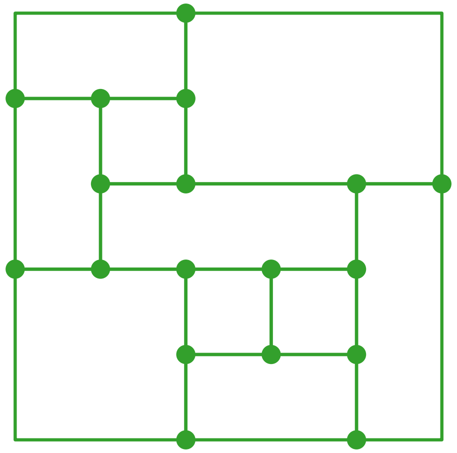
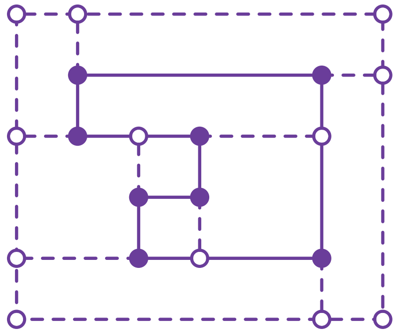
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

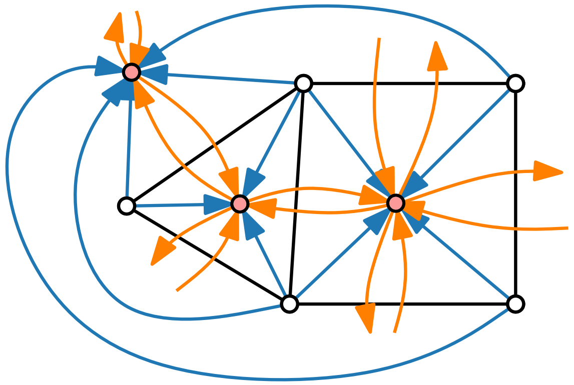
Visualization of Graphs



Lecture 5:
Orthogonal Layouts

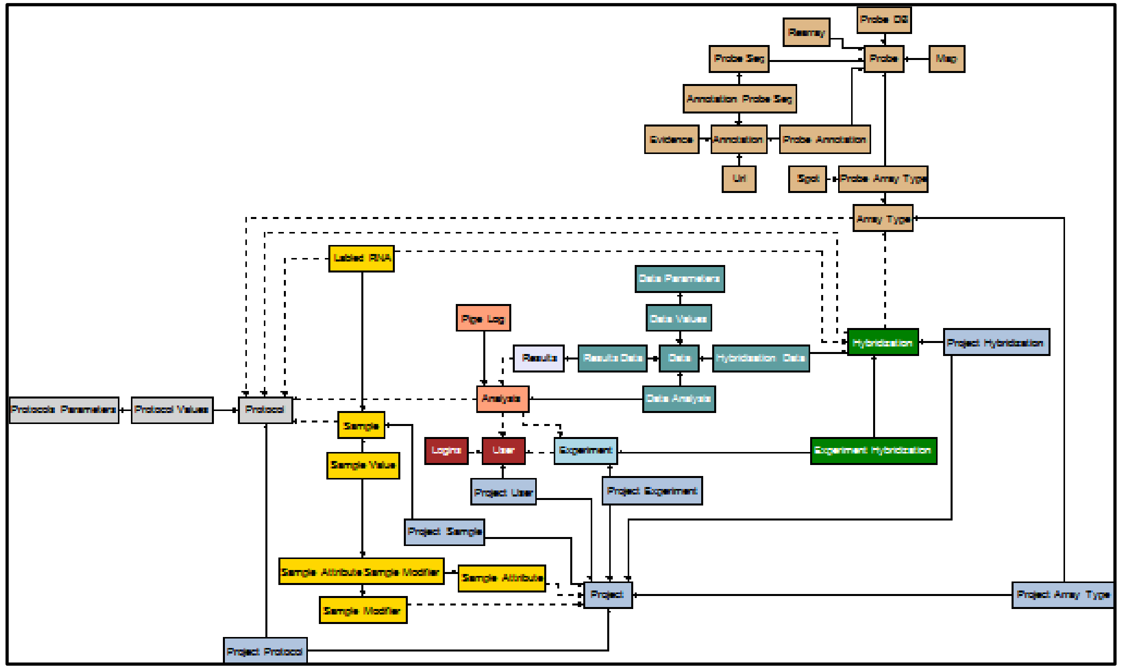


Part I:
Topolgy – Shape – Metric



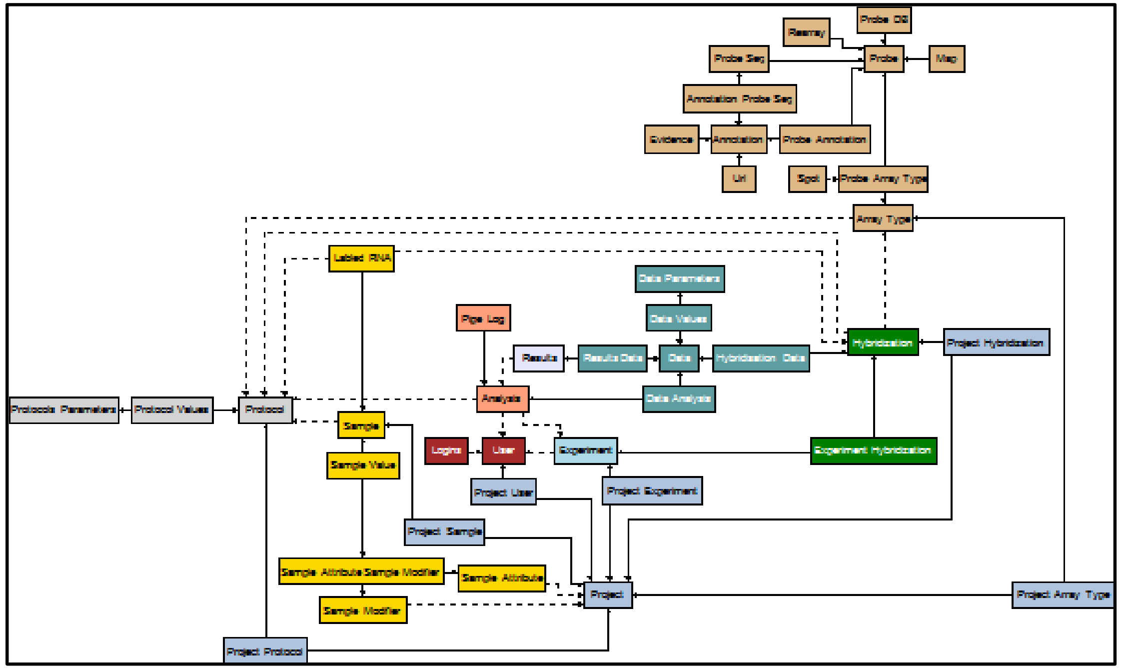
Jonathan Klawitter

Orthogonal Layout – Applications



ER diagram in OGDF

Orthogonal Layout – Applications



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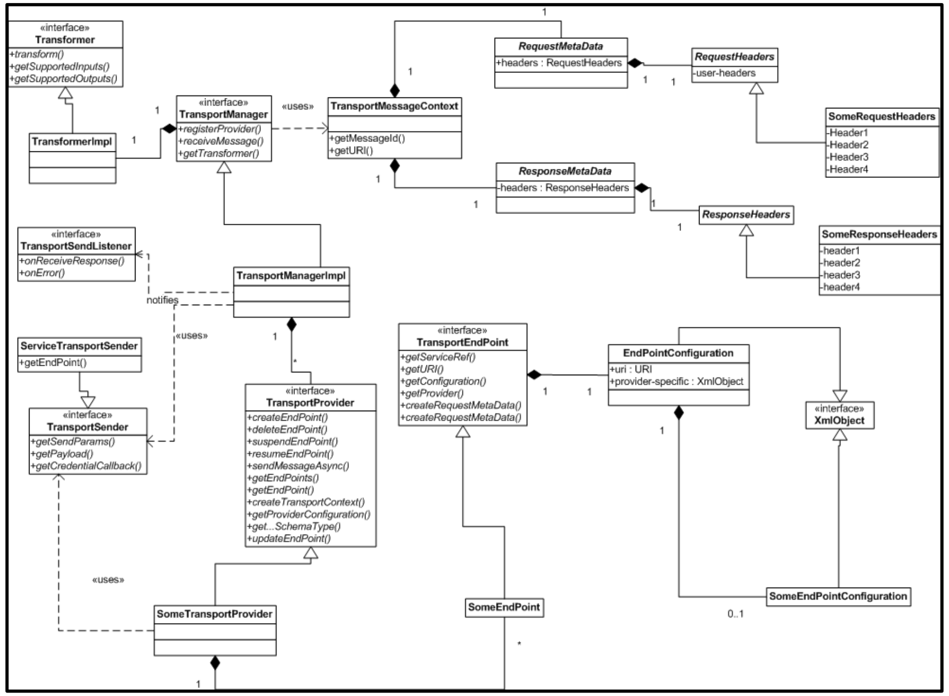
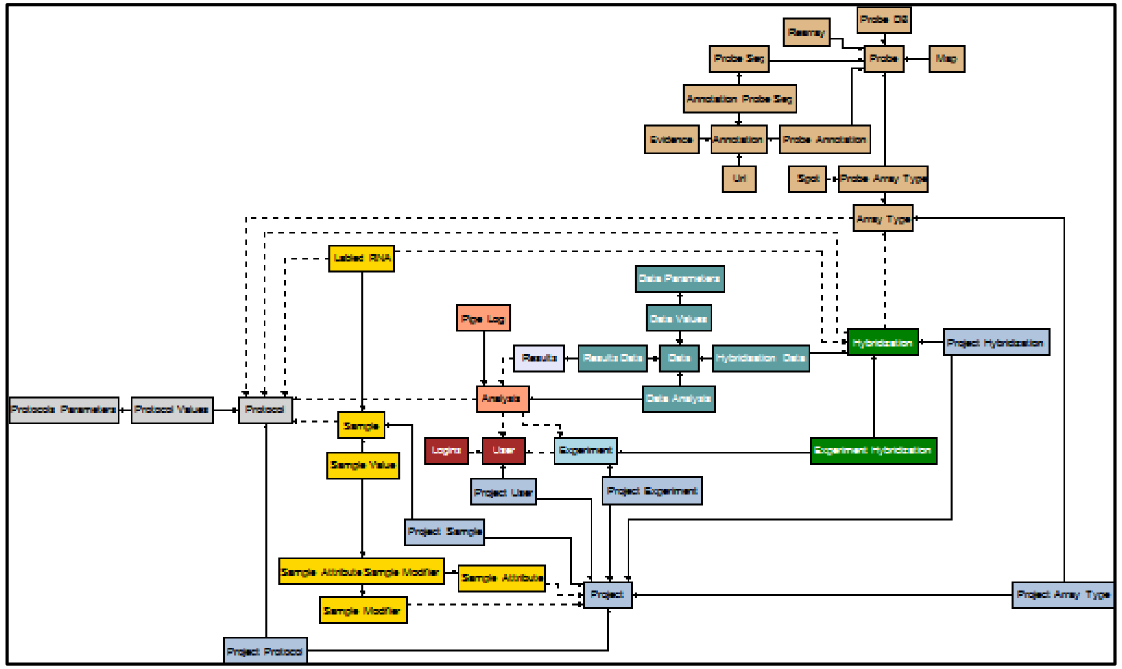


UML diagram by Oracle

Orthogonal Layout – Applications



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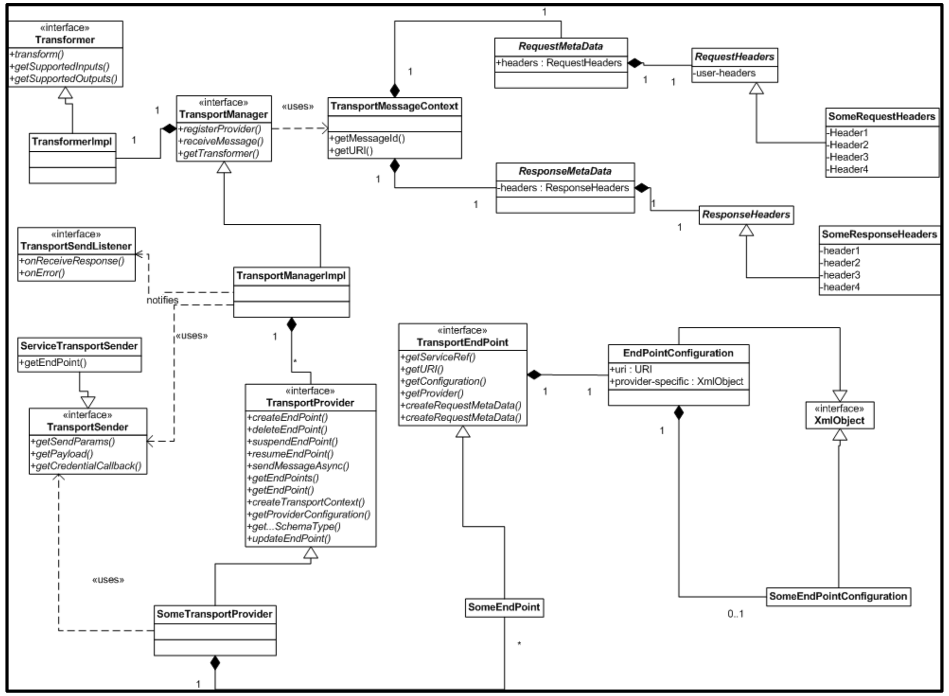

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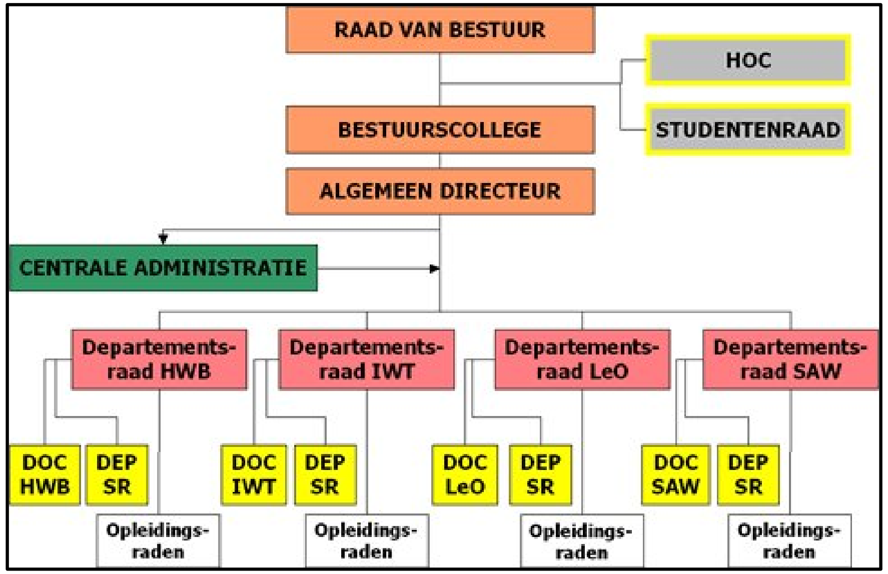

Organigram of HS Limburg

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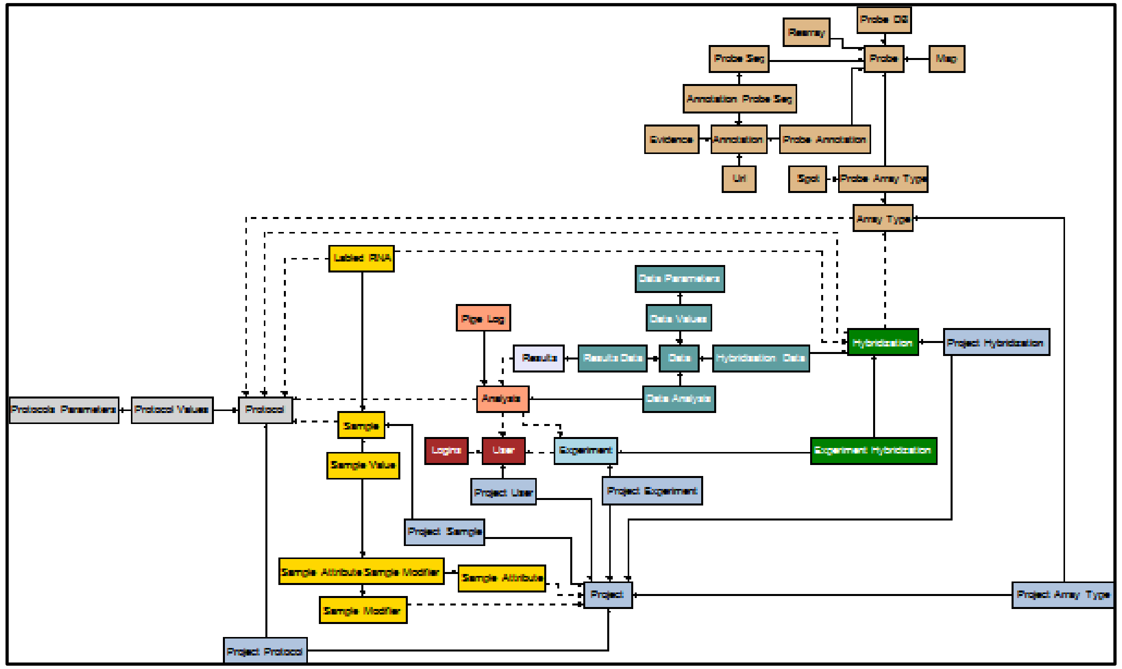

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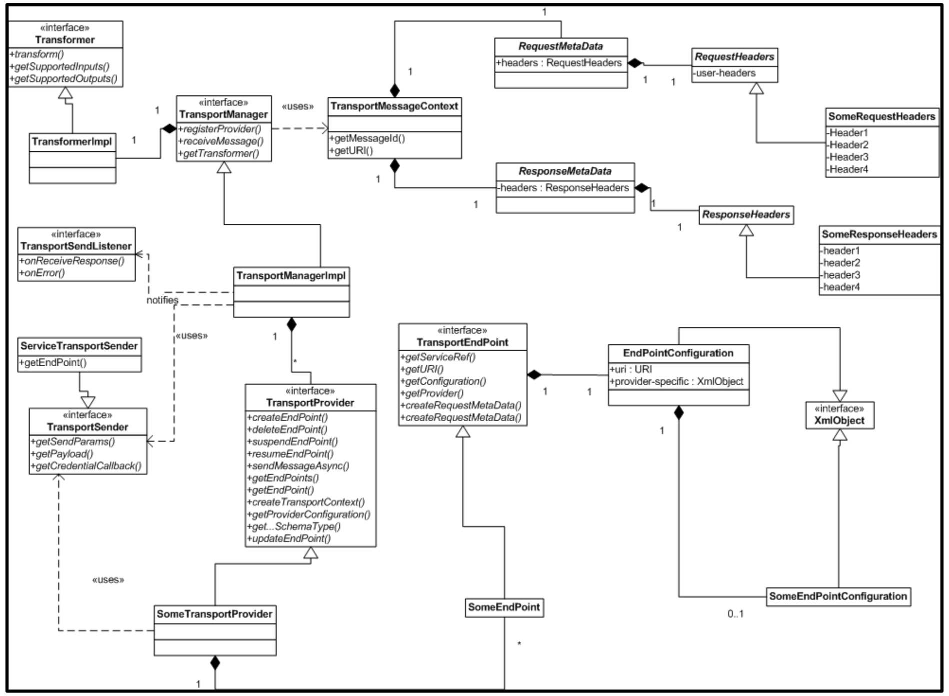

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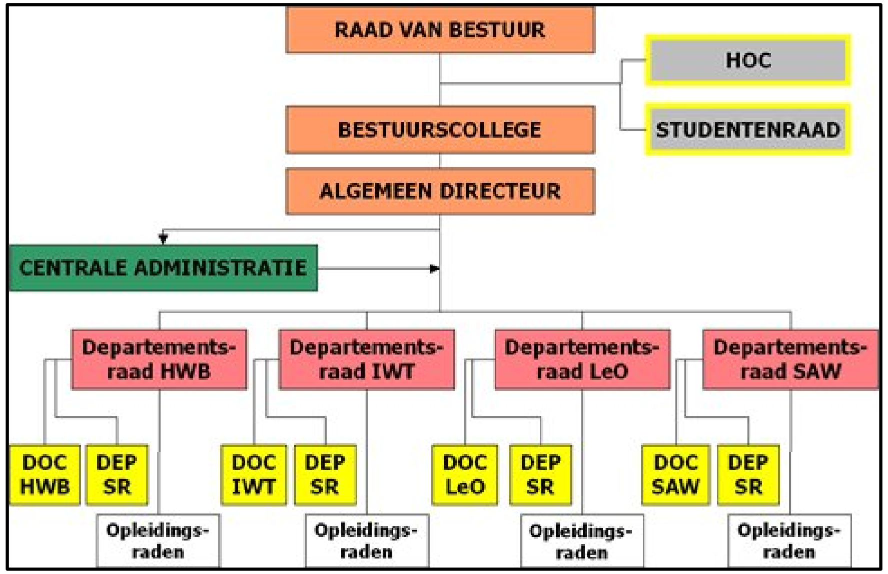

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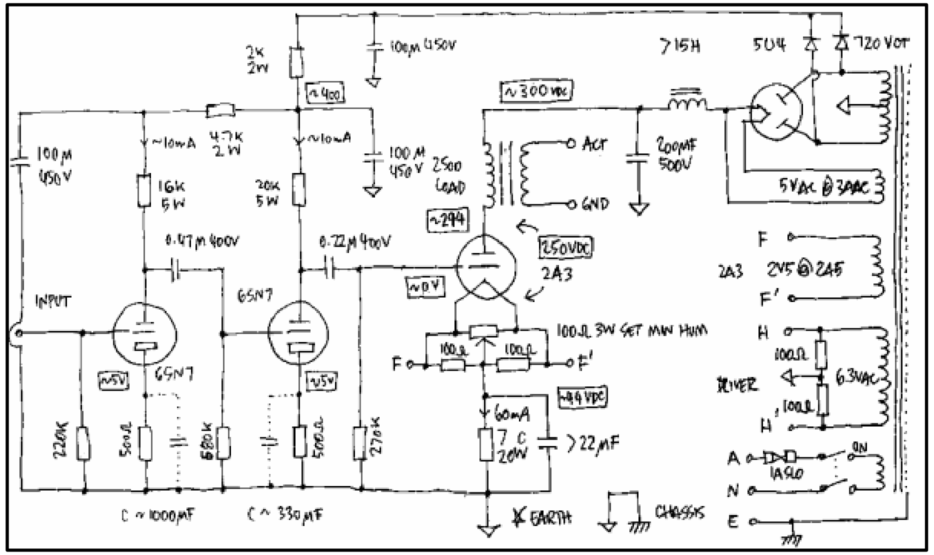

Circuit diagram by Jeff Atwood

Orthogonal Layout – Definition

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A drawing Γ of a graph $G = (V, E)$ is called **orthogonal** if

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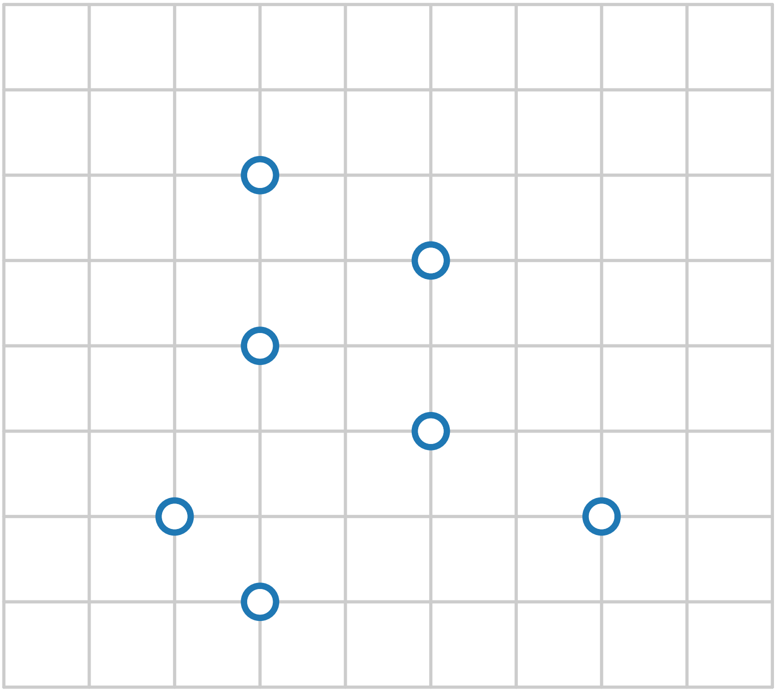

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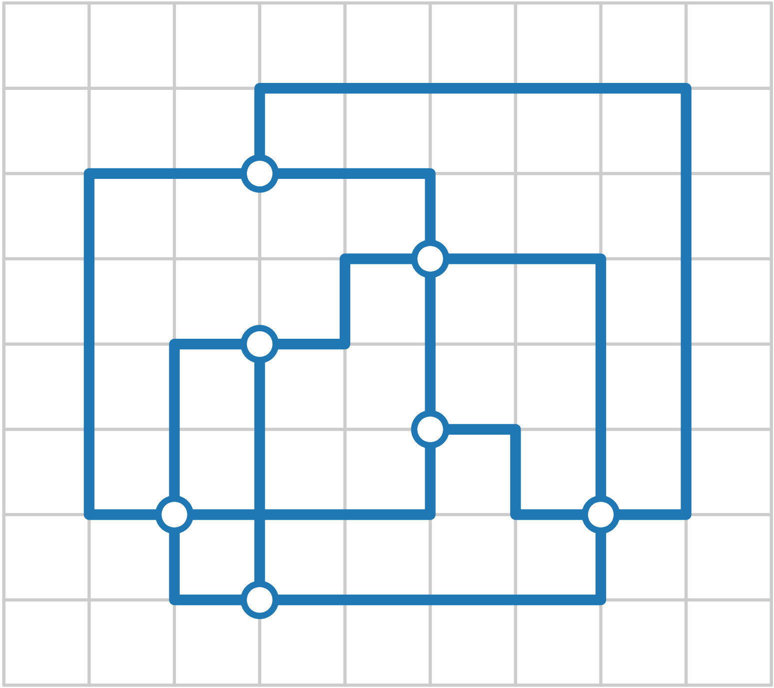

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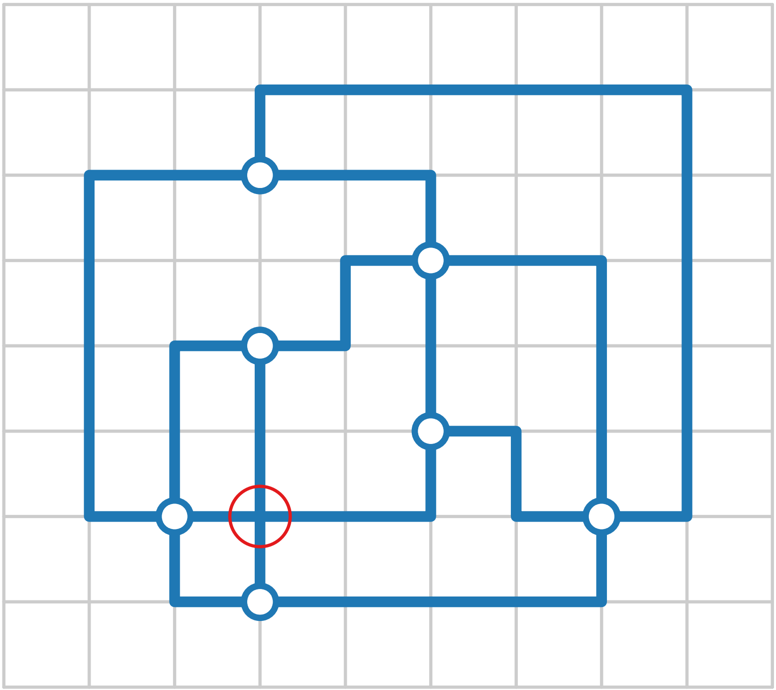

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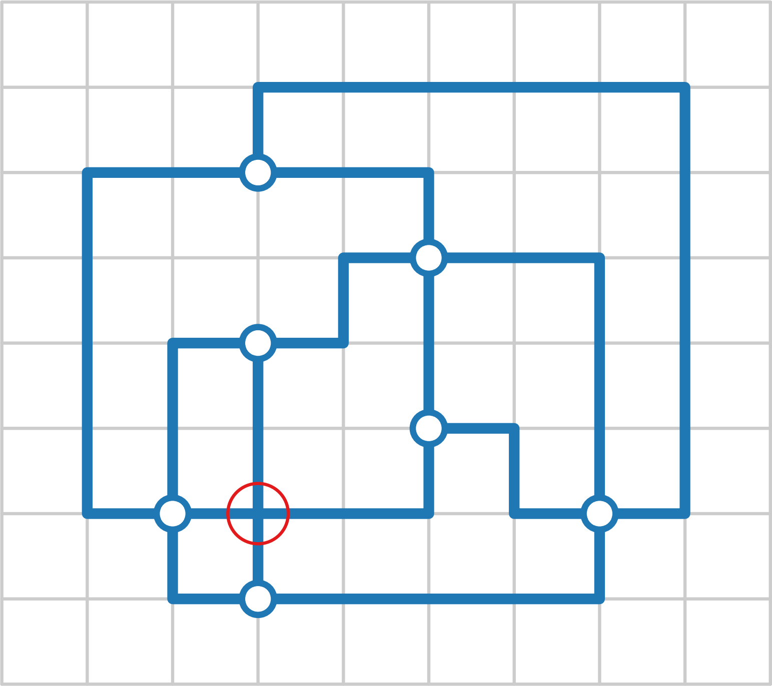

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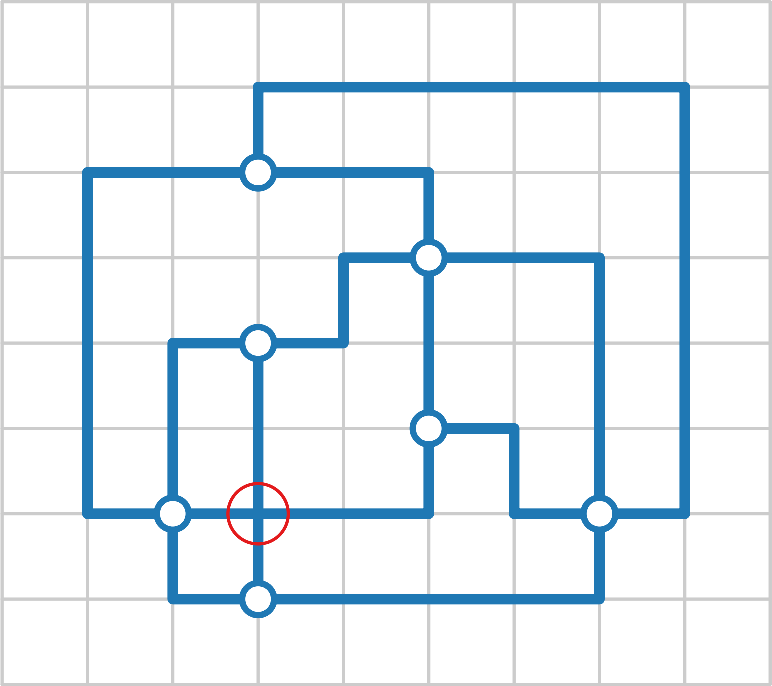

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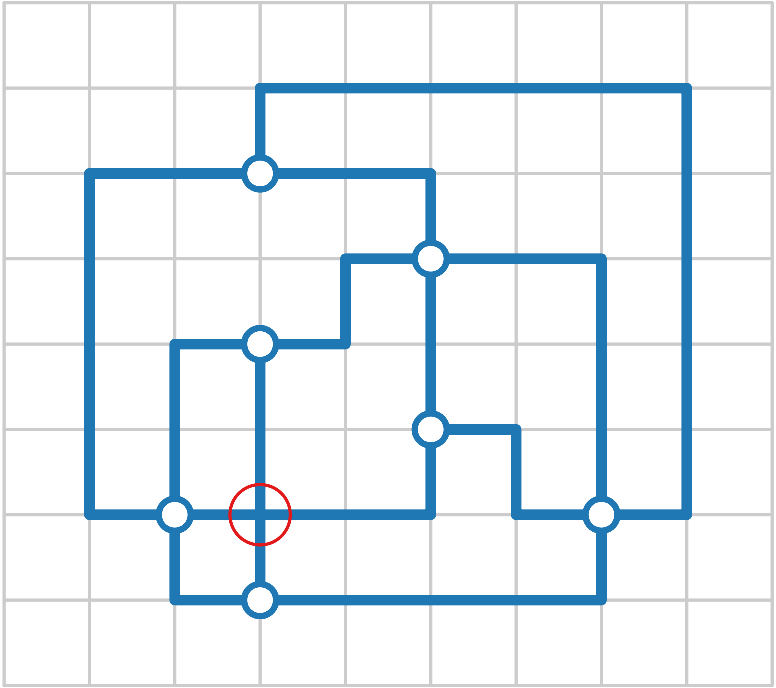

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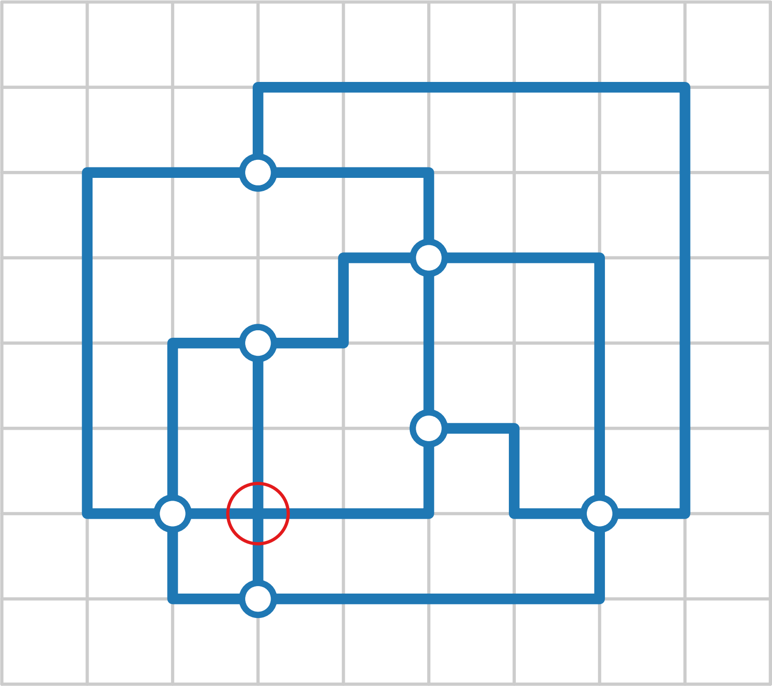

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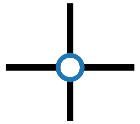
- Otherwise 

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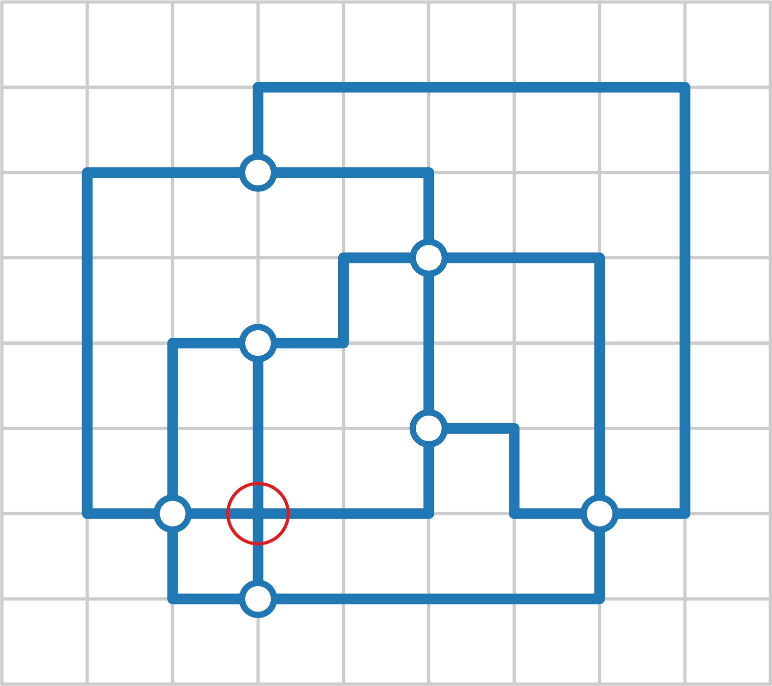

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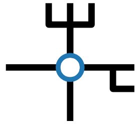

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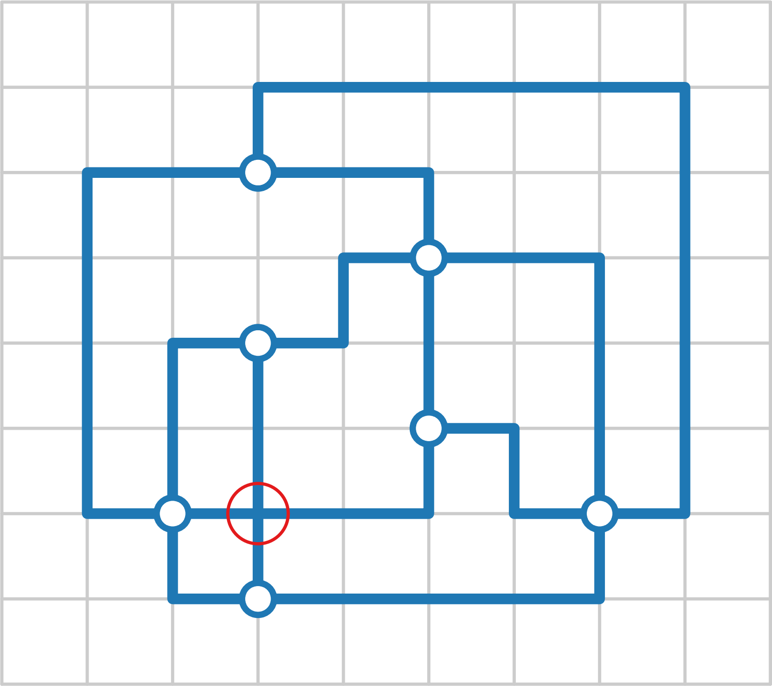

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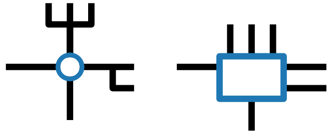
- Otherwise 

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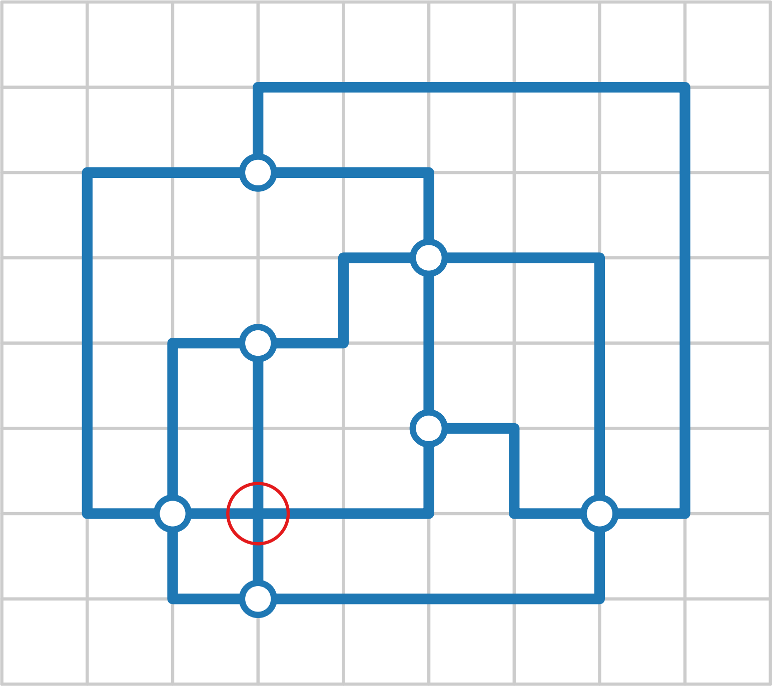

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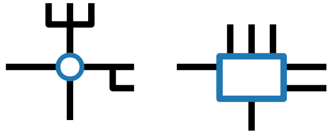
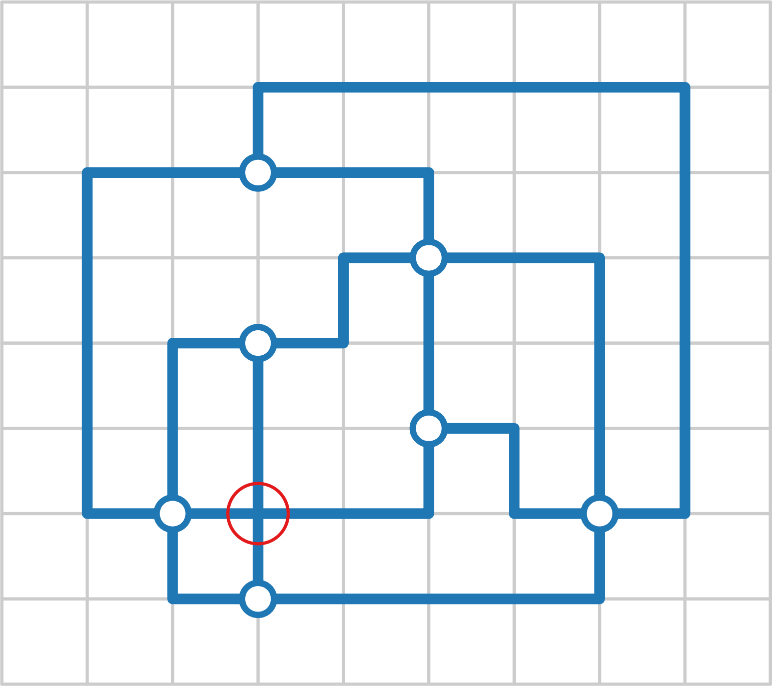
- Otherwise 

Planarization.

Orthogonal Layout – Definition



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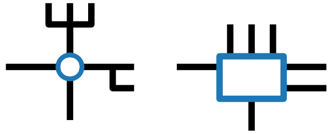
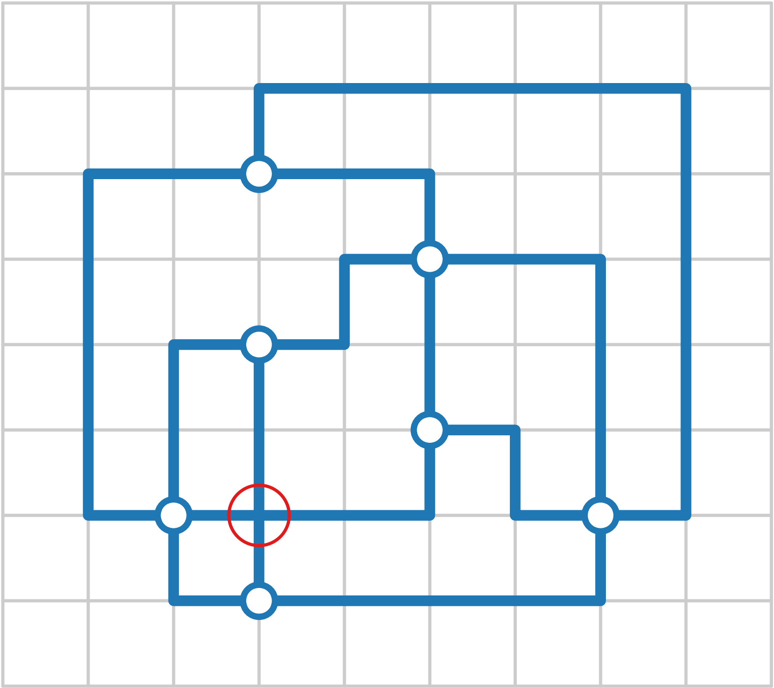


Planarization.

- Fix embedding

Orthogonal Layout – Definition



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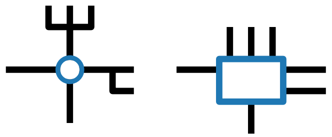

Planarization.

- Fix embedding
- Crossings become vertices

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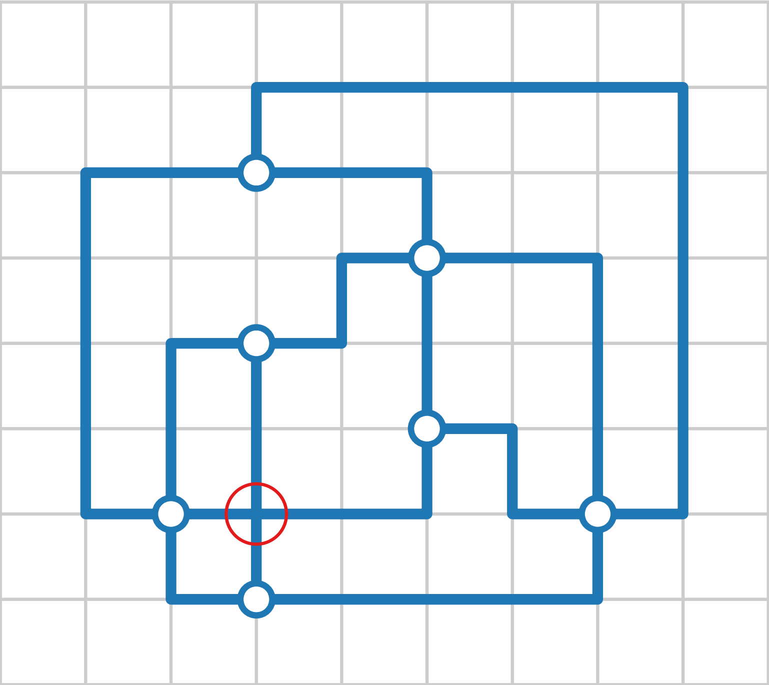

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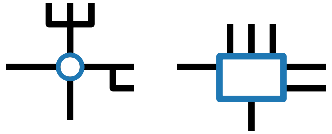

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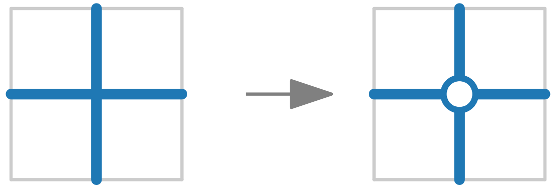

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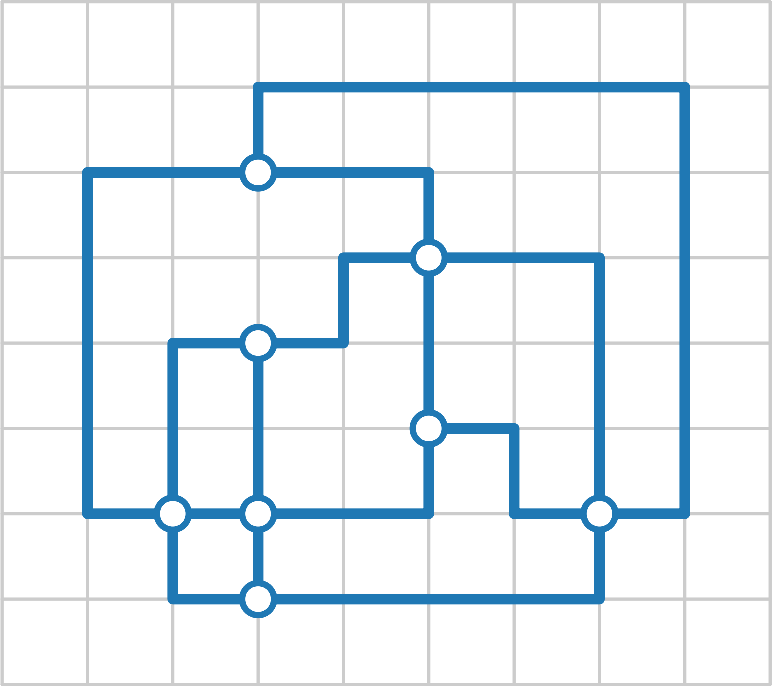

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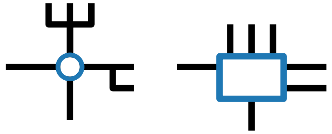

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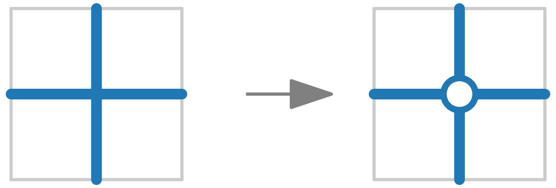

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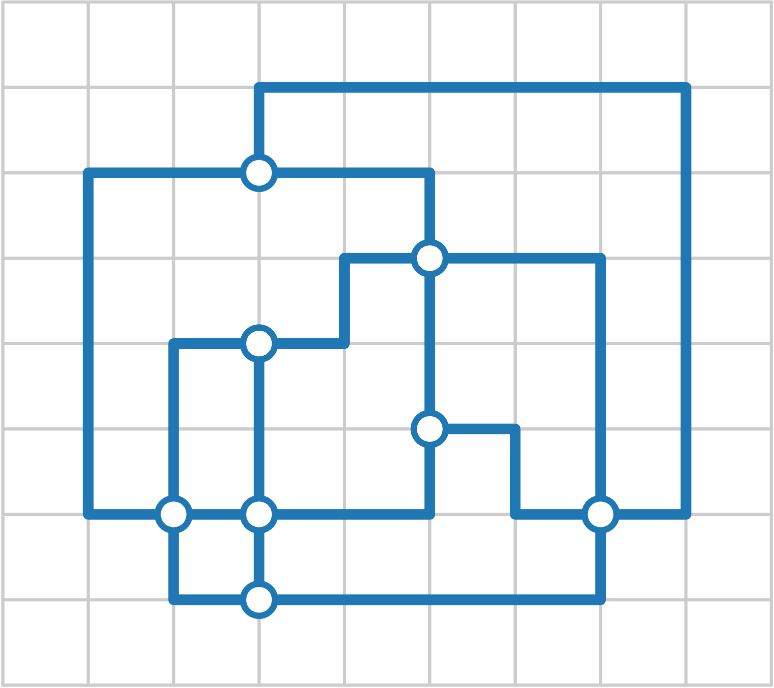

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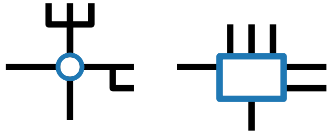

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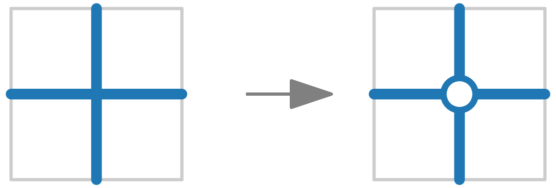
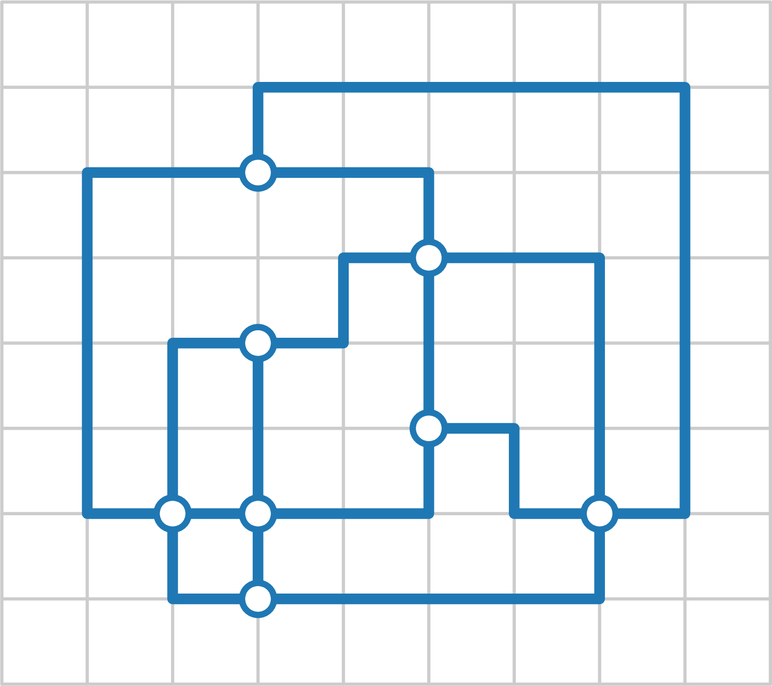


Aesthetic criteria.

Orthogonal Layout – Definition



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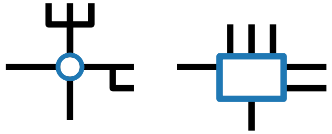

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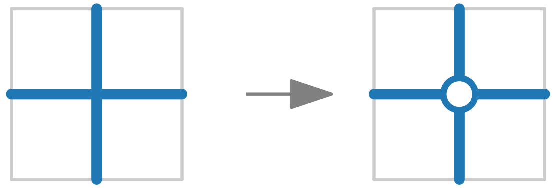

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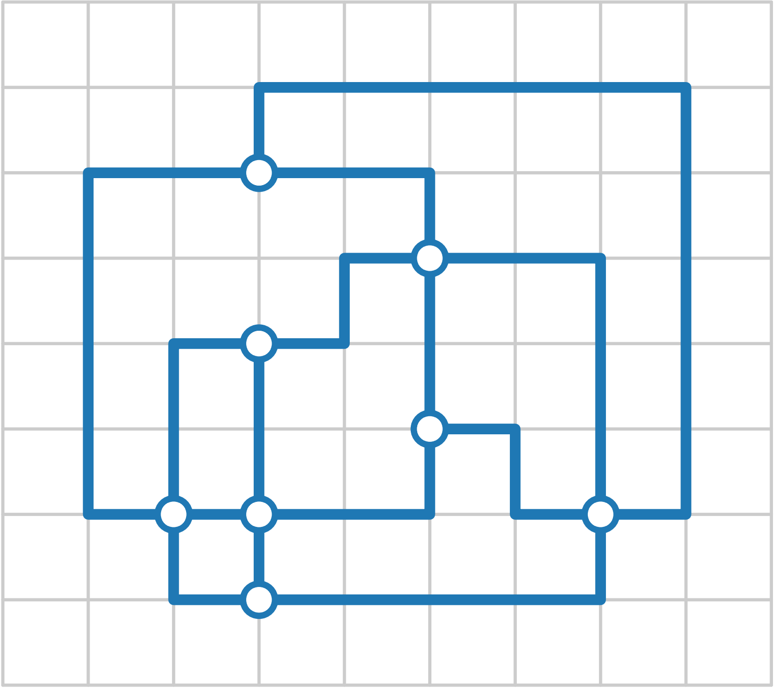
- Number of bends

Orthogonal Layout – Definition



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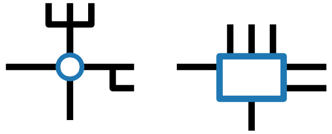

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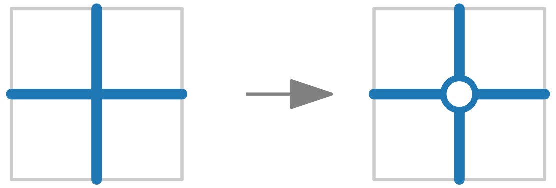
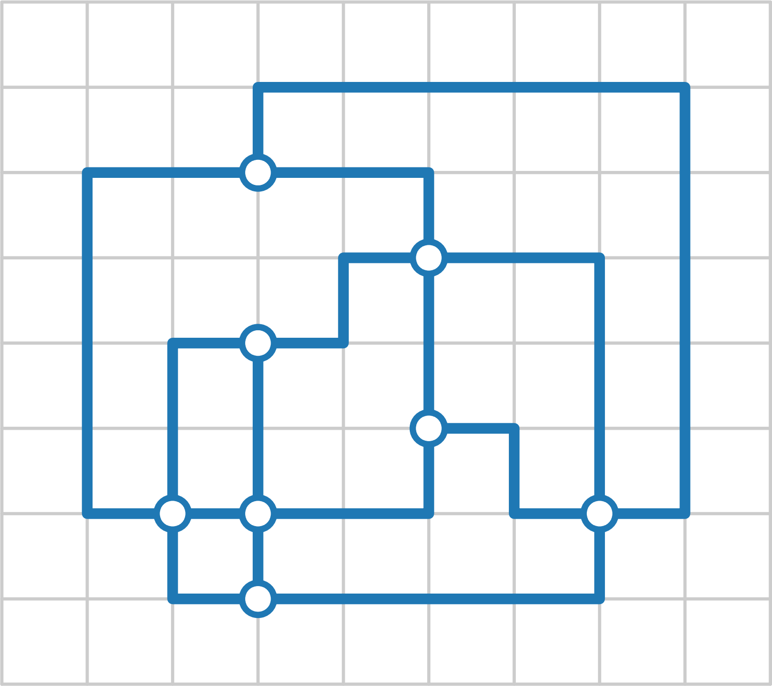


Aesthetic criteria.

- Number of bends
- Length of edges

Orthogonal Layout – Definition



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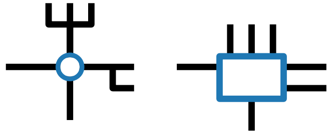

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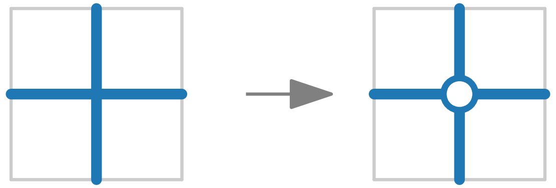
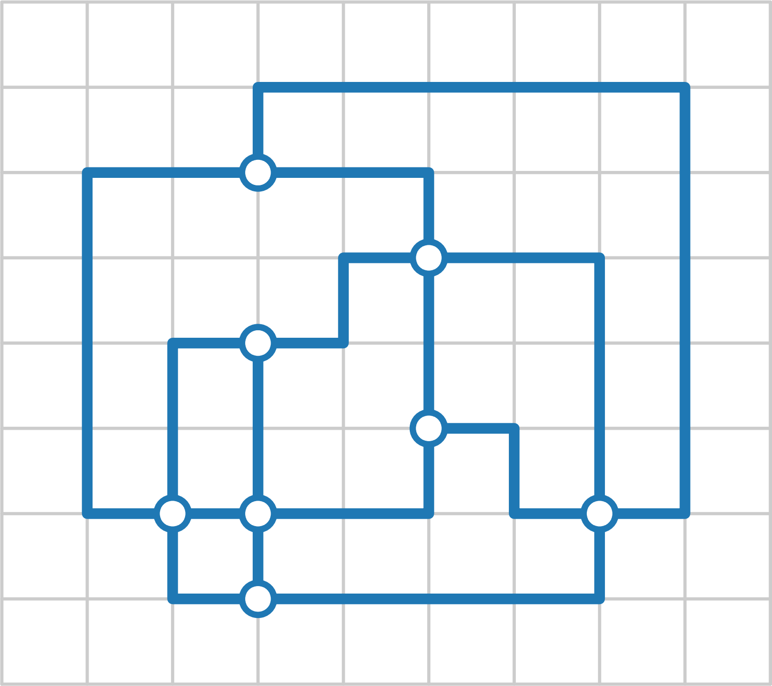


Aesthetic criteria.

- Number of bends
- Length of edges
- Width, height, area

Orthogonal Layout – Definition



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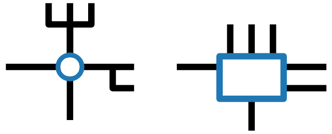

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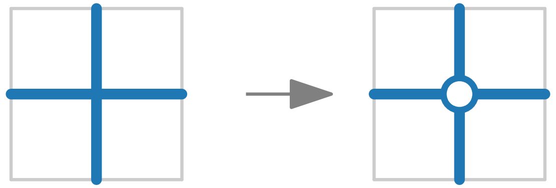

Aesthetic criteria.

- Number of bends
- Length of edges
- Width, height, area
- Monotonicity of edges
- ...

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

TOPOLOGY — SHAPE — METRICS

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

$V = \{v_1, v_2, v_3, v_4\}$
$E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$

TOPOLOGY – SHAPE – METRICS

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

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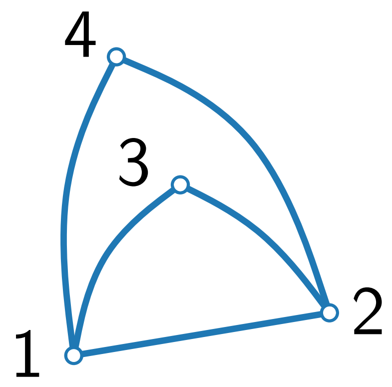
combinatorial
embedding/
planarization



TOPOLOGY — SHAPE — METRICS

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

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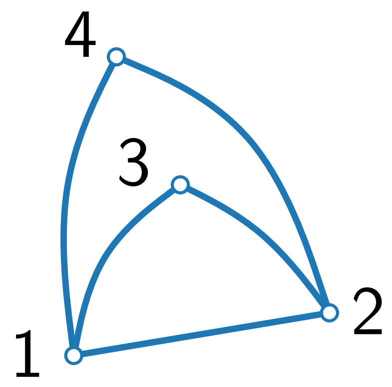

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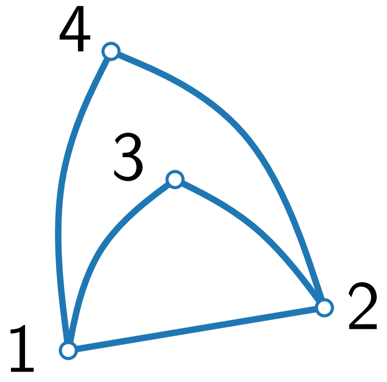

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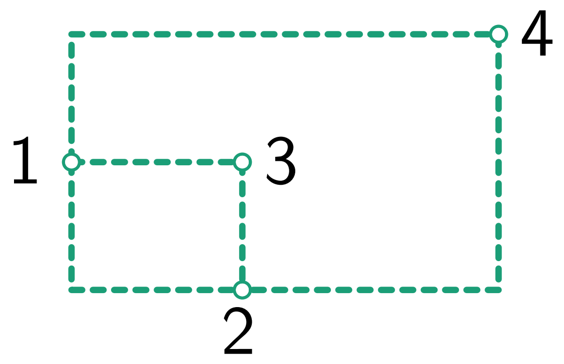

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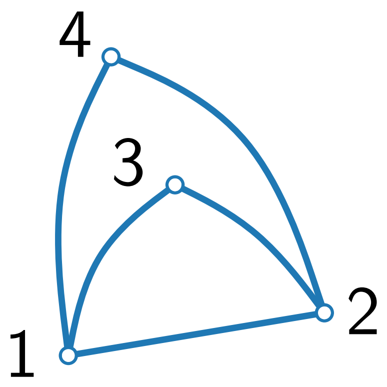

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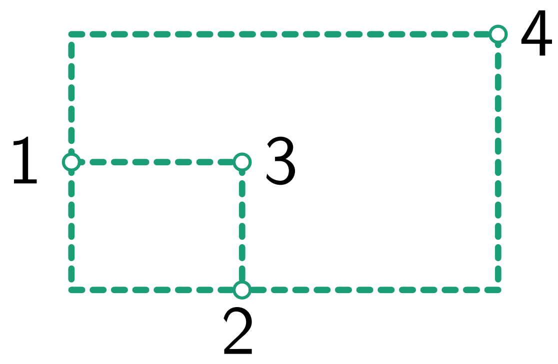

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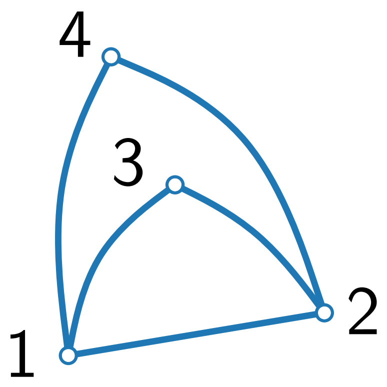

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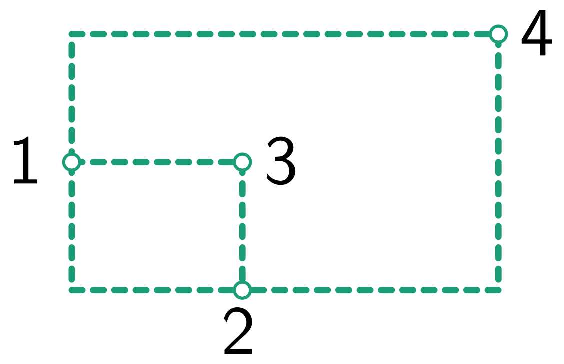
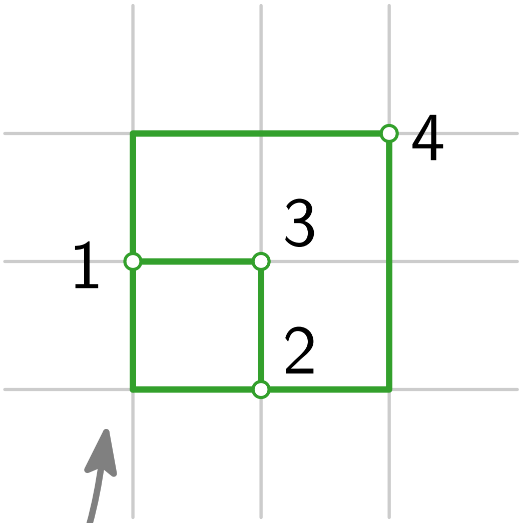


planar orthogonal drawing



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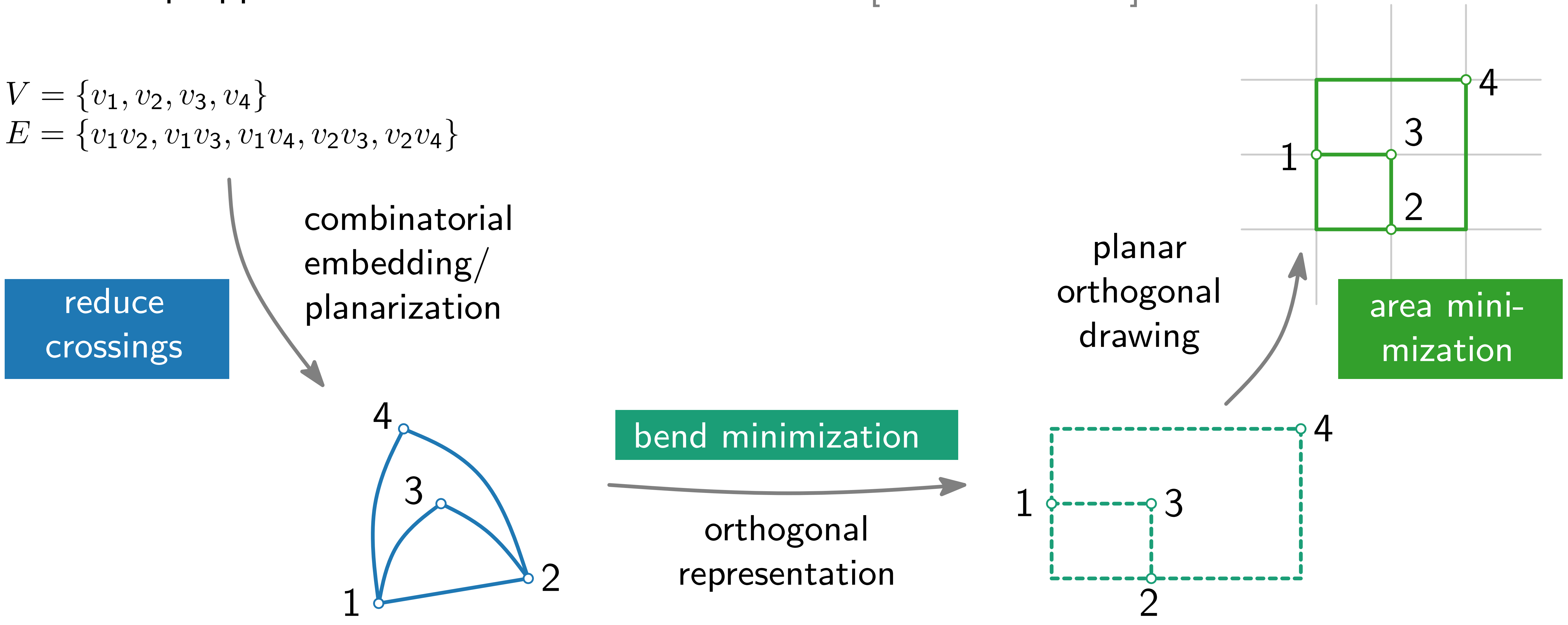

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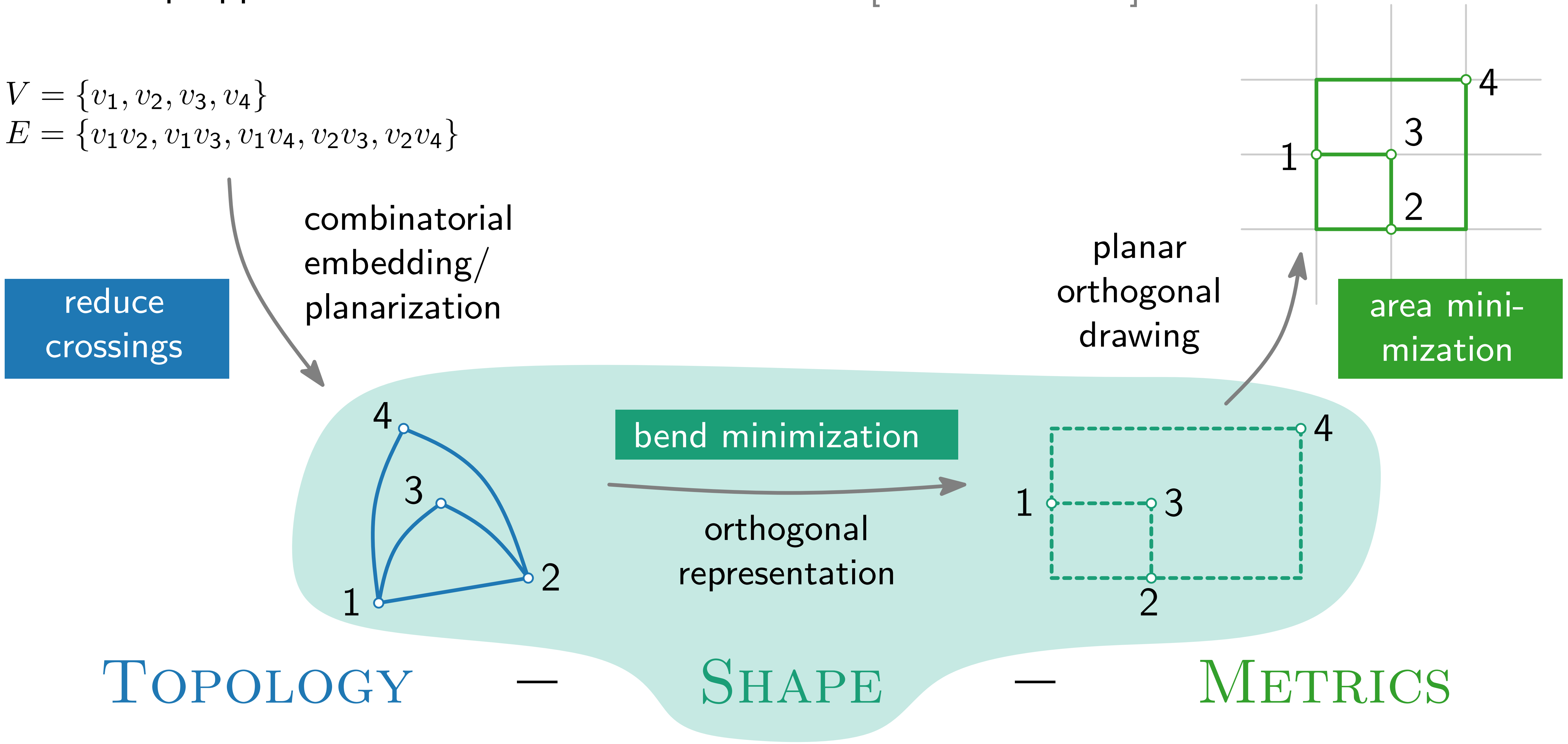
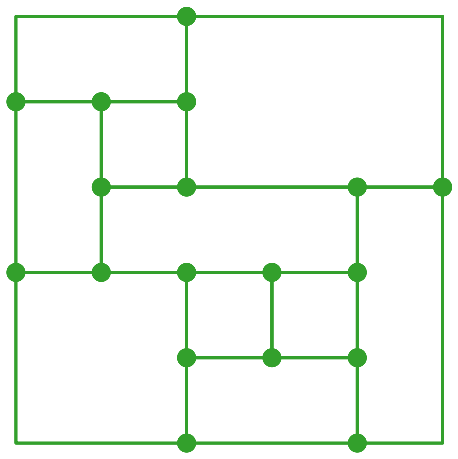
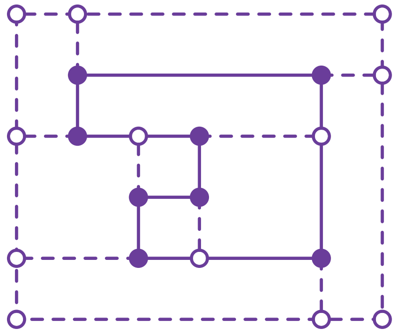


Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

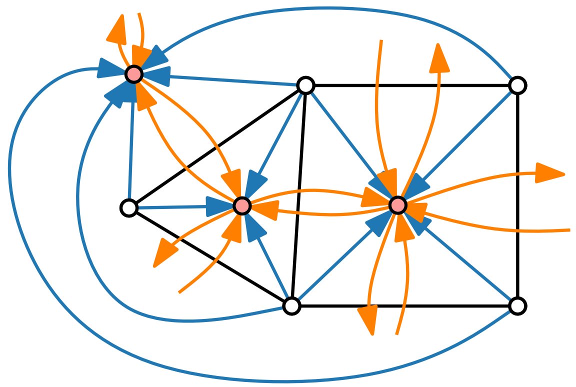
Visualization of Graphs



Lecture 5:
Orthogonal Layouts



Part II:
Orthogonal Representation



Jonathan Klawitter

Orthogonal Representation

Idea.
Describe orthogonal drawing combinatorically.

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Let $G = (V, E)$ be a plane graph with faces F and outer face f_0.

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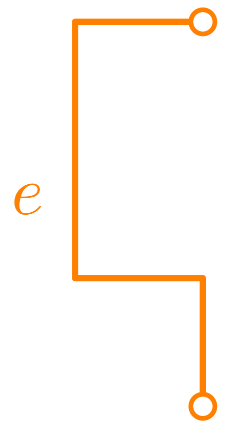
- Let e be an edge



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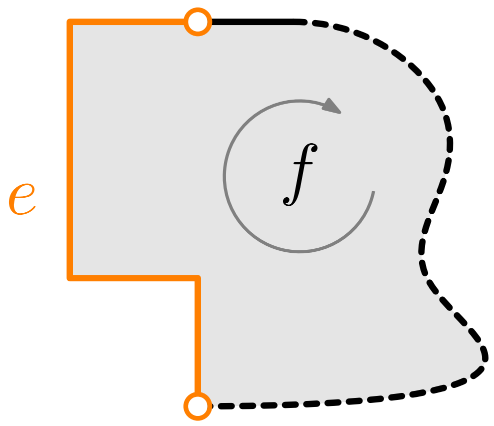

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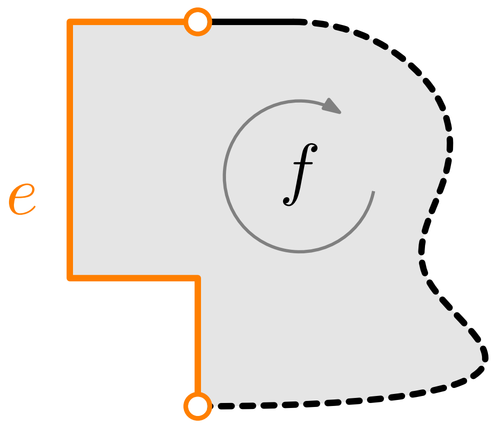

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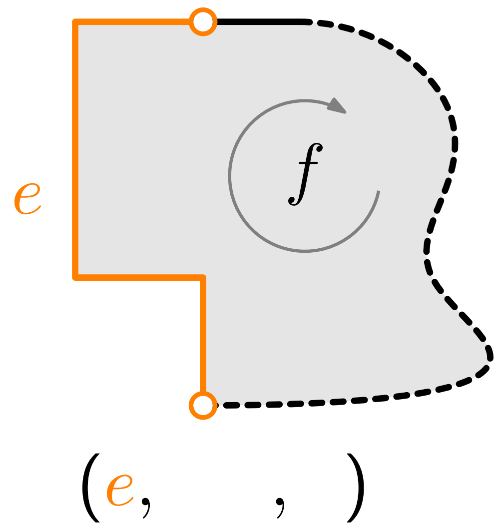

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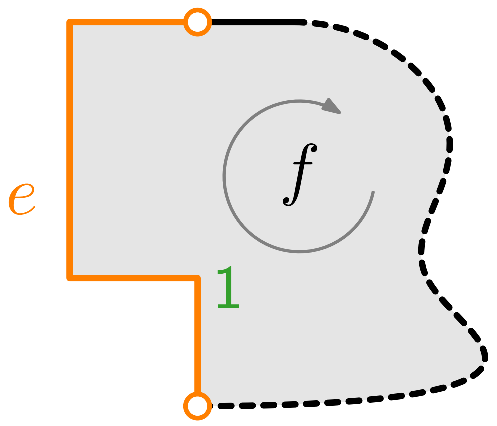

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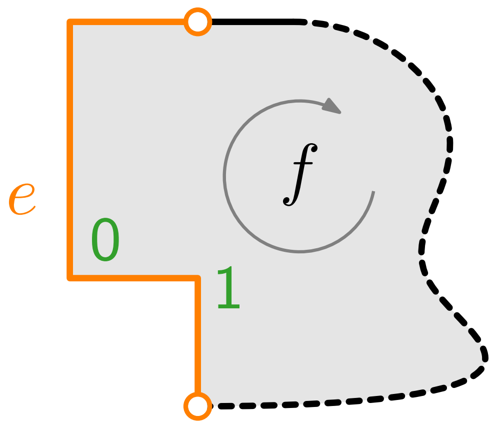

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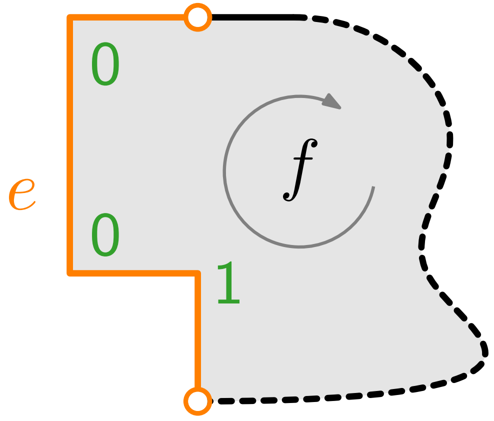

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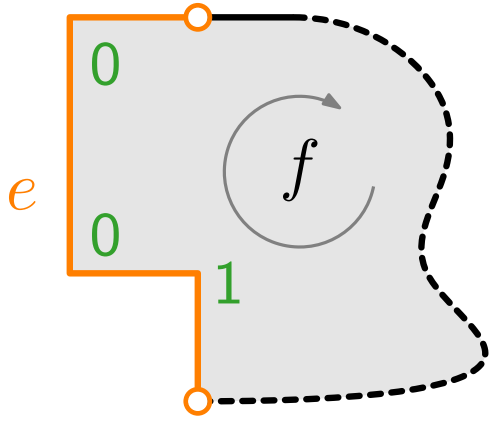

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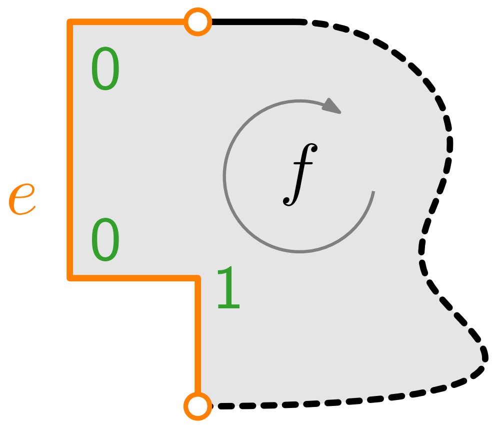

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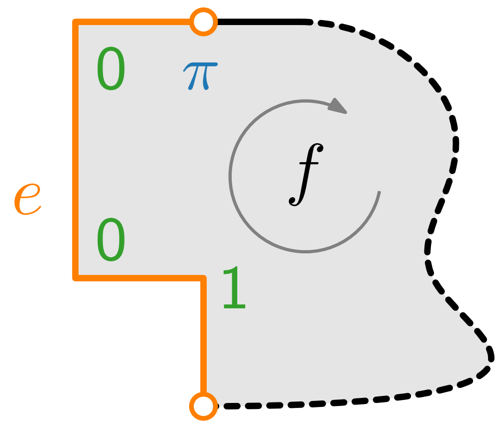

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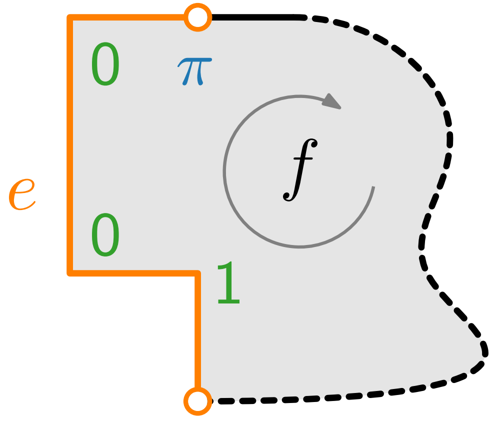

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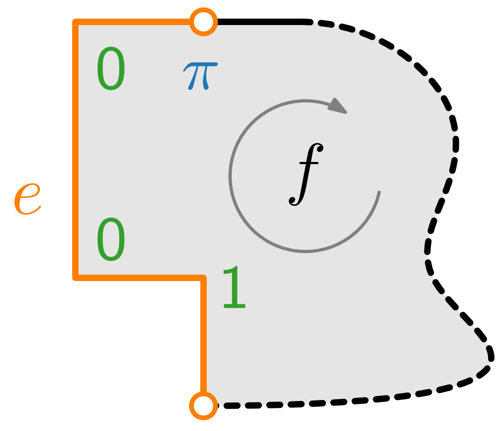

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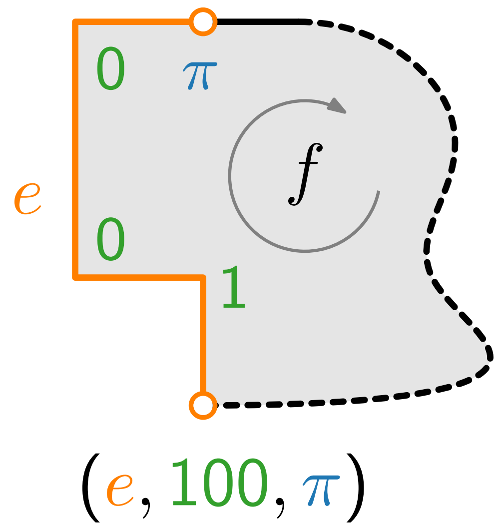
- An **orthogonal representation** $H(G)$ of G is defined as

$$H(G) = \{H(f) \mid f \in F\}.$$



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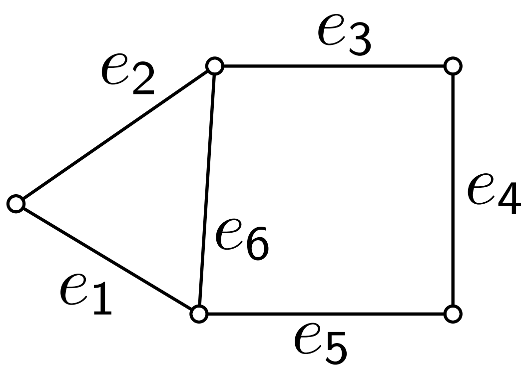

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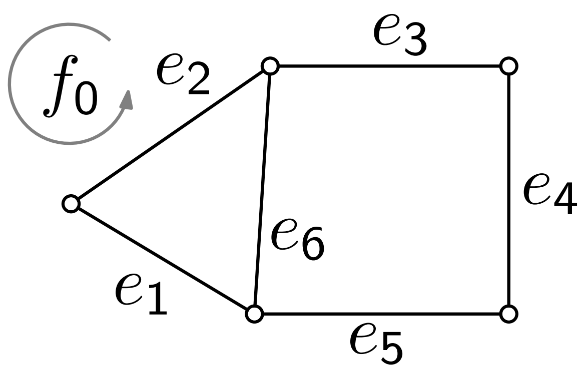

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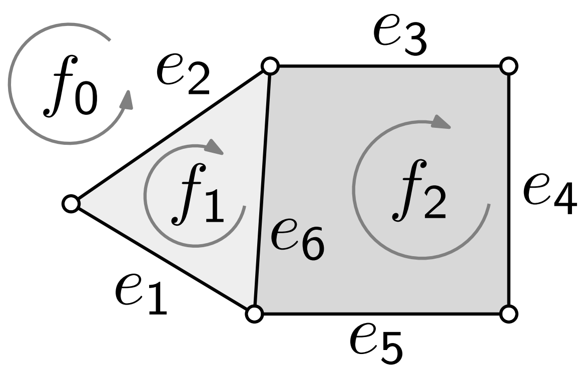

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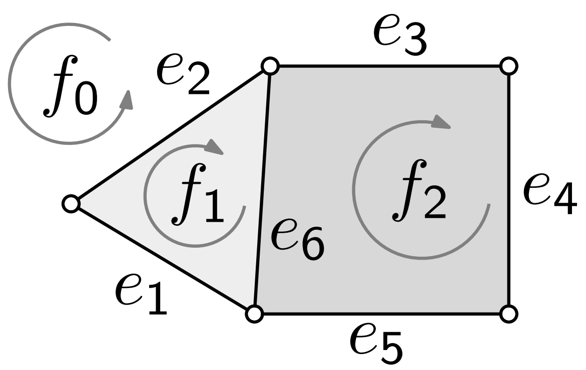

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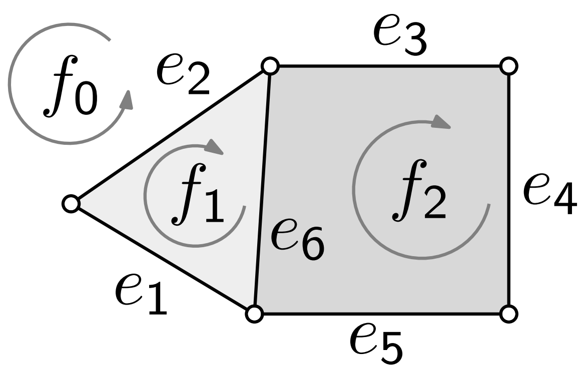

Combinatorial "drawing" of $H(G)$?

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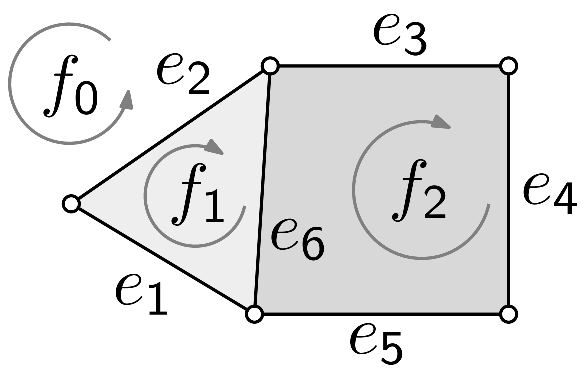
f_0



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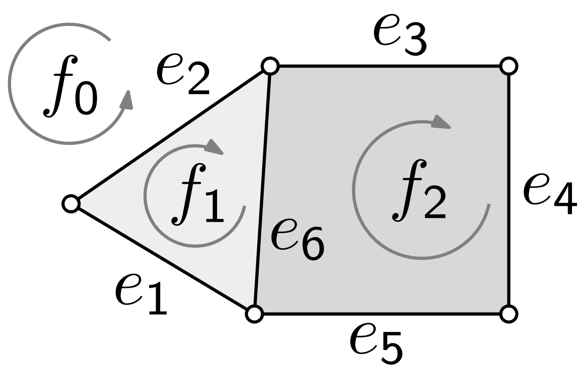

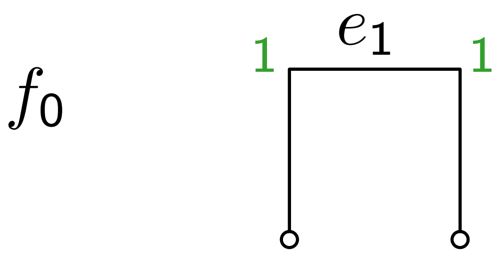

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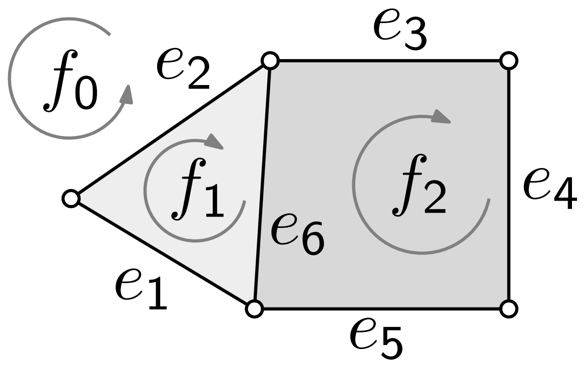

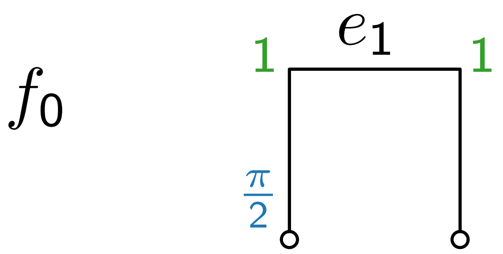

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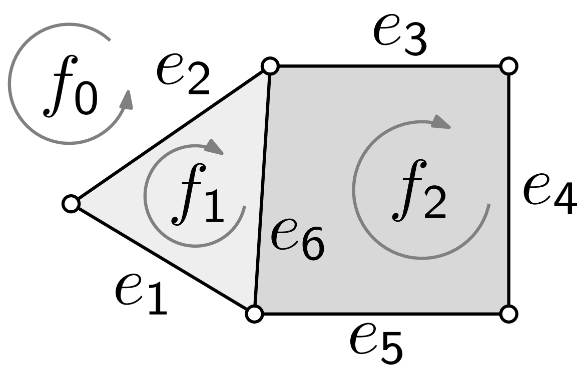

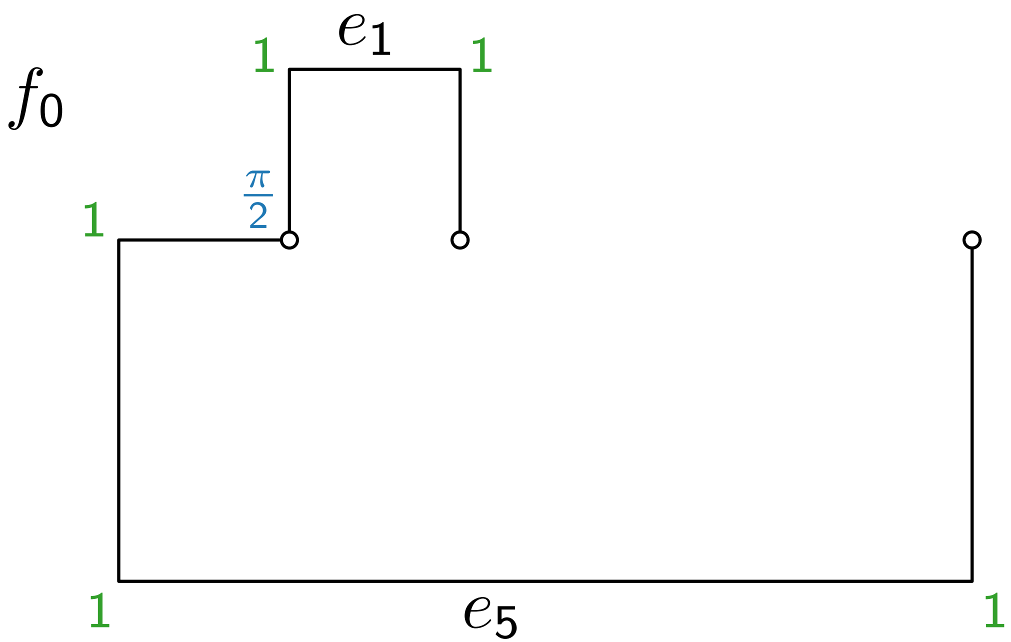

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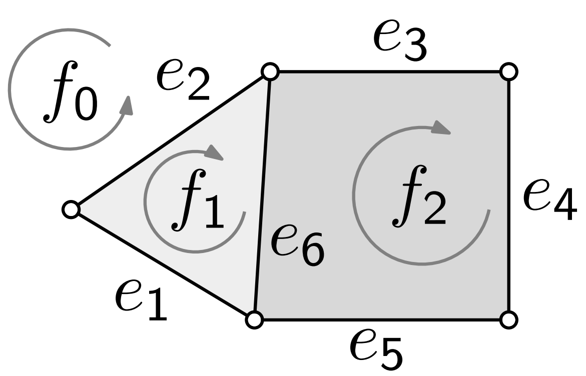
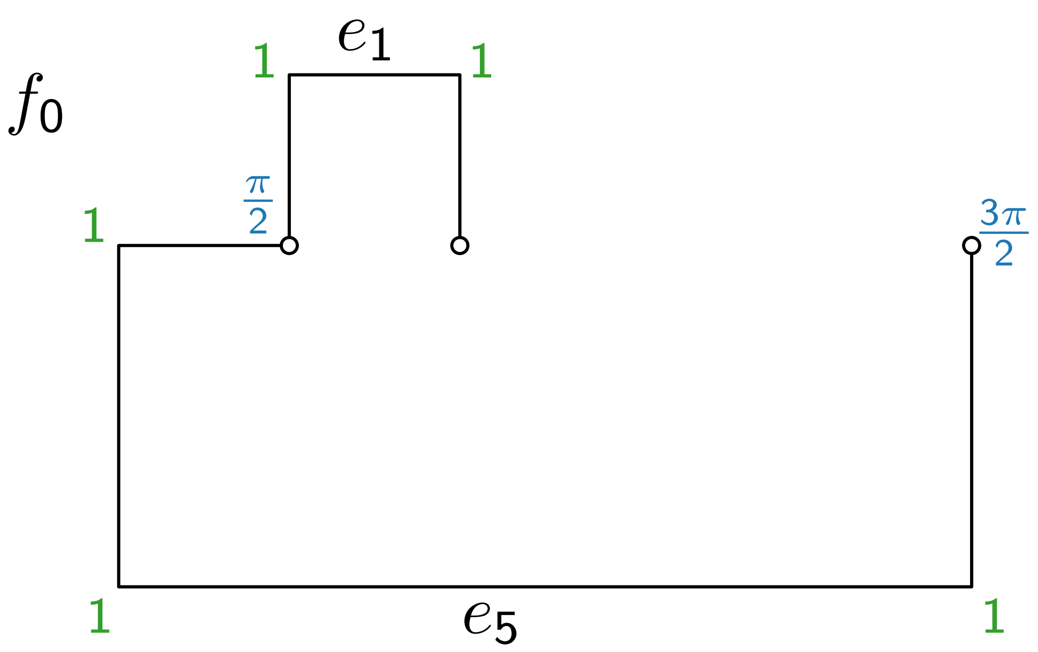
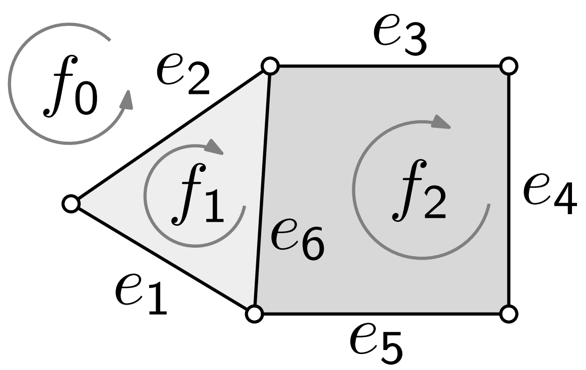
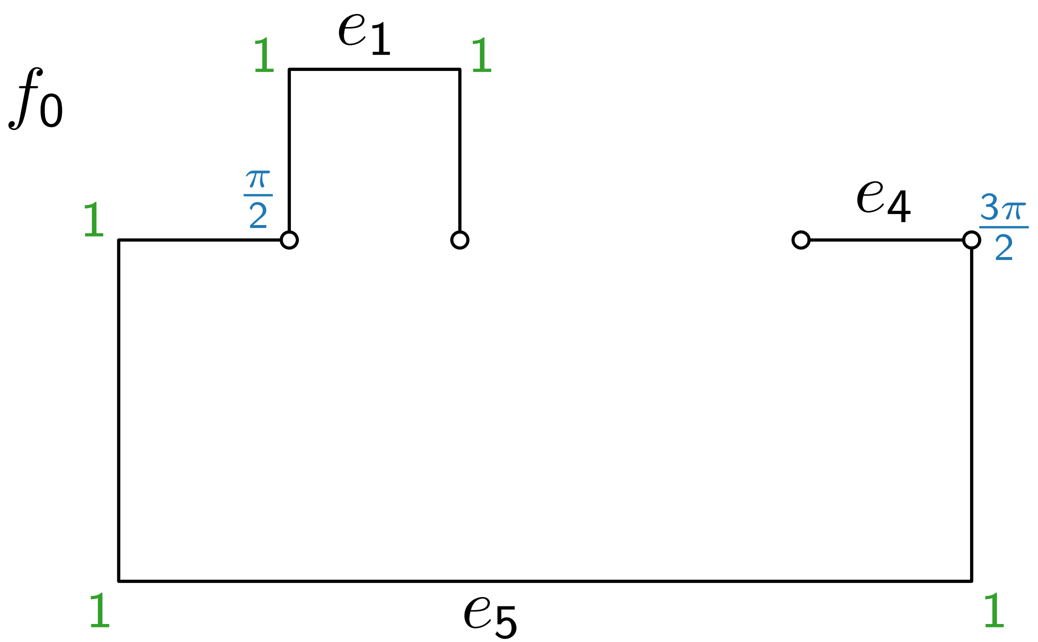




Orthogonal Representation – Example

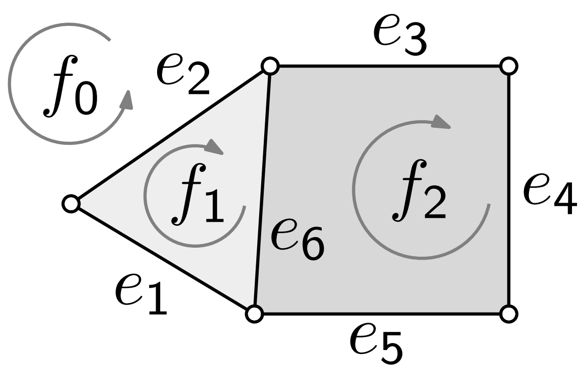
$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

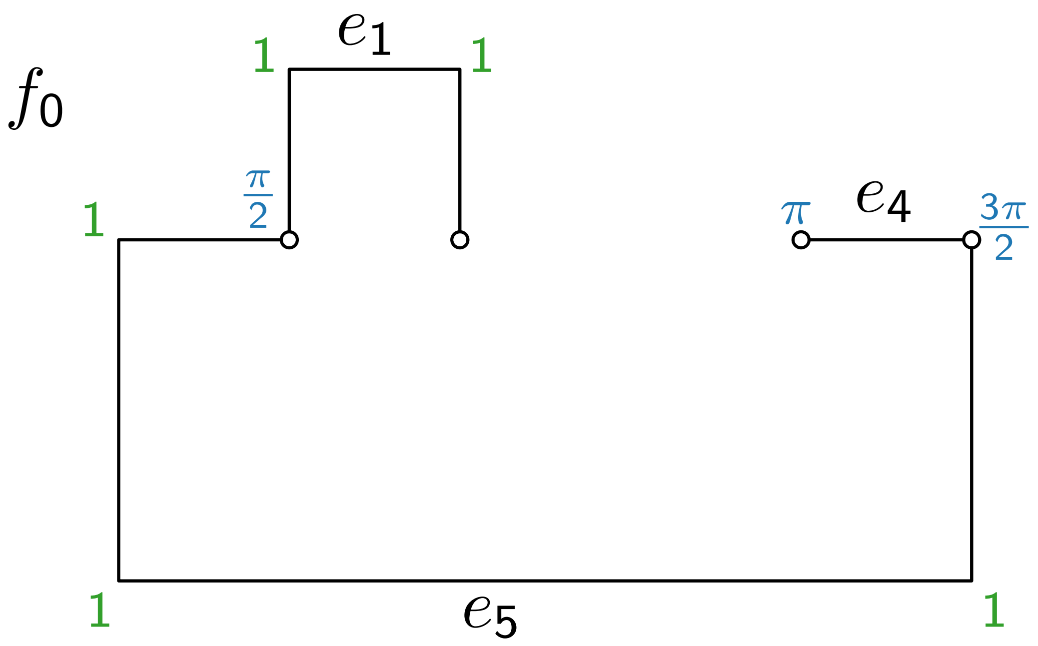




Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$





Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
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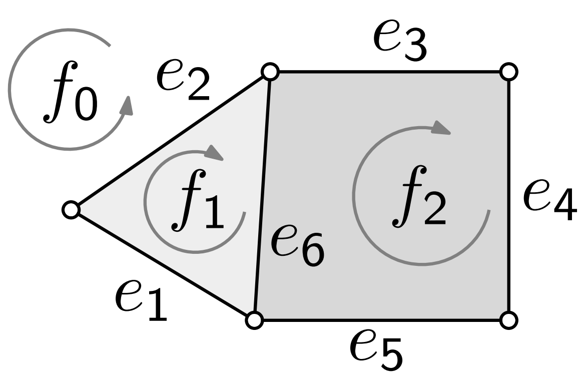
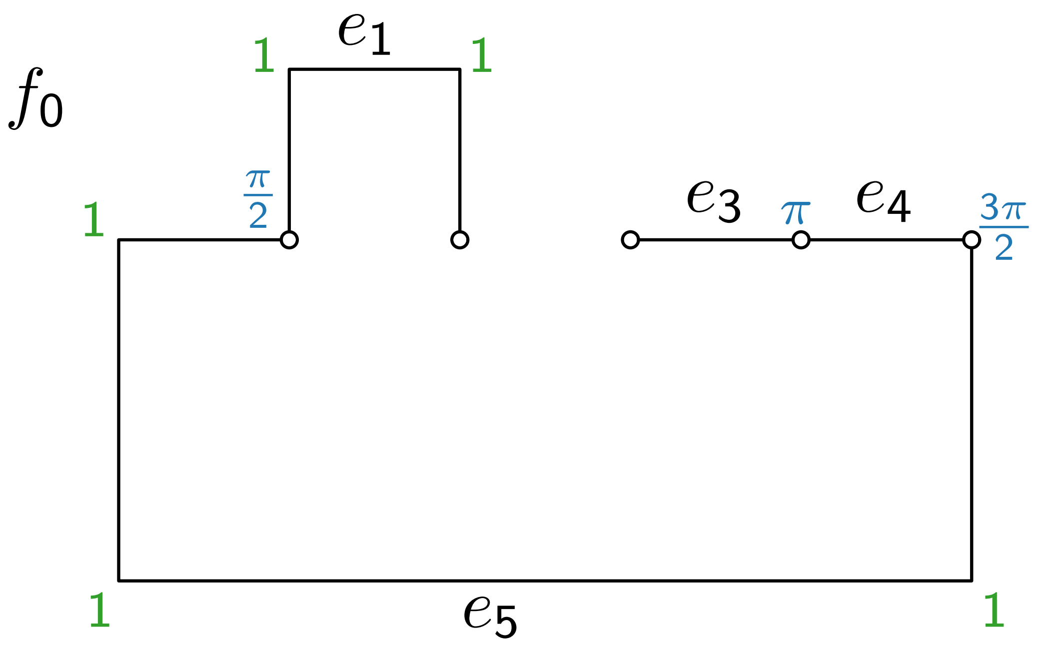
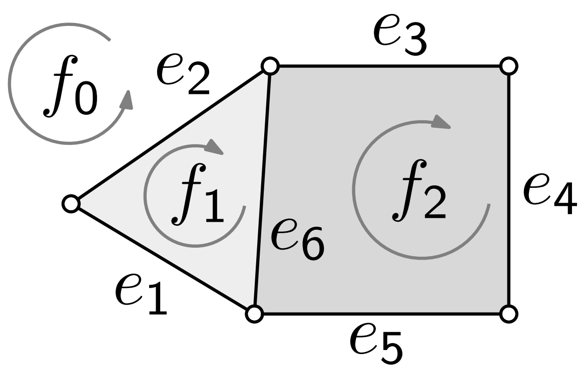
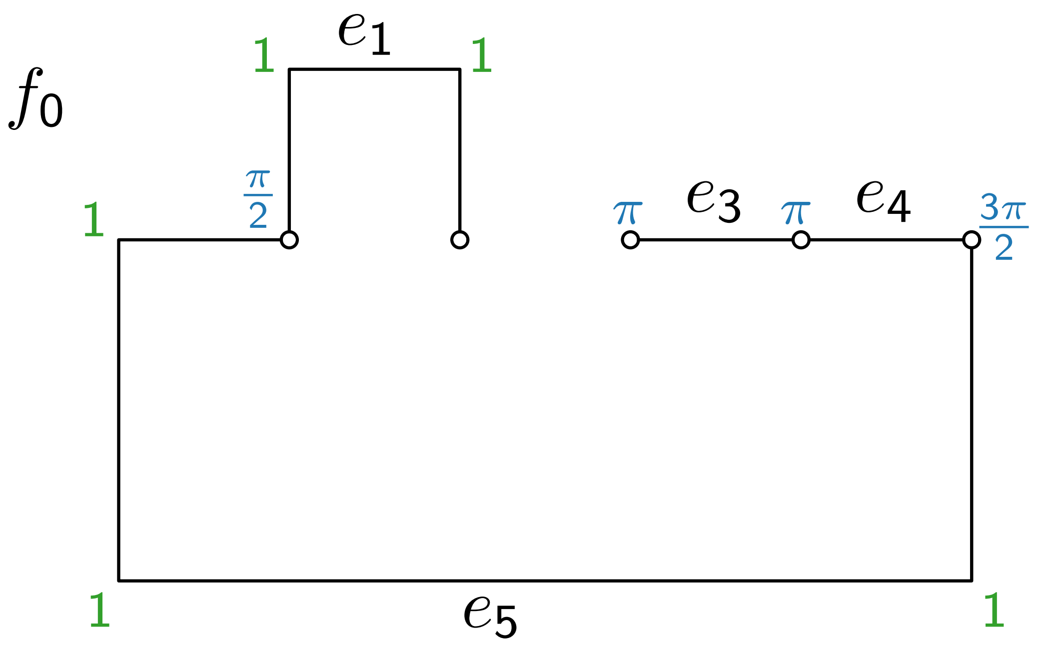




Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$





Orthogonal Representation – Example

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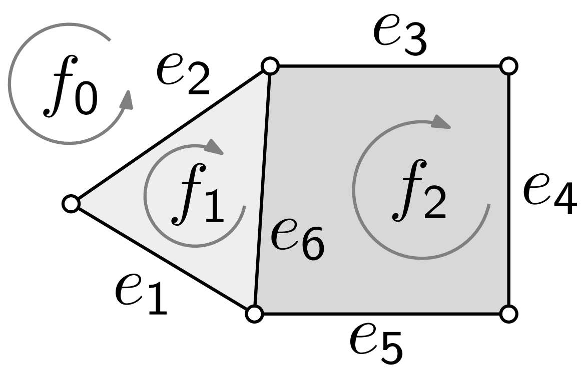

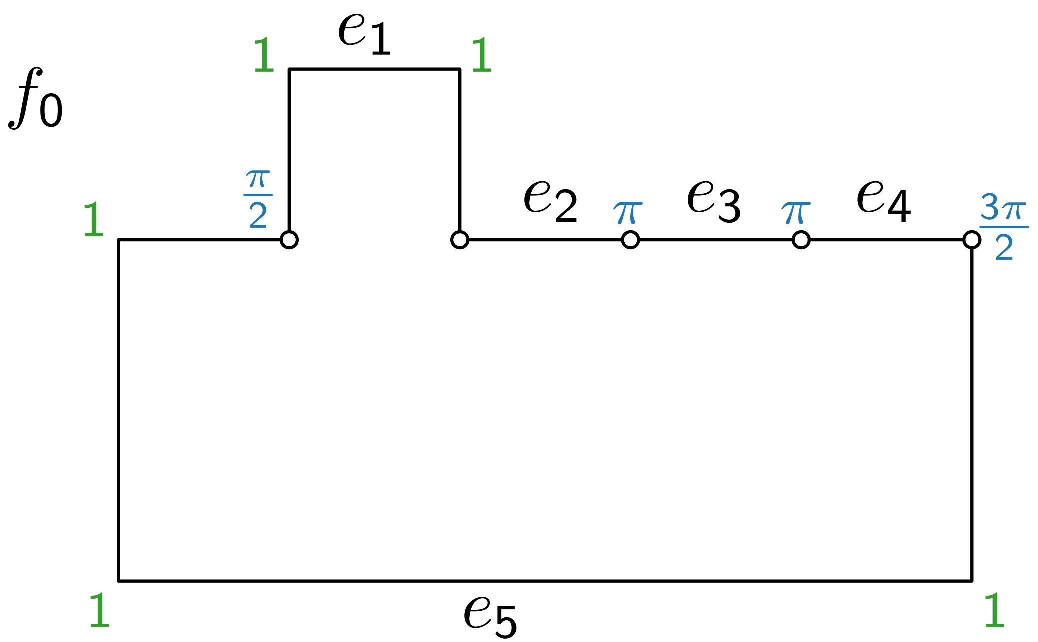

Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
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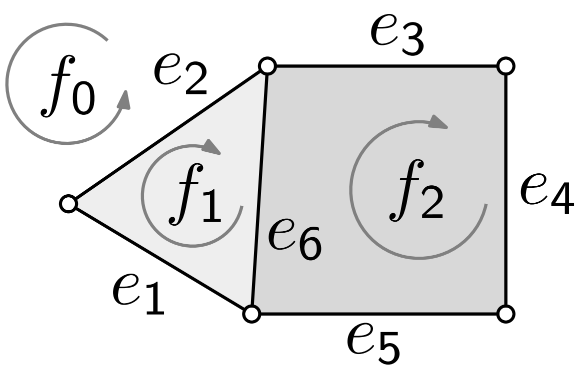

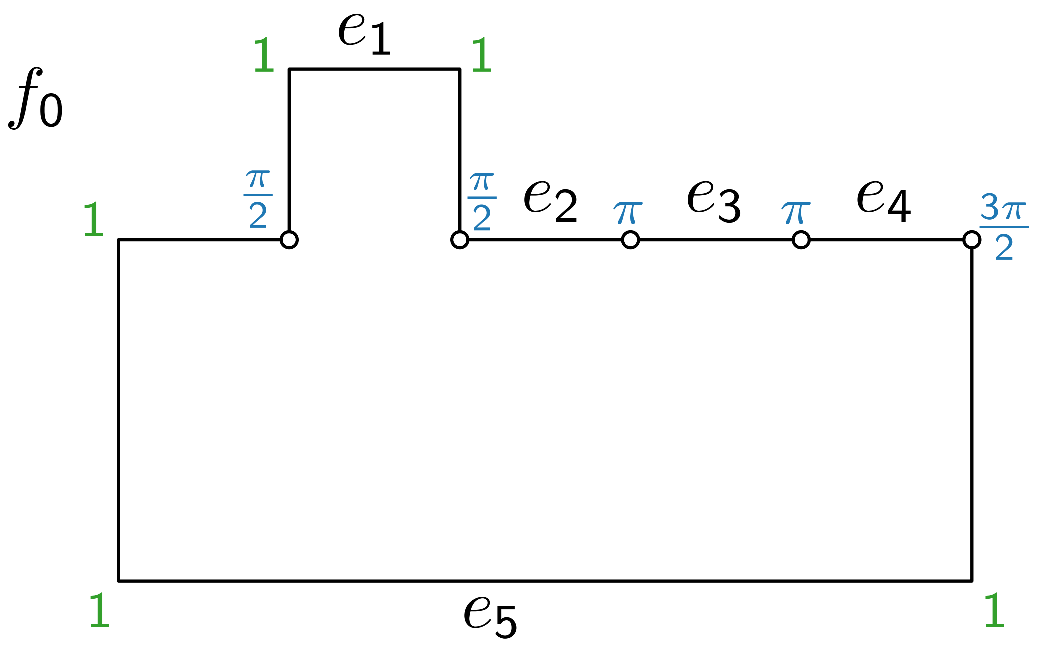

Orthogonal Representation – Example

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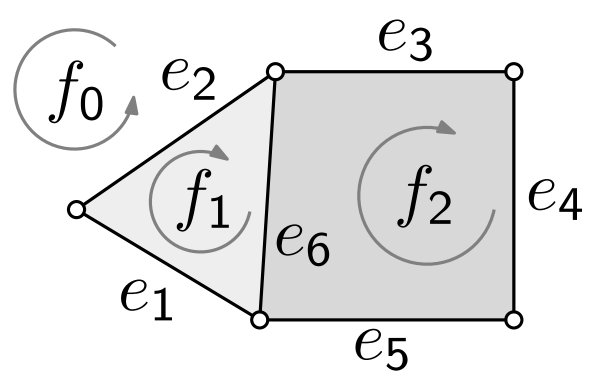
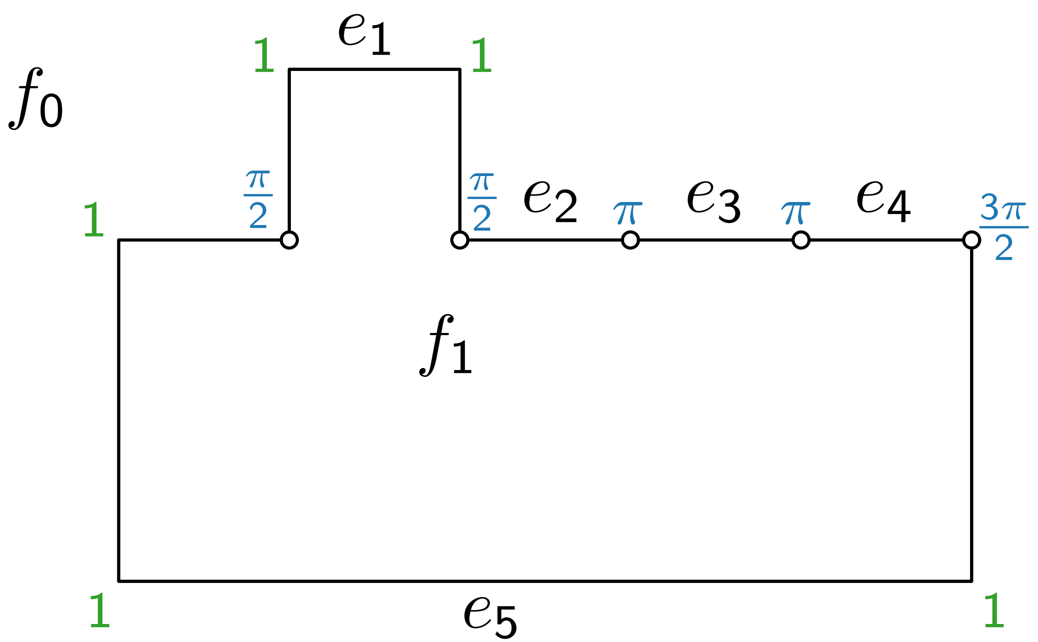
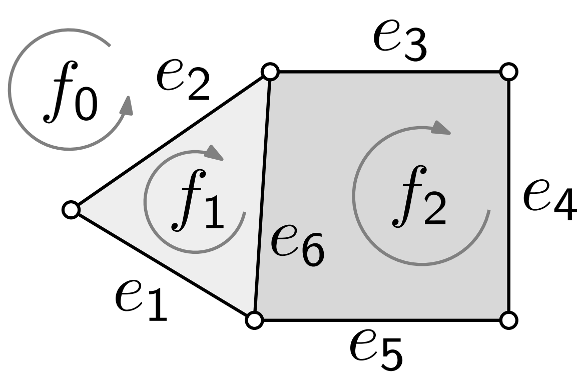
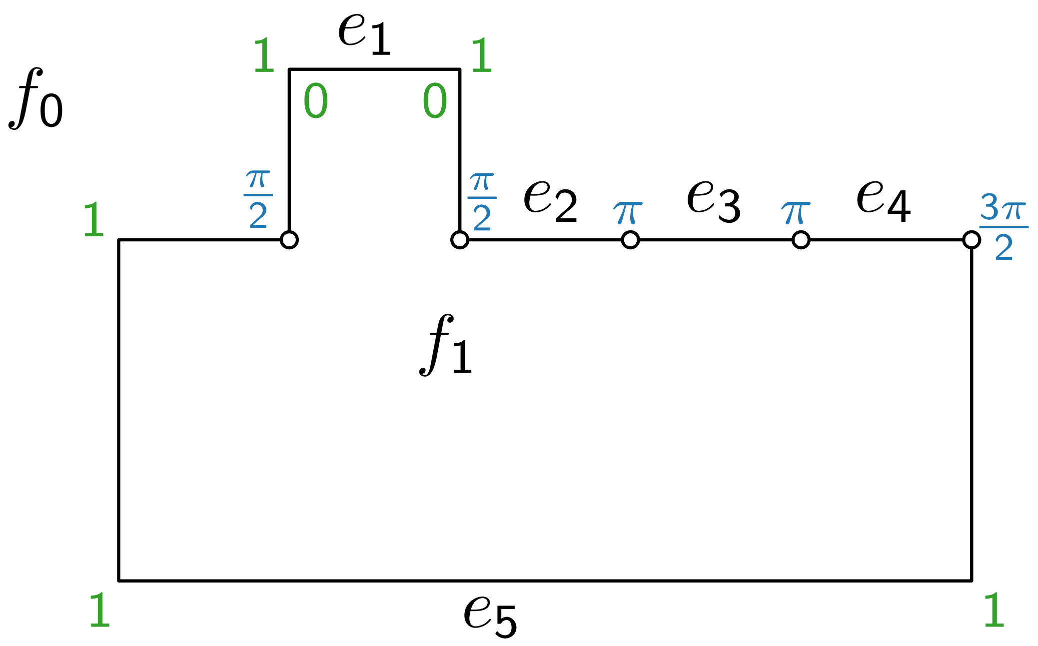




Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$





Orthogonal Representation – Example

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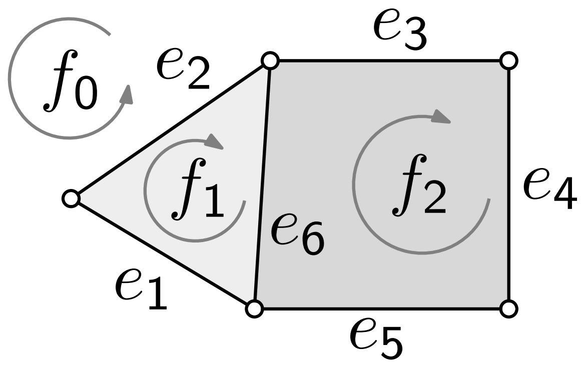
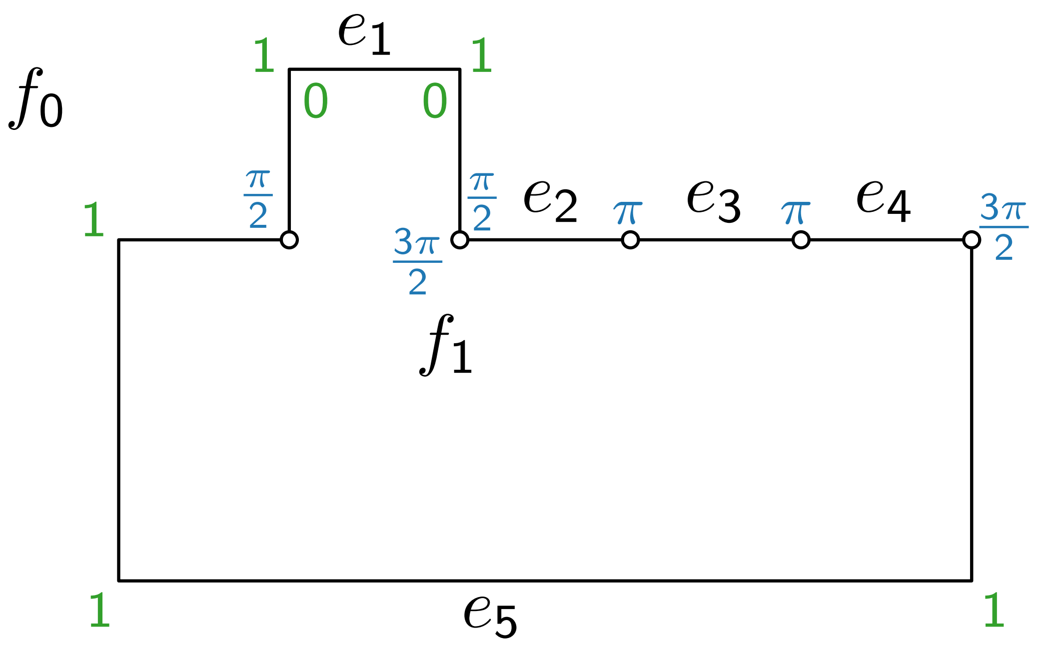
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$





Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
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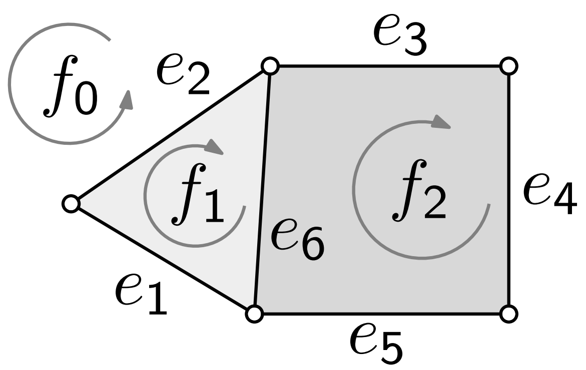
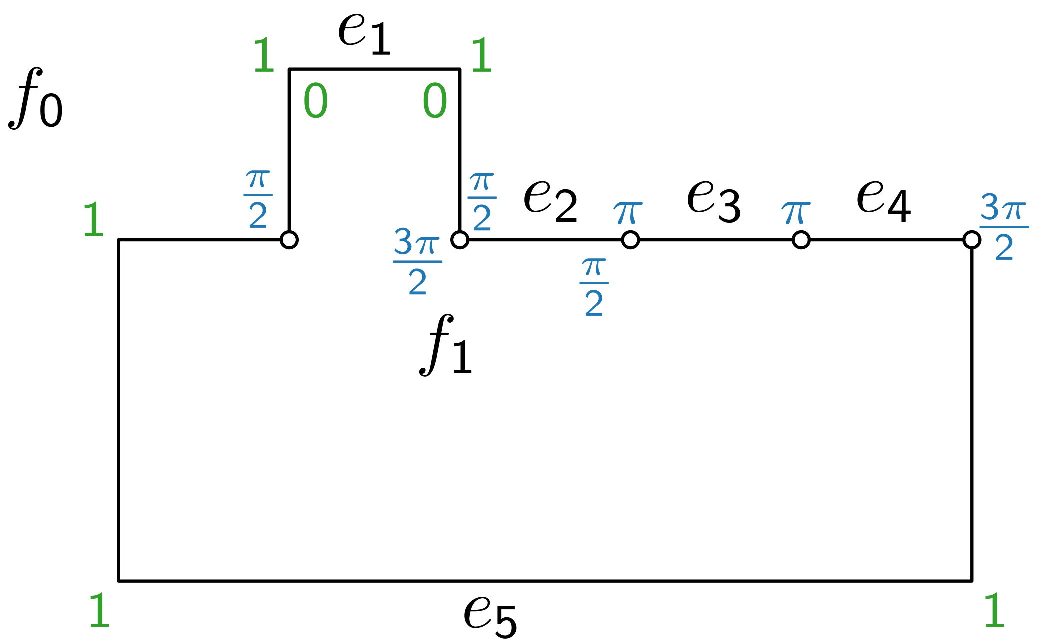
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$





Orthogonal Representation – Example

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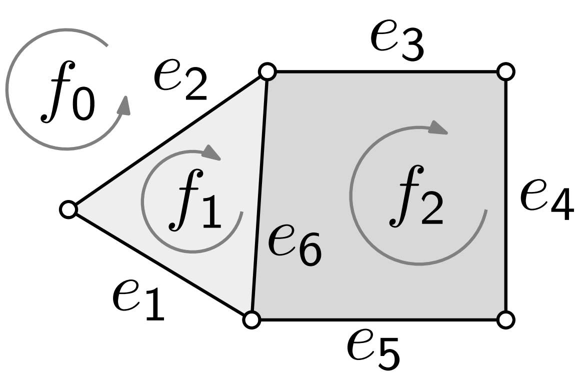

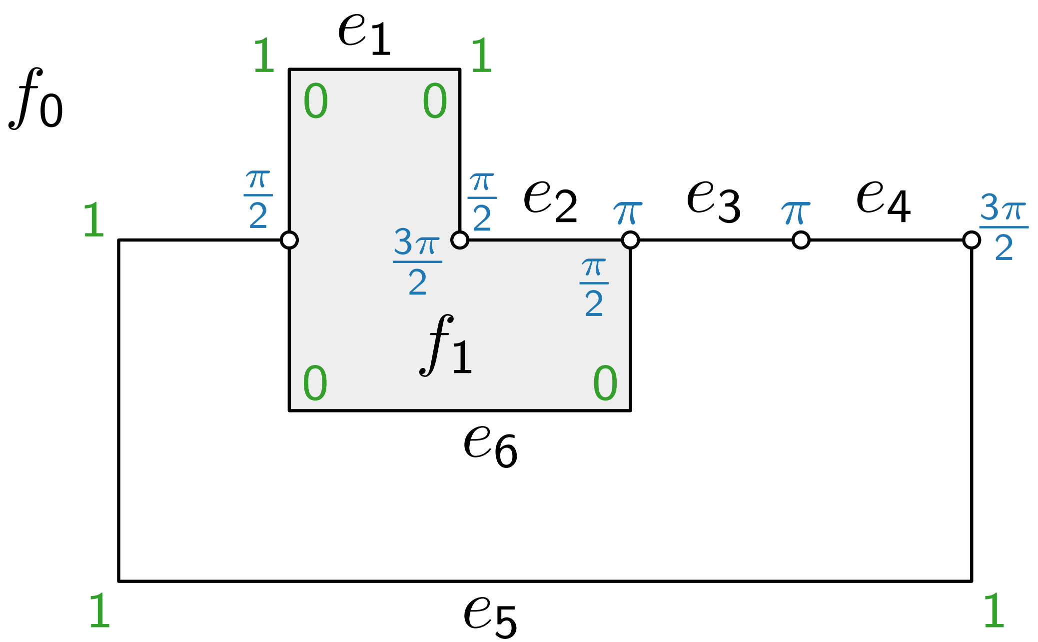

Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
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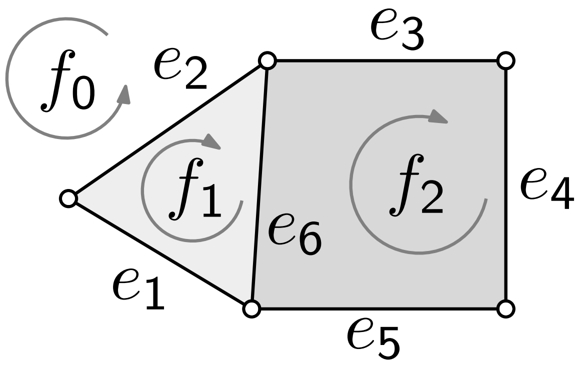

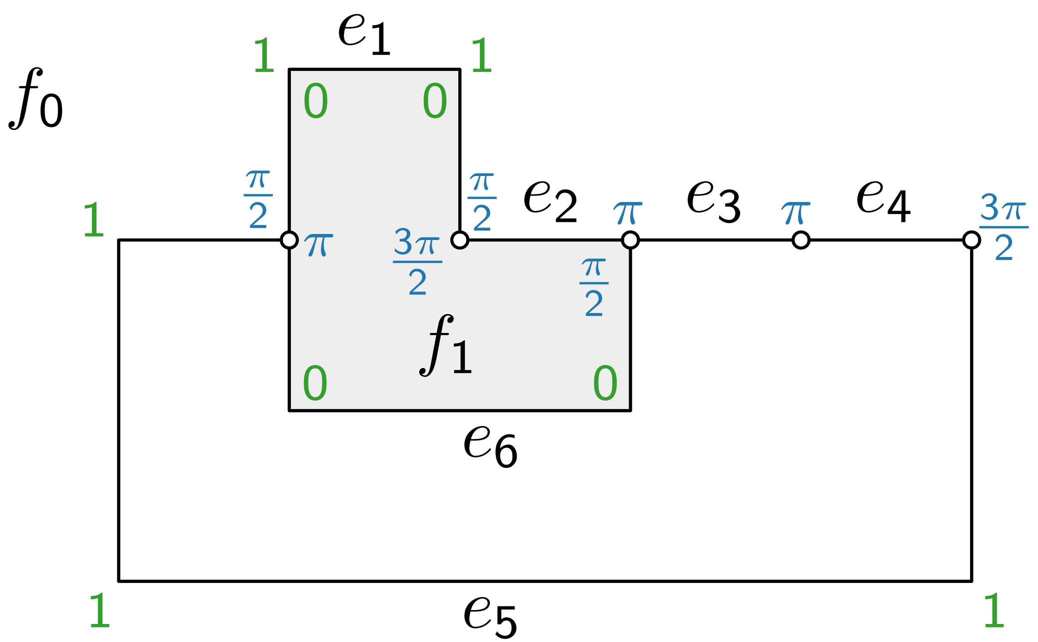

Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
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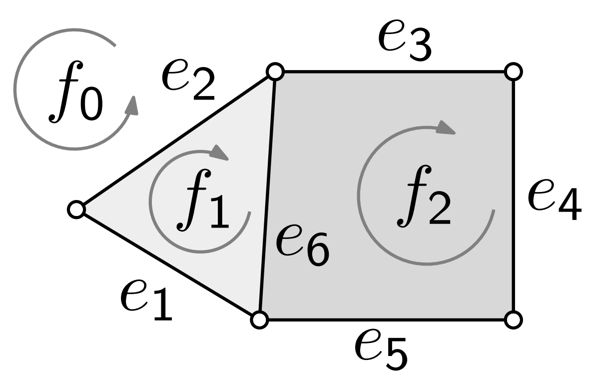

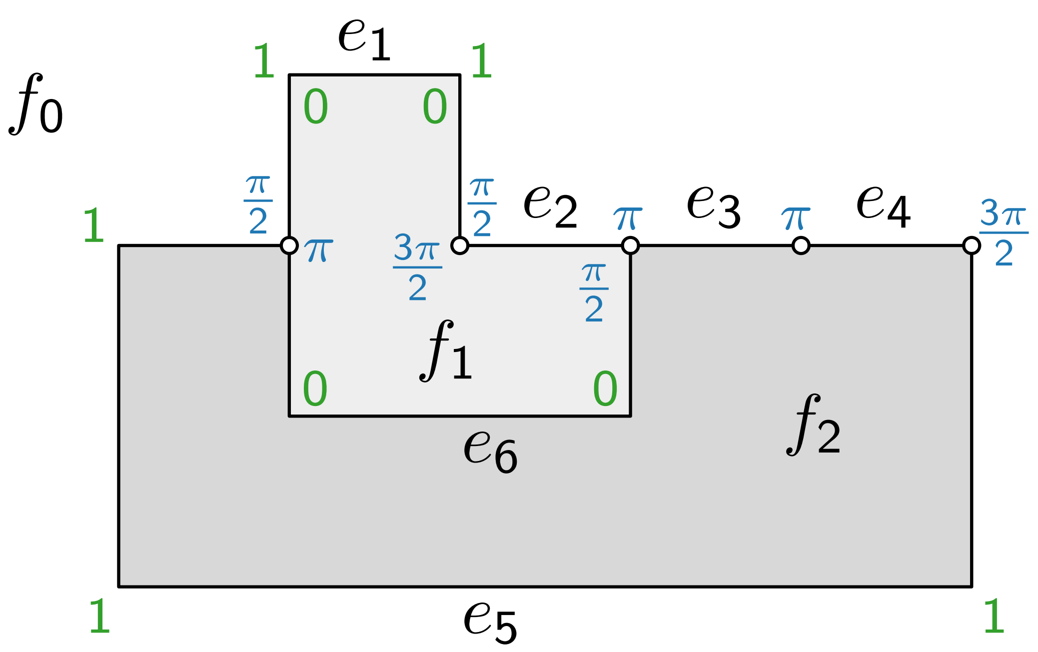

Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
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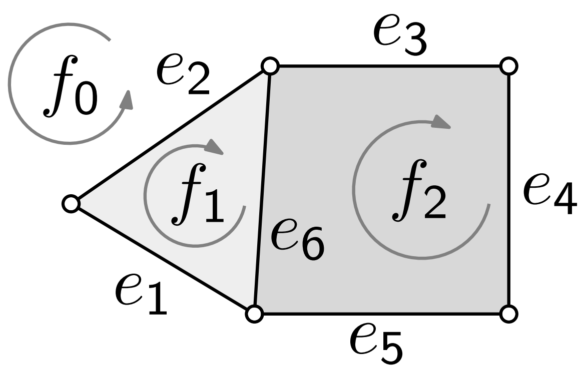

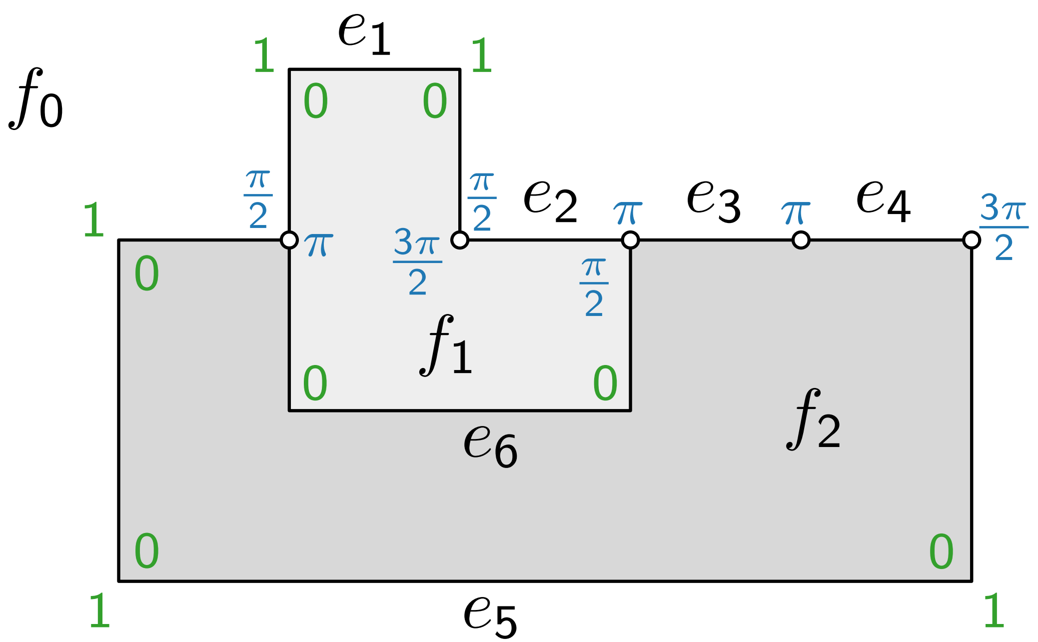

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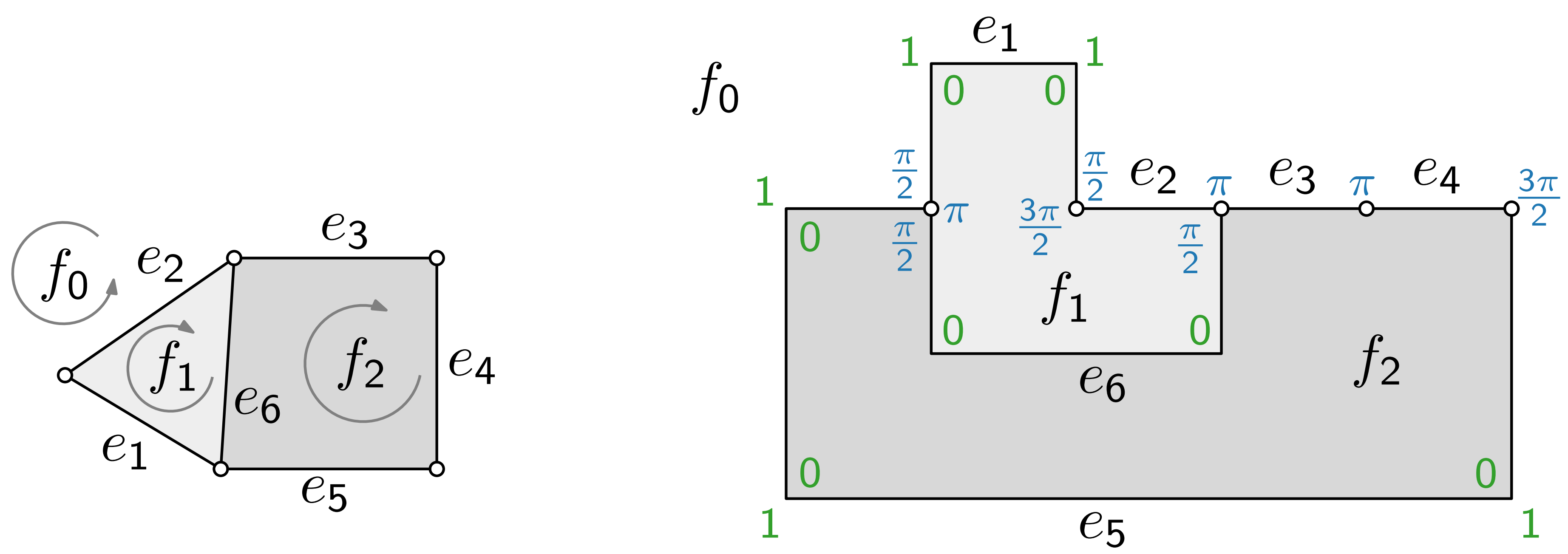

Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
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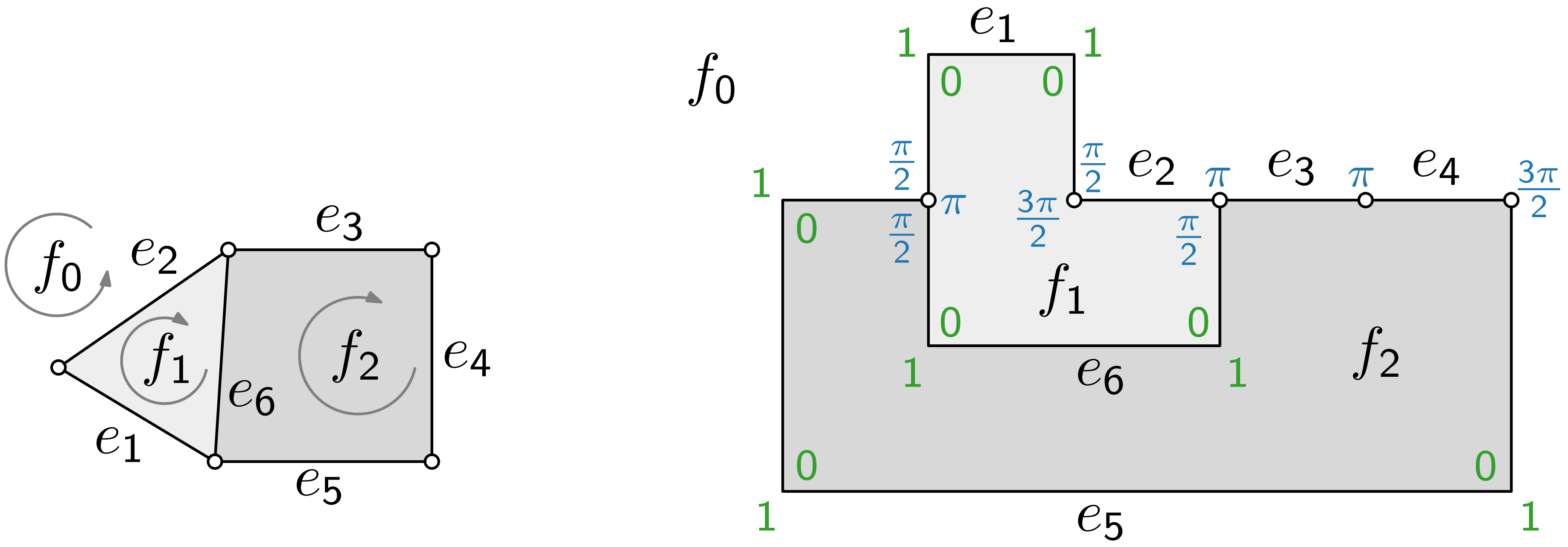

Orthogonal Representation – Example

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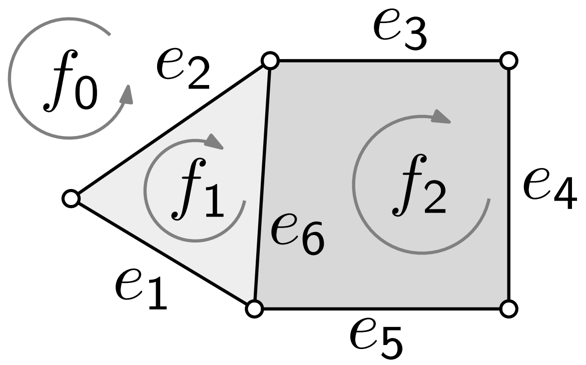

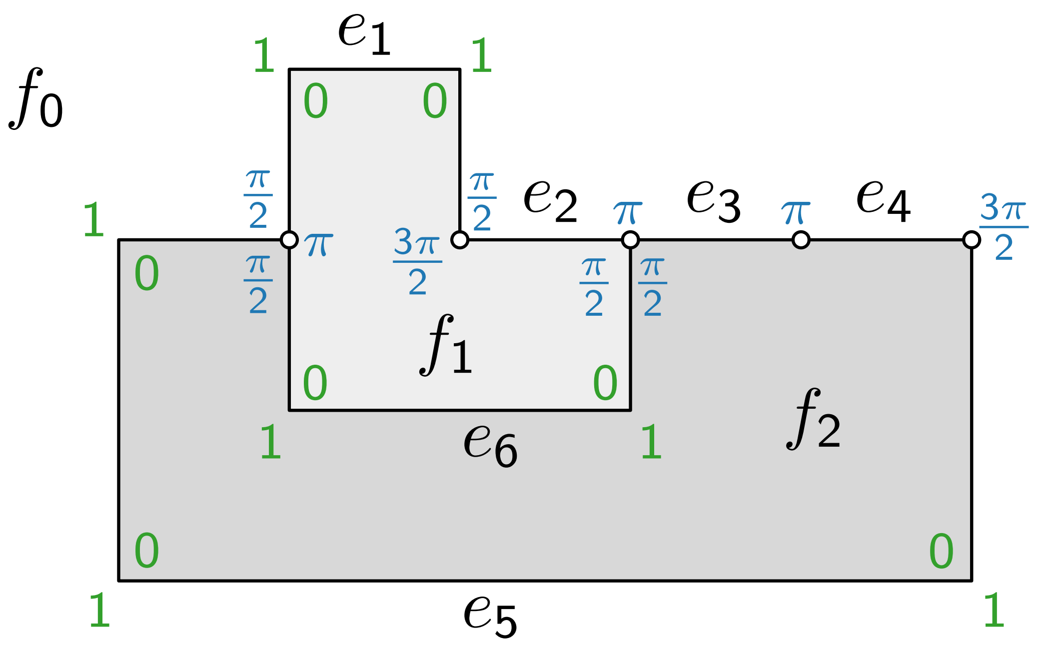

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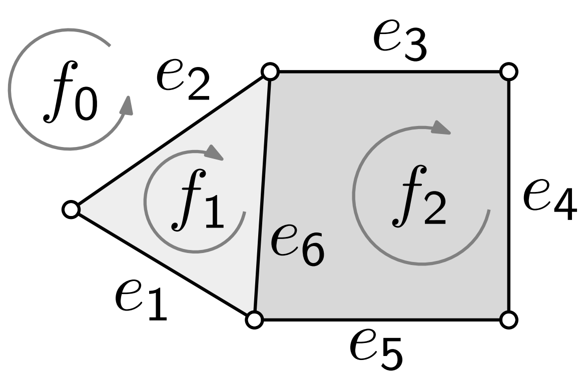

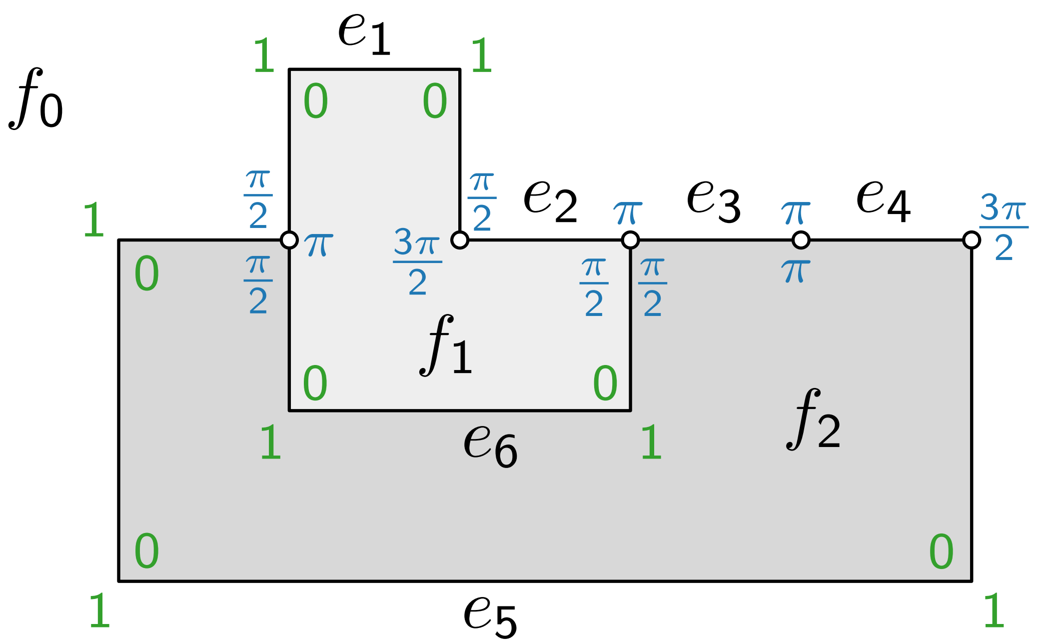

Orthogonal Representation – Example

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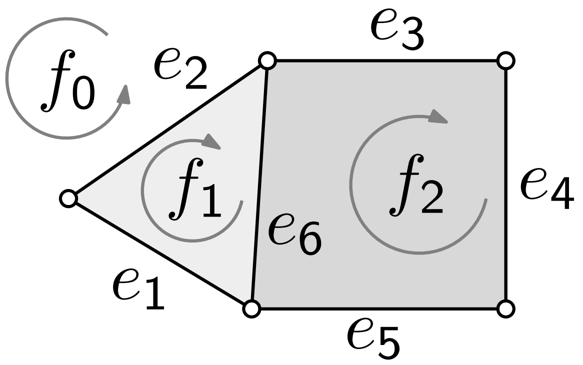

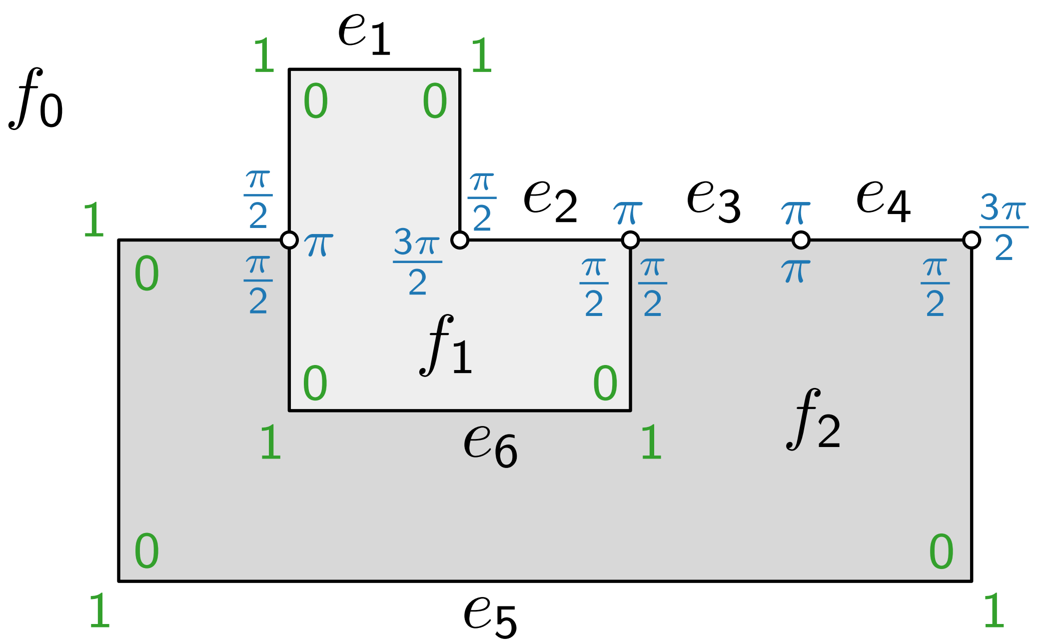

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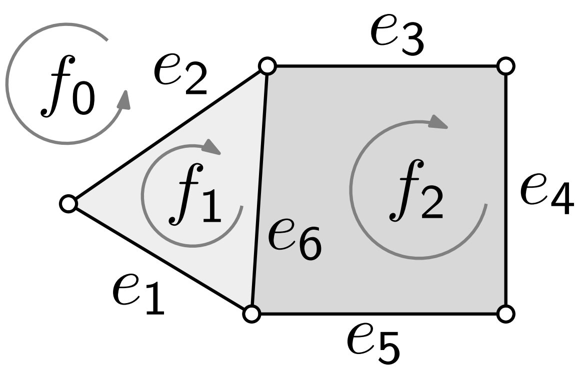
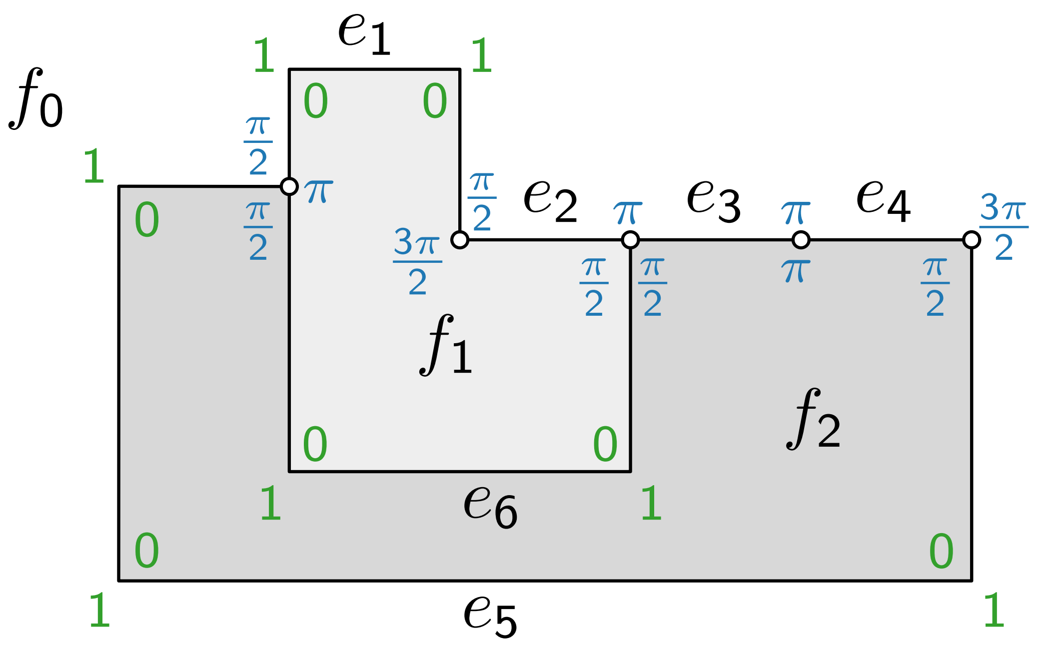
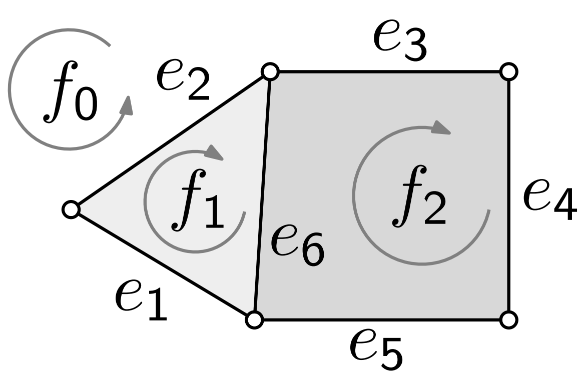
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

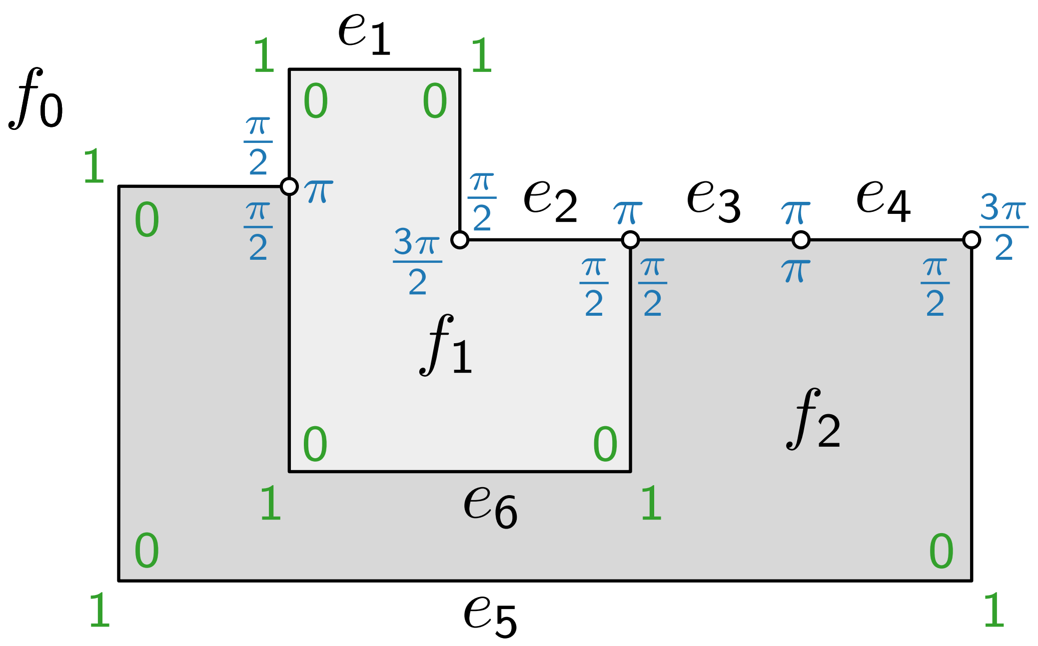




Orthogonal Representation – Example

$$H(f_0) = ((e_1, 11, \tfrac{\pi}{2}), (e_5, 111, \tfrac{3\pi}{2}), (e_4, \emptyset, \pi), (e_3, \emptyset, \pi), (e_2, \emptyset, \tfrac{\pi}{2}))$$
$$H(f_1) = ((e_1, 00, \tfrac{3\pi}{2}), (e_2, \emptyset, \tfrac{\pi}{2}), (e_6, 00, \pi))$$
$$H(f_2) = ((e_5, 000, \tfrac{\pi}{2}), (e_6, 11, \tfrac{\pi}{2}), (e_3, \emptyset, \pi), (e_4, \emptyset, \tfrac{\pi}{2}))$$

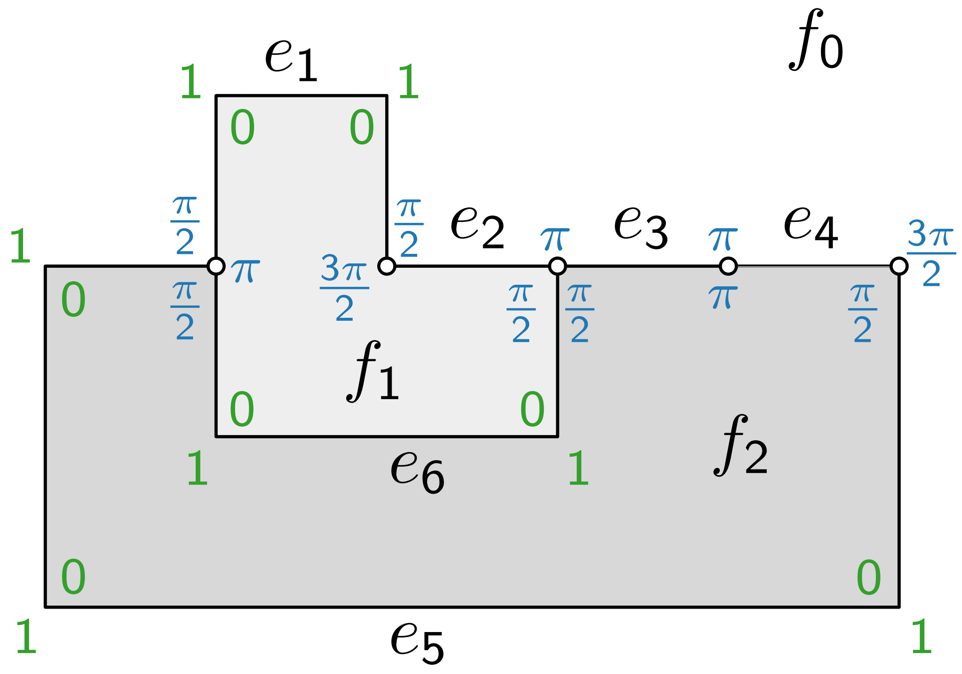




Concrete coordinates are not fixed yet!

Correctness of an Orthogonal Representation

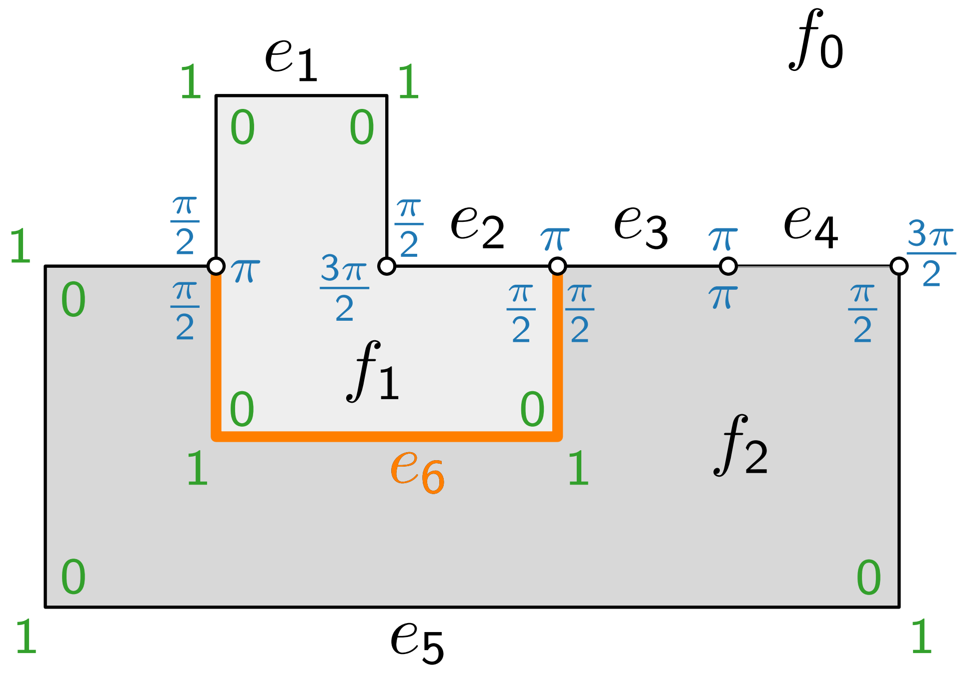
(H1) $H(G)$ corresponds to F, f_0.



Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.

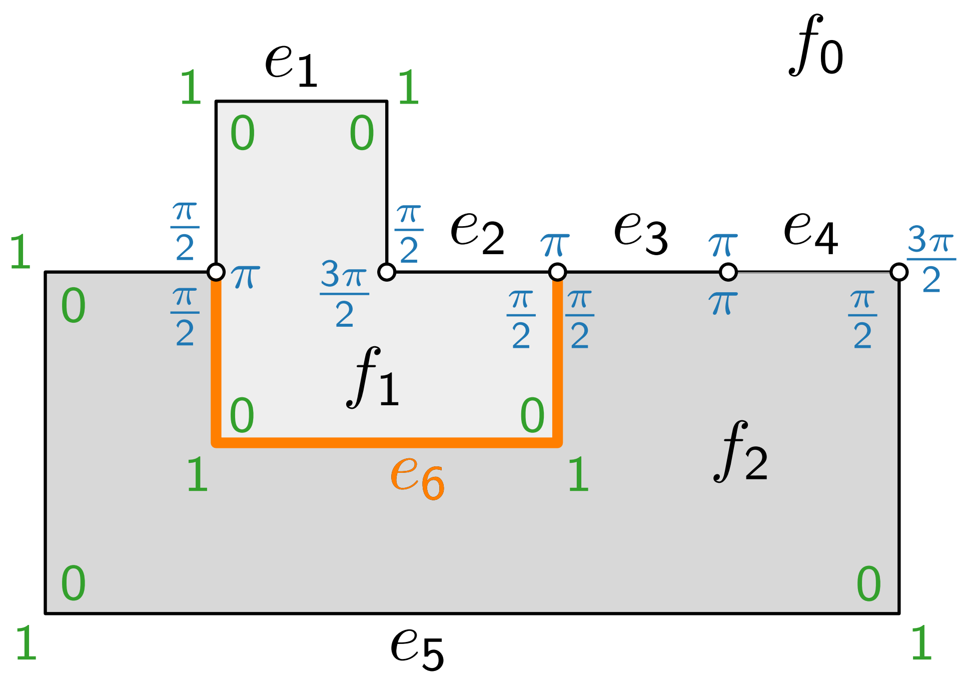
(H2) For each **edge** $\{u, v\}$ shared by faces f and g



Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u, v\}$ shared by faces f and g with $((u, v), \delta_1, \alpha_1) \in H(f)$ and $((v, u), \delta_2, \alpha_2) \in H(g)$



Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.

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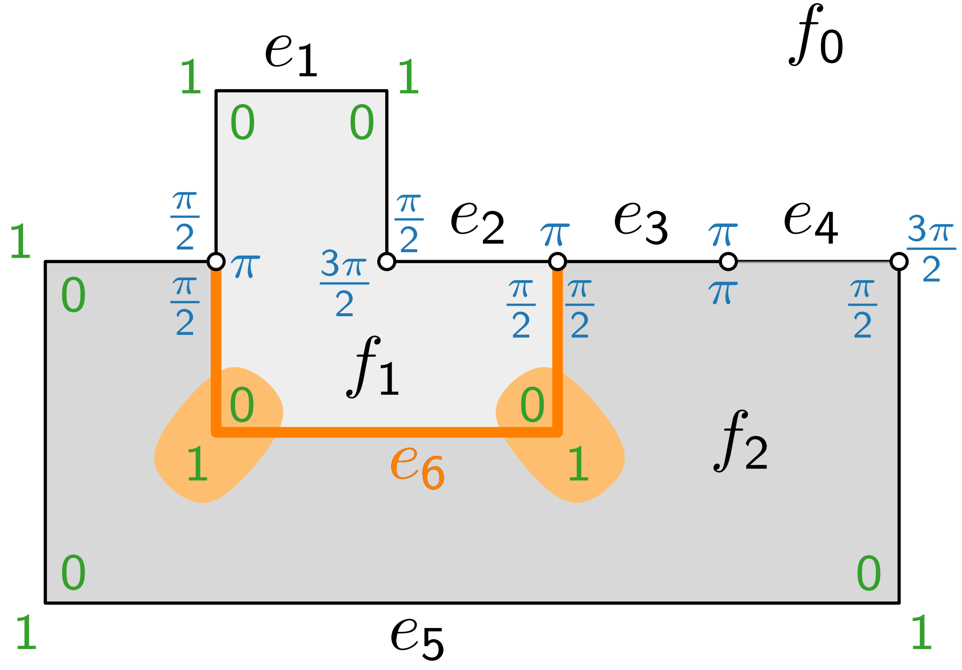

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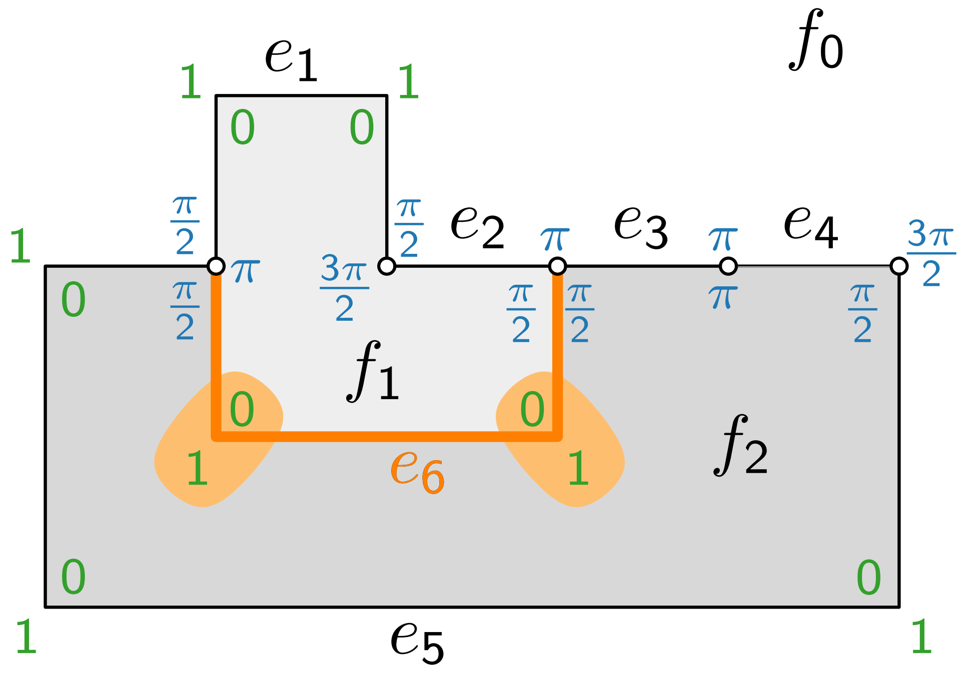

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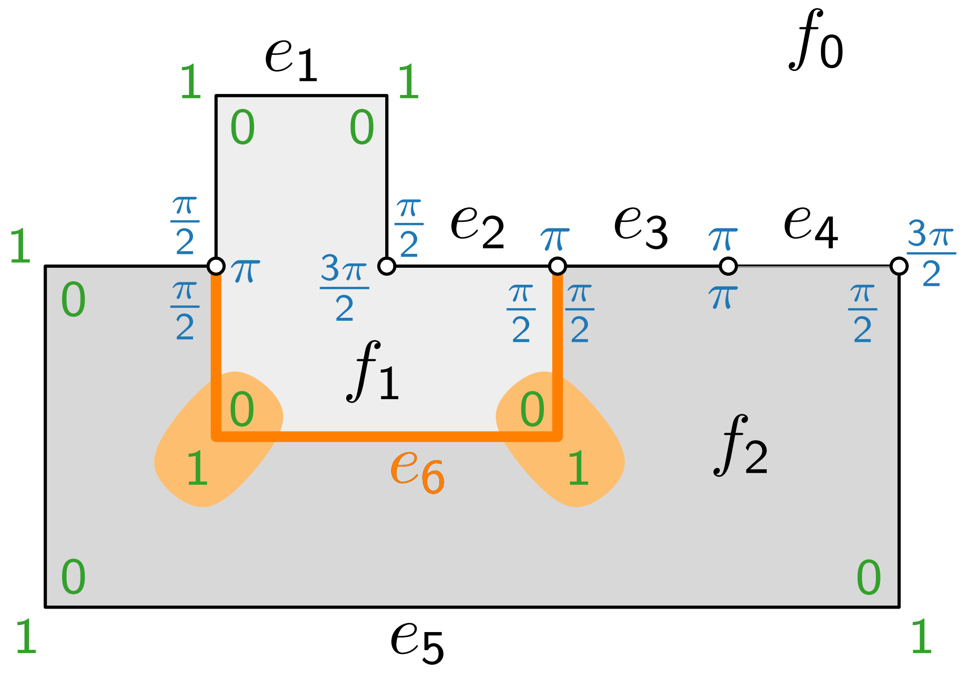
(H3) Let $|\delta|_0$ (resp. $|\delta|_1$) be the number of zeros (resp. ones) in δ and $r = (e, \delta, \alpha)$.
Let $C(r) := |\delta|_0 - |\delta|_1 + 2 - \alpha \cdot 2/\pi$.



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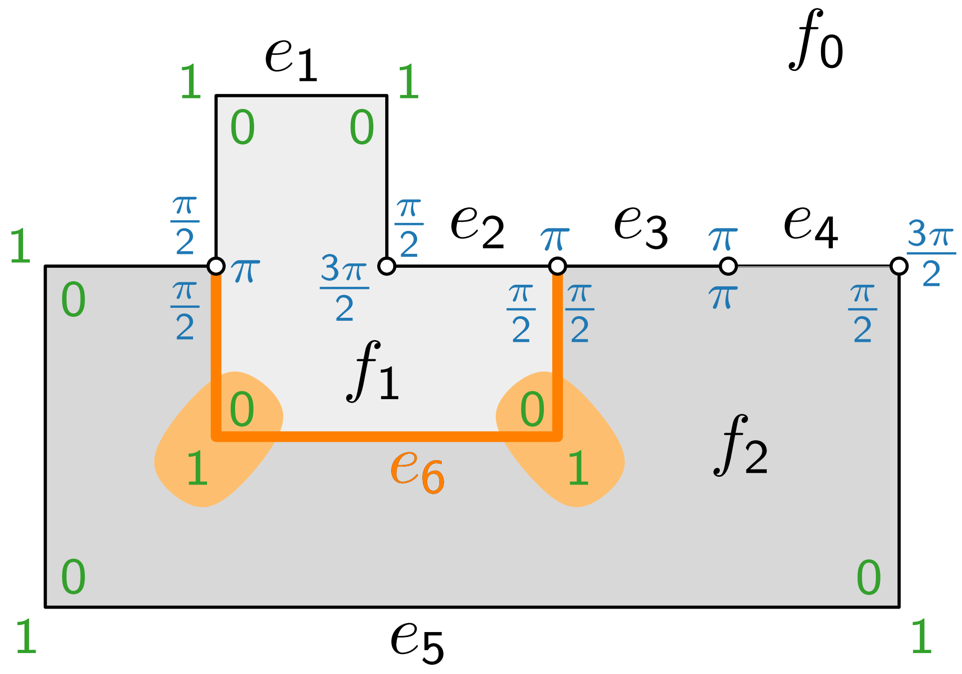
For each **face** f it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$



Correctness of an Orthogonal Representation

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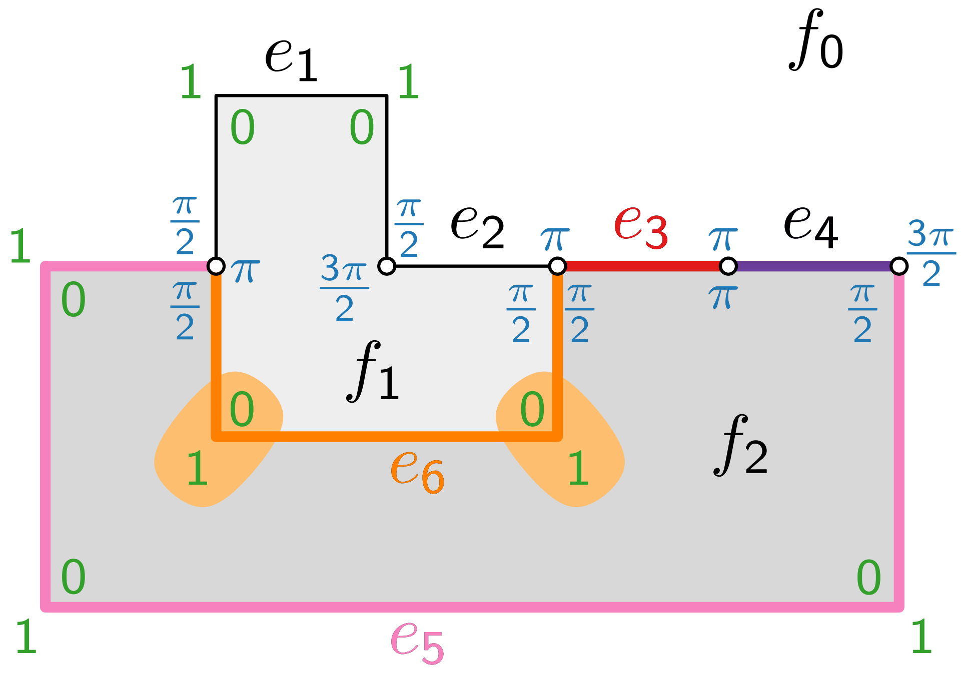

$C(e_3) = \quad - \quad + 2 - \quad =$

$C(e_4) = \quad - \quad + 2 - \quad =$

$C(e_5) = \quad - \quad + 2 - \quad =$

$C(e_6) = \quad - \quad + 2 - \quad =$

Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.

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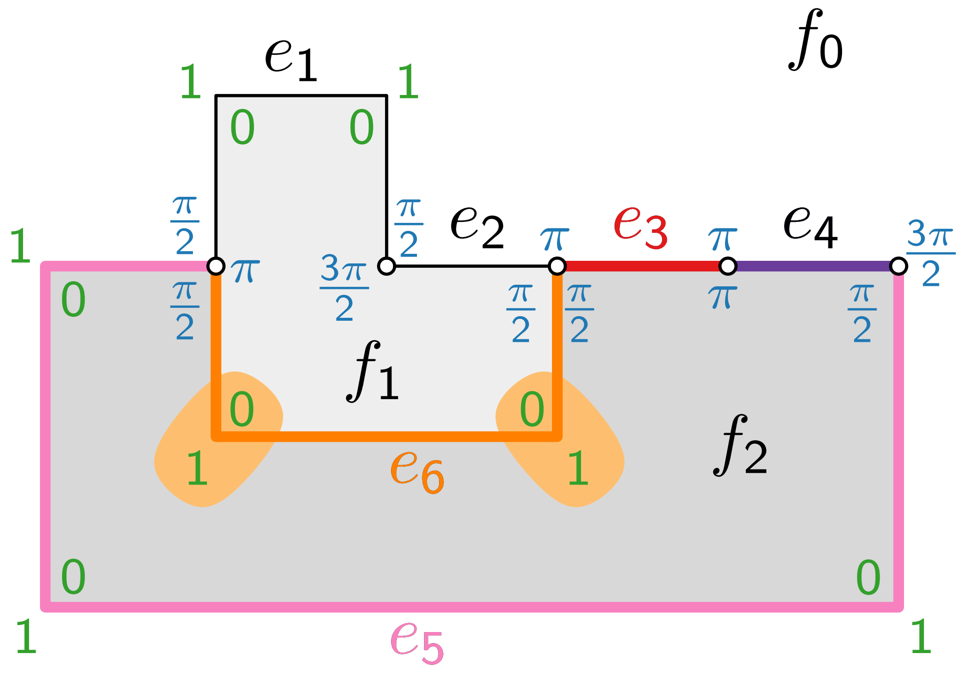

$C(e_3) = 0 - \quad + 2 - \quad =$

$C(e_4) = \quad - \quad + 2 - \quad =$

$C(e_5) = \quad - \quad + 2 - \quad =$

$C(e_6) = \quad - \quad + 2 - \quad =$

Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u, v\}$ shared by faces f and g with $((u,v), \delta_1, \alpha_1) \in H(f)$ and $((v,u), \delta_2, \alpha_2) \in H(g)$ sequence δ_1 is reversed and inverted δ_2.

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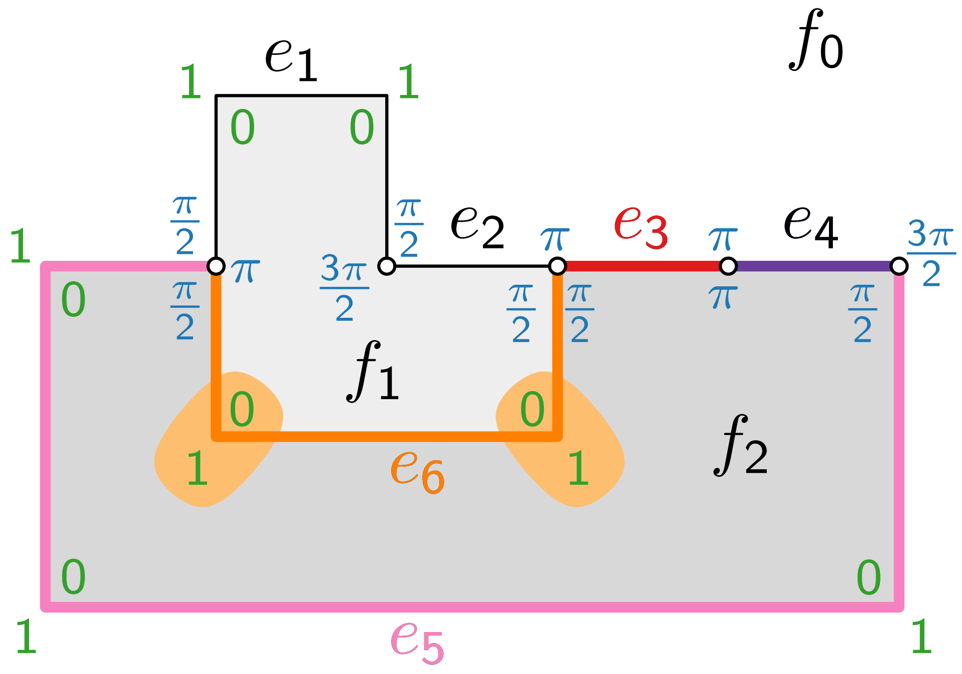

$C(e_3) = 0 - 0 + 2 - \quad =$

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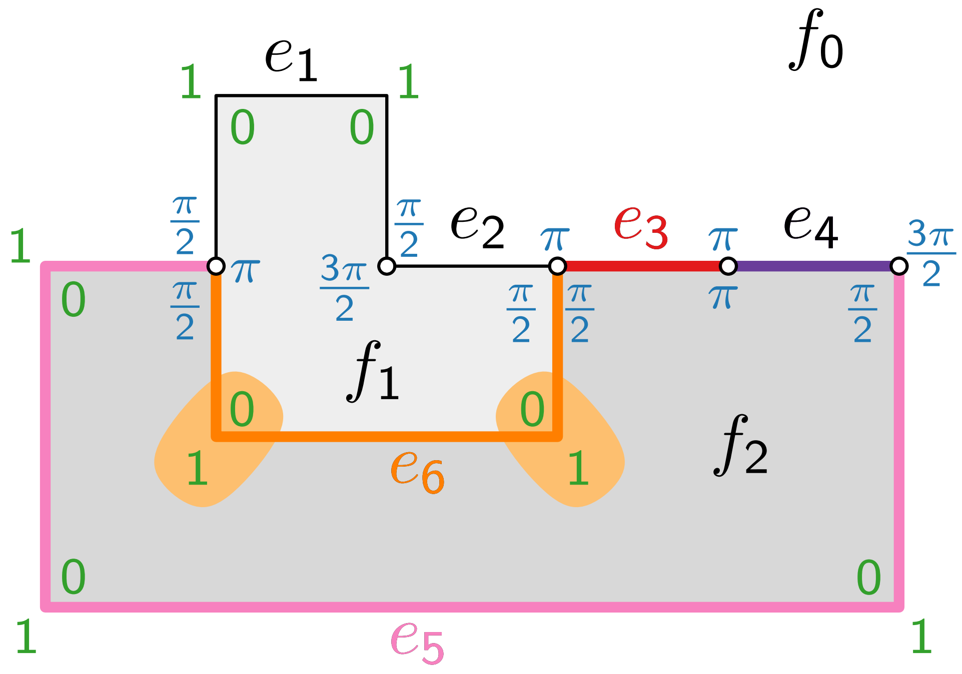

$$C(e_3) = 0 - 0 + 2 - \pi \cdot \frac{2}{\pi} =$$

$$C(e_4) = \quad - \quad + 2 - \quad =$$

$$C(e_5) = \quad - \quad + 2 - \quad =$$

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Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.

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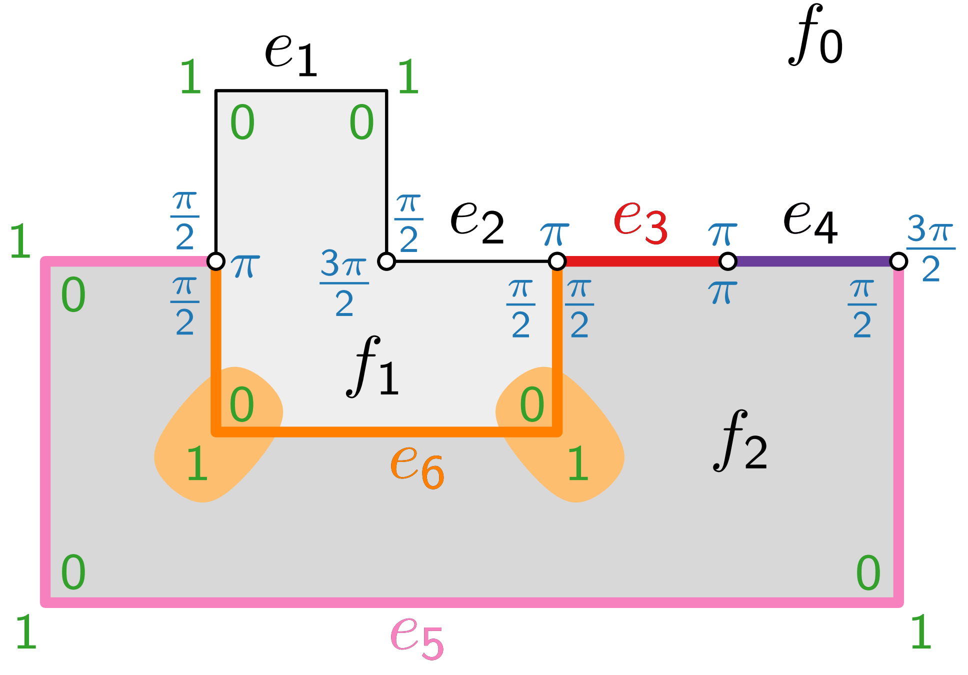

$C(e_3) = 0 - 0 + 2 - \pi \cdot \frac{2}{\pi} = 0$

$C(e_4) = \quad - \quad + 2 - \quad =$

$C(e_5) = \quad - \quad + 2 - \quad =$

$C(e_6) = \quad - \quad + 2 - \quad =$

Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.

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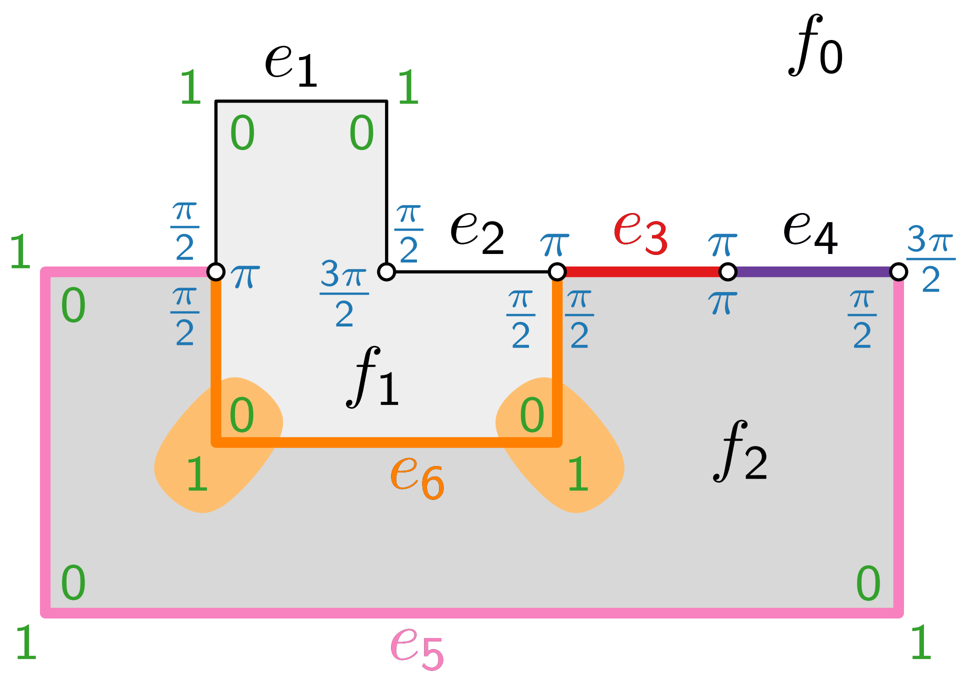

$$C(e_3) = 0 - 0 + 2 - \pi \cdot \frac{2}{\pi} = 0$$

$$C(e_4) = 0 - 0 + 2 - \frac{\pi}{2} \cdot \frac{2}{\pi} =$$

$$C(e_5) = \quad - \quad + 2 - \quad =$$

$$C(e_6) = \quad - \quad + 2 - \quad =$$

Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u,v\}$ shared by faces f and g with $((u,v),\delta_1,\alpha_1) \in H(f)$ and $((v,u),\delta_2,\alpha_2) \in H(g)$ sequence δ_1 is reversed and inverted δ_2.

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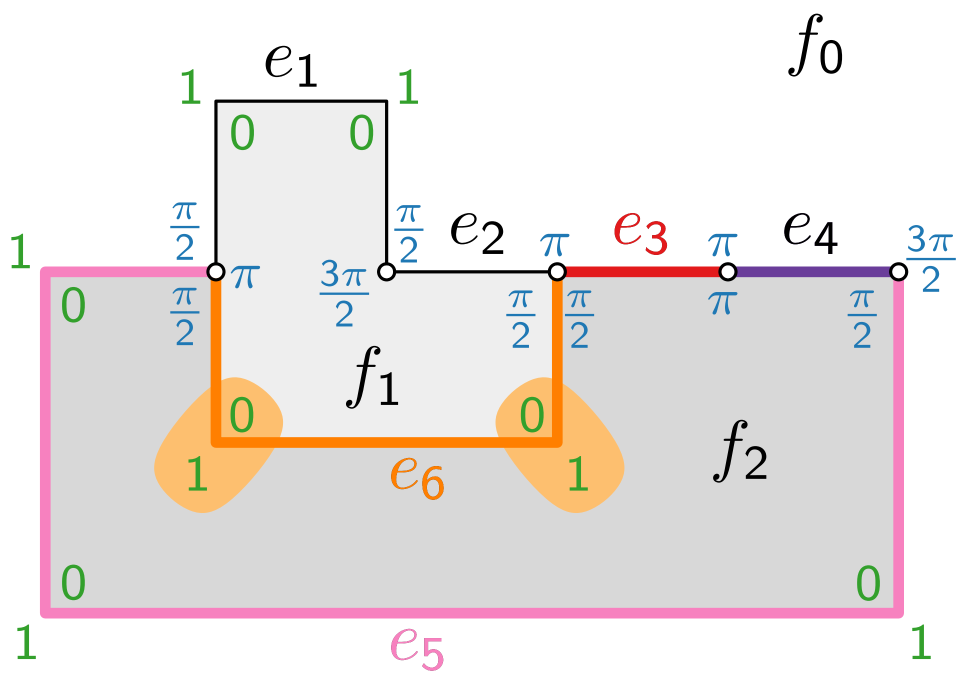

$C(e_3) = 0 - 0 + 2 - \pi \cdot \frac{2}{\pi} = 0$

$C(e_4) = 0 - 0 + 2 - \frac{\pi}{2} \cdot \frac{2}{\pi} = 1$

$C(e_5) = \quad - \quad + 2 - \quad =$

$C(e_6) = \quad - \quad + 2 - \quad =$

Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u, v\}$ shared by faces f and g with $((u, v), \delta_1, \alpha_1) \in H(f)$ and $((v, u), \delta_2, \alpha_2) \in H(g)$ sequence δ_1 is reversed and inverted δ_2.

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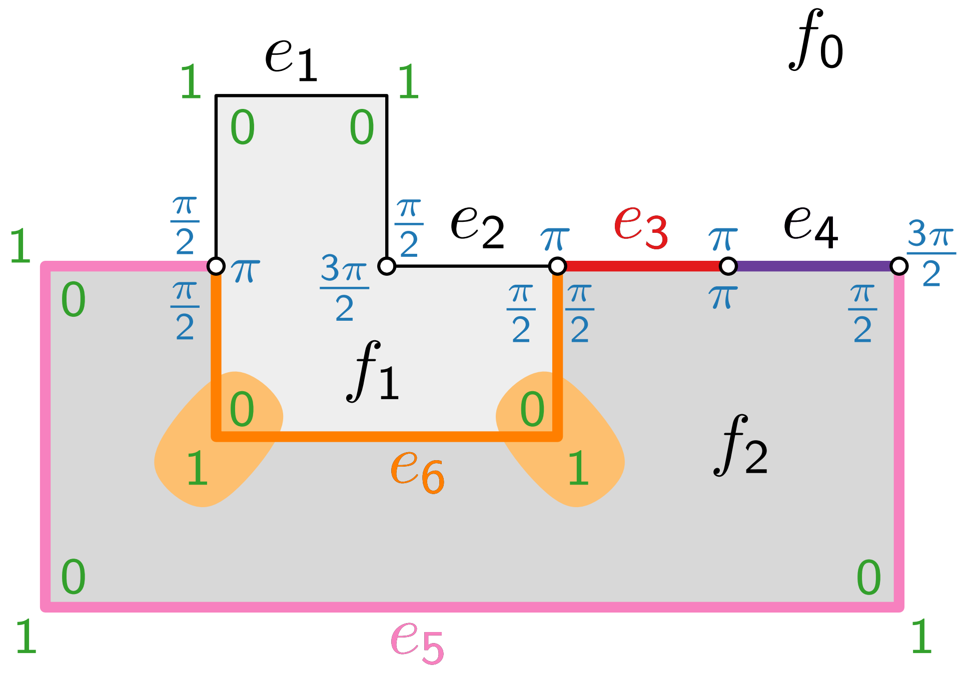

$C(e_3) = 0 - 0 + 2 - \pi \cdot \frac{2}{\pi} = 0$

$C(e_4) = 0 - 0 + 2 - \frac{\pi}{2} \cdot \frac{2}{\pi} = 1$

$C(e_5) = 3 - 0 + 2 - \quad =$

$C(e_6) = \quad - \quad + 2 - \quad =$

Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u,v\}$ shared by faces f and g with $((u,v),\delta_1,\alpha_1) \in H(f)$ and $((v,u),\delta_2,\alpha_2) \in H(g)$ sequence δ_1 is reversed and inverted δ_2.

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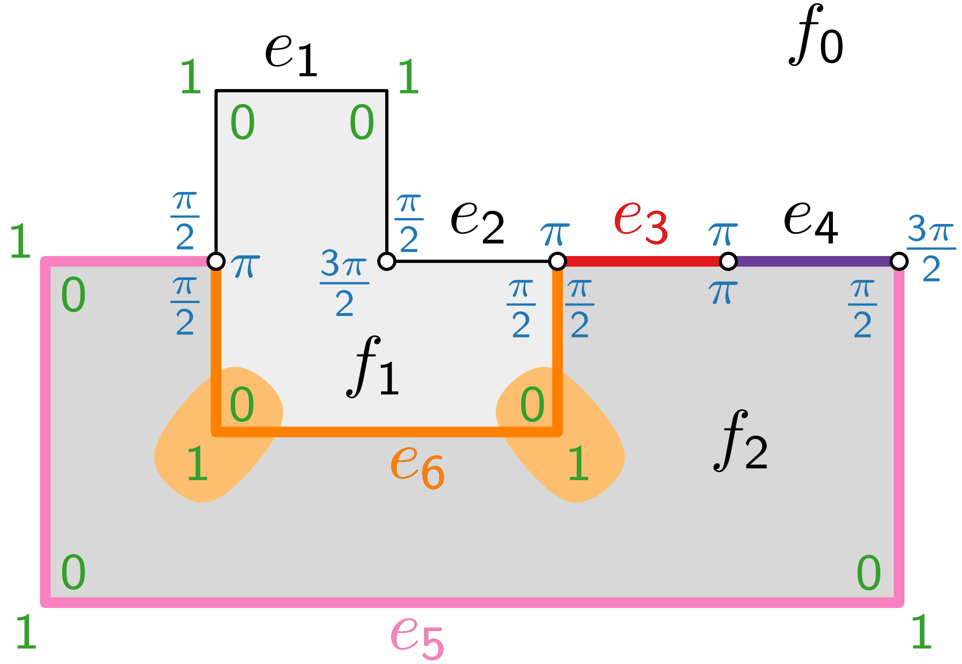

$$C(e_3) = 0 - 0 + 2 - \pi \cdot \frac{2}{\pi} = 0$$

$$C(e_4) = 0 - 0 + 2 - \frac{\pi}{2} \cdot \frac{2}{\pi} = 1$$

$$C(e_5) = 3 - 0 + 2 - \frac{\pi}{2} \cdot \frac{2}{\pi} =$$

$$C(e_6) = \quad - \quad + 2 - \quad =$$

Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u, v\}$ shared by faces f and g with $((u, v), \delta_1, \alpha_1) \in H(f)$ and $((v, u), \delta_2, \alpha_2) \in H(g)$ sequence δ_1 is reversed and inverted δ_2.

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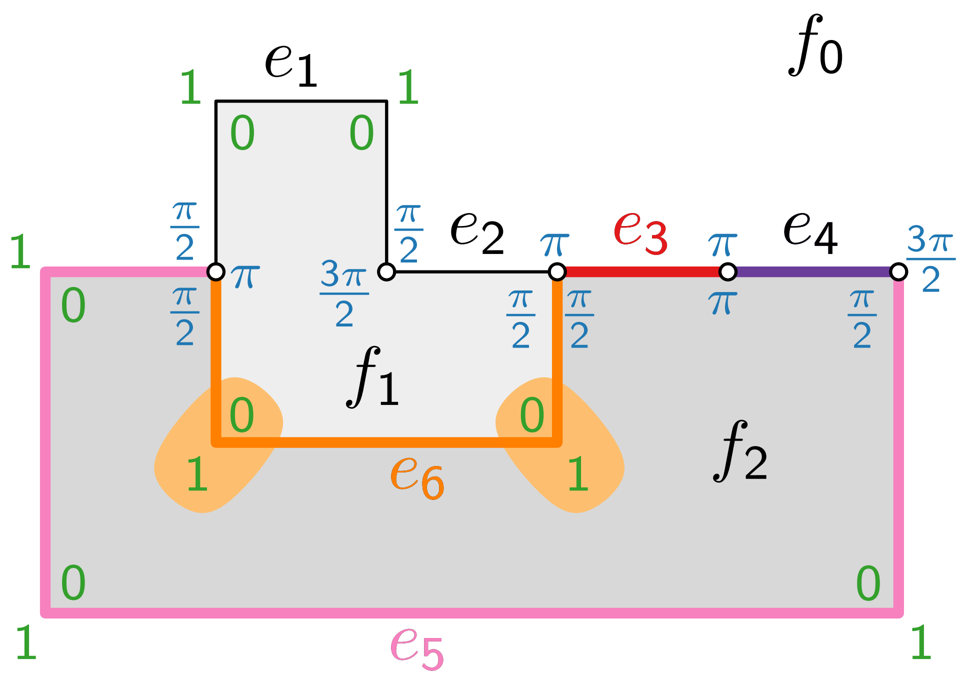

$$C(e_3) = 0 - 0 + 2 - \pi \cdot \frac{2}{\pi} = 0$$

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$$C(e_5) = 3 - 0 + 2 - \frac{\pi}{2} \cdot \frac{2}{\pi} = 4$$

$$C(e_6) = \quad - \quad + 2 - \quad =$$

Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u, v\}$ shared by faces f and g with $((u, v), \delta_1, \alpha_1) \in H(f)$ and $((v, u), \delta_2, \alpha_2) \in H(g)$ sequence δ_1 is reversed and inverted δ_2.

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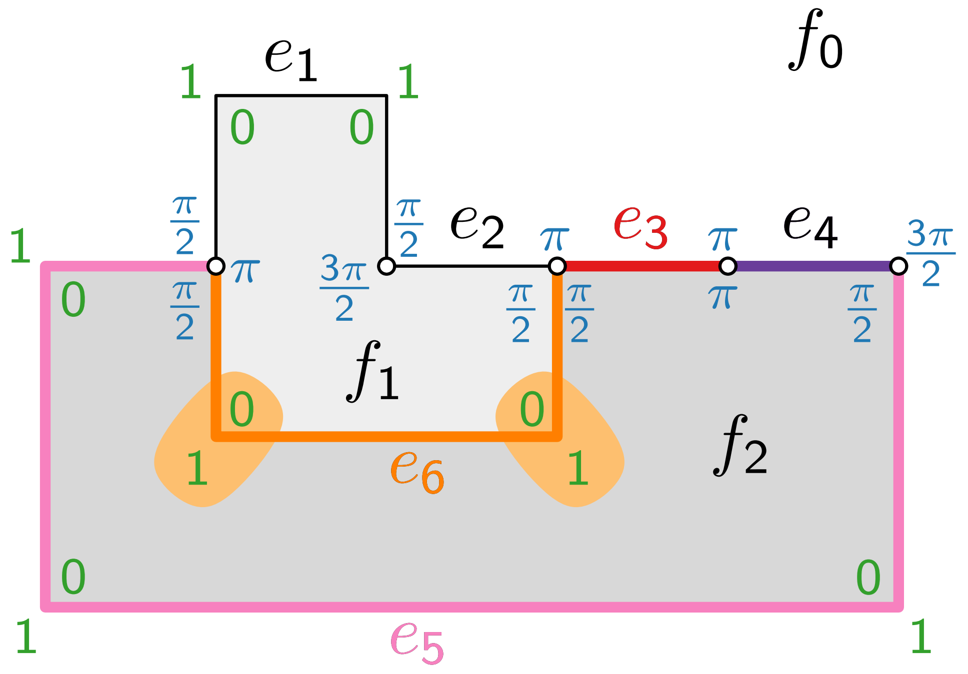

$$C(e_3) = 0 - 0 + 2 - \pi \cdot \tfrac{2}{\pi} = 0$$

$$C(e_4) = 0 - 0 + 2 - \tfrac{\pi}{2} \cdot \tfrac{2}{\pi} = 1$$

$$C(e_5) = 3 - 0 + 2 - \tfrac{\pi}{2} \cdot \tfrac{2}{\pi} = 4$$

$$C(e_6) = 0 - 2 + 2 - \tfrac{\pi}{2} \cdot \tfrac{2}{\pi} =$$

Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u, v\}$ shared by faces f and g with $((u, v), \delta_1, \alpha_1) \in H(f)$ and $((v, u), \delta_2, \alpha_2) \in H(g)$ sequence δ_1 is reversed and inverted δ_2.

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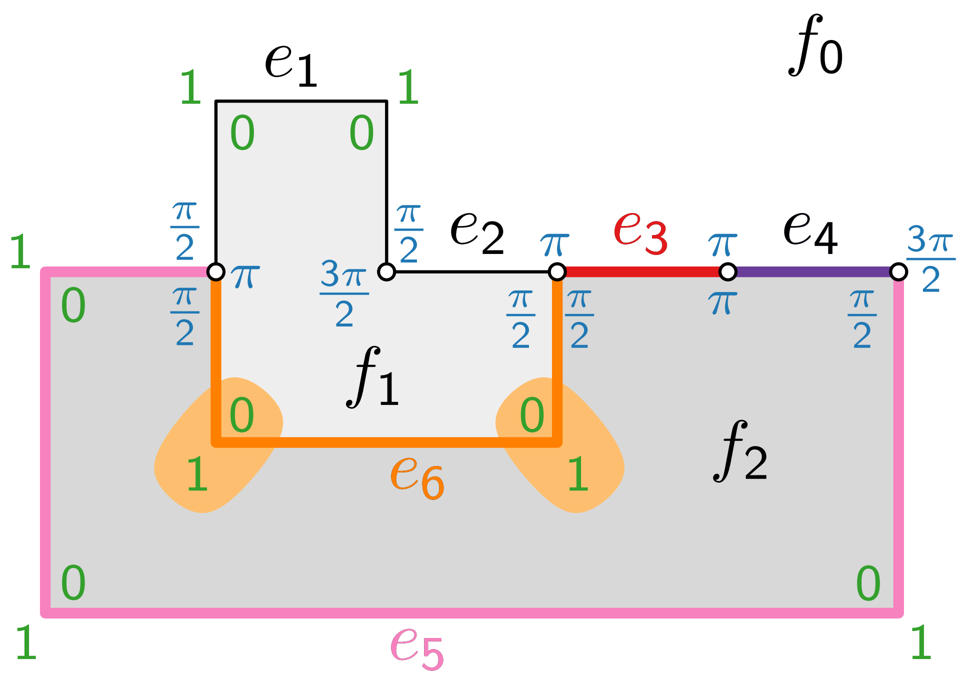

$$C(e_3) = 0 - 0 + 2 - \pi \cdot \frac{2}{\pi} = 0$$

$$C(e_4) = 0 - 0 + 2 - \frac{\pi}{2} \cdot \frac{2}{\pi} = 1$$

$$C(e_5) = 3 - 0 + 2 - \frac{\pi}{2} \cdot \frac{2}{\pi} = 4$$

$$C(e_6) = 0 - 2 + 2 - \frac{\pi}{2} \cdot \frac{2}{\pi} = -1$$

Correctness of an Orthogonal Representation

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u, v\}$ shared by faces f and g with $((u, v), \delta_1, \alpha_1) \in H(f)$ and $((v, u), \delta_2, \alpha_2) \in H(g)$ sequence δ_1 is reversed and inverted δ_2.

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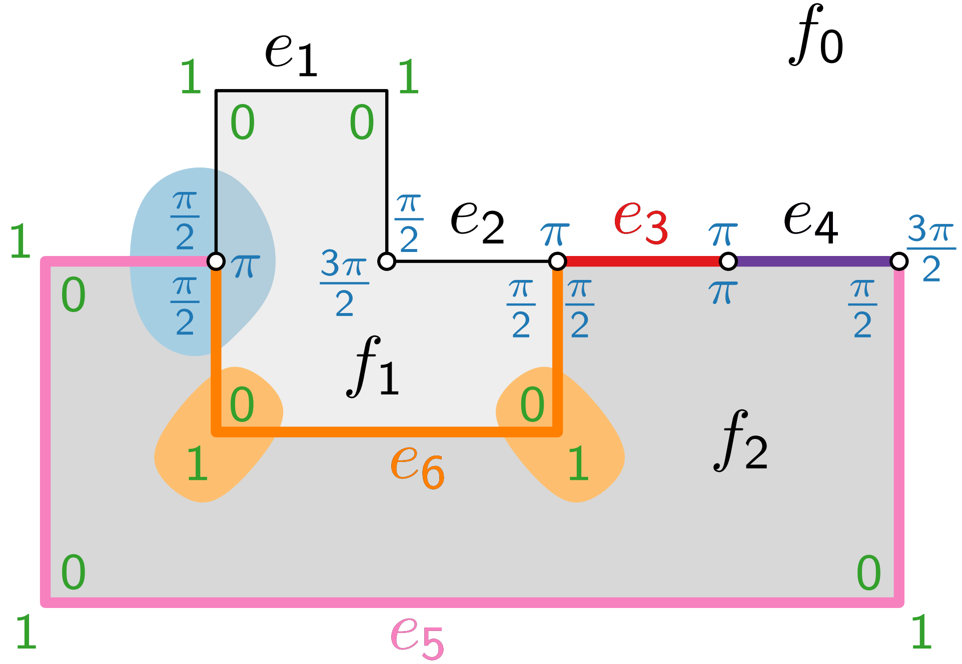
$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

(H4) For each **vertex** v the sum of incident angles is 2π.



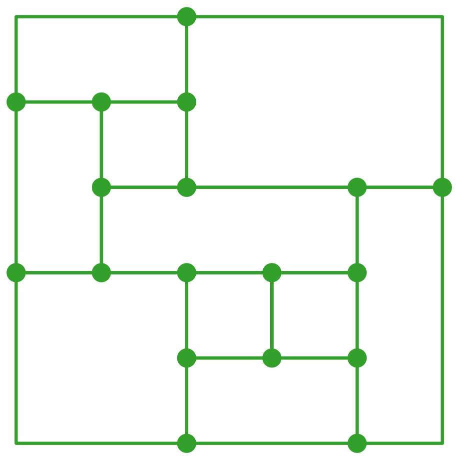
$$C(e_3) = 0 - 0 + 2 - \pi \cdot \frac{2}{\pi} = 0$$

$$C(e_4) = 0 - 0 + 2 - \frac{\pi}{2} \cdot \frac{2}{\pi} = 1$$

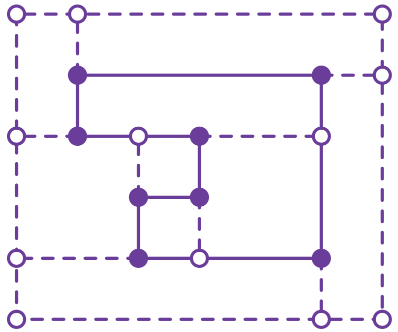
$$C(e_5) = 3 - 0 + 2 - \frac{\pi}{2} \cdot \frac{2}{\pi} = 4$$

$$C(e_6) = 0 - 2 + 2 - \frac{\pi}{2} \cdot \frac{2}{\pi} = -1$$

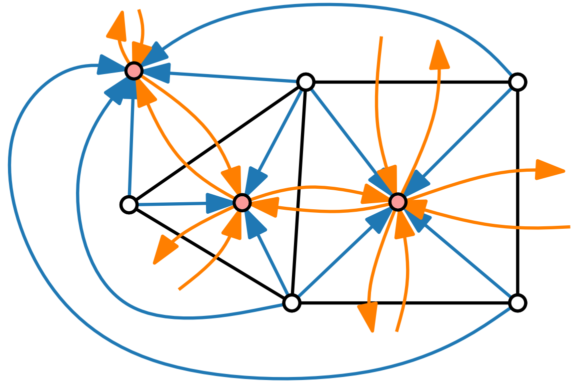
Visualization of Graphs



Lecture 5:
Orthogonal Layouts

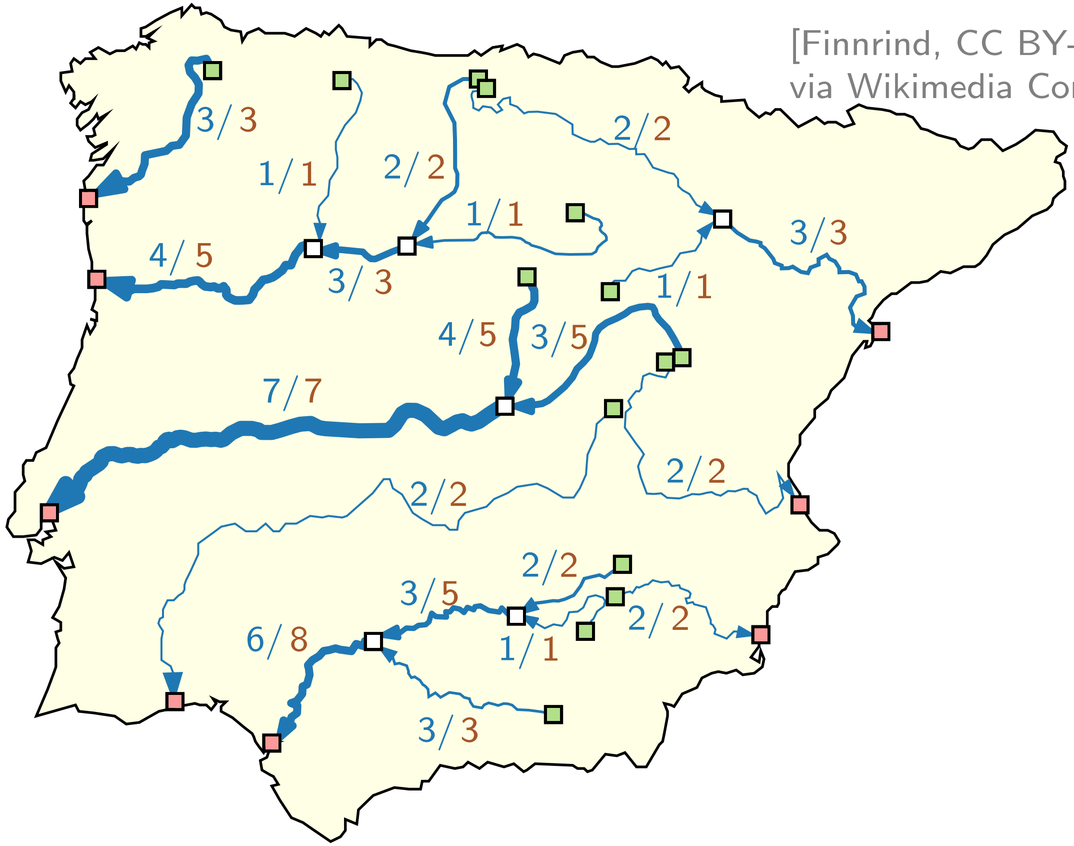


Part III:
Bend Minimization



Jonathan Klawitter

Reminder: s-t-Flow Networks



Flow network $(G = (V, E); S, T; u)$ with

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- *sources* $S \subseteq V$, *sinks* $T \subseteq V$
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A function $X \colon E \to \mathbb{R}_0^+$ is called S-T-**flow**, if:

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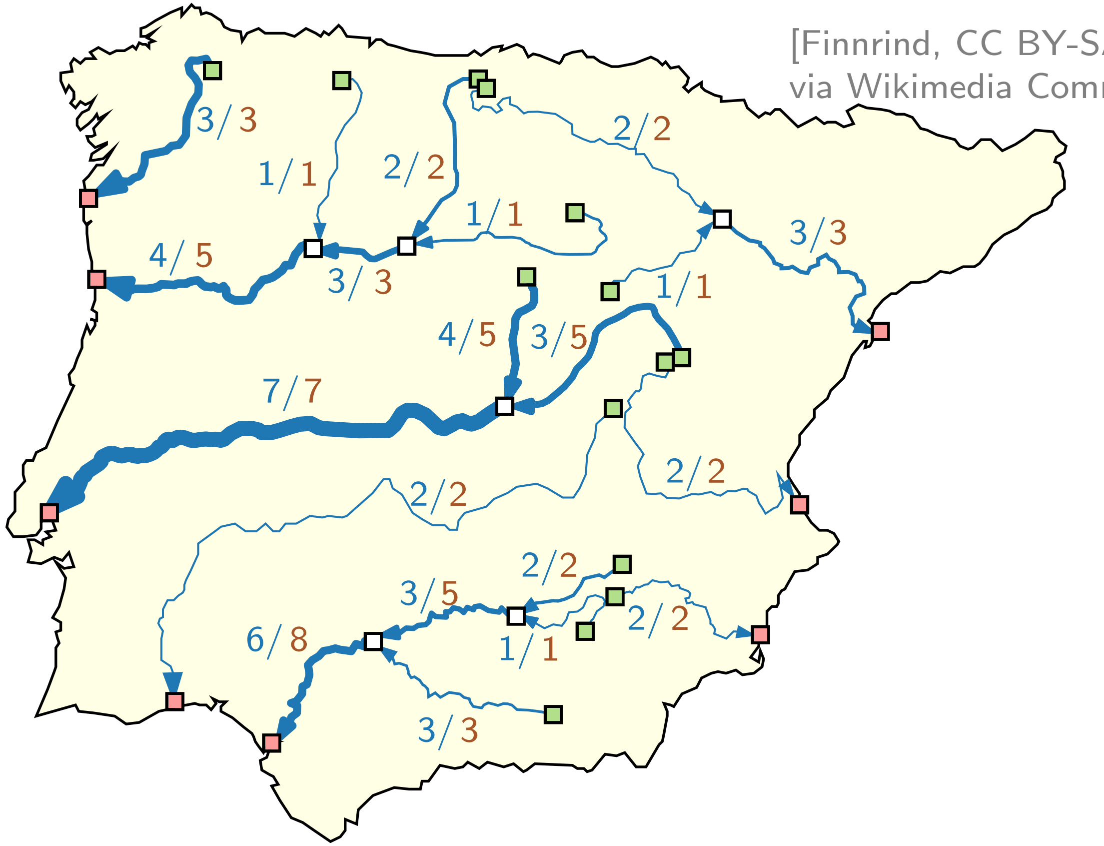

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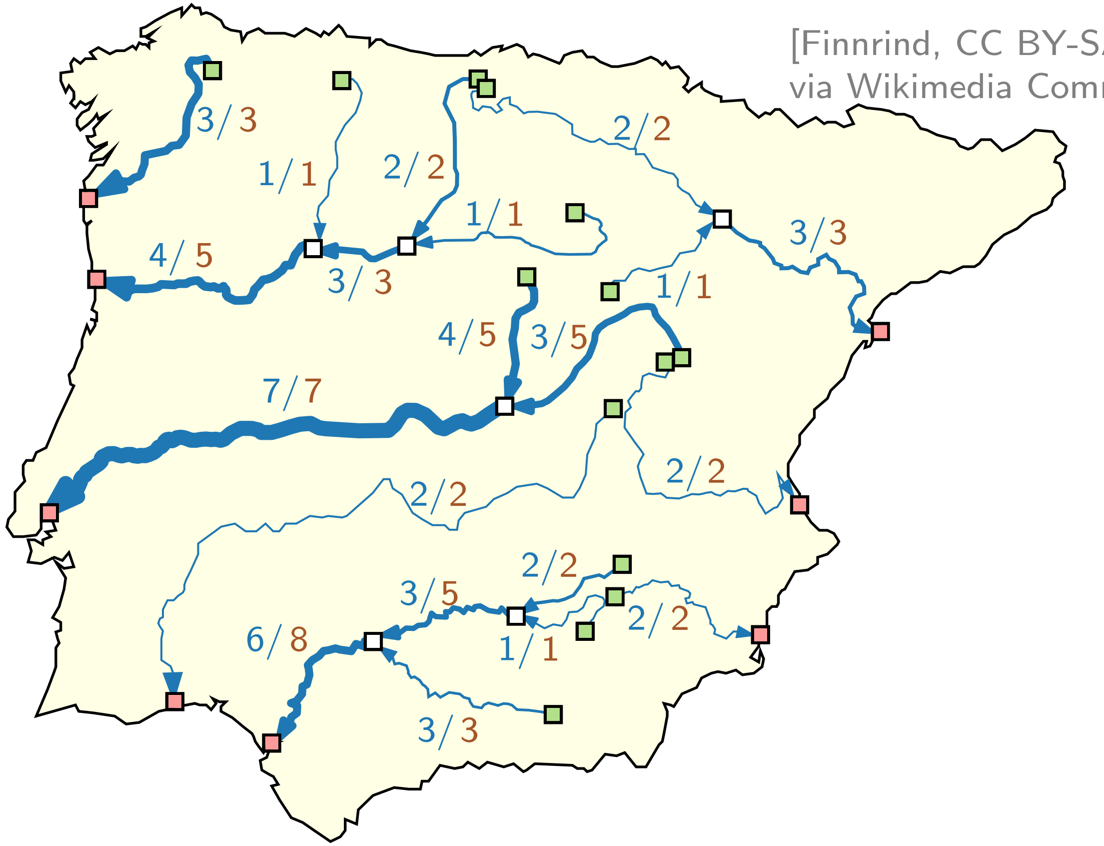

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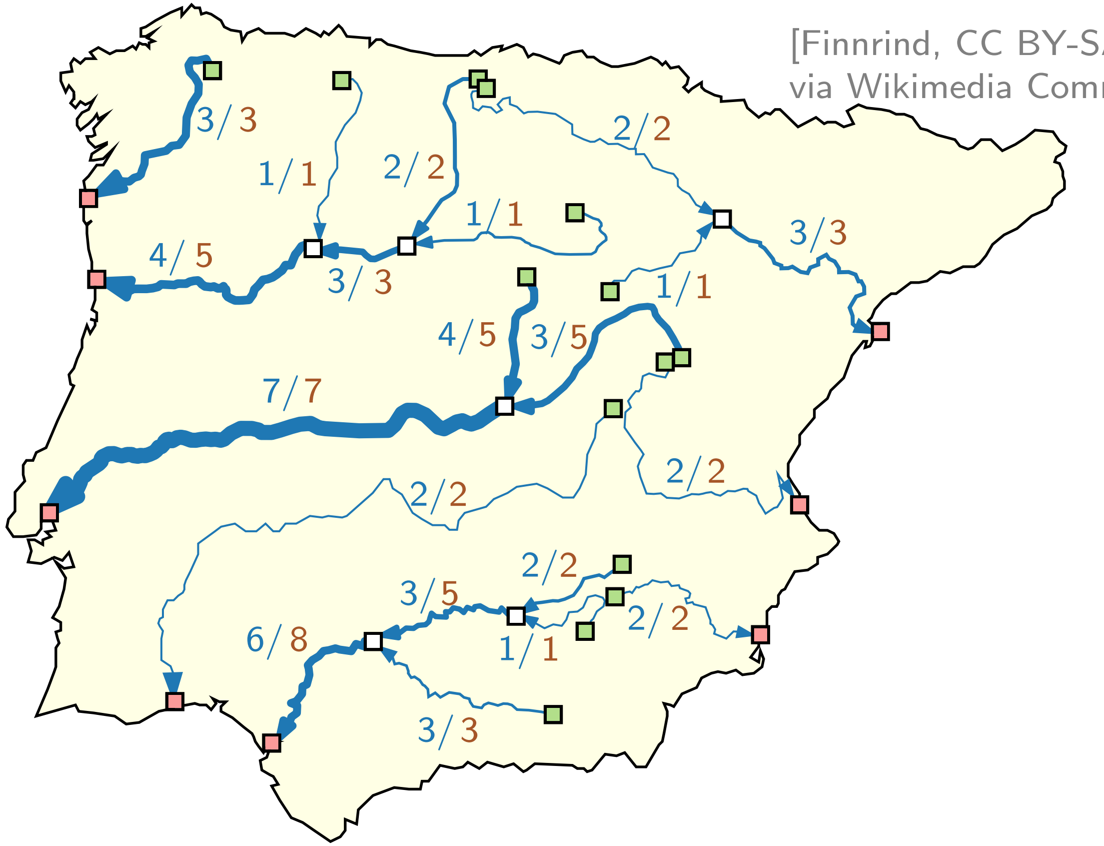

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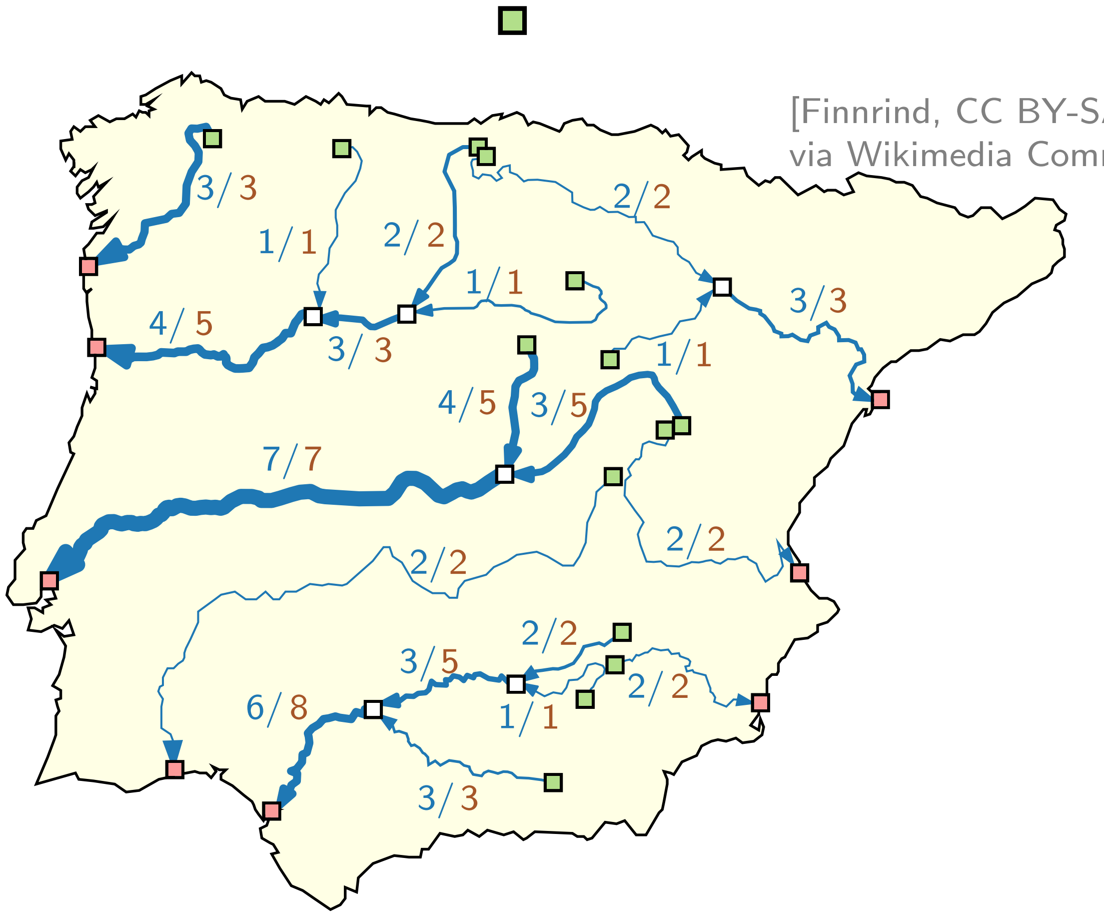

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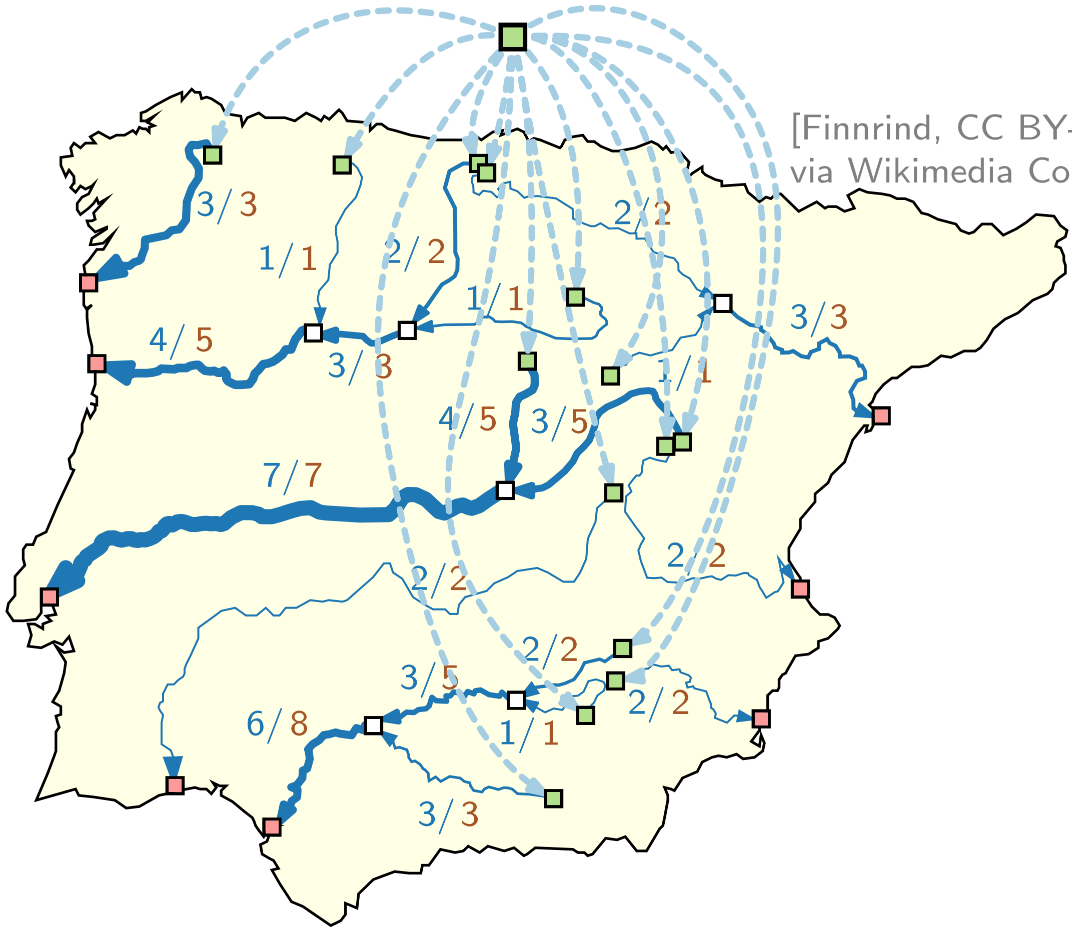

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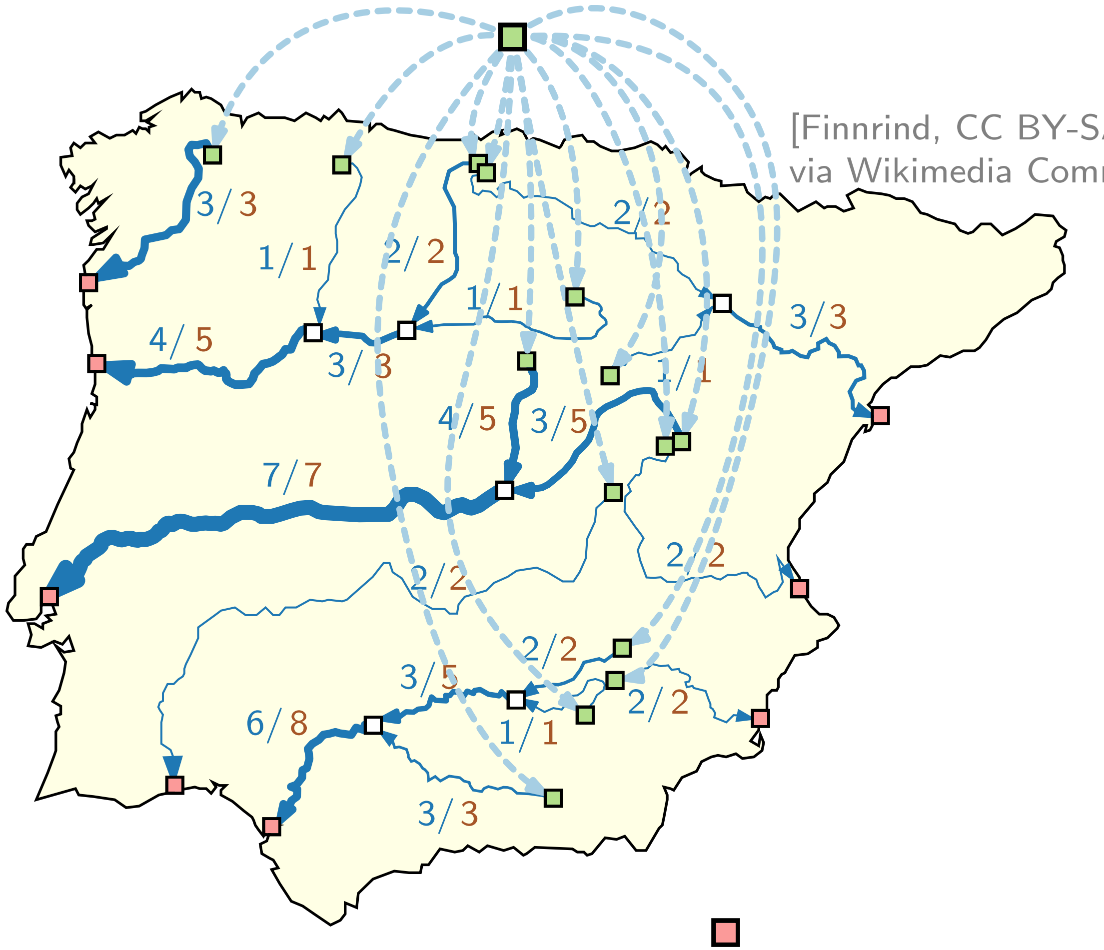

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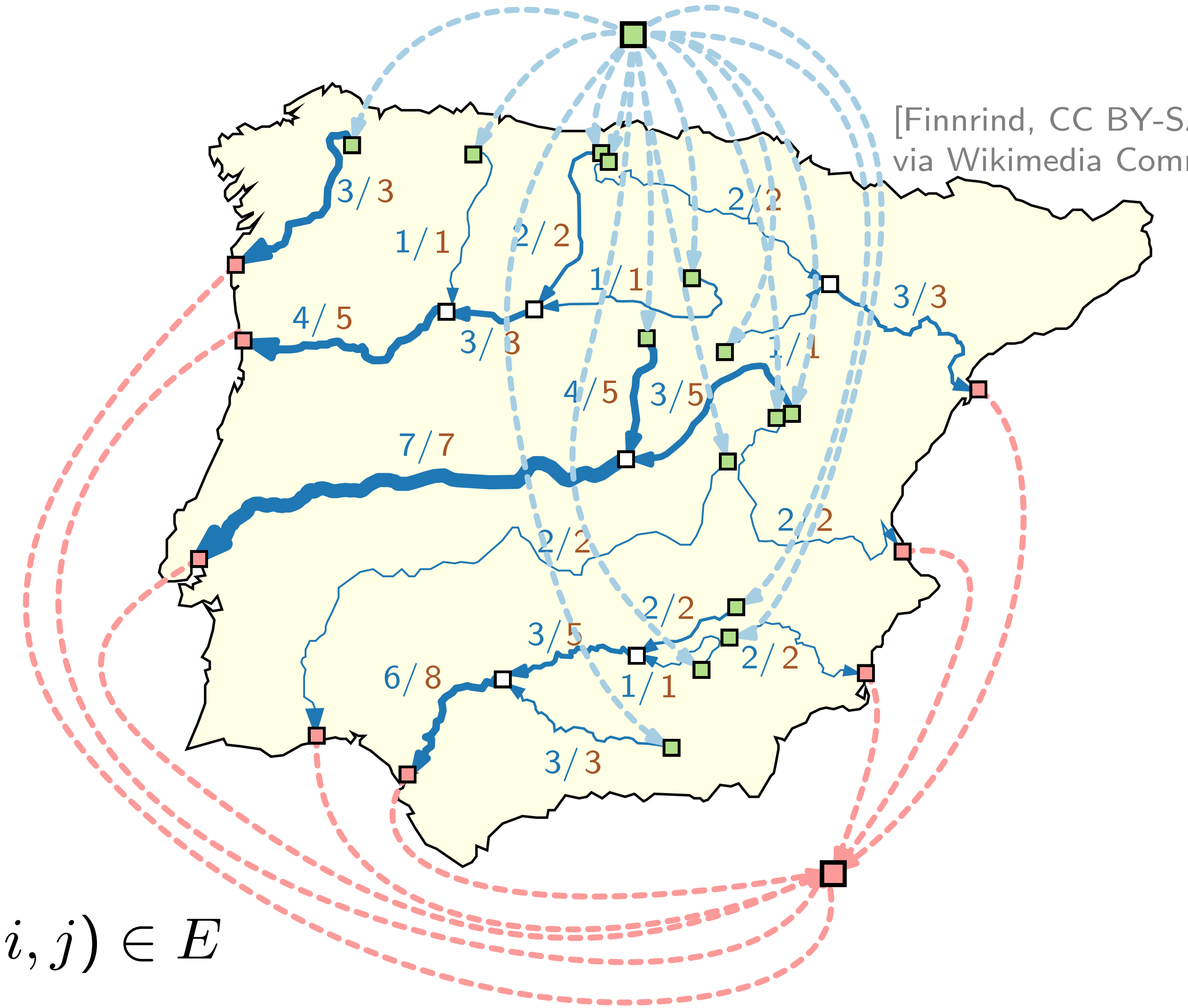

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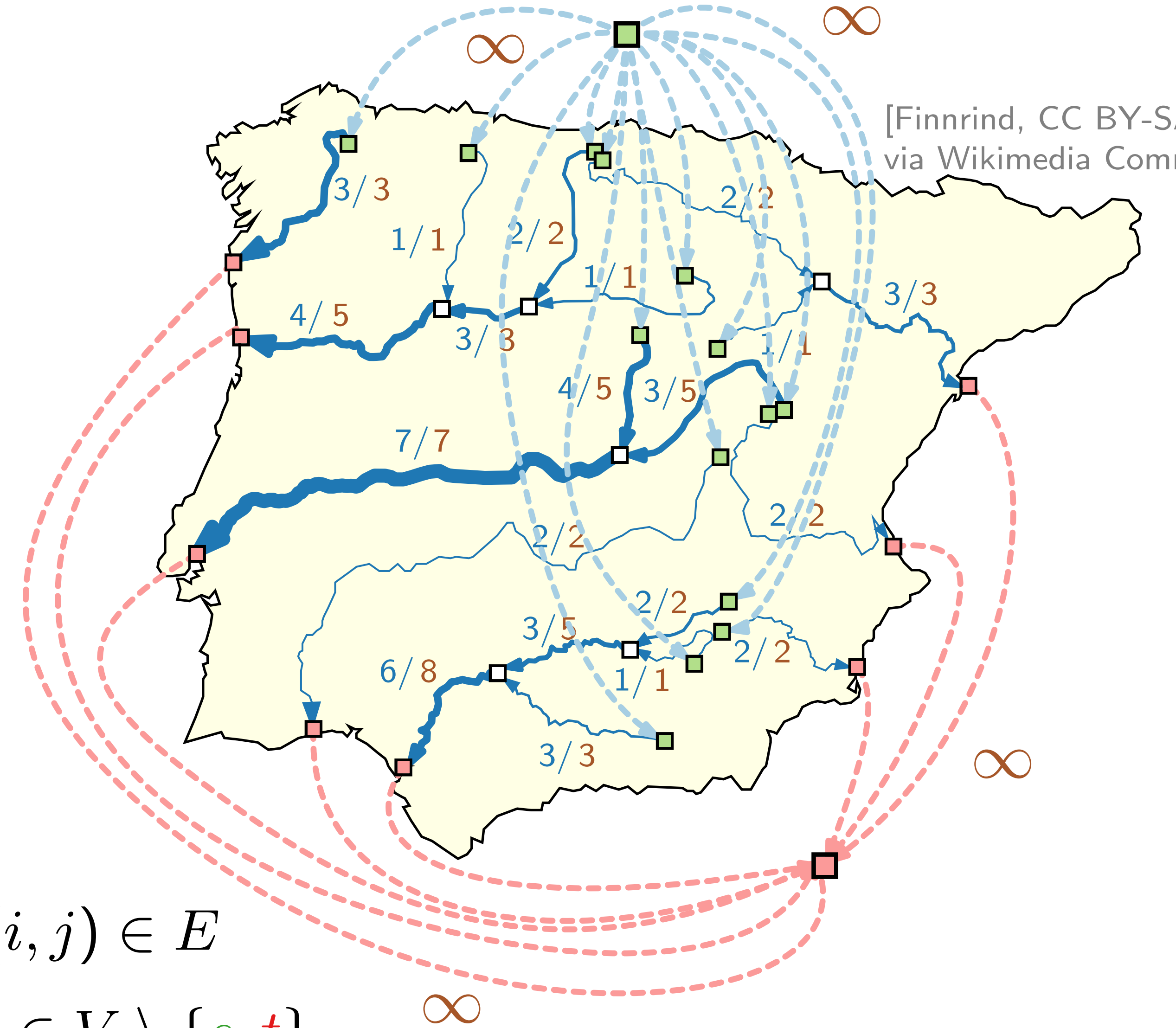

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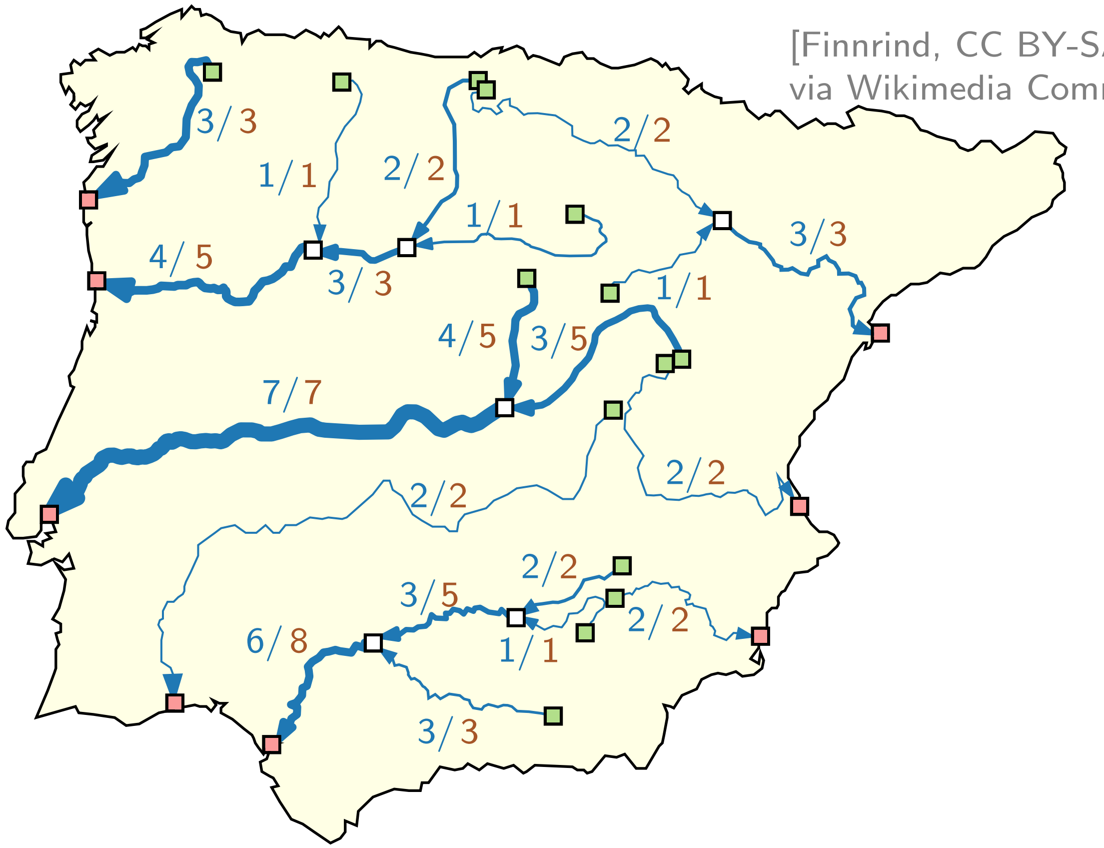
General Flow Network



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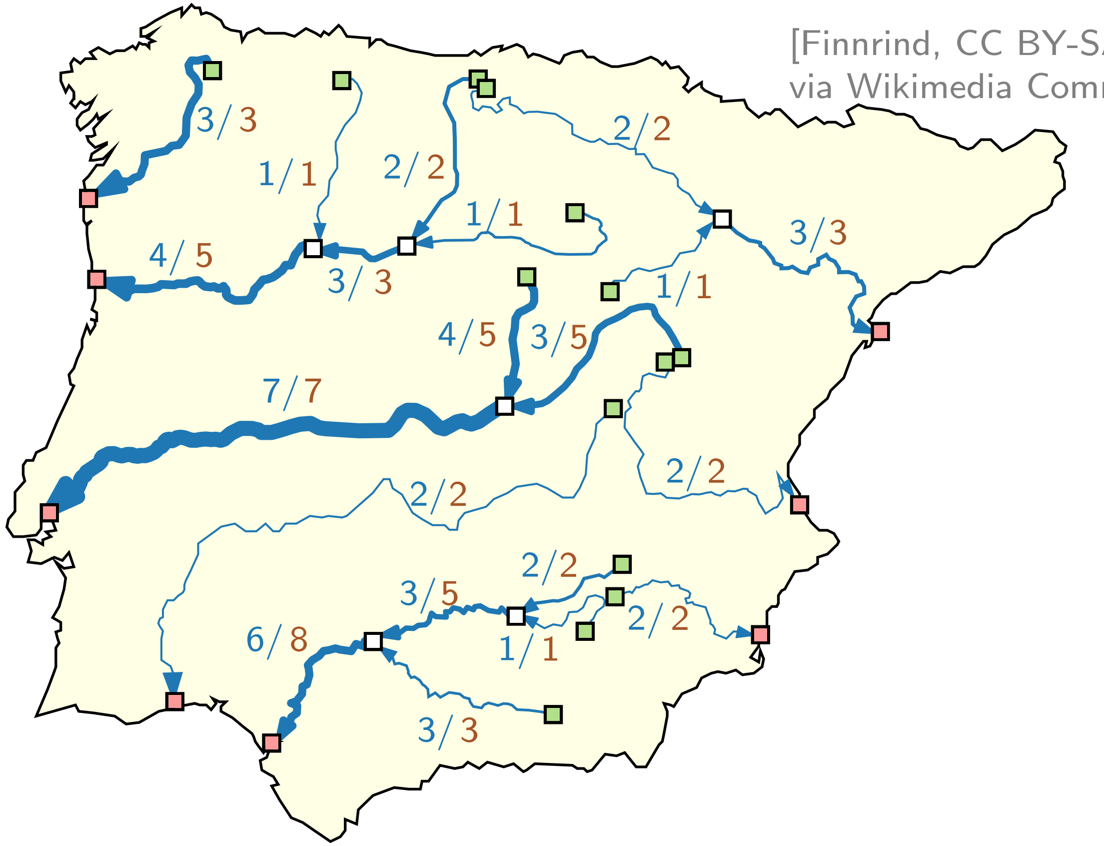

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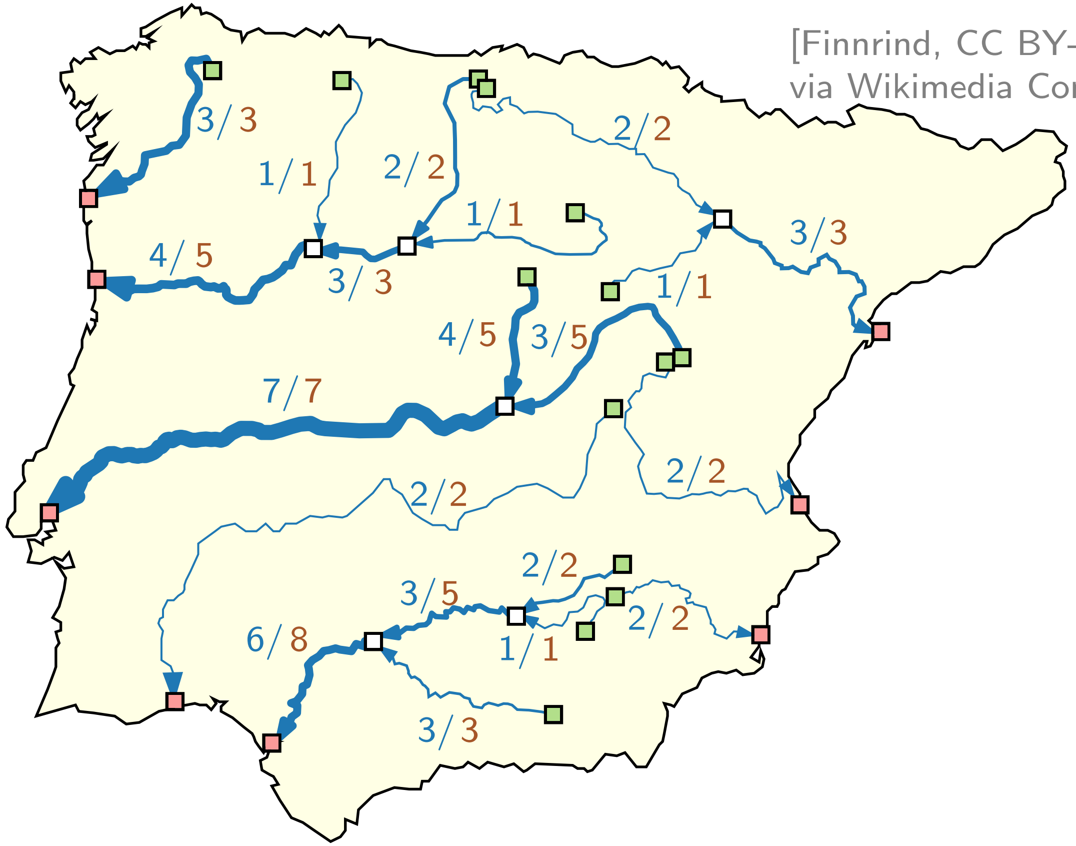

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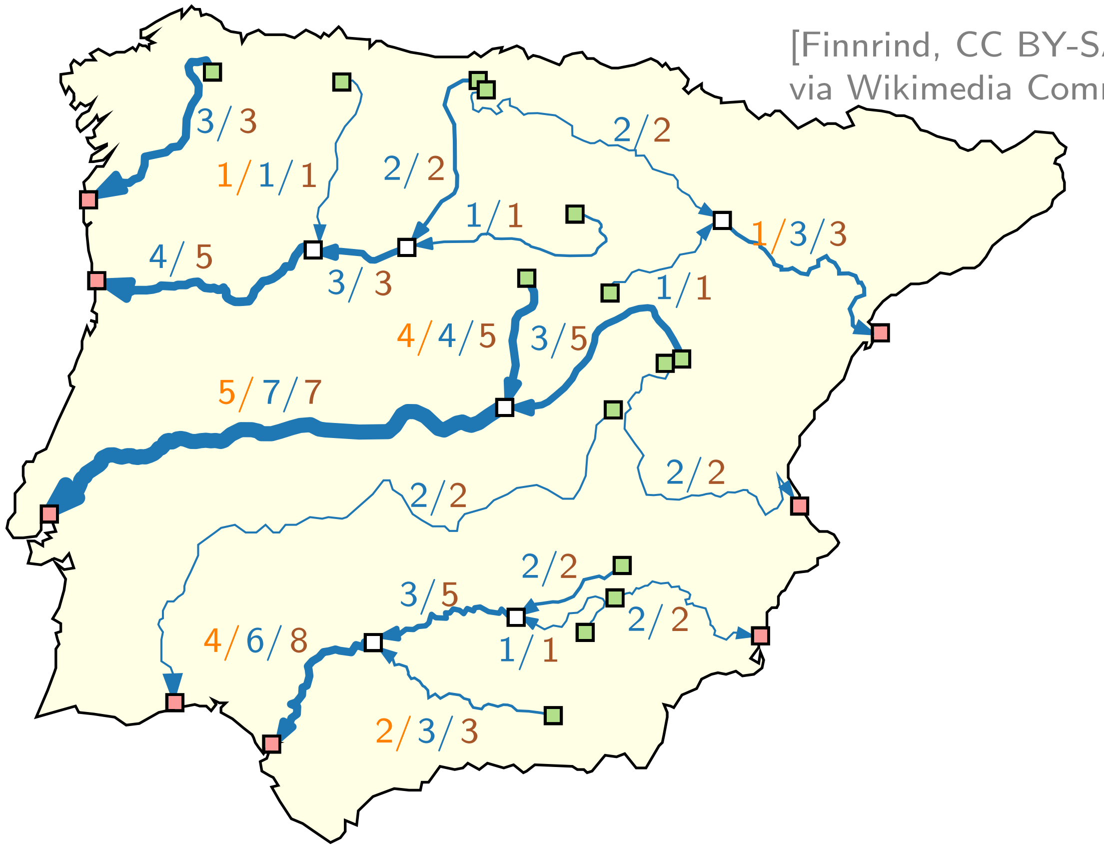

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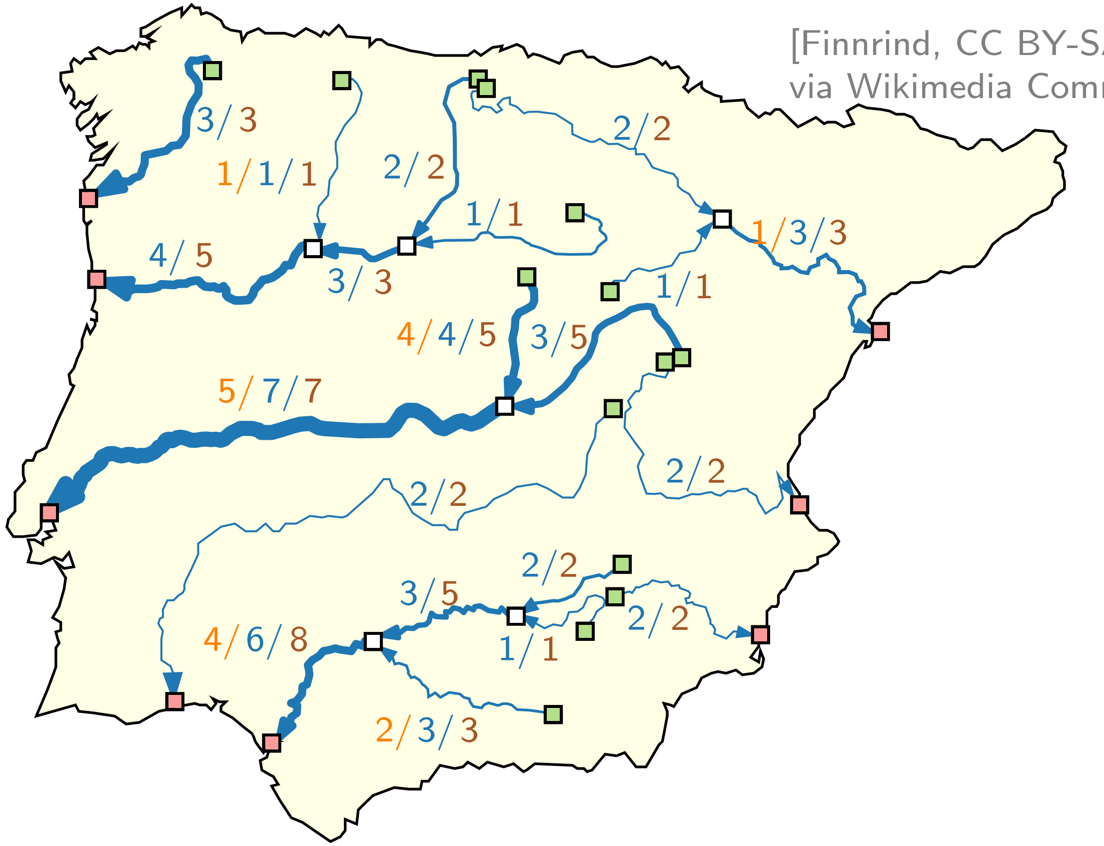

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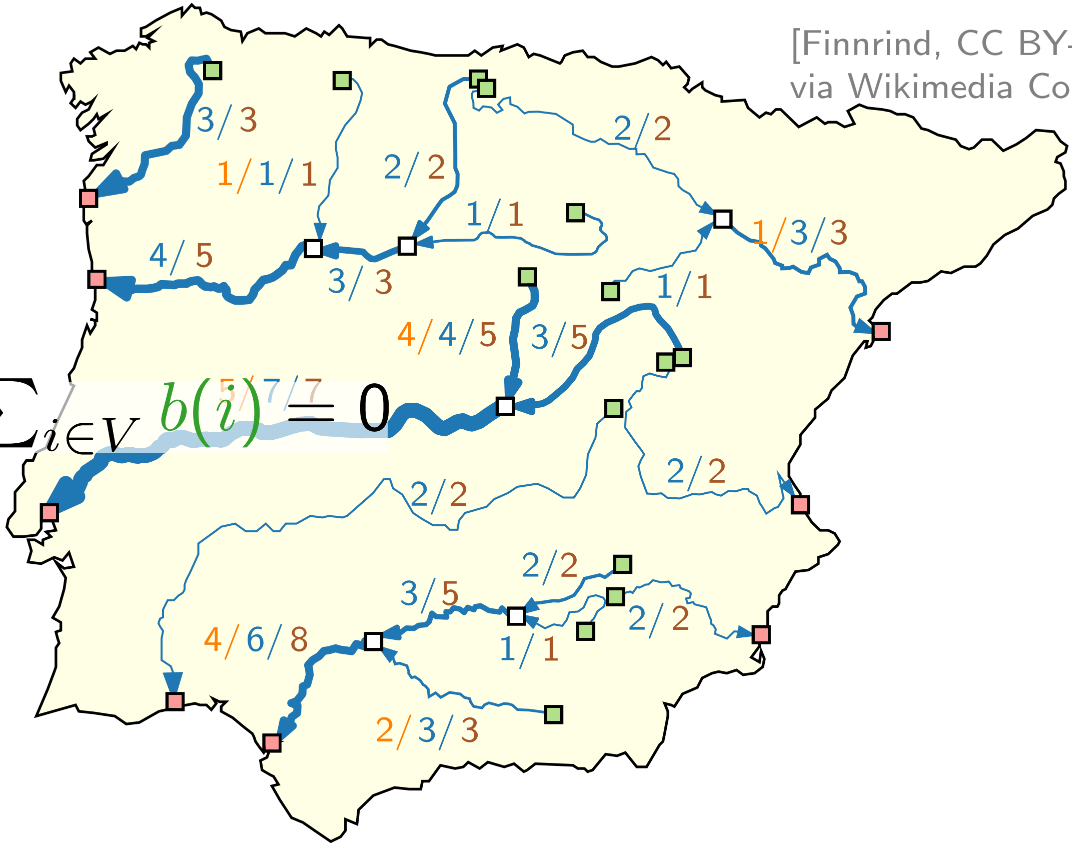

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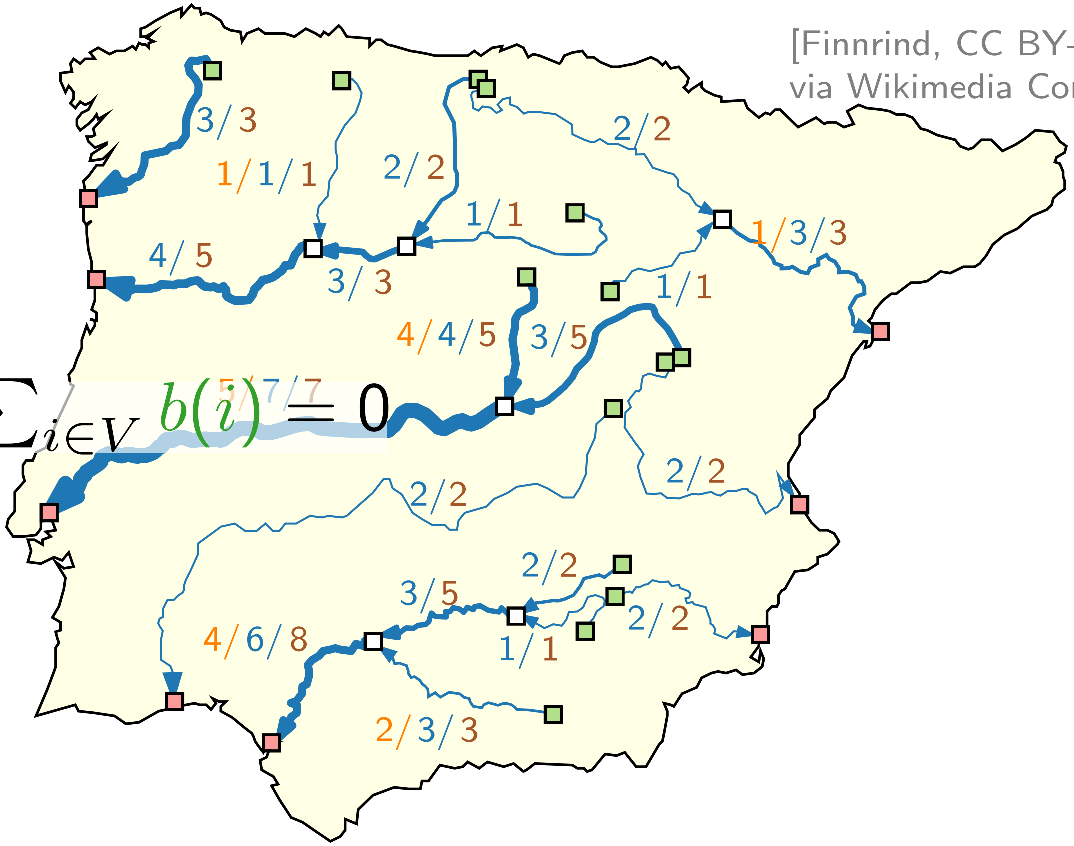

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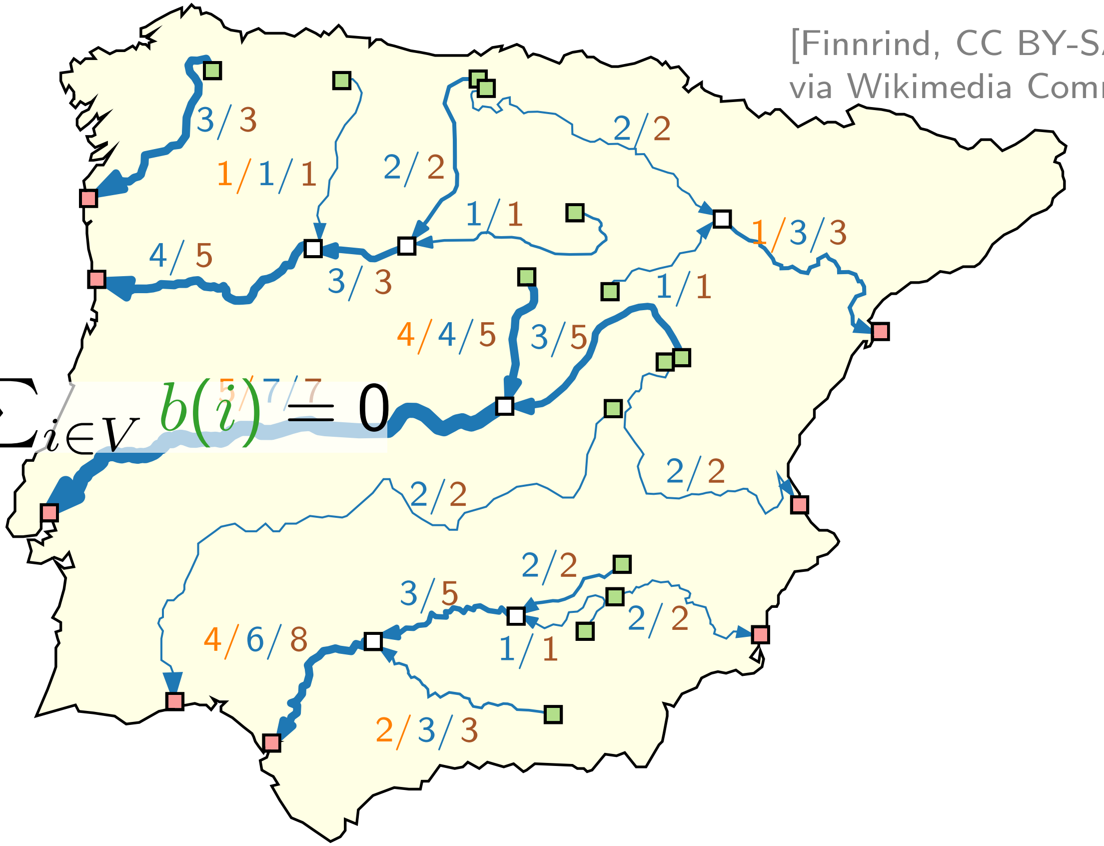

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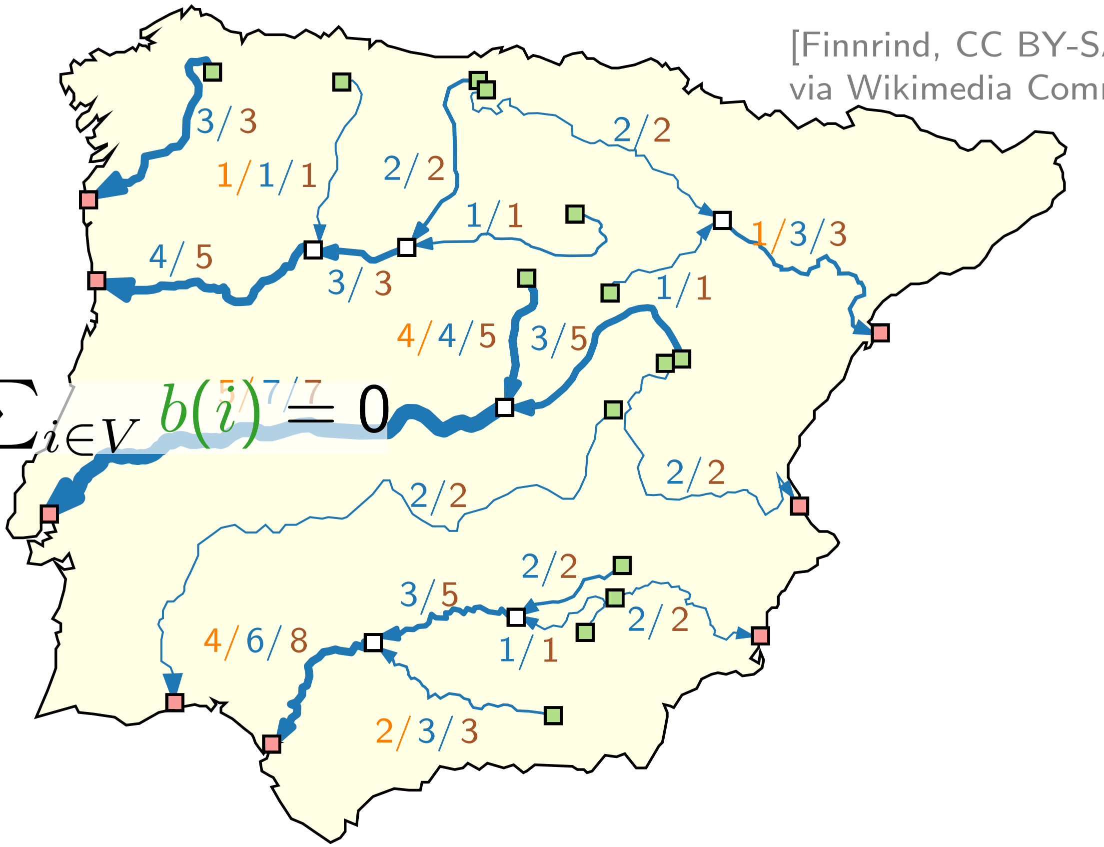

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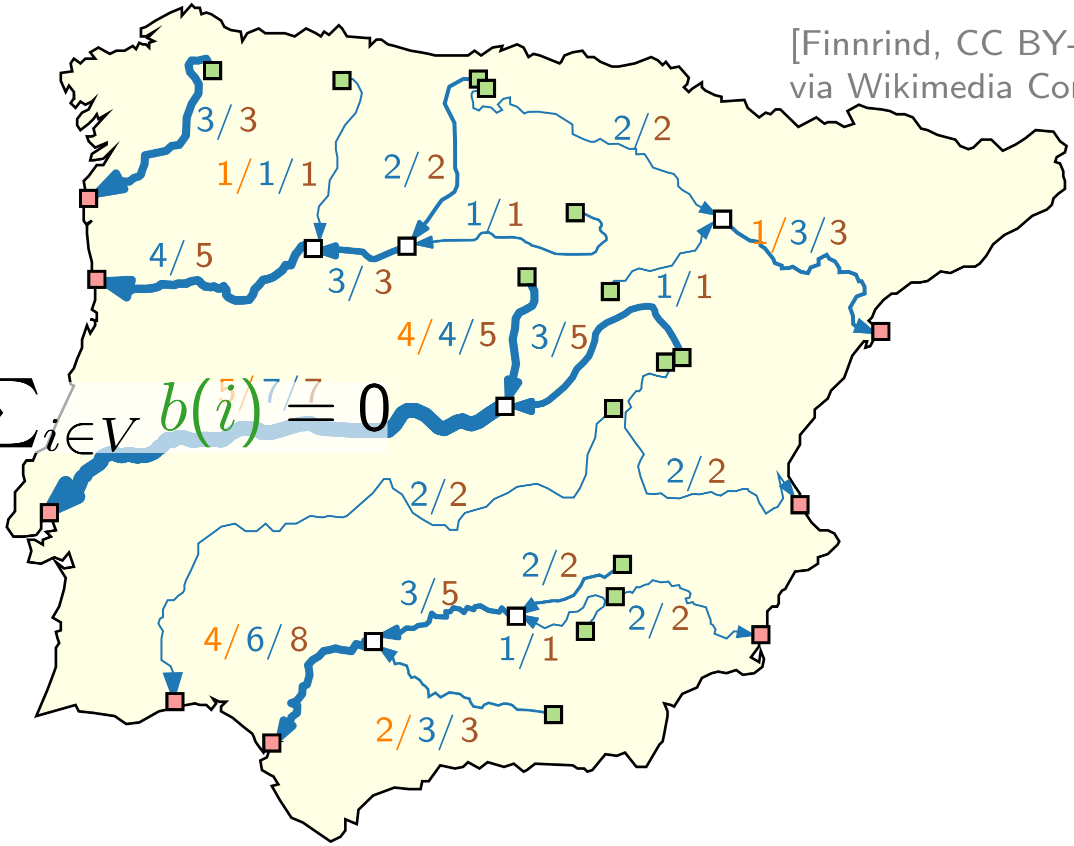

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A **minimum cost flow** is a valid flow where $\mathrm{cost}(X)$ is minimized.

General Flow Network – Algorithms

Polynomial Algorithms

#	Due to	Year	Running Time
1	Edmonds and Karp	1972	$O((n + m') \log U \, S(n, m, nC))$
2	Rock	1980	$O((n + m') \log U \, S(n, m, nC))$
3	Rock	1980	$O(n \log C \, M(n, m, U))$
4	Bland and Jensen	1985	$O(m \log C \, M(n, m, U))$
5	Goldberg and Tarjan	1987	$O(nm \log (n^2/m) \log (nC))$
6	Goldberg and Tarjan	1988	$O(nm \log n \log (nC))$
7	Ahuja, Goldberg, Orlin and Tarjan	1988	$O(nm \log \log U \log (nC))$

Strongly Polynomial Algorithms

#	Due to	Year	Running Time
1	Tardos	1985	$O(m^4)$
2	Orlin	1984	$O((n + m')^2 \log n \, S(n, m))$
3	Fujishige	1986	$O((n + m')^2 \log n \, S(n, m))$
4	Galil and Tardos	1986	$O(n^2 \log n \, S(n, m))$
5	Goldberg and Tarjan	1987	$O(nm^2 \log n \log(n^2/m))$
6	Goldberg and Tarjan	1988	$O(nm^2 \log^2 n)$
7	Orlin (this paper)	1988	$O((n + m') \log n \, S(n, m))$

$S(n, m) = O(m + n \log n)$		Fredman and Tarjan [1984]
$S(n, m, C) = O(\text{Min } (m + n\sqrt{\log C}),$		Ahuja, Mehlhorn, Orlin and Tarjan [1990]
$(m \log \log C))$		Van Emde Boas, Kaas and Zijlstra[1977]
$M(n, m) = O(\min (nm + n^{2+\varepsilon}, nm \log n)$ where ε is any fixed constant.		King, Rao, and Tarjan [1991]
$M(n, m, U) = O(nm \log (\frac{n}{m}\sqrt{\log U} + 2))$		Ahuja, Orlin and Tarjan [1989]

[Orlin 1991]

General Flow Network – Algorithms

Polynomial Algorithms

#	Due to	Year	Running Time
1	Edmonds and Karp	1972	$O((n + m') \log U\, S(n, m, nC))$
2	Rock	1980	$O((n + m') \log U\, S(n, m, nC))$
3	Rock	1980	$O(n \log C\, M(n, m, U))$
4	Bland and Jensen	1985	$O(m \log C\, M(n, m, U))$
5	Goldberg and Tarjan	1987	$O(nm \log (n^2/m) \log (nC))$
6	Goldberg and Tarjan	1988	$O(nm \log n \log (nC))$
7	Ahuja, Goldberg, Orlin and Tarjan	1988	$O(nm \log \log U \log (nC))$

Strongly Polynomial Algorithms

#	Due to	Year	Running Time
1	Tardos	1985	$O(m^4)$
2	Orlin	1984	$O((n + m')^2 \log n\, S(n, m))$
3	Fujishige	1986	$O((n + m')^2 \log n\, S(n, m))$
4	Galil and Tardos	1986	$O(n^2 \log n\, S(n, m))$
5	Goldberg and Tarjan	1987	$O(nm^2 \log n \log(n^2/m))$
6	Goldberg and Tarjan	1988	$O(nm^2 \log^2 n)$
7	Orlin (this paper)	1988	$O((n + m') \log n\, S(n, m))$

$S(n, m)$	=	$O(m + n \log n)$	Fredman and Tarjan [1984]
$S(n, m, C)$	=	$O(\text{Min}\,(m + n\sqrt{\log C}),$ $(m \log \log C))$	Ahuja, Mehlhorn, Orlin and Tarjan [1990] Van Emde Boas, Kaas and Zijlstra[1977]
$M(n, m)$	=	$O(\min\,(nm + n^{2+\varepsilon}, nm \log n)$ where ε is any fixed constant.	King, Rao, and Tarjan [1991]
$M(n, m, U)$	=	$O(nm \log (\frac{n}{m}\sqrt{\log U} + 2))$	Ahuja, Orlin and Tarjan [1989]

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Theorem. [Orlin 1991]
The minimum cost flow problem can be solved in $O(n^2 \log^2 n + m^2 \log n)$ time.

General Flow Network – Algorithms

Polynomial Algorithms

#	Due to	Year	Running Time
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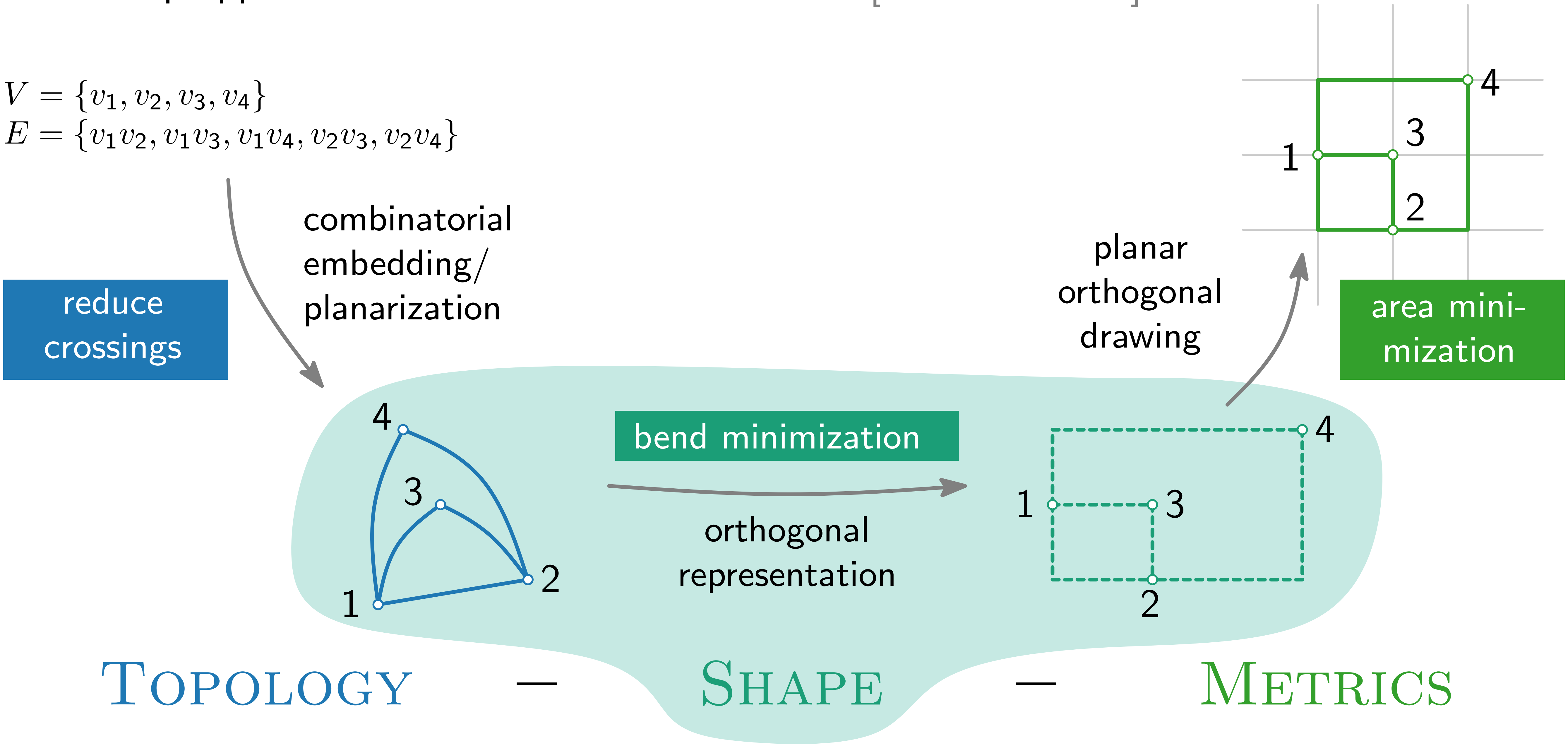
The minimum cost flow problem can be solved in $O(n^2 \log^2 n + m^2 \log n)$ time.

Theorem. [Cornelsen & Karrenbauer 2011]
The minimum cost flow problem for planar graphs with bounded costs and faze sizes can be solved in $O(n^{3/2})$ time.

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

$$V = \{v_1, v_2, v_3, v_4\}$$
$$E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$$



Bend Minimization with Given Embedding

Geometric bend minimization.

Given:

Find:

Bend Minimization with Given Embedding

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Given: ■ Plane graph $G = (V, E)$ with maximum degree 4

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Given:
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Given:
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Find: Orthogonal drawing with minimum number of bends that preserves the embedding.

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Compare with the following variation.

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Find: **Orthogonal representation** $H(G)$ with minimum number of bends that preserves the embedding.

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- faces $\xrightarrow{\measuredangle}$ neighbouring faces ($\#$ bends toward the neighbour)

Flow Network for Bend Minimization

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u, v\}$ shared by faces f and g, sequence δ_1 is reversed and inverted δ_2.

(H3) For each **face** f it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

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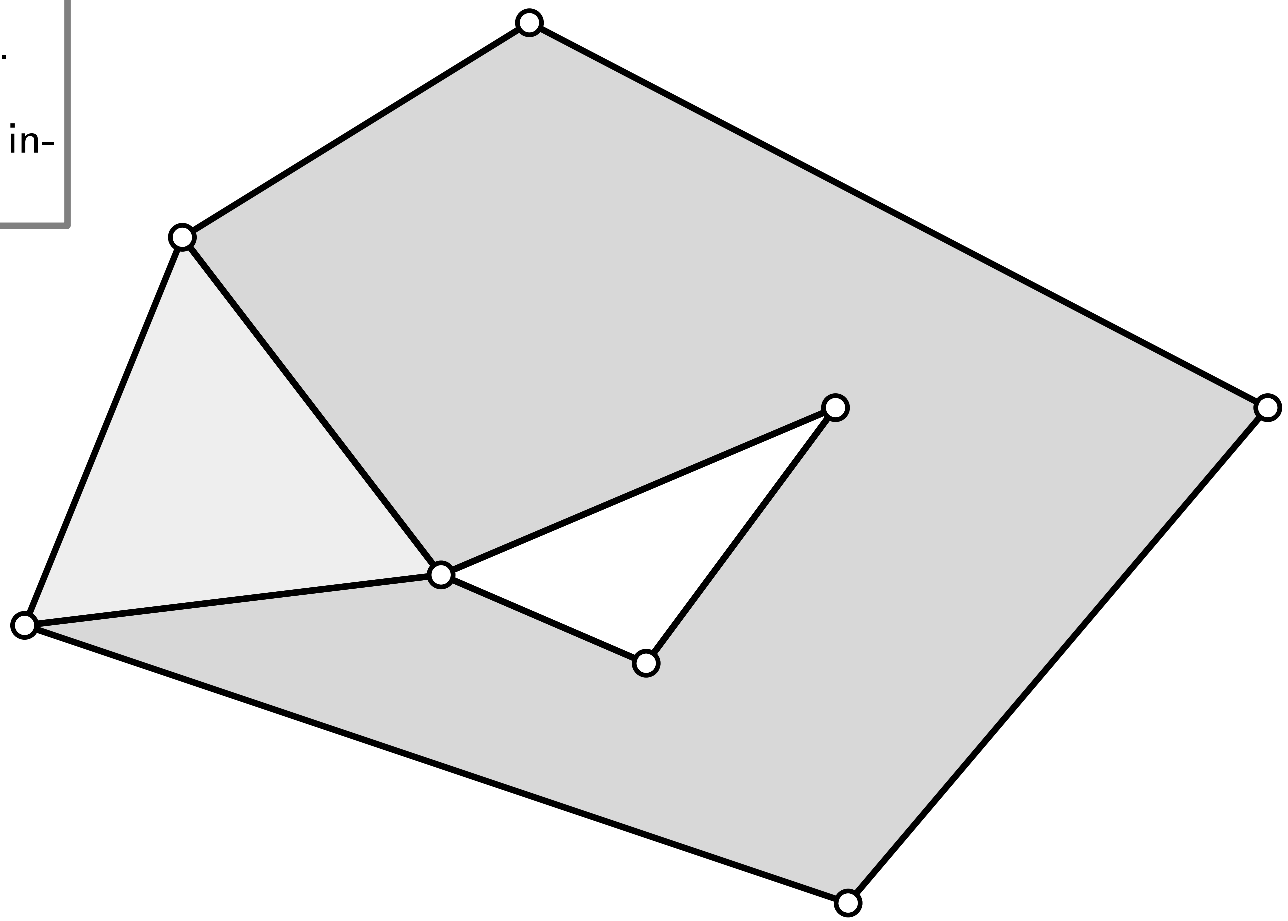

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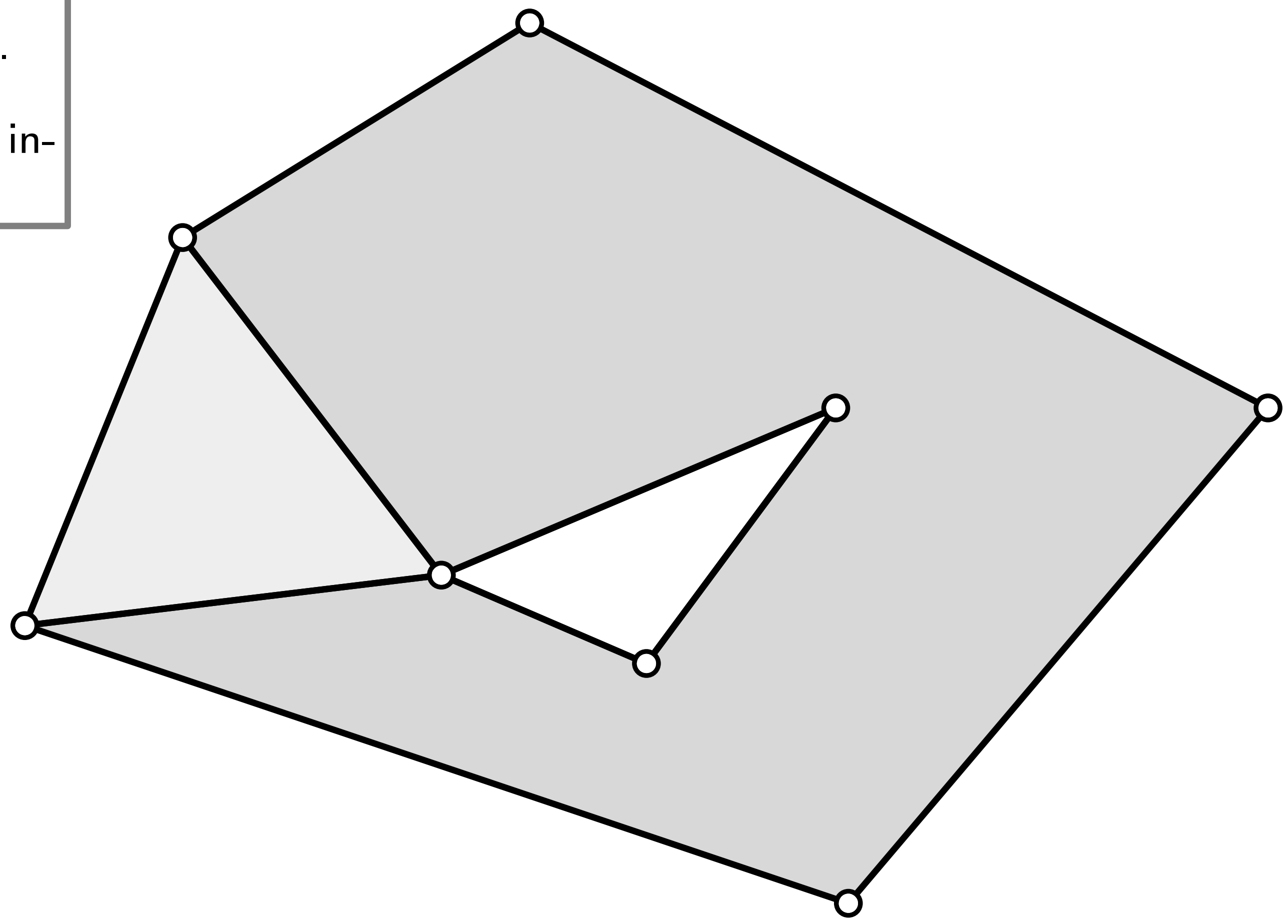

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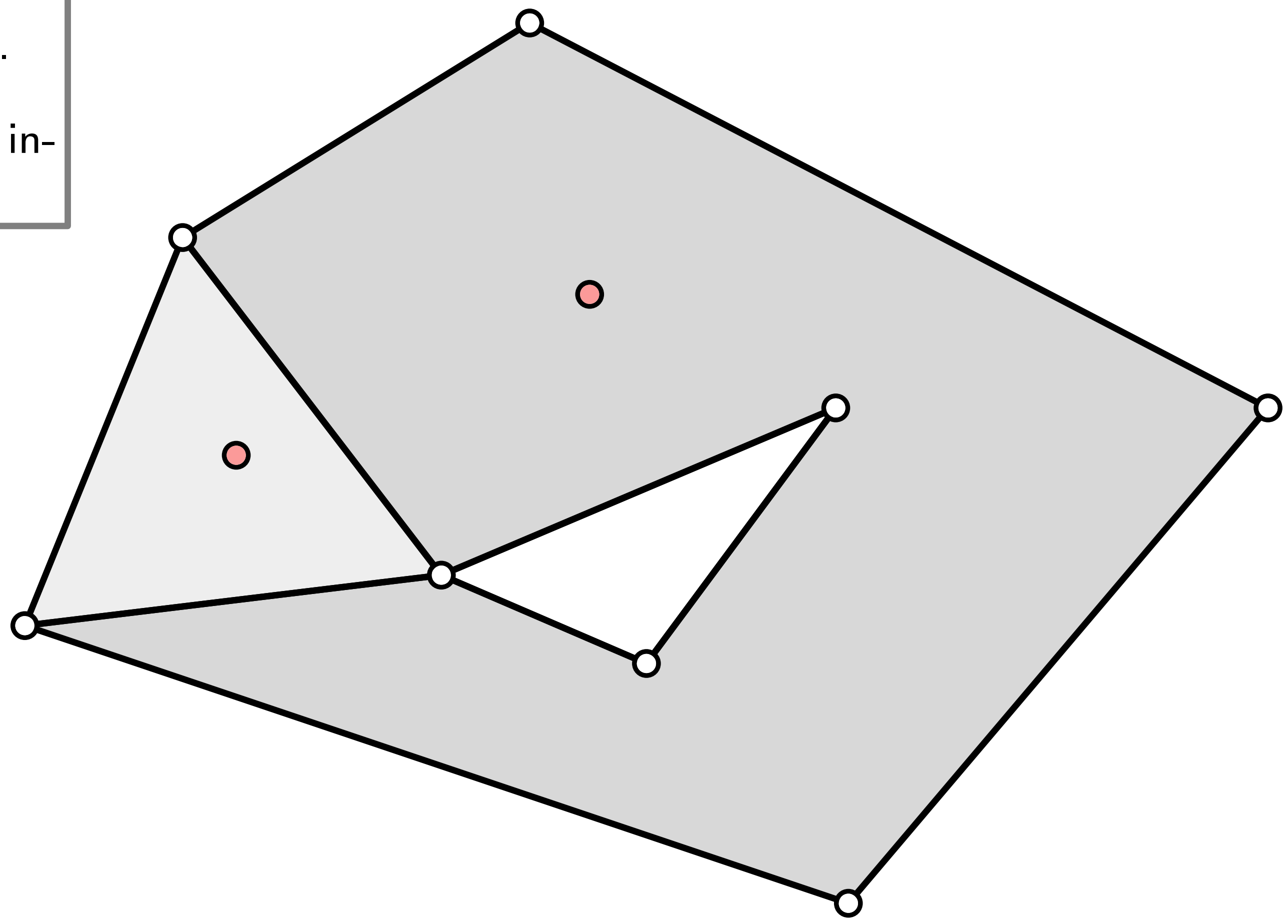

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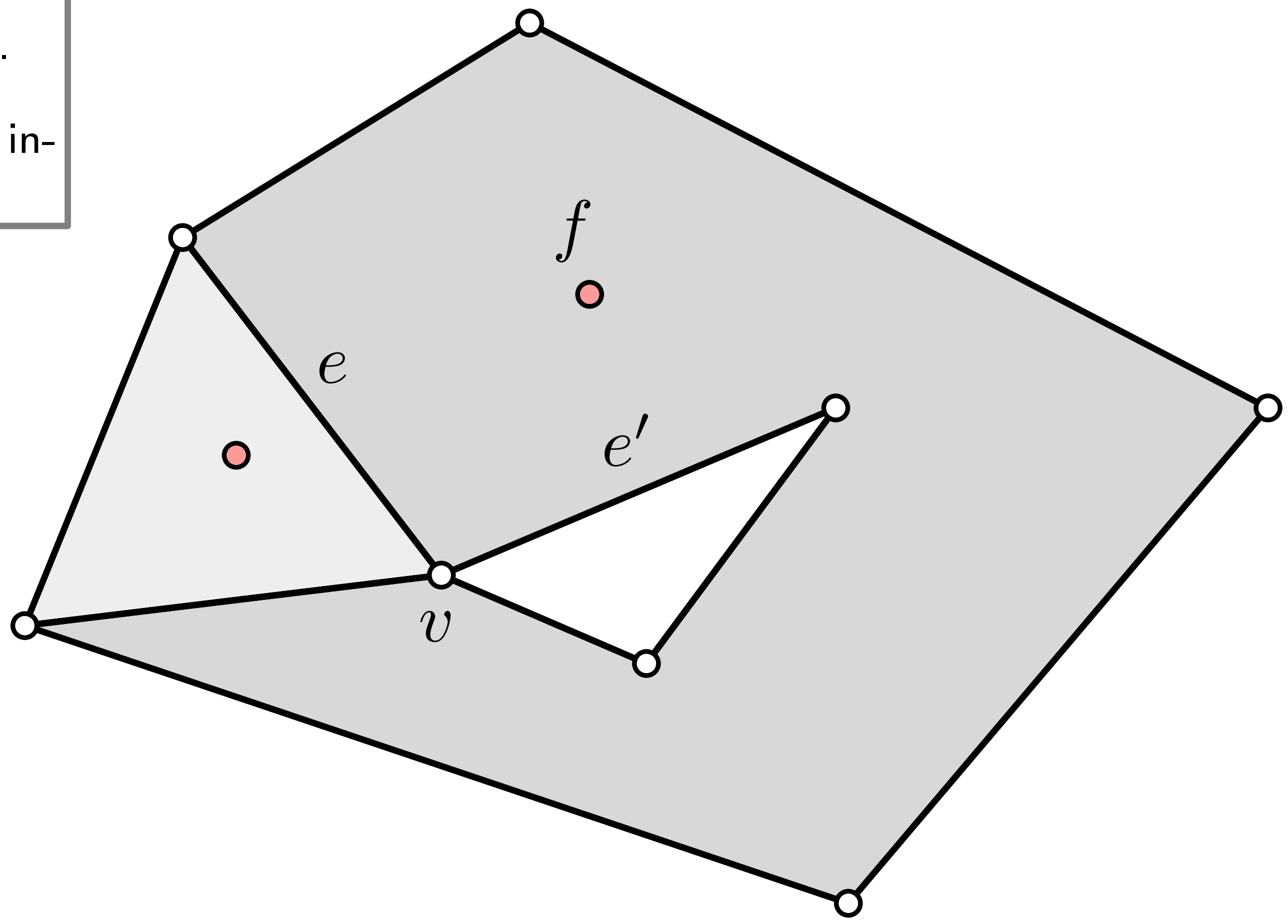

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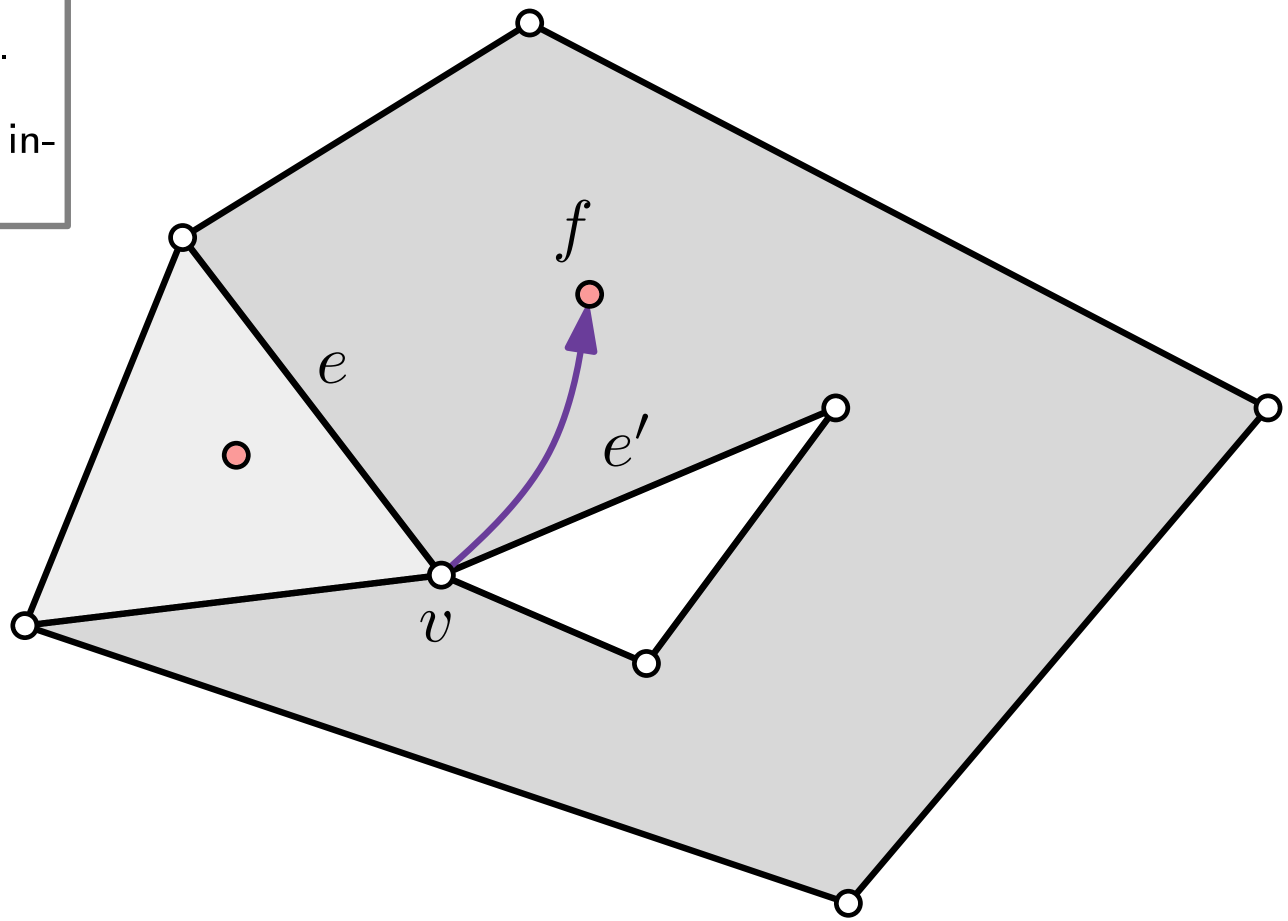

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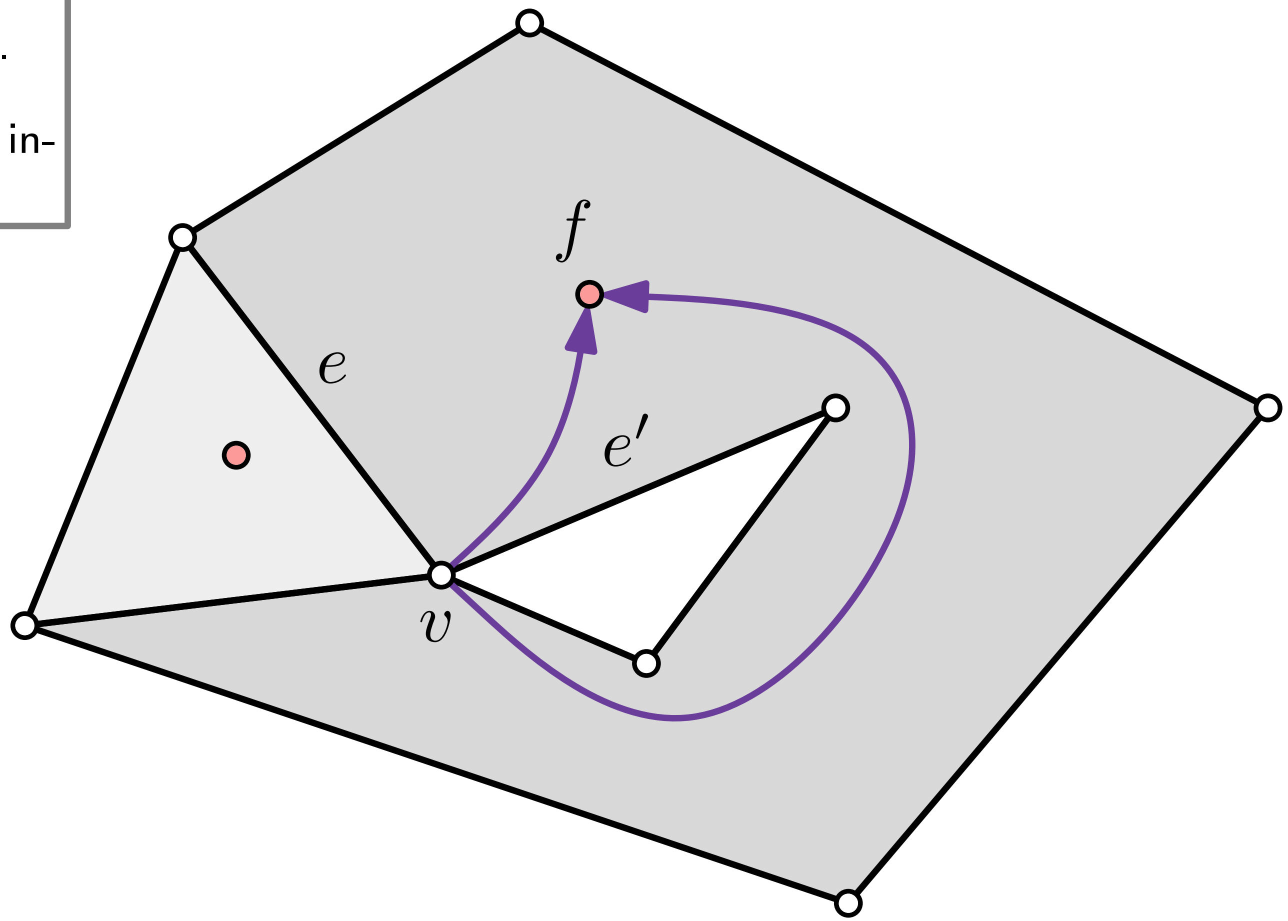

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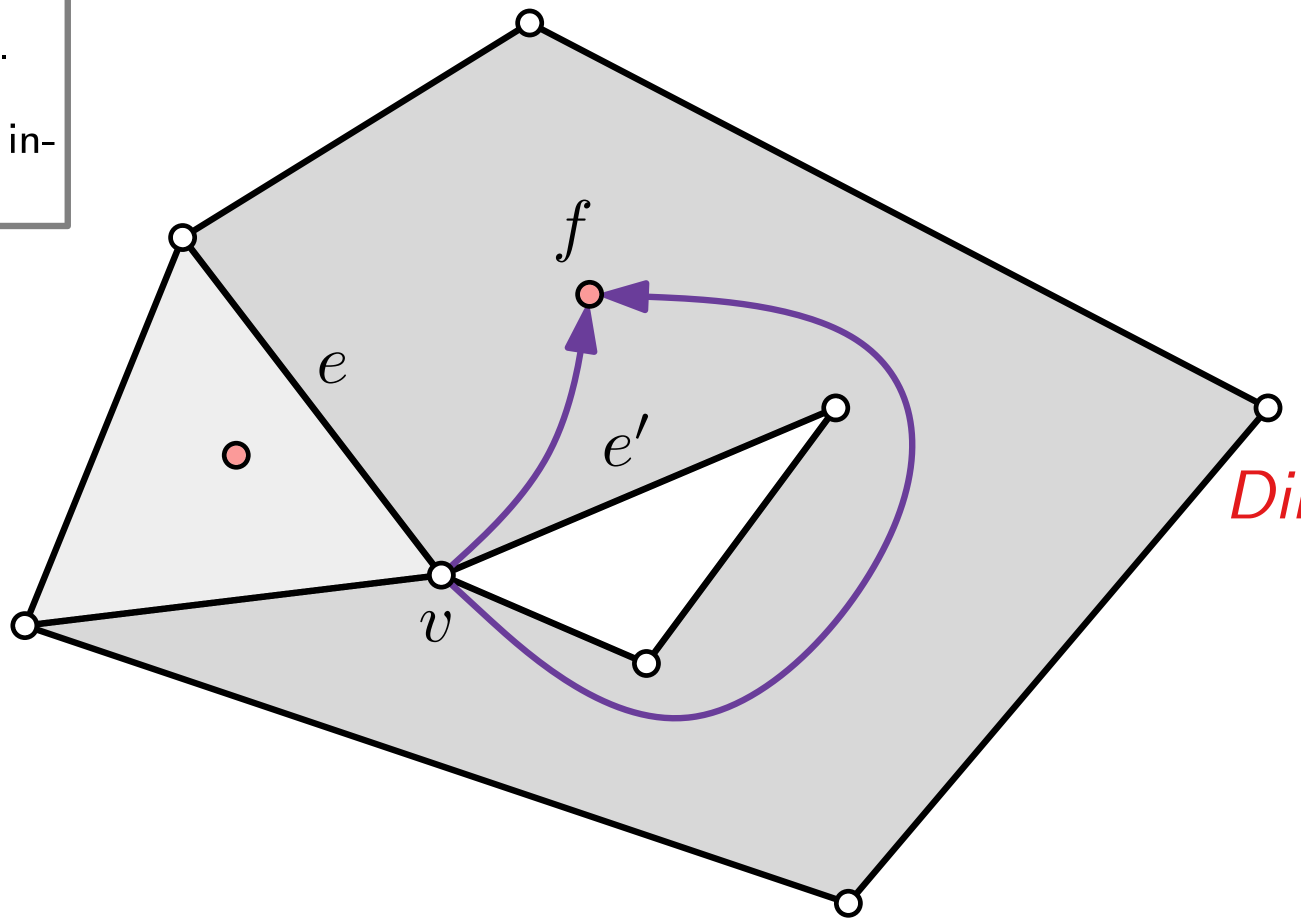

Directed multigraph!

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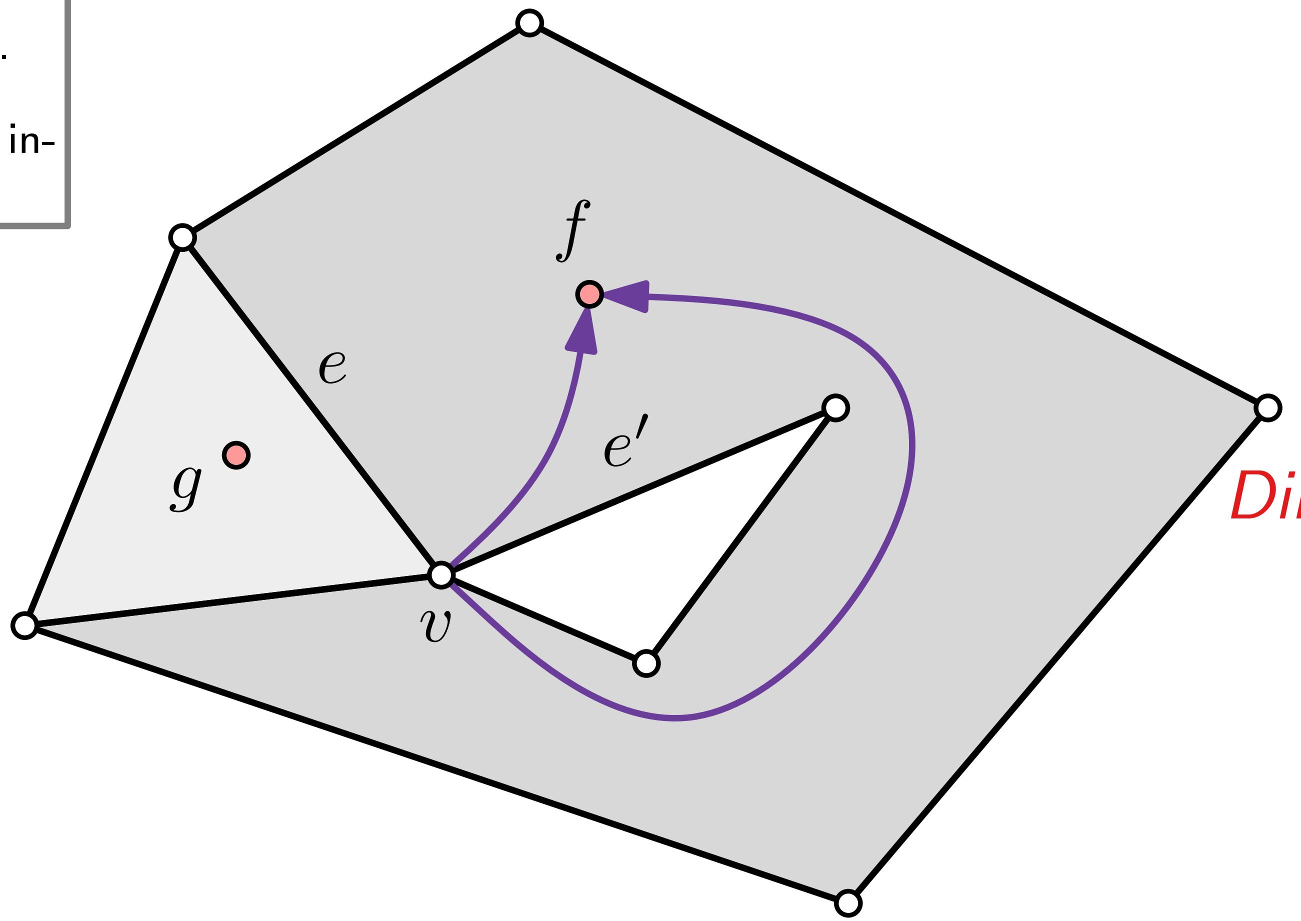

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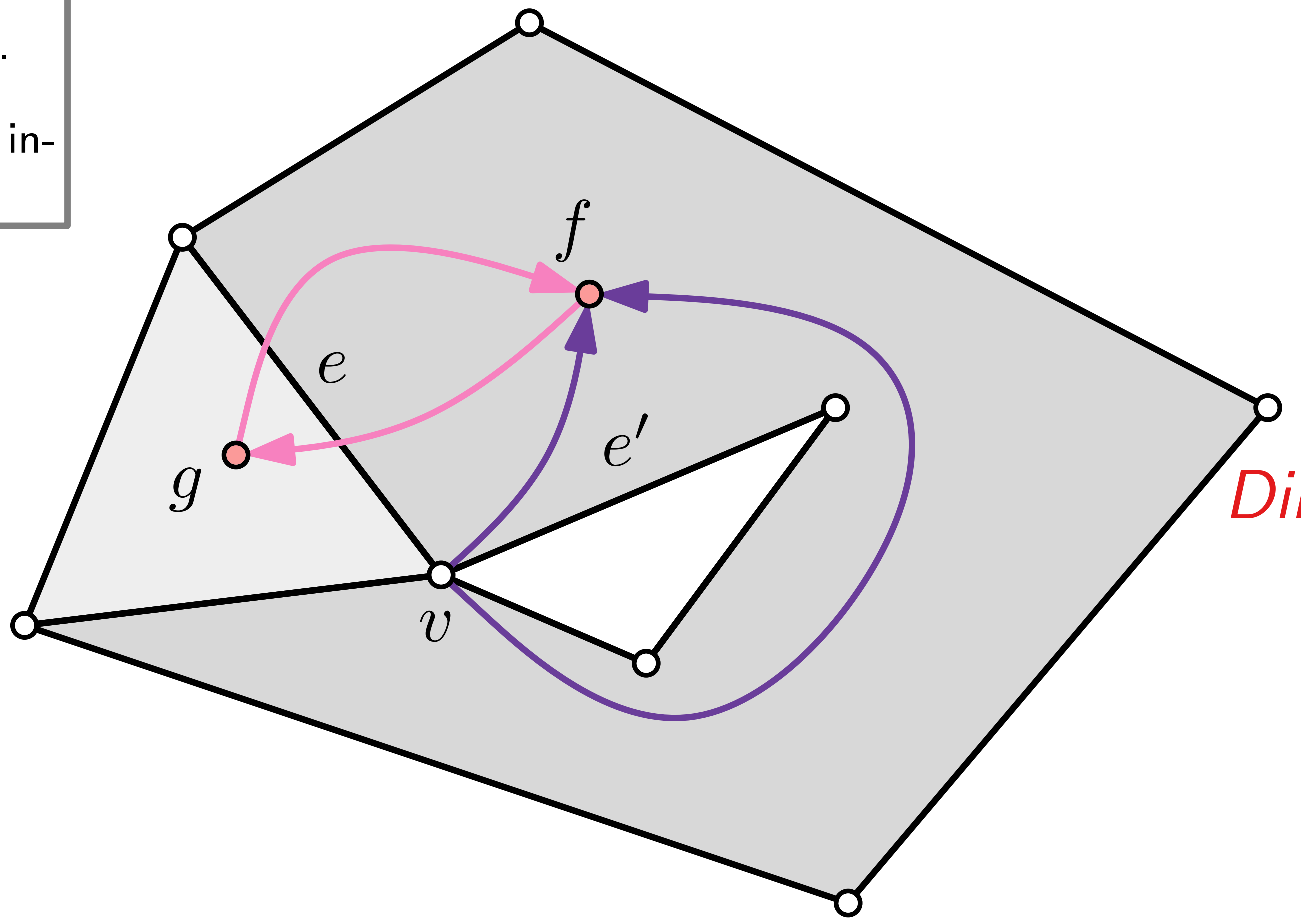

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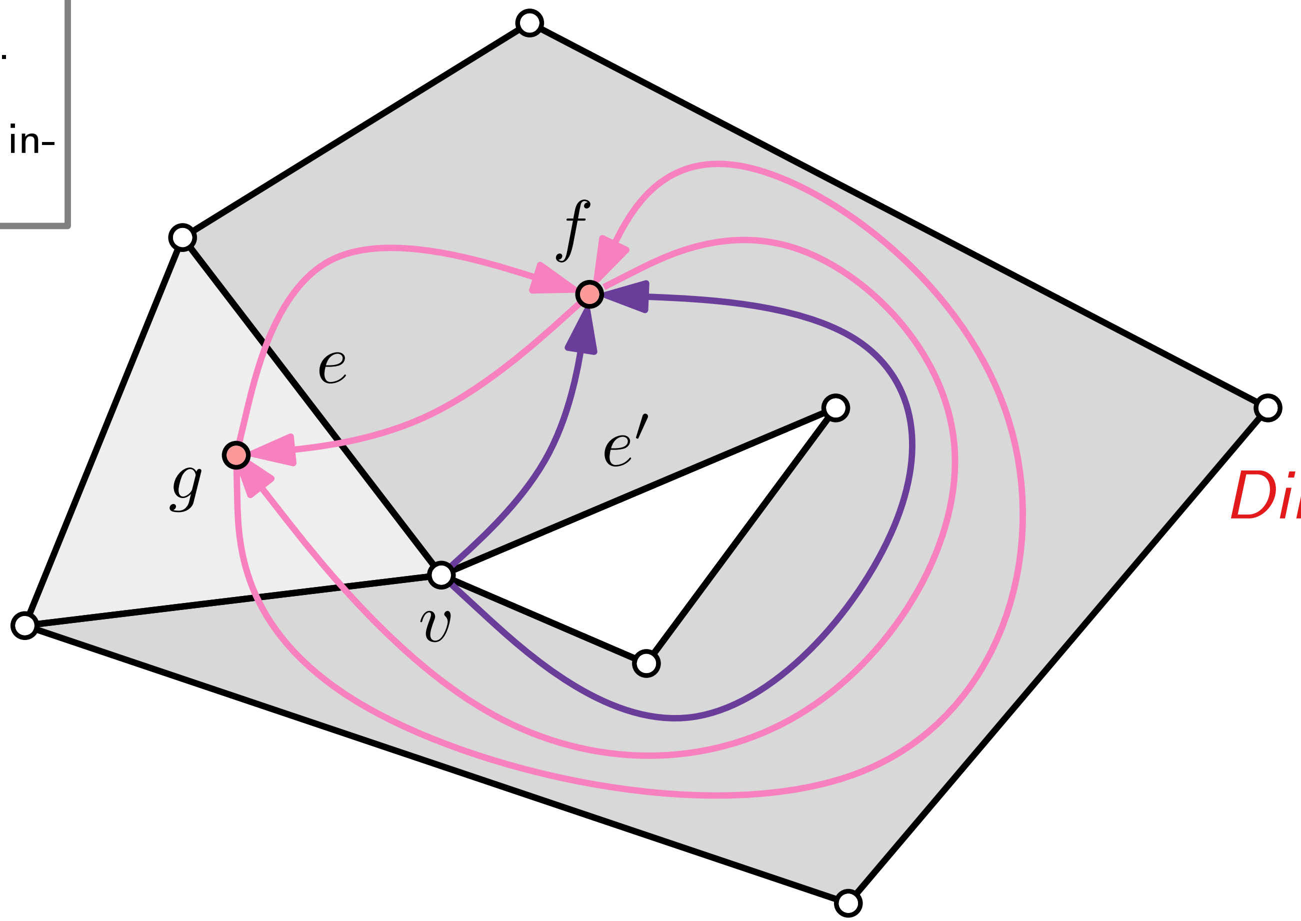

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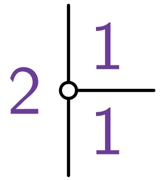

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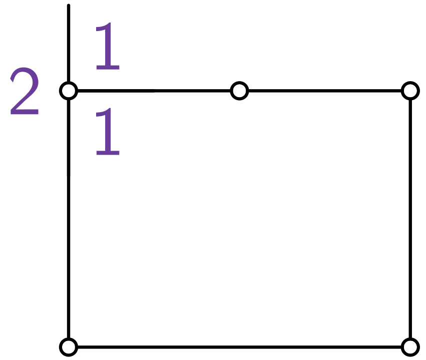

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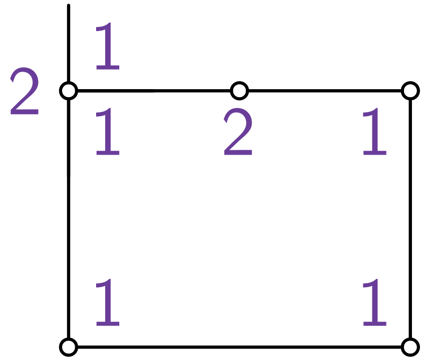

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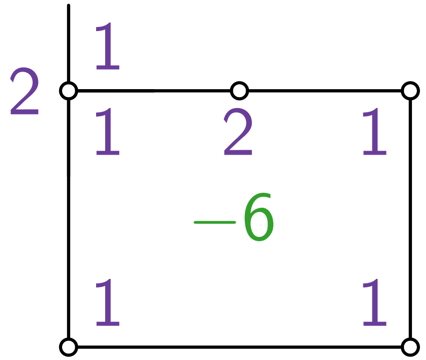

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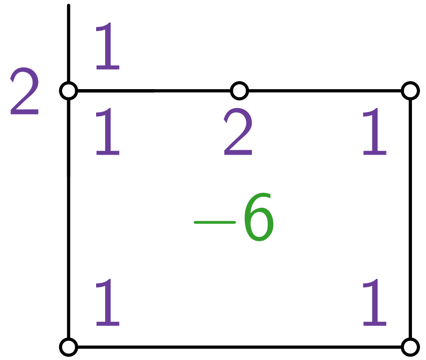

Flow Network for Bend Minimization

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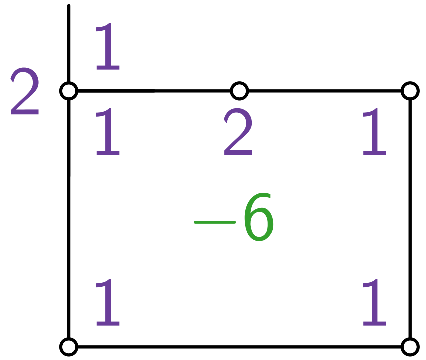

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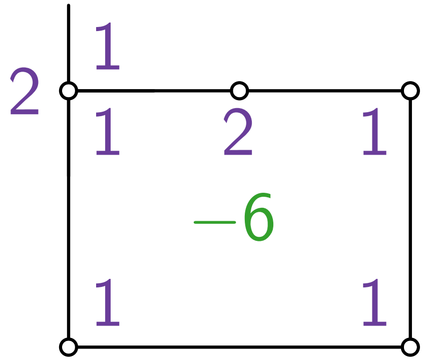

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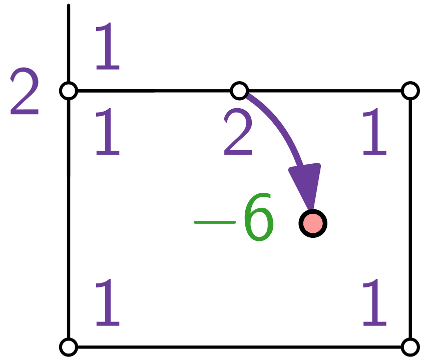

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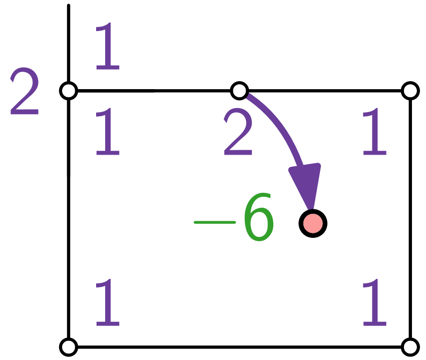

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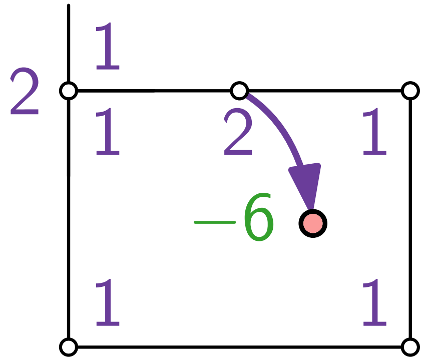

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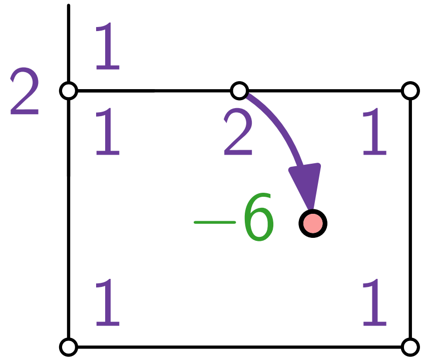

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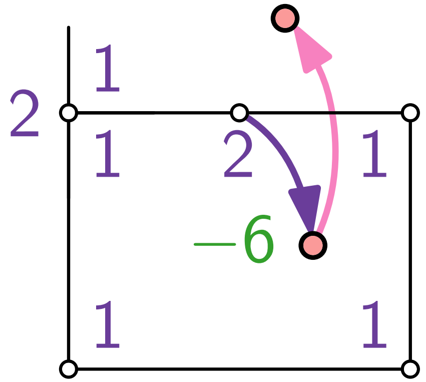

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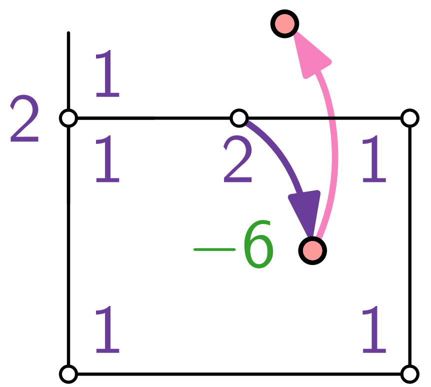

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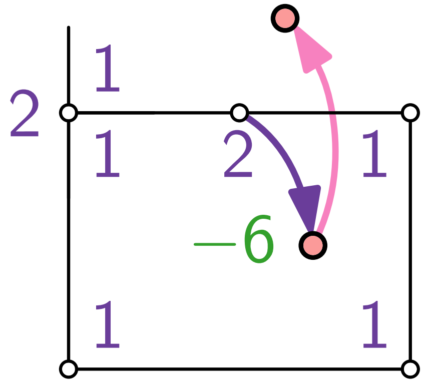

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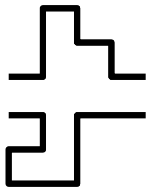

We model only the *number* of bends. Why is it enough?

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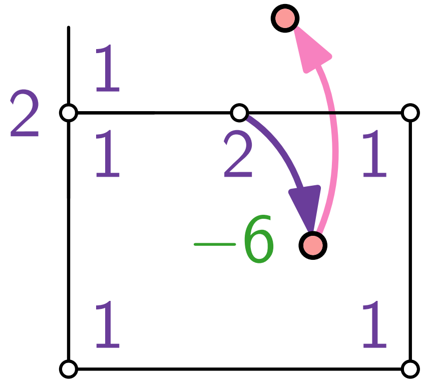

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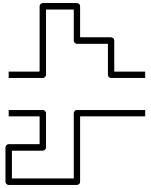

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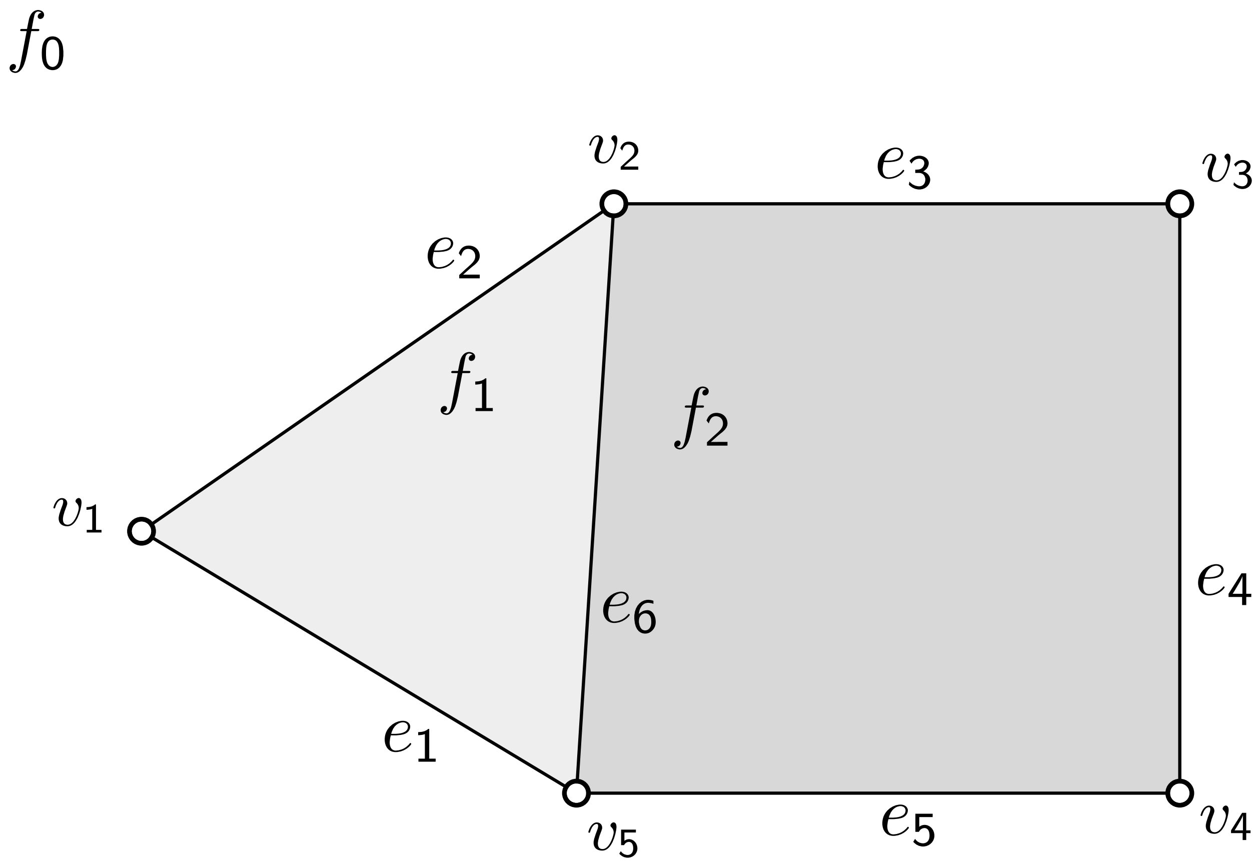
→ Exercise

Flow Network Example



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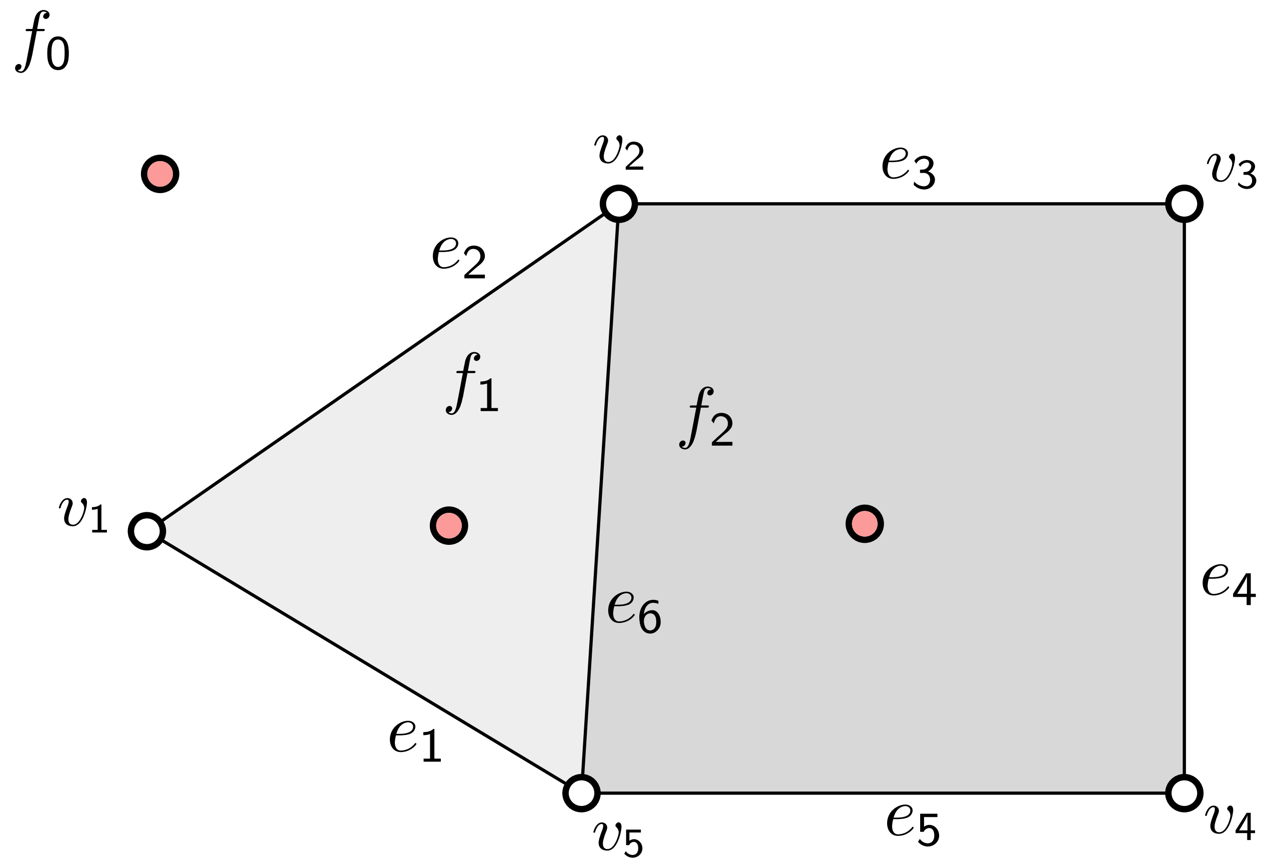

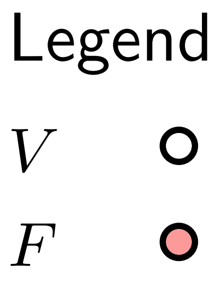

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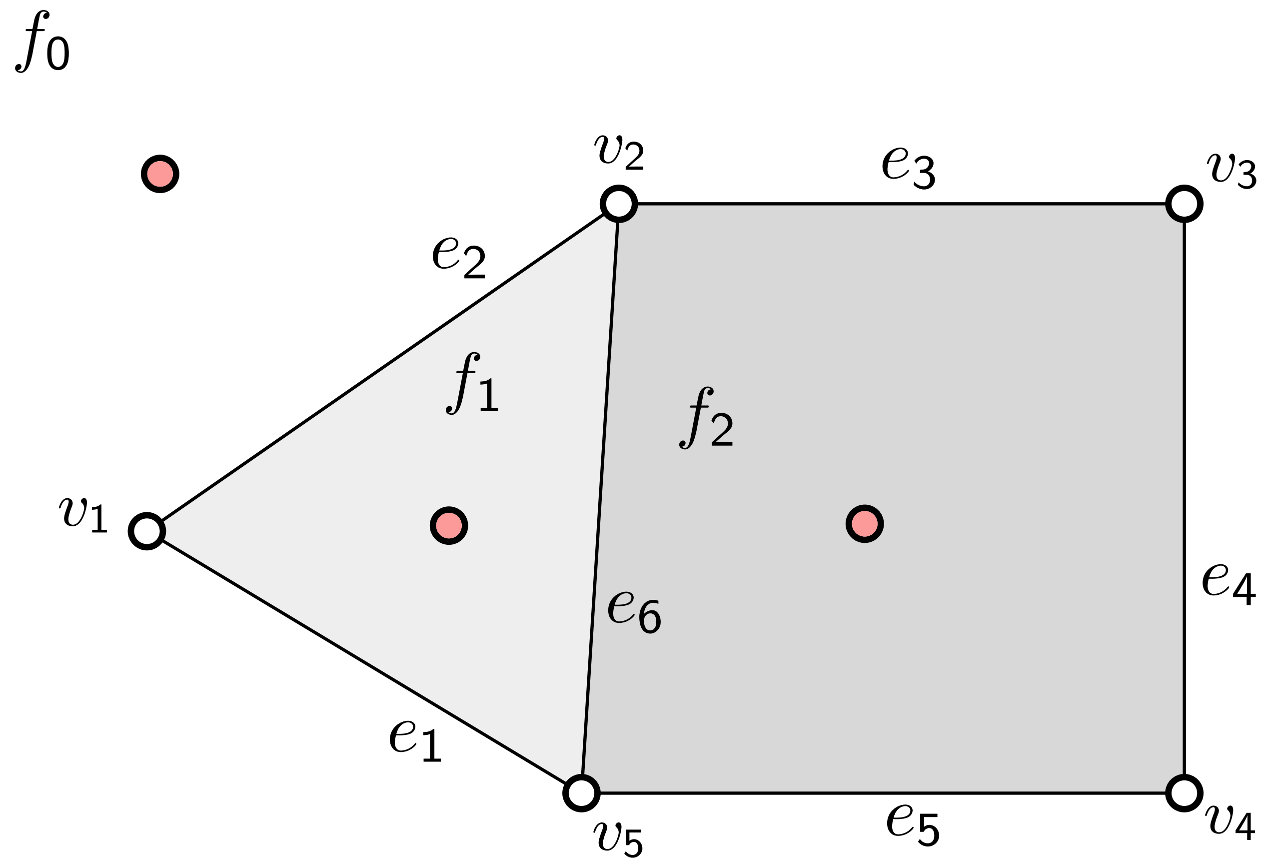

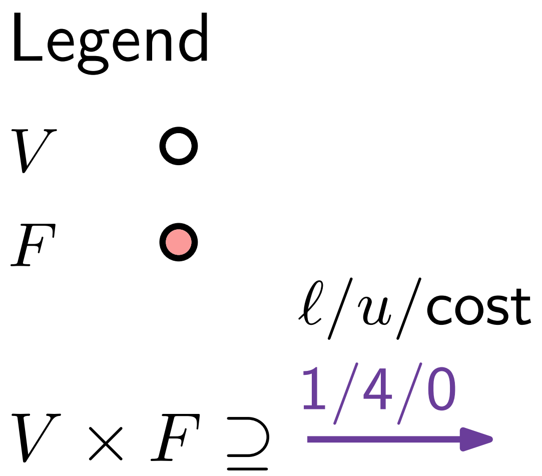

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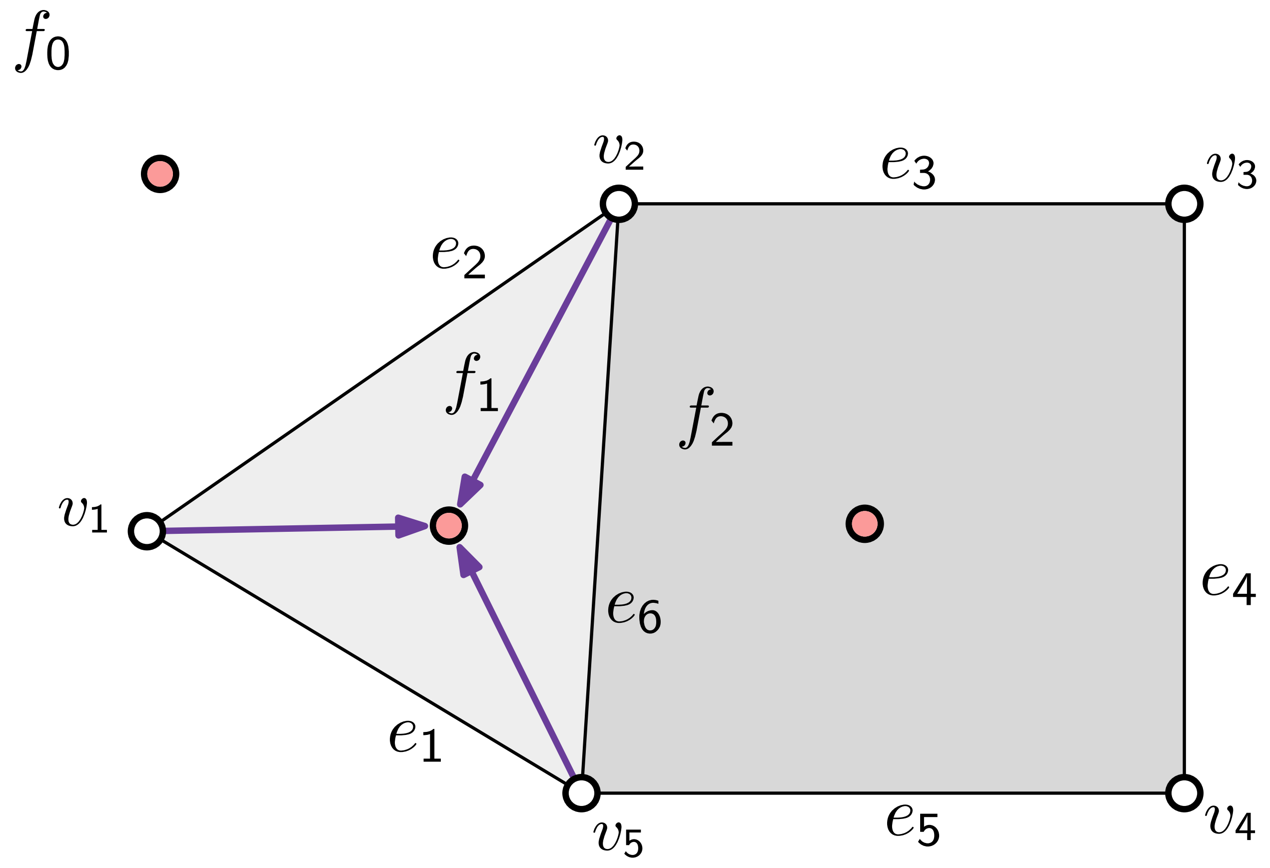


Legend

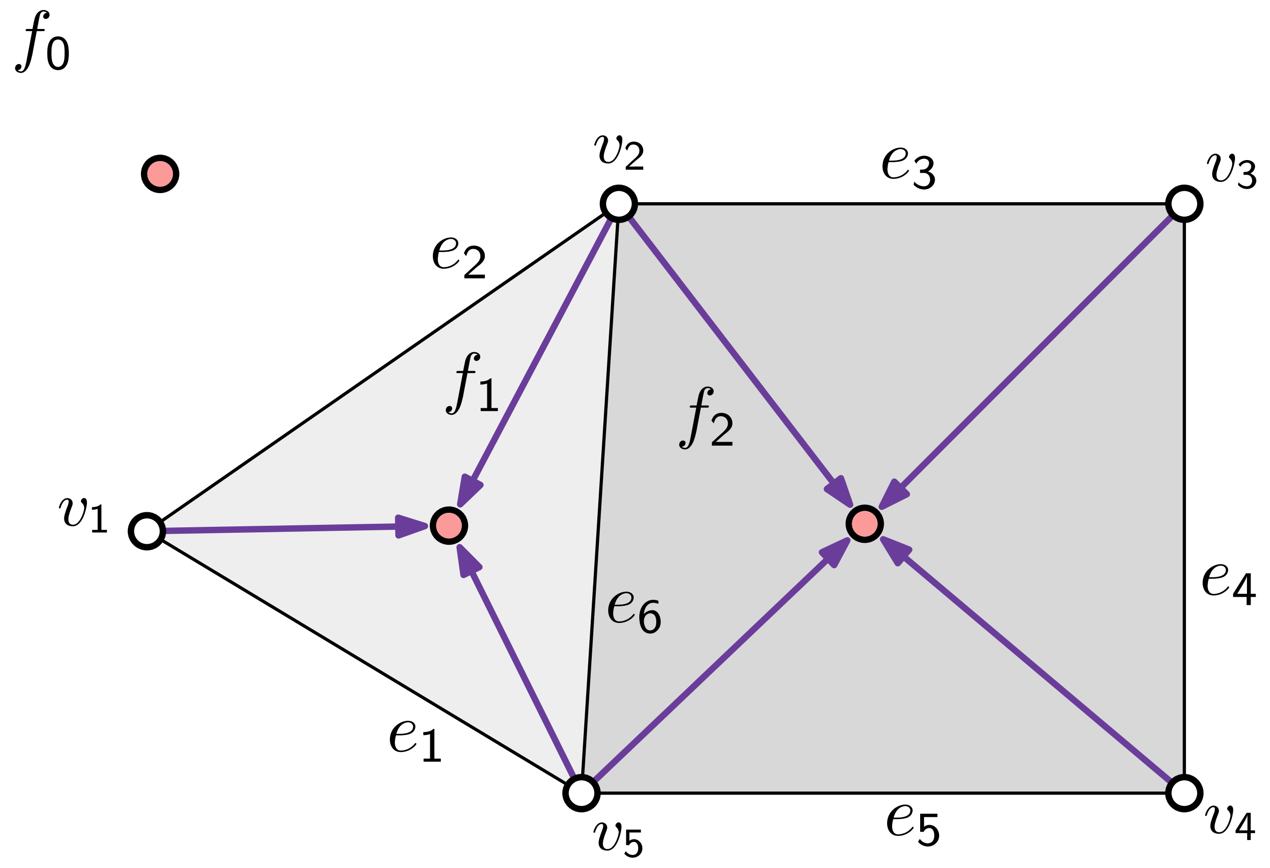
V ○

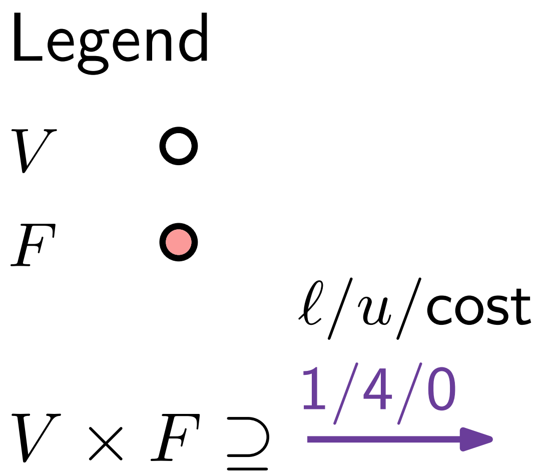
F ●

$\ell/u/$cost

$V \times F \supseteq$ $\xrightarrow{1/4/0}$

Flow Network Example





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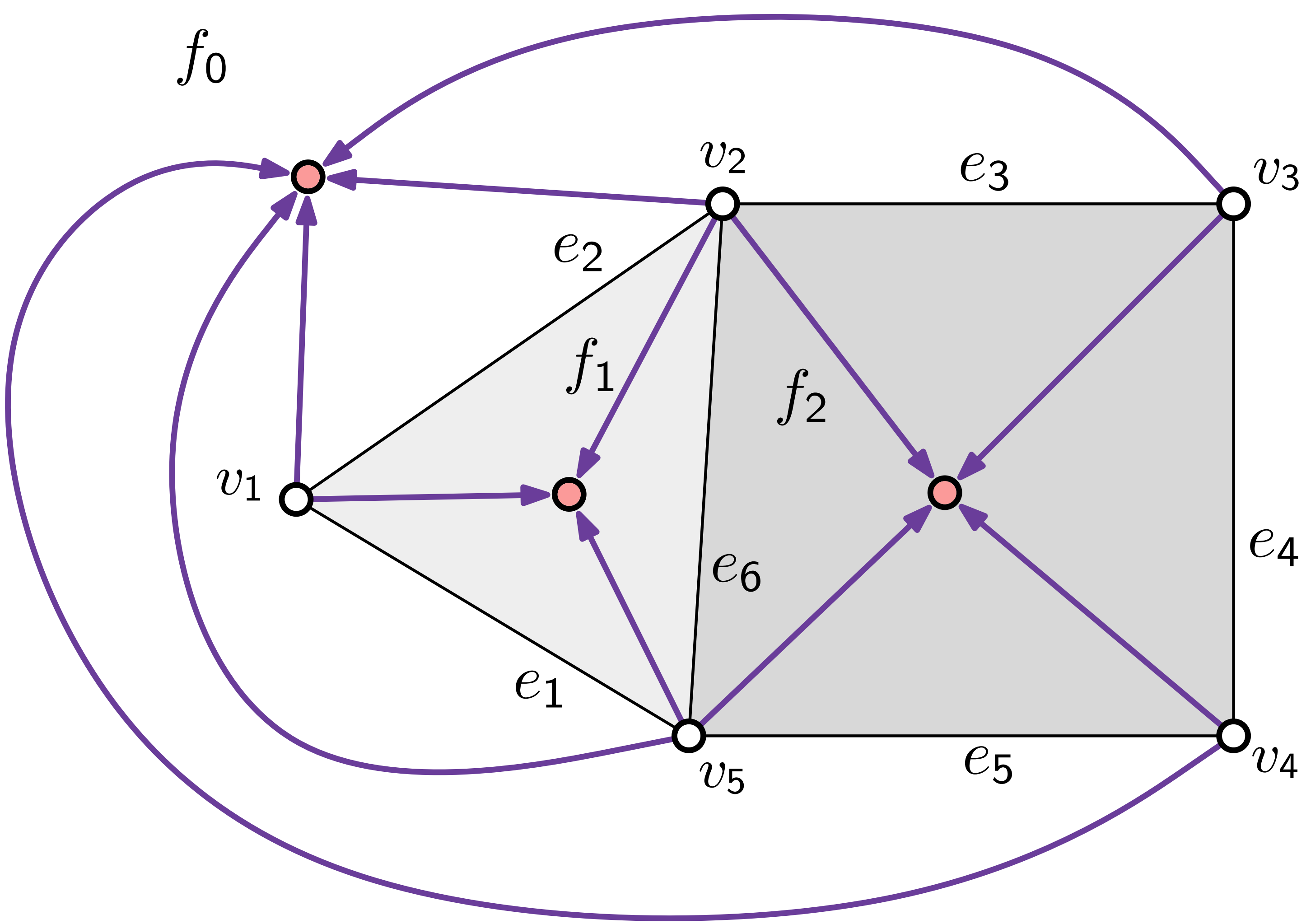

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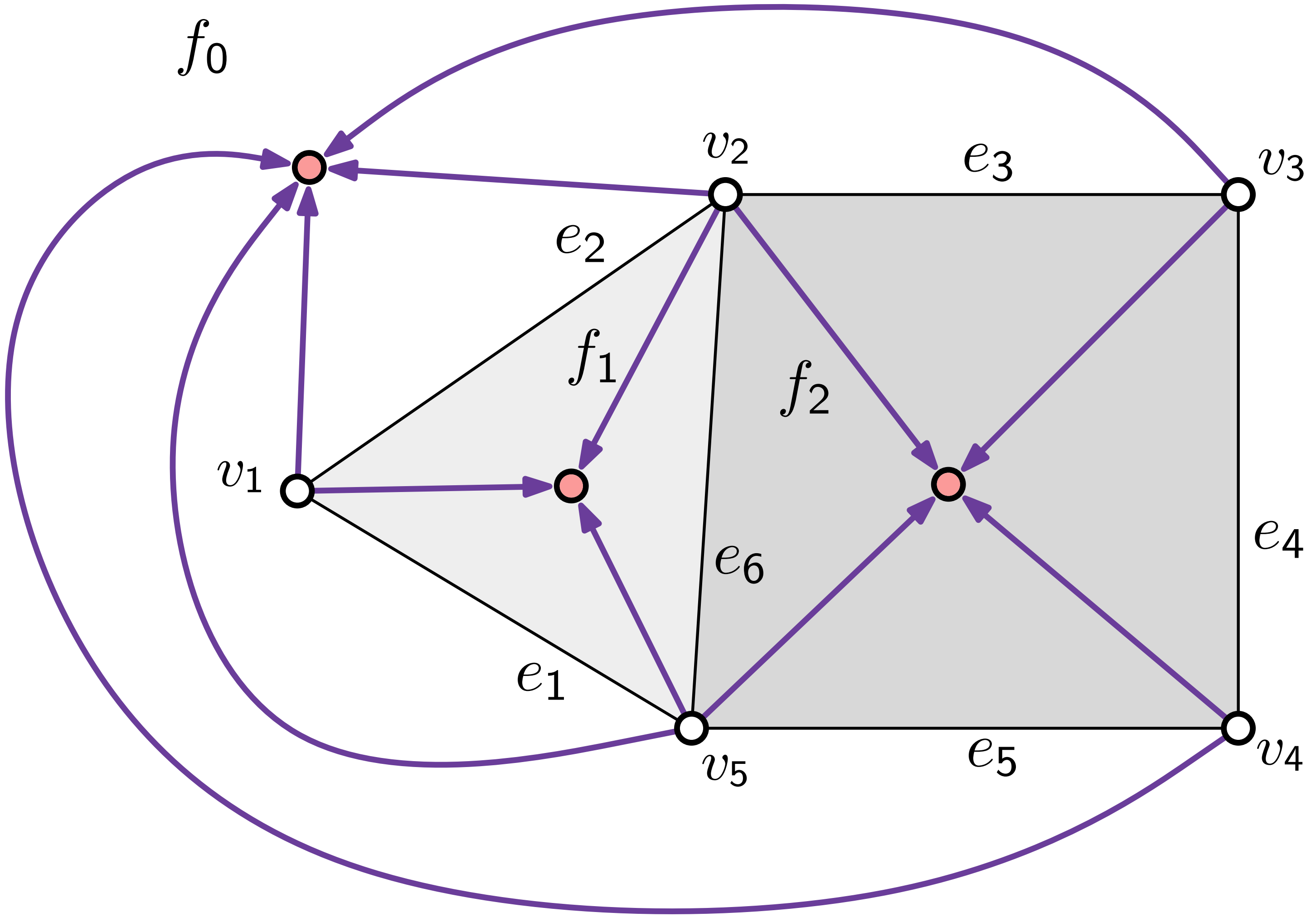

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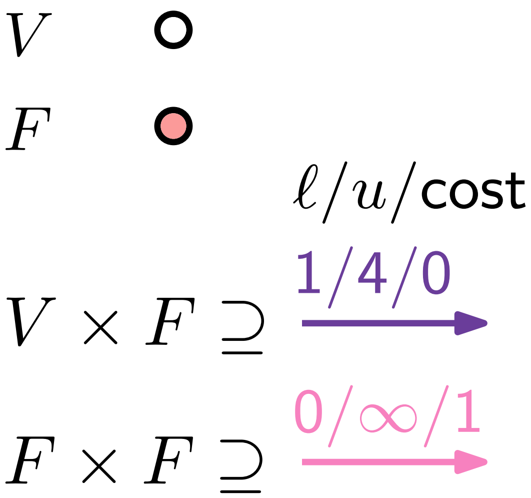

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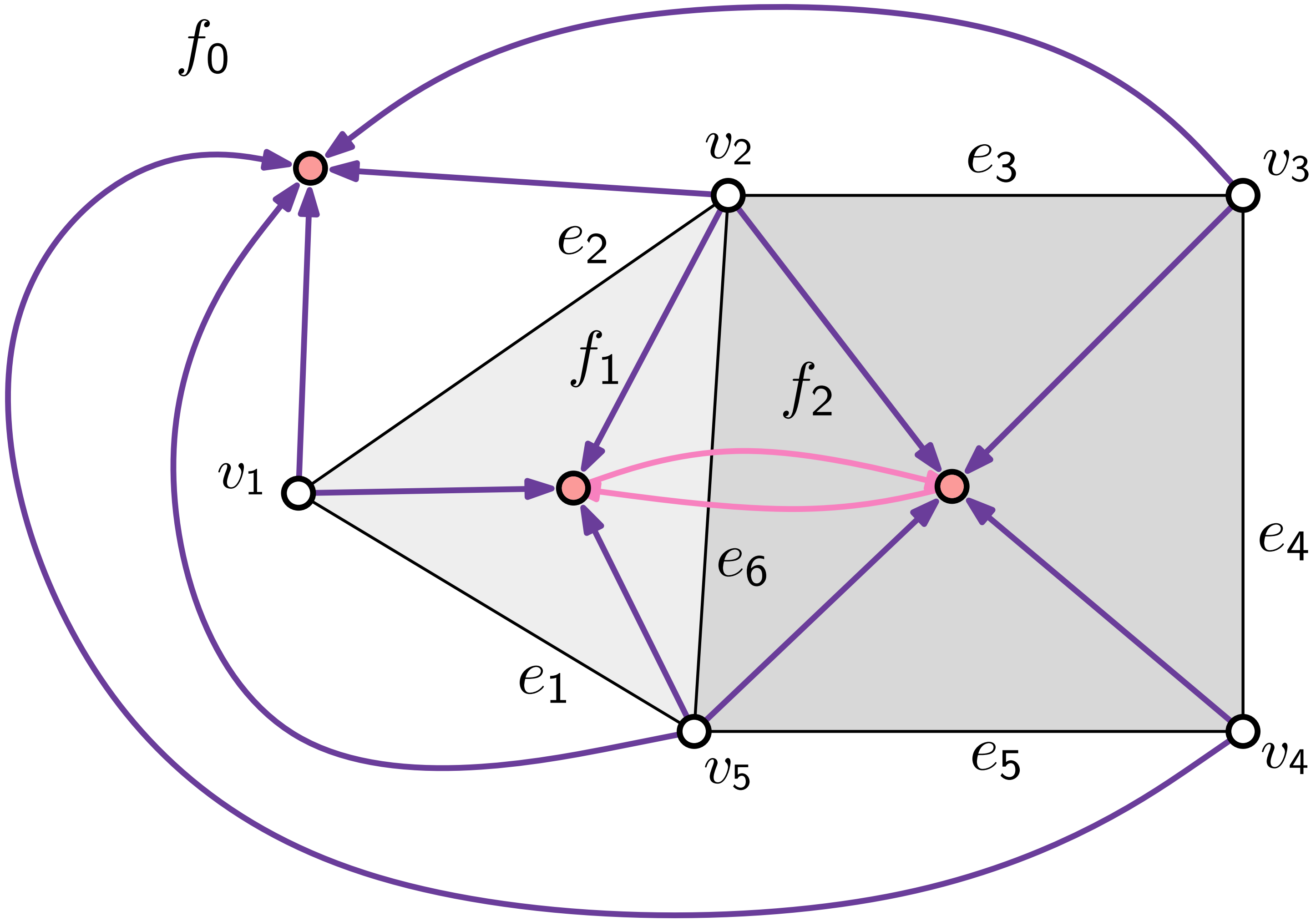

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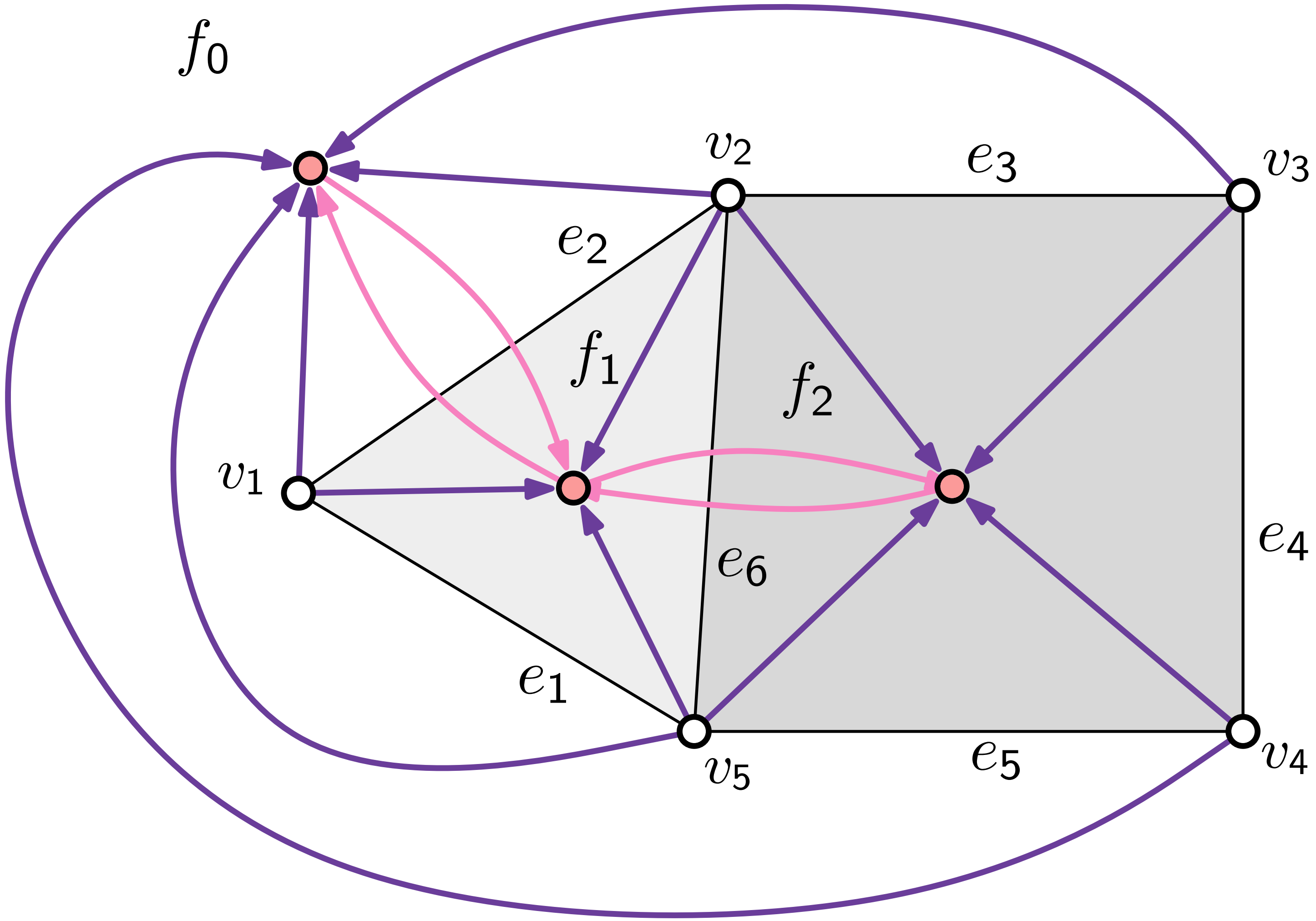

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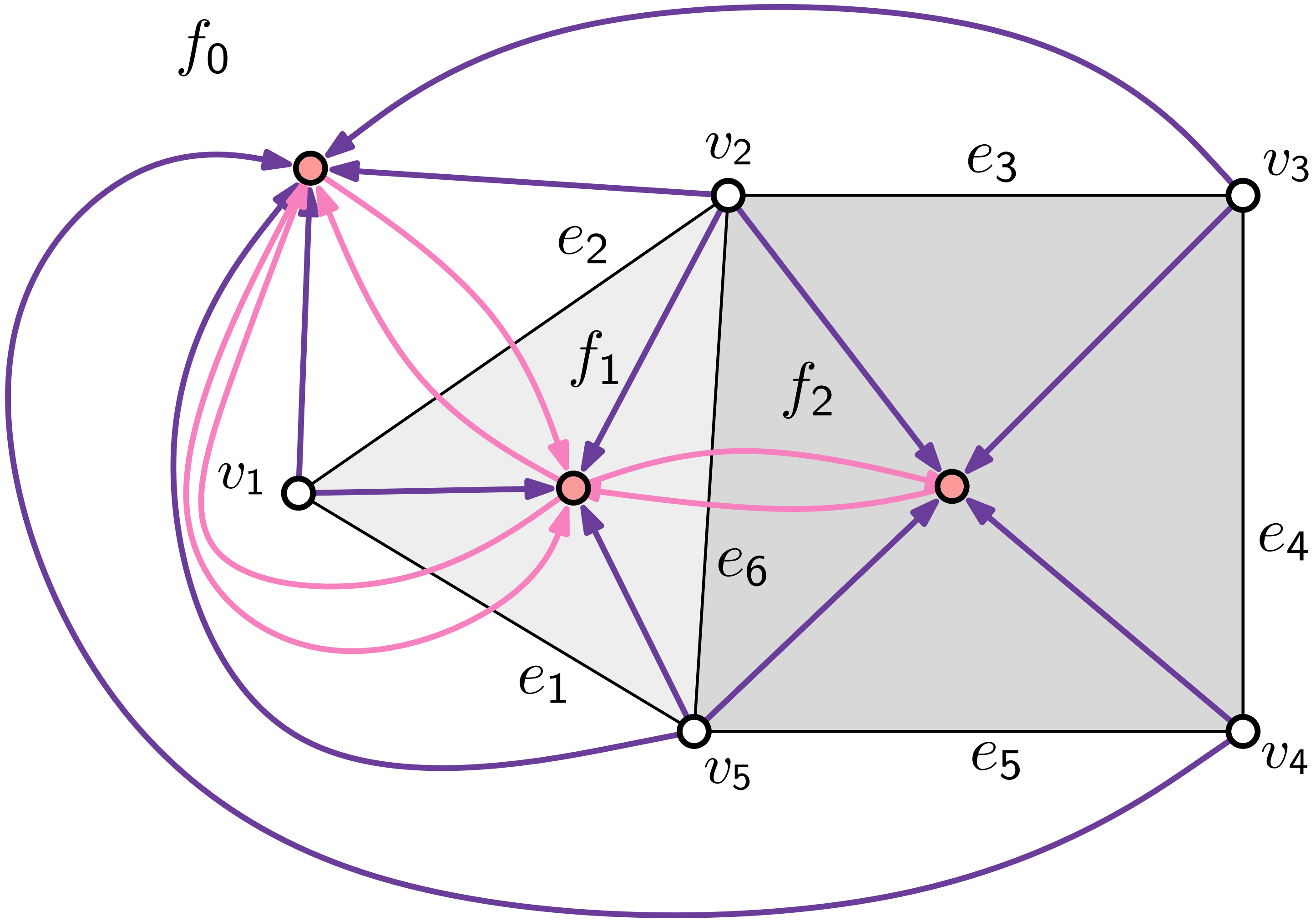

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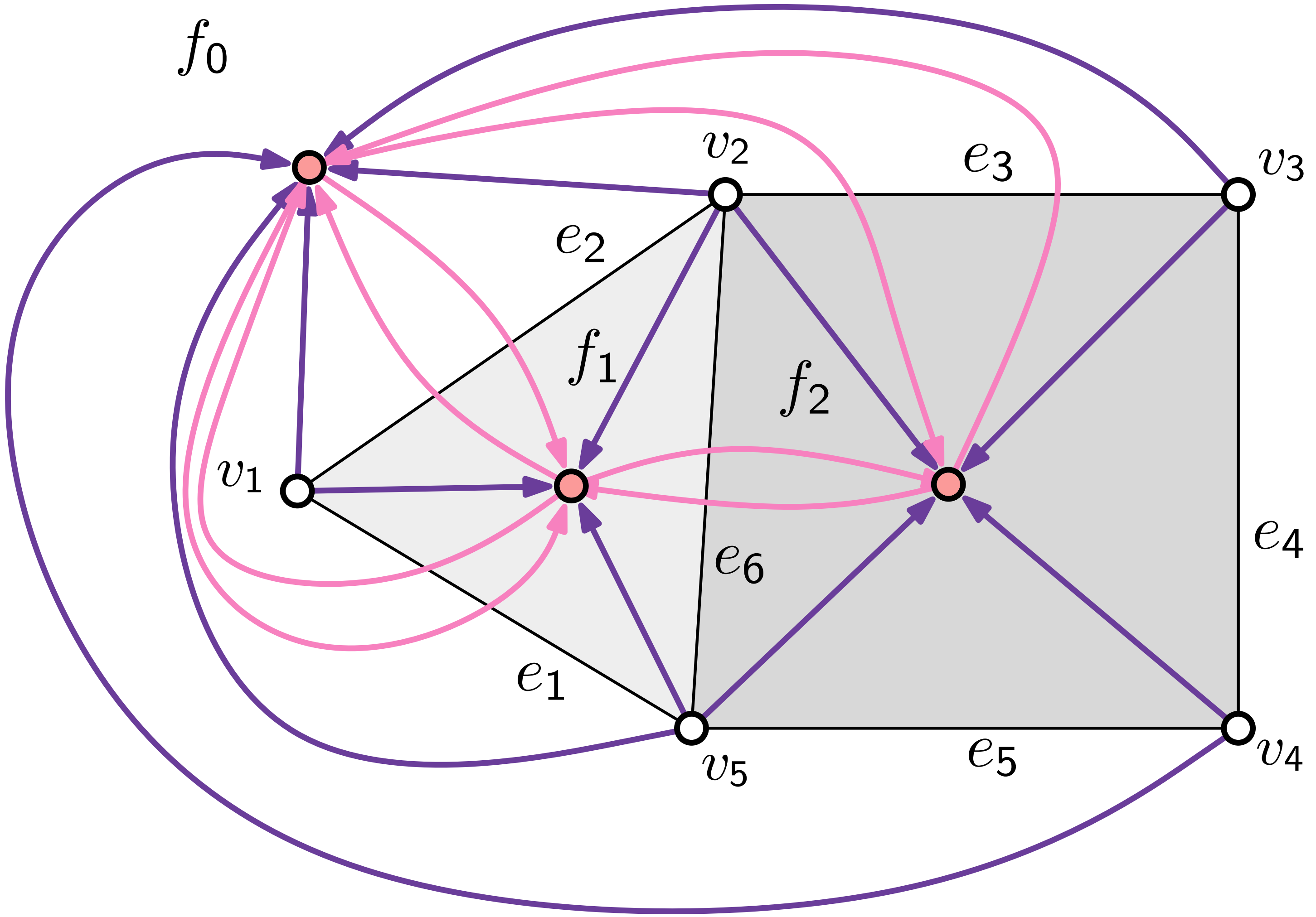

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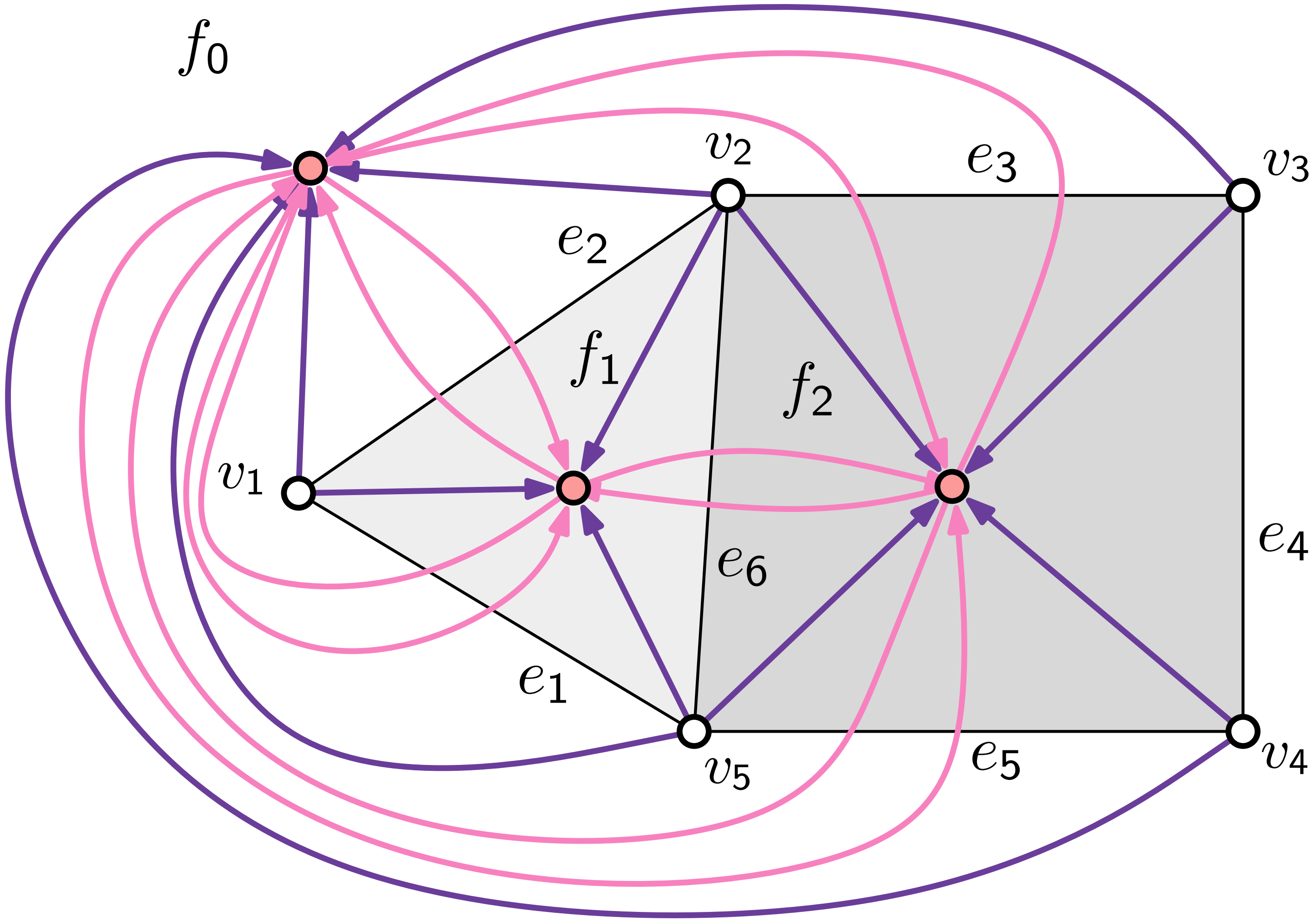

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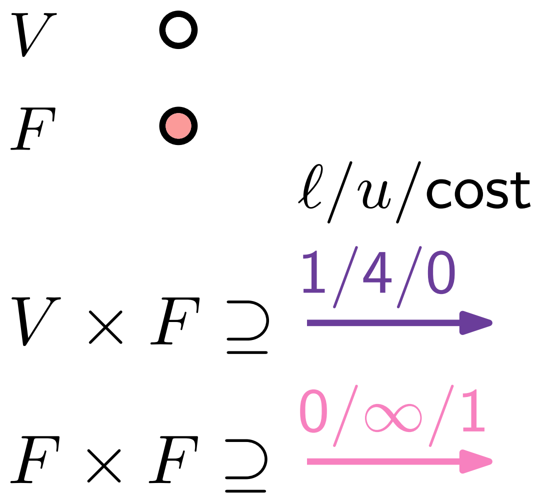

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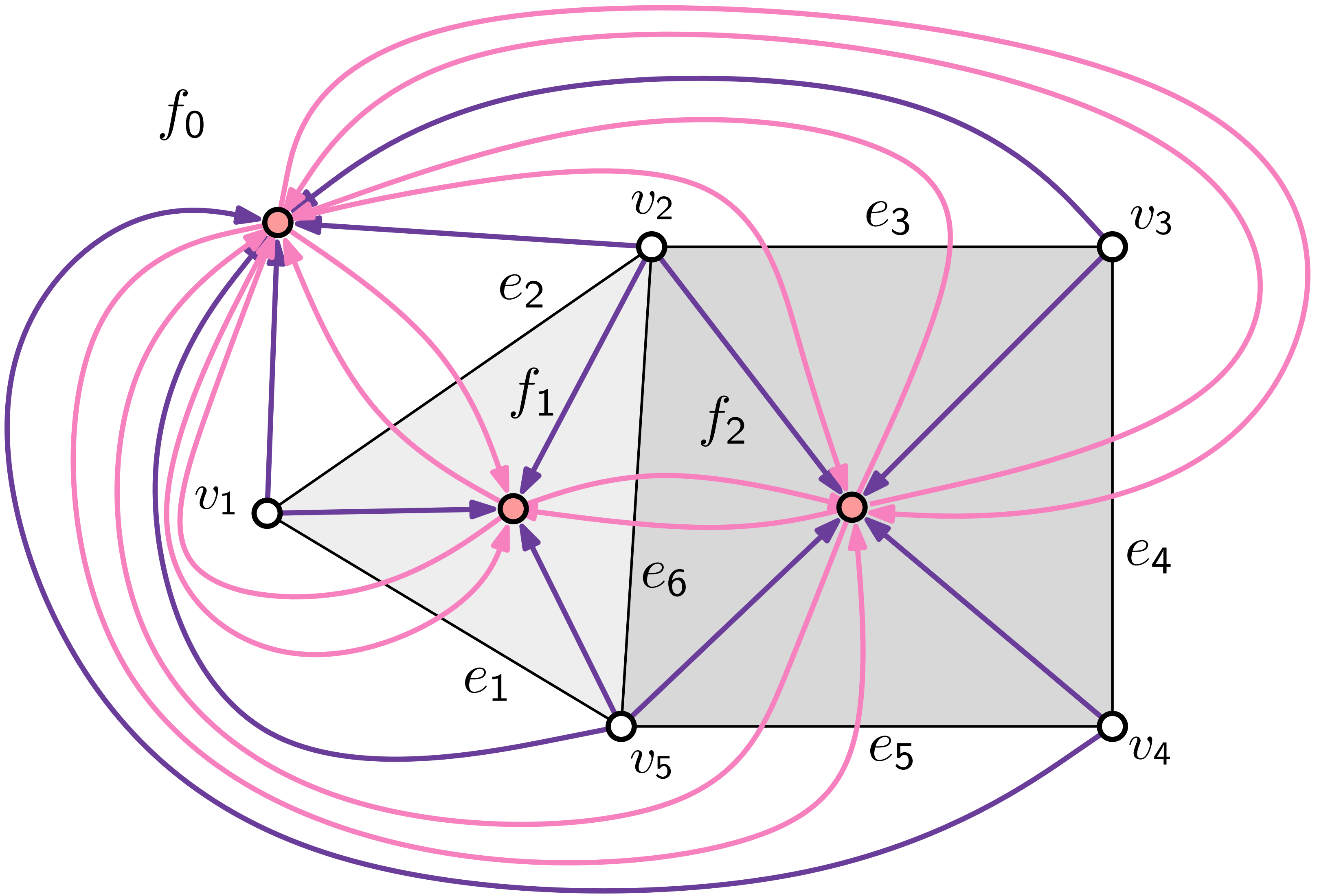

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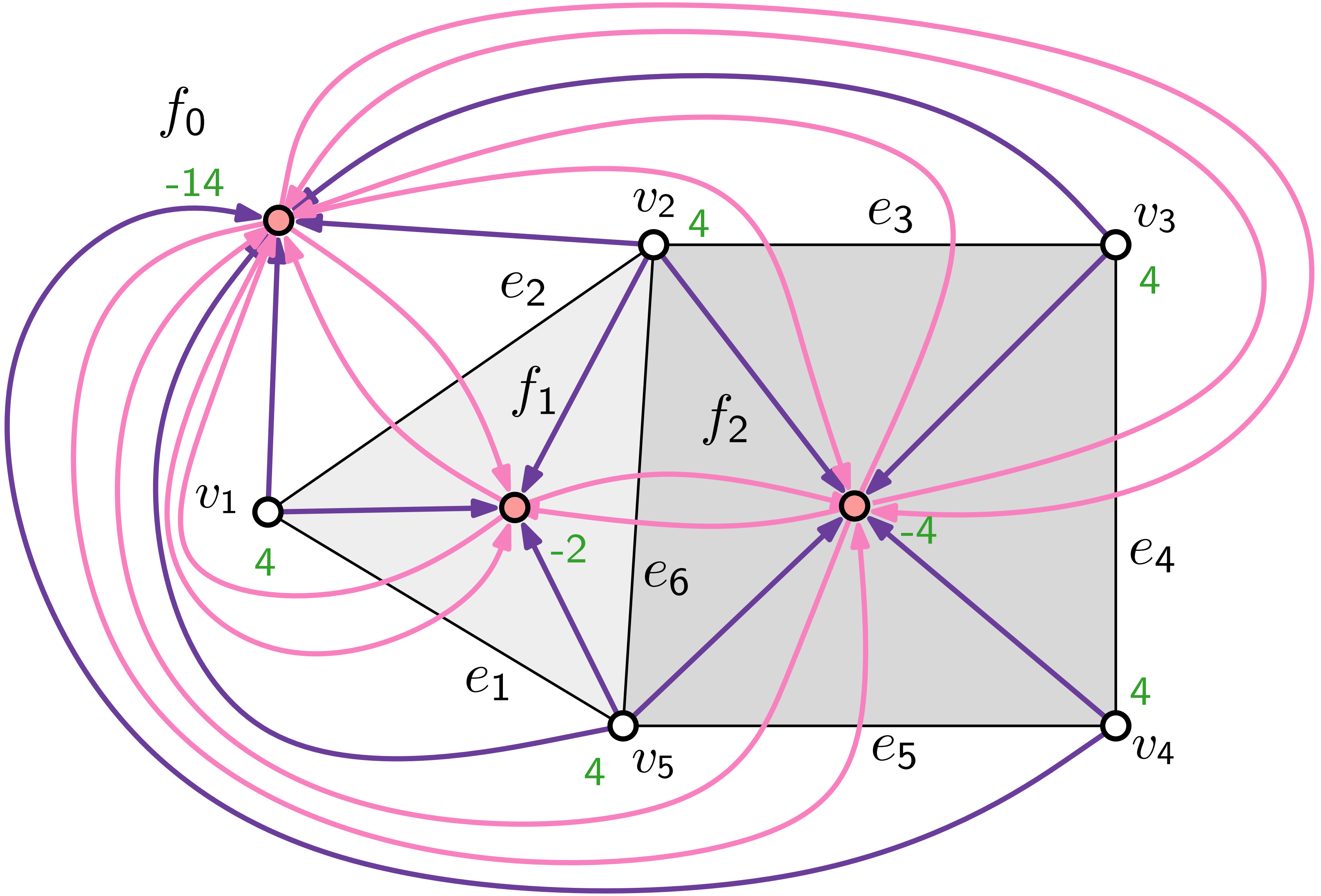


Legend

V ○

F ●

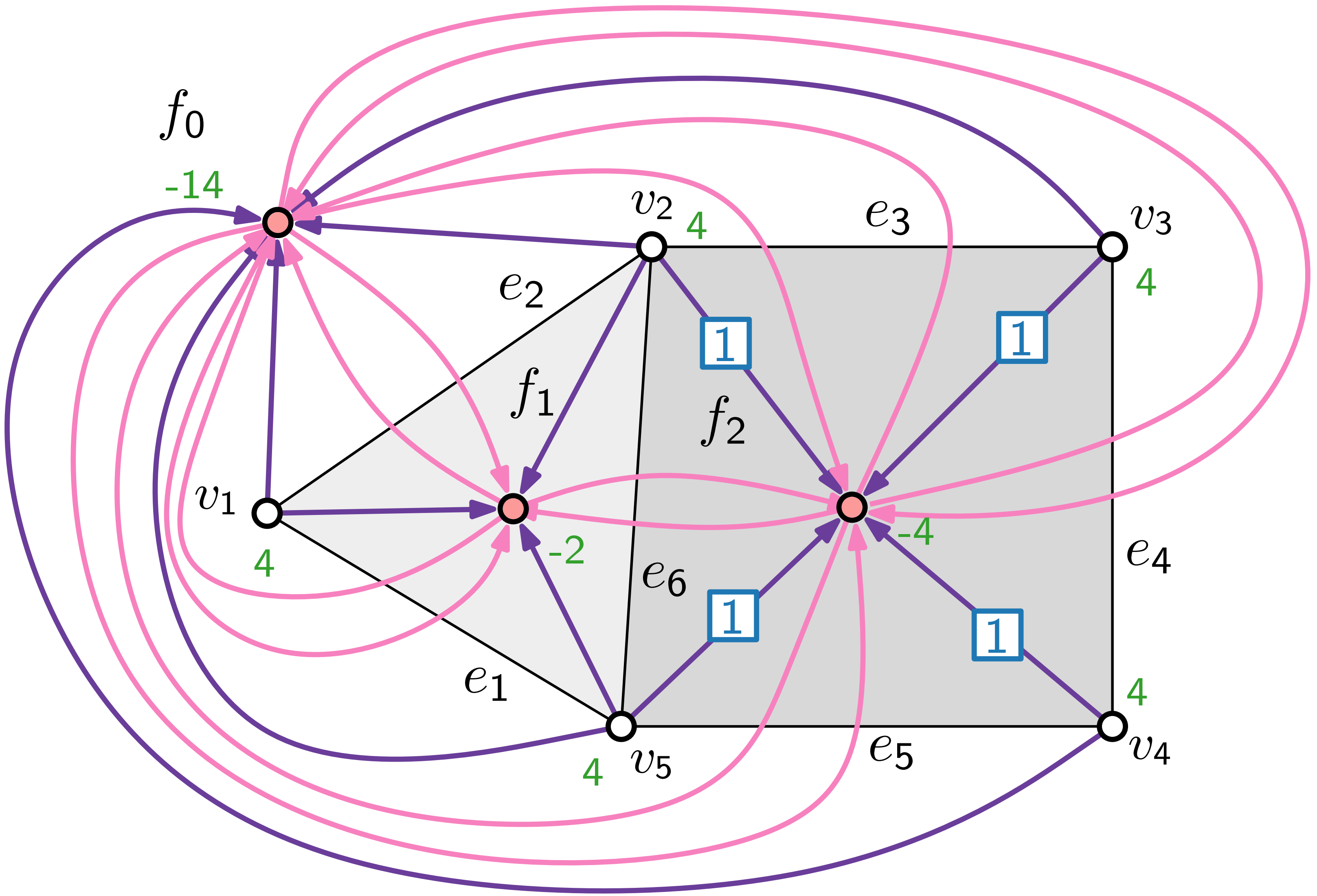
$\ell/u/\mathrm{cost}$

$V \times F \supseteq$ 1/4/0 →

$F \times F \supseteq$ $0/\infty/1$ →

$4 = b$-value

Flow Network Example



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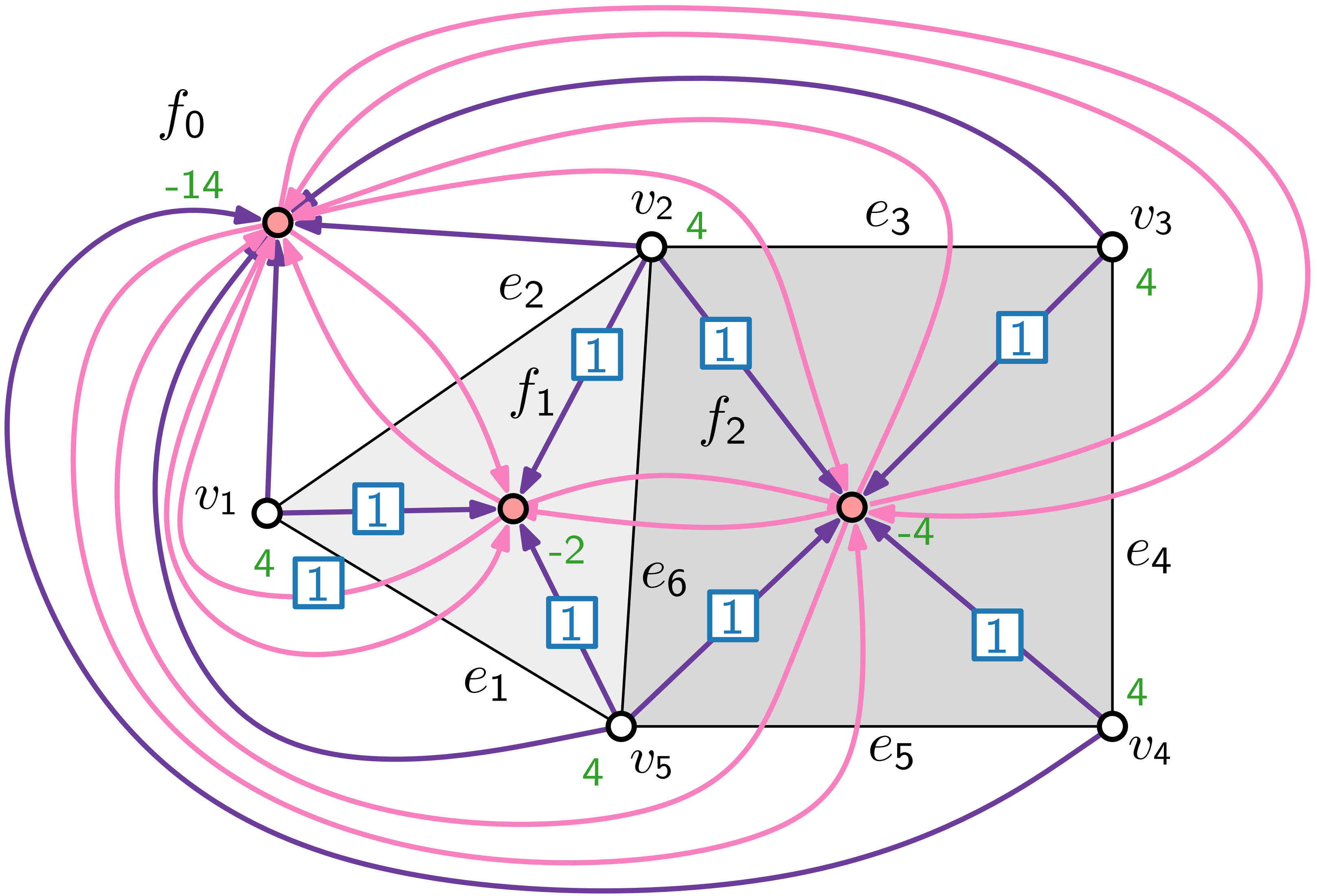

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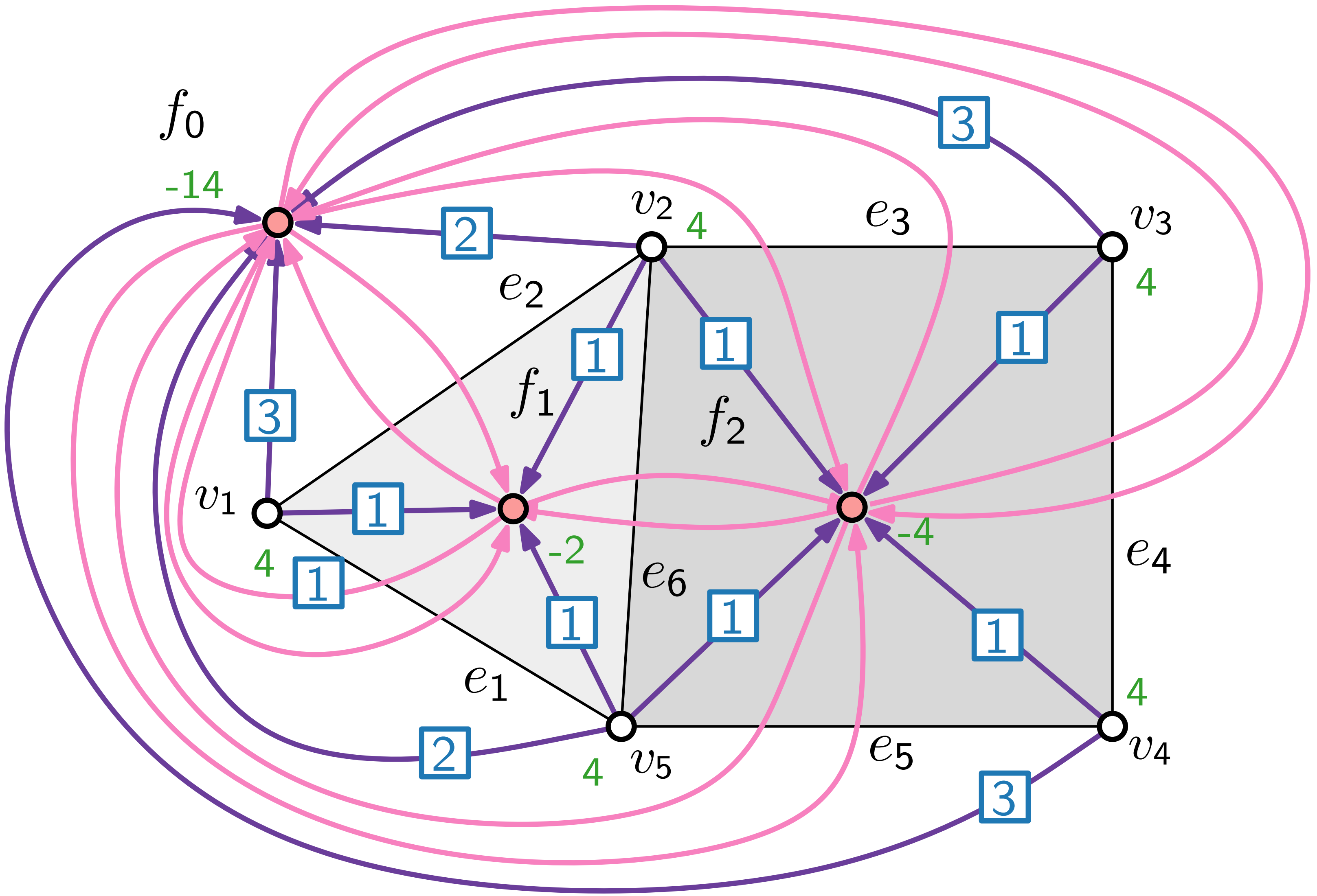

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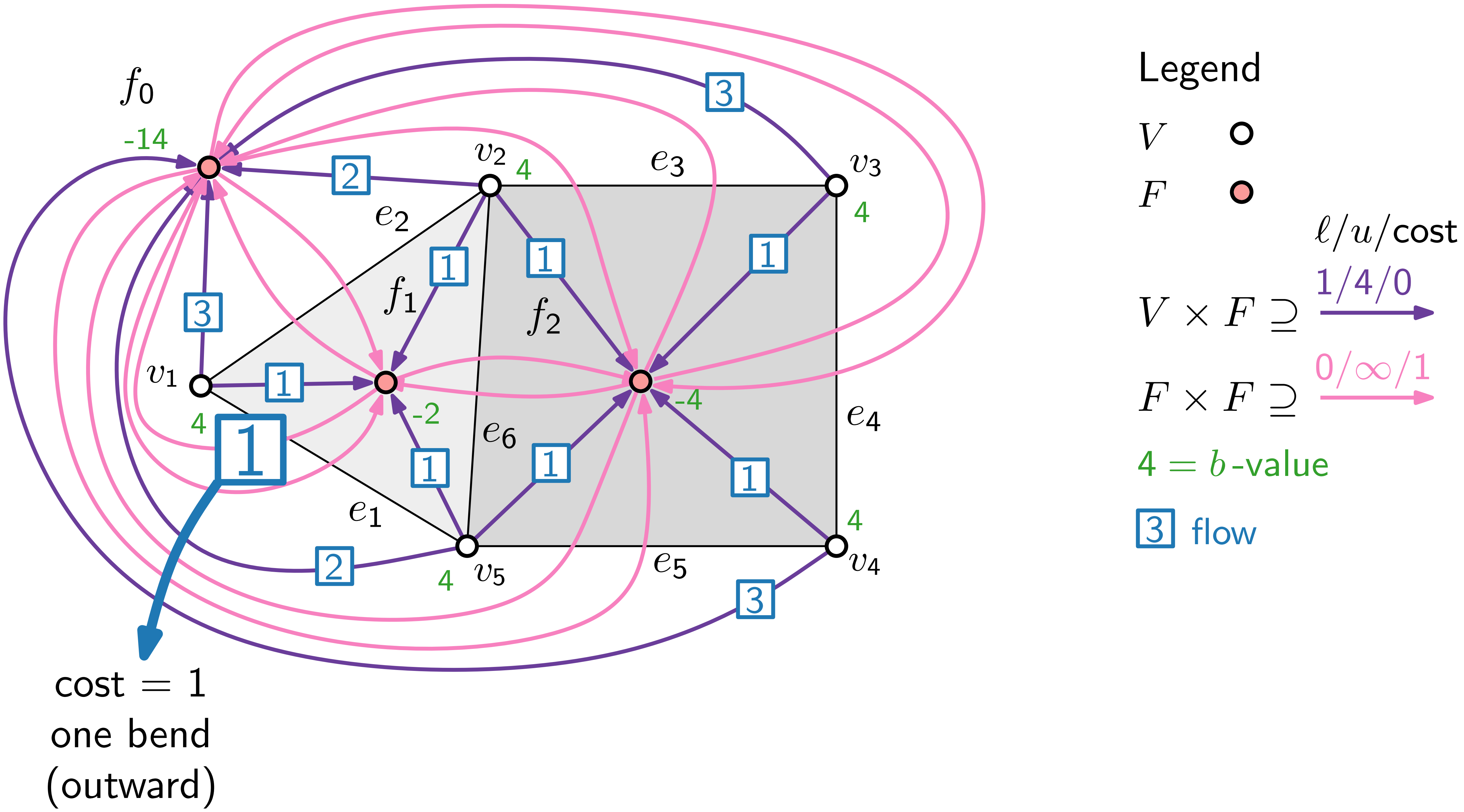

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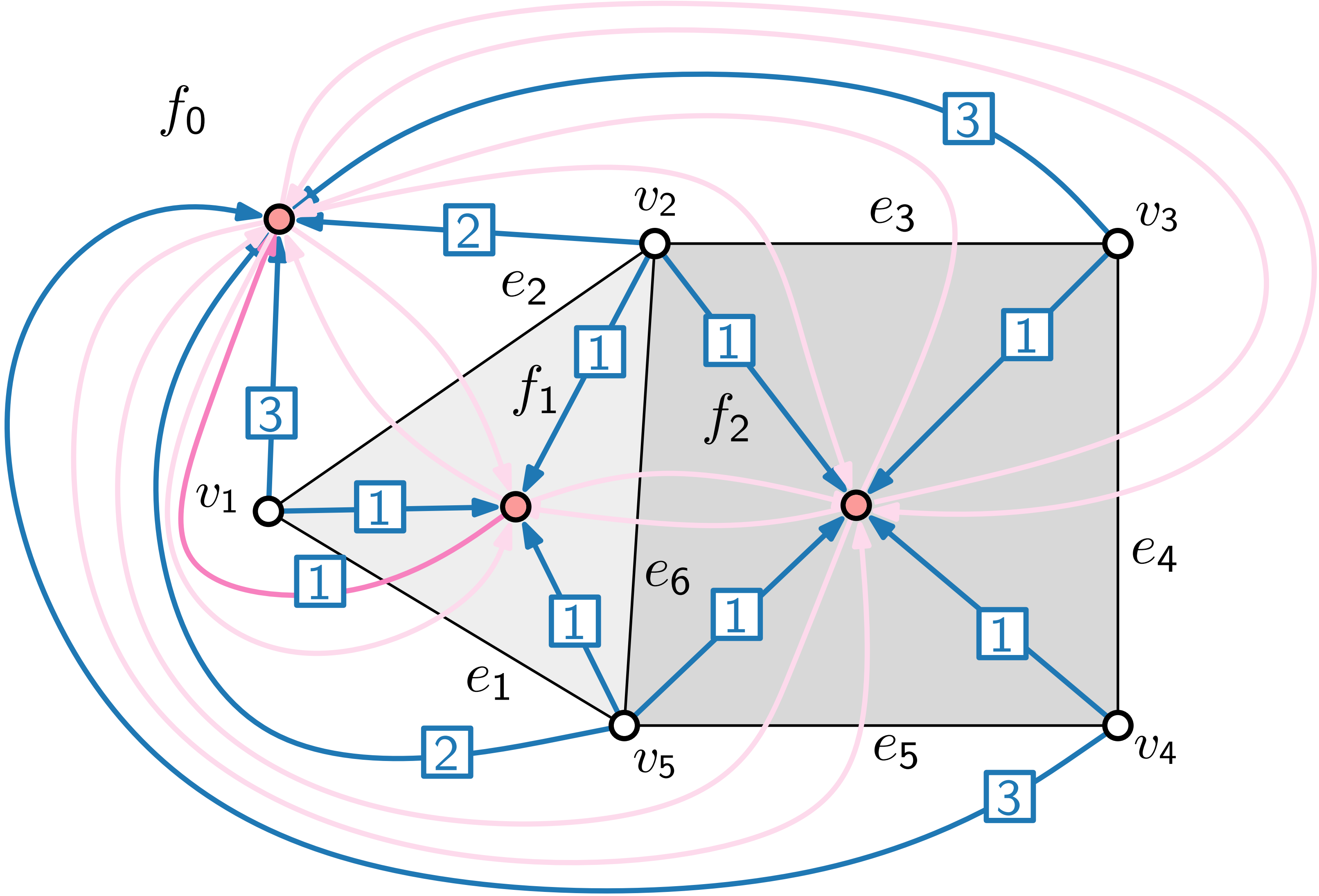

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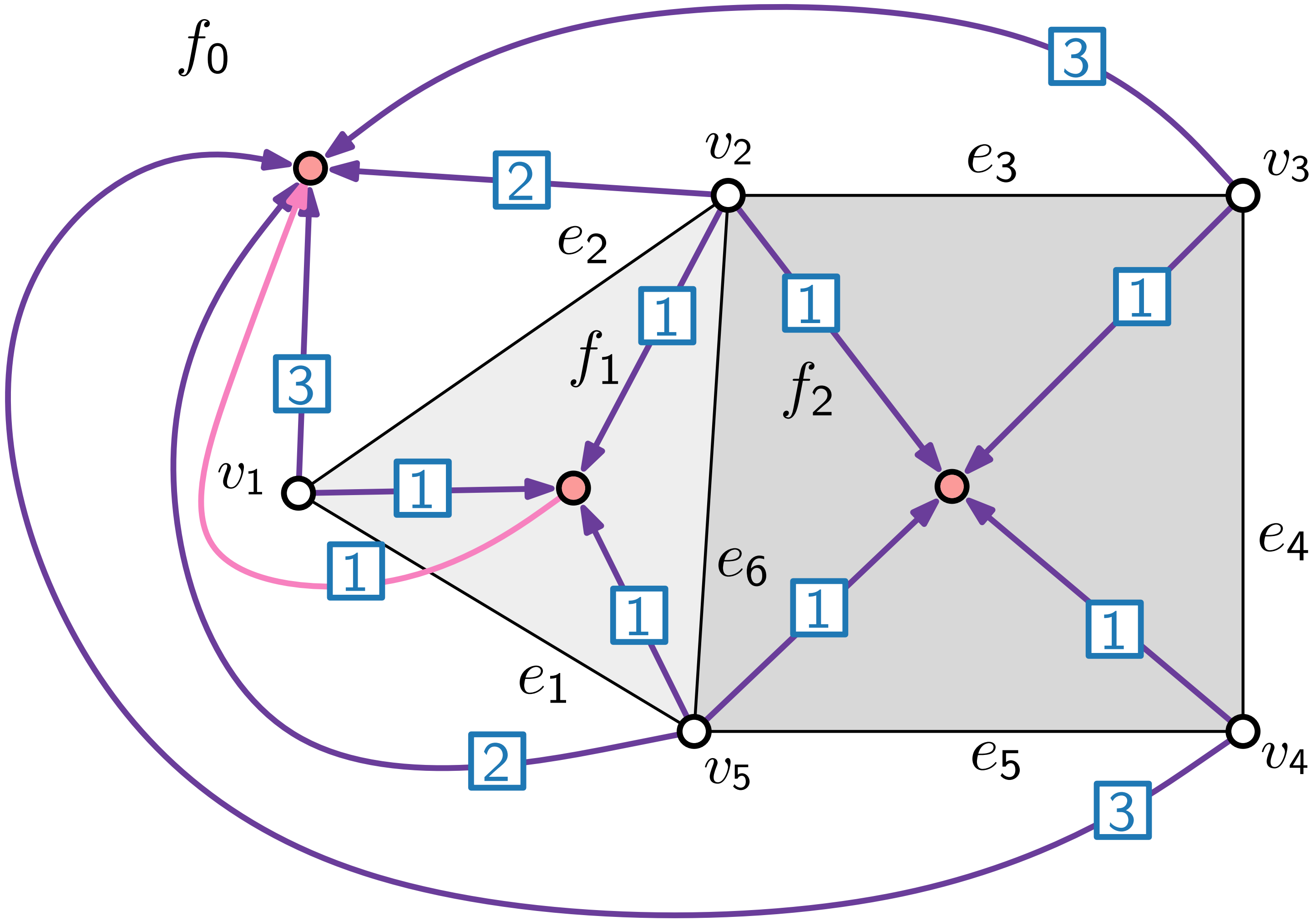

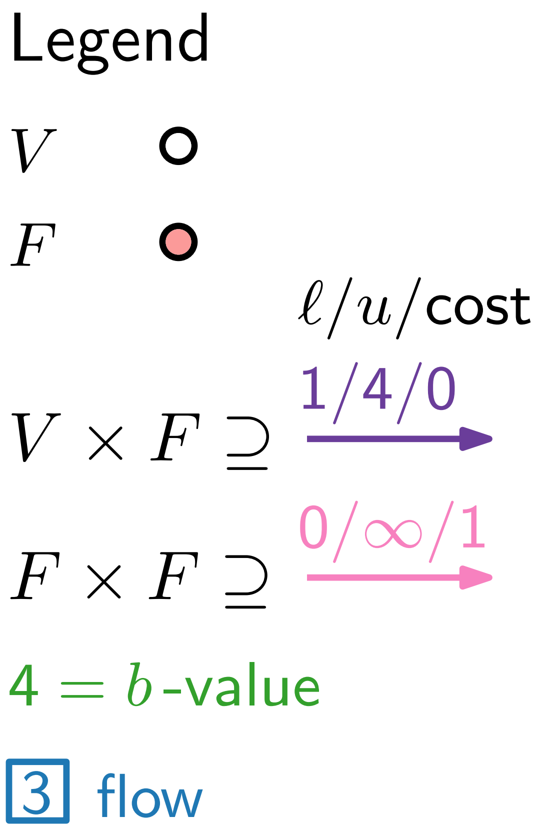

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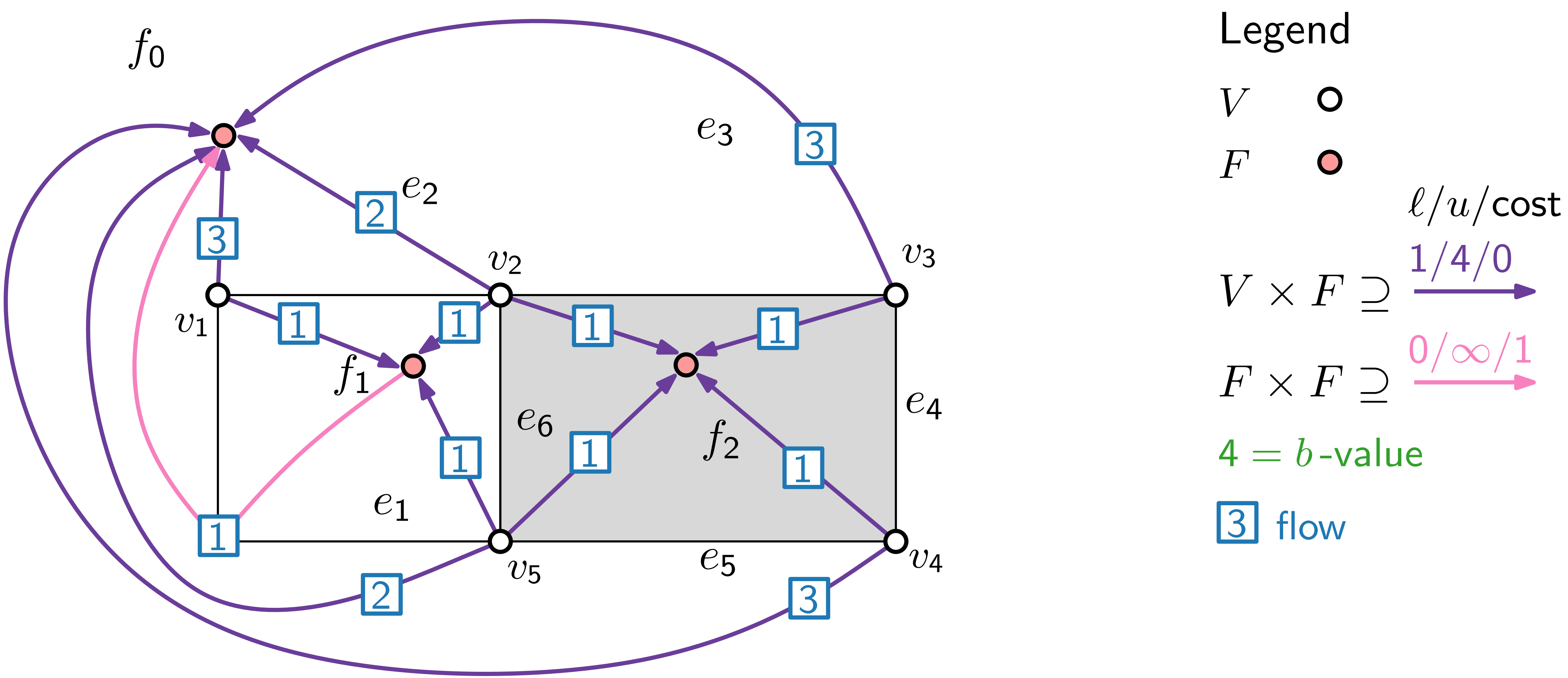

Bend Minimization – Result

Theorem. [Tamassia '87]
A plane graph (G, F, f_0) has a valid orthogonal representation $H(G)$ with k bends iff the flow network $N(G)$ has a valid flow X with cost k.

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A plane graph (G, F, f_0) has a valid orthogonal representation $H(G)$ with k bends iff the flow network $N(G)$ has a valid flow X with cost k.

(H1) $H(G)$ corresponds to F, f_0.

(H2) For each **edge** $\{u, v\}$ shared by faces f and g, sequence δ_1 is reversed and inverted δ_2.

(H3) For each **face** f it holds that:

$$\sum_{r \in H(f)} C(r) = \begin{cases} -4 & \text{if } f = f_0 \\ +4 & \text{otherwise.} \end{cases}$$

(H4) For each **vertex** v the sum of incident angles is 2π.

Proof.

$\Leftarrow$ Given valid flow X in $N(G)$ with cost k.
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- Show properties (H1)–(H4).

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(H3) Angle sum of $f = \pm 4$ ✓ Exercise.

(H4) Total angle at each vertex $= 2\pi$ ✓

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- $b(v) = 4 \quad \forall v \in V$
- $b(f) = -2\,\deg_G(f) + \begin{cases} -4 & \text{if } f = f_0, \\ +4 & \text{otherwise} \end{cases}$
- $\ell(v, f) := 1 \leq X(v, f) \leq 4 =: u(v, f)$
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- From Theorem follows that the combinatorial orthogonal bend minimization problem for plane graphs can be solved using an algorithm for the Min-Cost-Flow problem.

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Theorem. [Garg & Tamassia 2001]
Bend Minimization without a given combinatorial embedding is an NP-hard problem.

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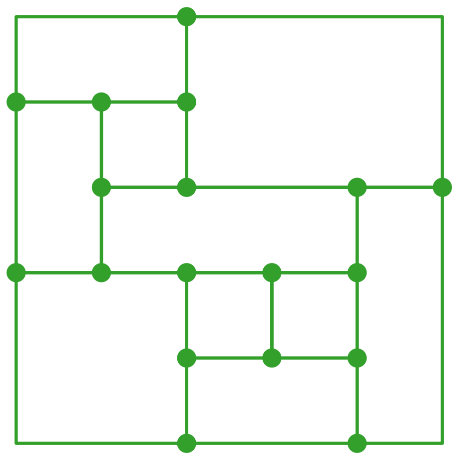
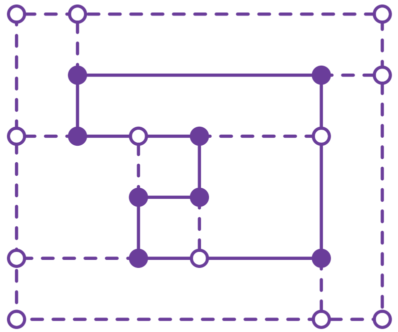
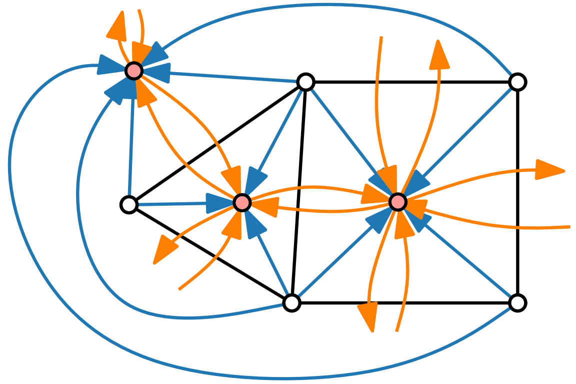
Visualization of Graphs



Lecture 5:
Orthogonal Layouts



Part IV:
Area Minimization

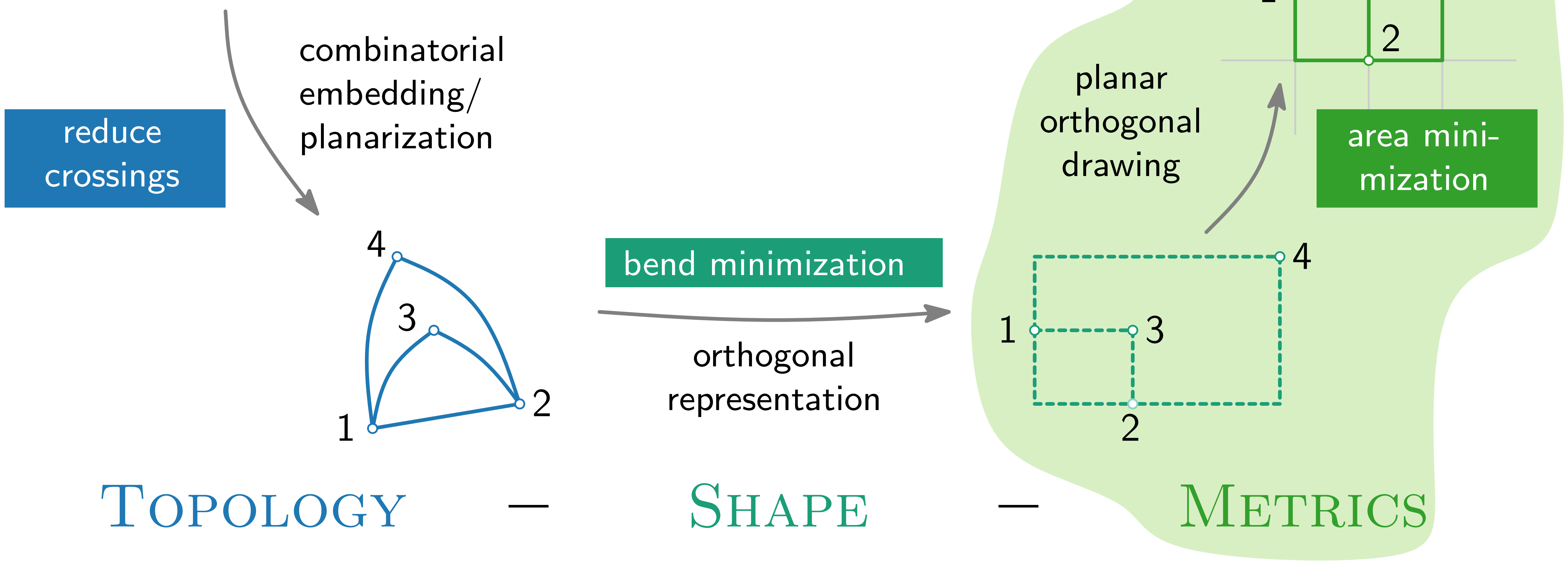


Jonathan Klawitter

Topology – Shape – Metrics

Three-step approach: [Tamassia 1987]

$V = \{v_1, v_2, v_3, v_4\}$
$E = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_2v_4\}$



Compaction

Compaction problem.

Given:

Find:

Compaction

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Special case.

All faces are rectangles.

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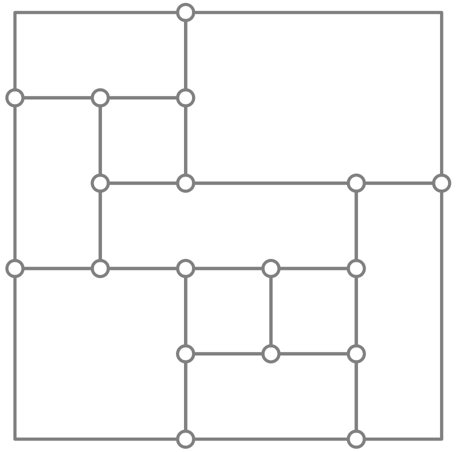
Idea.

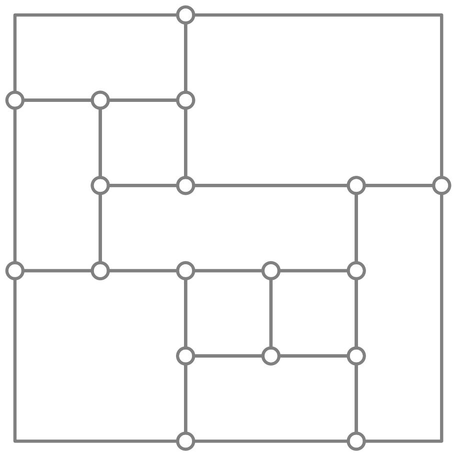
- Formulate flow network for horizontal/vertical compaction

Flow Network for Edge Length Assignment



Flow Network for Edge Length Assignment

Definition.
Flow Network $N_{\text{hor}} = ((W_{\text{hor}}, E_{\text{hor}}); b; \ell; u; \text{cost})$



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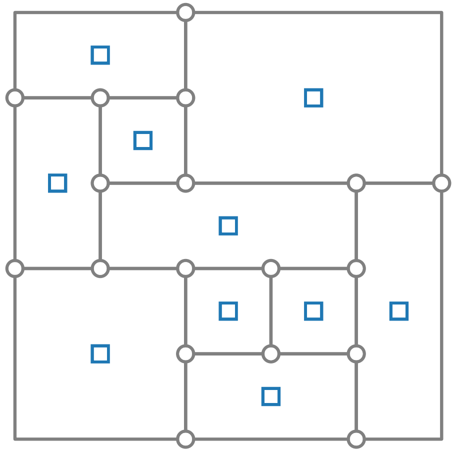

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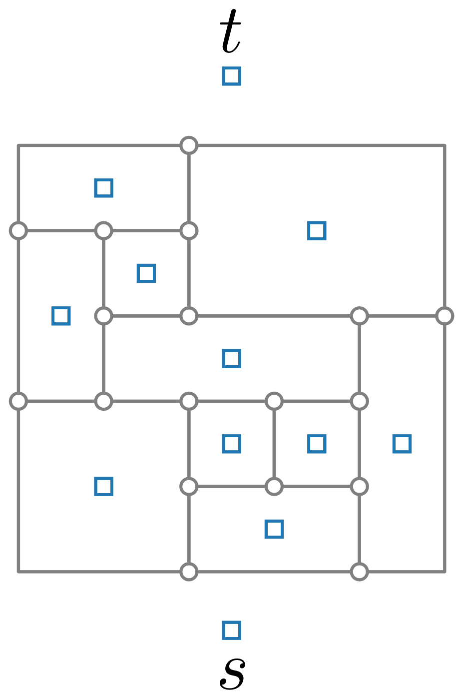

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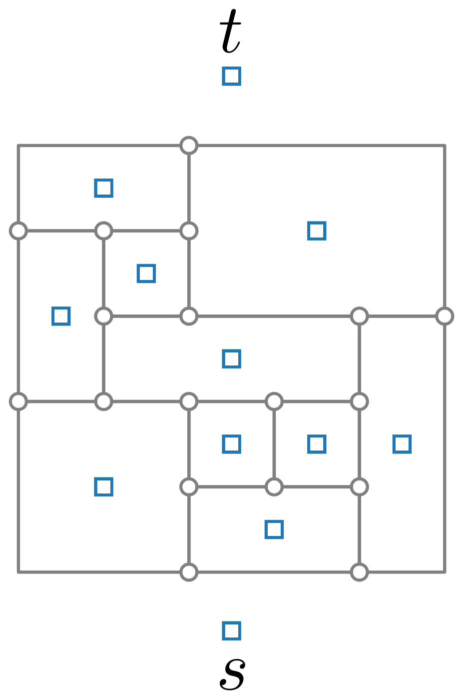
- $W_{\text{hor}} = F \setminus \{f_0\} \cup \{s, t\}$
- $E_{\text{hor}} = \{(f, g) \mid f, g$ share a *horizontal* segment and f lies *below* $g\}$



Flow Network for Edge Length Assignment

Definition.

Flow Network $N_{\mathrm{hor}} = ((W_{\mathrm{hor}}, E_{\mathrm{hor}}); b; \ell; u; \mathrm{cost})$

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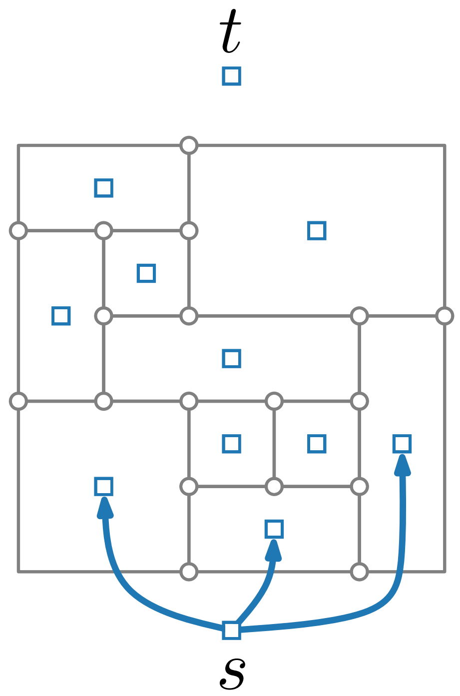

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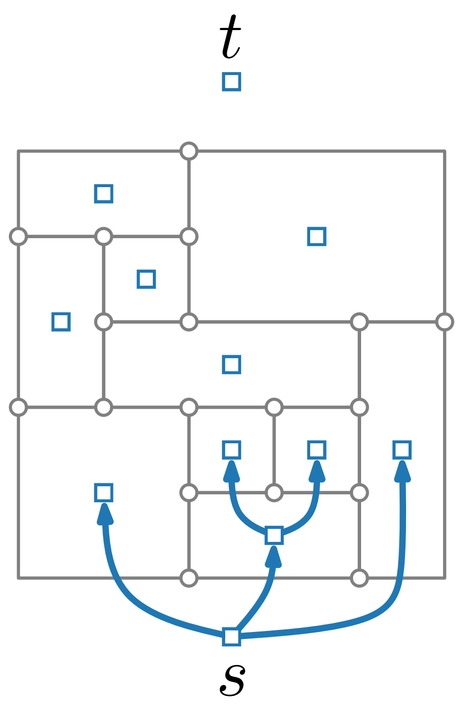

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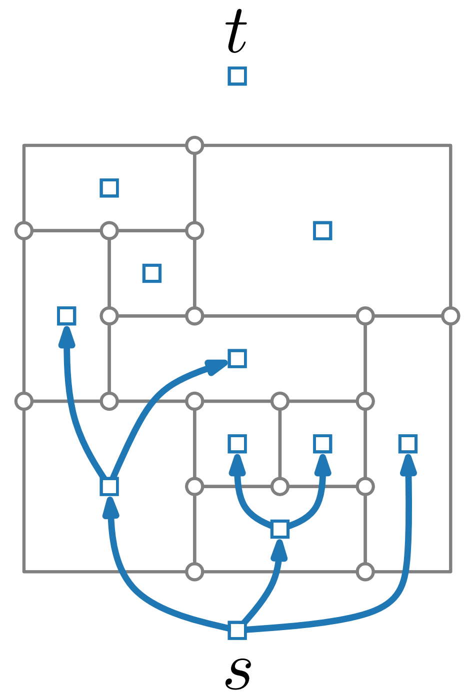

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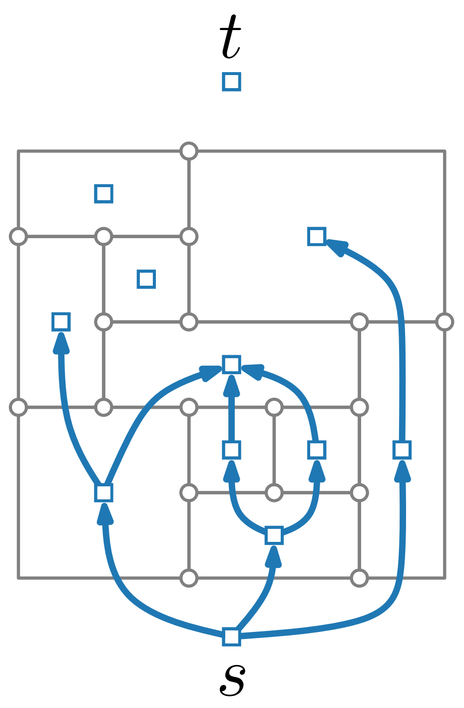

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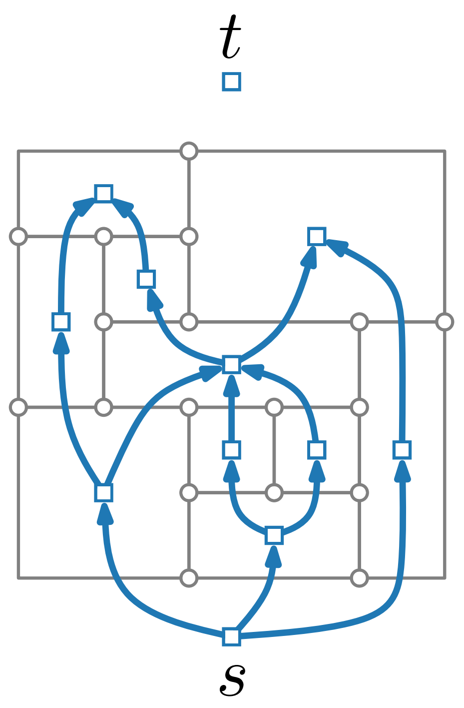

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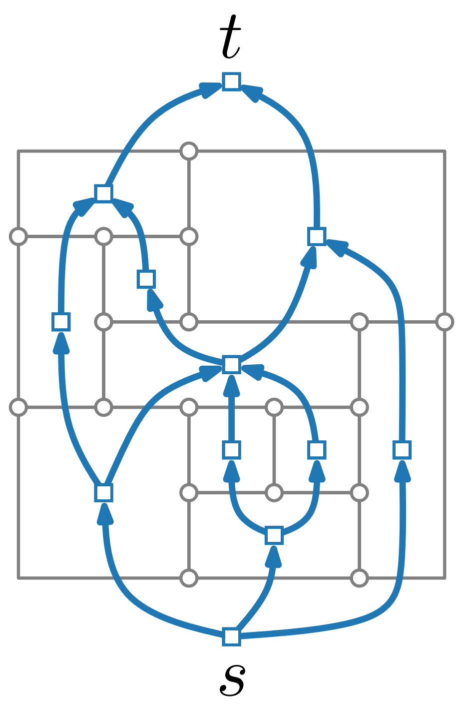

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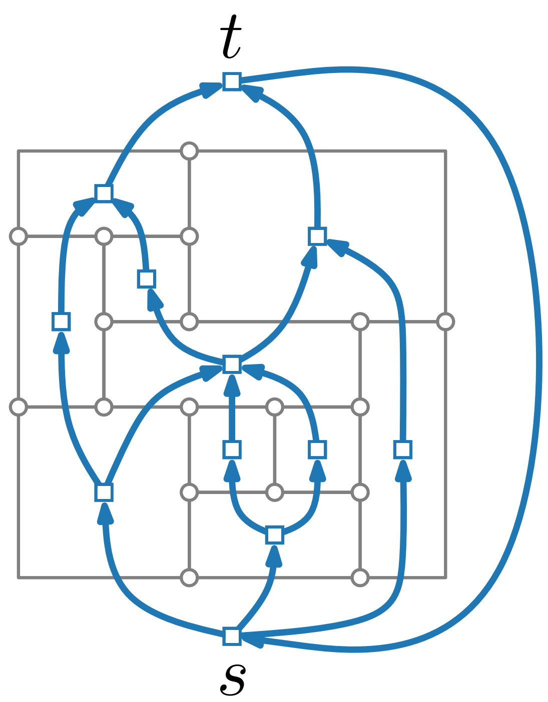
- $E_{\text{hor}} = \{(f, g) \mid f, g$ share a *horizontal* segment and f lies *below* $g\} \cup \{(t, s)\}$



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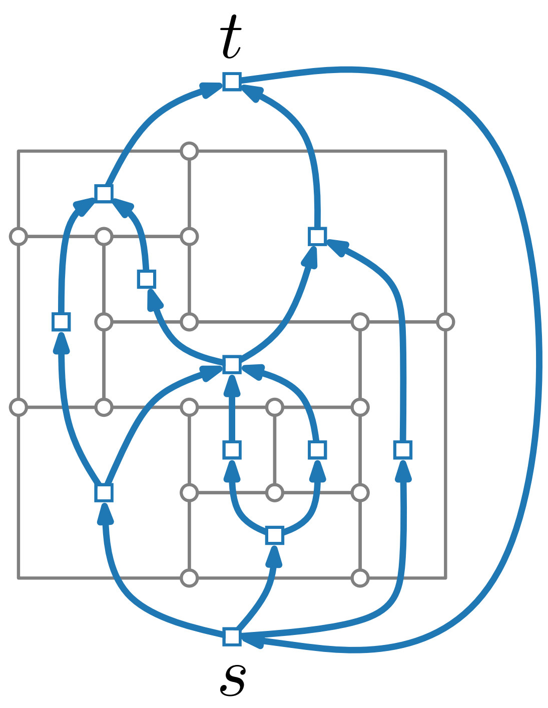
- $W_{\text{hor}} = F \setminus \{f_0\} \cup \{s, t\}$ $\square$
- $E_{\text{hor}} = \{(f, g) \mid f, g$ share a *horizontal* segment and f lies *below* $g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \quad \forall a \in E_{\text{hor}}$



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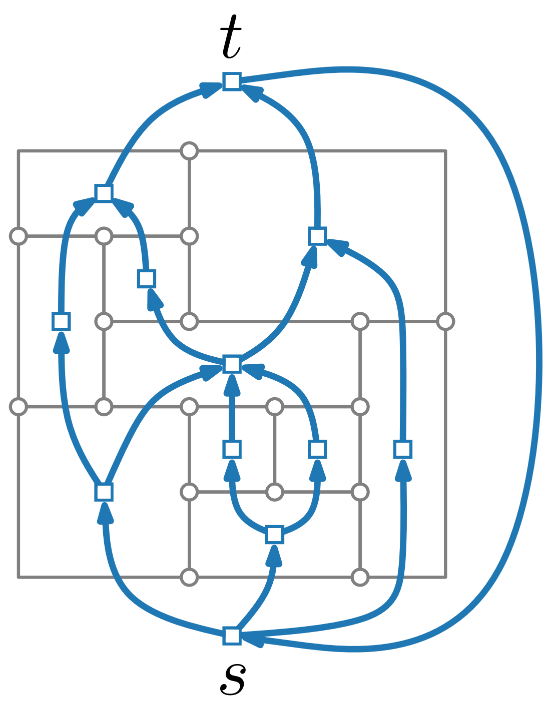
- $\ell(a) = 1 \quad \forall a \in E_{\text{hor}}$
- $u(a) = \infty \quad \forall a \in E_{\text{hor}}$



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- $\ell(a) = 1 \quad \forall a \in E_{\text{hor}}$
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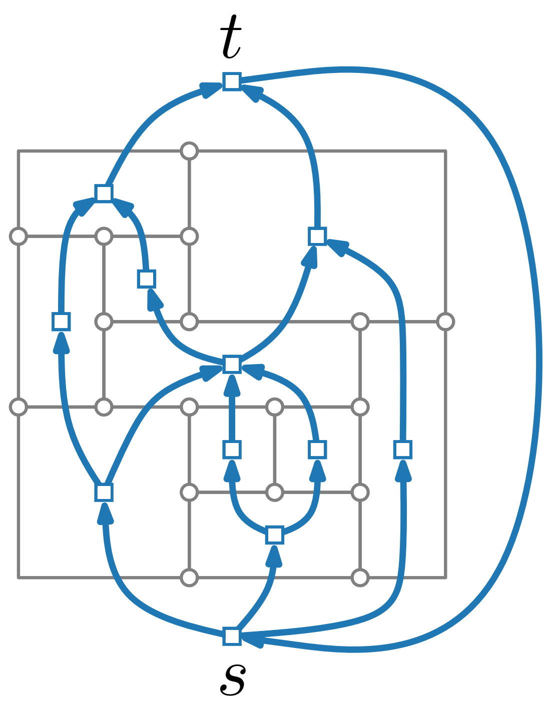

Flow Network for Edge Length Assignment

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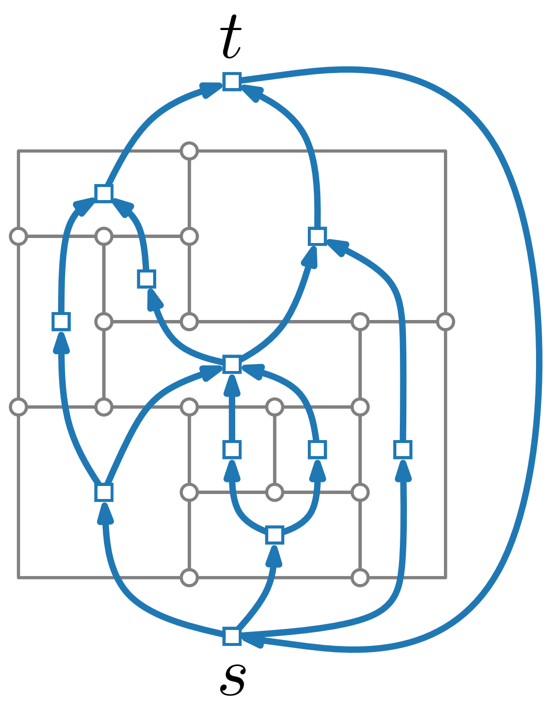
- $E_{\text{hor}} = \{(f, g) \mid f, g$ share a *horizontal* segment and f lies *below* $g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \quad \forall a \in E_{\text{hor}}$
- $u(a) = \infty \quad \forall a \in E_{\text{hor}}$
- $\text{cost}(a) = 1 \quad \forall a \in E_{\text{hor}}$
- $b(f) = 0 \quad \forall f \in W_{\text{hor}}$

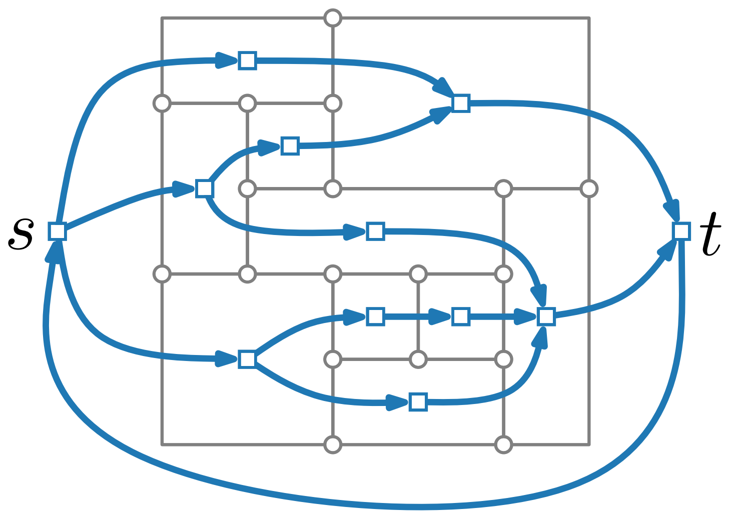


Flow Network for Edge Length Assignment

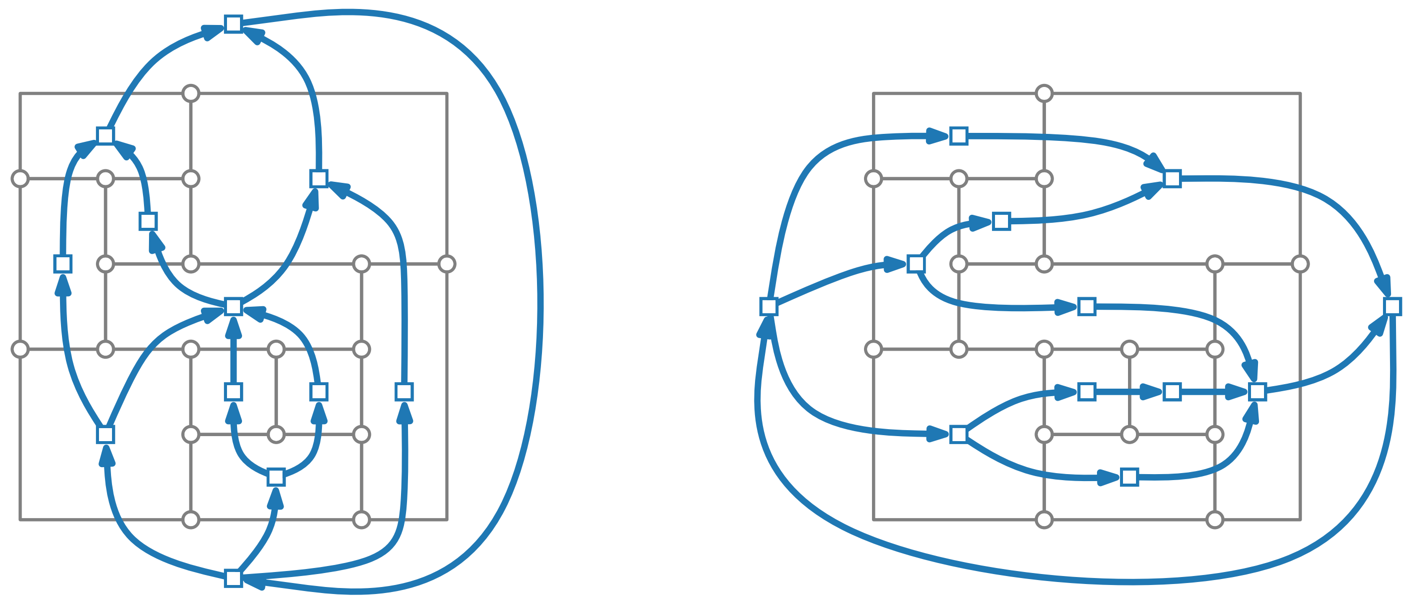
Definition.

Flow Network $N_{\text{ver}} = ((W_{\text{ver}}, E_{\text{ver}}); b; \ell; u; \text{cost})$

- $W_{\text{ver}} = F \setminus \{f_0\} \cup \{s, t\}$ □
- $E_{\text{ver}} = \{(f, g) \mid f, g$ share a *vertical* segment and f lies to the *left* of $g\} \cup \{(t, s)\}$
- $\ell(a) = 1 \quad \forall a \in E_{\text{ver}}$
- $u(a) = \infty \quad \forall a \in E_{\text{ver}}$
- $\text{cost}(a) = 1 \quad \forall a \in E_{\text{ver}}$
- $b(f) = 0 \quad \forall f \in W_{\text{ver}}$

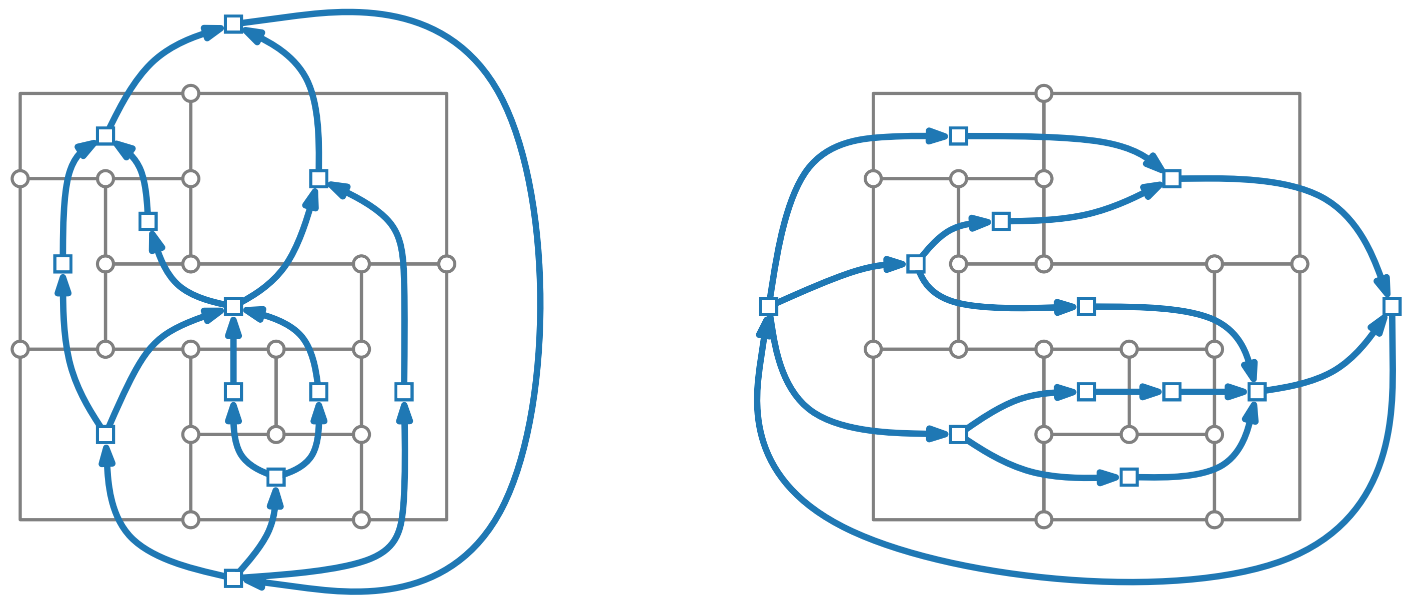


Compaction – Result



Theorem.
Valid min-cost-flows for N_{hor} and N_{ver} exists iff corresponding edge lenghts induce orthogonal drawing.

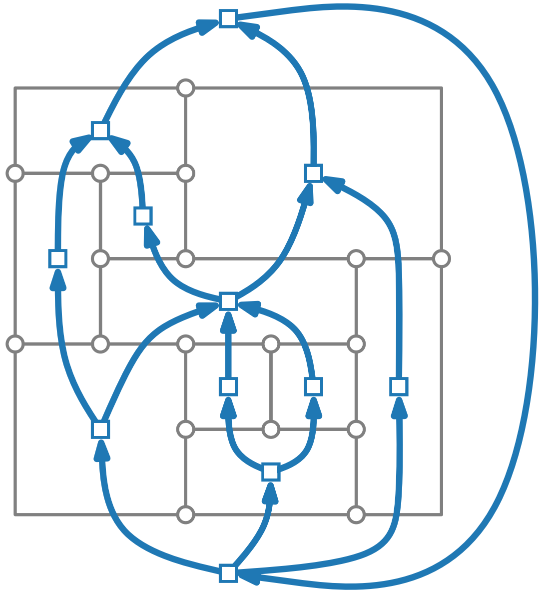
Compaction – Result

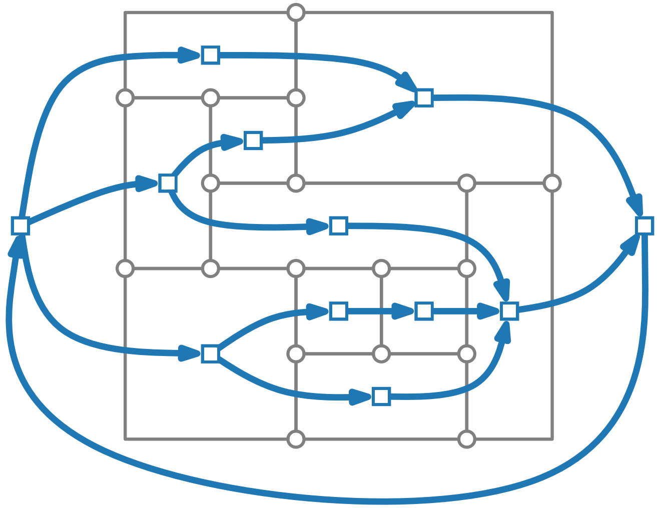


Theorem.
Valid min-cost-flows for N_{hor} and N_{ver} exists iff corresponding edge lenghts induce orthogonal drawing.

What values of the drawing represent the following?

Compaction – Result



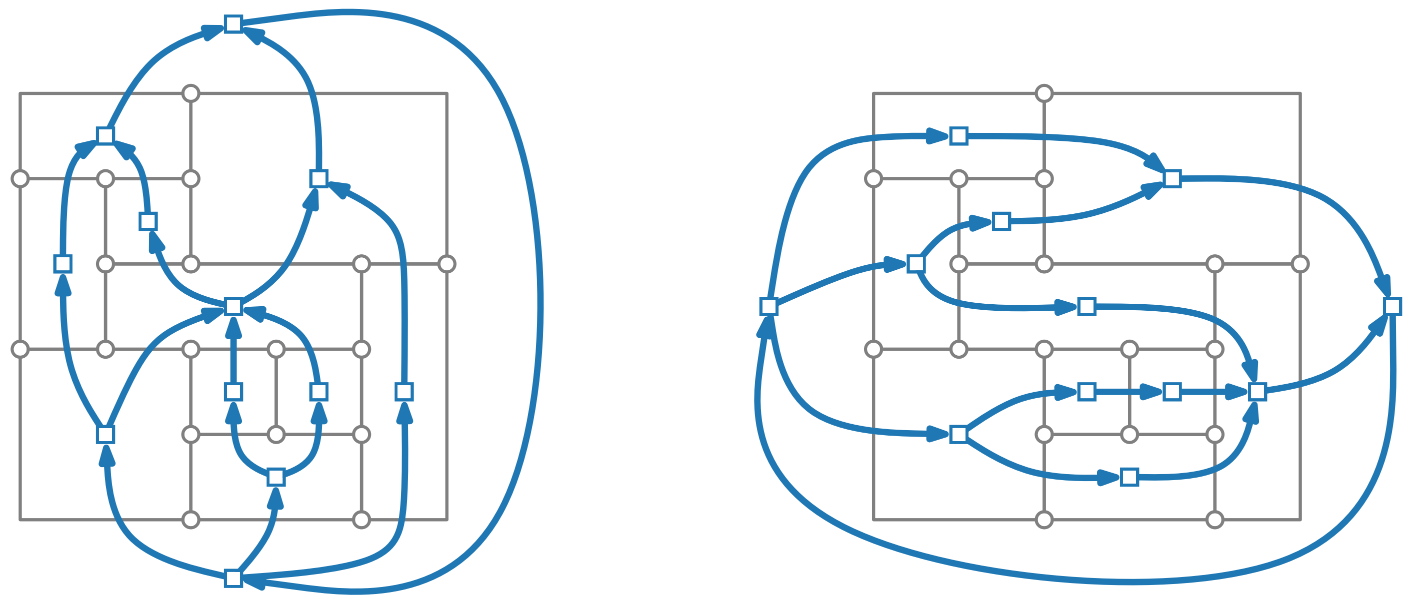


Theorem.
Valid min-cost-flows for N_{hor} and N_{ver} exists iff corresponding edge lenghts induce orthogonal drawing.

What values of the drawing represent the following?

- $|X_{\text{hor}}(t,s)|$ and $|X_{\text{ver}}(t,s)|$?

Compaction – Result

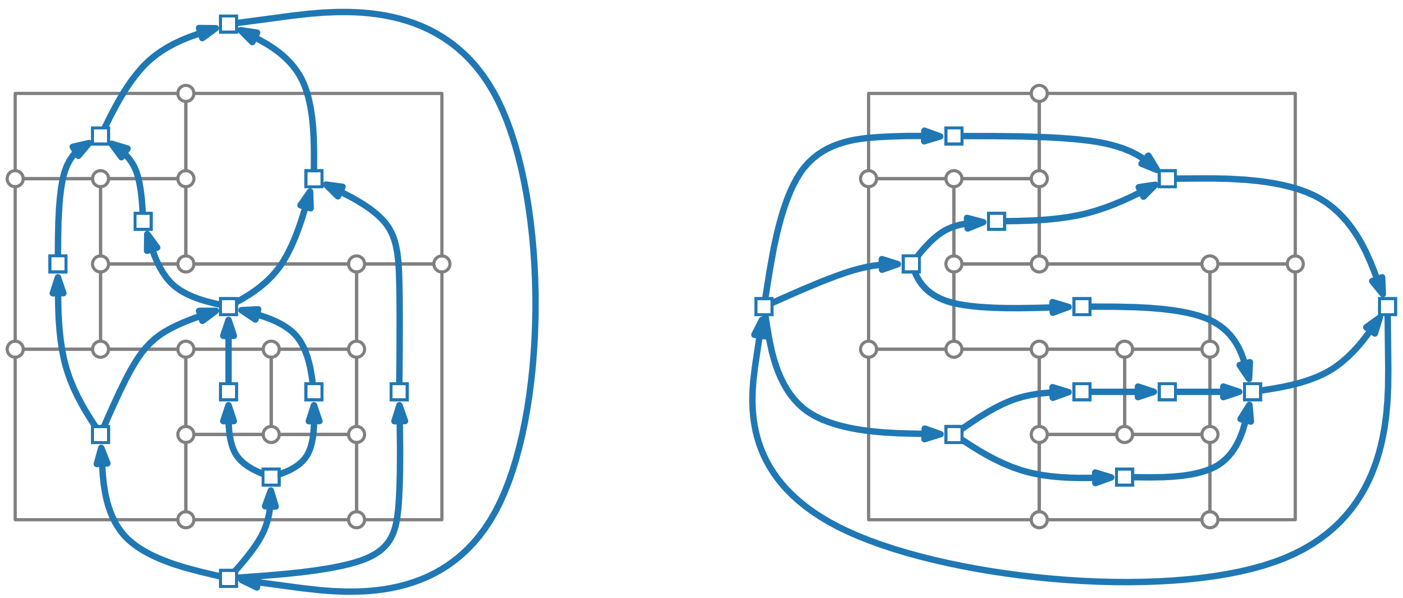


Theorem.
Valid min-cost-flows for N_{hor} and N_{ver} exists iff corresponding edge lenghts induce orthogonal drawing.

What values of the drawing represent the following?

- $|X_{\text{hor}}(t,s)|$ and $|X_{\text{ver}}(t,s)|$? width and height of drawing

Compaction – Result

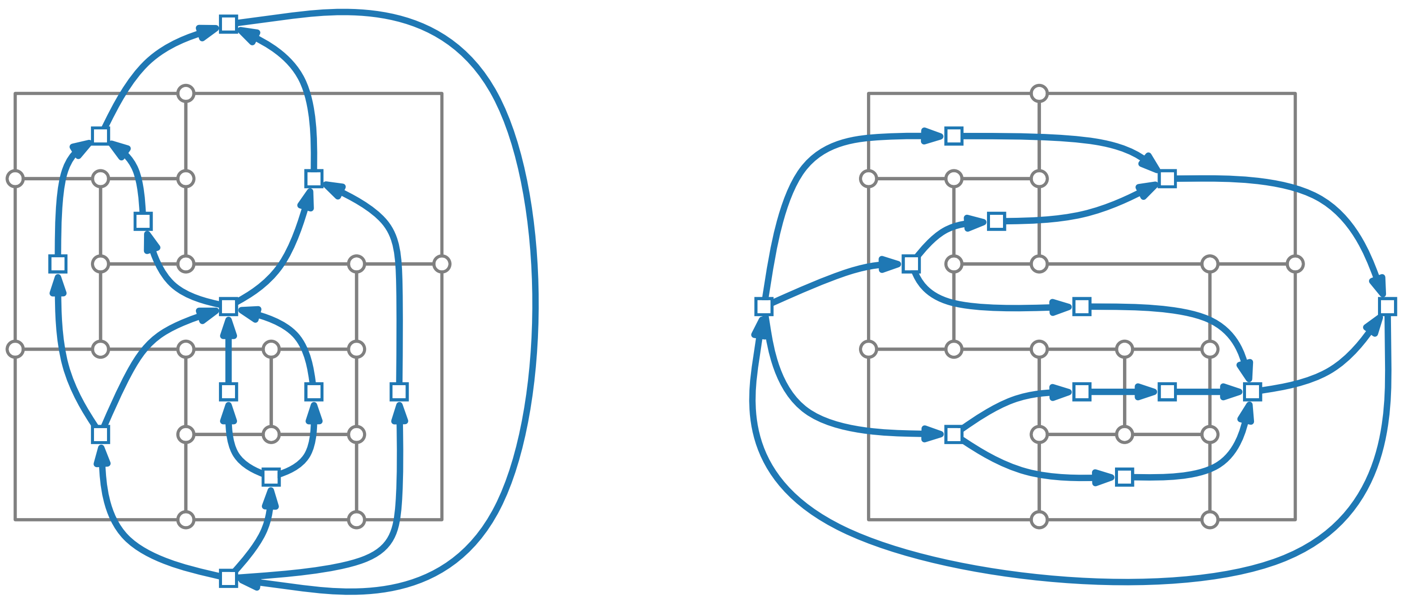


Theorem.
Valid min-cost-flows for N_{hor} and N_{ver} exists iff corresponding edge lenghts induce orthogonal drawing.

What values of the drawing represent the following?

- $|X_{\text{hor}}(t, s)|$ and $|X_{\text{ver}}(t, s)|$? width and height of drawing
- $\sum_{e \in E_{\text{hor}}} X_{\text{hor}}(e) + \sum_{e \in E_{\text{ver}}} X_{\text{ver}}(e)$

Compaction – Result

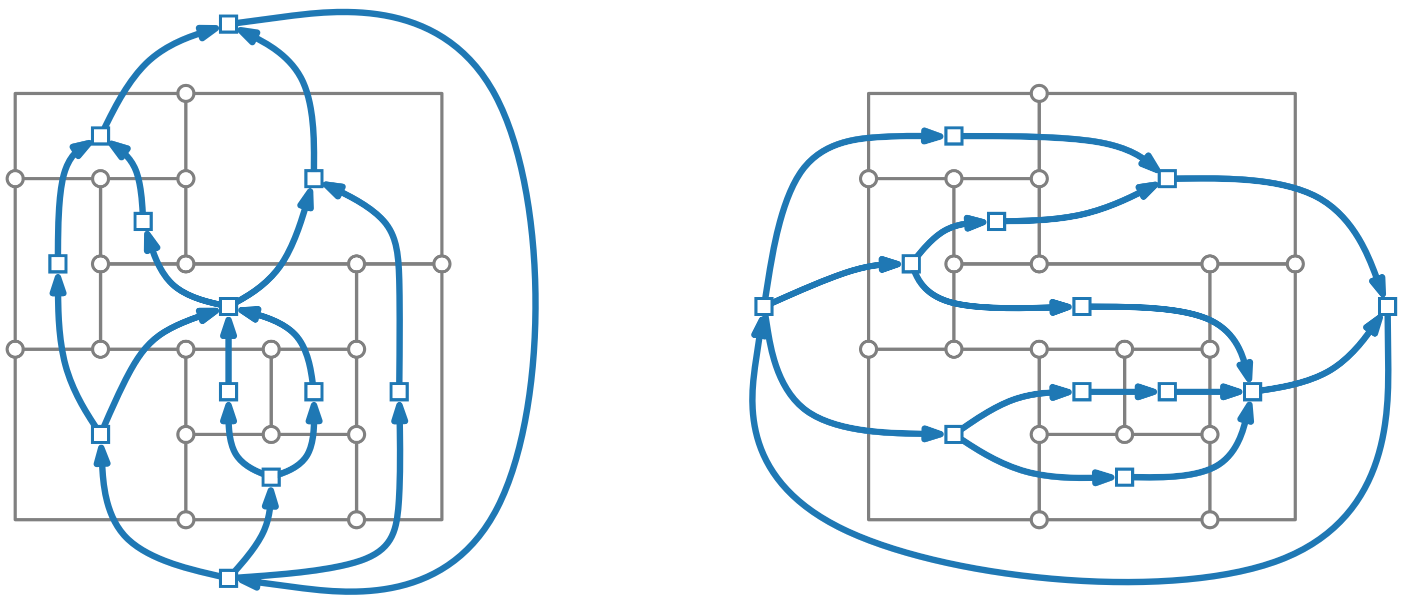


Theorem.
Valid min-cost-flows for N_{hor} and N_{ver} exists iff corresponding edge lenghts induce orthogonal drawing.

What values of the drawing represent the following?

- $|X_{\text{hor}}(t, s)|$ and $|X_{\text{ver}}(t, s)|$? width and height of drawing
- $\sum_{e \in E_{\text{hor}}} X_{\text{hor}}(e) + \sum_{e \in E_{\text{ver}}} X_{\text{ver}}(e)$ total edge length

Compaction – Result



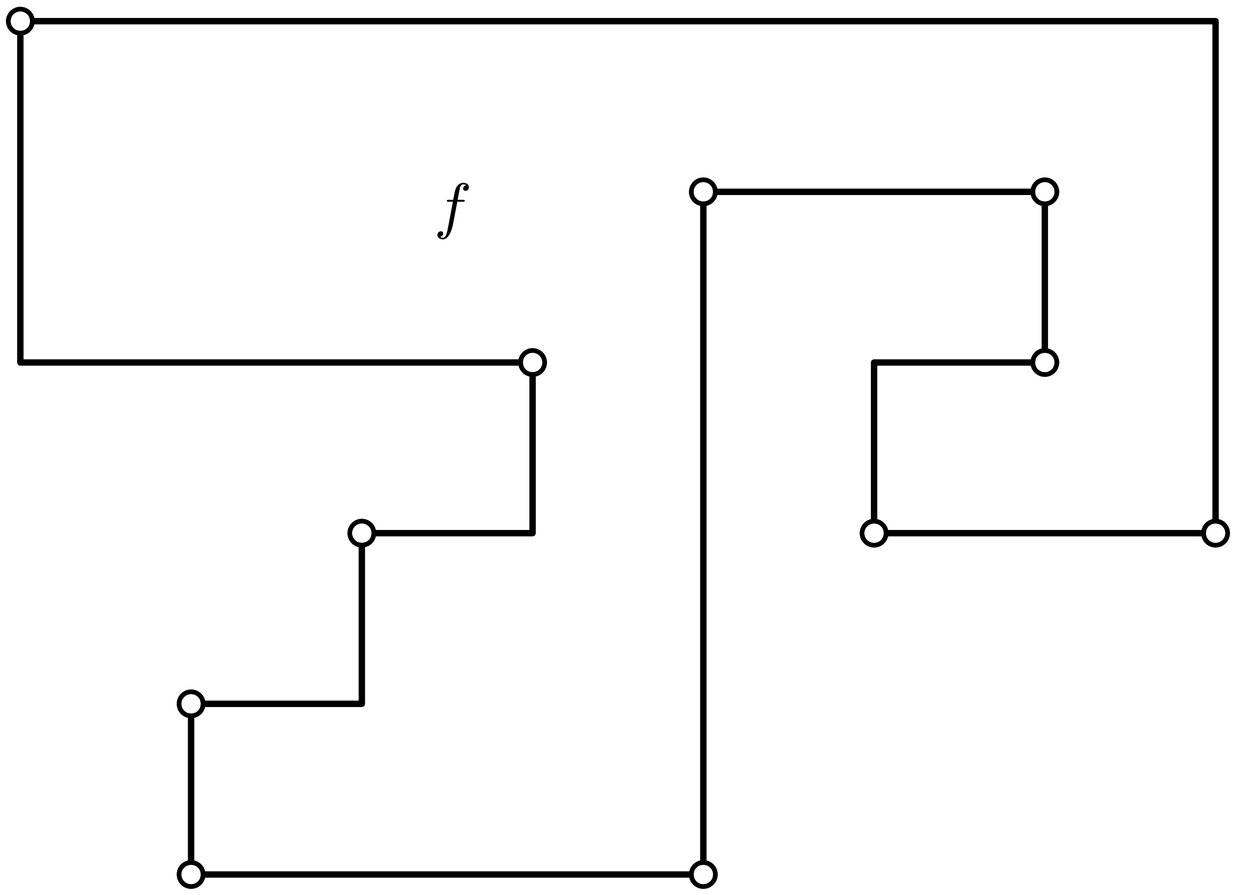
What if not all faces rectangular?

Theorem.
Valid min-cost-flows for N_{hor} and N_{ver} exists iff corresponding edge lenghts induce orthogonal drawing.

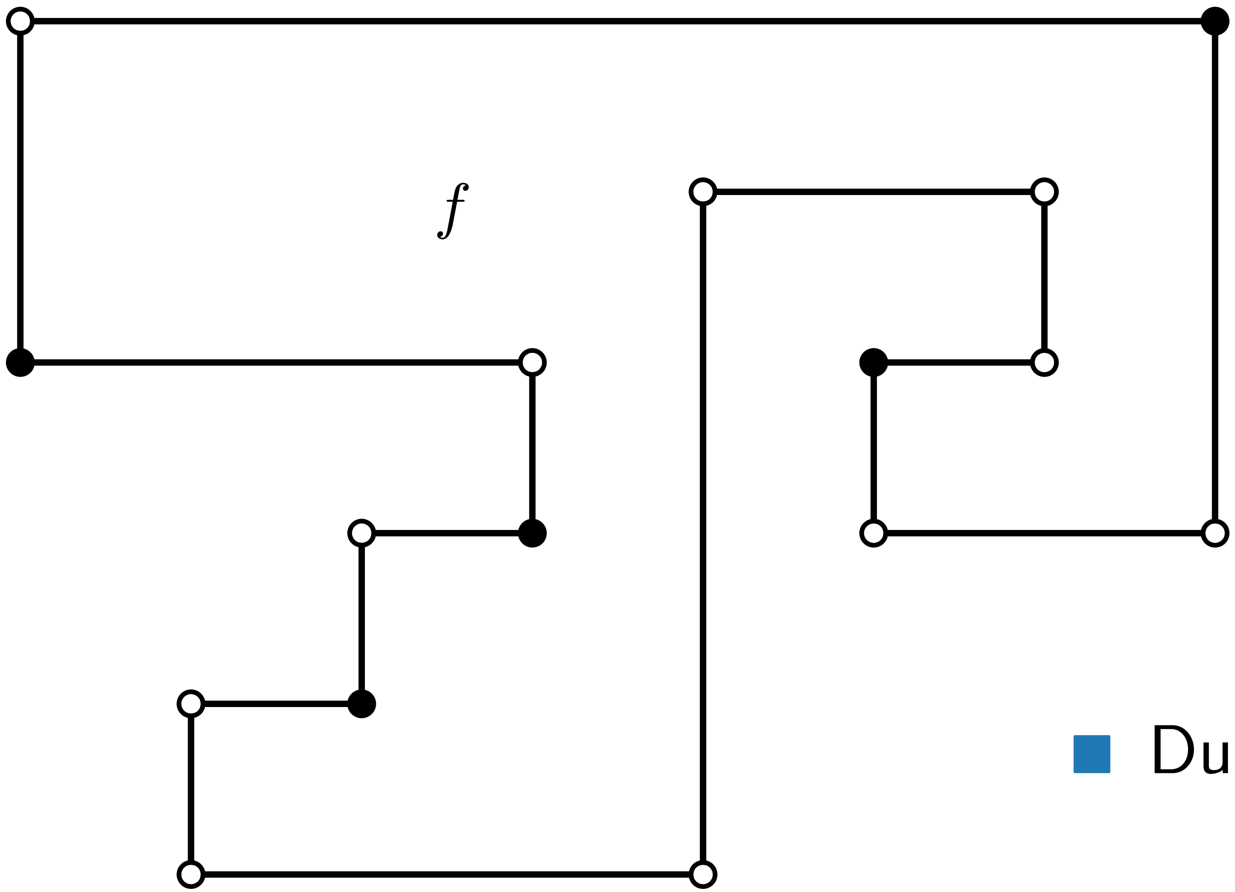
What values of the drawing represent the following?

- $|X_{\text{hor}}(t,s)|$ and $|X_{\text{ver}}(t,s)|$? width and height of drawing
- $\sum_{e \in E_{\text{hor}}} X_{\text{hor}}(e) + \sum_{e \in E_{\text{ver}}} X_{\text{ver}}(e)$ total edge length

Refinement of (G, H) – Inner Face

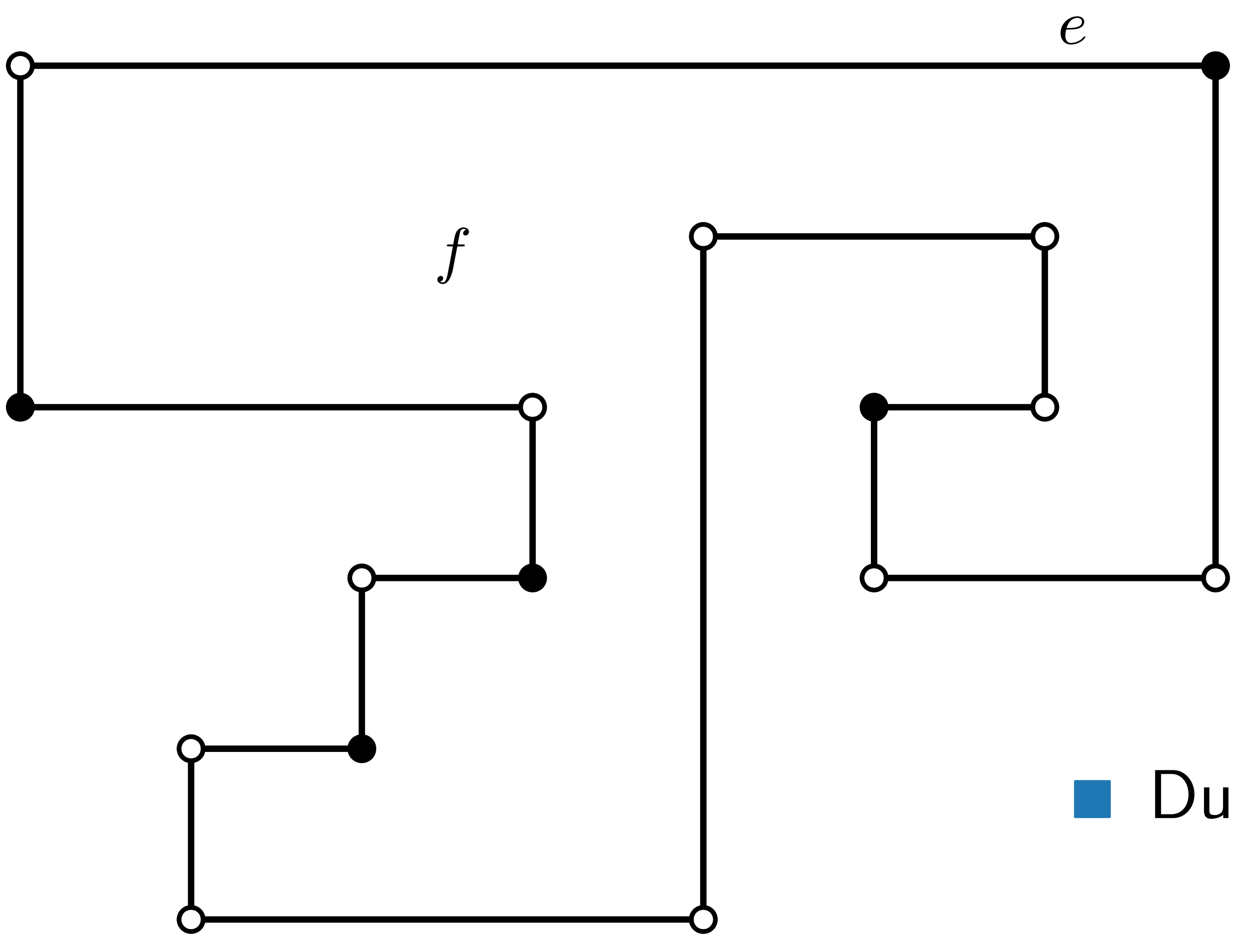


Refinement of (G, H) – Inner Face



- Dummy vertices for bends

Refinement of (G, H) – Inner Face



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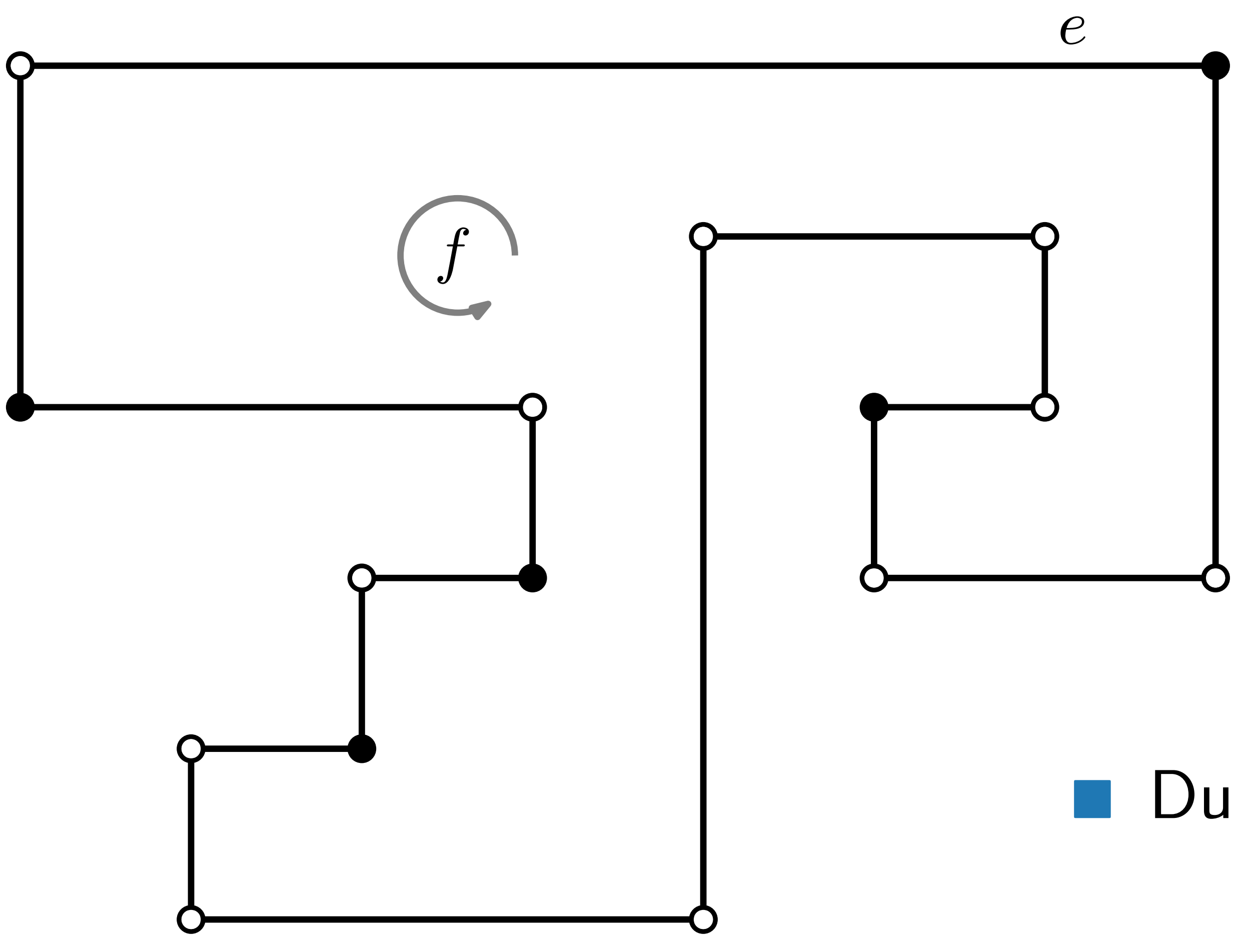

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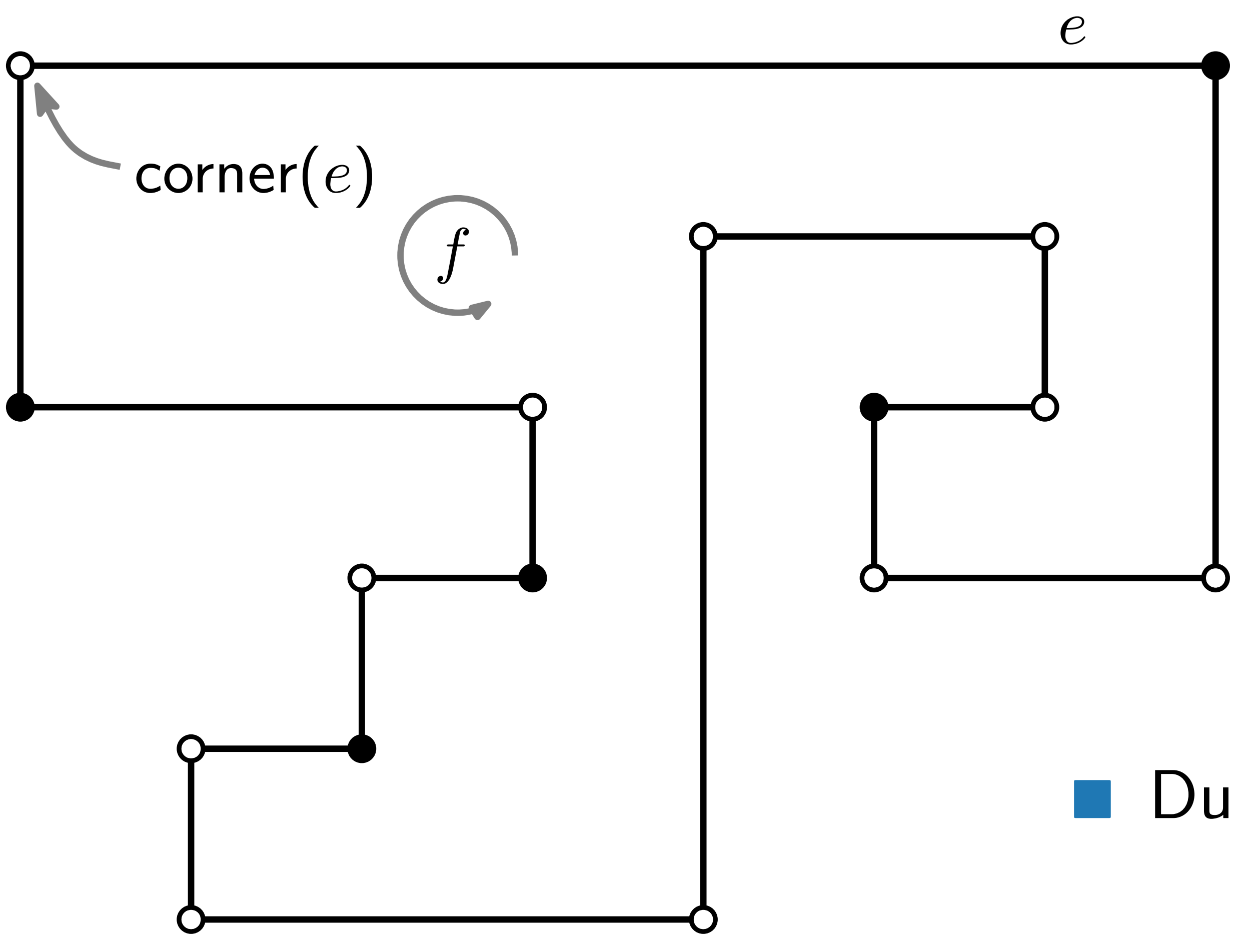

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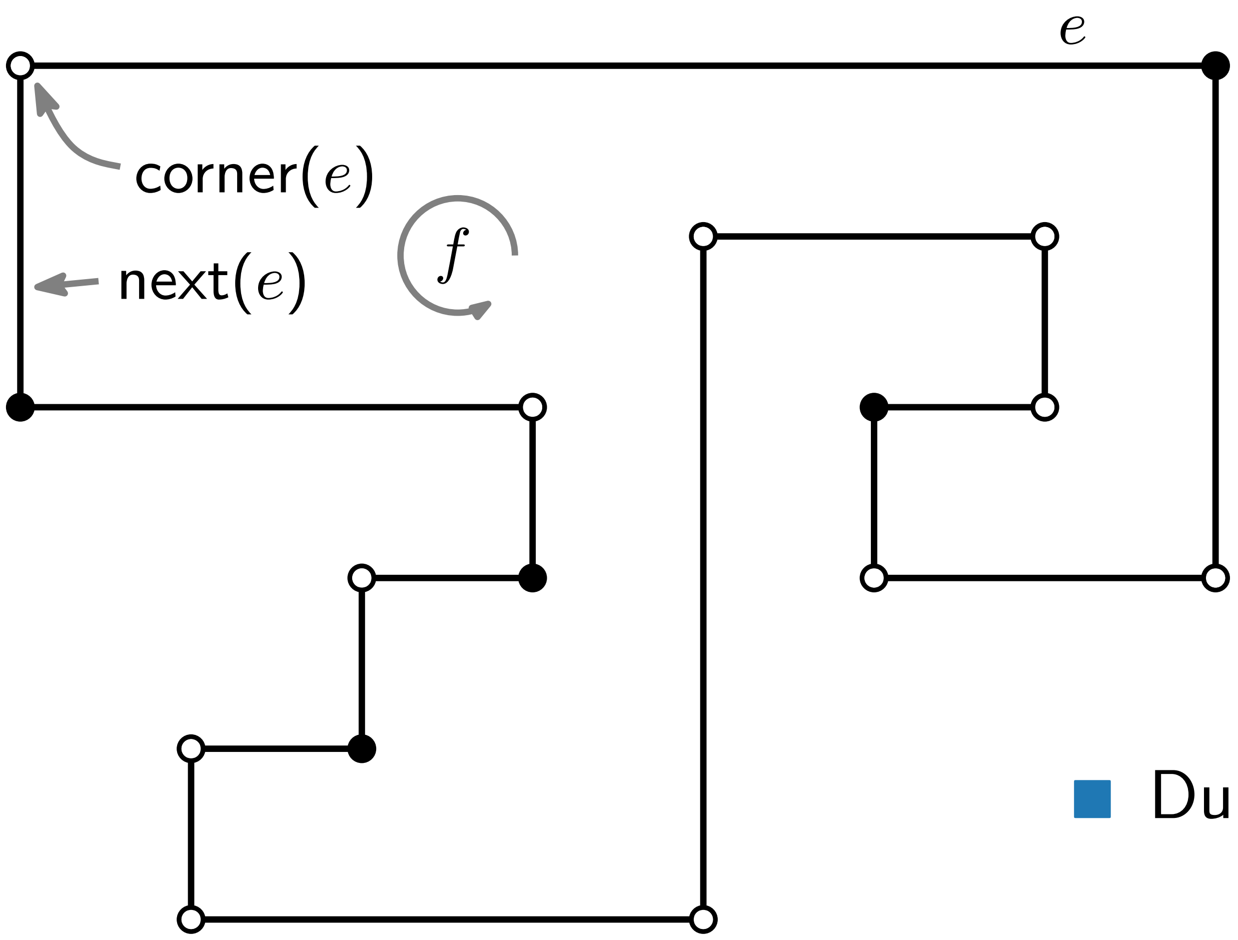

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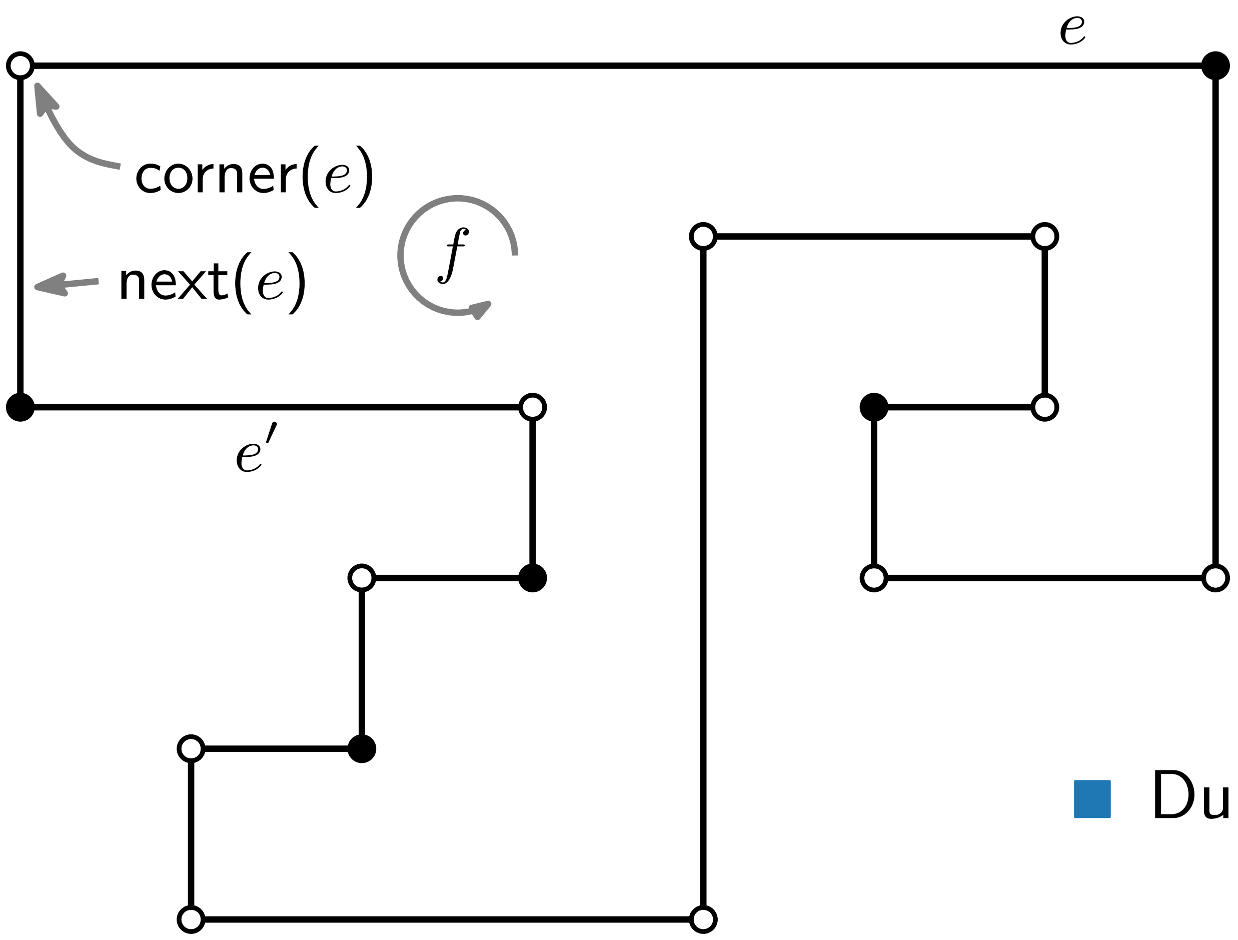

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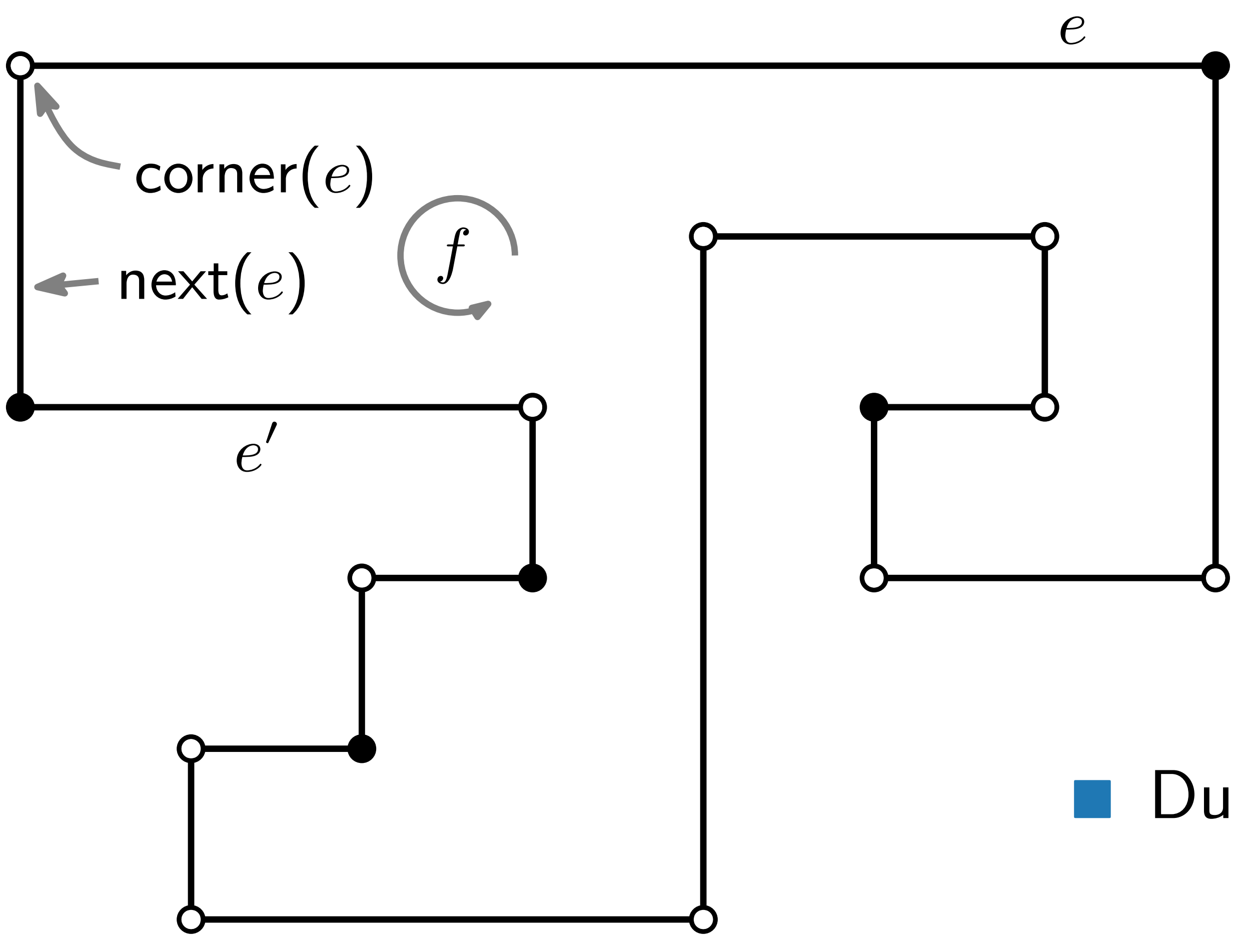
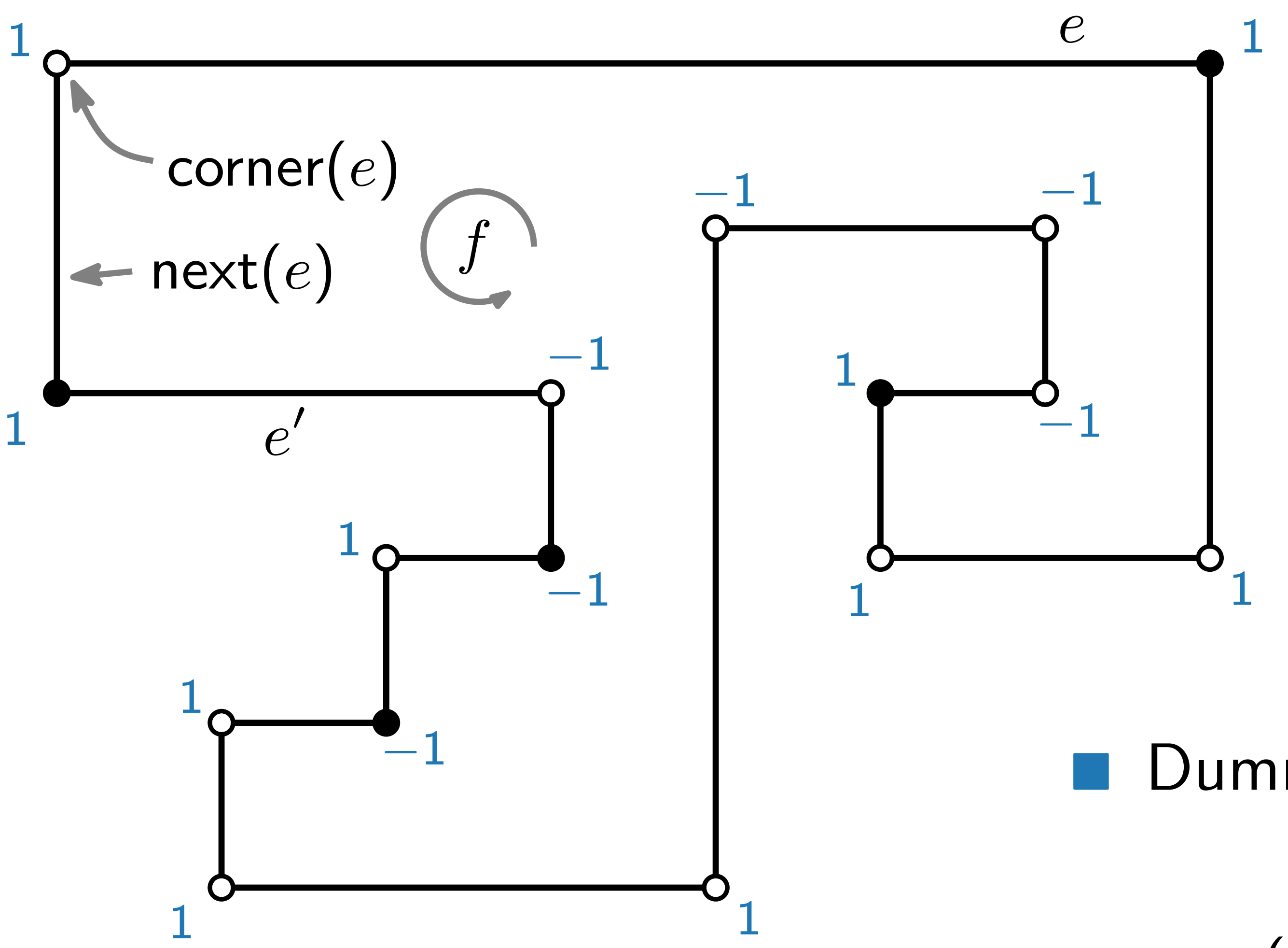


- Dummy vertices for bends
- turn(e) $= \begin{cases} 1 & \text{left turn} \\ 0 & \text{no turn} \\ -1 & \text{right turn} \end{cases}$

Refinement of (G, H) – Inner Face



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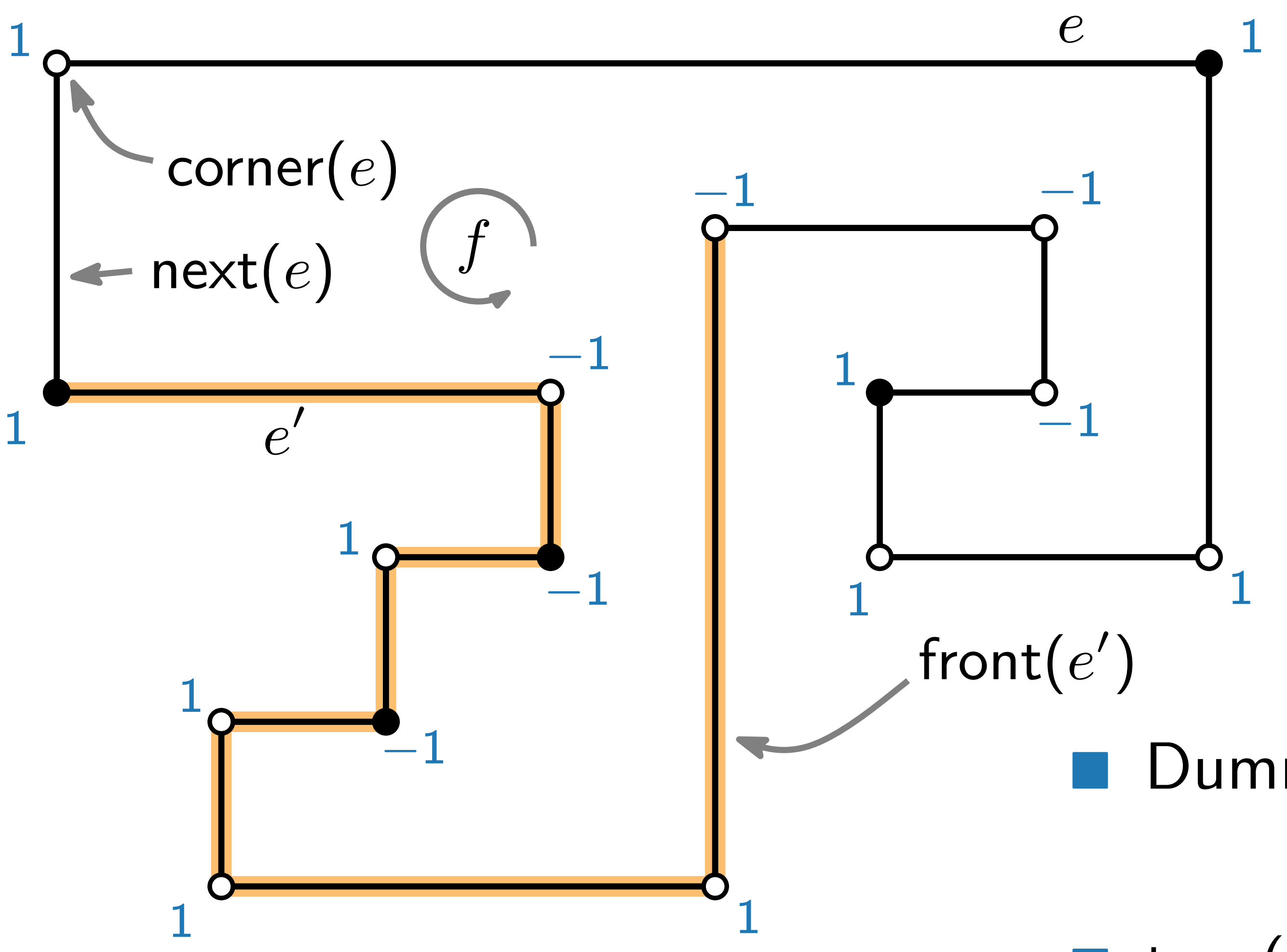

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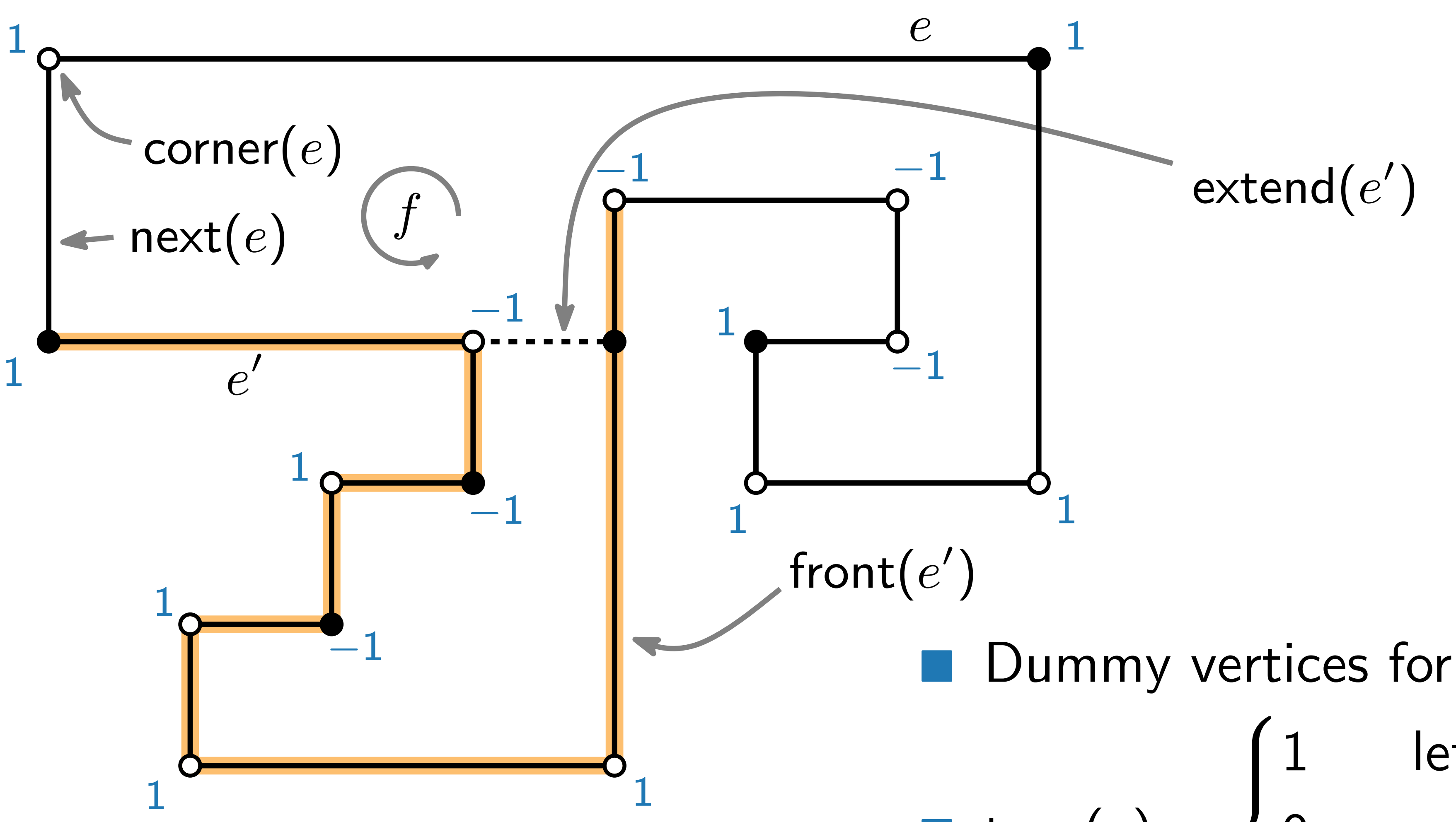

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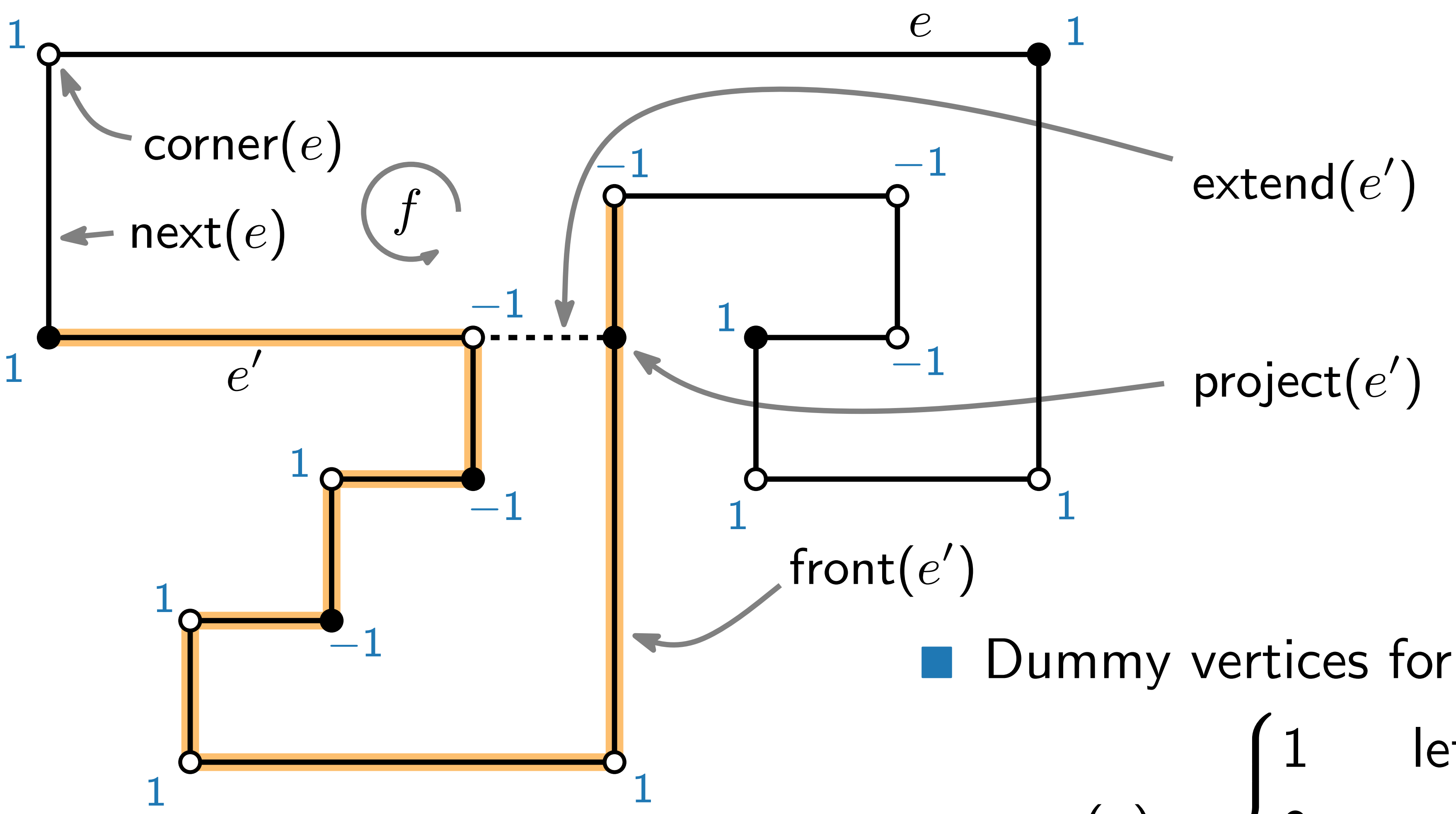

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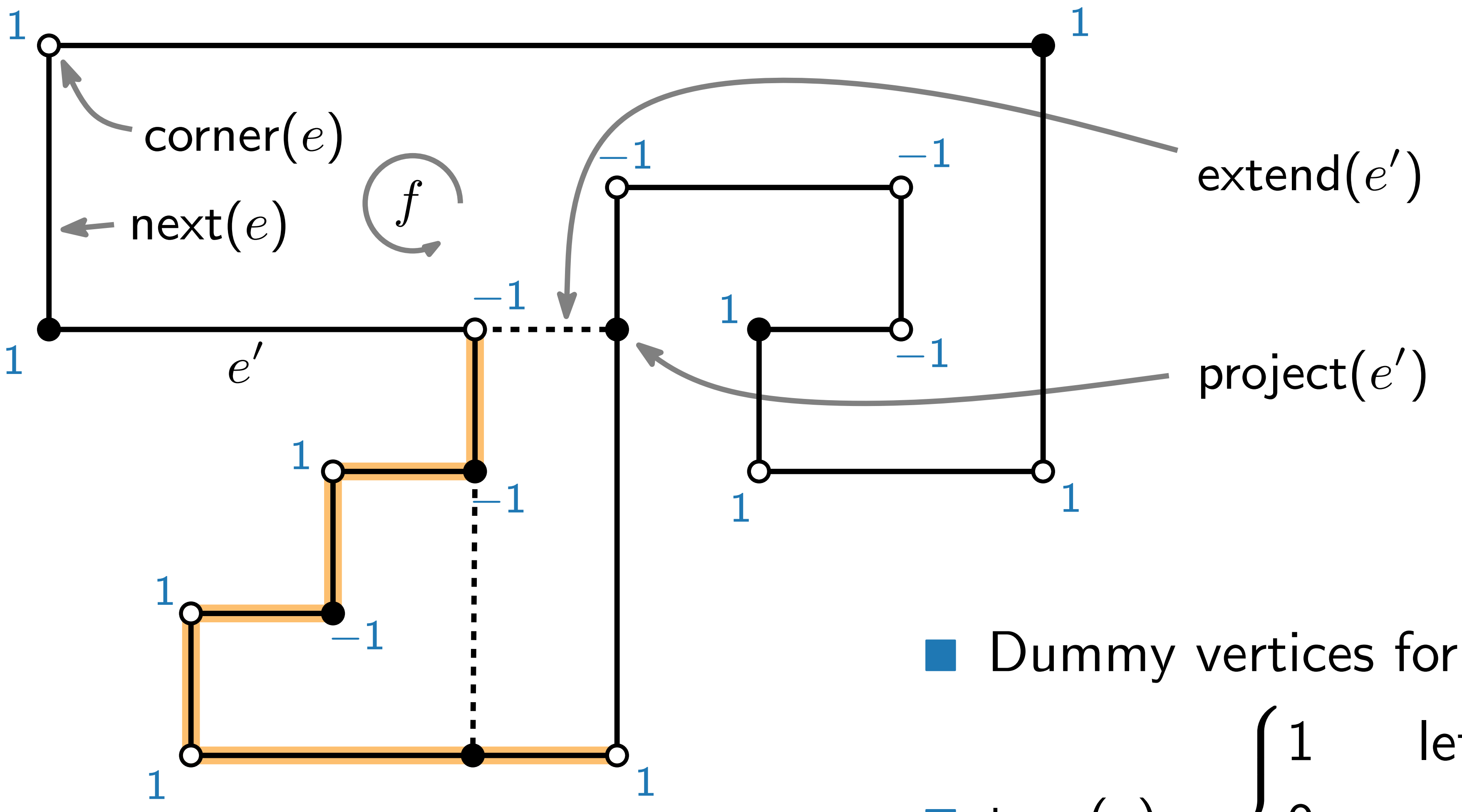

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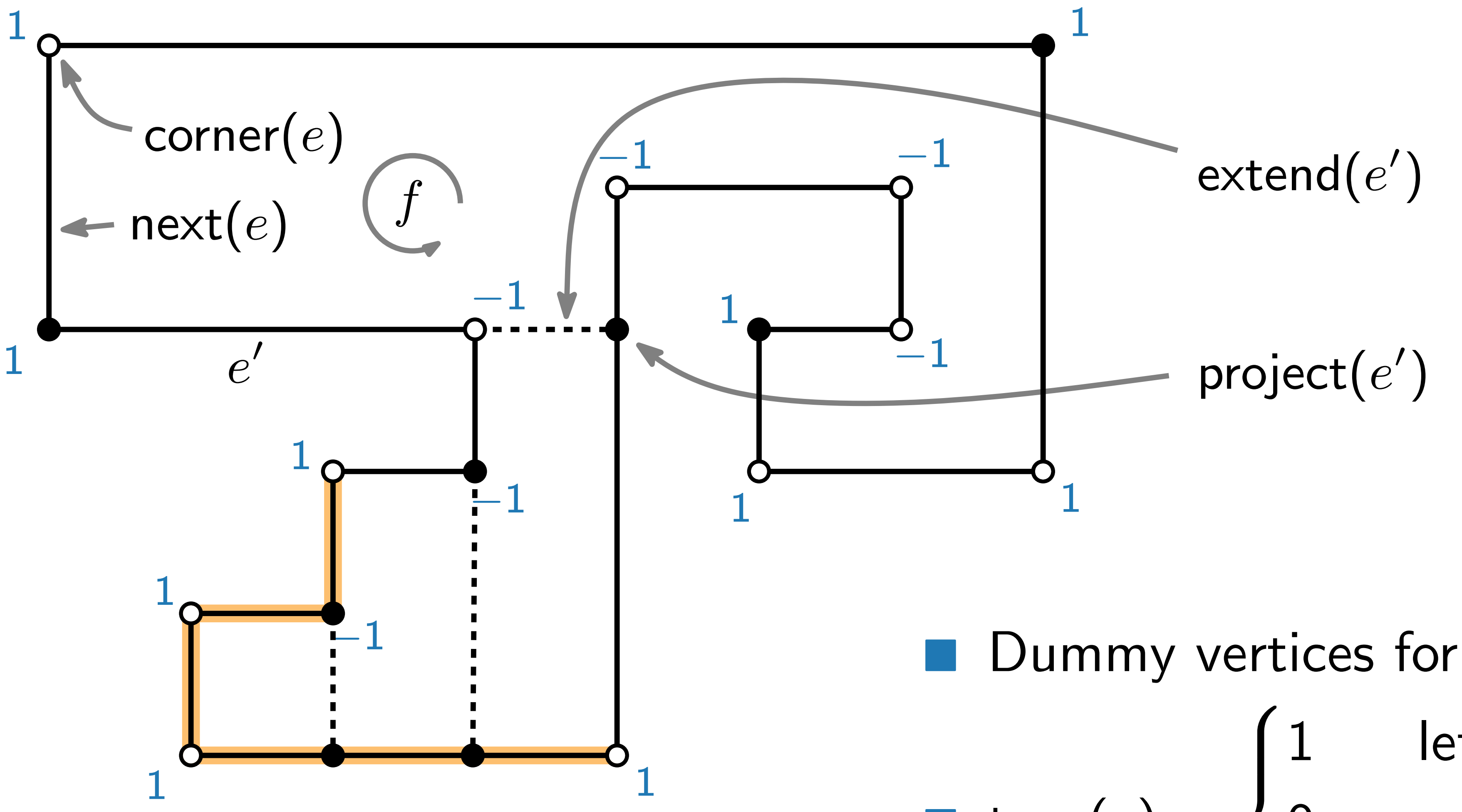

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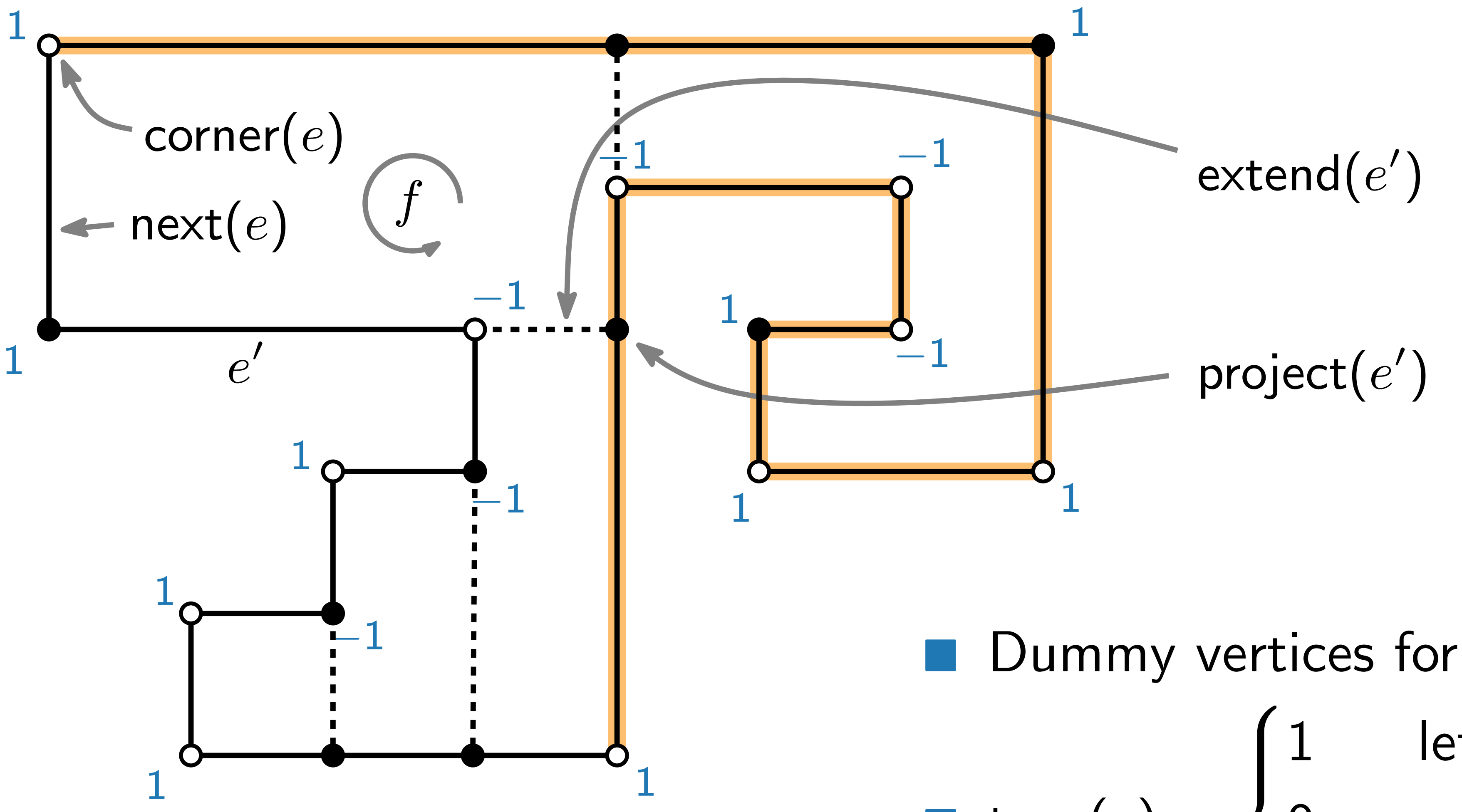

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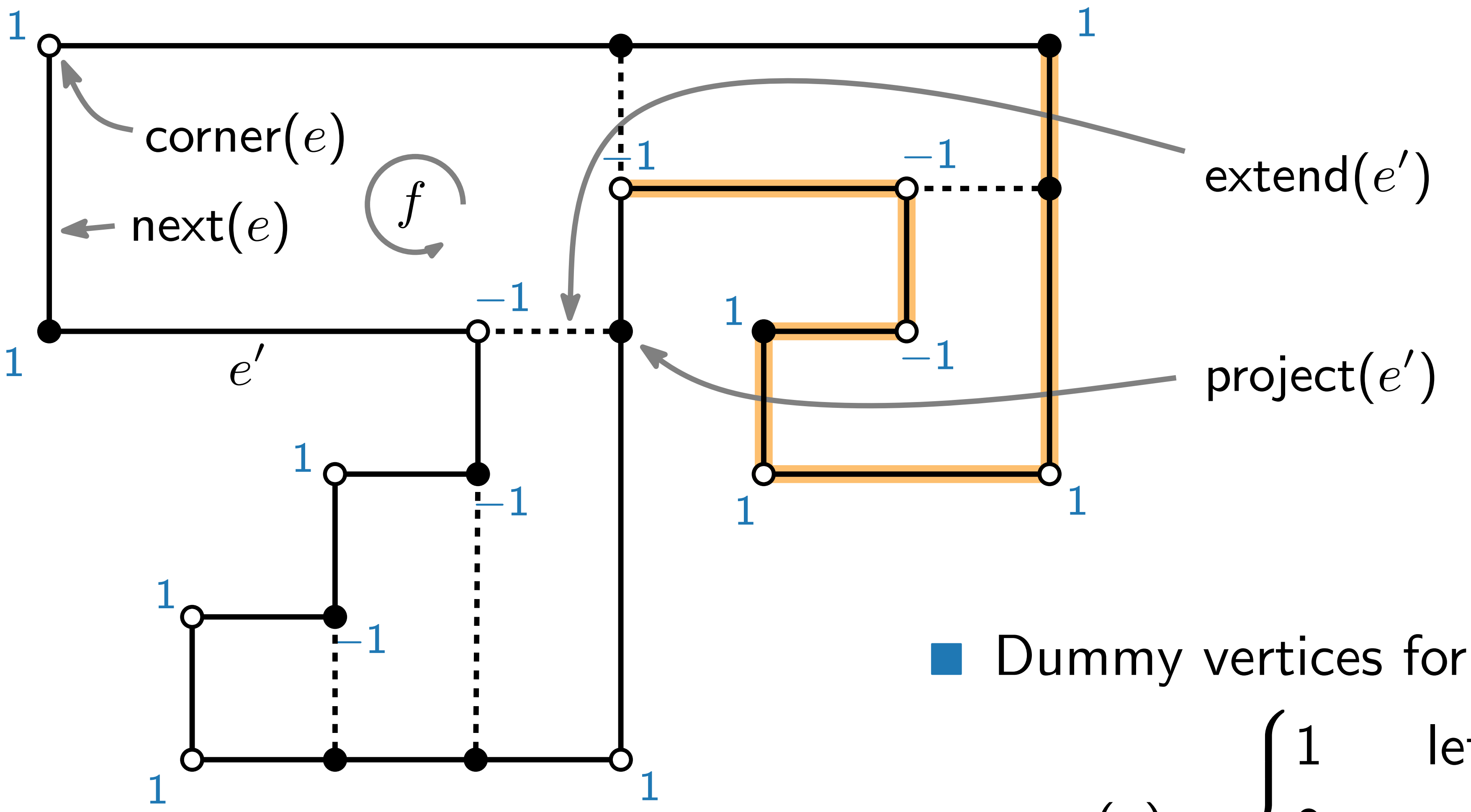

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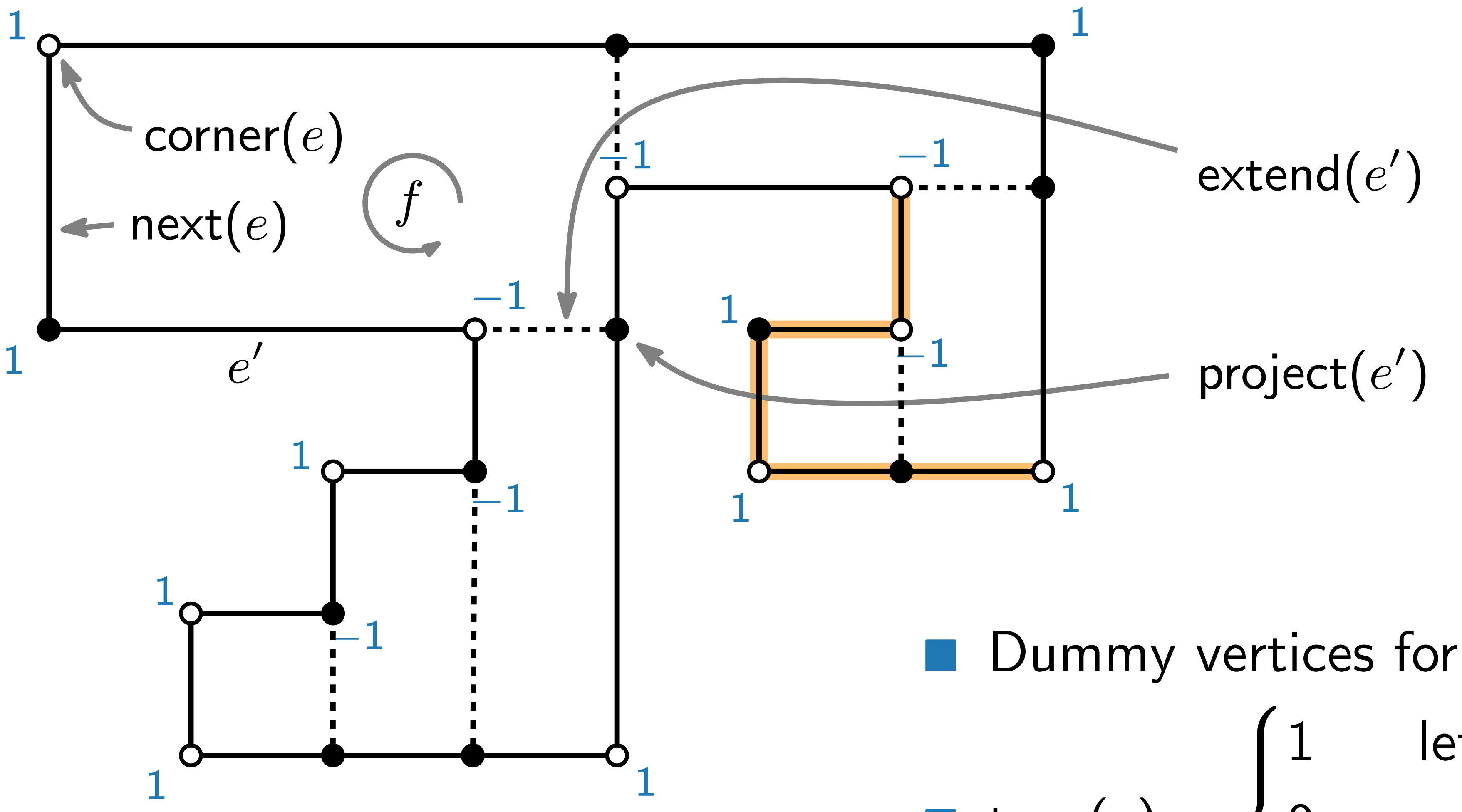

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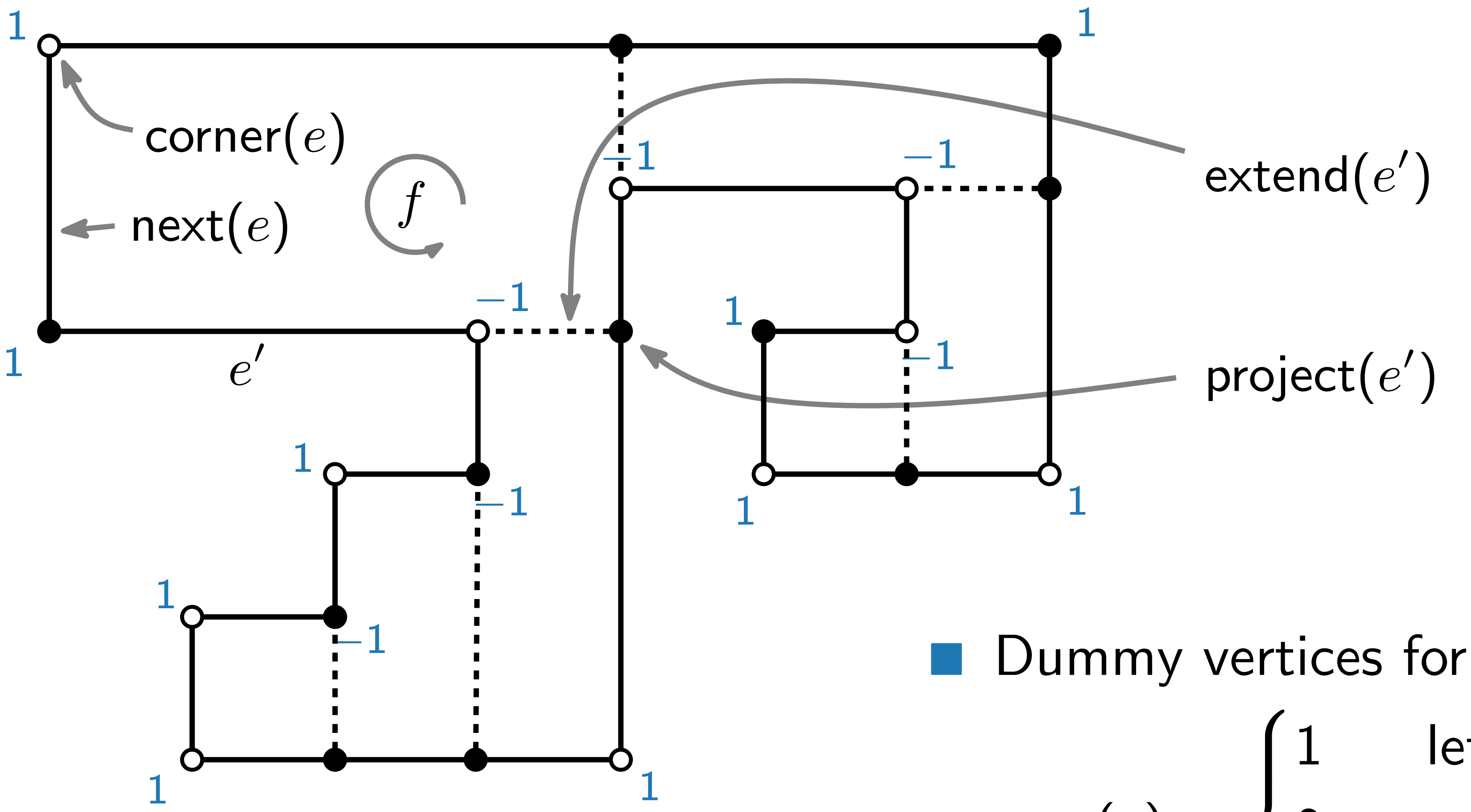

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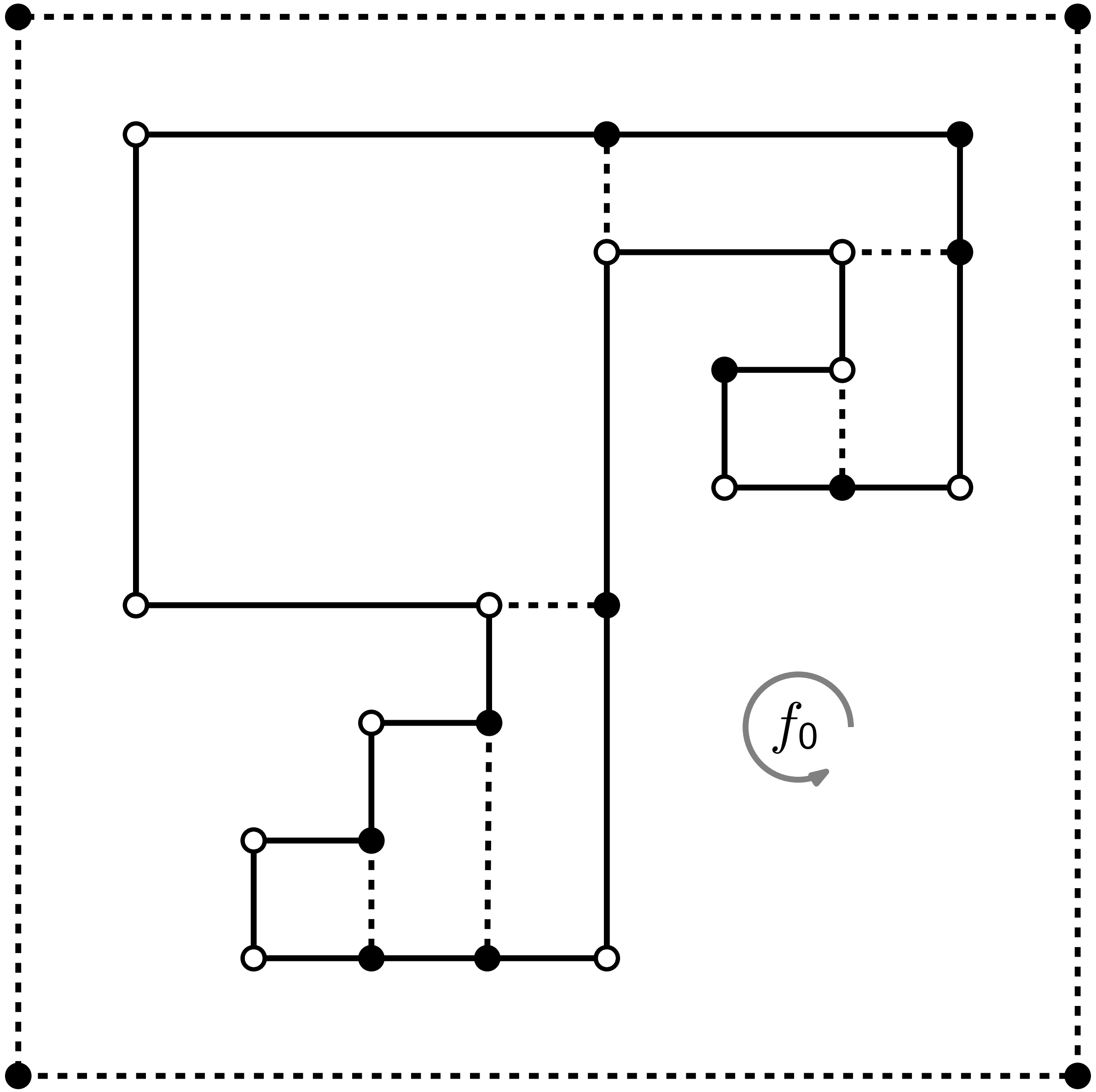
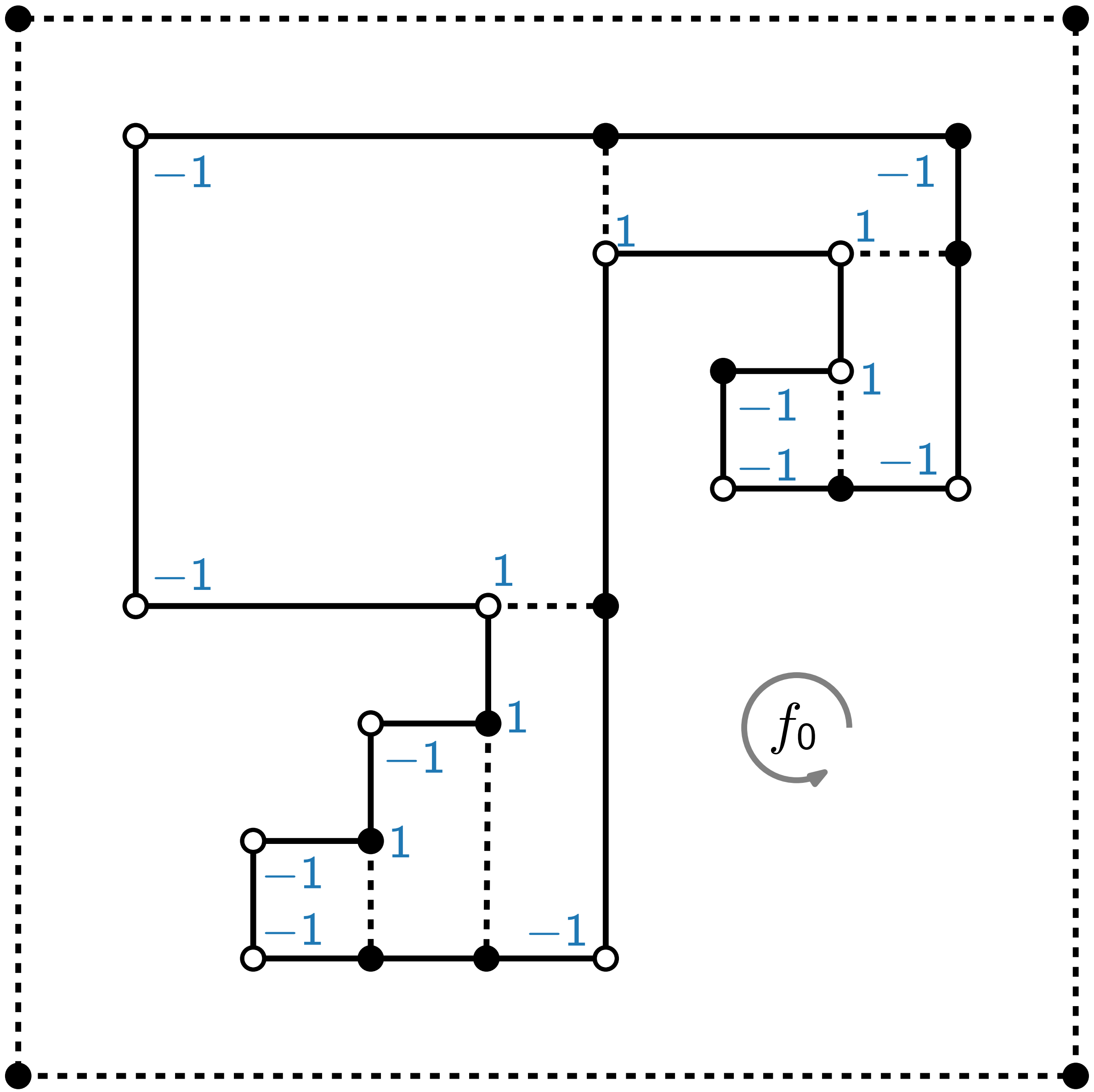
Refinement of (G, H) – Outer Face



Refinement of (G, H) – Outer Face



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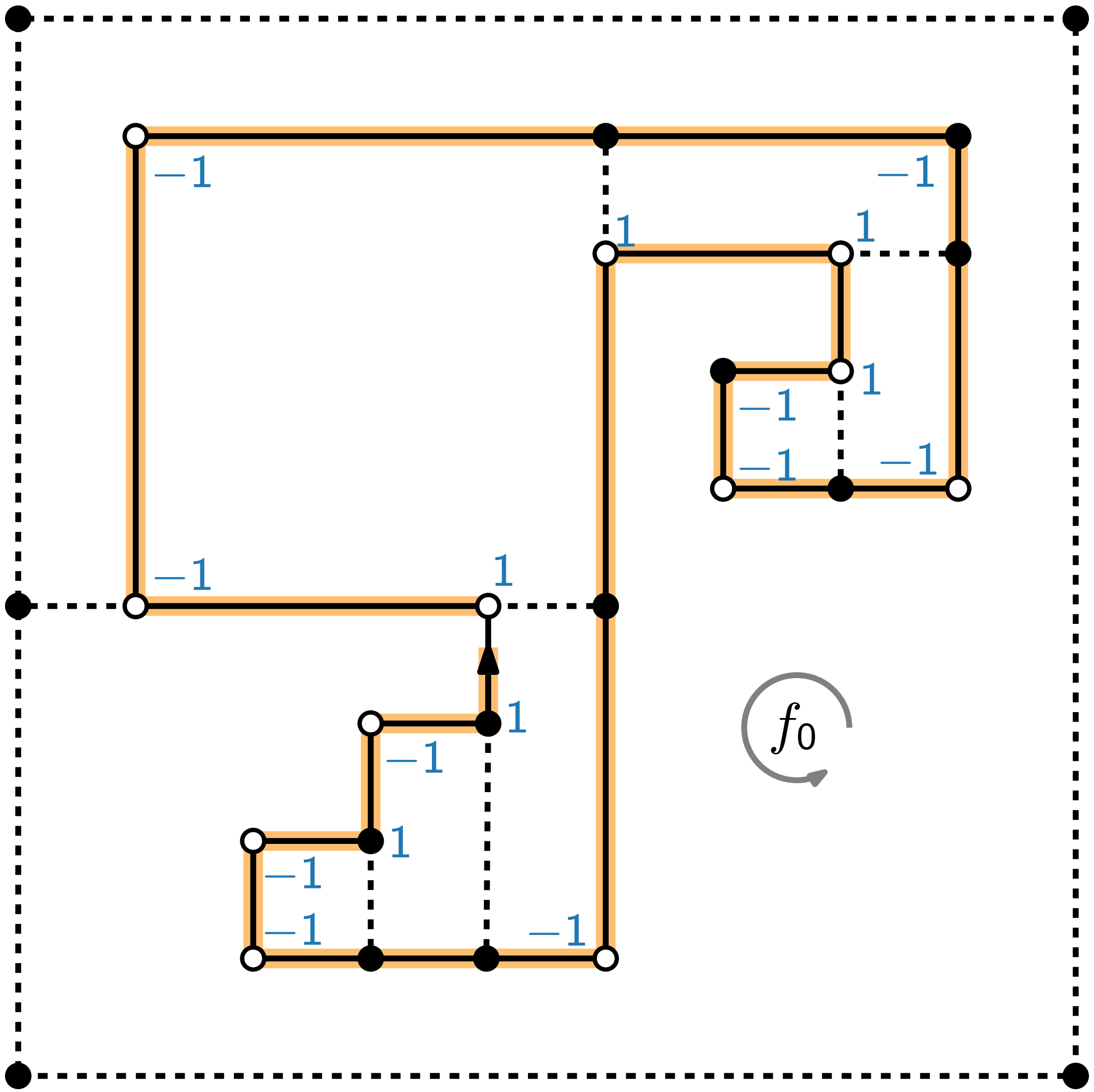

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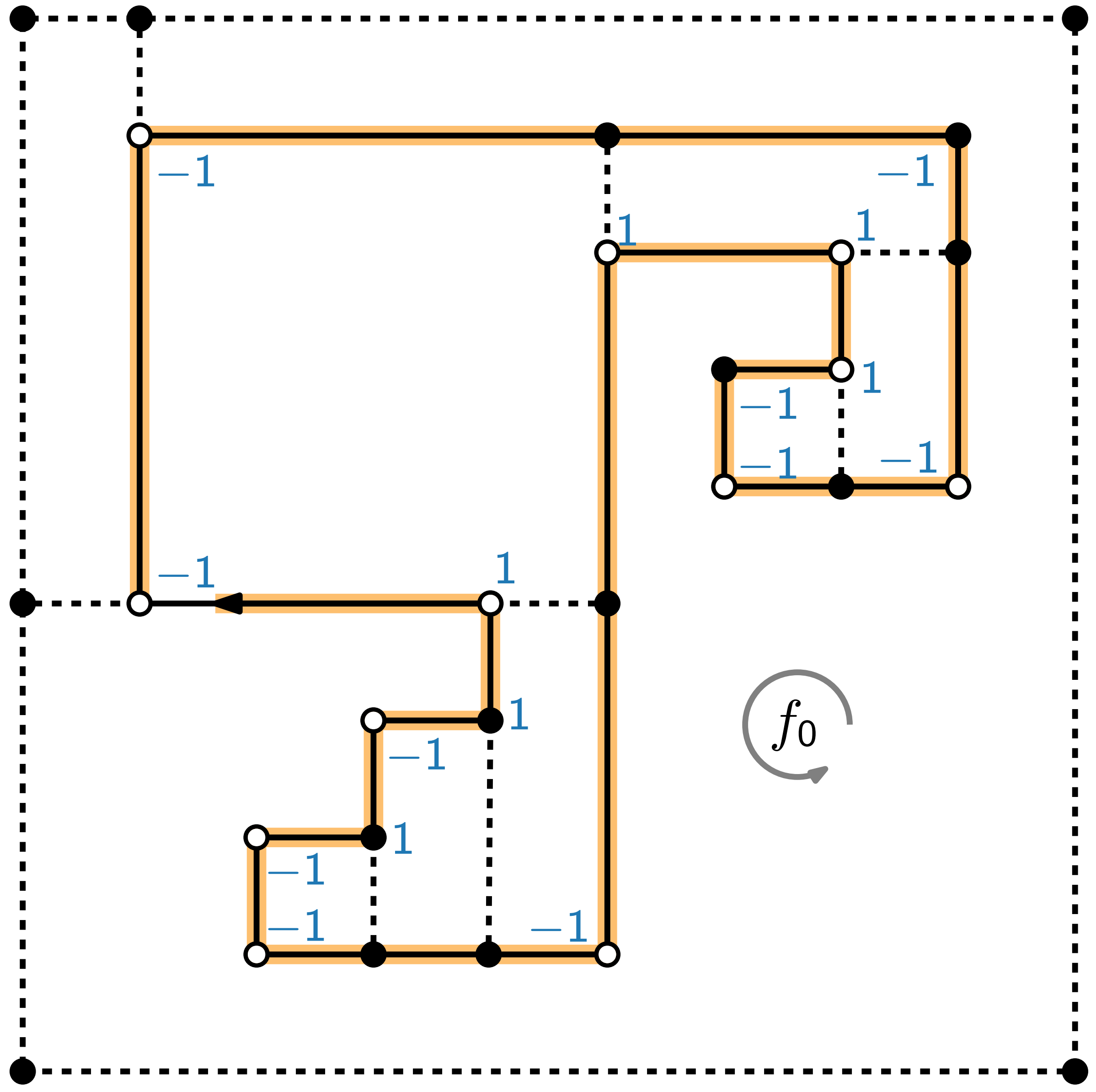

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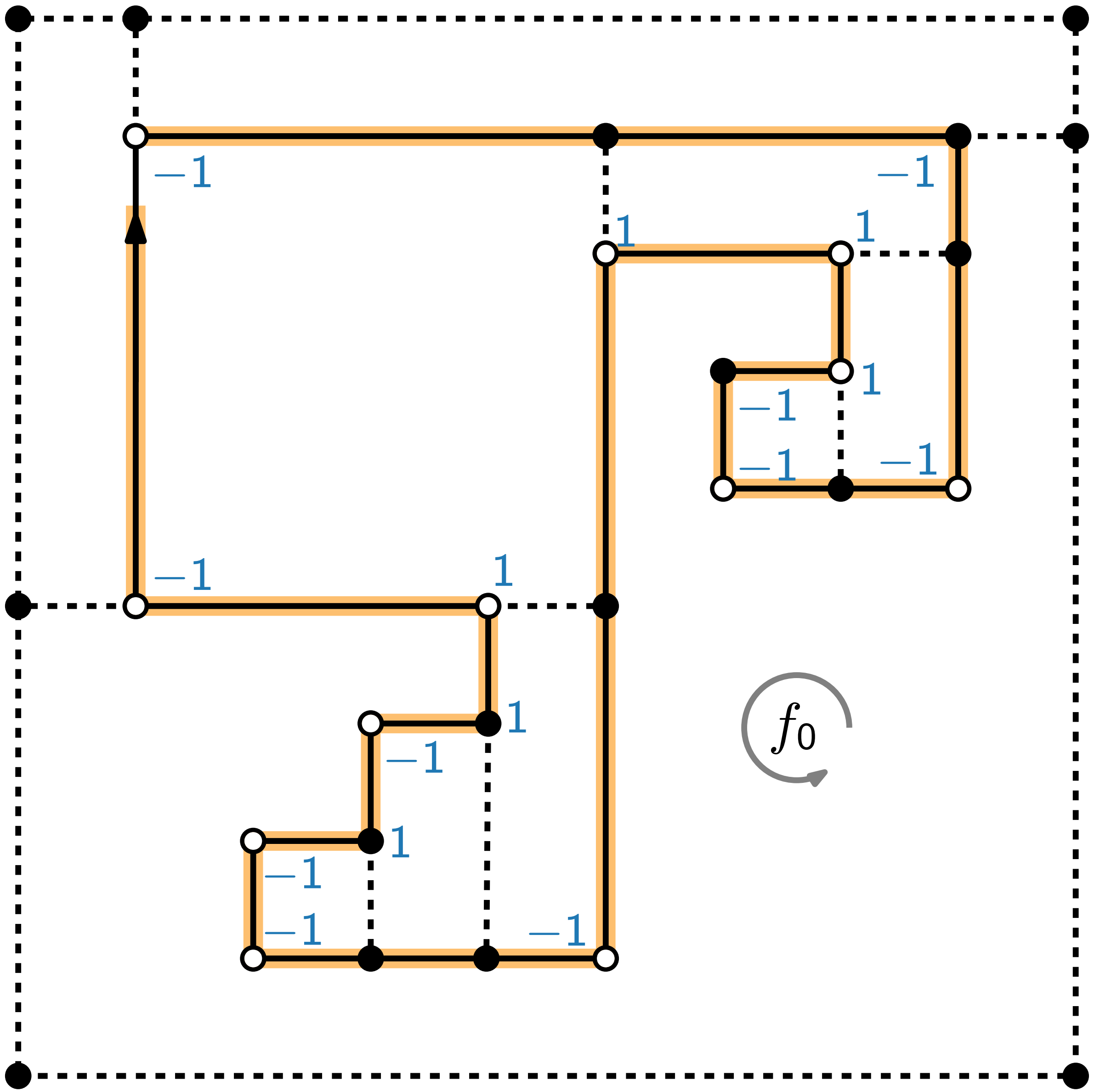
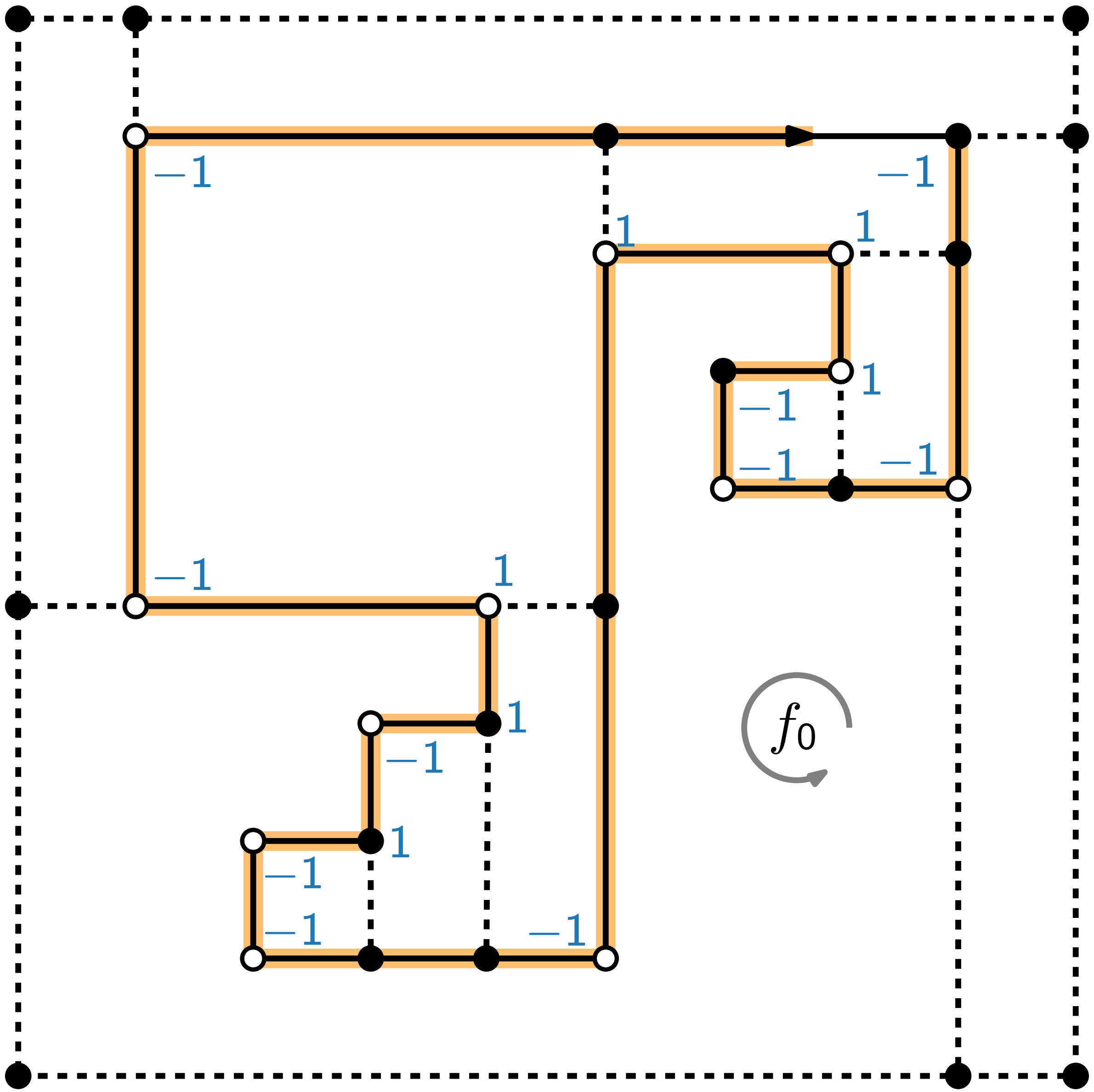
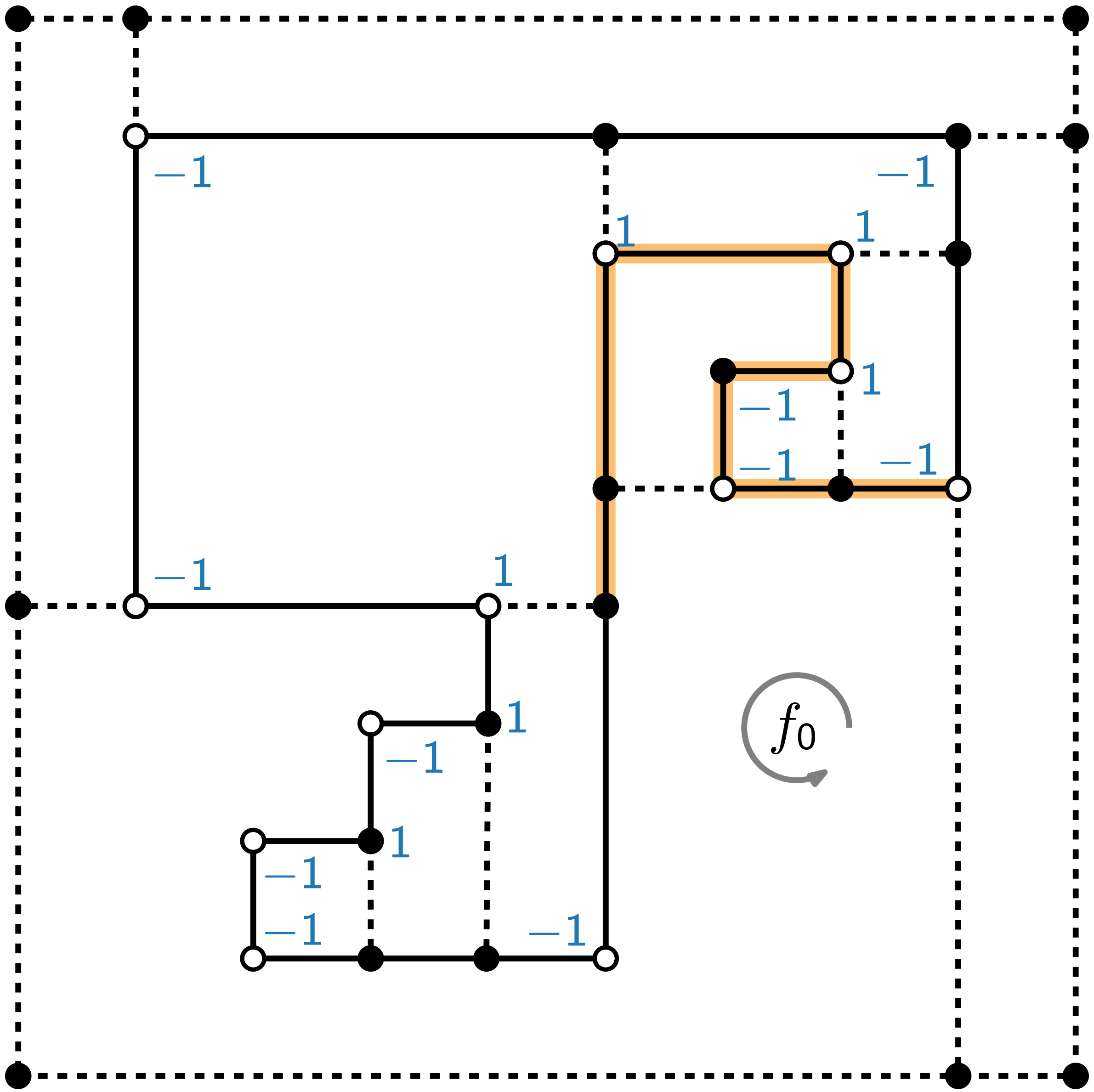


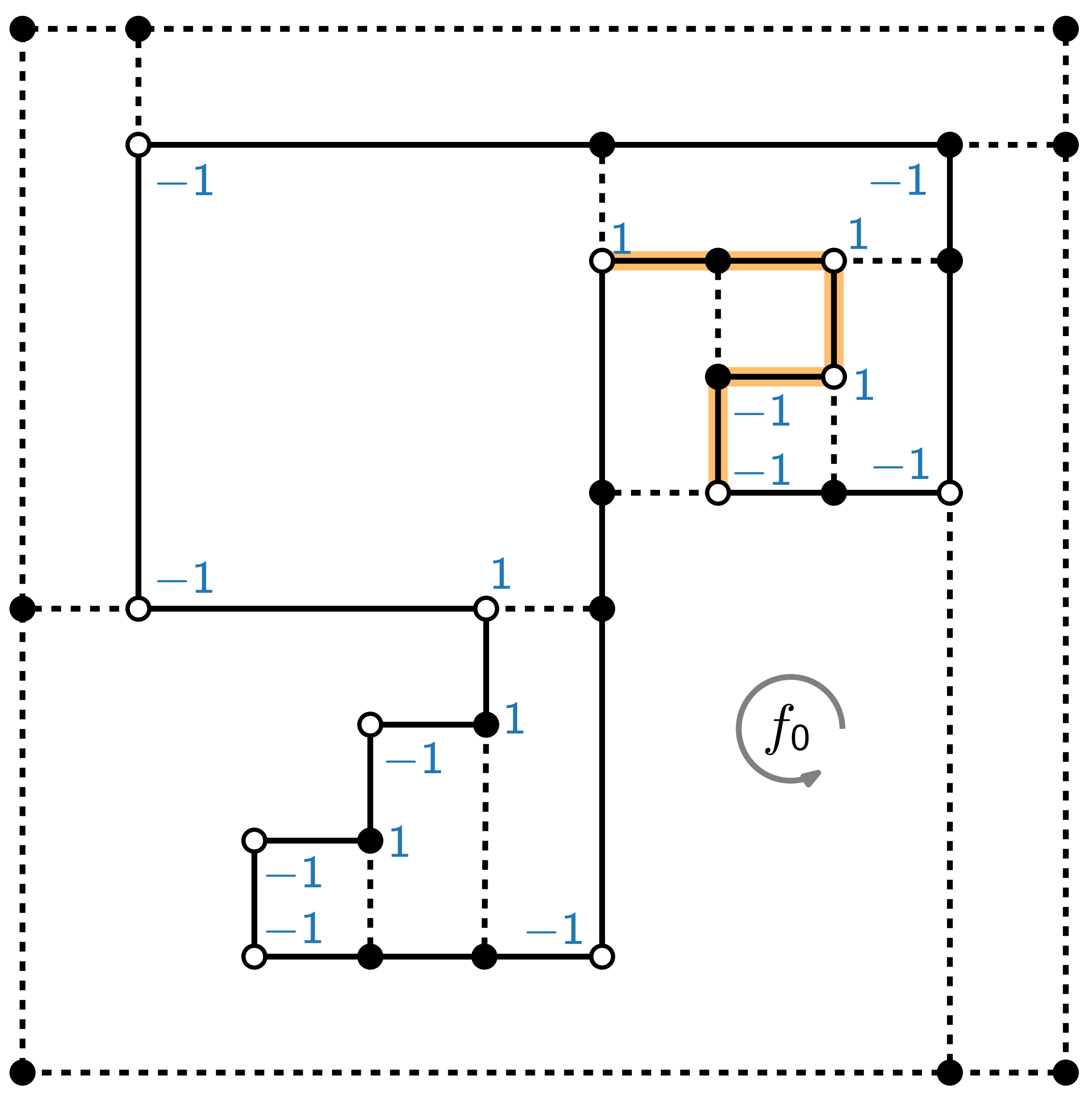
Refinement of (G, H) – Outer Face



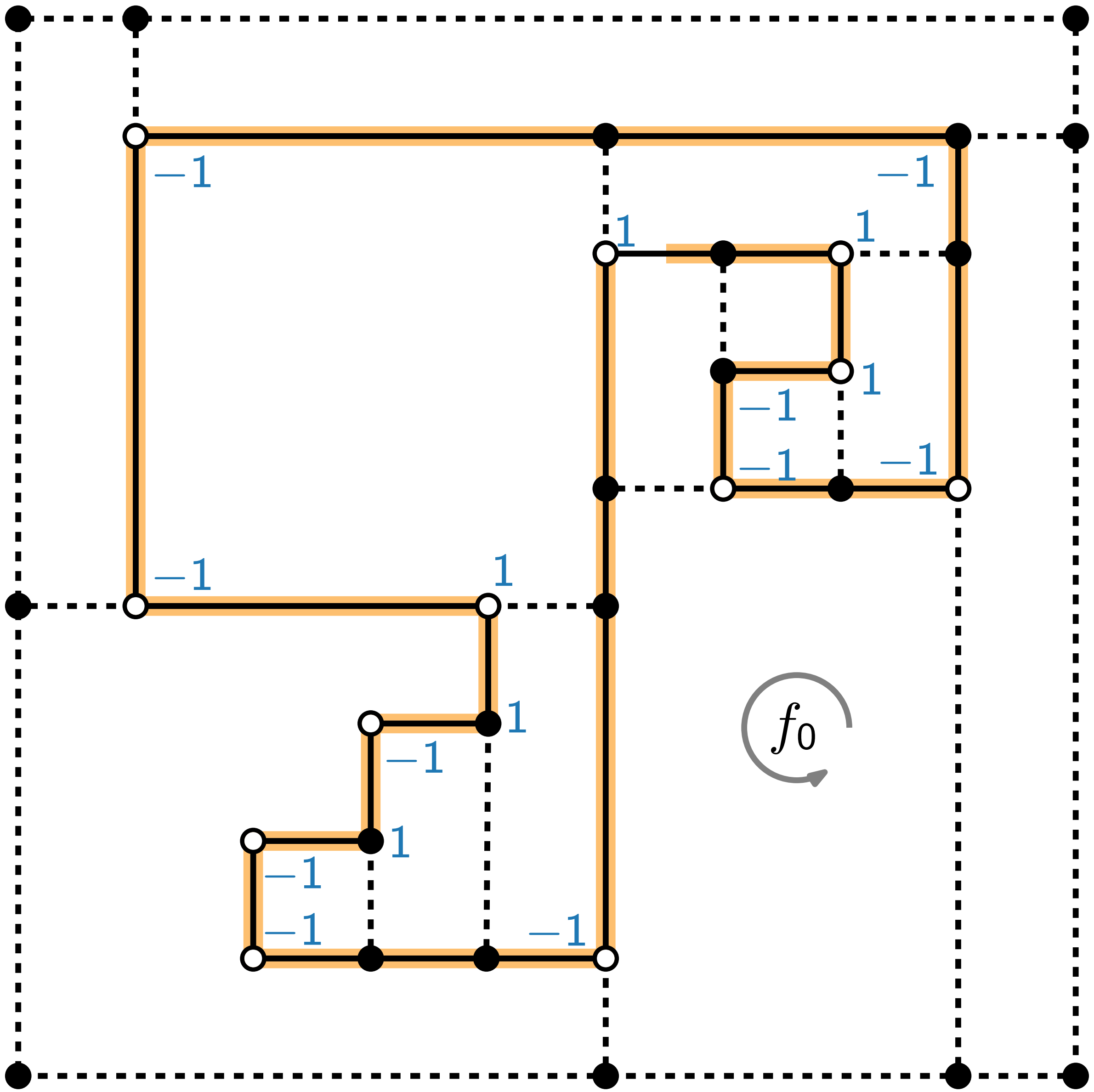
Refinement of (G, H) – Outer Face



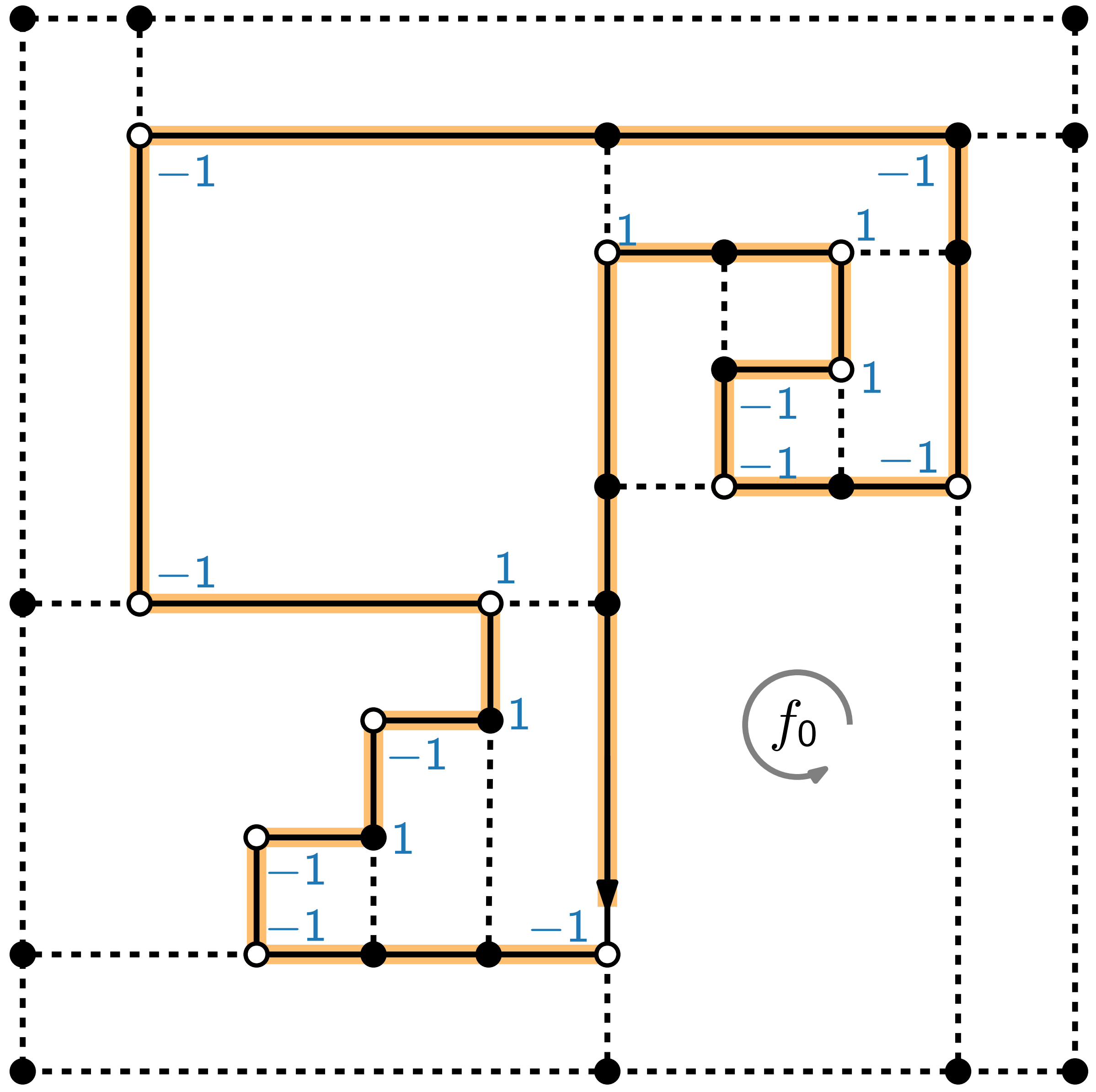
Refinement of (G, H) – Outer Face



Refinement of (G, H) – Outer Face



Refinement of (G, H) – Outer Face



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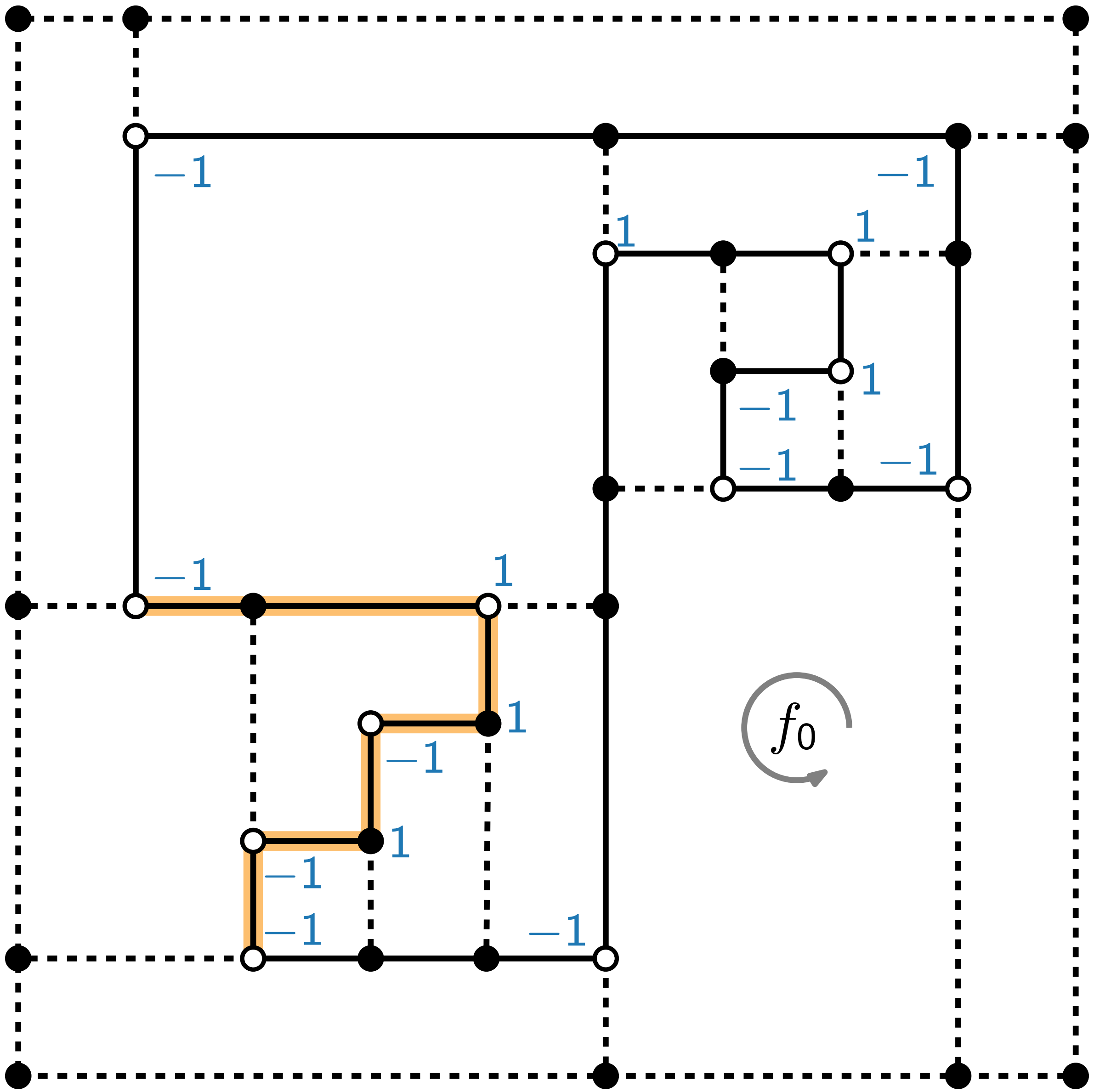

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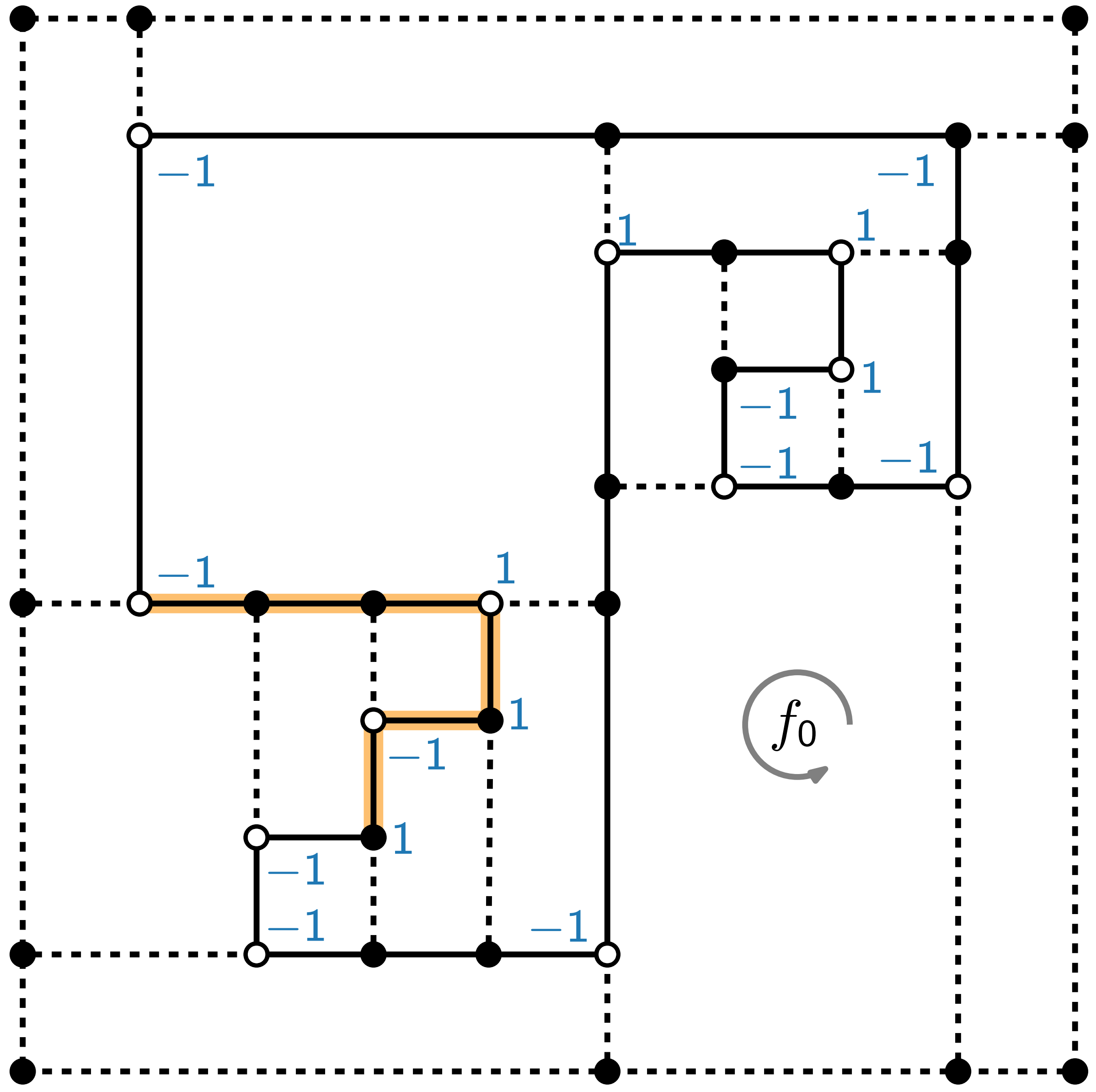

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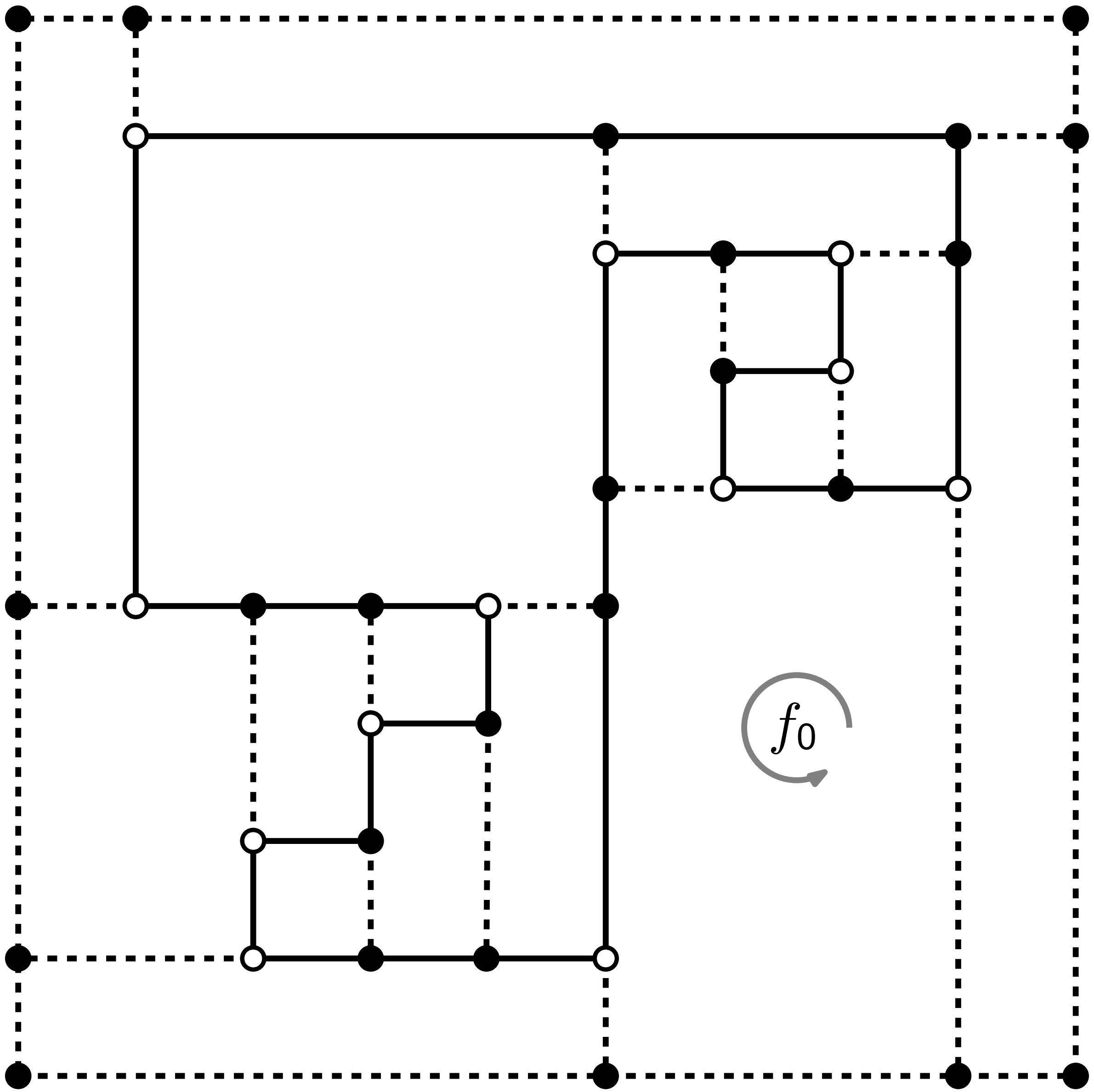
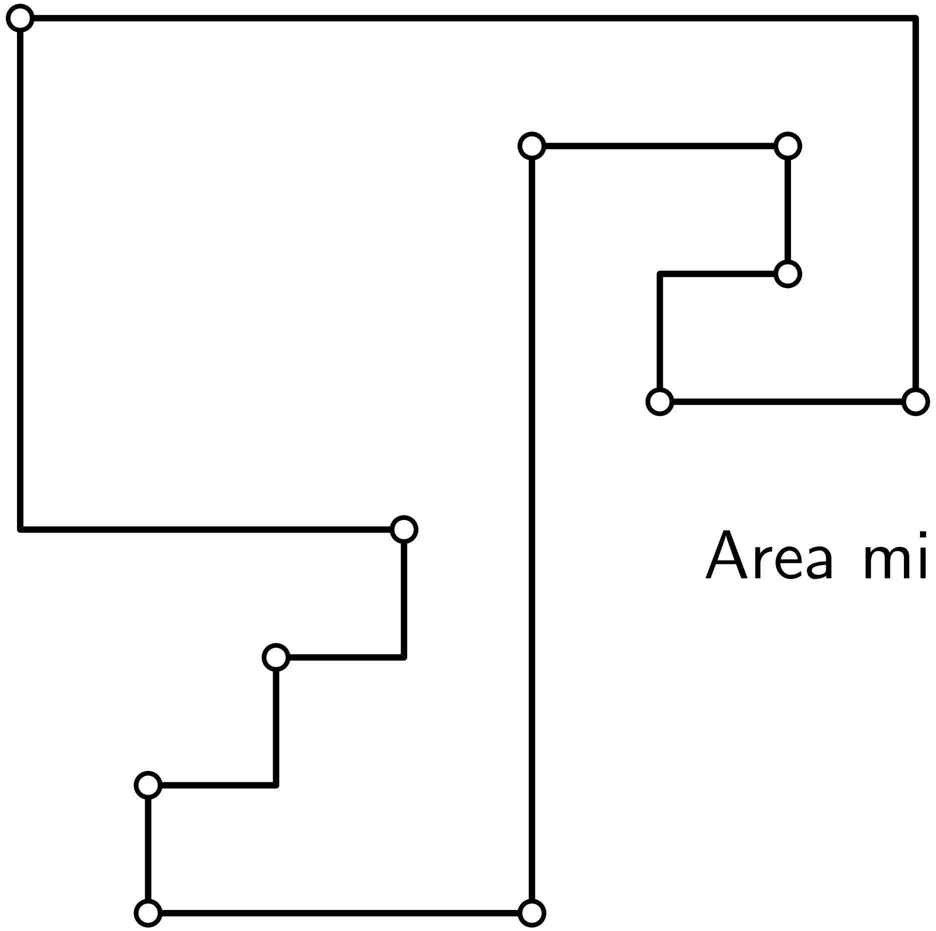
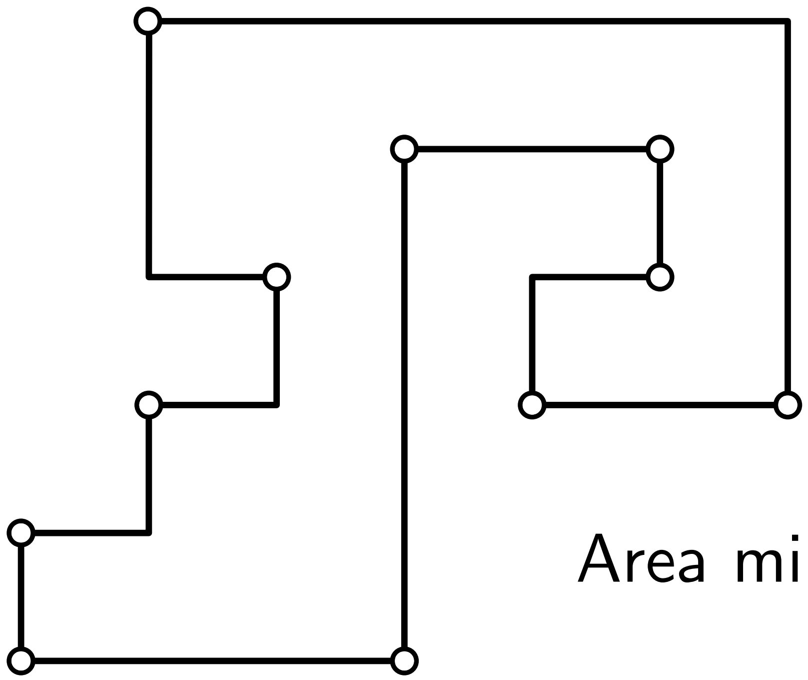


Refinement of (G, H) – Outer Face



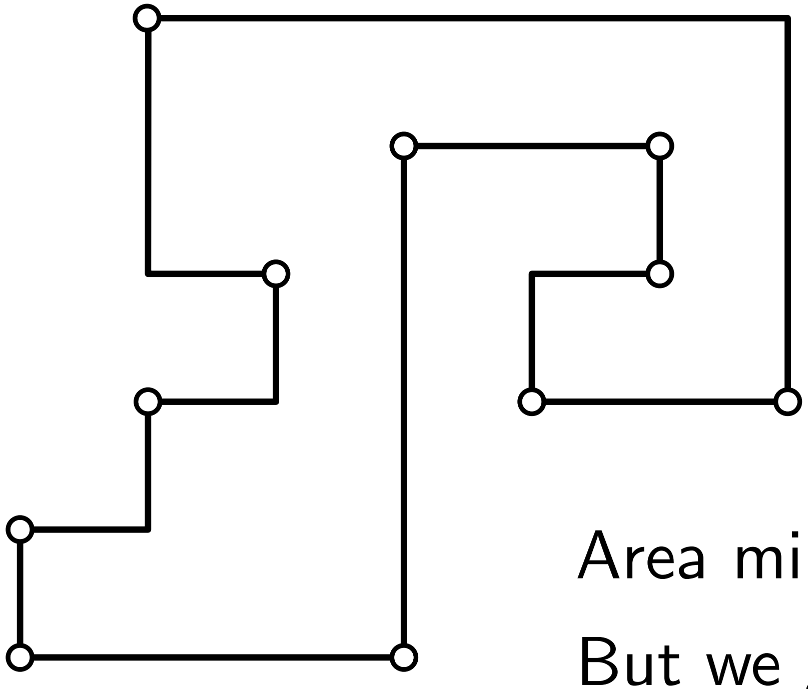
Area minimized?

Refinement of (G, H) – Outer Face



Area minimized? No!

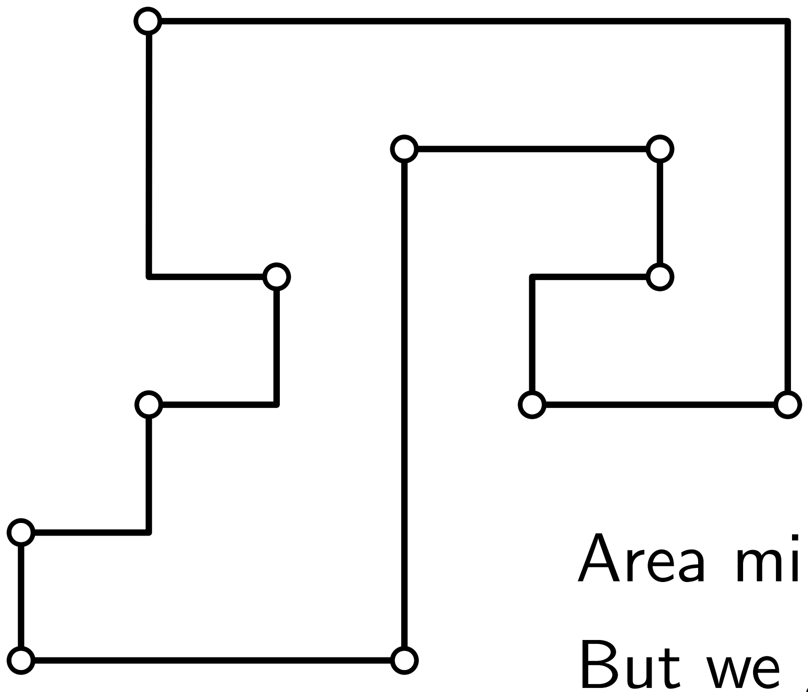
Refinement of (G, H) – Outer Face



Area minimized? No!

But we get bound $O((n+b)^2)$ on the area.

Refinement of (G, H) – Outer Face



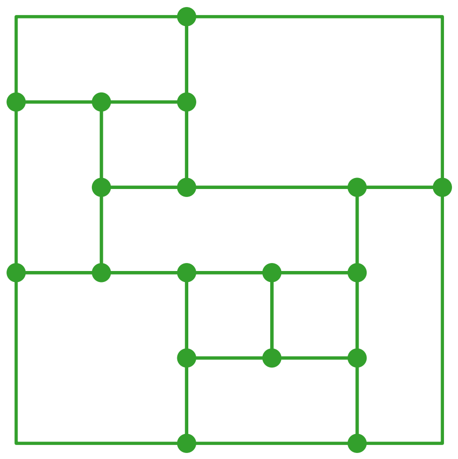
Area minimized? No!

But we get bound $O((n+b)^2)$ on the area.

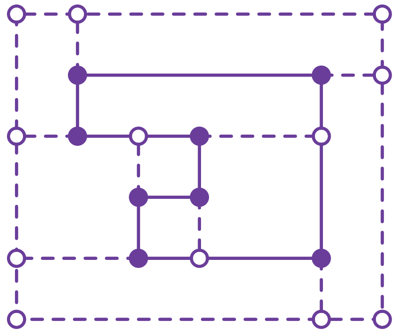
Theorem. [Patrignani 2001]
Compaction for given orthogonal representation is in general NP-hard.

Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

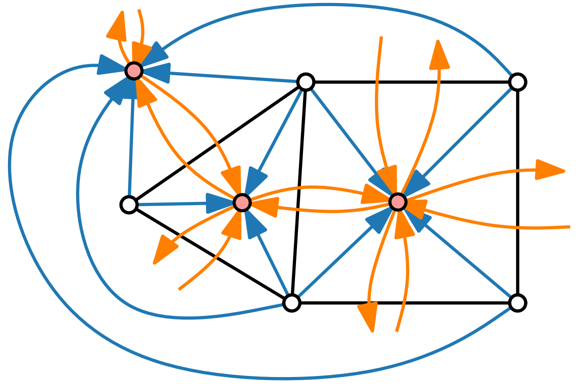
Visualization of Graphs



Lecture 5:
Orthogonal Layouts

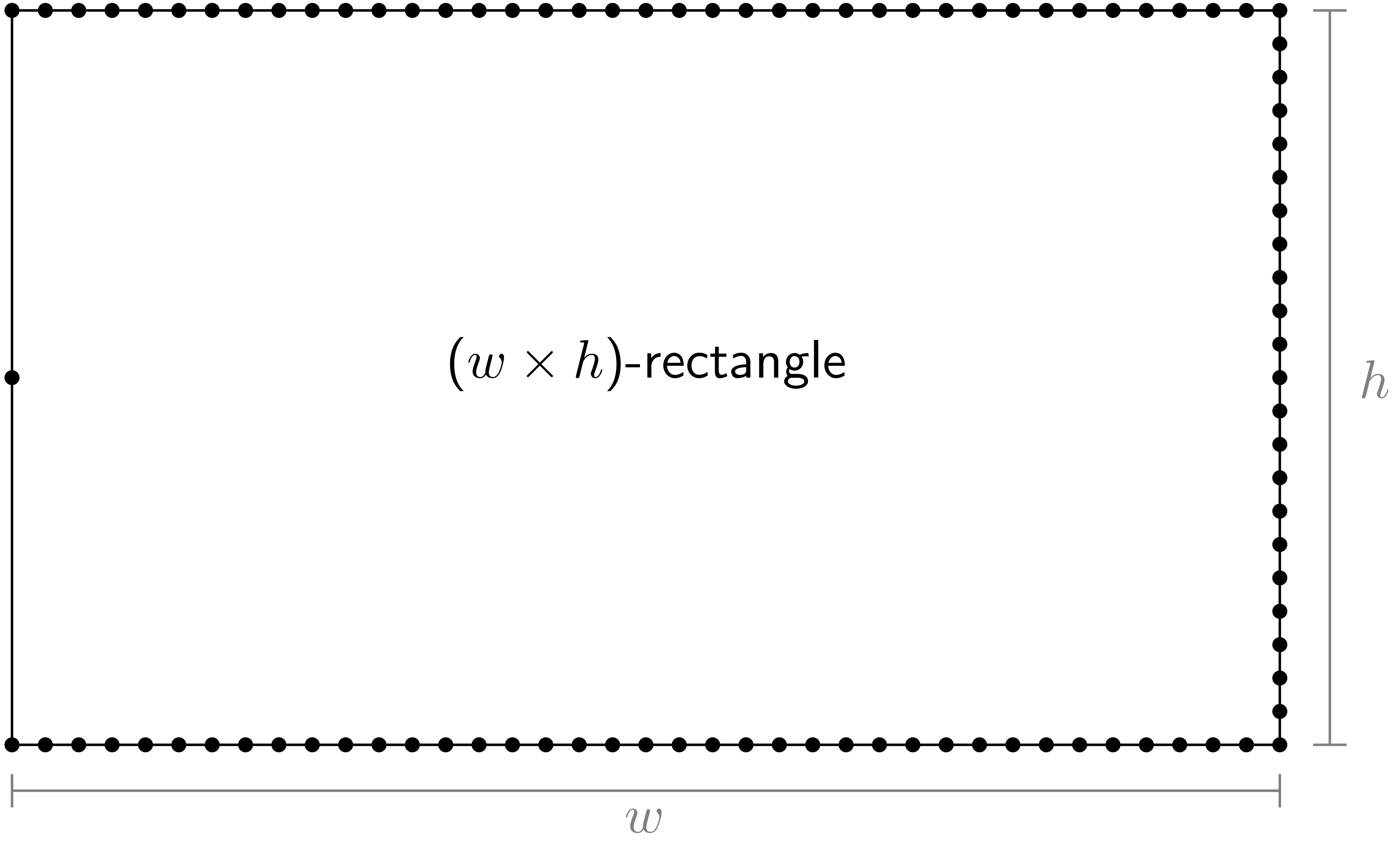


Part V:
NP-hardness



Jonathan Klawitter

Boundary, belt, and "piston" gadget



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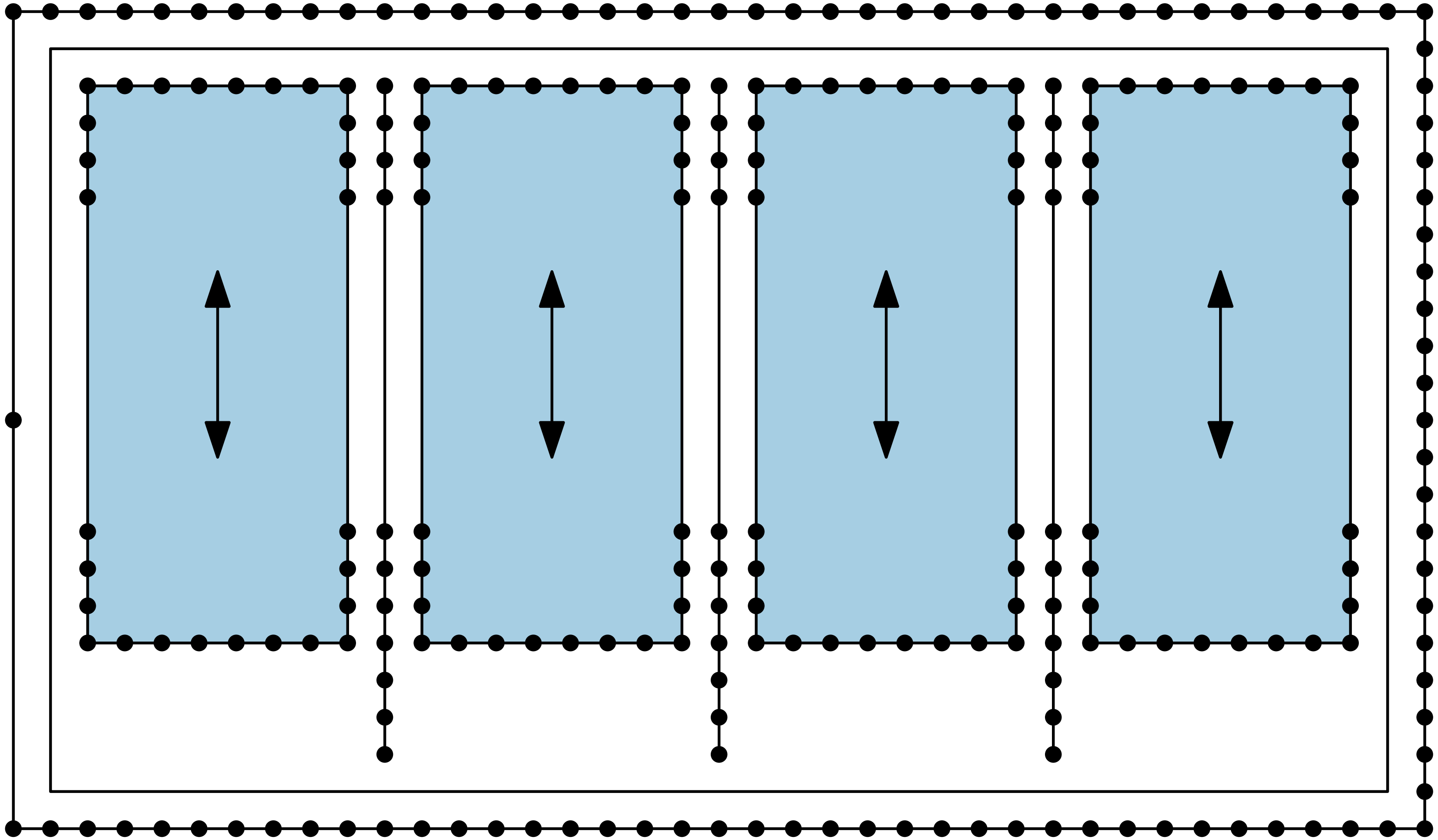

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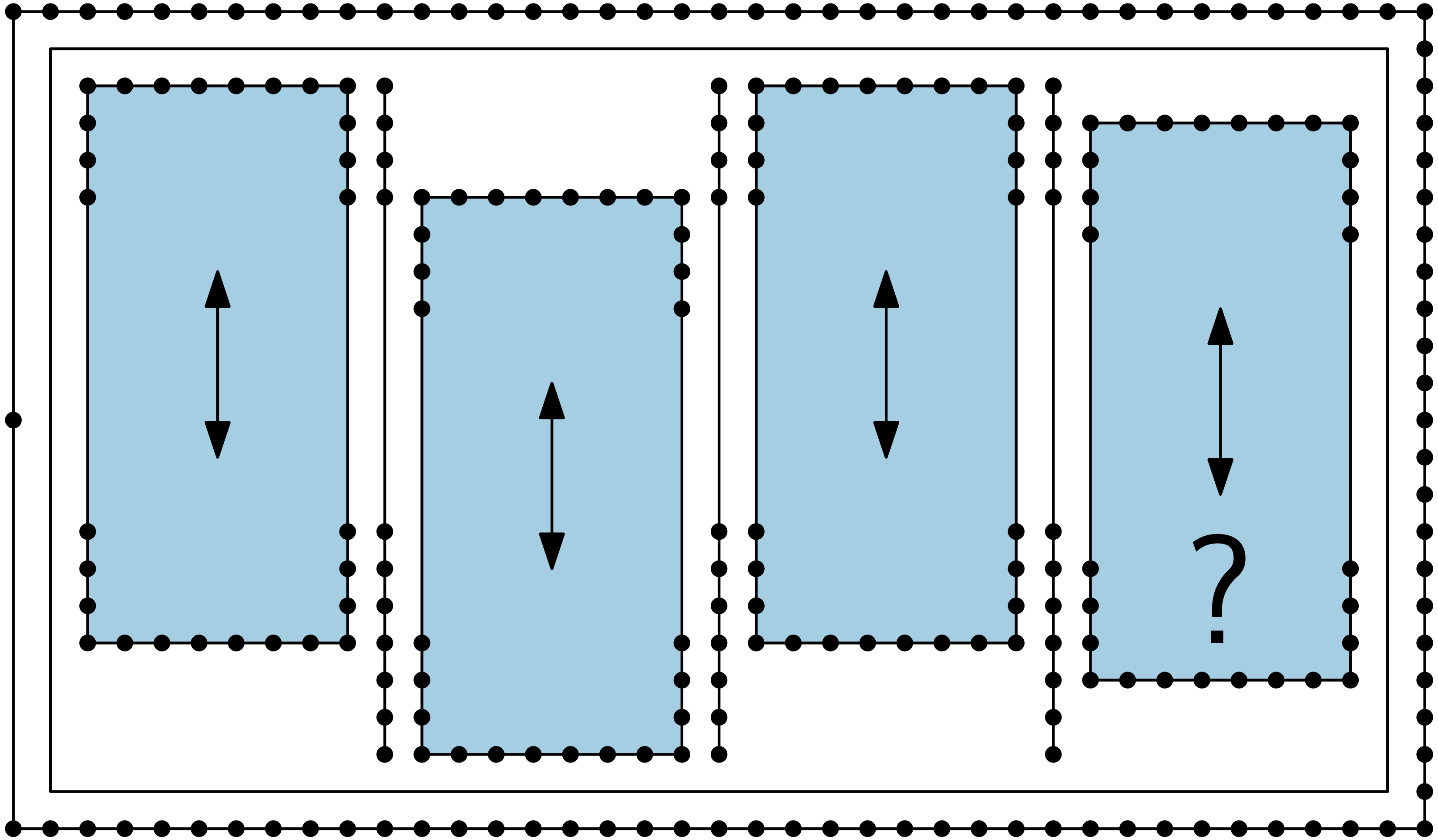

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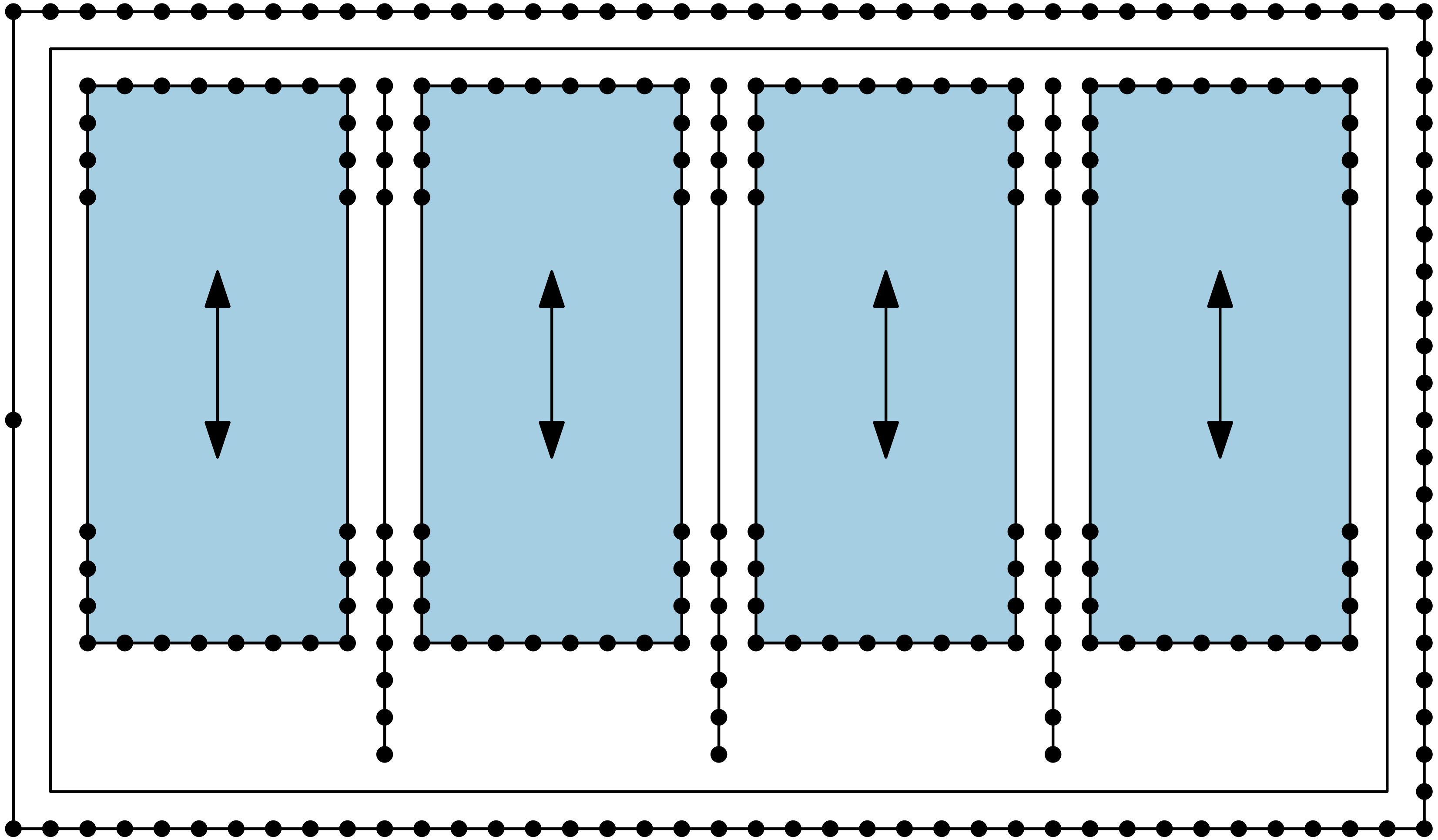

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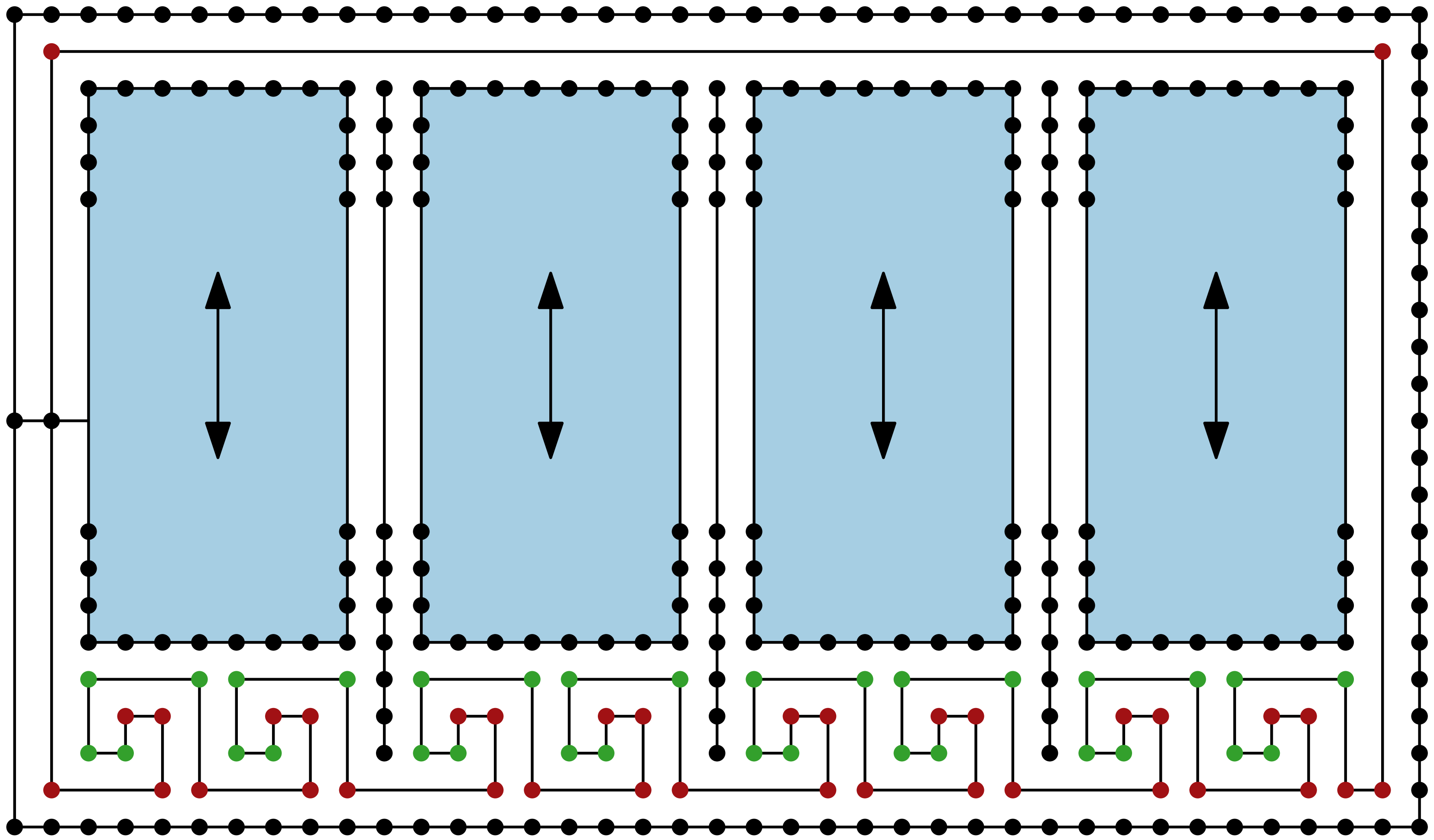

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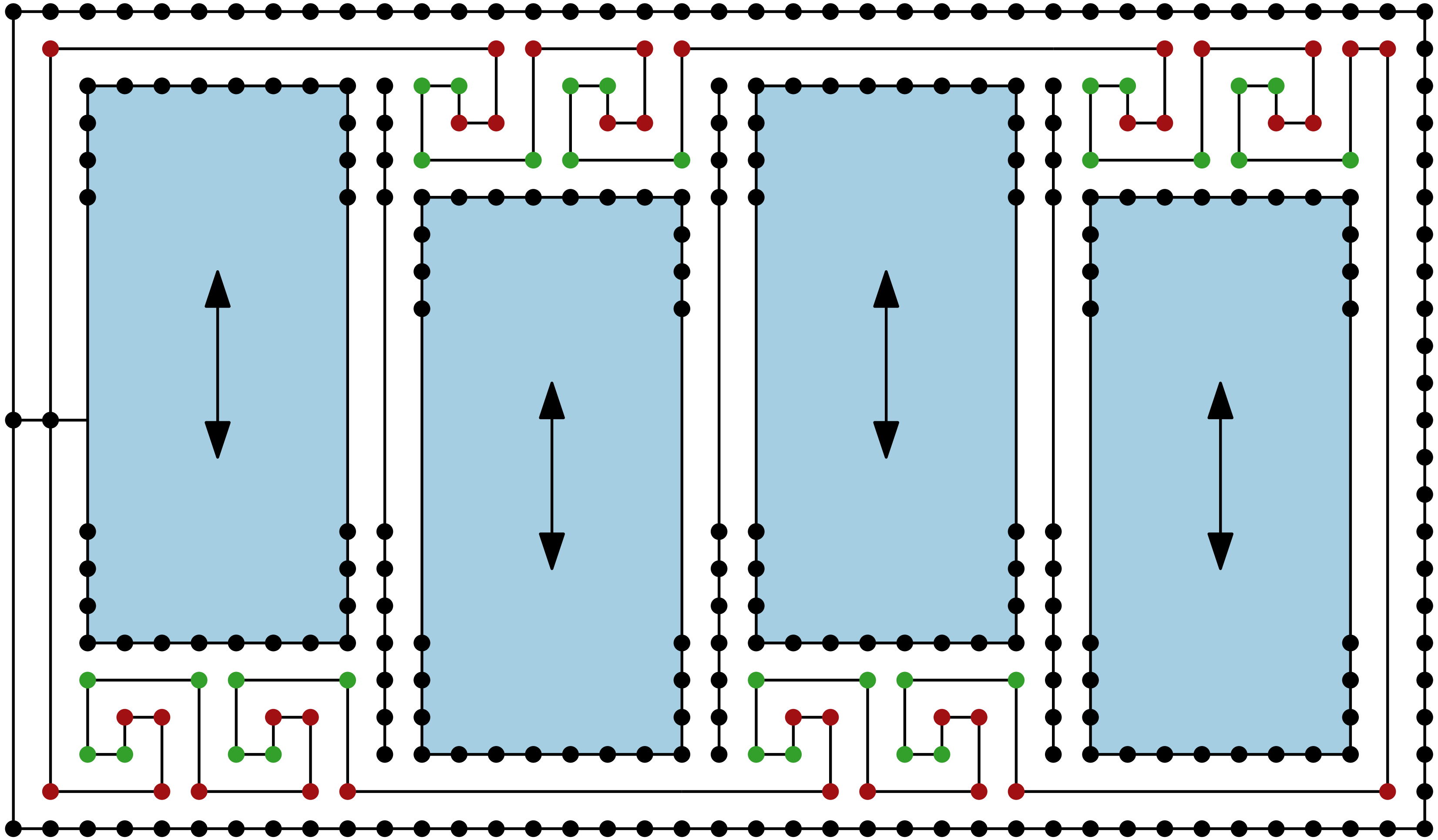

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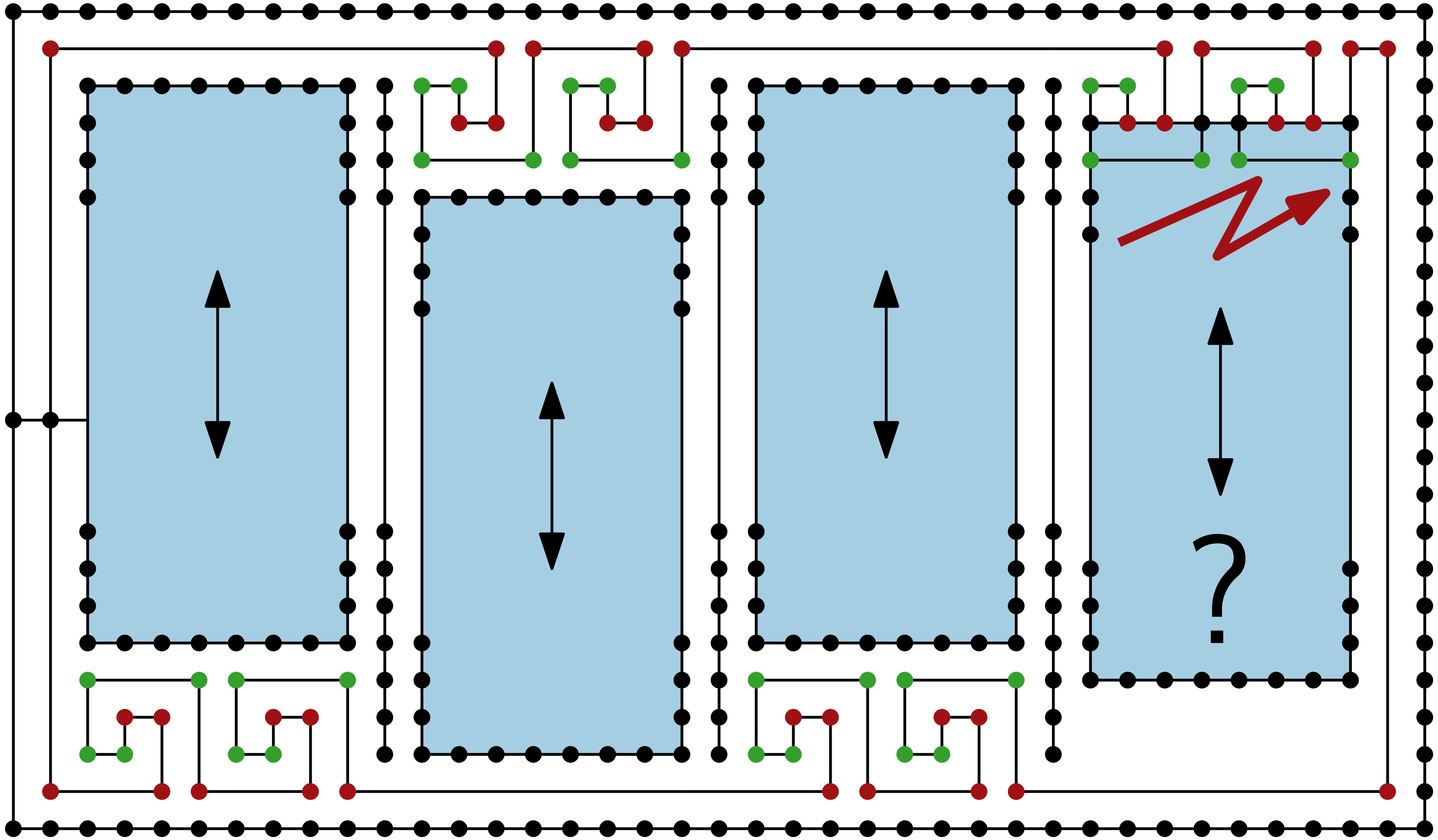
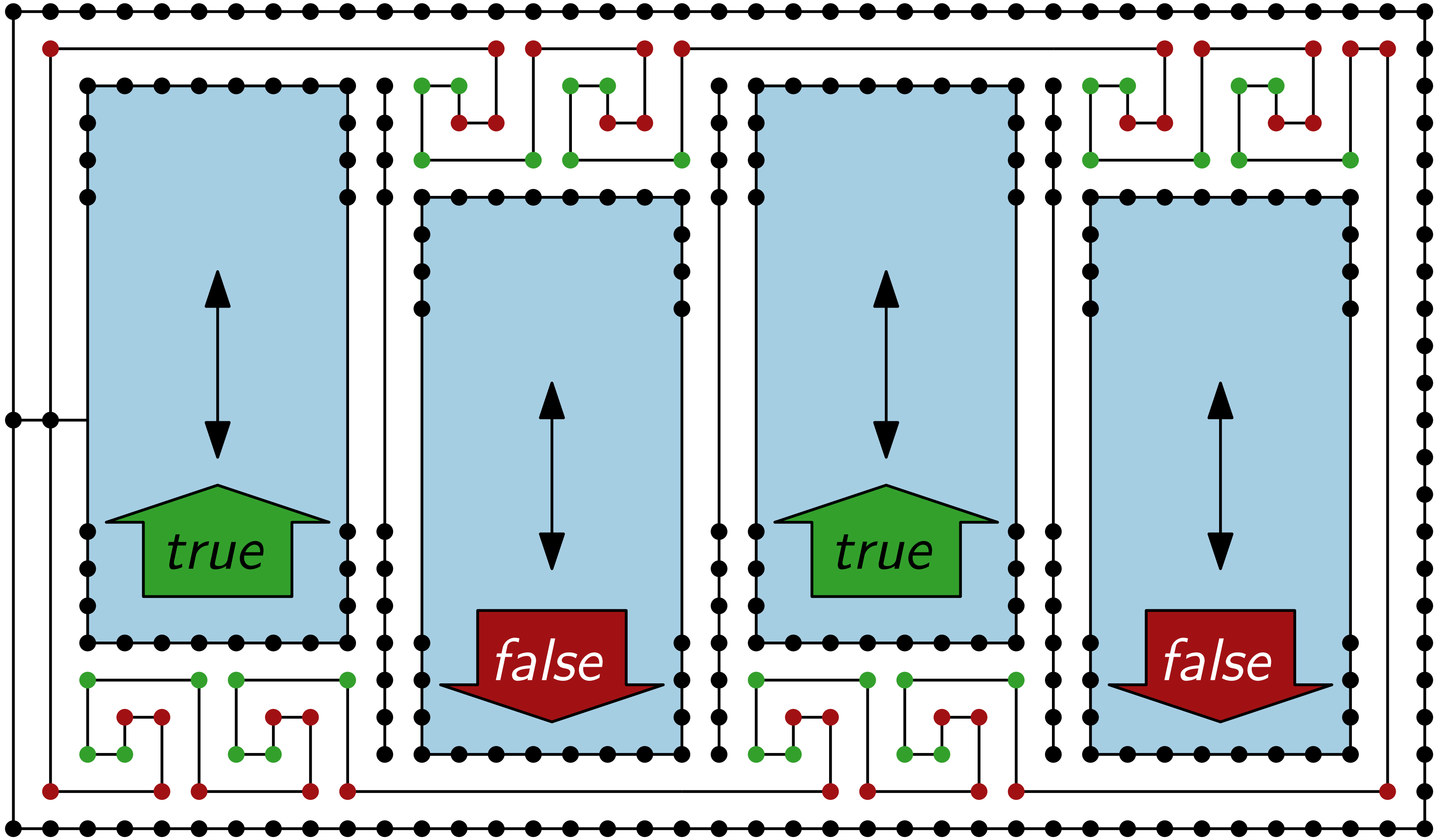
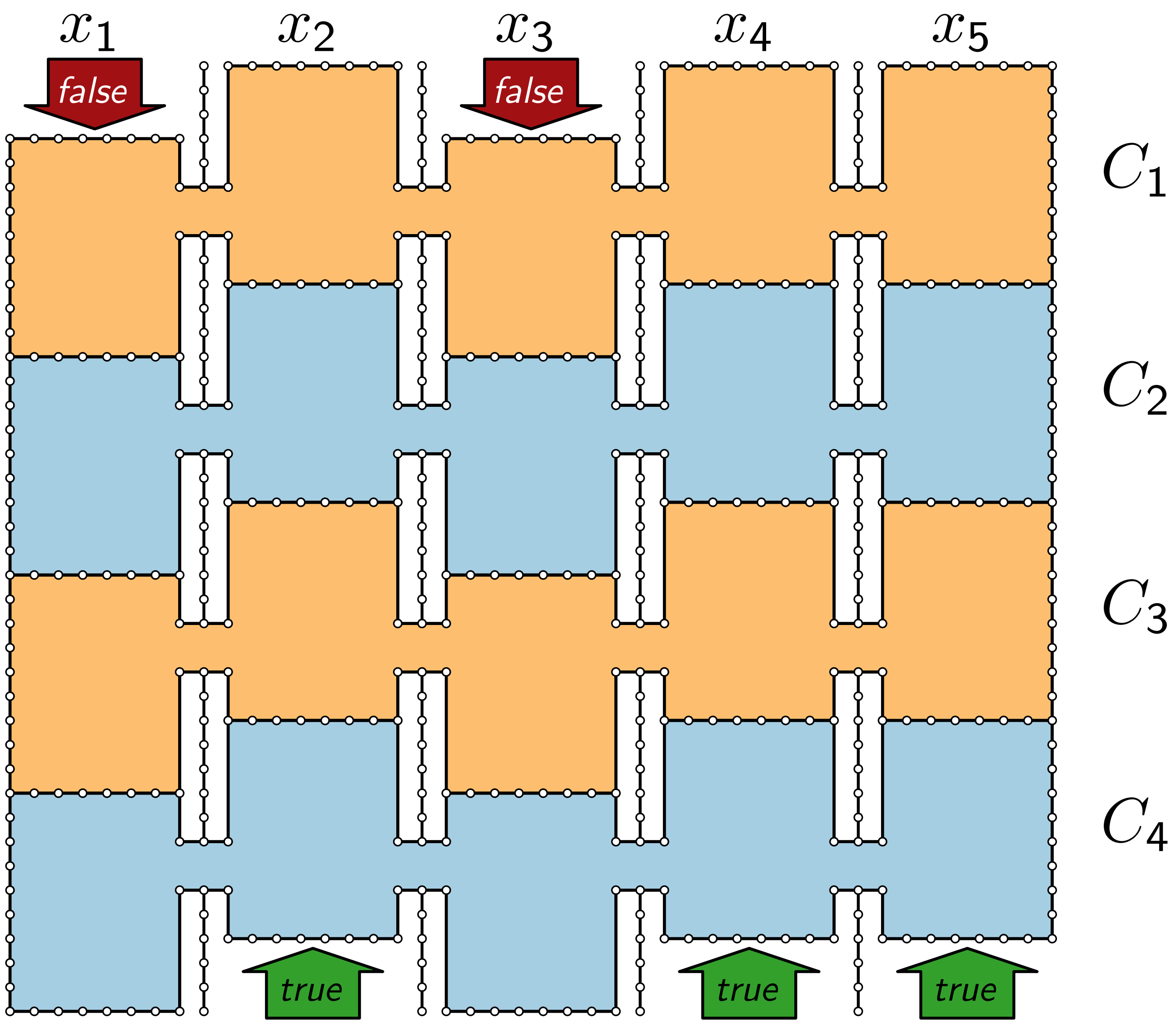


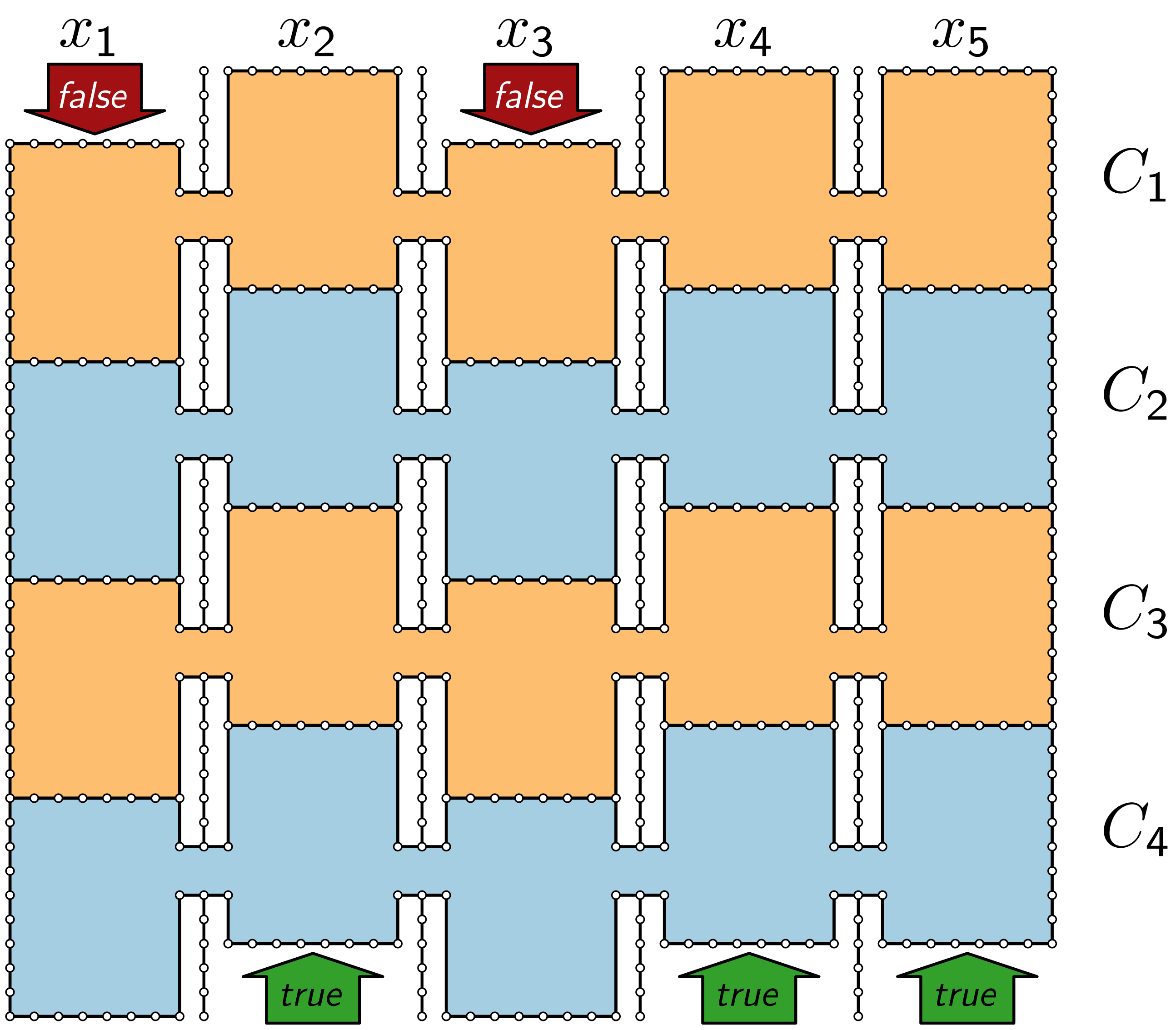
Boundary, belt, and "piston" gadget



Clause gadgets

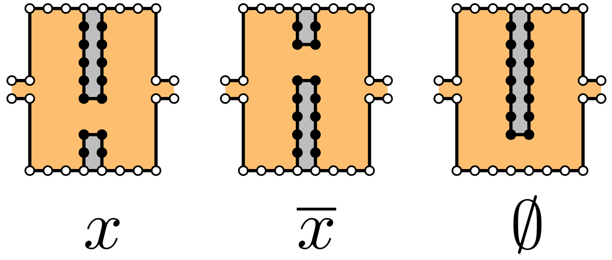


Clause gadgets

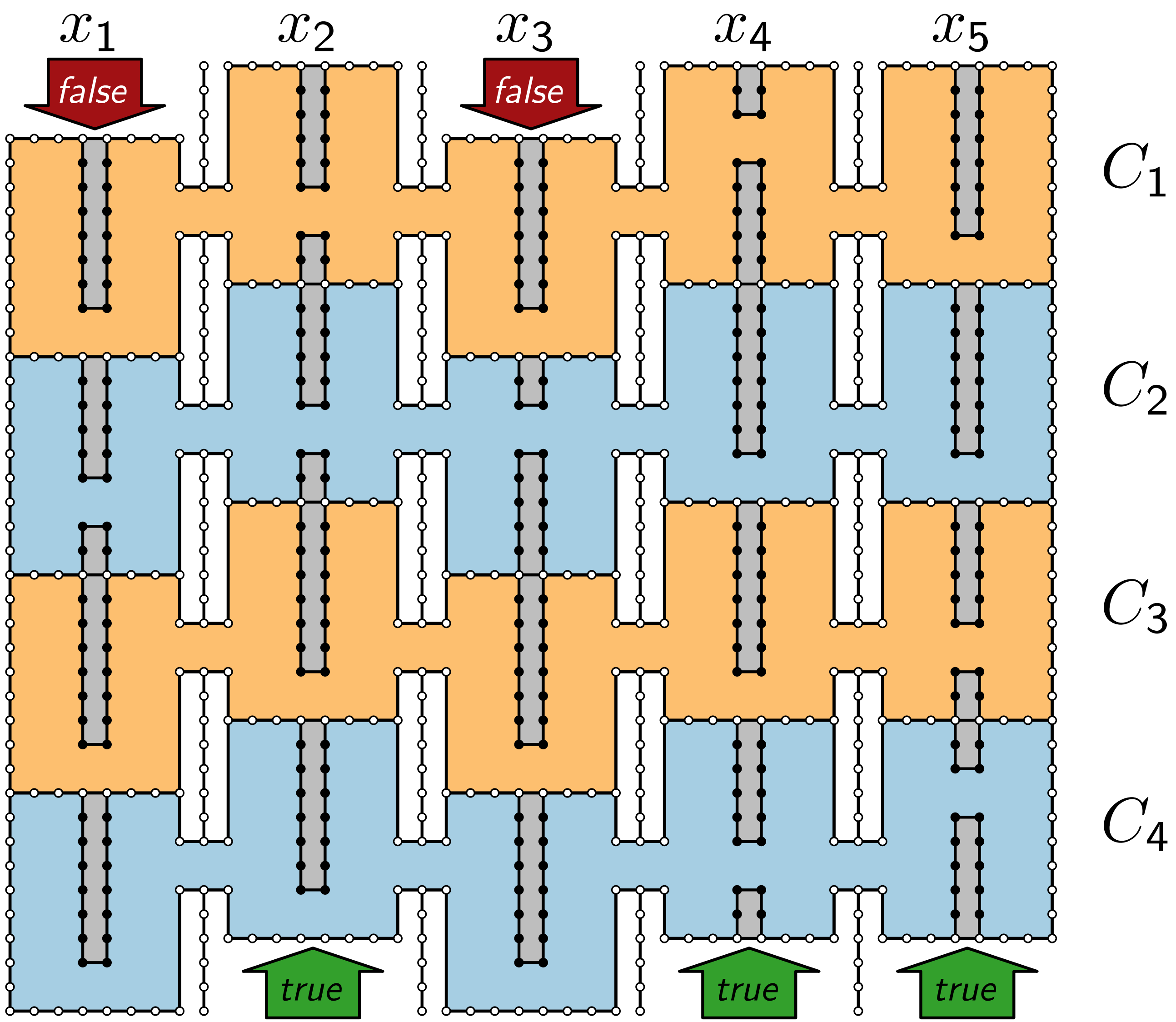


Example:

$C_1 = x_2 \vee \overline{x_4}$
$C_2 = x_1 \vee x_2 \vee \overline{x_3}$
$C_3 = x_5$
$C_4 = x_4 \vee \overline{x_5}$



Clause gadgets



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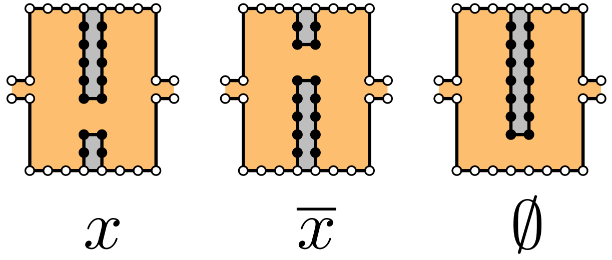
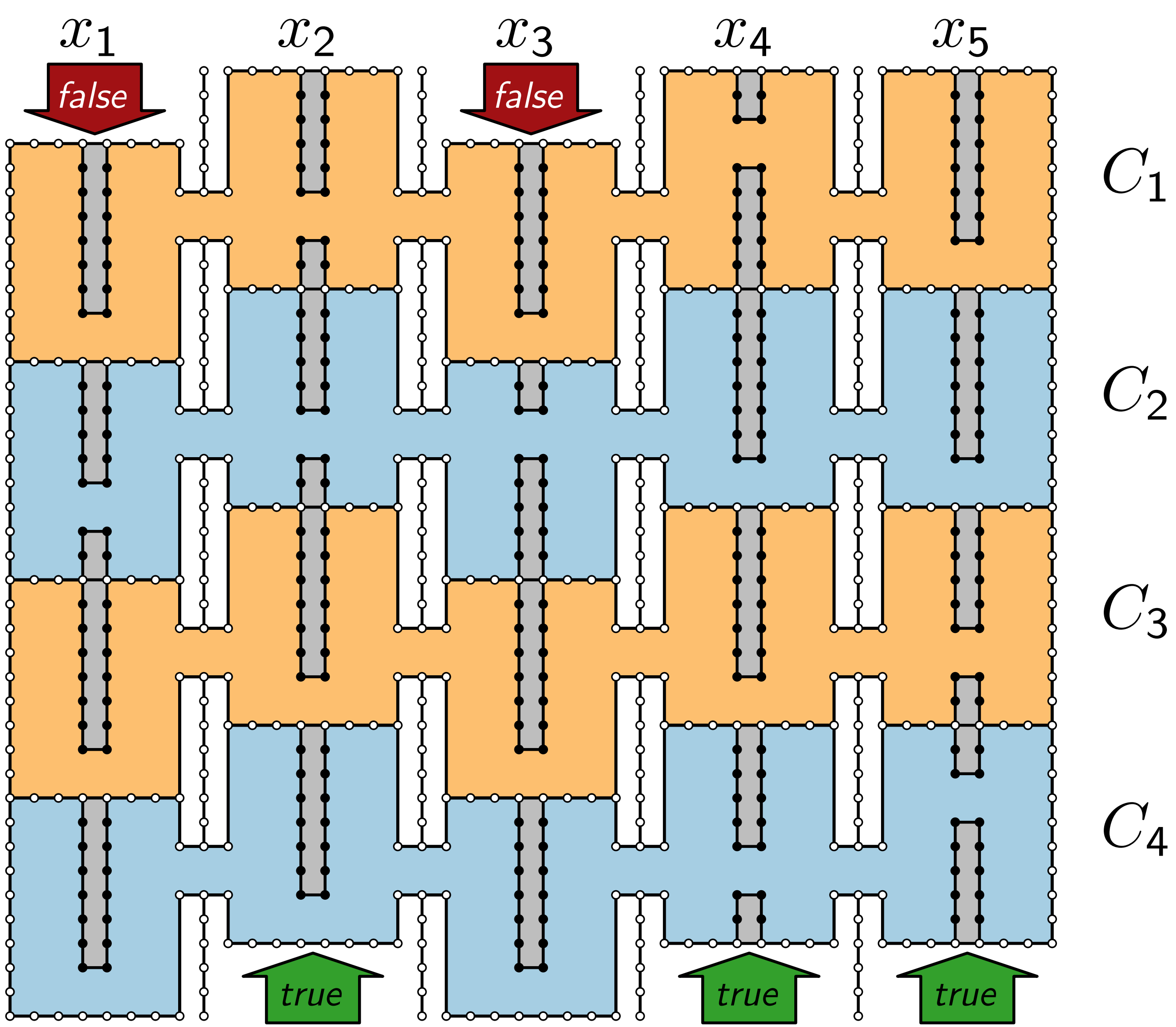


Clause gadgets



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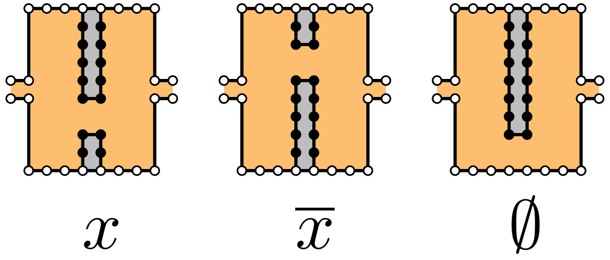
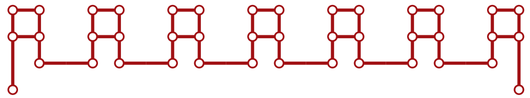
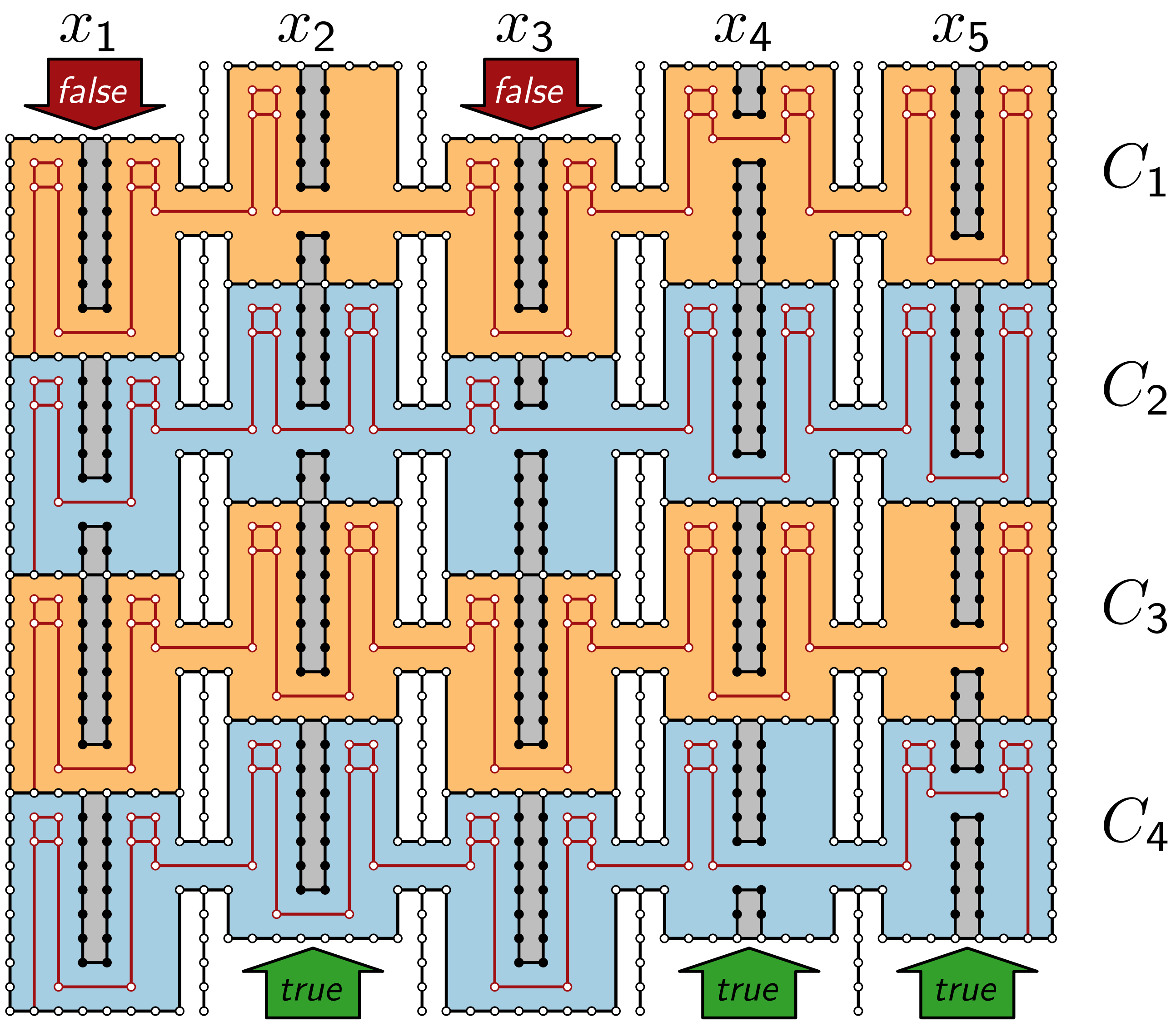




insert $(2n-1)$-chain through each clause

Clause gadgets



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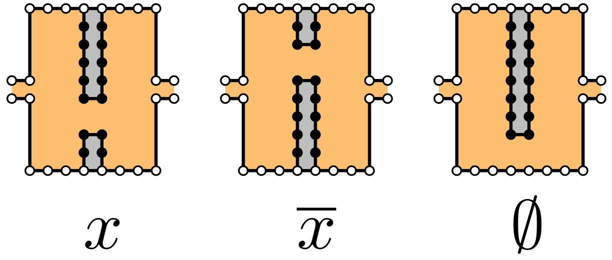

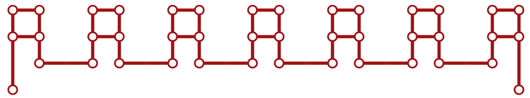

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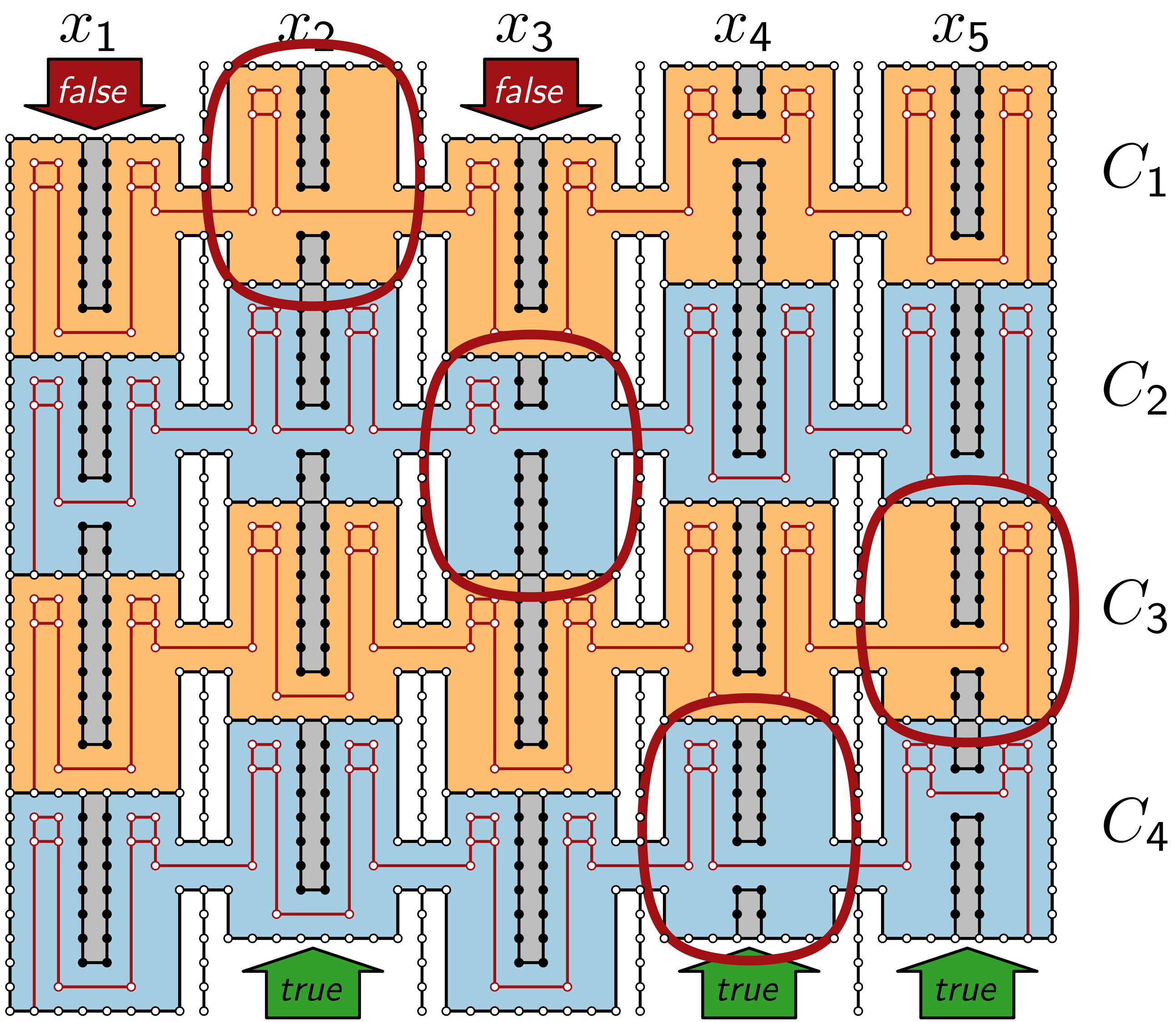

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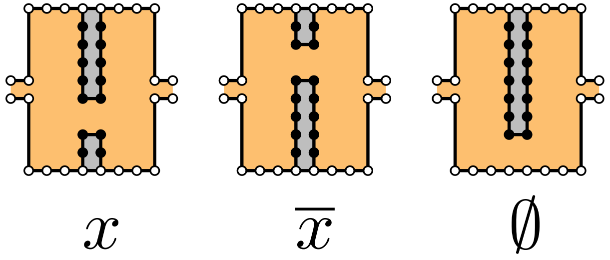
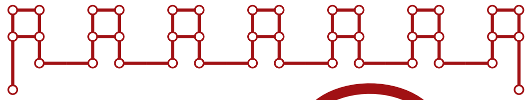
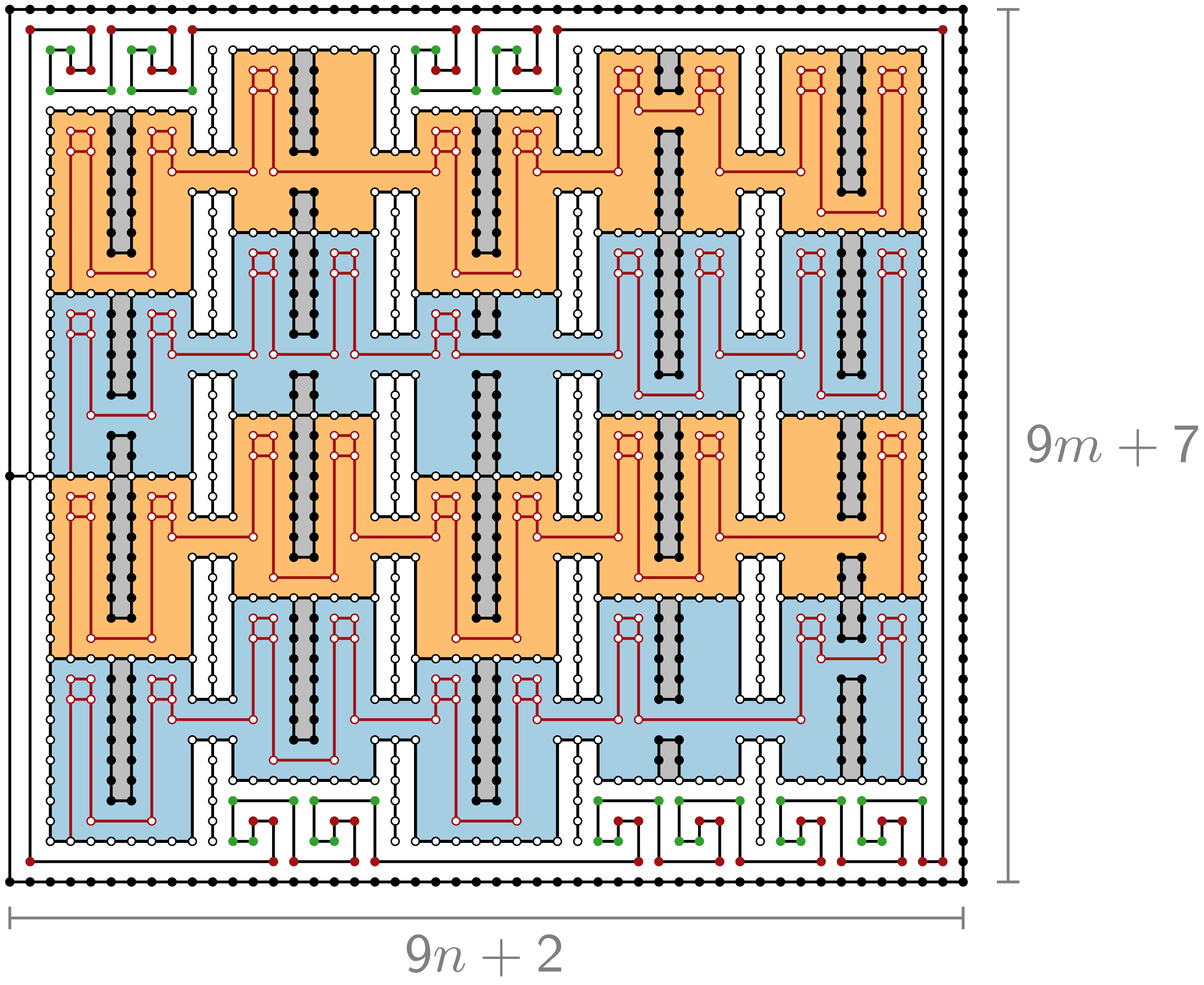


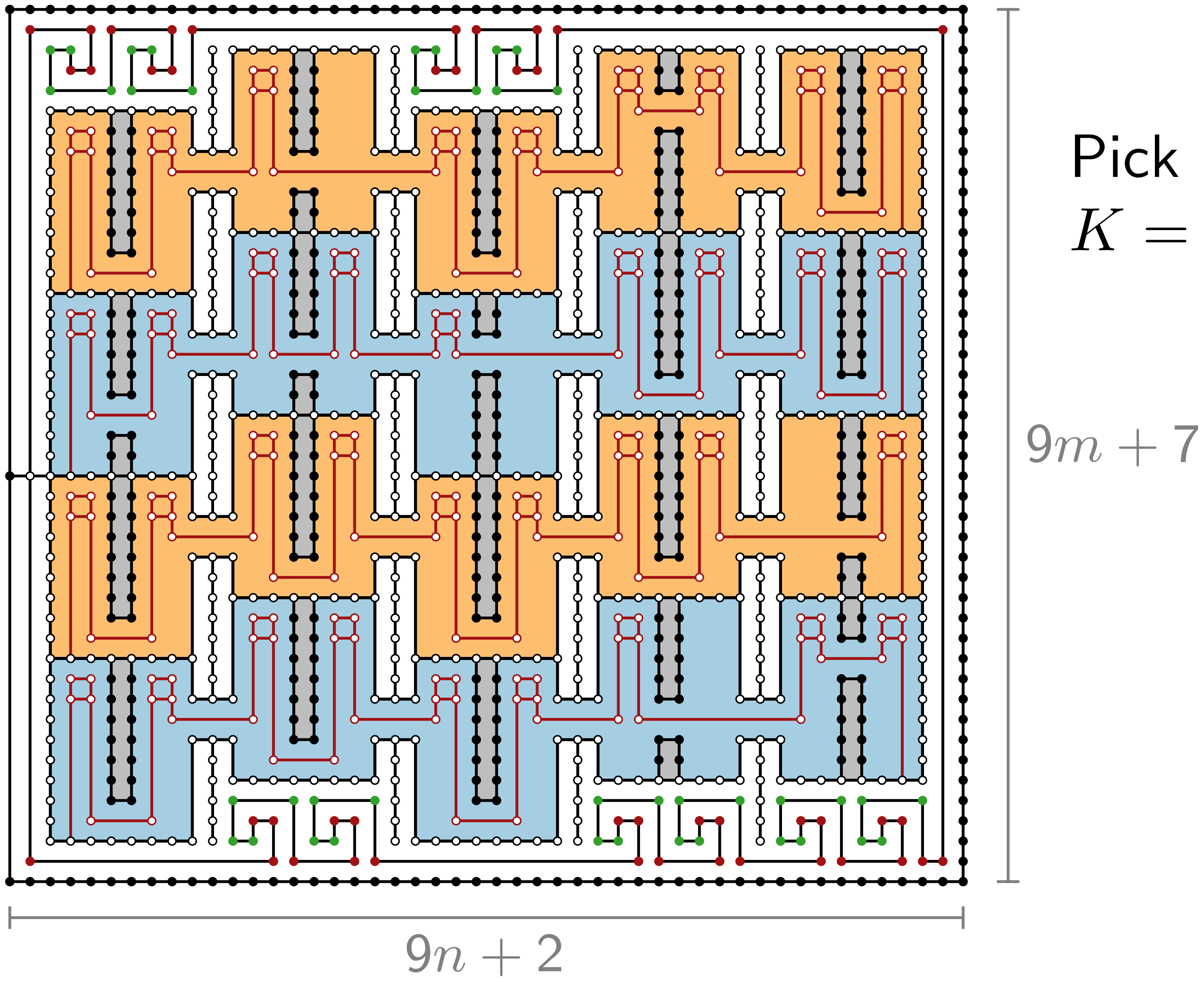


insert $(2n - 1)$-chain through each clause

Complete reduction

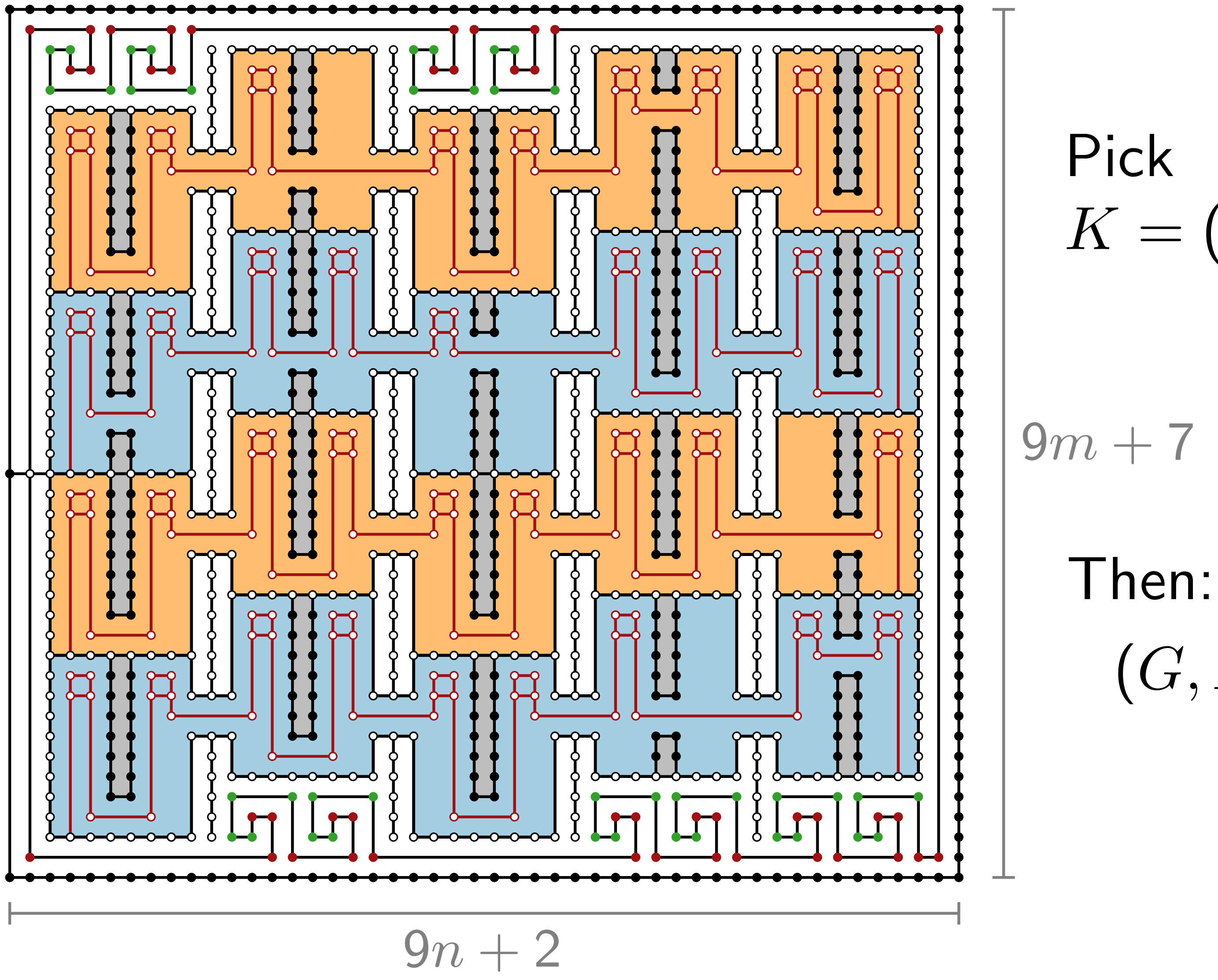


Complete reduction



Pick
$K = (9n + 2) \cdot (9m + 7)$

Complete reduction



Pick
$K = (9n + 2) \cdot (9m + 7)$

Then:

(G, H) has an area K drawing
$\Leftrightarrow$
Φ satisfiable

$\square$

Literature

- [GD Ch. 5] for detailed explanation
- [Tamassia 1987] “On embedding a graph in the grid with the minmum number of bends”
  original paper on flow for bend minimisation
- [Patrignani 2001] “On the complexity of orthogonal compaction”
  NP-hardness proof of compactification