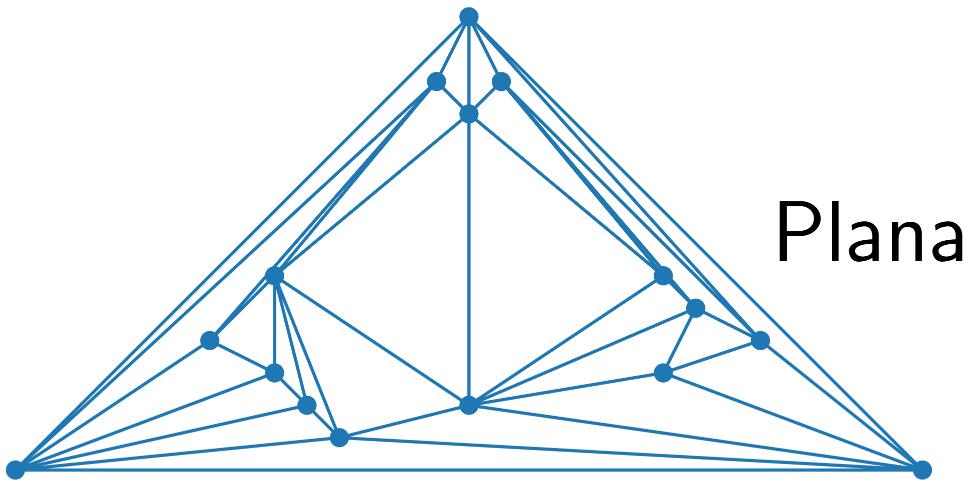
Julius-Maximilians-UNIVERSITÄT WÜRZBURG

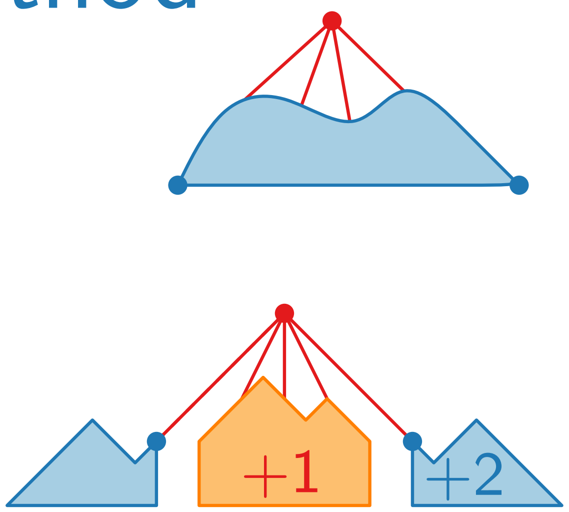
# Visualization of Graphs

## Lecture 3:
## Straight-Line Drawings of Planar Graphs I: Canonical Ordering and Shift Method

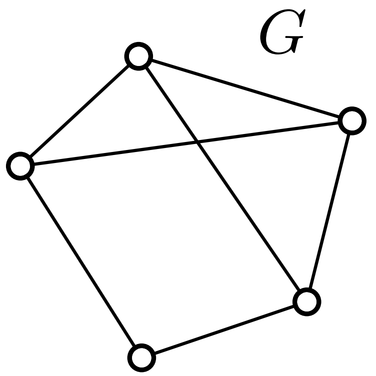


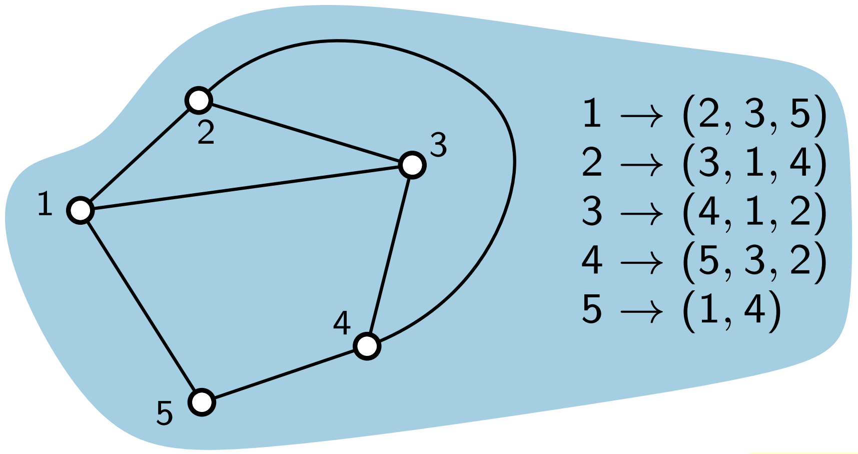
### Part I:
### Planar Straight-Line Drawings

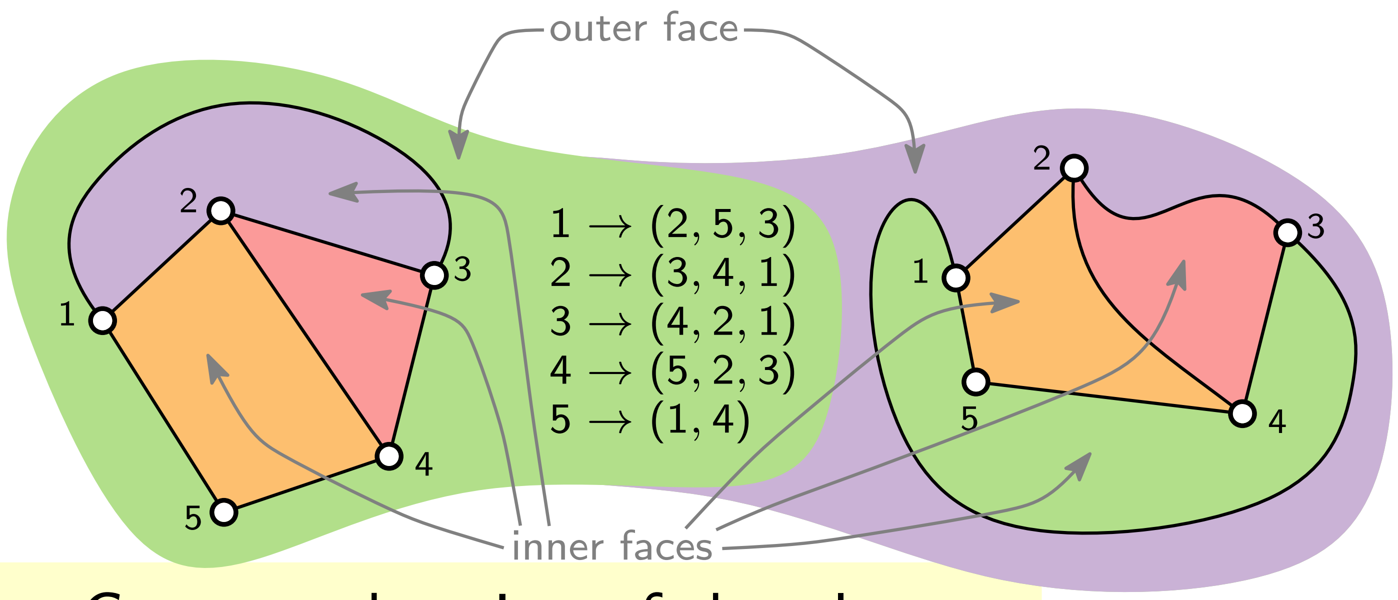
Jonathan Klawitter



# Planar Graphs




$G$ is **planar**:
it can be drawn in such a way that no edges cross each other.

**planar embedding**:
Clockwise orientation of adjacent vertices around each vertex.

A planar graph can have many planar embeddings.

A planar embedding can have many planar drawings!

**faces:** Connected region of the plane bounded by edges

**Euler's polyhedra formula.**

#faces - #edges + #vertices = #conn.comp. + 1

$f - m + n = c + 1$

**Proof.** By induction on $m$:

$m = 0 \Rightarrow f = 1$ and $c = n$

$\Rightarrow 1 - 0 + n = n + 1$ ✓

$m > 1 \Rightarrow$ remove 1 edge $e$ $\Rightarrow m - 1$

$\Rightarrow c + 1$

$\Rightarrow f - 1$

# Properties of Planar Graphs

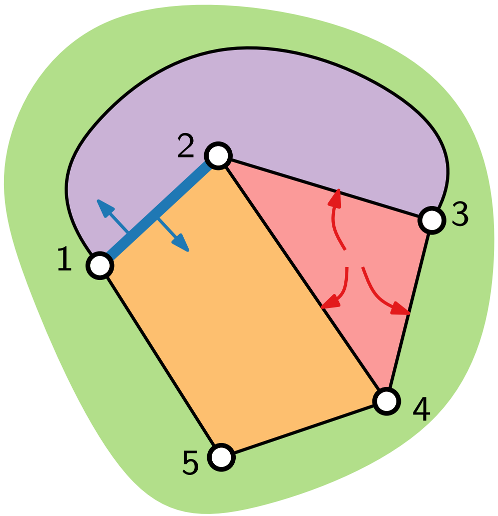
**Euler's polyhedra formula.**

#faces - #edges + #vertices = #conn.comp. + 1

$f \quad - \quad m \quad + \quad n \quad = \quad c \quad + 1$

**Theorem.** $G$ simple planar graph with $n \geq 3$.

1. $m \leq 3n - 6$
2. $f \leq 2n - 4$
3. There is a vertex of degree at most five




**Proof.** 1. Every edge incident to $\leq 2$ faces
Every face incident to $\geq 3$ edges
$\Rightarrow 3f \leq 2m$
$\Rightarrow 6 \leq 3c + 3 \leq 3f - 3m + 3n \leq 2m - 3m + 3n = 3n - m$
$\Rightarrow m \leq 3n - 6$

2. $3f \leq 2m \leq 6n - 12 \Rightarrow f \leq 2n - 4$

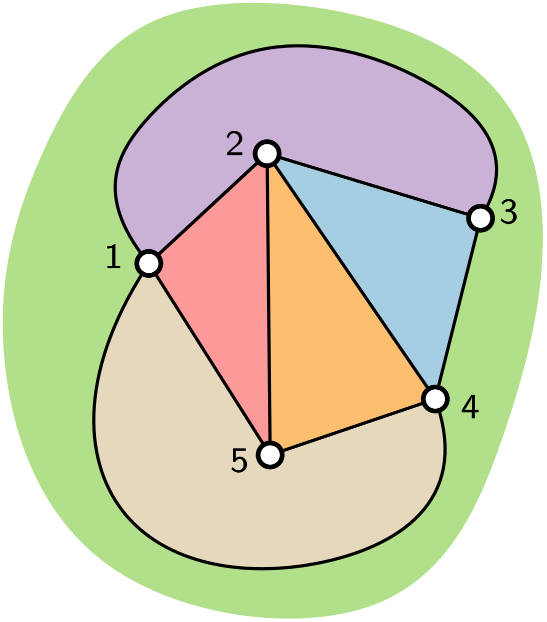
3. $\sum_{v \in V} \deg(v) = 2m \leq 6n - 12$
$\Rightarrow \min_{v \in V} \deg(v) \leq 1/n \sum_{v \in V} \deg(v) < 6$

**Handshaking-Lemma.**
$\sum_{v \in V} \deg(v) = 2|E|$

# Triangulations

with planar embedding

A **plane (inner) triangulation** is a plane graph where every (inner) face is a triangle.
A **maximal planar graph** is a planar graph where adding any edge would destroy planarity.

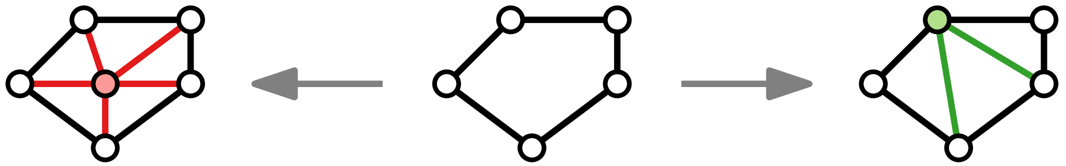



**Observation.**
A maximal plane graph is a plane triangulation.

**Lemma.**
A plane triangulation is at least 3-connected and thus has a unique planar embedding.

We focus on plane triangulations:

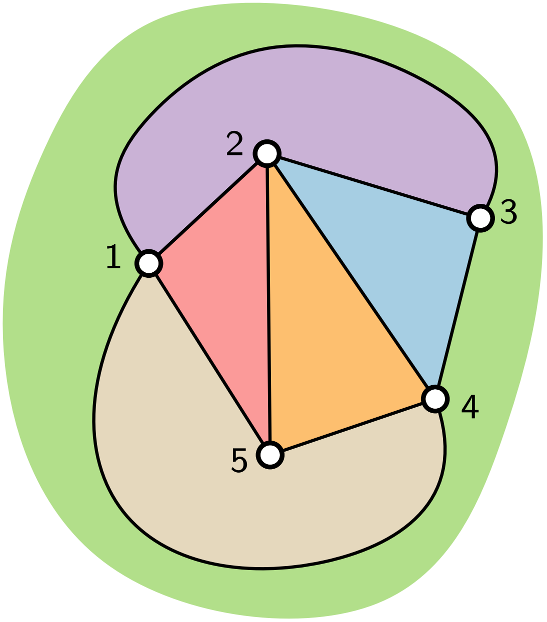
**Lemma.**
Every plane graph is subgraph of a plane triangulation.



# Triangulations

with planar embedding

A **plane (inner) triangulation** is a plane graph where every (inner) face is a triangle.
A **maximal planar graph** is a planar graph where adding any edge would destroy planarity.

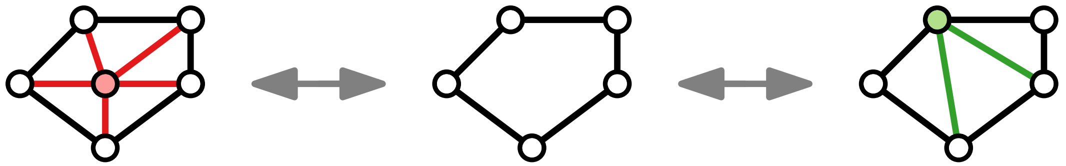



**Observation.**
A maximal plane graph is a plane triangulation.

**Lemma.**
A plane triangulation is at least 3-connected and thus has a unique planar embedding.

We focus on plane triangulations:

**Lemma.**
Every plane graph is subgraph of a plane triangulation.



# Motivation

- Why planar and straight-line?

[Bennett, Ryall, Spaltzeholz and Gooch '07]

**The Aesthetics of Graph Visualization**

**3.2. Edge Placement Heuristics**

By far the most agreed-upon edge placement heuristic is to *minimize the number of edge crossings* in a graph [BMRW98, Har98, DH96, Pur02, TR05, TBB88]. The importance of avoiding edge crossings has also been extensively validated in terms of user preference and performance (see Section 4). Similarly, based on perceptual principles, it is beneficial to *minimize the number of edge bends* within a graph [Pur02, TR05, TBB88]. Edge bends make edges more difficult to follow because an edge with a sharp bend is more likely to be perceived as two separate objects. This leads to the heuristic of *keeping edge bends uniform* with respect to the bend's position on the edge and its angle [TR05]. If an edge must be bent to satisfy other aesthetic criteria, the angle of the bend should be as little as possible, and the bend placement should evenly divide the edge.

## Drawing conventions

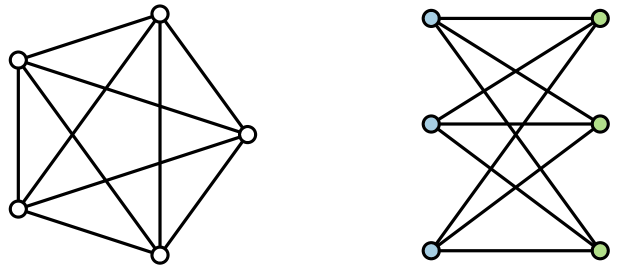
- No crossings $\Rightarrow$ planar
- No bends $\Rightarrow$ straight-line

## Drawing aestethics

- Area

# Towards Straight-Line Drawings

**Theorem.** [Kuratowski 1930]
$G$ planar $\Leftrightarrow$
neither $K_5$ nor $K_{3,3}$ minor of $G$



$K_5$ $K_{3,3}$

**Characterization**

**Theorem.** [Hopcroft & Tarjan 1974]
Let $G$ be a graph with $n$ vertices. There is an $\mathcal{O}(n)$-time algorithm to test whether $G$ is planar.

**Recognition**

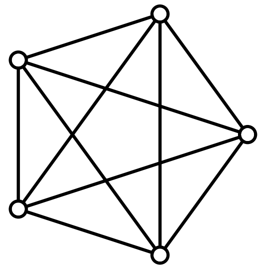
Also computes a planar embedding in $\mathcal{O}(n)$.

**Theorem.** [Wagner 1936, Fáry 1948, Stein 1951]
Every planar graph has an planar drawing where the edges are straight-line segments.

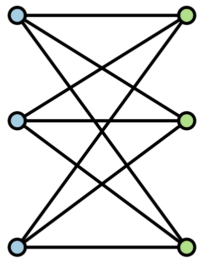
**Drawing**

# Towards Straight-Line Drawings

**Theorem.** [Kuratowski 1930]
$G$ planar $\Leftrightarrow$
neither $K_5$ nor $K_{3,3}$ minor of $G$



$K_5$



$K_{3,3}$

**Characterization**

**Theorem.** [Hopcroft & Tarjan 1974]
Let $G$ be a graph with $n$ vertices. There is an $\mathcal{O}(n)$-time algorithm to test whether $G$ is planar.

**Recognition**

Also computes a planar embedding in $\mathcal{O}(n)$.

**Theorem.** [Wagner 1936, Fáry 1948, Stein 1951]
Every planar graph has an planar drawing where the edges are straight-line segments.

**Drawing**

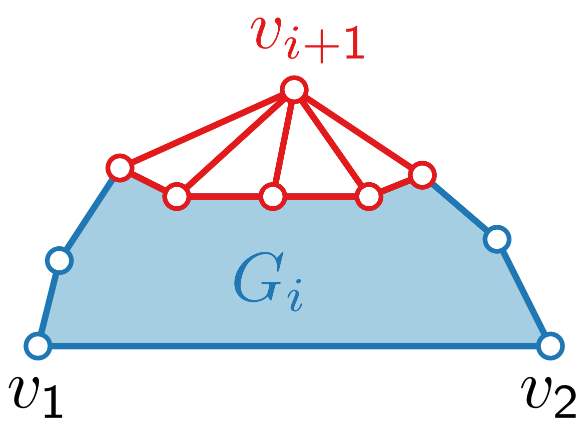
The algorithms implied by this theory produce drawings with area **not** bounded by any polynomial on $n$.

# Planar straight-line drawings

**Theorem.** [De Fraysseix, Pach, Pollack '90]
Every $n$-vertex planar graph has a planar straight-line drawing of size $(2n-4) \times (n-2)$.

**Idea.**

- Start with singe edge $(v_1, v_2)$. Let this be $G_2$.
- To obtain $G_{i+1}$, add $v_{i+1}$ to $G_i$ so that neighbours of $v_{i+1}$ are on the outer face of $G_i$.
- Neighbours of $v_{i+1}$ in $G_i$ have to form path of length at least two.

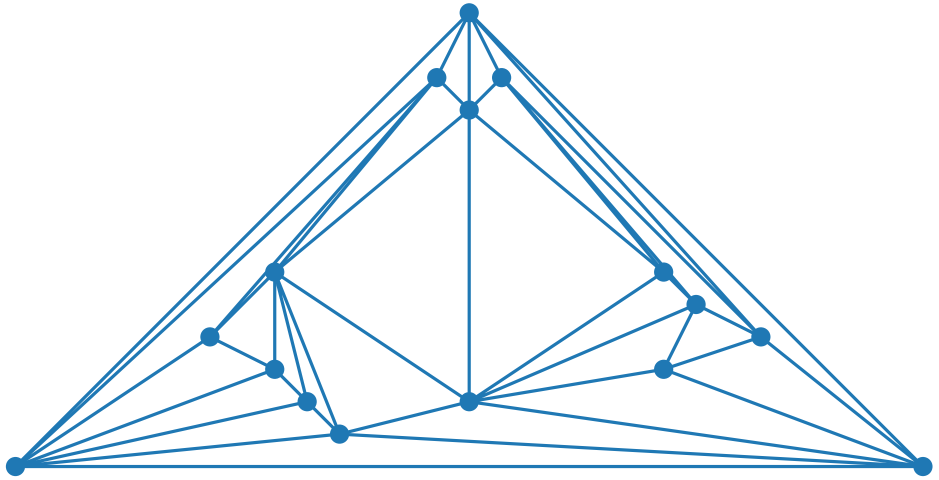



**Theorem.** [Schnyder '90]
Every $n$-vertex planar graph has a planar straight-line drawing of size $(n-2) \times (n-2)$.

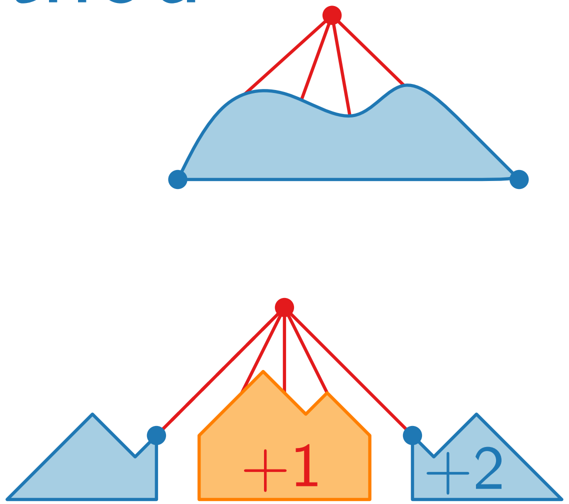
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

# Visualization of Graphs

## Lecture 3:
## Straight-Line Drawings of Planar Graphs I:
## Canonical Ordering and Shift Method



### Part II:
### Canonical Order




Jonathan Klawitter

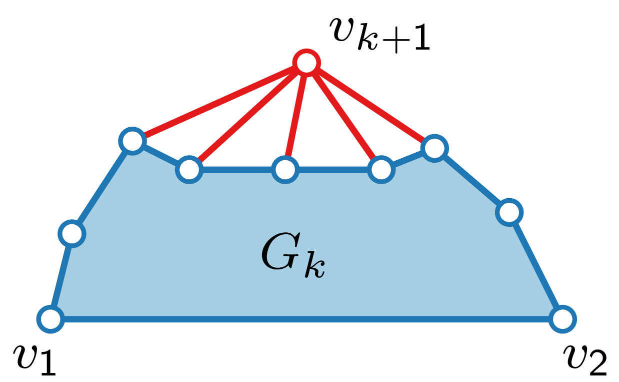
# Canonical Order – Definition

**Definition.**
Let $G = (V, E)$ be a triangulated plane graph on $n \geq 3$ vertices.
An order $\pi = (v_1, v_2, \ldots, v_n)$ is called a **canonical order**, if the following conditions hold for each $k$, $3 \leq k \leq n$:

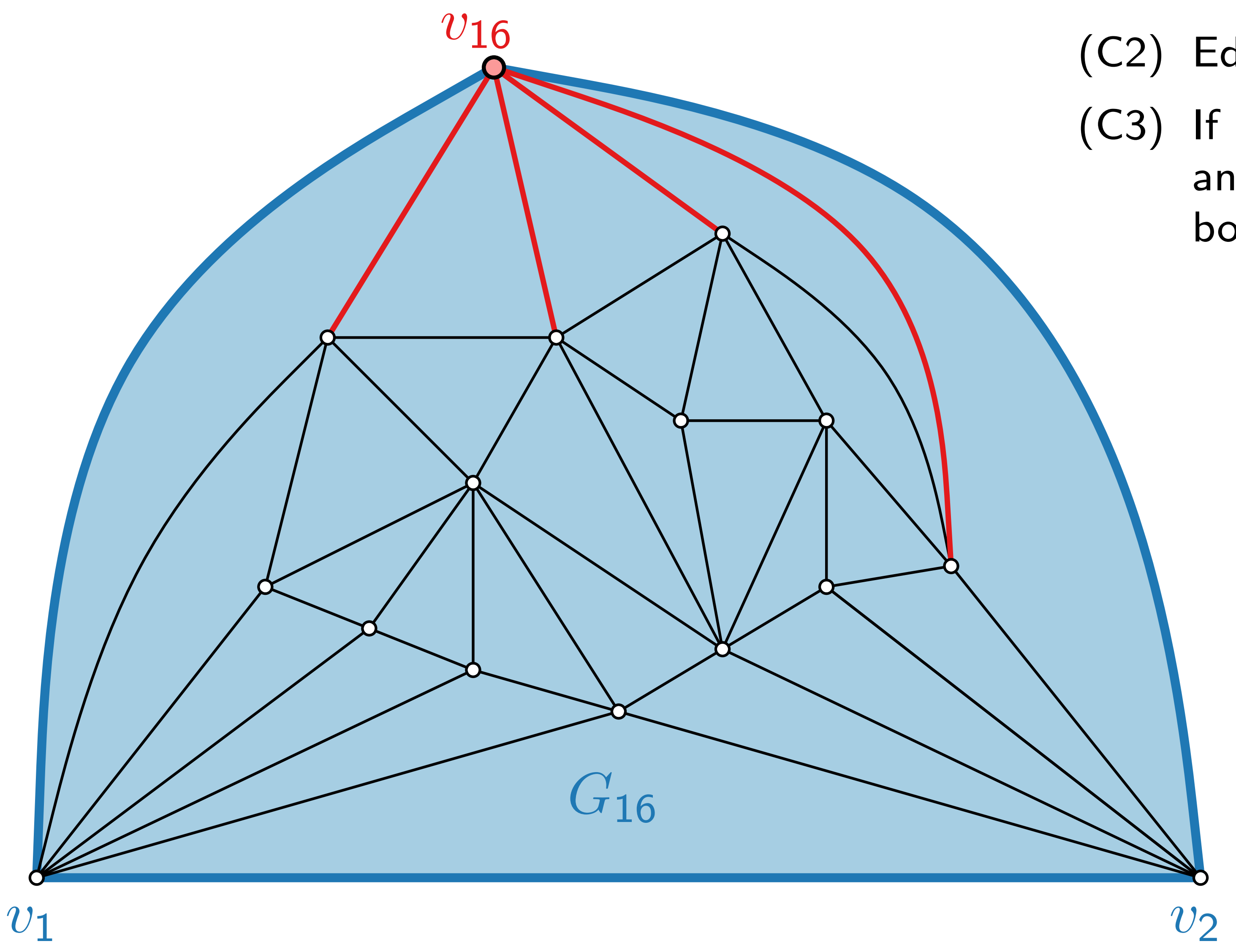
(C1) Vertices $\{v_1, \ldots v_k\}$ induce a biconnected internally triangulated graph; call it $G_k$.

(C2) Edge $(v_1, v_2)$ belongs to the outer face of $G_k$.

(C3) If $k < n$ then vertex $v_{k+1}$ lies in the outer face of $G_k$, and all neighbors of $v_{k+1}$ in $G_k$ appear on the boundary of $G_k$ consecutively.



# Canonical Order – Example

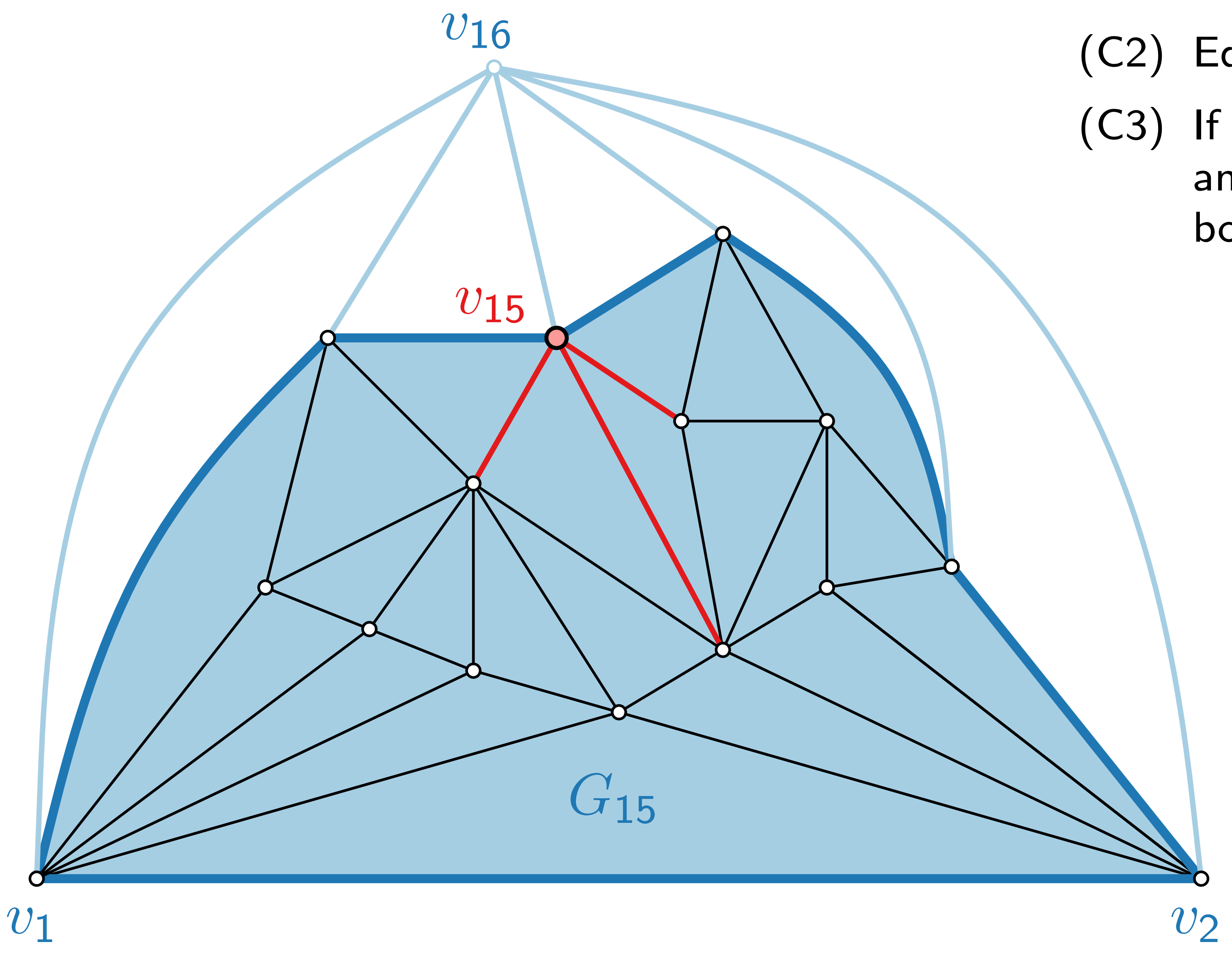



(C1) Vertices $\{v_1, \dots v_k\}$ induce a biconnected internally triangulated graph; call it $G_k$.

(C2) Edge $(v_1, v_2)$ belongs to the outer face of $G_k$.

(C3) If $k < n$ then vertex $v_{k+1}$ lies in the outer face of $G_k$, and all neighbors of $v_{k+1}$ in $G_k$ appear on the boundary of $G_k$ consecutively.

# Canonical Order – Example

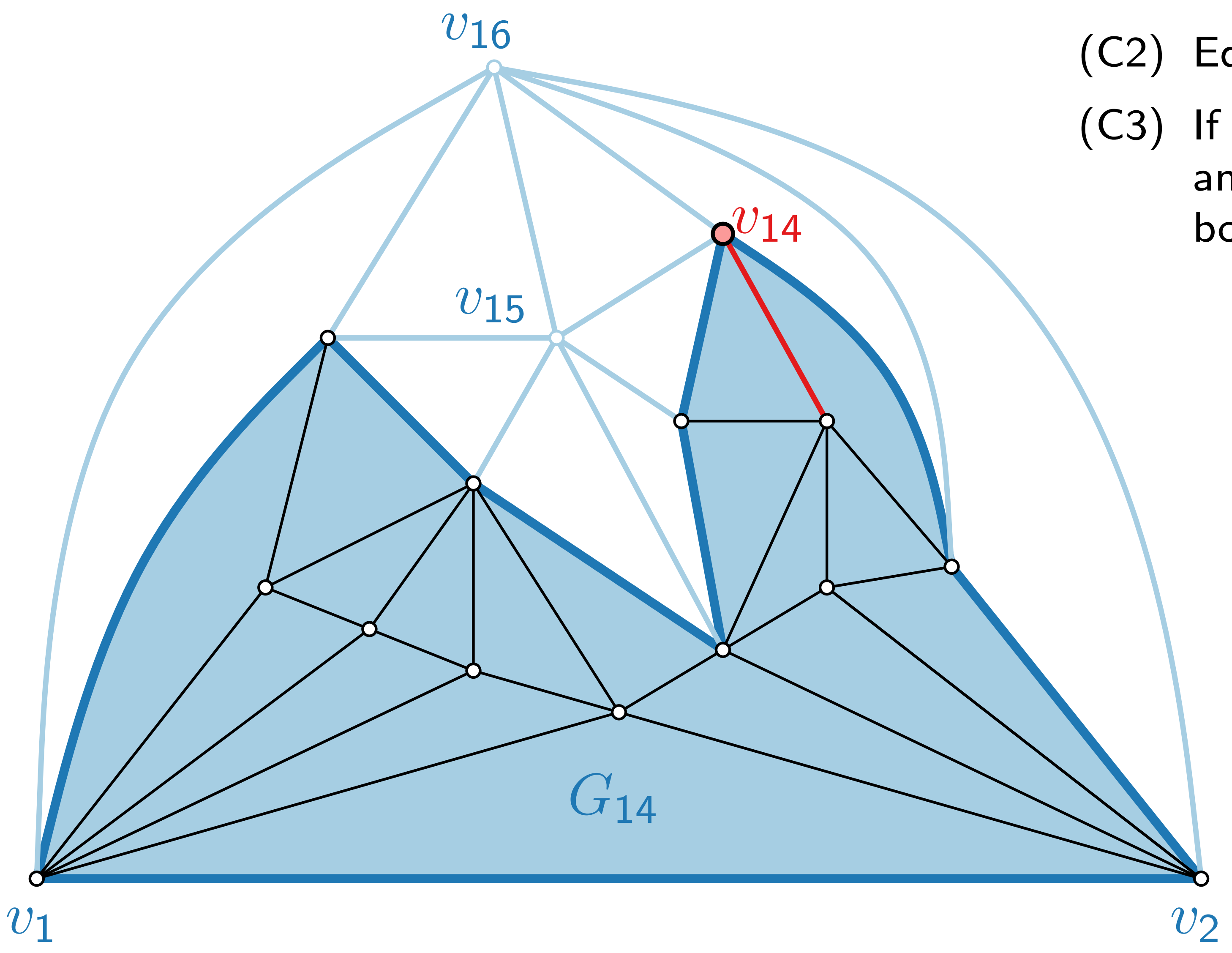



(C1) Vertices $\{v_1, \ldots v_k\}$ induce a biconnected internally triangulated graph; call it $G_k$.

(C2) Edge $(v_1, v_2)$ belongs to the outer face of $G_k$.

(C3) If $k < n$ then vertex $v_{k+1}$ lies in the outer face of $G_k$, and all neighbors of $v_{k+1}$ in $G_k$ appear on the boundary of $G_k$ consecutively.

# Canonical Order – Example

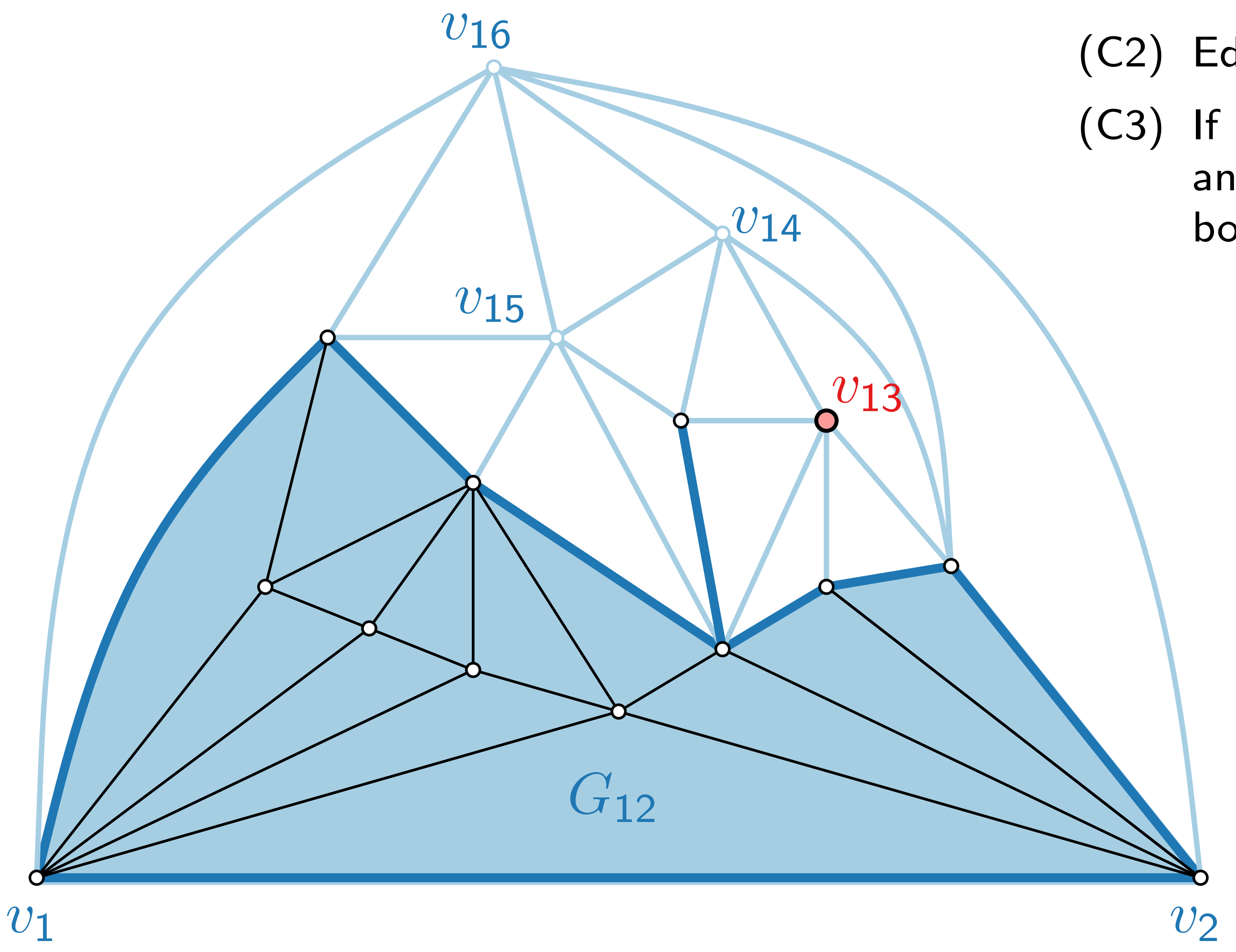



(C1) Vertices $\{v_1, \ldots v_k\}$ induce a biconnected internally triangulated graph; call it $G_k$.

(C2) Edge $(v_1, v_2)$ belongs to the outer face of $G_k$.

(C3) If $k < n$ then vertex $v_{k+1}$ lies in the outer face of $G_k$, and all neighbors of $v_{k+1}$ in $G_k$ appear on the boundary of $G_k$ consecutively.

# Canonical Order – Example

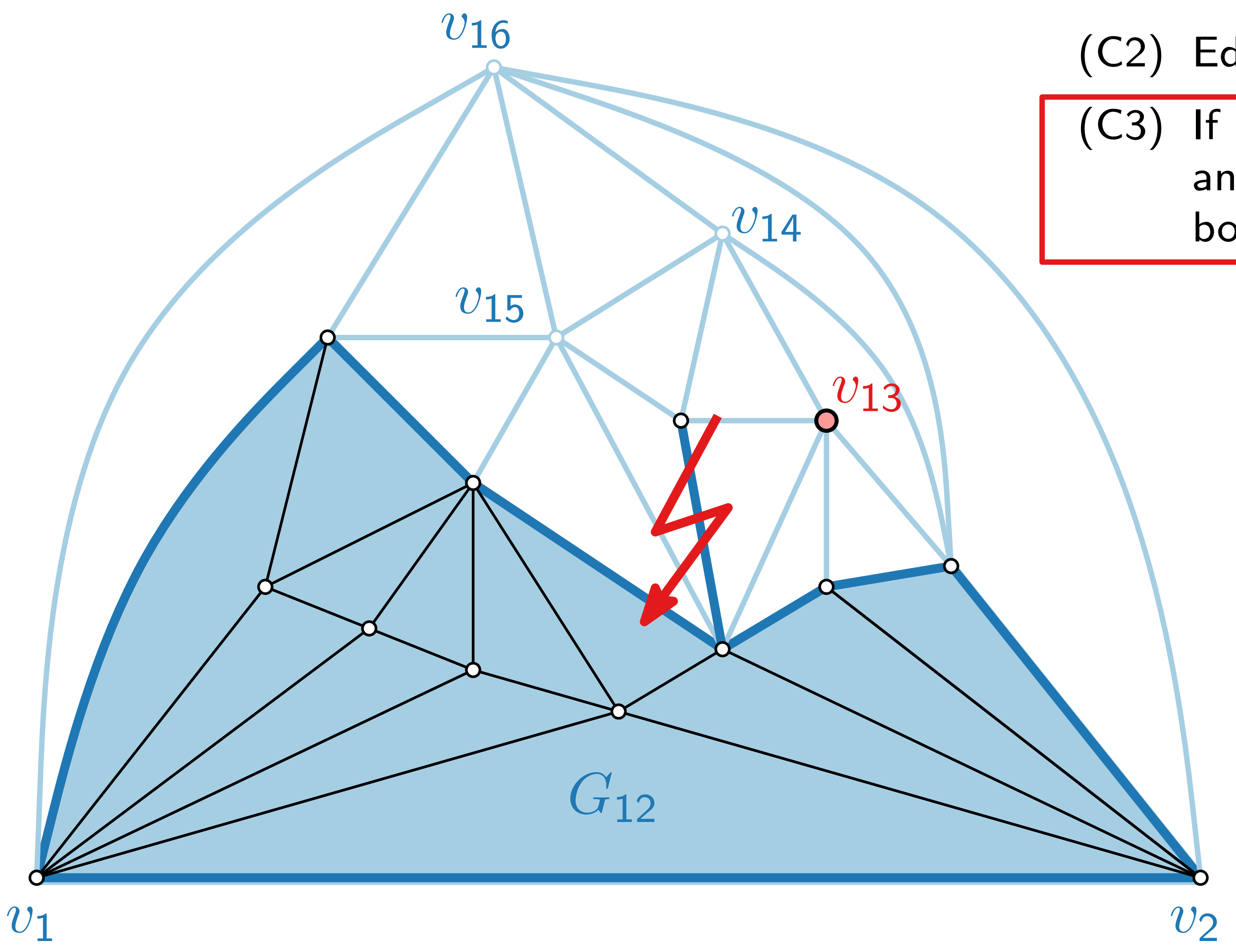



(C1) Vertices $\{v_1, \dots v_k\}$ induce a biconnected internally triangulated graph; call it $G_k$.

(C2) Edge $(v_1, v_2)$ belongs to the outer face of $G_k$.

(C3) If $k < n$ then vertex $v_{k+1}$ lies in the outer face of $G_k$, and all neighbors of $v_{k+1}$ in $G_k$ appear on the boundary of $G_k$ consecutively.

# Canonical Order – Example

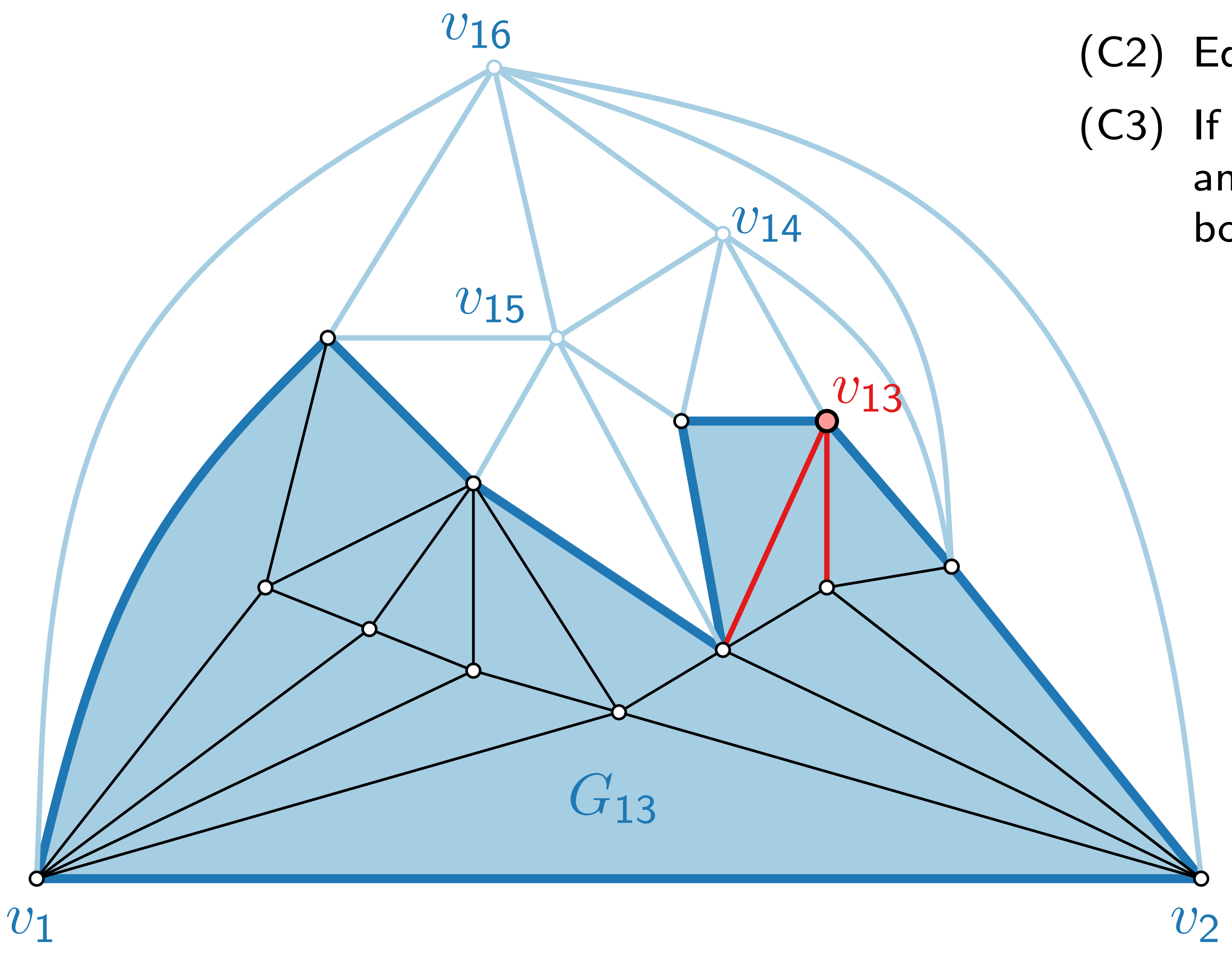



(C1) Vertices $\{v_1, \ldots v_k\}$ induce a biconnected internally triangulated graph; call it $G_k$.

(C2) Edge $(v_1, v_2)$ belongs to the outer face of $G_k$.

(C3) If $k < n$ then vertex $v_{k+1}$ lies in the outer face of $G_k$, and all neighbors of $v_{k+1}$ in $G_k$ appear on the boundary of $G_k$ consecutively.

# Canonical Order – Example

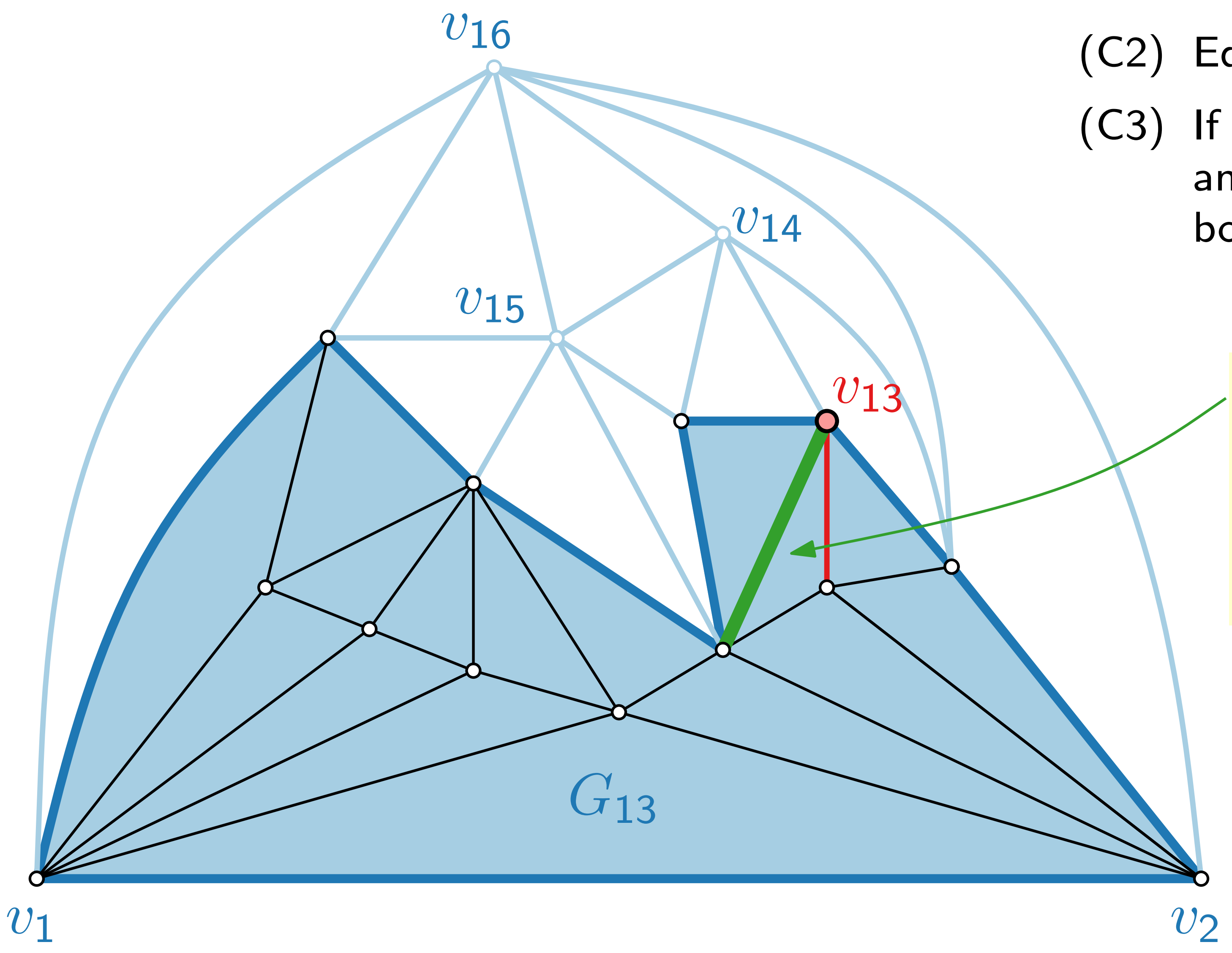



(C1) Vertices $\{v_1, \ldots v_k\}$ induce a biconnected internally triangulated graph; call it $G_k$.

(C2) Edge $(v_1, v_2)$ belongs to the outer face of $G_k$.

(C3) If $k < n$ then vertex $v_{k+1}$ lies in the outer face of $G_k$, and all neighbors of $v_{k+1}$ in $G_k$ appear on the boundary of $G_k$ consecutively.

# Canonical Order – Example



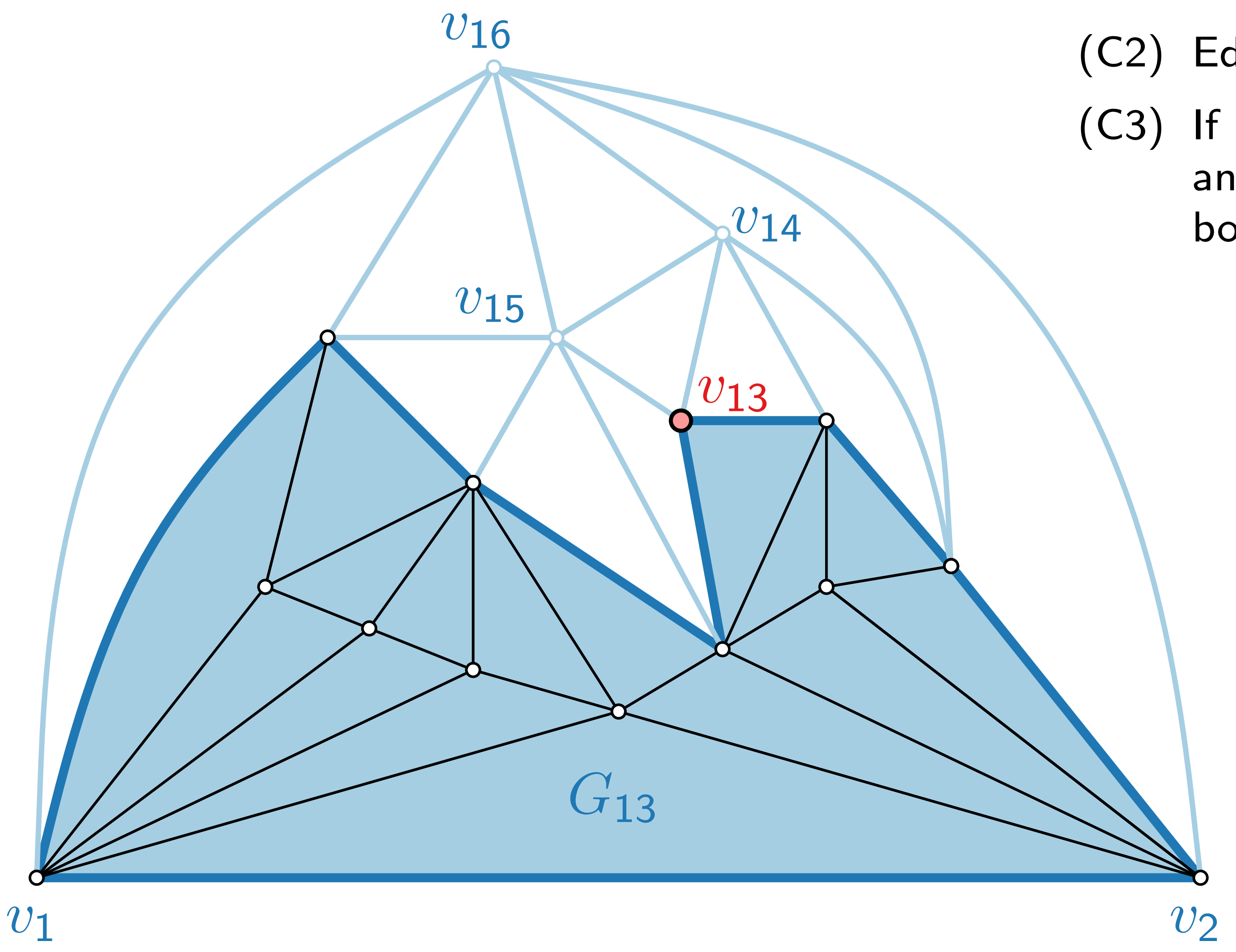

(C1) Vertices $\{v_1, \ldots v_k\}$ induce a biconnected internally triangulated graph; call it $G_k$.

(C2) Edge $(v_1, v_2)$ belongs to the outer face of $G_k$.

(C3) If $k < n$ then vertex $v_{k+1}$ lies in the outer face of $G_k$, and all neighbors of $v_{k+1}$ in $G_k$ appear on the boundary of $G_k$ consecutively.

**chord**
edge joining two nonadjacent vertices in a cycle

# Canonical Order – Example

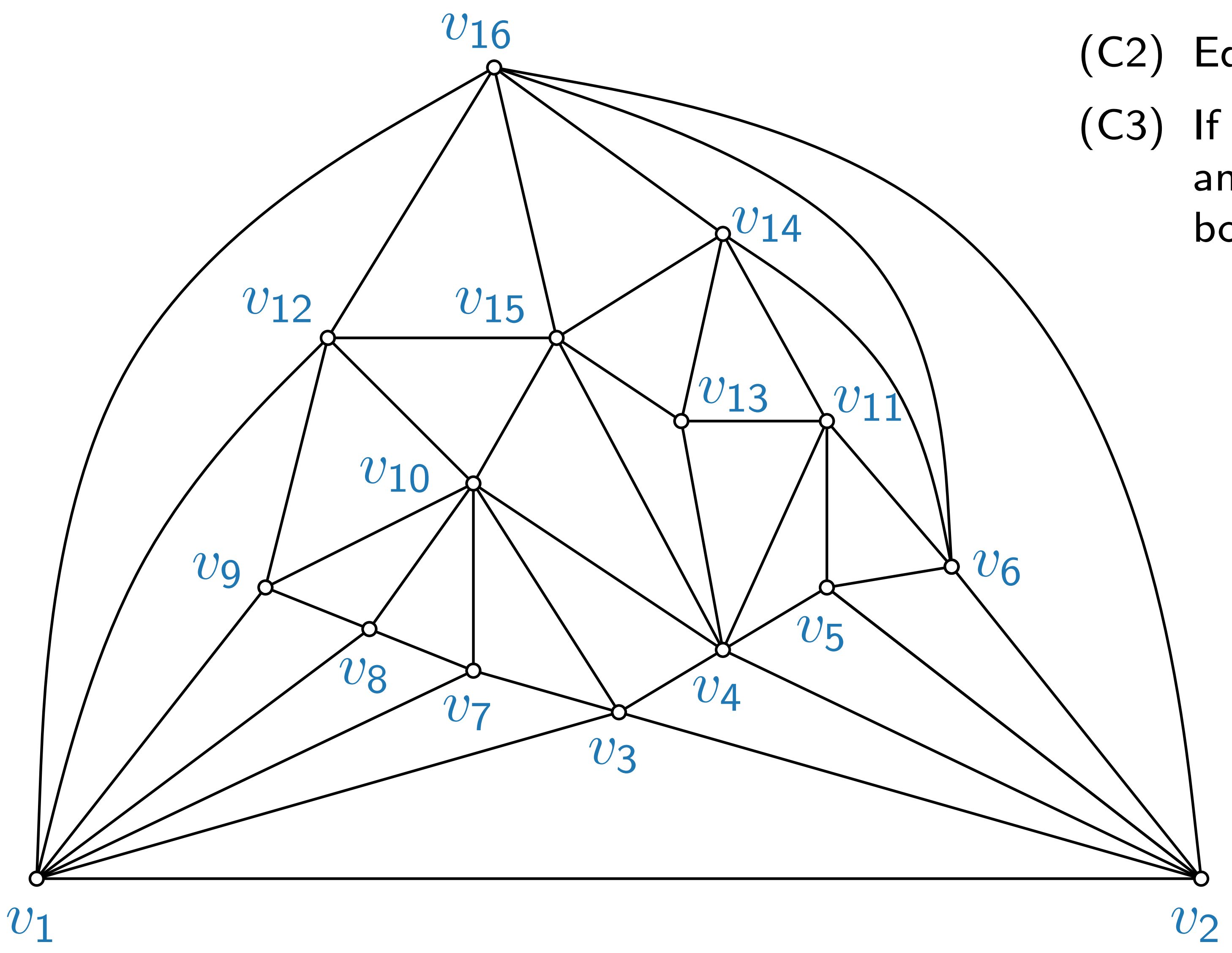



(C1) Vertices $\{v_1, \ldots v_k\}$ induce a biconnected internally triangulated graph; call it $G_k$.

(C2) Edge $(v_1, v_2)$ belongs to the outer face of $G_k$.

(C3) If $k < n$ then vertex $v_{k+1}$ lies in the outer face of $G_k$, and all neighbors of $v_{k+1}$ in $G_k$ appear on the boundary of $G_k$ consecutively.

# Canonical Order – Example




(C1) Vertices $\{v_1, \ldots v_k\}$ induce a biconnected internally triangulated graph; call it $G_k$.

(C2) Edge $(v_1, v_2)$ belongs to the outer face of $G_k$.

(C3) If $k < n$ then vertex $v_{k+1}$ lies in the outer face of $G_k$, and all neighbors of $v_{k+1}$ in $G_k$ appear on the boundary of $G_k$ consecutively.

# Canonical Order – Existence

**Lemma.**
Every triangulated plane graph has a canonical order.

(C1) $G_k$ biconnected and internally triangulated

(C2) $(v_1, v_2)$ on outer face of $G_k$

(C3) $k < n \Rightarrow v_{k+1}$ in outer face of $G_k$, neighbors of $v_{k+1}$ in $G_k$ consecutive on boundary

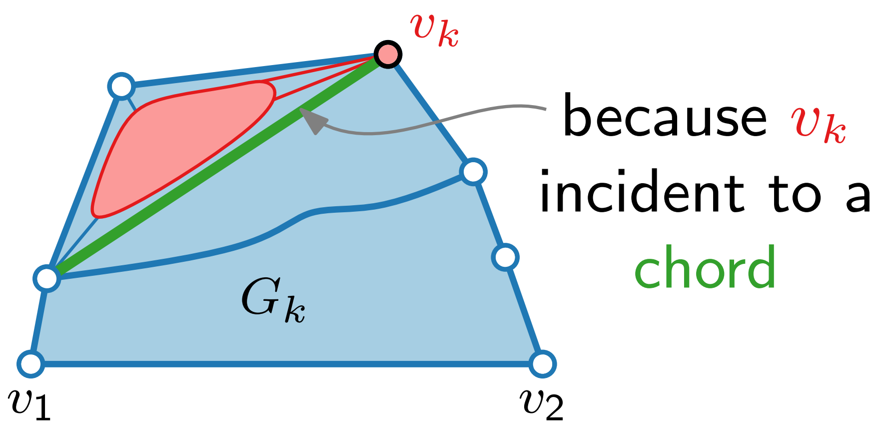
Base Case:
Let $G_n = G$, and let $v_1, v_2, v_n$ be the vertices of the outer face of $G_n$. Conditions (C1) – (C3) hold.

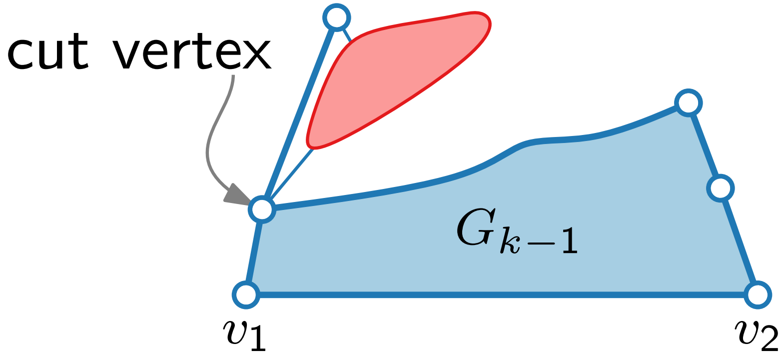
Induction hypothesis:
Vertices $v_{n-1}, \ldots, v_{k+1}$ have been chosen such that conditions (C1) – (C3) hold for $k + 1 \leq i \leq n$.

Induction step: Consider $G_k$. We search for $v_k$.

Have to show:

1. $v_k$ not incident to chord is sufficient
2. Such $v_k$ exists

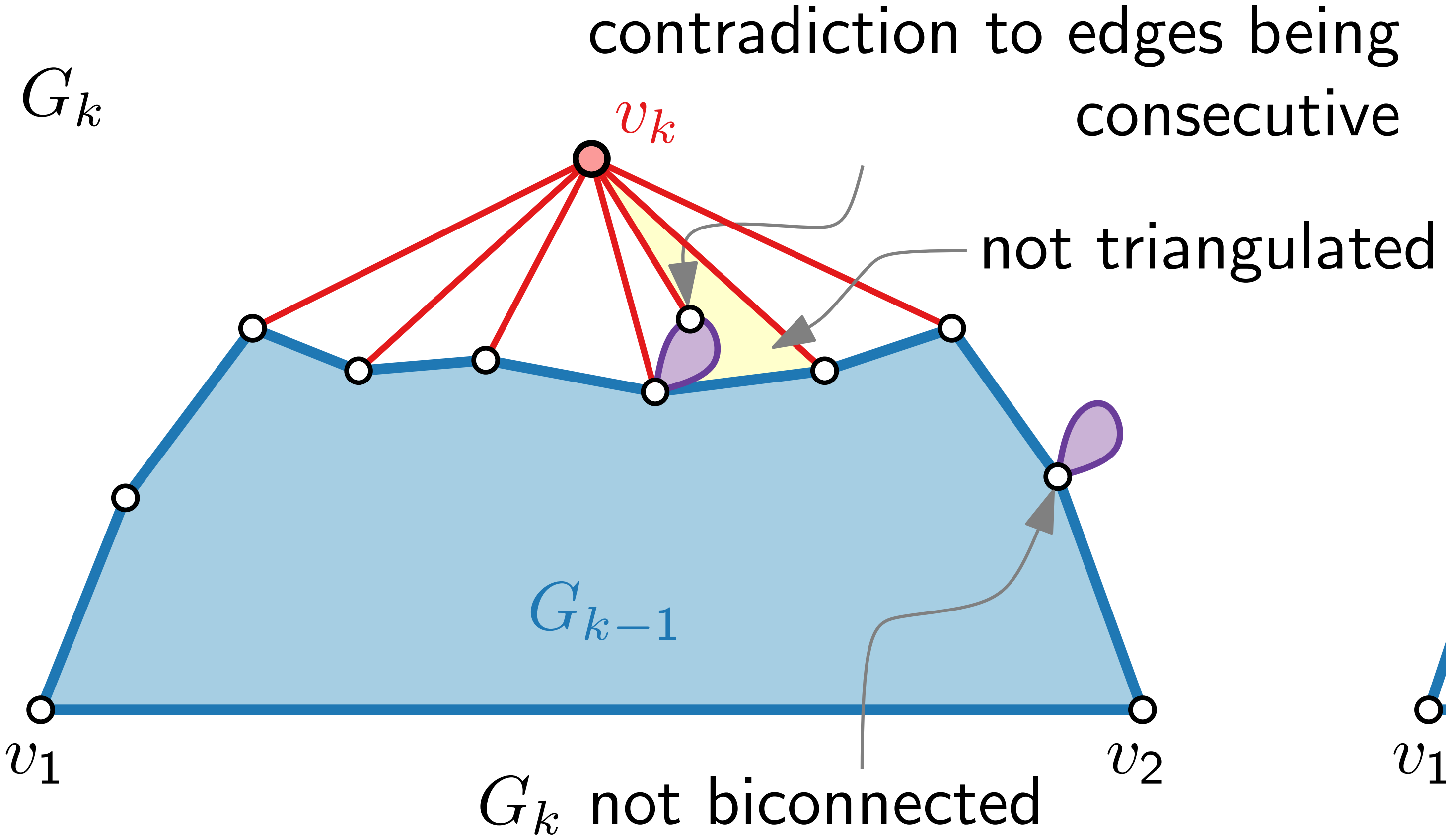


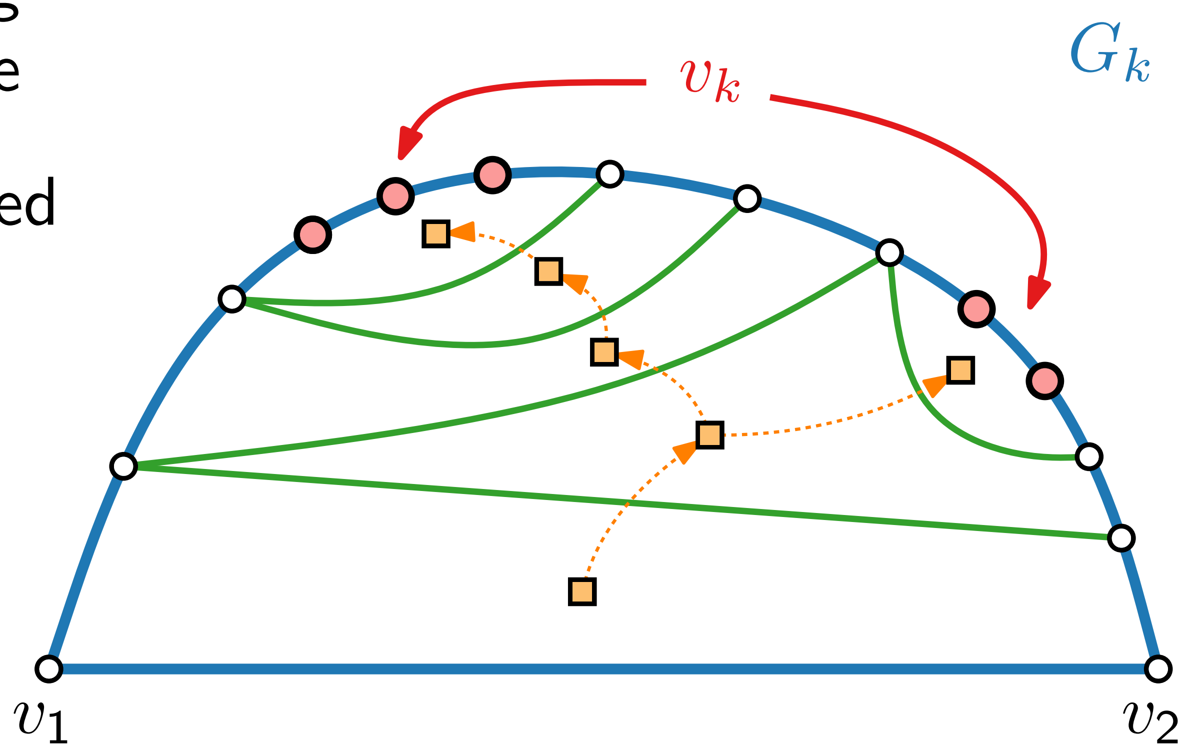



# Canonical Order – Existence

**Claim 1.**
If $v_k$ is not incident to a chord, then $G_{k-1}$ is biconnected.

**Claim 2.**
There exists a vertex in $G_k$ that is not incident to a chord as choice for $v_k$.




This completes proof of Lemma. □

# Canonical Order – Implementation

outer face

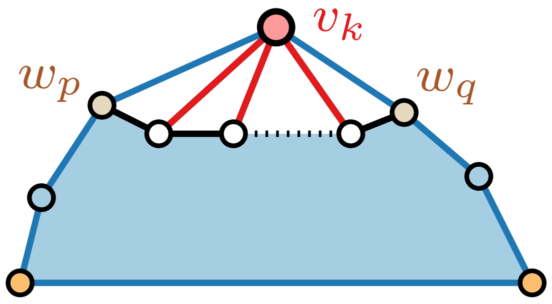
CanonicalOrder($G = (V, E), (v_1, v_2, v_n)$)

**forall** $v \in V$ **do**
  chords($v$) $\leftarrow$ 0; out($v$) $\leftarrow$ false; mark($v$) $\leftarrow$ false

mark($v_1$), mark($v_2$), out($v_1$), out($v_2$), out($v_n$) $\leftarrow$ true

**for** $k = n$ **to** 3 **do**
  choose $v$ such that mark($v$) $=$ false, out($v$) $=$ true,
   and chords($v$) $= 0$ *// keep list with candidates*
  $v_k \leftarrow v$; mark($v$) $\leftarrow$ true
  *// Let $w_1 = v_1, w_2, \ldots, w_{t-1}, w_t = v_2$ denote the boundary of $G_{k-1}$ and let $w_p, \ldots, w_q$ be the unmarked neighbors of $v_k$*
  out($w_i$) $\leftarrow$ true for all $p < i < q$ *// $O(n)$ in total*
  update number of chords for $w_i$
  and its neighbours *// $O(m) = O(n)$ in total*

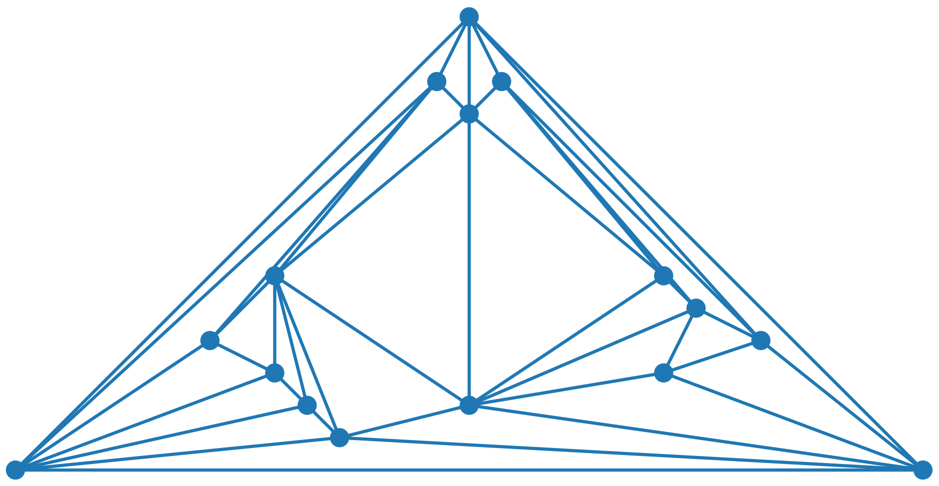
- chord($v$):
  \# chords adjacent to $v$
- out($v$) $=$ true iff $v$ is currently outer vertex
- mark($v$) $=$ true iff $v$ has received its number




**Lemma.**
Algorithm CanonicalOrder computes a canonical order of a plane graph in $\mathcal{O}(n)$ time.

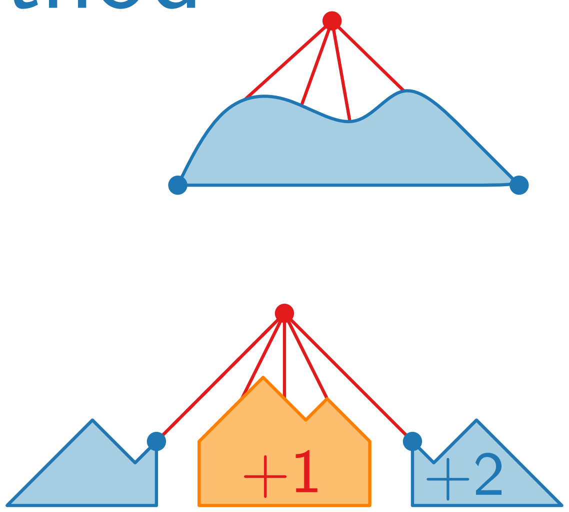
# Visualization of Graphs

## Lecture 3:
## Straight-Line Drawings of Planar Graphs I:
## Canonical Ordering and Shift Method



### Part III:
### Shift Method

Jonathan Klawitter

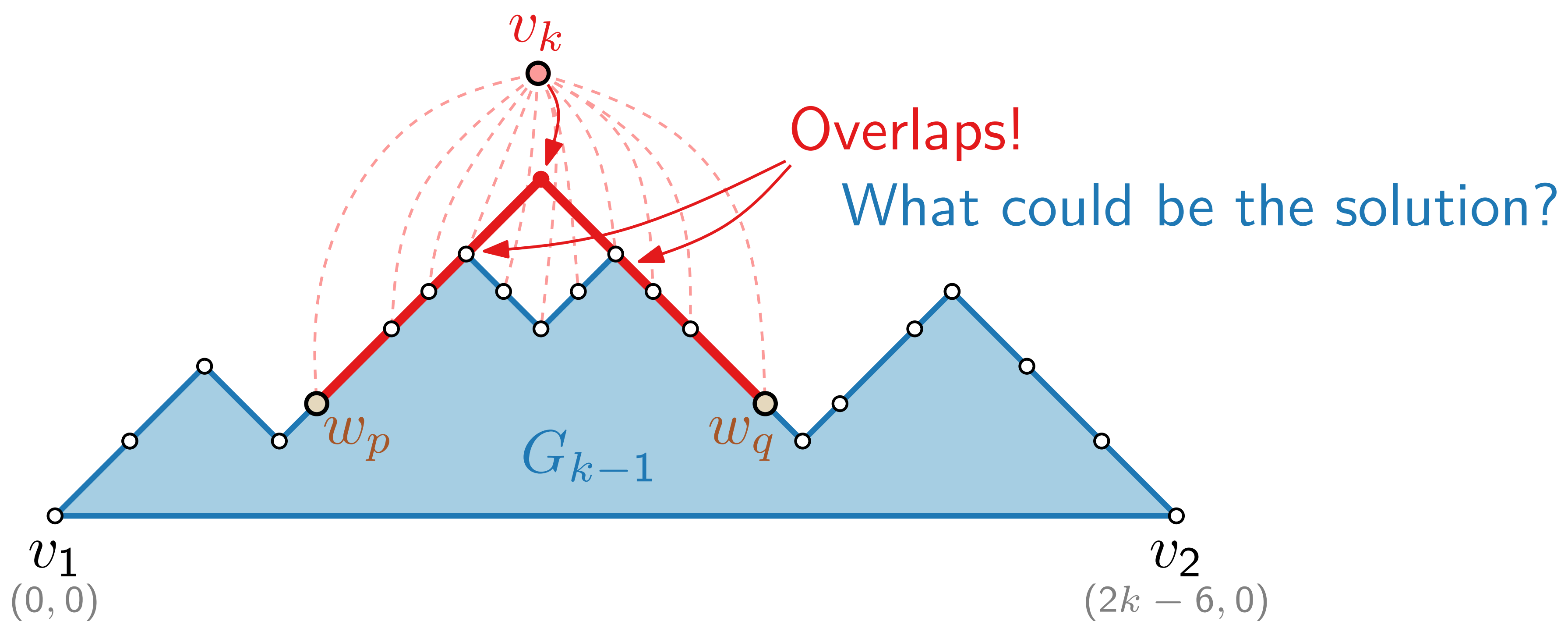


# Shift Method – Idea

**Drawing invariants:**

$G_{k-1}$ is drawn such that

- $v_1$ is on $(0,0)$, $v_2$ is on $(2k-6,0)$,
- boundary of $G_{k-1}$ (minus edge $(v_1,v_2)$) is drawn $x$-monotone,
- each edge of the boundary of $G_{k-1}$ (minus edge $(v_1,v_2)$) is drawn with slopes $\pm 1$.



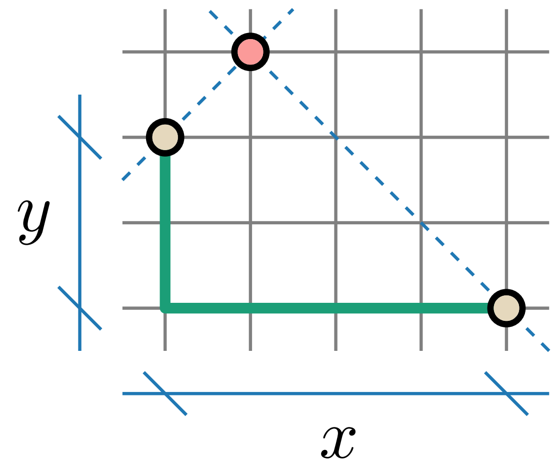
# Shift Method – Idea

**Drawing invariants:**

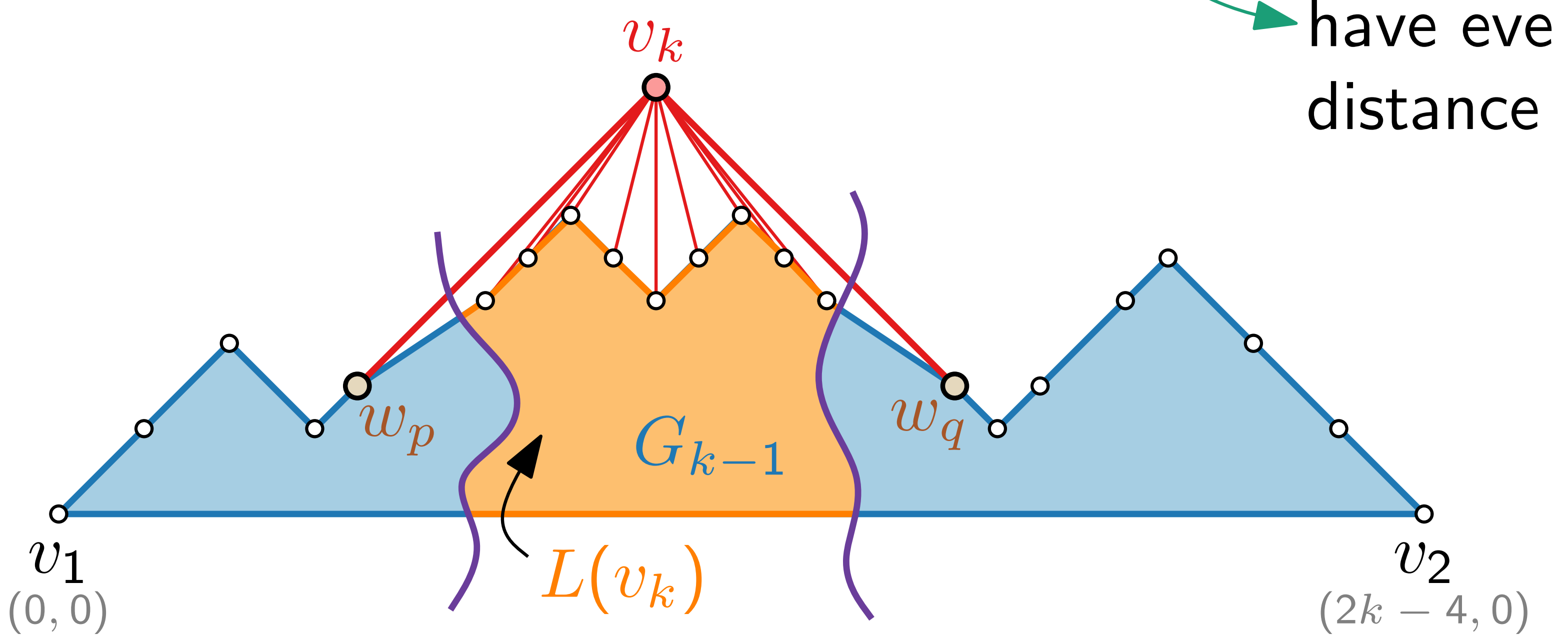
$G_{k-1}$ is drawn such that

- $v_1$ is on $(0,0)$, $v_2$ is on $(2k-6,0)$,
- boundary of $G_{k-1}$ (minus edge $(v_1,v_2)$) is drawn $x$-monotone,
- each edge of the boundary of $G_{k-1}$ (minus edge $(v_1,v_2)$) is drawn with slopes $\pm 1$.

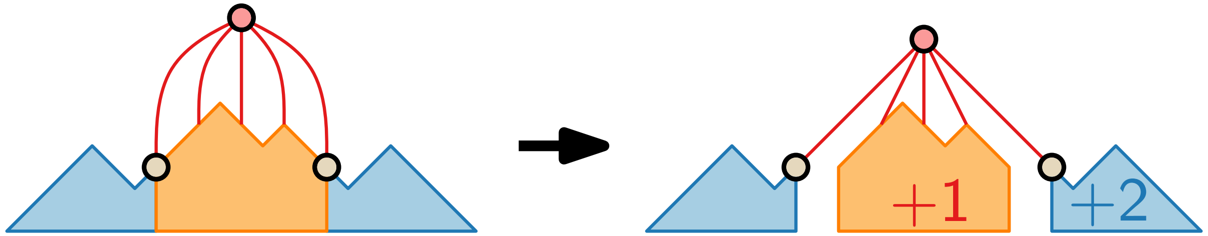
**Does $v_k$ land on grid?**

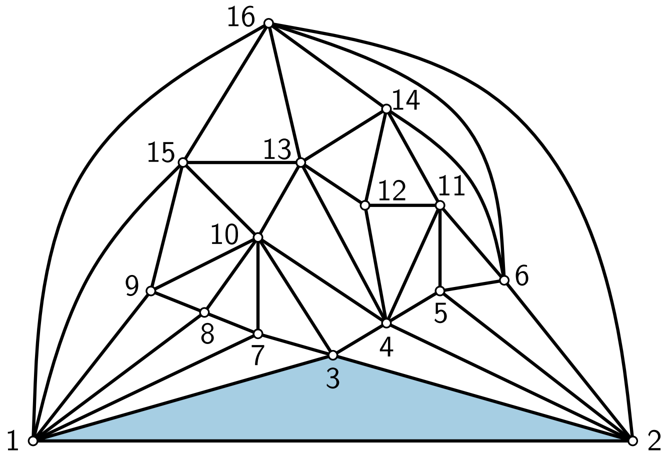


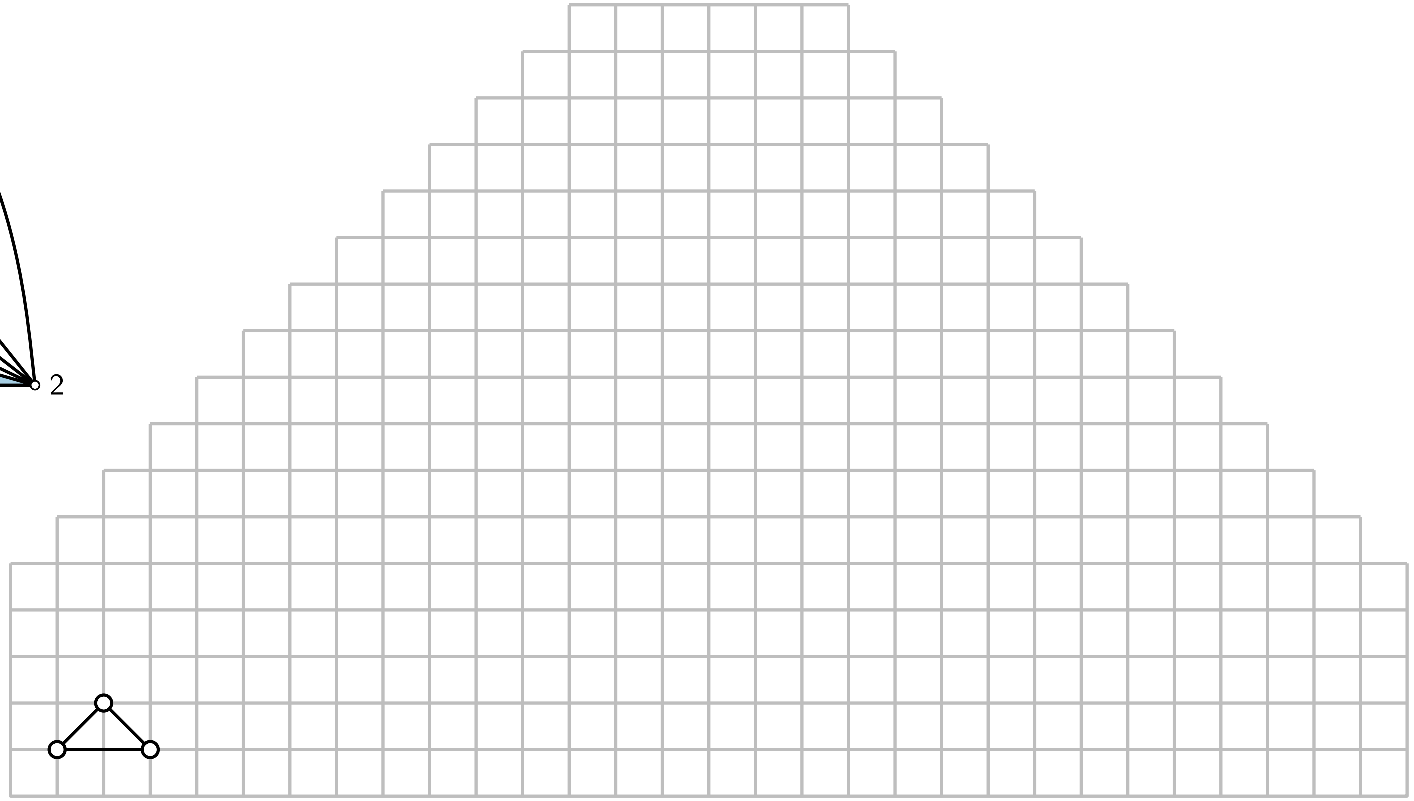

yes, beause $w_p$ and $w_q$ have even Manhattan distance



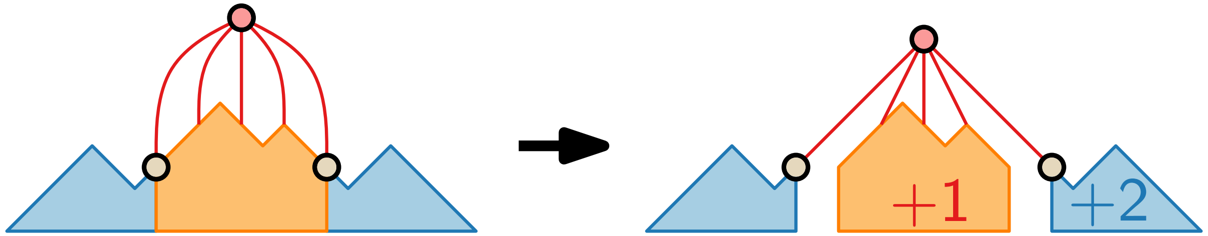
# Shift Method – Example

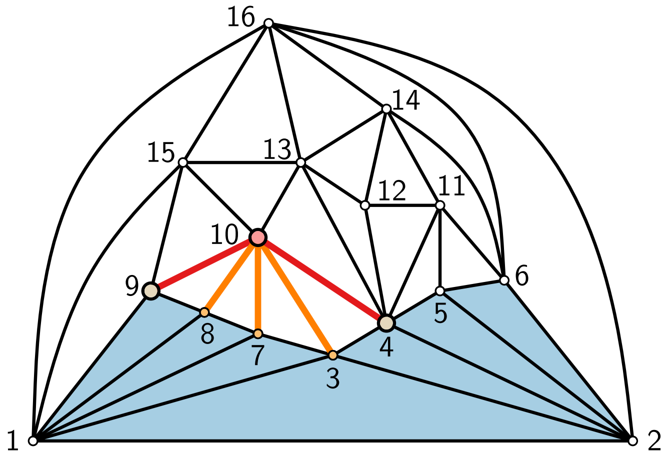


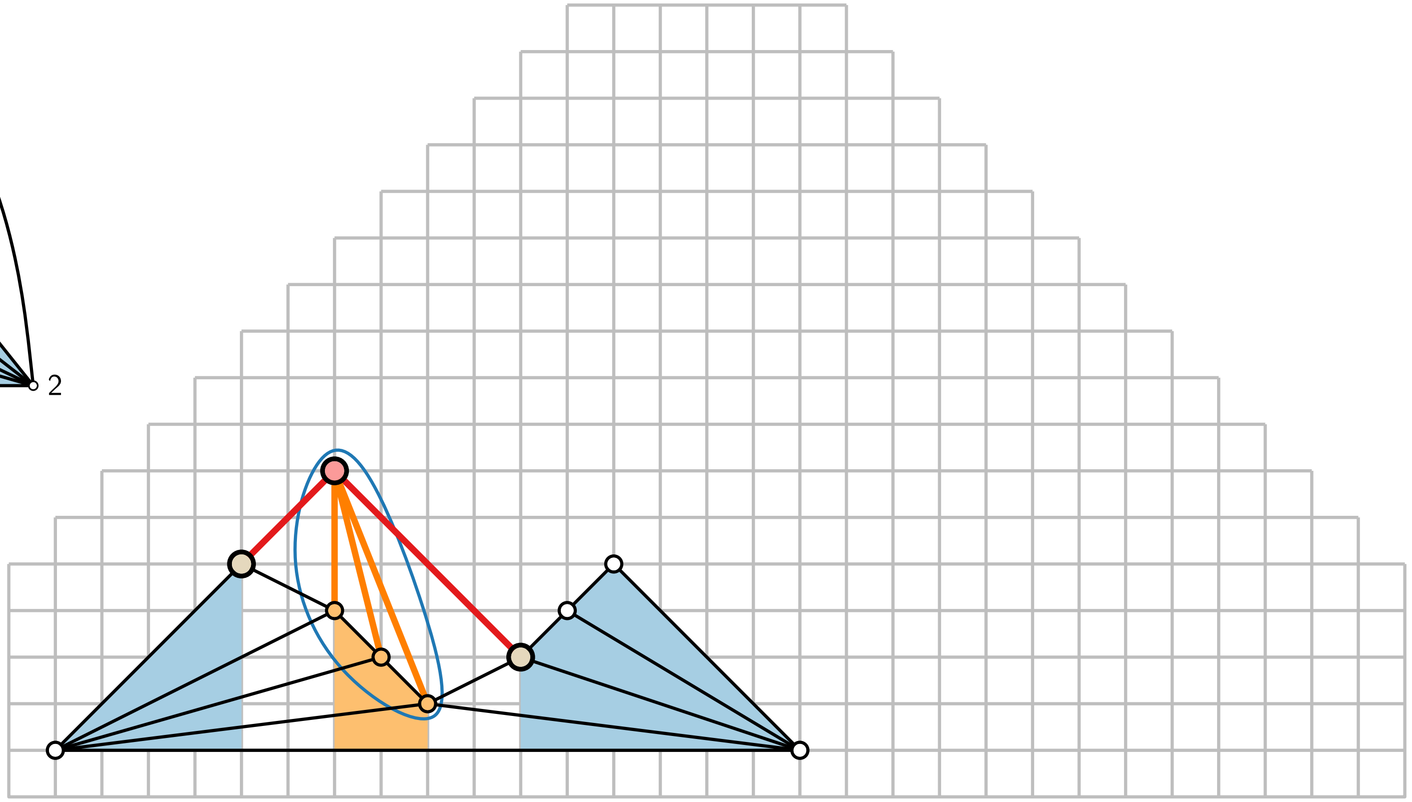



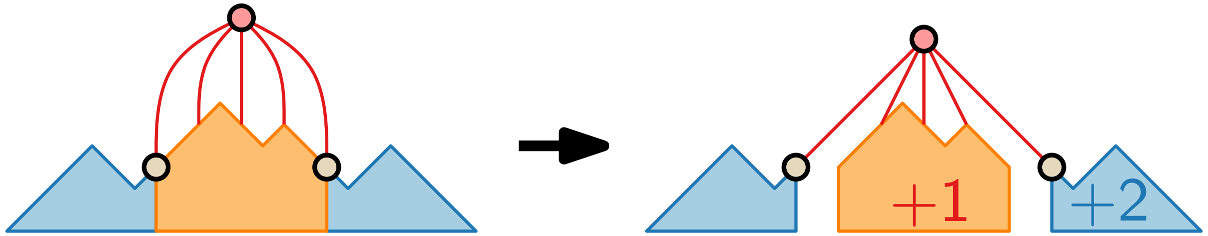
# Shift Method – Example

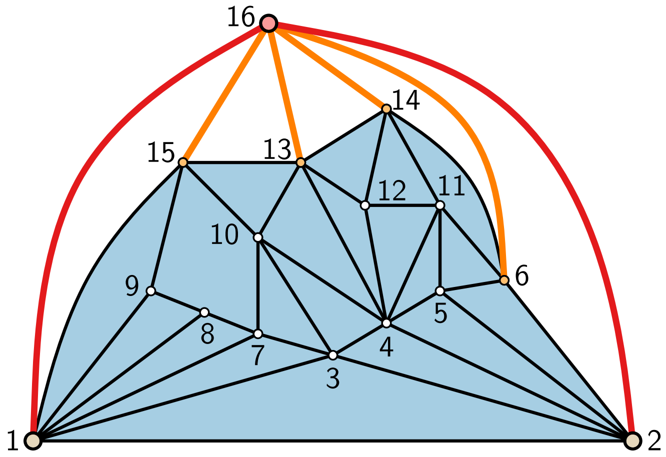


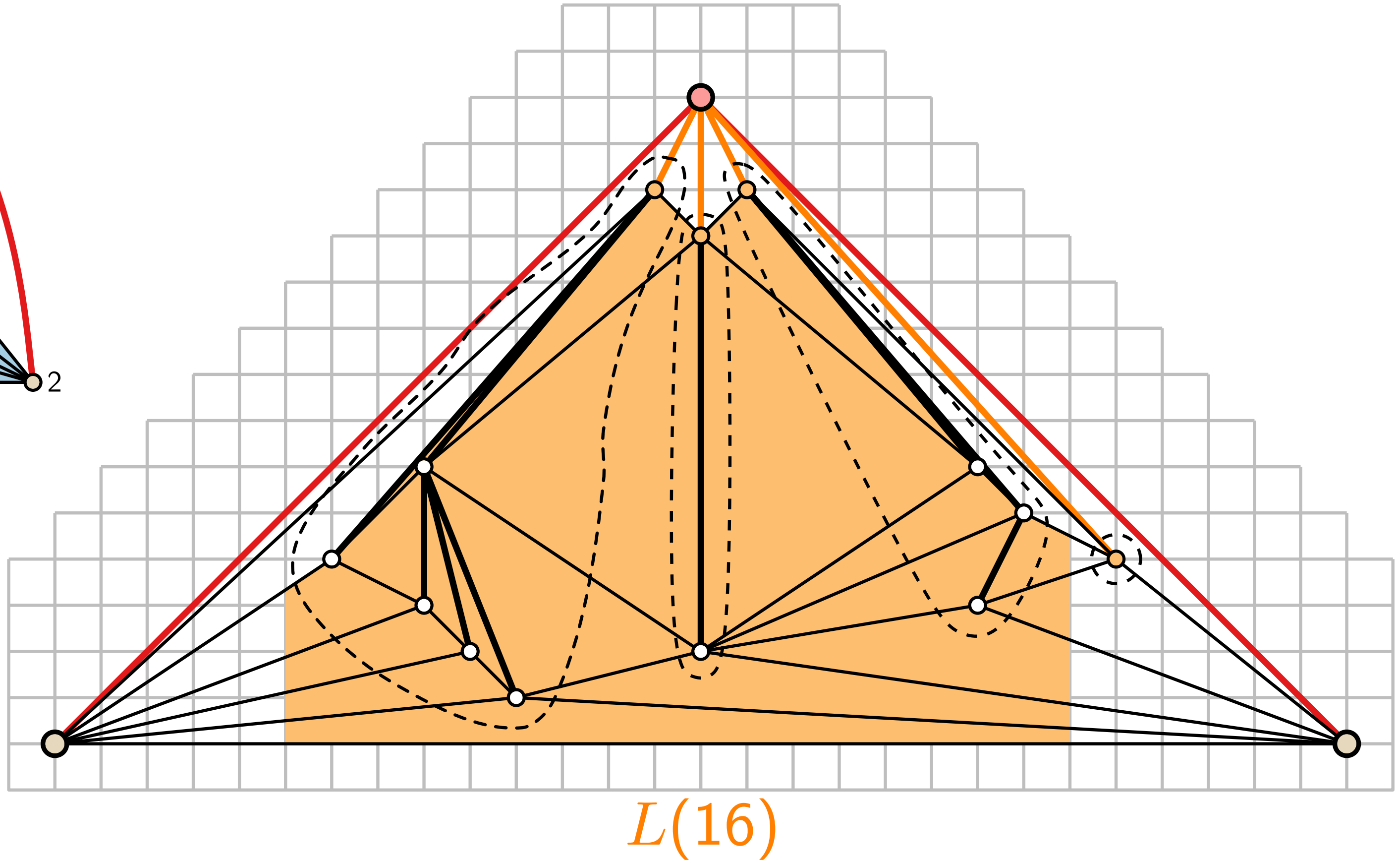



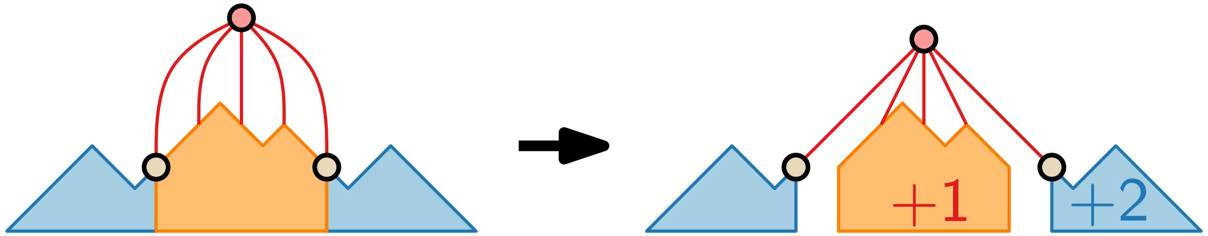
# Shift Method – Example

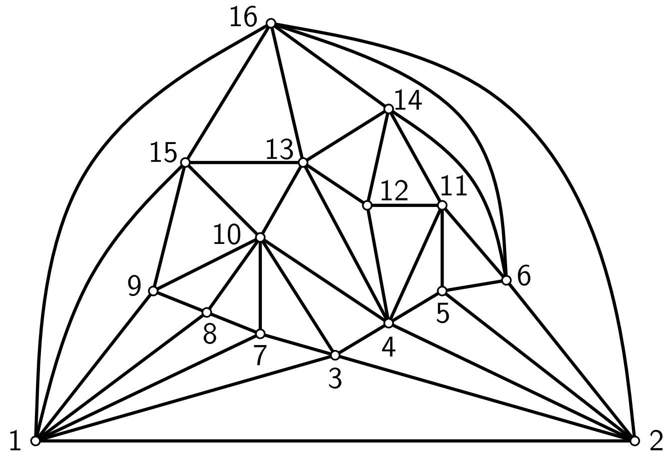


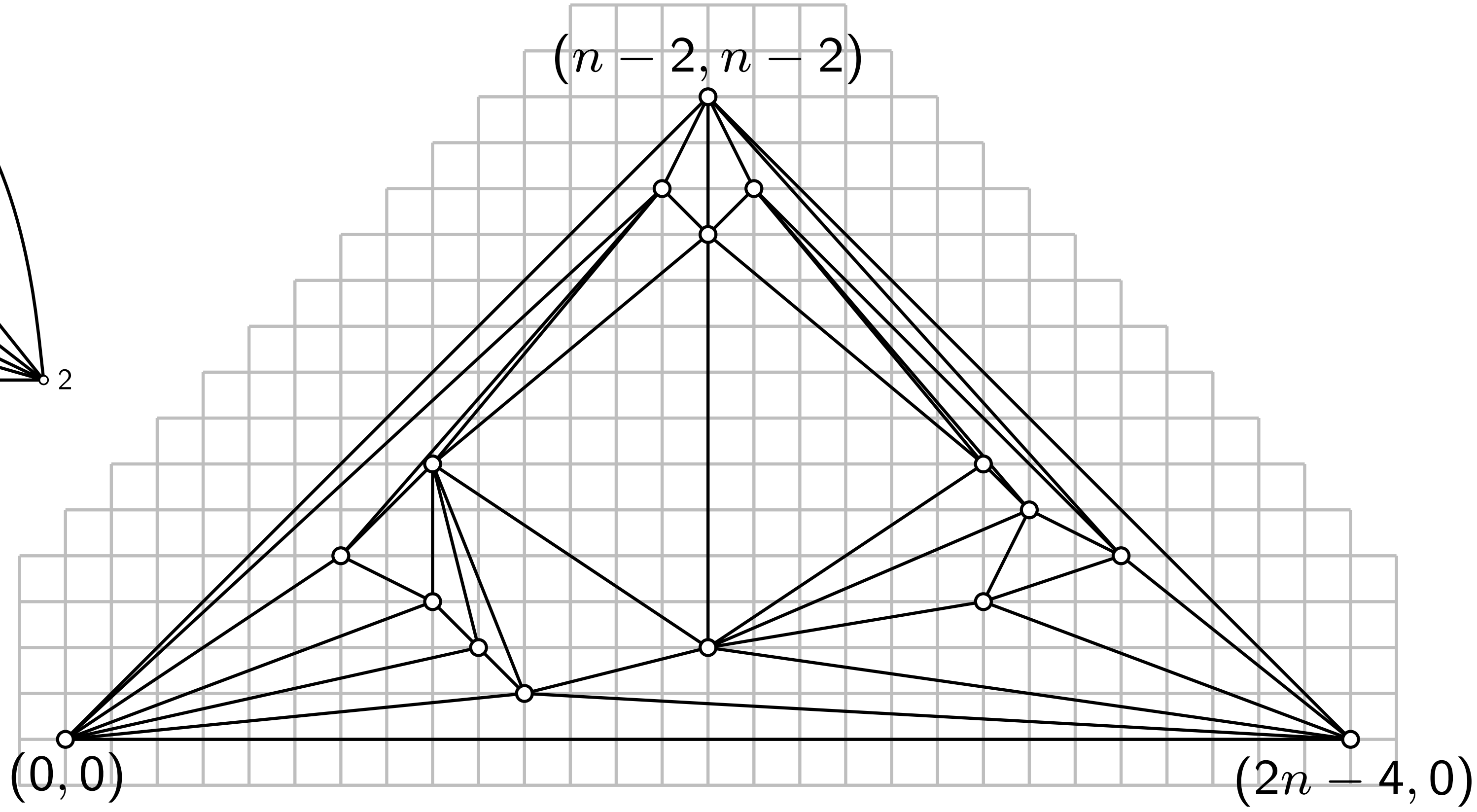



# Shift Method – Example



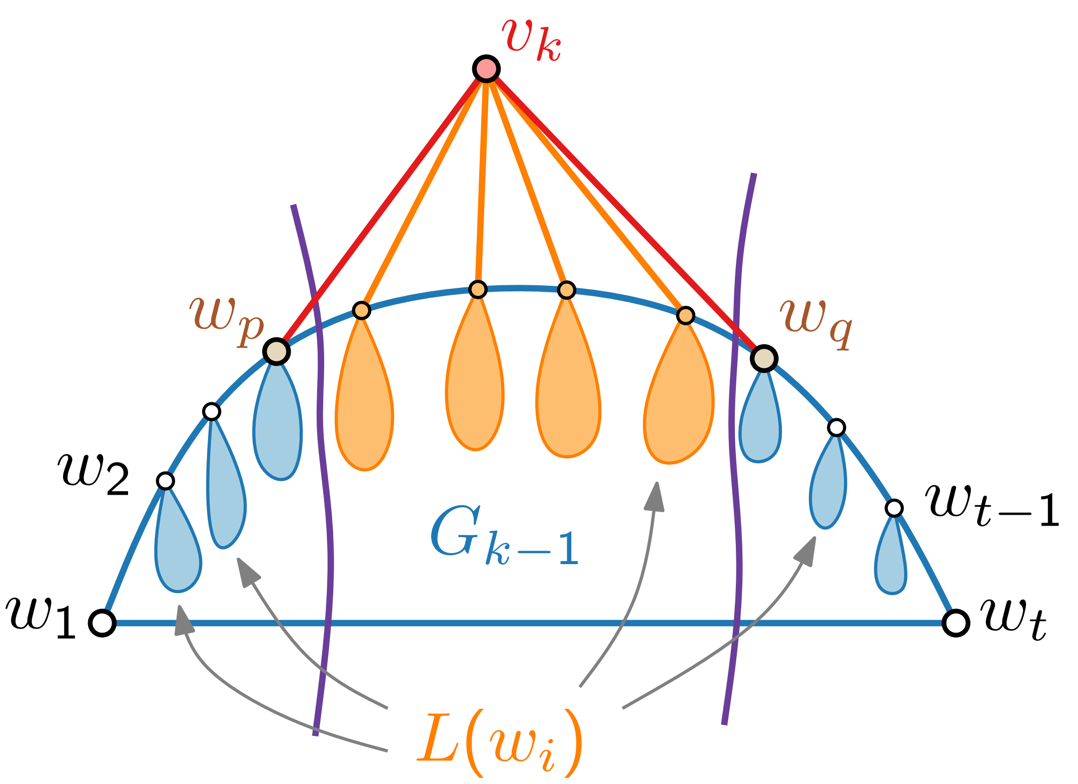



# Shift Method – Planarity

**Observations.**

- Each internal vertex is covered exactly once.
- Covering relation defines a tree in $G$
- and a forest in $G_i$, $1 \leq i \leq n-1$.




**Lemma.**
Let $0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_t \in \mathbb{N}$, such that $\delta_q - \delta_p \geq 2$ and even.
If we shift $L(w_i)$ by $\delta_i$ to the right, then we get a planar straight-line drawing.

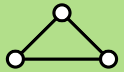
Proof by induction:
If $G_{k-1}$ is drawn planar and straight-line, then so is $G_k$.

# Shift Method – Pseudocode

Let $v_1, \ldots, v_n$ be a canonical order of $G$

**for** $i = 1$ to 3 **do**

  $L(v_i) \leftarrow \{v_i\}$

$P(v_1) \leftarrow (0,0)$; $P(v_2) \leftarrow (2,0)$, $P(v_3) \leftarrow (1,1)$ 

**for** $i = 4$ to $n$ **do**

  Let $w_1 = v_1, w_2, \ldots, w_{t-1}, w_t = v_2$
  denote the boundary of $G_{i-1}$
  and let $w_p, \ldots, w_q$ be the neighbours of $v_i$

  **for** $\forall v \in \cup_{j=p+1}^{q-1} L(w_j)$ **do** // $\mathcal{O}(n^2)$ *in total*

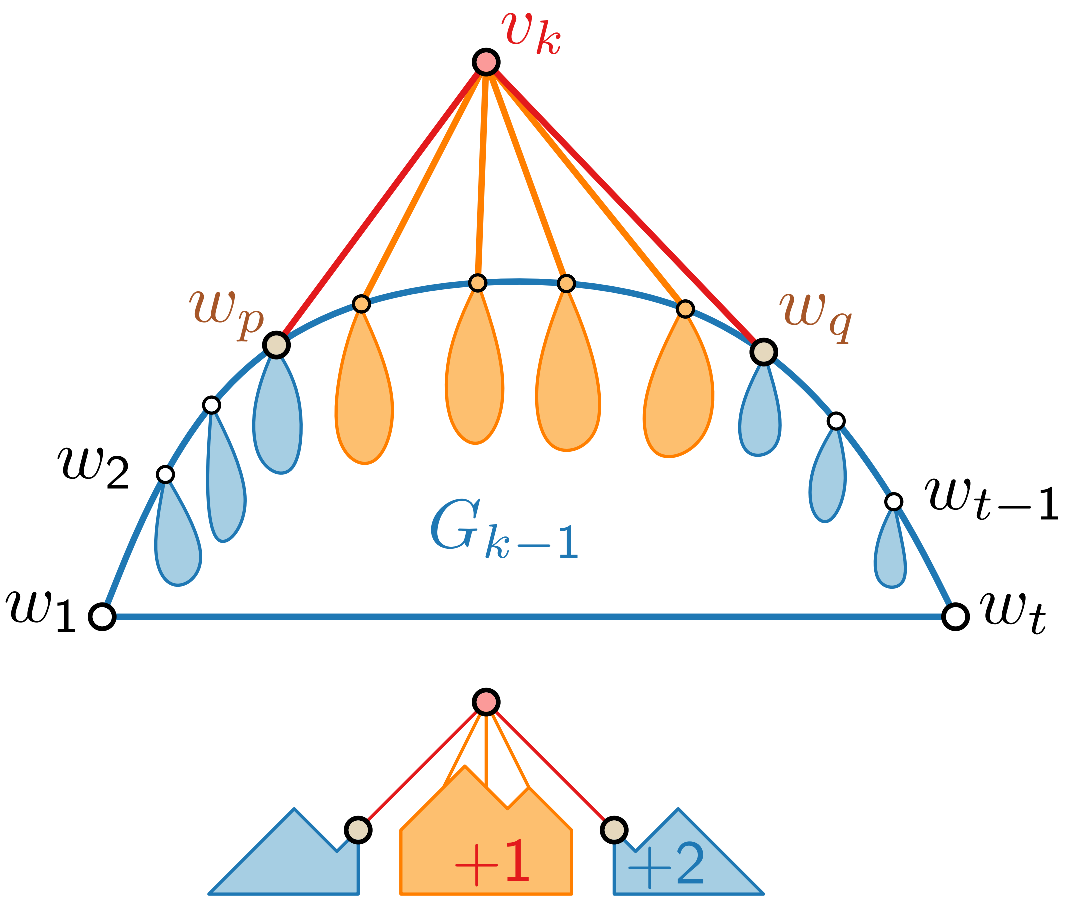
    $x(v) \leftarrow x(v) + 1$

  **for** $\forall v \in \cup_{j=q}^{t} L(w_j)$ **do** // $\mathcal{O}(n^2)$ *in total*

    $x(v) \leftarrow x(v) + 2$

  $P(v_i) \leftarrow$ intersection of $+1/-1$ diagonals
  through $P(w_p)$ and $P(w_q)$

  $L(v_i) \leftarrow \cup_{j=p+1}^{q-1} L(w_j) \cup \{v_i\}$

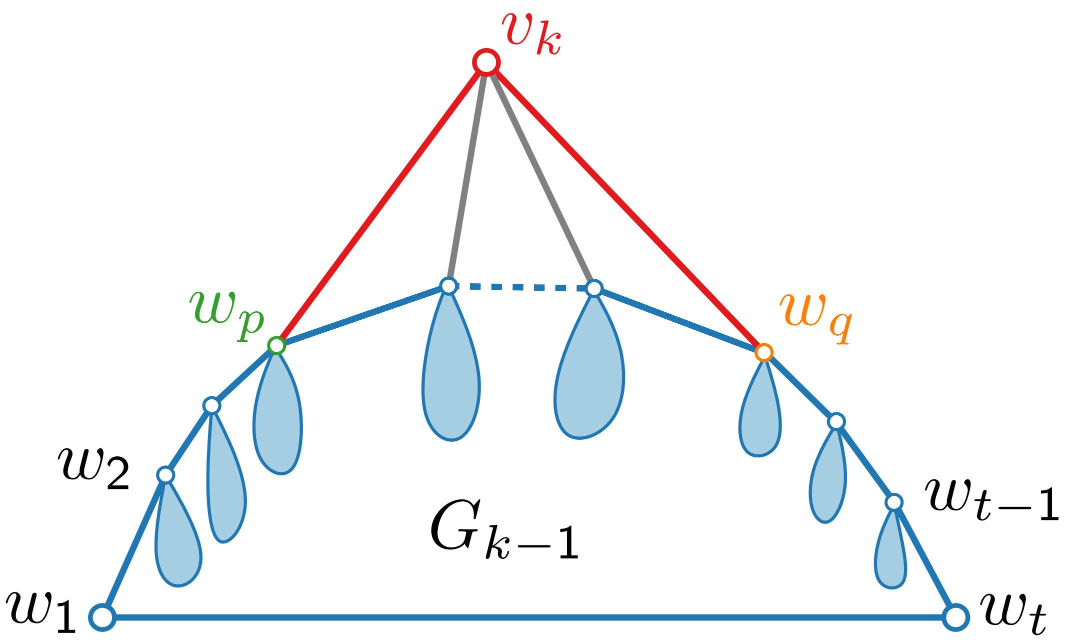



**Running Time?**

# Shift Method – Linear Time Implementation

**Idea 1.**
To compute $x(v_k)$ & $y(v_k)$,
we only need $y(w_p)$ and $y(w_q)$ and $x(w_q) - x(w_p)$

**Idea 2.**
Instead of storing explicit x-coordinates,
we store x distances.

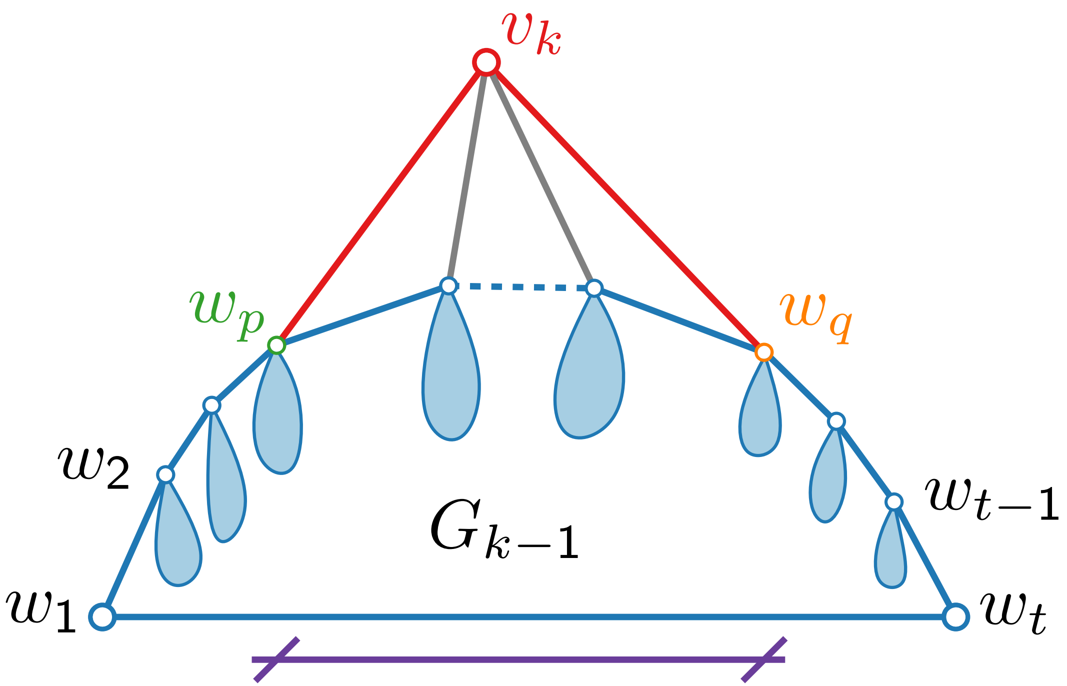



(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

(2) $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$

# Shift Method – Linear Time Implementation

**Idea 1.**
To compute $x(v_k)$ & $y(v_k)$,
we only need $y(w_p)$ and $y(w_q)$ and $x(w_q) - x(w_p)$




**Idea 2.**
Instead of storing explicit x-coordinates,
we store x distances.

After x distance for $v_n$ computed, use preorder traversal to compute all x-coordinates.

(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

(2) $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$

(3) $x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$

# Shift Method – Linear Time Implementation

**Relative x distance tree.**

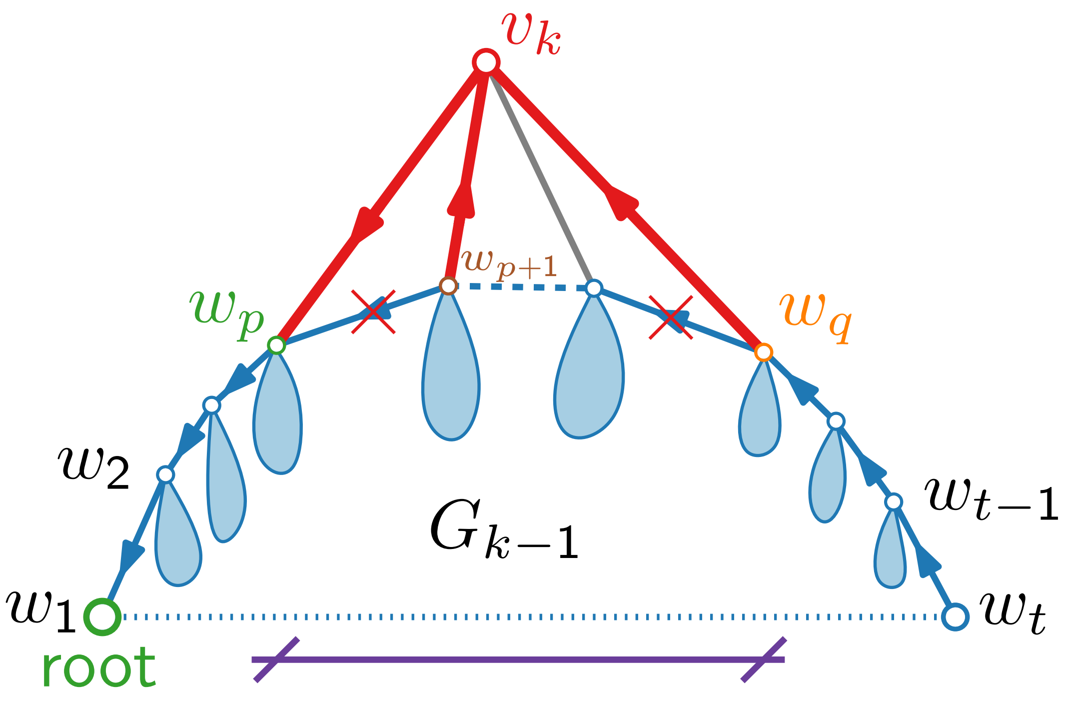
For each vertex $v$ store

- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$

**Calculations.**

- $\Delta_x(w_{p+1})$++, $\Delta_x(w_q)$++
- $\Delta_x(w_p, w_q) = \Delta_x(w_{p+1}) + \ldots + \Delta_x(w_q)$
- $\Delta_x(v_k)$ by (3)
- $y(v_k)$ by (2)
- $\Delta_x(w_q) = \Delta_x(w_p, w_q) - \Delta_x(v_k)$
- $\Delta_x(w_{p+1}) = \Delta_x(w_{p+1}) - \Delta_x(v_k)$

$\mathcal{O}(n)$ in total




(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

(2) $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$

(3) $x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$

# Literature

- [PGD Ch. 4.2] for detailed explanation of shift method
- [de Fraysseix, Pach, Pollack 1990] “How to draw a planar graph on a grid” – original paper on shift method