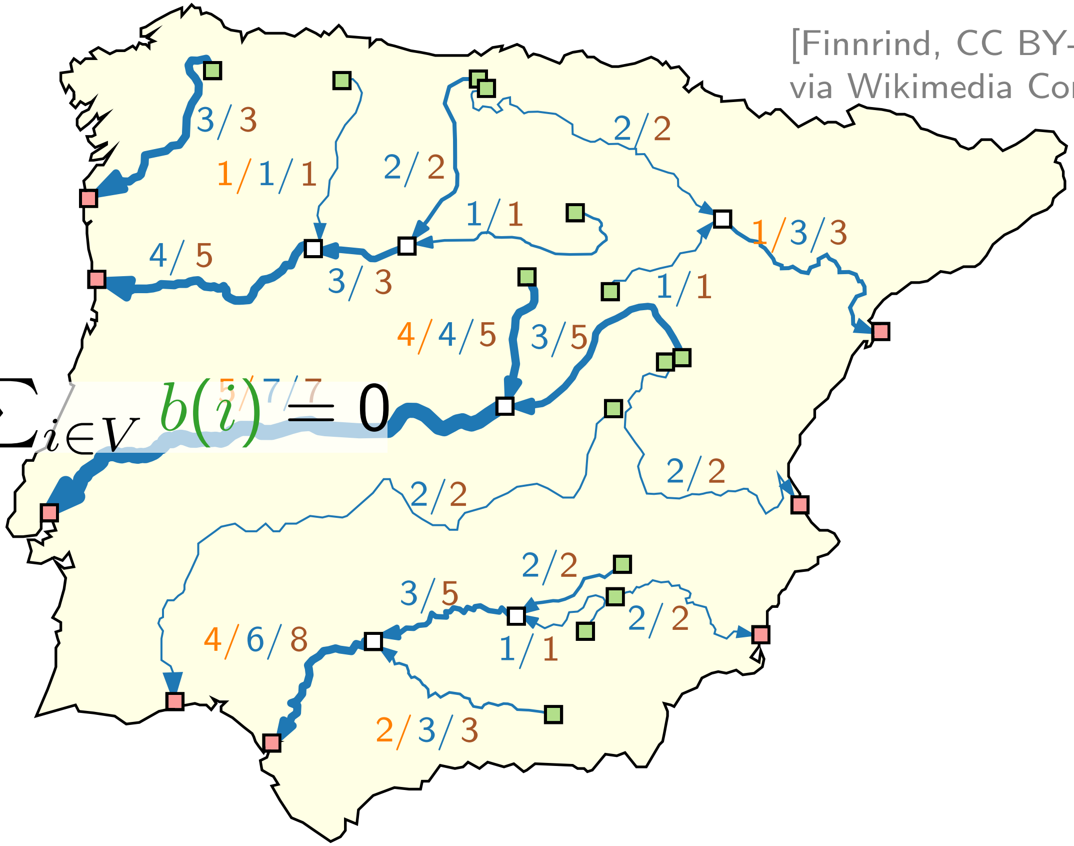
$$\sum_{(i,j)\in E} X(i,j) - \sum_{(j,i)\in E} X(j,i) = 0 \qquad \forall i \in V \setminus \{s,t\}$$

A **maximum** $s$-$t$-flow is an $s$-$t$-flow where $\sum_{(s,j)\in E} X(s,j)$ is maximized.

# General Flow Network

**Flow network** $(G = (V, E); b; \ell; u)$ with

- directed graph $G = (V, E)$
- node *production/consumption* $b\colon V \to \mathbb{R}$ with $\sum_{i \in V} b(i) = 0$
- edge *lower bound* $\ell\colon E \to \mathbb{R}_0^+$
- edge *capacity* $u\colon E \to \mathbb{R}_0^+$






A function $X\colon E \to \mathbb{R}_0^+$ is called **valid flow**, if:

$$\ell(i,j) \le X(i,j) \le u(i,j) \qquad \forall (i,j) \in E$$

$$\sum_{(i,j)\in E} X(i,j) - \sum_{(j,i)\in E} X(j,i) = b(i) \qquad \forall i \in V$$

- *Cost function* $\mathrm{cost}\colon E \to \mathbb{R}_0^+$ and $\mathrm{cost}(X) := \sum_{(i,j)\in E} \mathrm{cost}(i,j) \cdot X(i,j)$

A **minimum cost flow** is a valid flow where $\mathrm{cost}(X)$ is minimized.

# General Flow Network – Algorithms

**Polynomial Algorithms**

| # | Due to | Year | Running Time |
|---|---|---|---|
| 1 | Edmonds and Karp | 1972 | O((n + m') log U S(n, m, nC)) |
| 2 | Rock | 1980 | O((n + m') log U S(n, m, nC)) |
| 3 | Rock | 1980 | O(n log C M(n, m, U)) |
| 4 | Bland and Jensen | 1985 | O(m log C M(n, m, U)) |
| 5 | Goldberg and Tarjan | 1987 | O(nm log $(n^2/m)$ log (nC)) |
| 6 | Goldberg and Tarjan | 1988 | O(nm log n log (nC)) |
| 7 | Ahuja, Goldberg, Orlin and Tarjan | 1988 | O(nm log log U log (nC)) |

**Strongly Polynomial Algorithms**

| # | Due to | Year | Running Time |
|---|---|---|---|
| 1 | Tardos | 1985 | $O(m^4)$ |
| 2 | Orlin | 1984 | $O((n + m')^2$ log n S(n, m)) |
| 3 | Fujishige | 1986 | $O((n + m')^2$ log n S(n, m)) |
| 4 | Galil and Tardos | 1986 | $O(n^2$ log n S(n, m)) |
| 5 | Goldberg and Tarjan | 1987 | $O(nm^2$ log n log$(n^2/m))$ |
| 6 | Goldberg and Tarjan | 1988 | $O(nm^2 \log^2 n)$ |
| 7 | Orlin (this paper) | 1988 | O((n + m') log n S(n, m)) |

| | | | |
|---|---|---|---|
| S(n, m) | = | O( m + n log n) | Fredman and Tarjan [1984] |
| S(n, m, C) | = | O( Min (m + n$\sqrt{\log C}$), (m log log C)) | Ahuja, Mehlhorn, Orlin and Tarjan [1990] Van Emde Boas, Kaas and Zijlstra[1977] |
| M(n, m) | = | O(min (nm + $n^{2+\varepsilon}$, nm log n) where ε is any fixed constant. | King, Rao, and Tarjan [1991] |
| M(n, m, U) | = | O(nm log ($\frac{n}{m}\sqrt{\log U}$ + 2)) | Ahuja, Orlin and Tarjan [1989] |

[Orlin 1991]

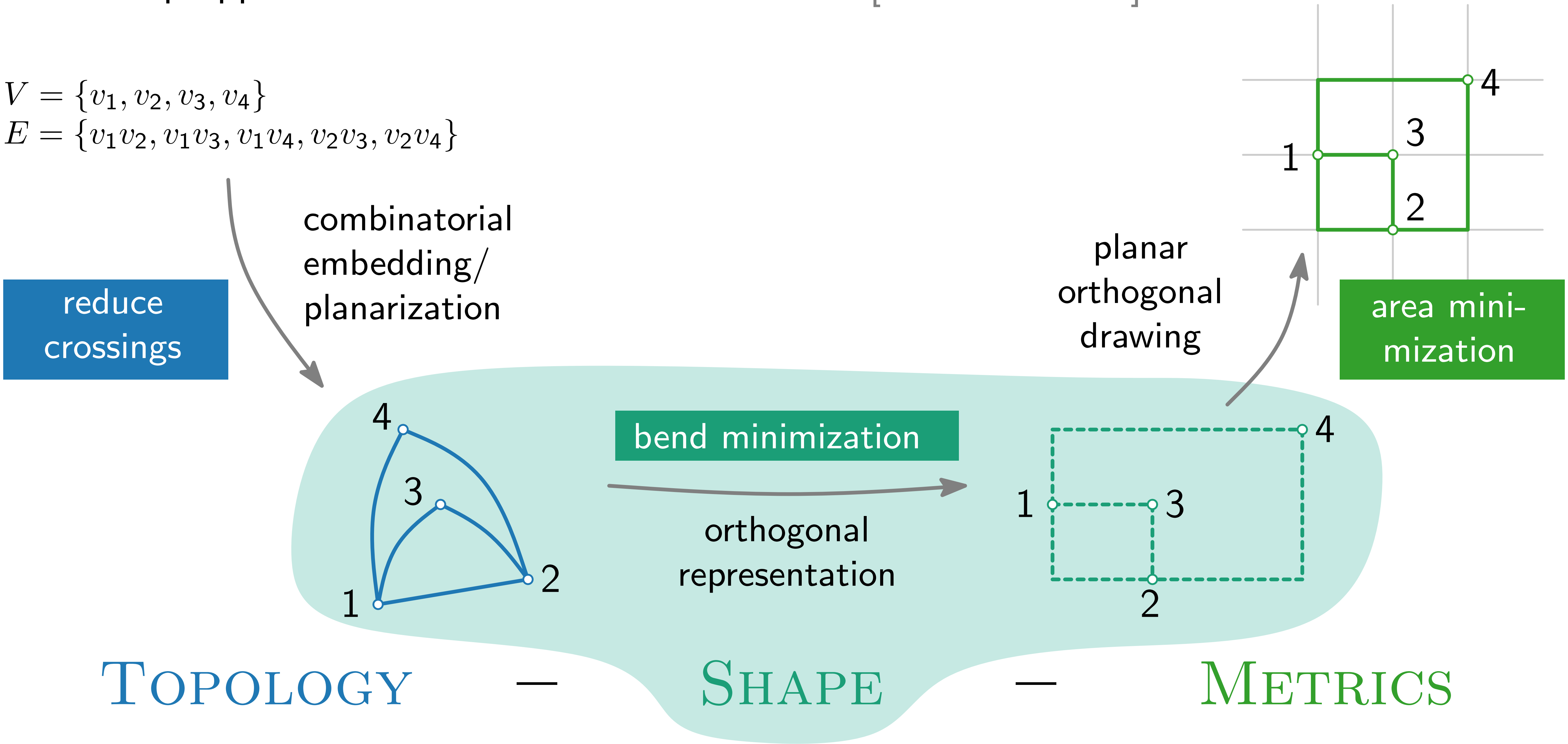
**Theorem.** [Orlin 1991]
The minimum cost flow problem can be solved in $O(n^2 \log^2 n + m^2 \log n)$ time.

**Theorem.** [Cornelsen & Karrenbauer 2011]
The minimum cost flow problem for planar graphs with bounded costs and faze sizes can be solved in $O(n^{3/2})$ time.

# Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

$V = \{v_1, v_2, v_3, v_4\}$
$E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$



# Bend Minimization with Given Embedding

**Geometric bend minimization.**

Given:
- Plane graph $G = (V, E)$ with maximum degree 4
- Combinatorial embedding $F$ and outer face $f_0$

Find: Orthogonal drawing with minimum number of bends that preserves the embedding.

Compare with the following variation.

**Combinatorial bend minimization.**

Given:
- Plane graph $G = (V, E)$ with maximum degree 4
- Combinatorial embedding $F$ and outer face $f_0$

Find: **Orthogonal representation** $H(G)$ with minimum number of bends that preserves the embedding.

# Combinatorial Bend Minimization

**Combinatorial bend minimization.**

Given:
- Plane graph $G = (V, E)$ with maximum degree 4
- Combinatorial embedding $F$ and outer face $f_0$

Find: **Orthogonal representation** $H(G)$ with minimum number of bends that preserves the embedding

**Idea.**

Formulate as a network flow problem:

- a unit of flow $= \measuredangle \frac{\pi}{2}$
- vertices $\xrightarrow{\measuredangle}$ faces ($\# \measuredangle \frac{\pi}{2}$ per face)
- faces $\xrightarrow{\measuredangle}$ neighbouring faces ($\#$ bends toward the neighbour)

# Flow Network for Bend Minimization

(H1) $H(G)$ corresponds to $F$, $f_0$.

(H2) For each **edge** $\{u, v\}$ shared by faces $f$ and $g$, sequence $\delta_1$ is reversed and inverted $\delta_2$.

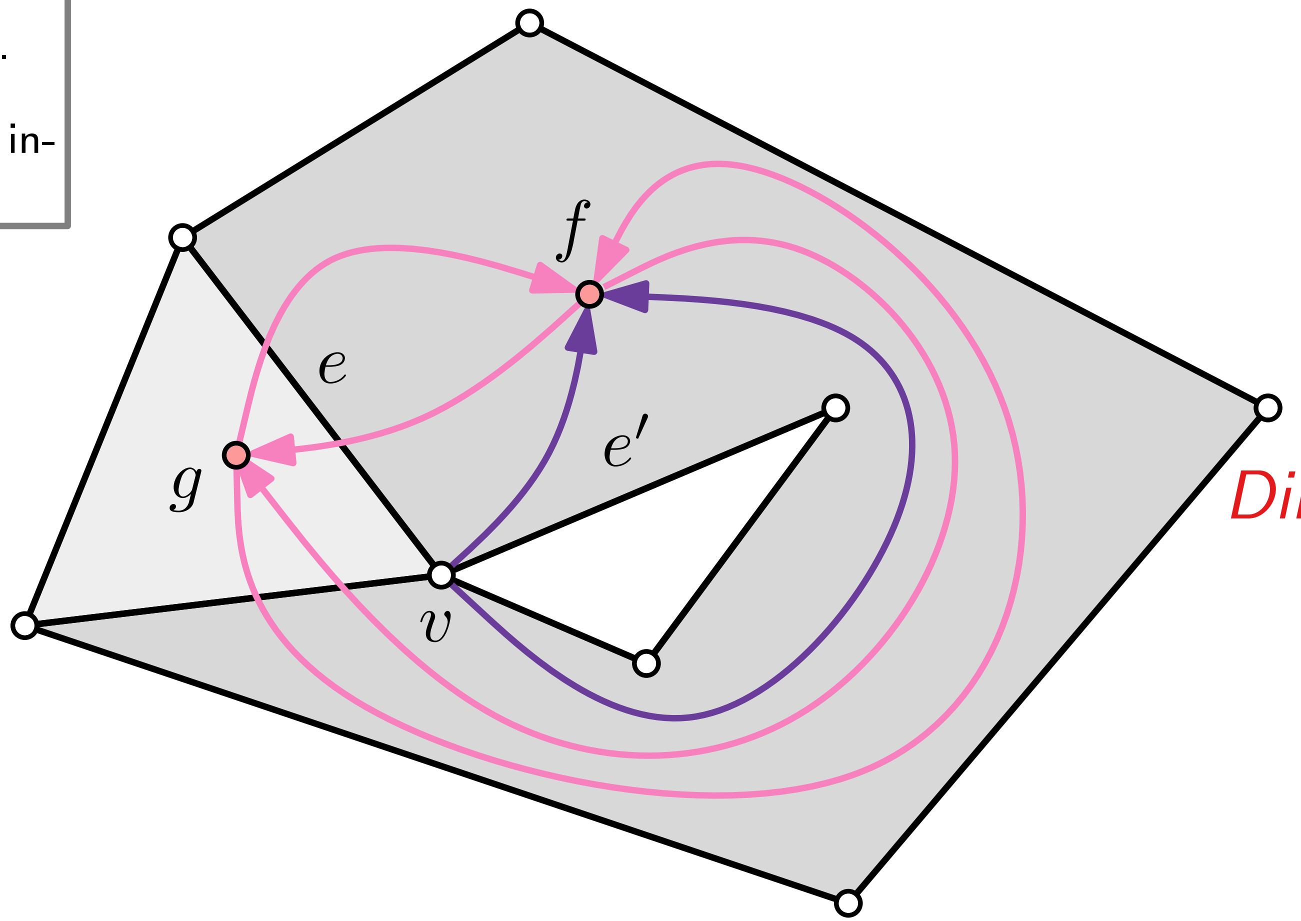
(H3) For each **face** $f$ it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

(H4) For each **vertex** $v$ the sum of incident angles is $2\pi$.

Define flow network $N(G) = ((V \cup F, E); b; \ell; u; \text{cost})$:

- $E = \{(v, f)_{ee'} \in V \times F \mid v \text{ between edges } e, e' \text{ of } \partial f\} \cup \{(f, g)_e \in F \times F \mid f, g \text{ have common edge } e\}$




*Directed multigraph!*

# Flow Network for Bend Minimization

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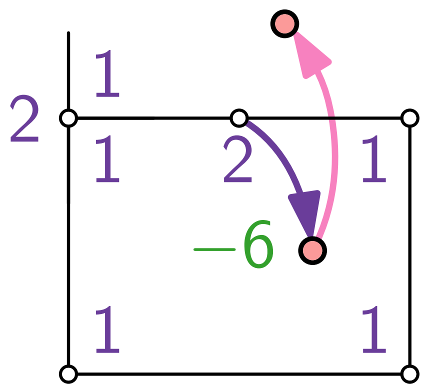
- $b(v) = 4 \quad \forall v \in V$
- $b(f) = -2\deg_G(f) + \begin{cases} -4 & \text{if } f = f_0, \\ +4 & \text{otherwise} \end{cases}$

$\Rightarrow \sum_w b(w) = 0$ (Euler)

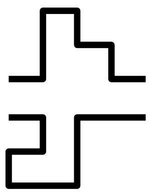



$\forall (v, f) \in E, v \in V, f \in F$ $\qquad \ell(v, f) := 1 \leq X(v, f) \leq 4 =: u(v, f)$

$\text{cost}(v, f) = 0$

$\forall (f, g) \in E, f, g \in F$ $\qquad \ell(f, g) := 0 \leq X(f, g) \leq \infty =: u(f, g)$

$\text{cost}(f, g) = 1$



We model only the *number* of bends. Why is it enough?

# Flow Network for Bend Minimization

(H1) $H(G)$ corresponds to $F$, $f_0$.

(H2) For each **edge** $\{u, v\}$ shared by faces $f$ and $g$, sequence $\delta_1$ is reversed and inverted $\delta_2$.

(H3) For each **face** $f$ it holds that:

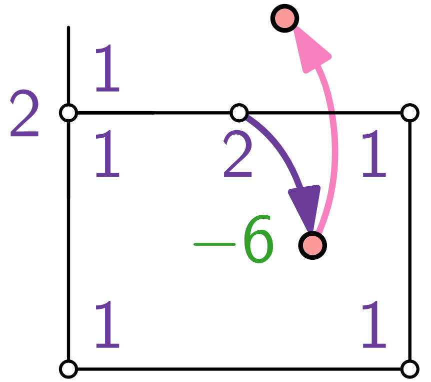
$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

(H4) For each **vertex** $v$ the sum of incident angles is $2\pi$.

Define flow network $N(G) = ((V \cup F, E); b; \ell; u; \text{cost})$:

- $E = \{(v, f)_{ee'} \in V \times F \mid v \text{ between edges } e, e' \text{ of } \partial f\} \cup \{(f, g)_e \in F \times F \mid f, g \text{ have common edge } e\}$
- $b(v) = 4 \quad \forall v \in V$
- $b(f) = -2\deg_G(f) + \begin{cases} -4 & \text{if } f = f_0, \\ +4 & \text{otherwise} \end{cases}$

$\Rightarrow \sum_w b(w) = 0$ (Euler)

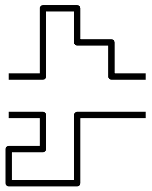



$\forall (v, f) \in E, v \in V, f \in F \qquad \ell(v, f) := 1 \leq \boxed{X(v, f)} \leq 4 =: u(v, f)$

$\text{cost}(v, f) = 0$

$\forall (f, g) \in E, f, g \in F \qquad \ell(f, g) := 0 \leq \boxed{X(f, g)} \leq \infty =: u(f, g)$

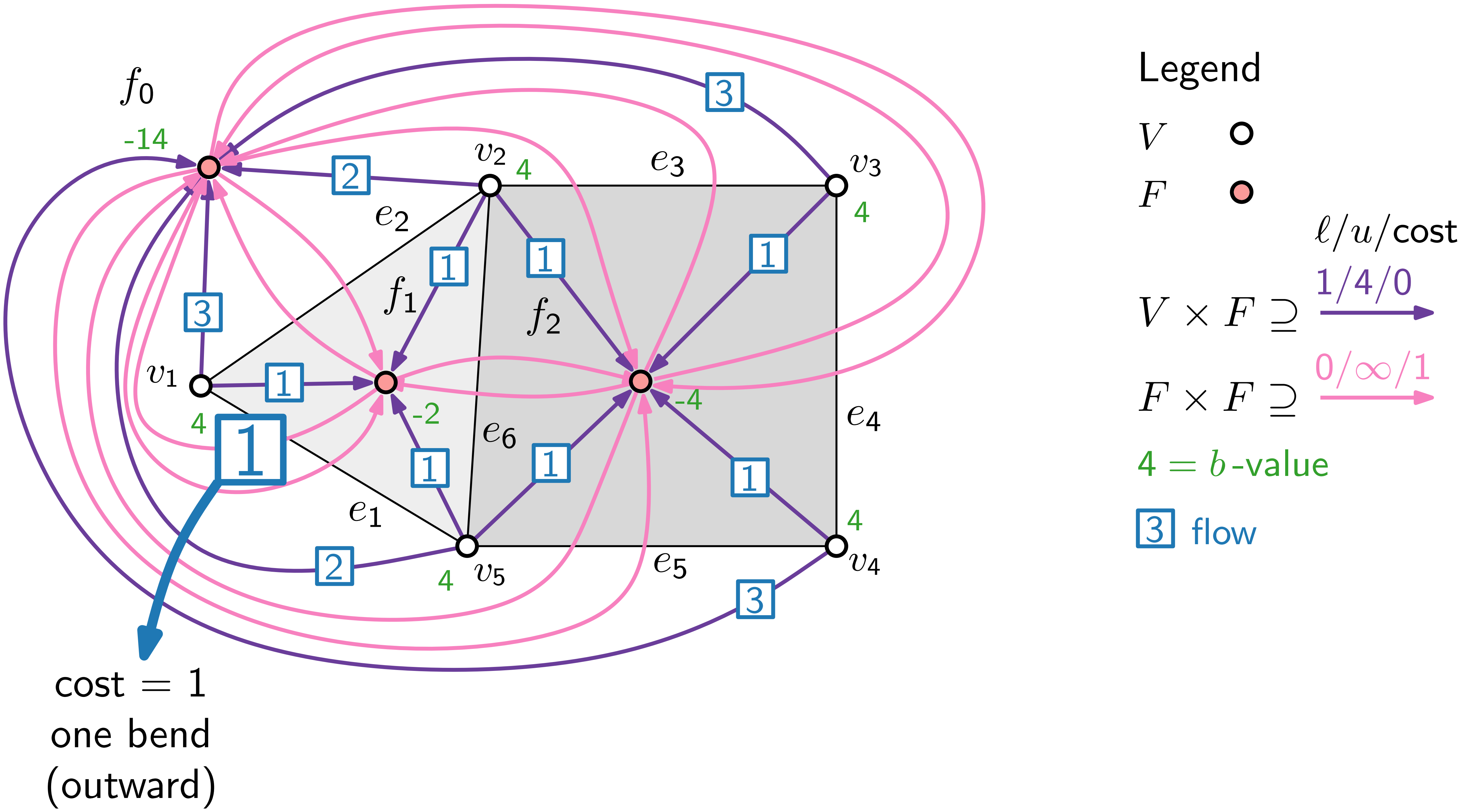
$\text{cost}(f, g) = 1$



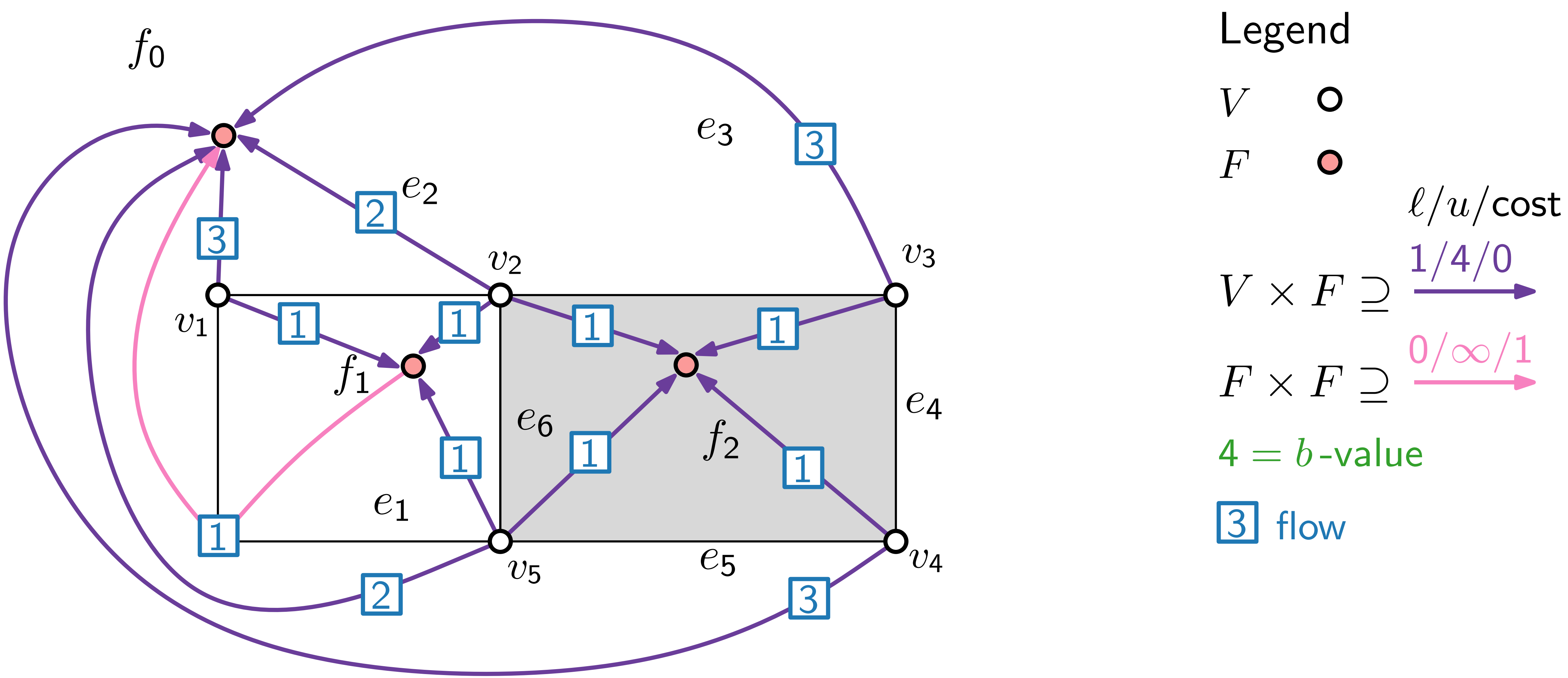
We model only the *number* of bends.
Why is it enough?

→ Exercise

# Flow Network Example



# Flow Network Example



# Bend Minimization – Result

**Theorem.** [Tamassia '87]
A plane graph $(G, F, f_0)$ has a valid orthogonal representation $H(G)$ with $k$ bends iff the flow network $N(G)$ has a valid flow $X$ with cost $k$.

(H1) $H(G)$ corresponds to $F$, $f_0$.

(H2) For each **edge** $\{u, v\}$ shared by faces $f$ and $g$, sequence $\delta_1$ is reversed and inverted $\delta_2$.

(H3) For each **face** $f$ it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

(H4) For each **vertex** $v$ the sum of incident angles is $2\pi$.

**Proof.**

$\Leftarrow$ Given valid flow $X$ in $N(G)$ with cost $k$.
Construct orthogonal representation $H(G)$ with $k$ bends.

- Transform from flow to orthogonal description.
- Show properties (H1)–(H4).

(H1) H(G) matches $F, f_0$ ✓

(H2) Bend order inverted and reversed on opposite sides ✓

(H3) Angle sum of $f = \pm 4$ ✓ Exercise.

(H4) Total angle at each vertex $= 2\pi$ ✓

# Bend Minimization – Result

**Theorem.** [Tamassia '87]
A plane graph $(G, F, f_0)$ has a valid orthogonal representation $H(G)$ with $k$ bends iff the flow network $N(G)$ has a valid flow $X$ with cost $k$.

- $b(v) = 4 \quad \forall v \in V$
- $b(f) = -2\,\deg_G(f) + \begin{cases} -4 & \text{if } f = f_0, \\ +4 & \text{otherwise} \end{cases}$
- $\ell(v, f) := 1 \leq X(v, f) \leq 4 =: u(v, f)$
  $\text{cost}(v, f) = 0$
  $\ell(f, g) := 0 \leq X(f, g) \leq \infty =: u(f, g)$
  $\text{cost}(f, g) = 1$

**Proof.**

$\Rightarrow$ Given an orthogonal representation $H(G)$ with $k$ bends. Construct valid flow $X$ in $N(G)$ with cost $k$.

- Define flow $X\colon E \to \mathbb{R}_0^+$.
- Show that $X$ is a valid flow and has cost $k$.

(N1) $X(vf) = 1/2/3/4$ ✓

(N2) $X(fg) = |\delta_{fg}|_0$, $(e, \delta_{fg}, x)$ describes $e \stackrel{*}{=} fg$ from $f$ ✓

(N3) capacities, deficit/demand coverage ✓

(N4) cost $= k$ ✓

# Bend Minimization – Remarks

- From Theorem follows that the combinatorial orthogonal bend minimization problem for plane graphs can be solved using an algorithm for the Min-Cost-Flow problem.

**Theorem.** [Garg & Tamassia 1996]
The minimum cost flow problem for planar graphs with bounded costs and vertex degrees can be solved in $O(n^{7/4}\sqrt{\log n})$ time.

**Theorem.** [Cornelsen & Karrenbauer 2011]
The minimum cost flow problem for planar graphs with bounded costs and faze sizes can be solved in $O(n^{3/2})$ time.

**Theorem.** [Garg & Tamassia 2001]
Bend Minimization without a given combinatorial embedding is an NP-hard problem.

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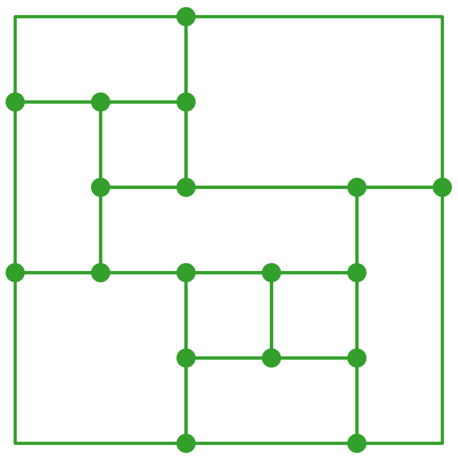
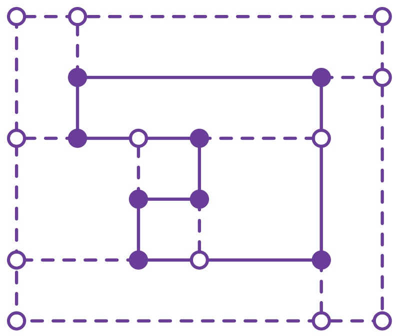
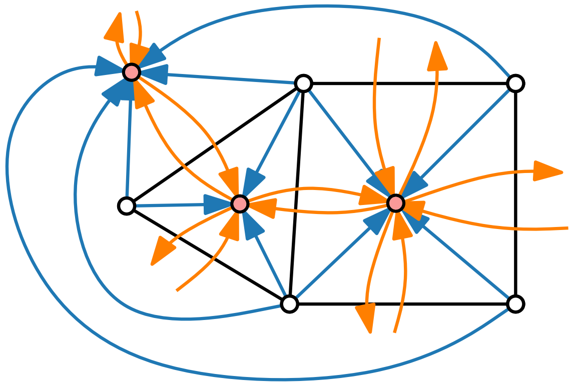
# Visualization of Graphs





## Lecture 5:
## Orthogonal Layouts

### Part IV:
### Area Minimization



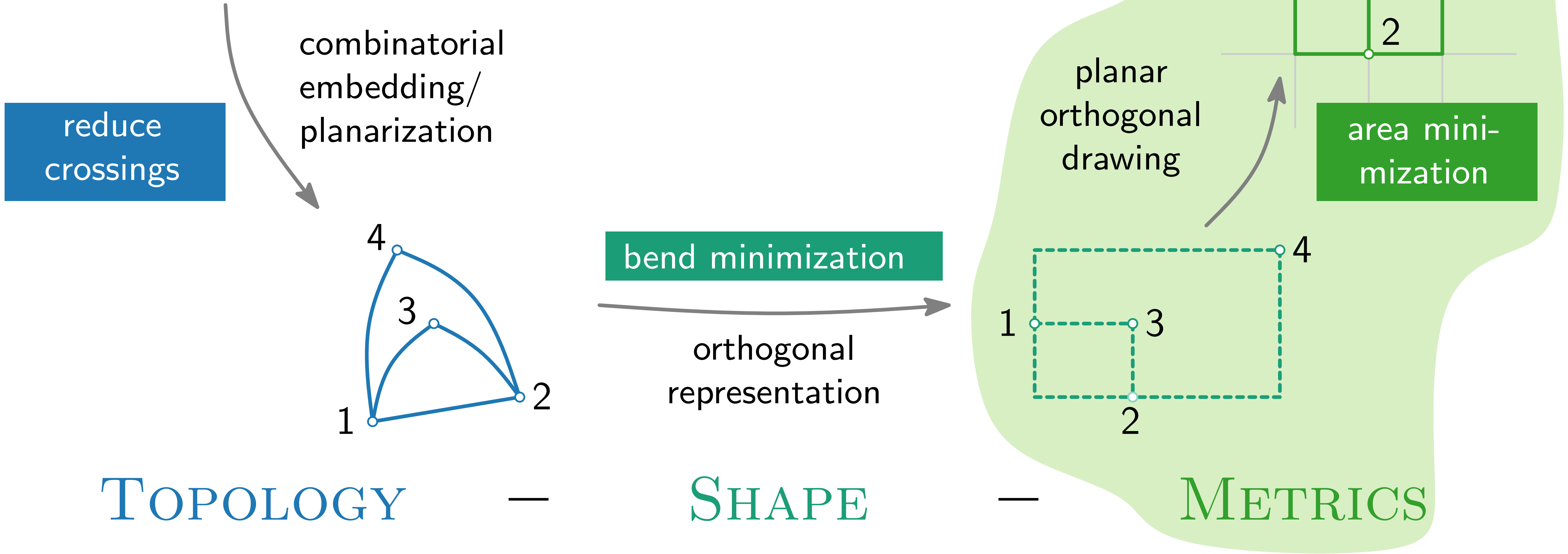
Jonathan Klawitter

# Topology – Shape – Metrics

Three-step approach:

[Tamassia 1987]

$$V = \{v_1, v_2, v_3, v_4\}$$
$$E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$$



# Compaction

**Compaction problem.**

Given:
- Plane graph $G = (V, E)$ with maximum degree 4
- Orthogonal representation $H(G)$

Find: Compact orthogonal layout of $G$ that realizes $H(G)$

**Special case.**

All faces are rectangles.

$\rightarrow$ Guarantees possible
- minimum total edge length
- minimum area

**Properties.**

- bends only on the outer face
- opposite sides of a face have the same length

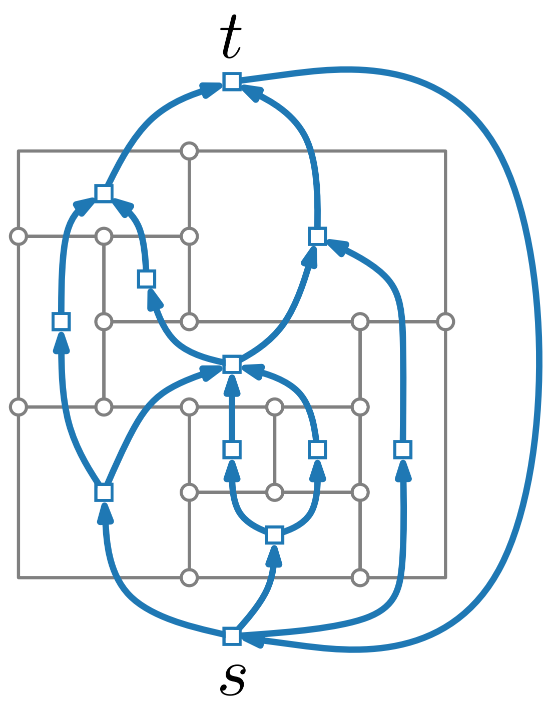
**Idea.**

- Formulate flow network for horizontal/vertical compaction

# Flow Network for Edge Length Assignment

**Definition.**
Flow Network $N_{\text{hor}} = ((W_{\text{hor}}, E_{\text{hor}}); b; \ell; u; \text{cost})$

- $W_{\text{hor}} = F \setminus \{f_0\} \cup \{s, t\}$ $\square$
- $E_{\text{hor}} = \{(f, g) \mid f, g$ share a *horizontal* segment and $f$ lies *below* $g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \quad \forall a \in E_{\text{hor}}$
- $u(a) = \infty \quad \forall a \in E_{\text{hor}}$
- $\text{cost}(a) = 1 \quad \forall a \in E_{\text{hor}}$
- $b(f) = 0 \quad \forall f \in W_{\text{hor}}$

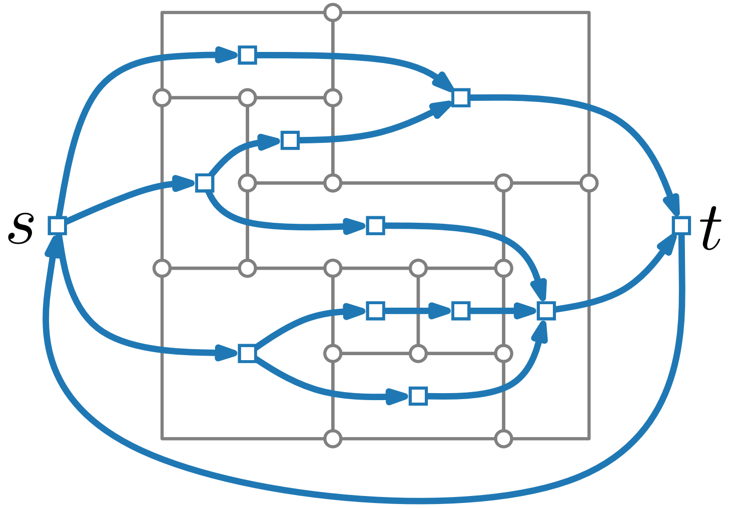


# Flow Network for Edge Length Assignment

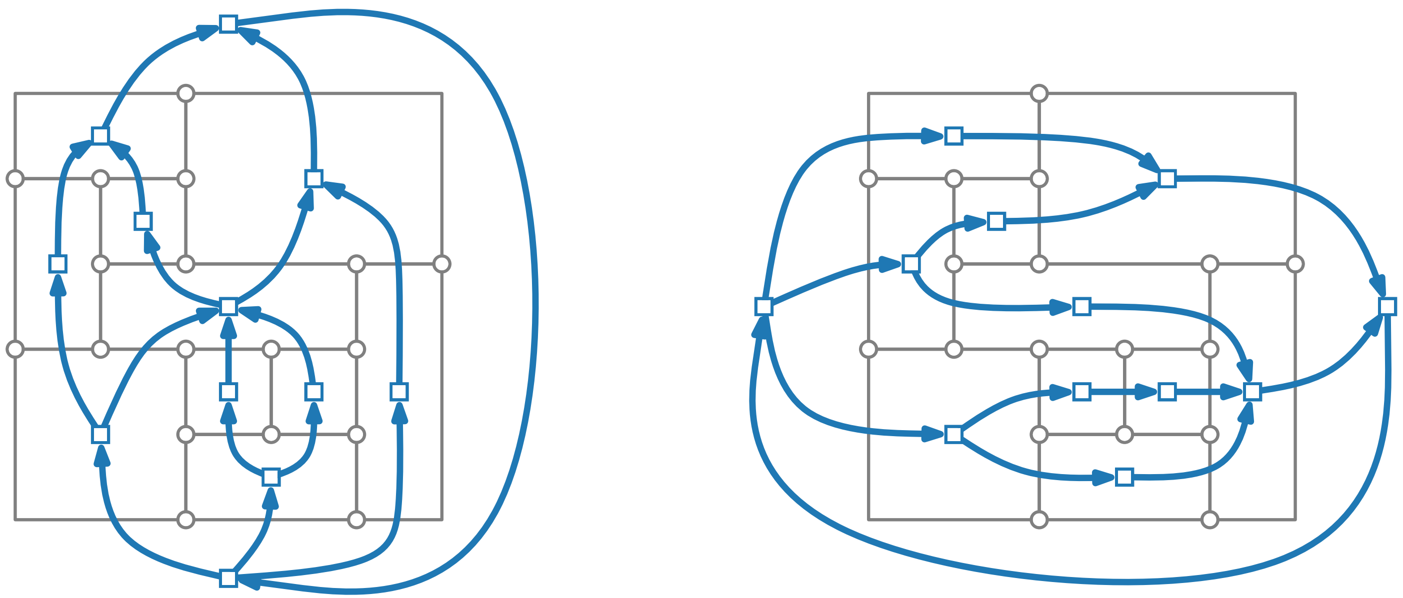
**Definition.**

Flow Network $N_{\text{ver}} = ((W_{\text{ver}}, E_{\text{ver}}); b; \ell; u; \text{cost})$

- $W_{\text{ver}} = F \setminus \{f_0\} \cup \{s, t\}$
- $E_{\text{ver}} = \{(f, g) \mid f, g$ share a *vertical* segment and $f$ lies to the *left* of $g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \quad \forall a \in E_{\text{ver}}$
- $u(a) = \infty \quad \forall a \in E_{\text{ver}}$
- $\text{cost}(a) = 1 \quad \forall a \in E_{\text{ver}}$
- $b(f) = 0 \quad \forall f \in W_{\text{ver}}$



# Compaction – Result



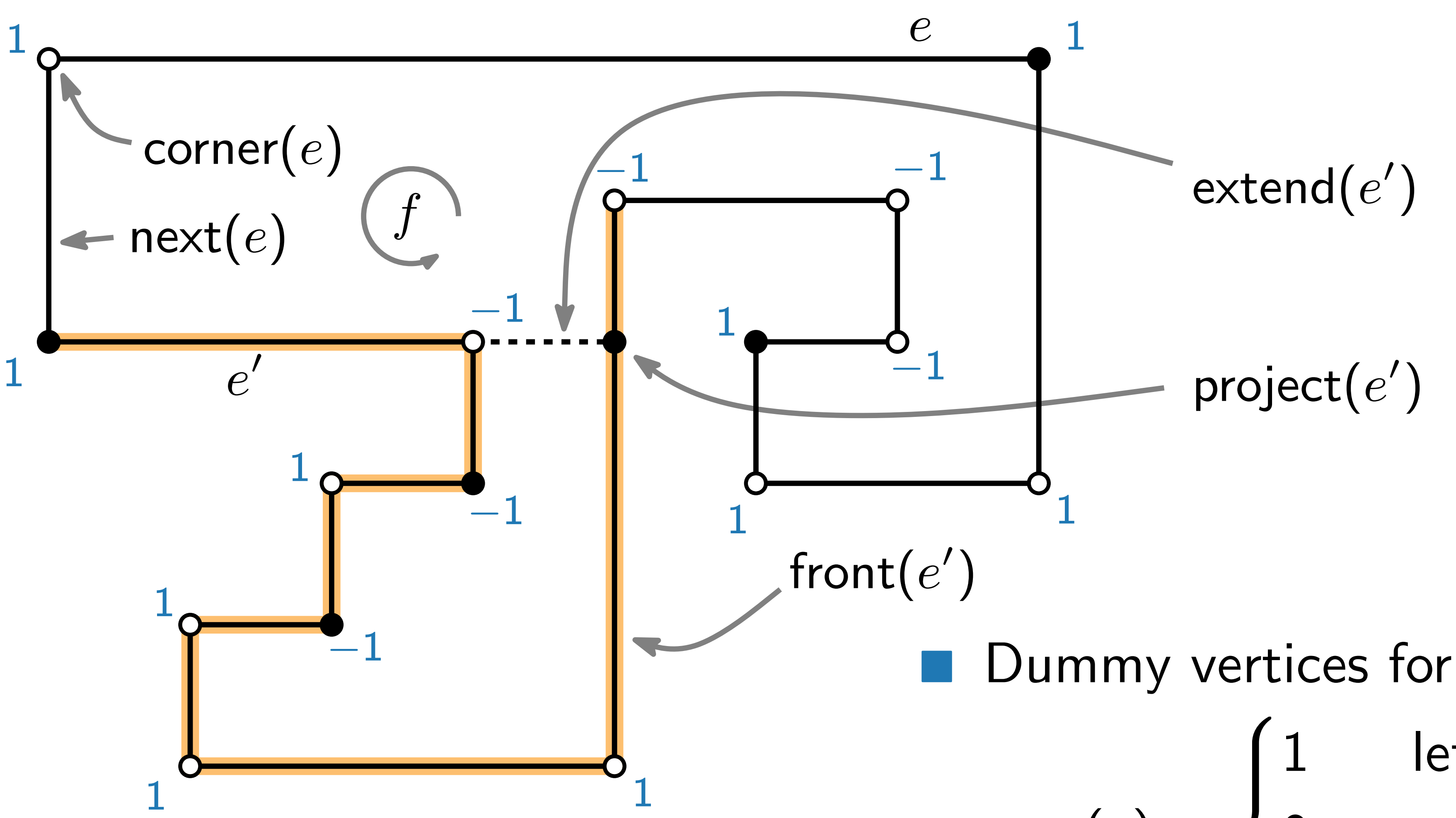
What if not all faces rectangular?

**Theorem.**
Valid min-cost-flows for $N_{\text{hor}}$ and $N_{\text{ver}}$ exists iff corresponding edge lenghts induce orthogonal drawing.

What values of the drawing represent the following?

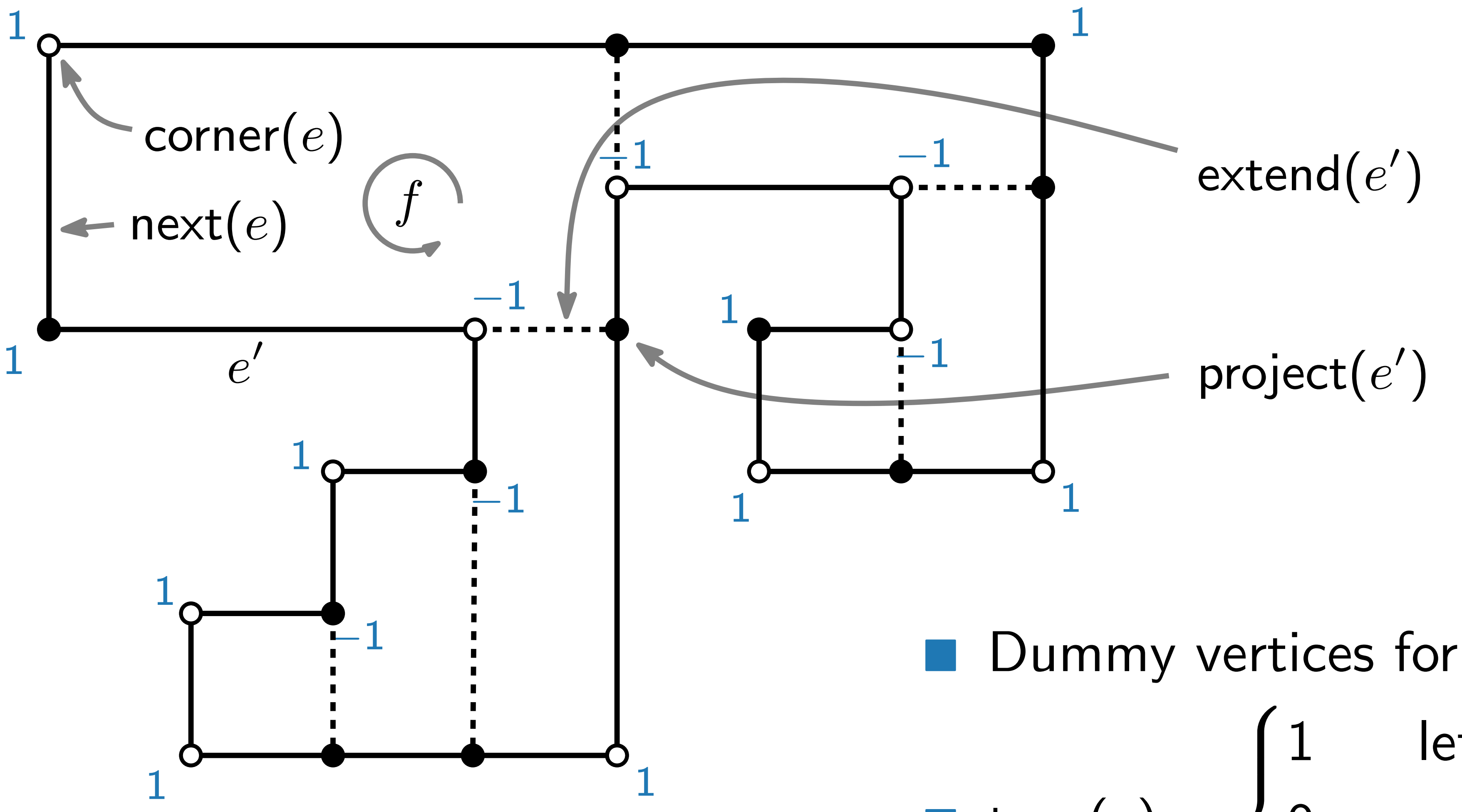
- $|X_{\text{hor}}(t,s)|$ and $|X_{\text{ver}}(t,s)|$? width and height of drawing
- $\sum_{e \in E_{\text{hor}}} X_{\text{hor}}(e) + \sum_{e \in E_{\text{ver}}} X_{\text{ver}}(e)$ total edge length

# Refinement of $(G, H)$ – Inner Face




- Dummy vertices for bends
- $\text{turn}(e) = \begin{cases} 1 & \text{left turn} \\ 0 & \text{no turn} \\ -1 & \text{right turn} \end{cases}$

# Refinement of $(G, H)$ – Inner Face

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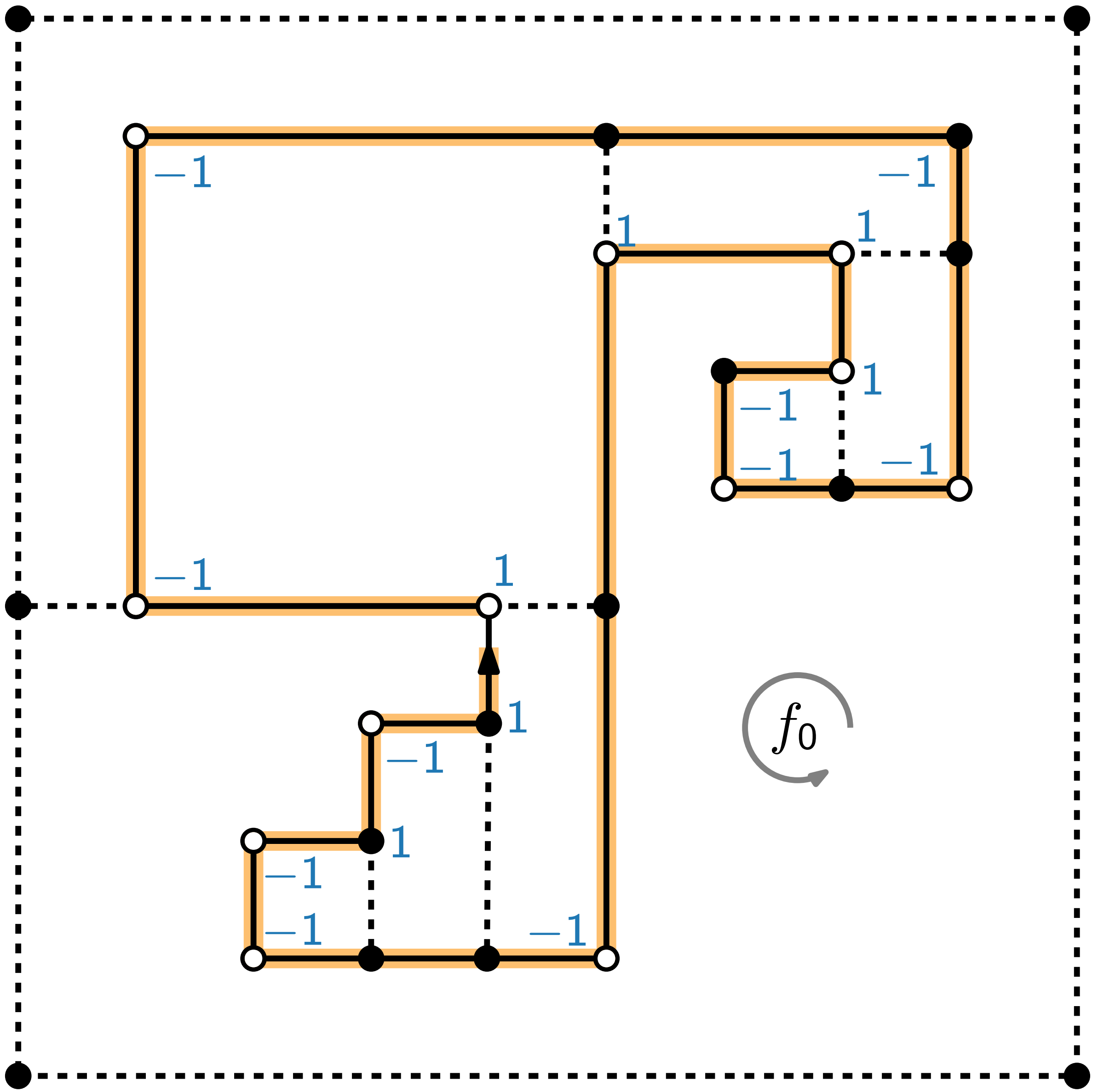
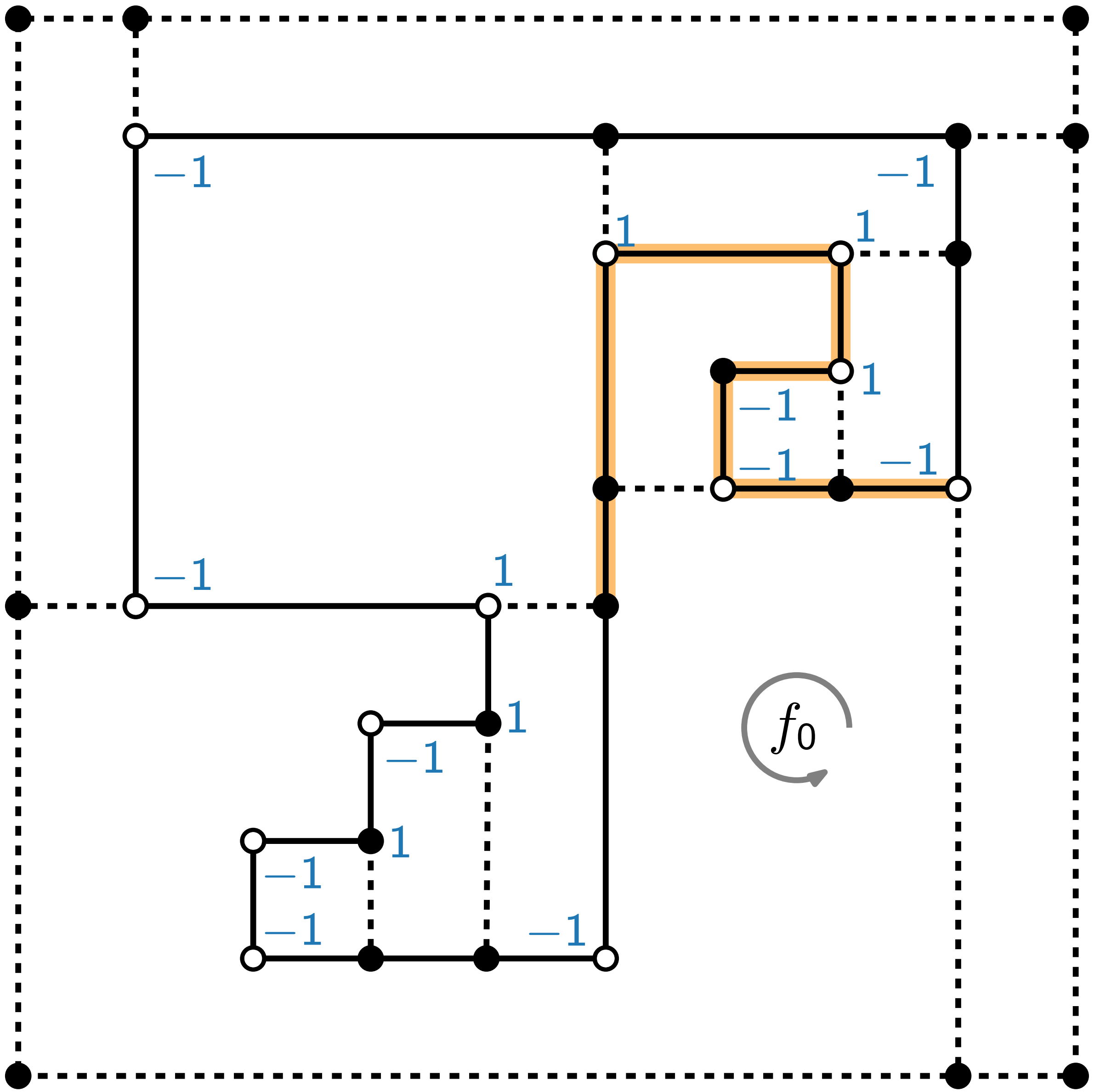


# Refinement of $(G, H)$ – Outer Face



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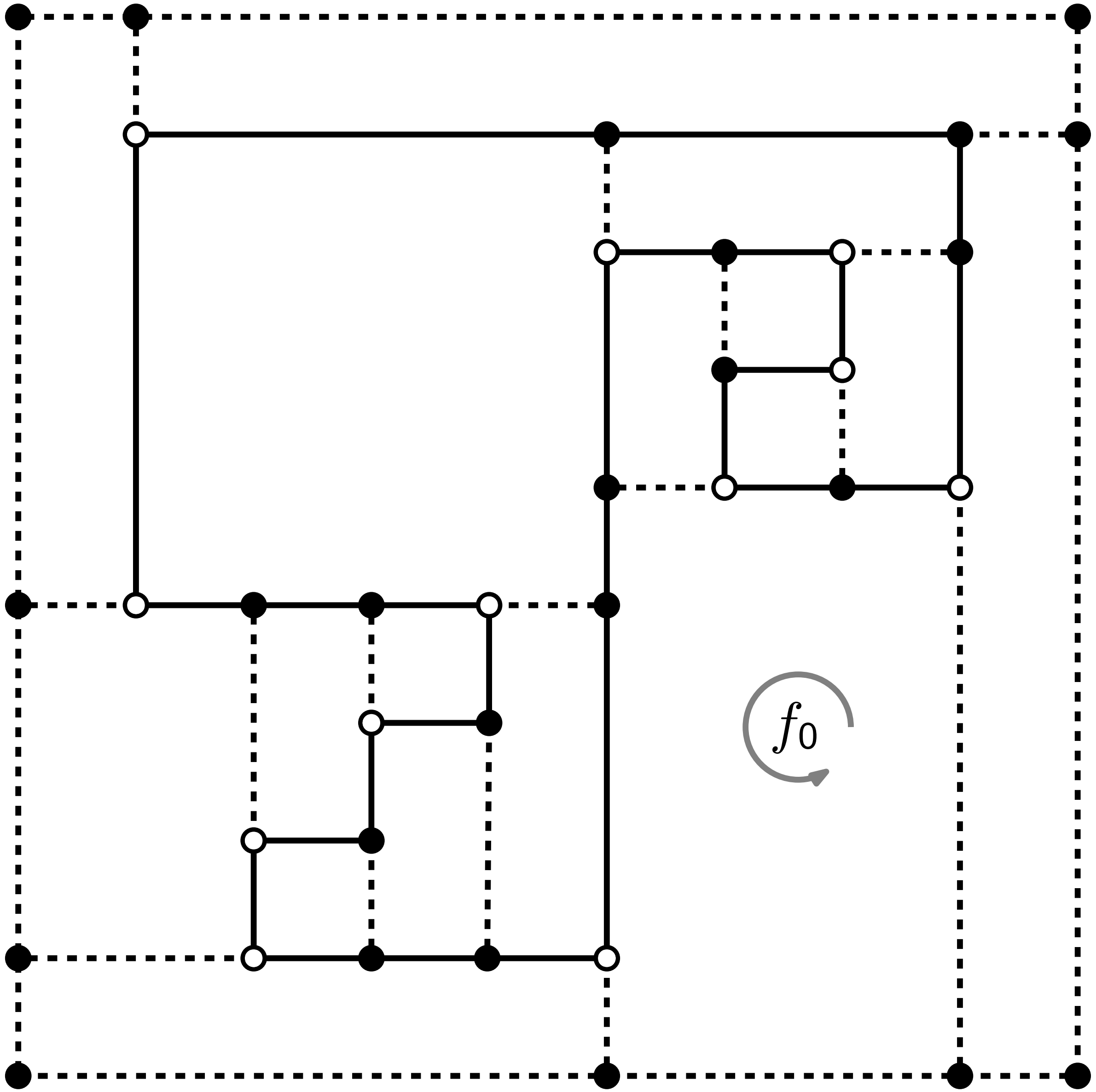
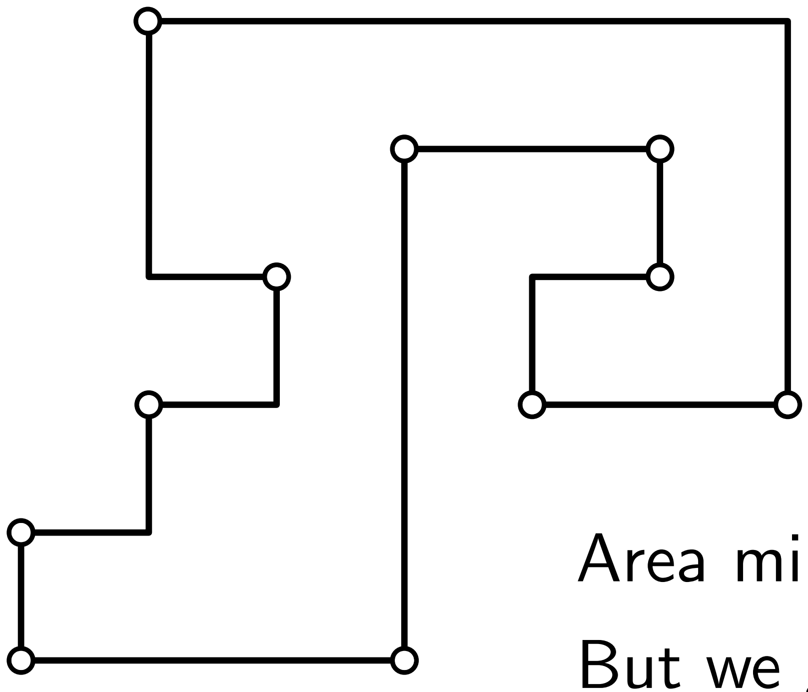


# Refinement of $(G, H)$ – Outer Face

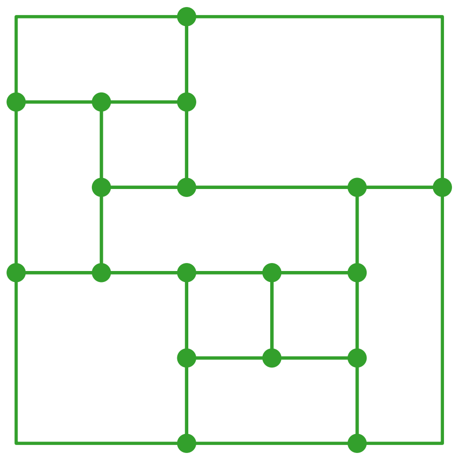


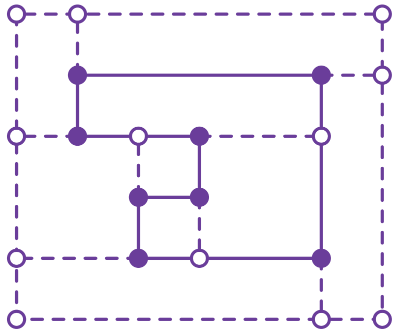
Area minimized? No!

But we get bound $O((n+b)^2)$ on the area.

**Theorem.** [Patrignani 2001]
Compaction for given orthogonal representation is in general NP-hard.

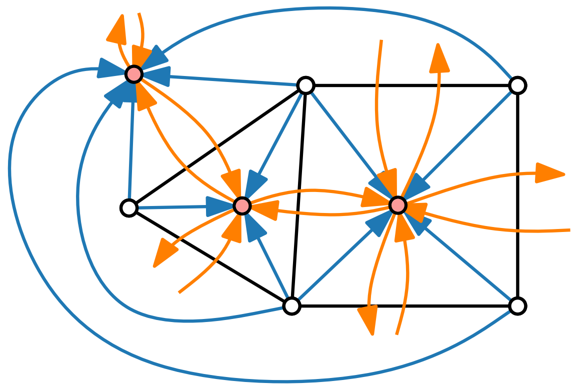
# Visualization of Graphs





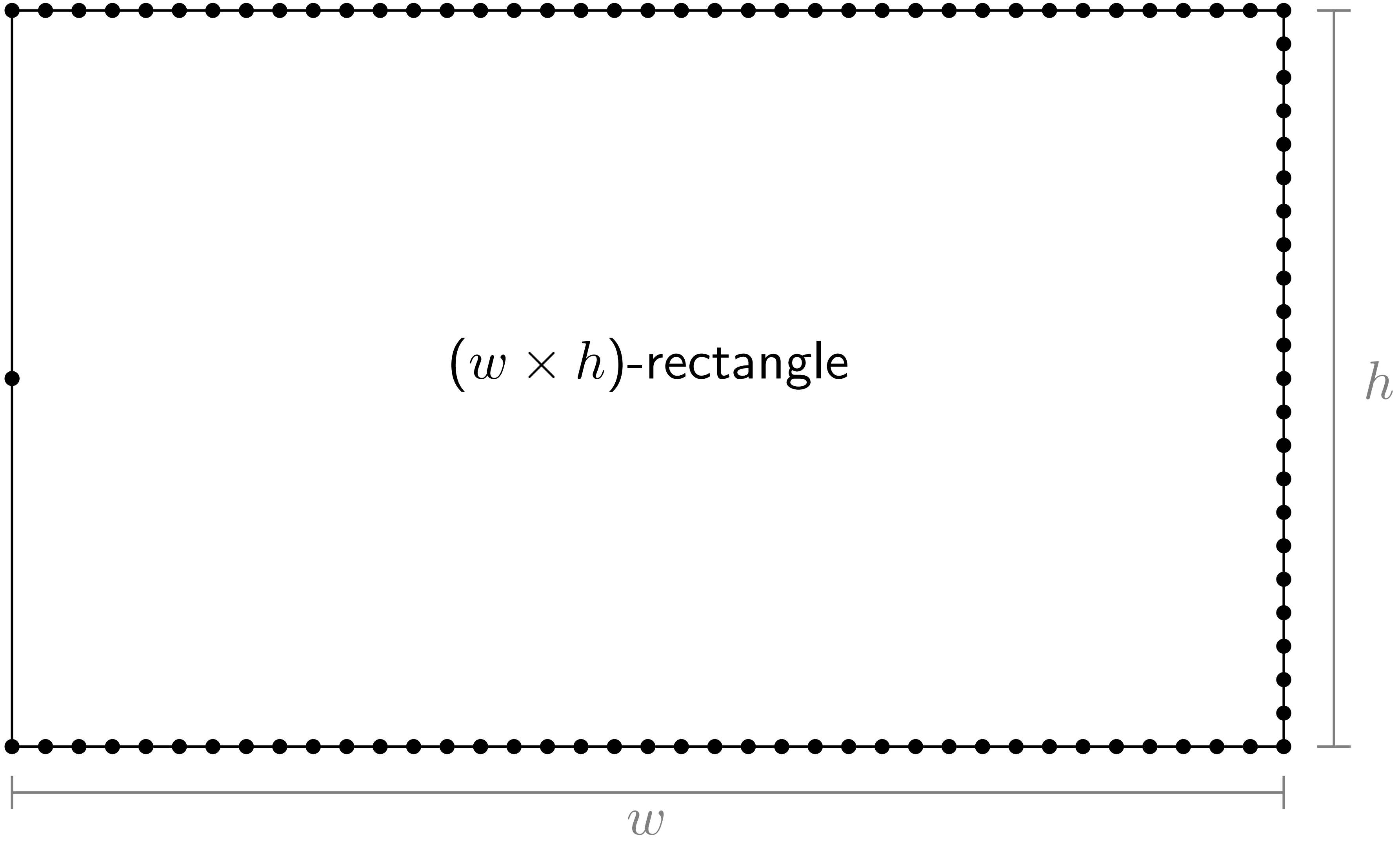
## Lecture 5: Orthogonal Layouts

### Part V: NP-hardness

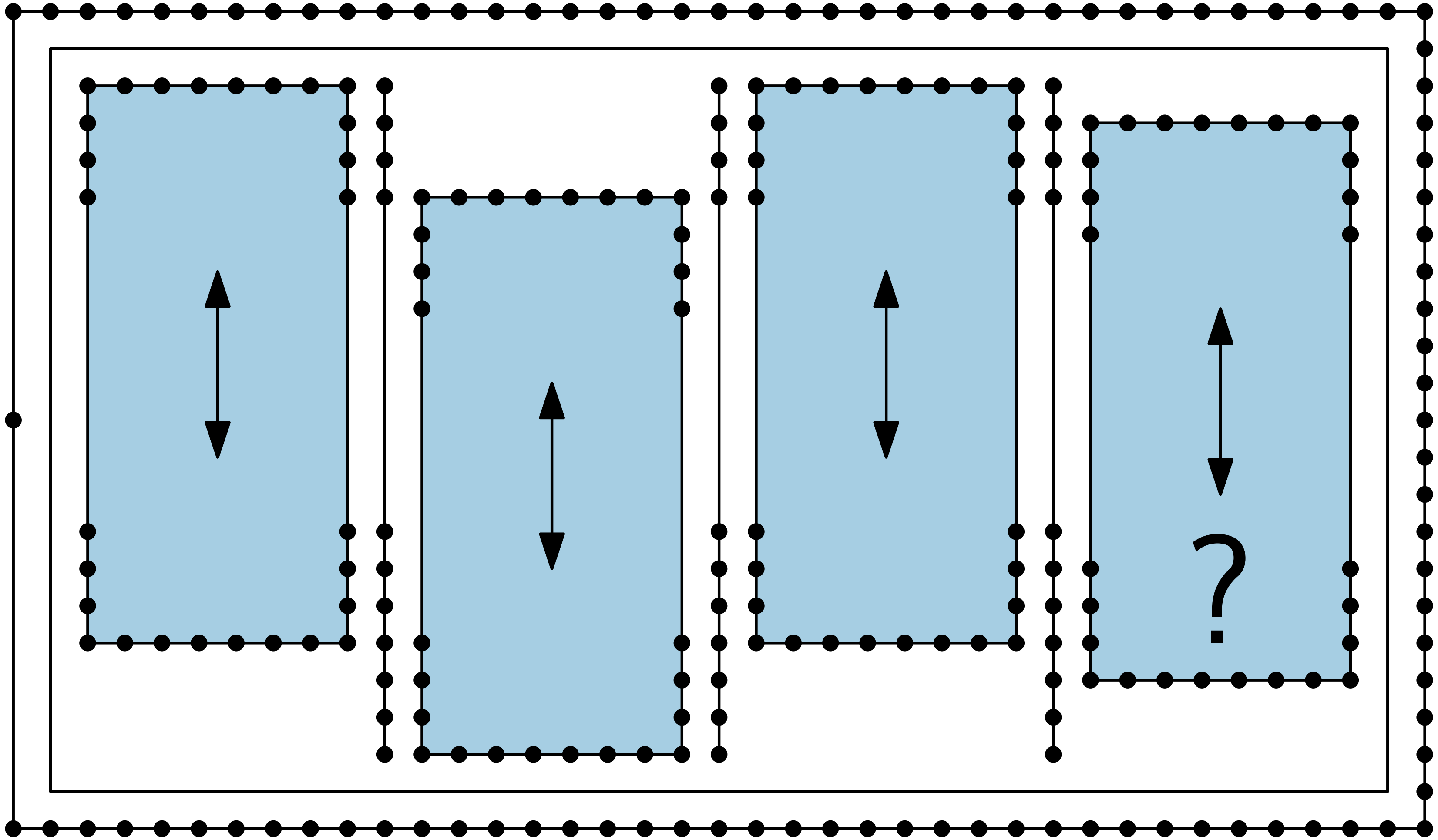


Jonathan Klawitter

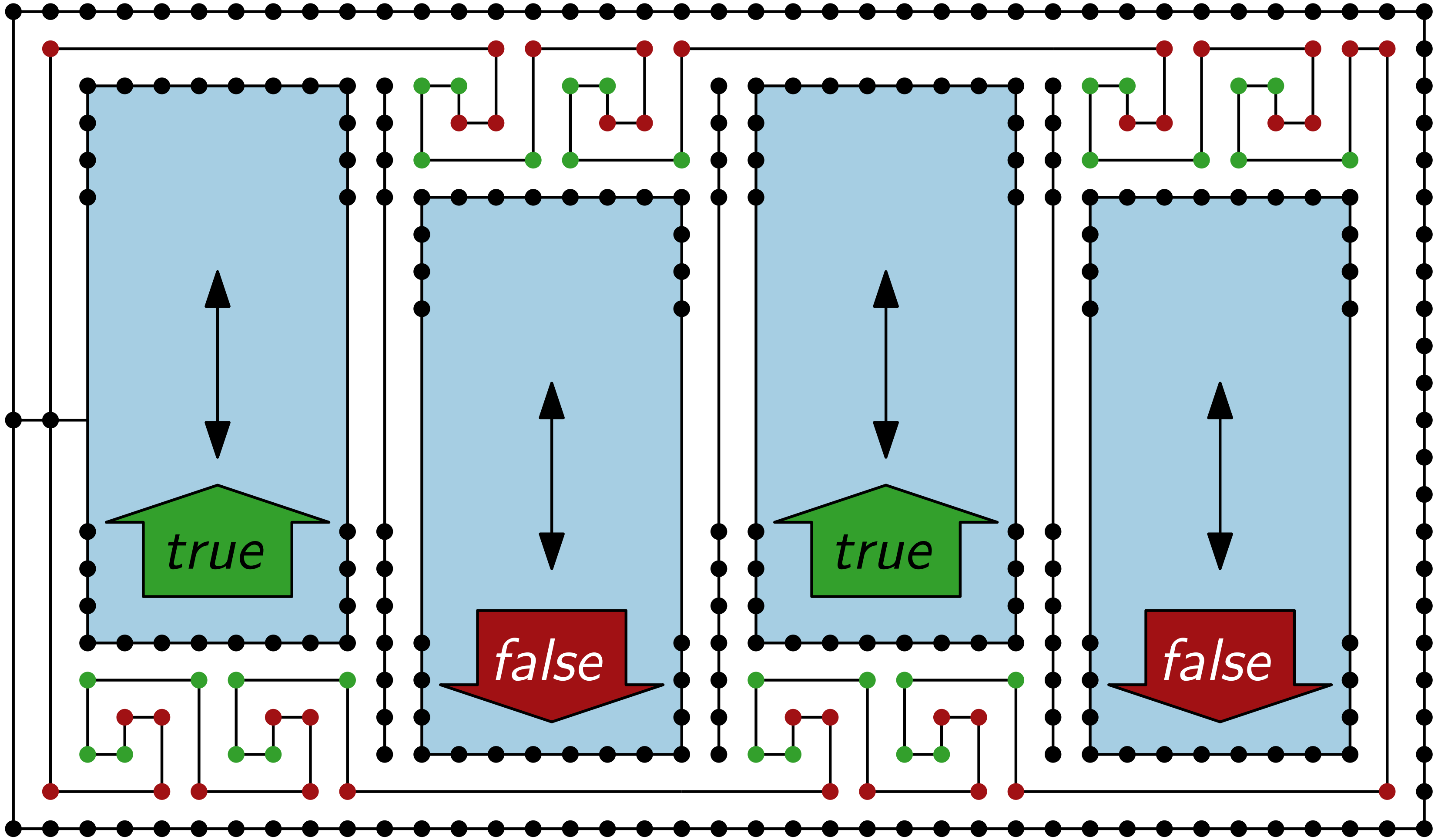
# Boundary, belt, and “piston” gadget



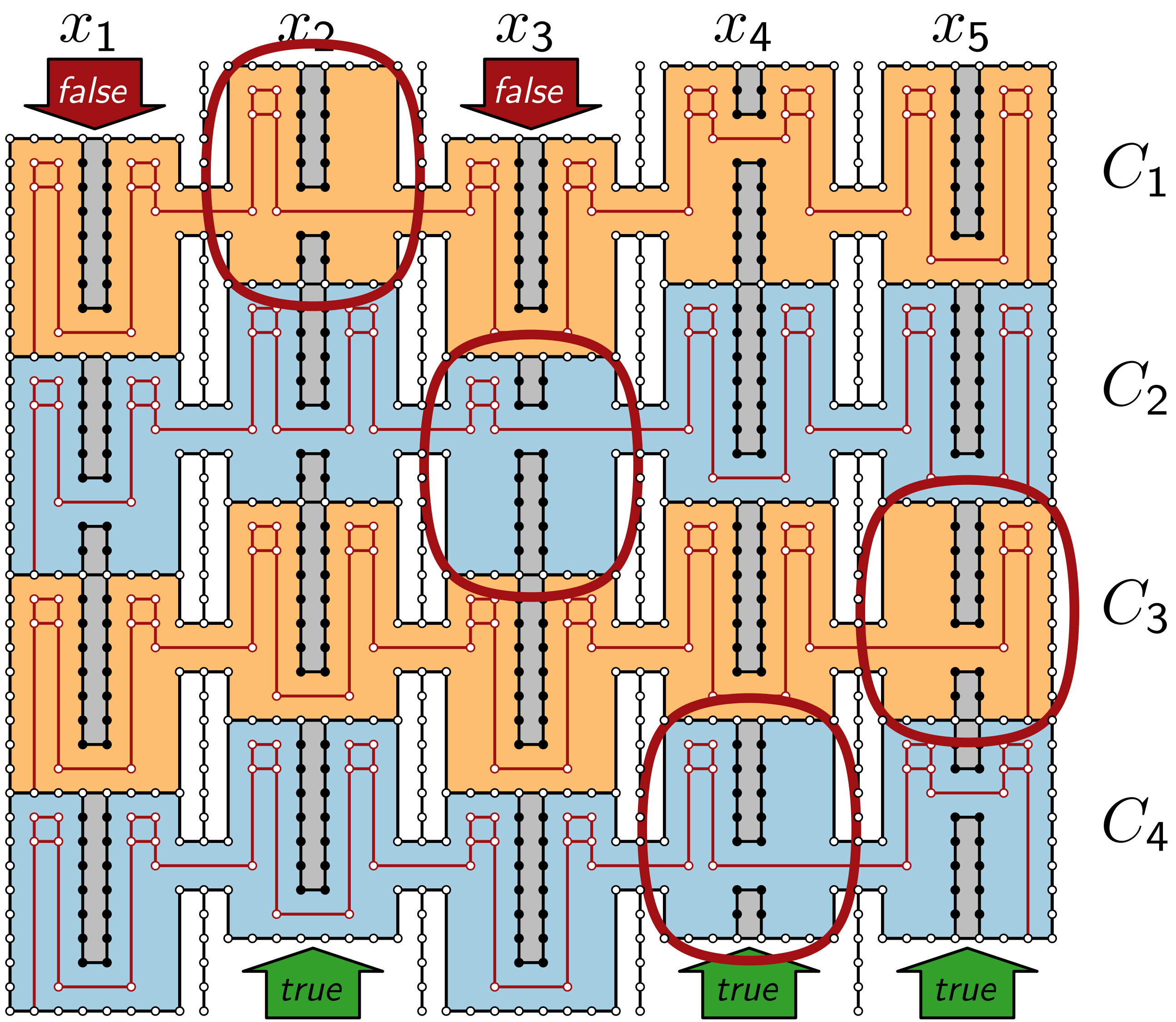
# Boundary, belt, and “piston” gadget



# Boundary, belt, and "piston" gadget



# Clause gadgets



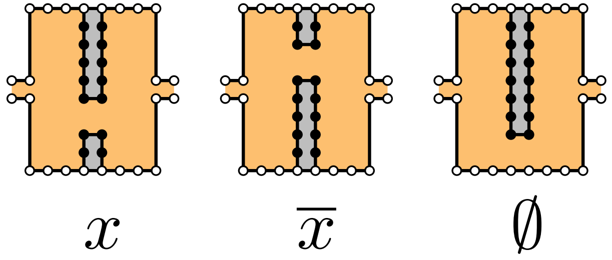

Example:

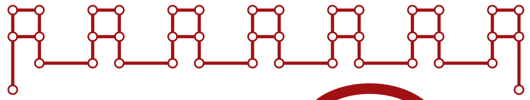
$C_1 = x_2 \vee \overline{x_4}$

$C_2 = x_1 \vee x_2 \vee \overline{x_3}$

$C_3 = x_5$

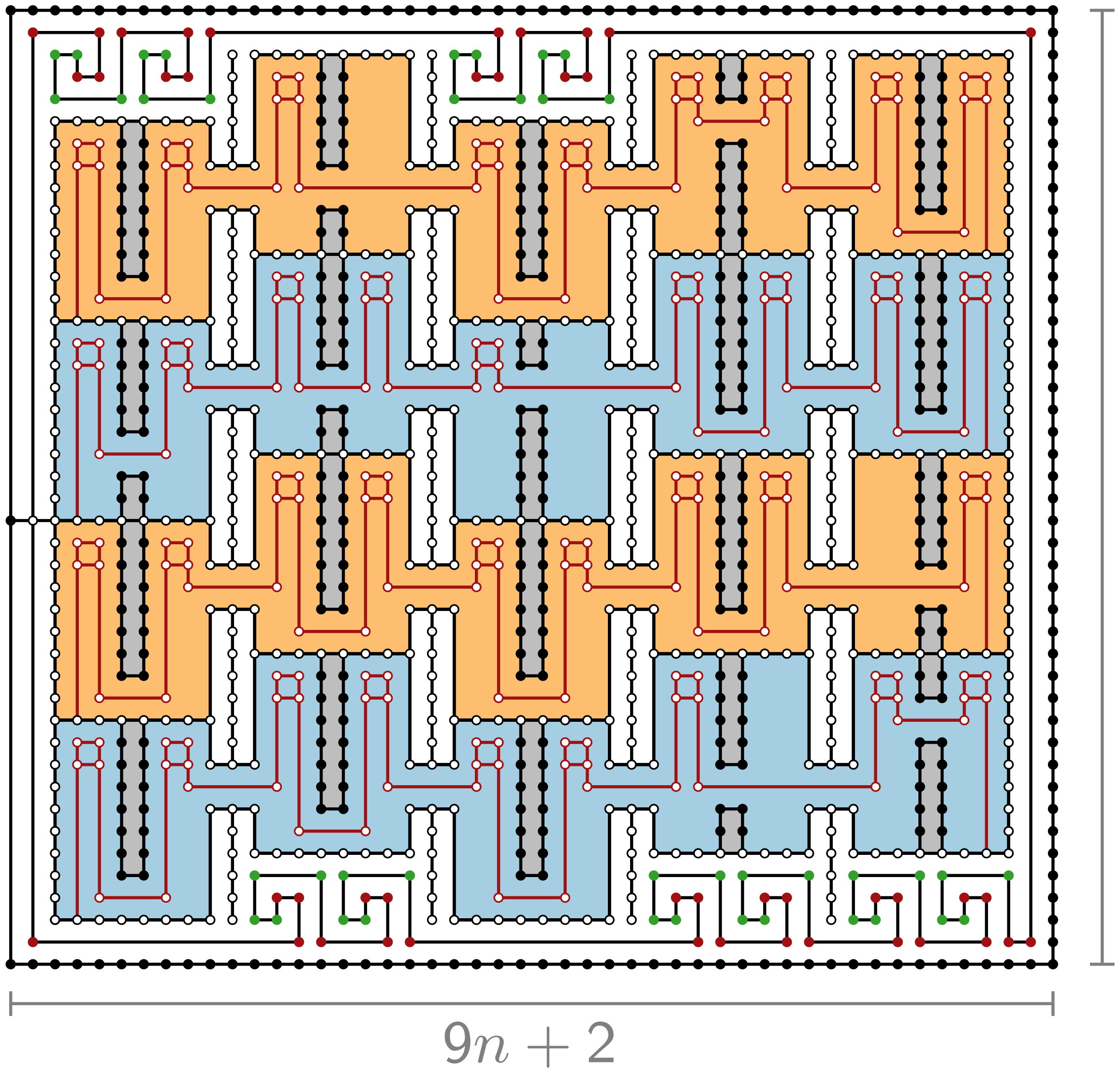
$C_4 = x_4 \vee \overline{x_5}$



insert $(2n - 1)$-chain through each clause

# Complete reduction




Pick
$K = (9n + 2) \cdot (9m + 7)$

Then:

$(G, H)$ has an area $K$ drawing
$\Leftrightarrow$
$\Phi$ satisfiable

$\square$

# Literature

- [GD Ch. 5] for detailed explanation
- [Tamassia 1987] "On embedding a graph in the grid with the minmum number of bends"
  original paper on flow for bend minimisation
- [Patrignani 2001] "On the complexity of orthogonal compaction"
  NP-hardness proof of compactification