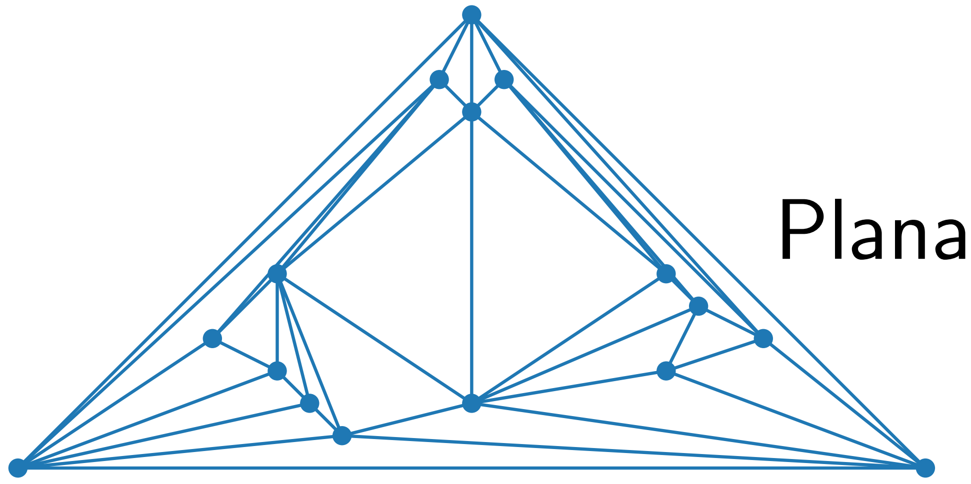
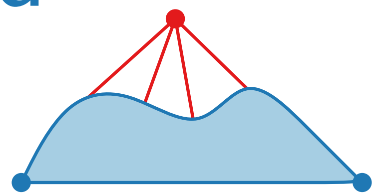
Visualization of Graphs

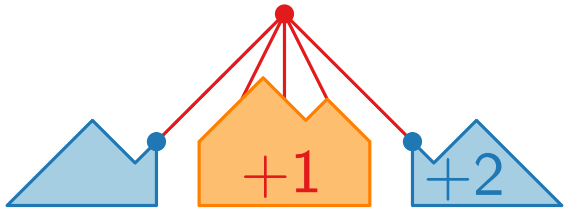
Lecture 3:
Straight-Line Drawings of Planar Graphs I:
Canonical Ordering and Shift Method



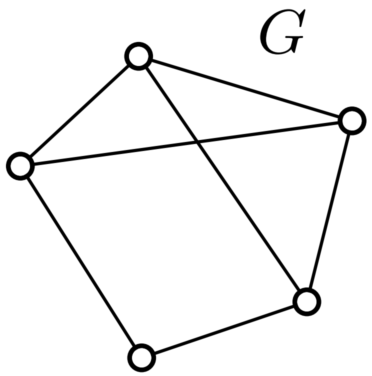
Part I:
Planar Straight-Line Drawings

Jonathan Klawitter





Planar Graphs



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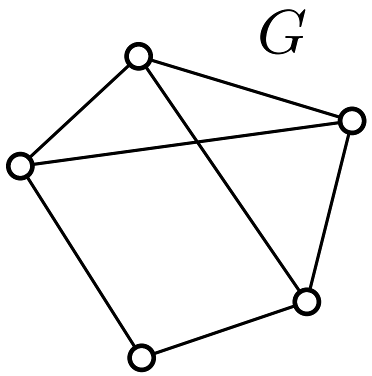

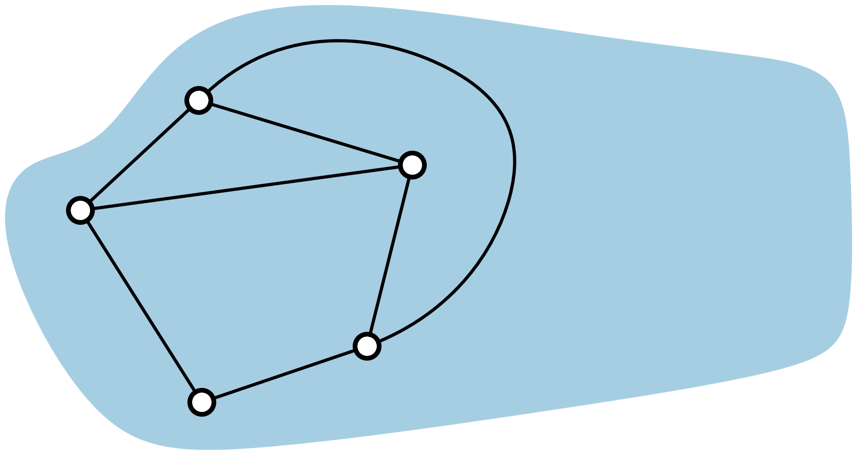

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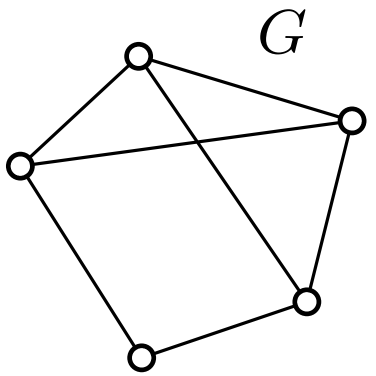

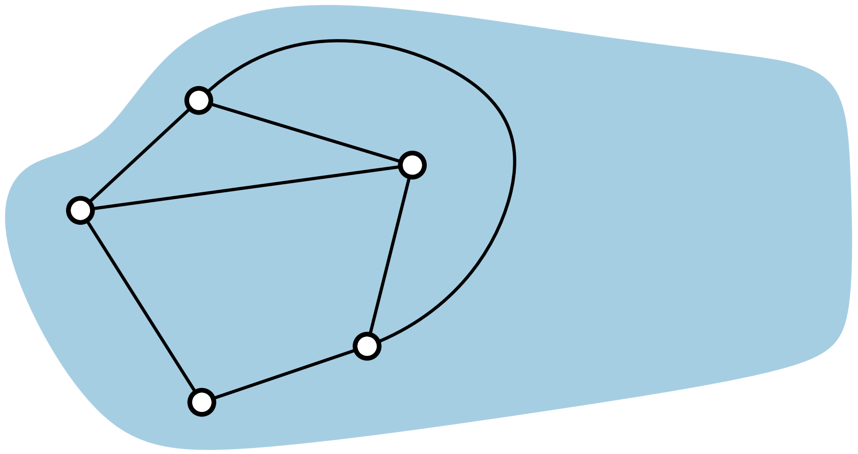

G is **planar**:
it can be drawn in such a way that no edges cross each other.

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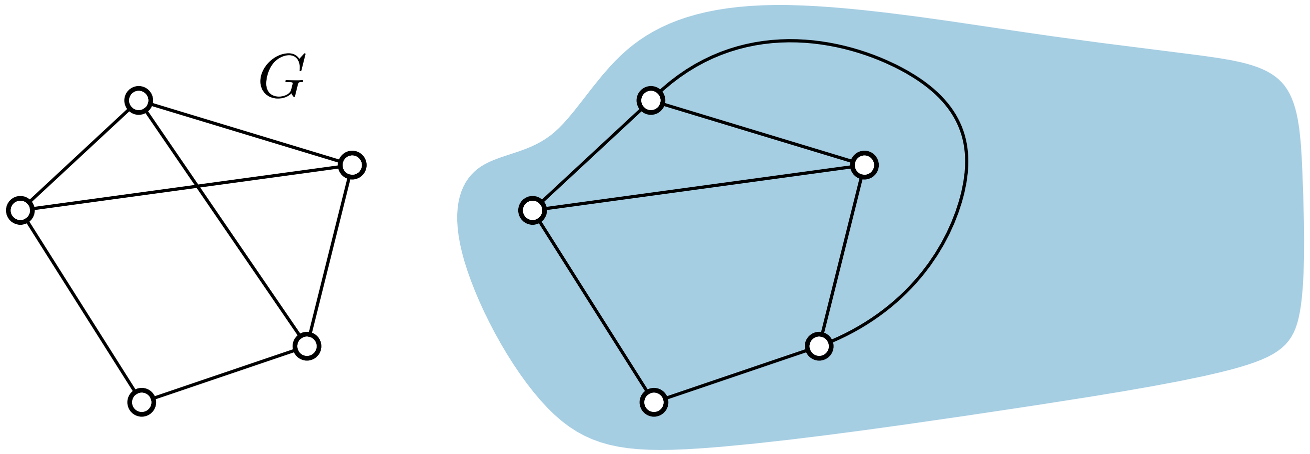

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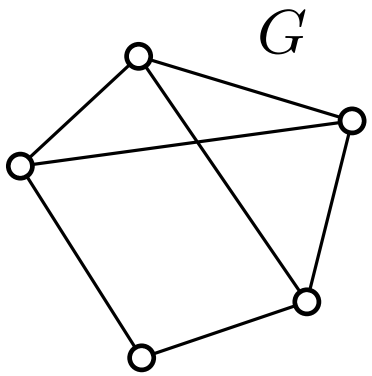

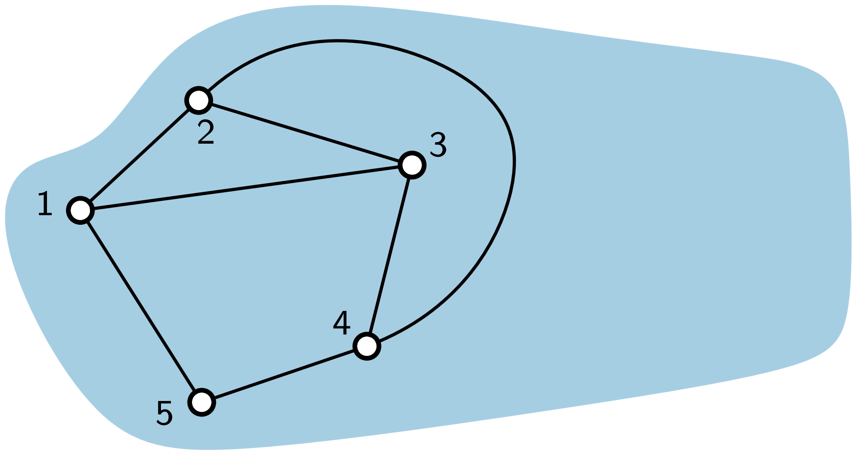

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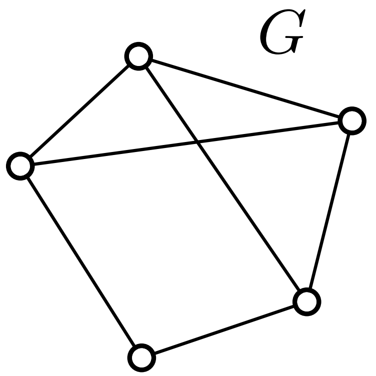

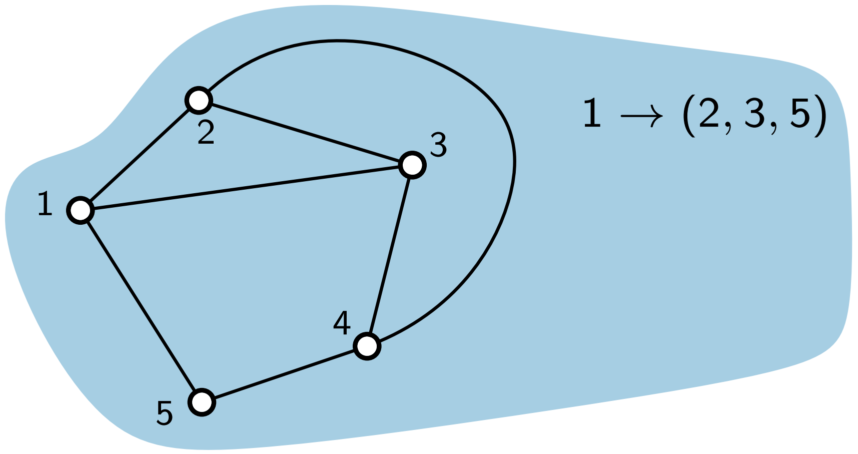

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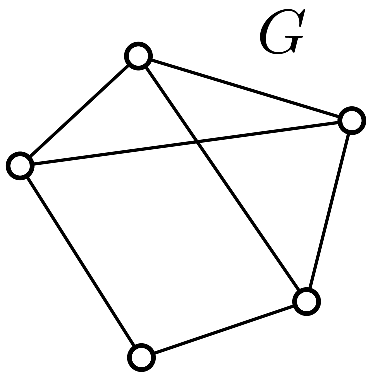

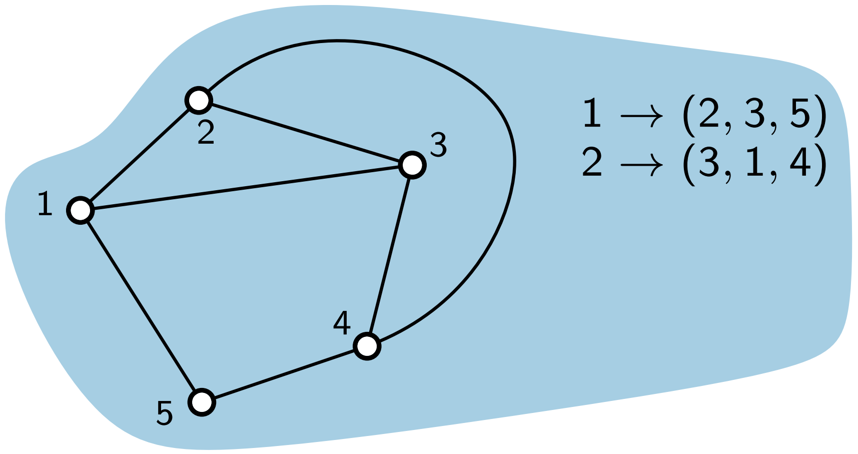

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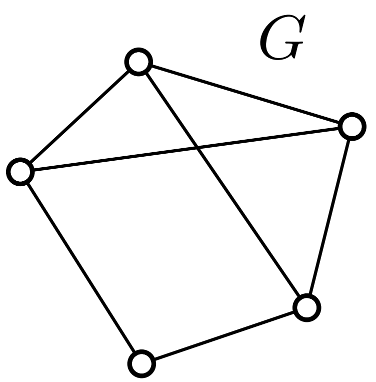

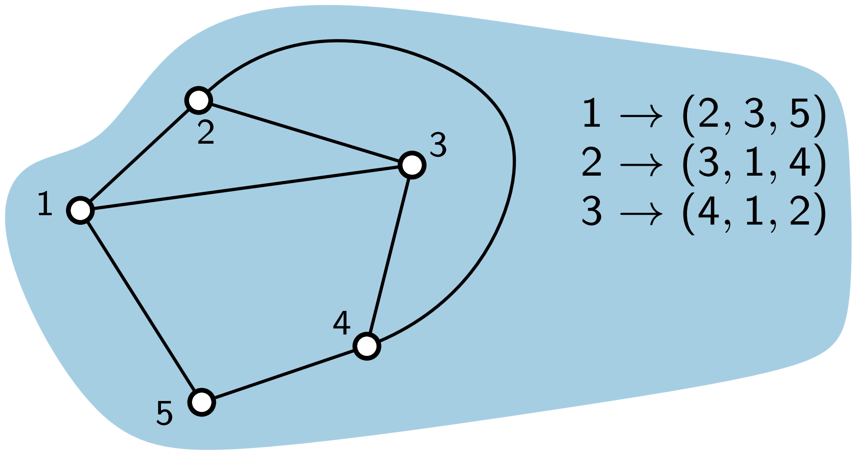

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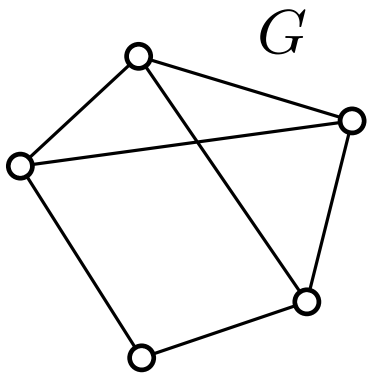

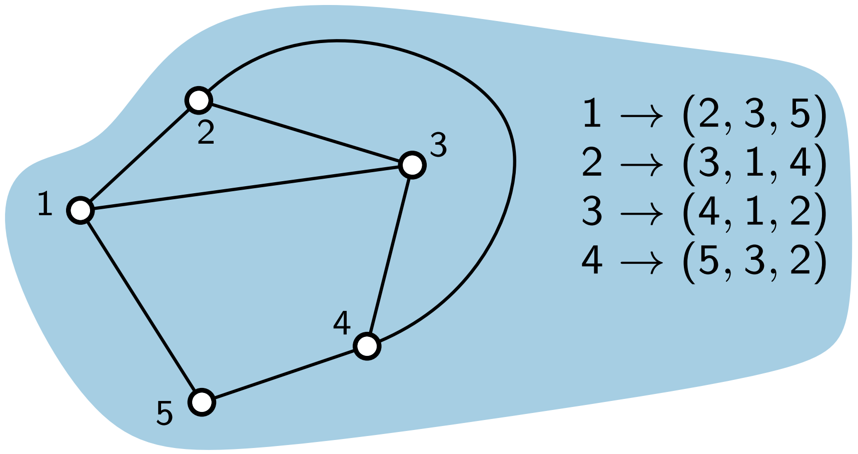

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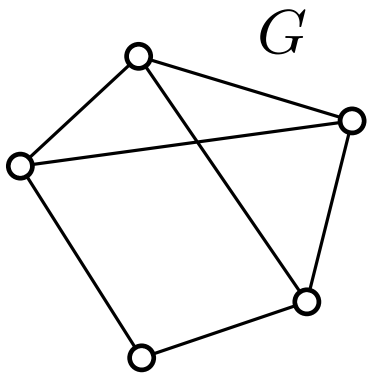

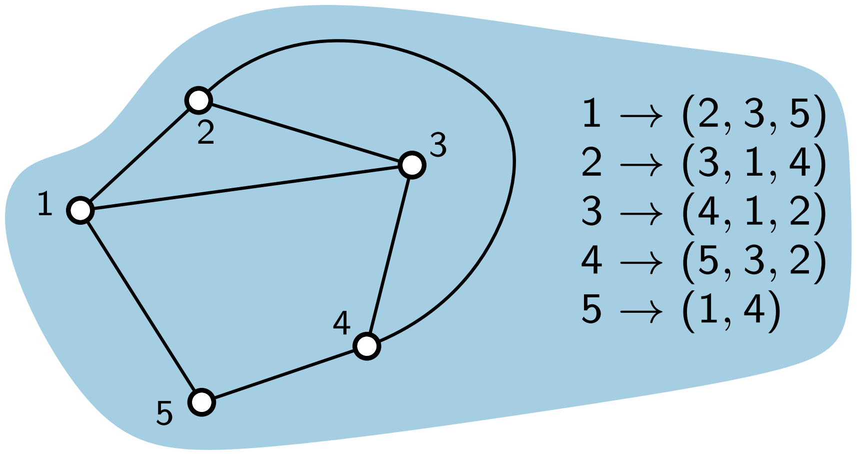

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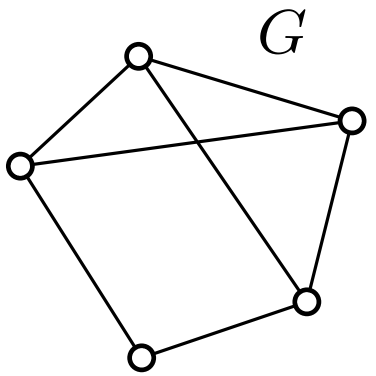

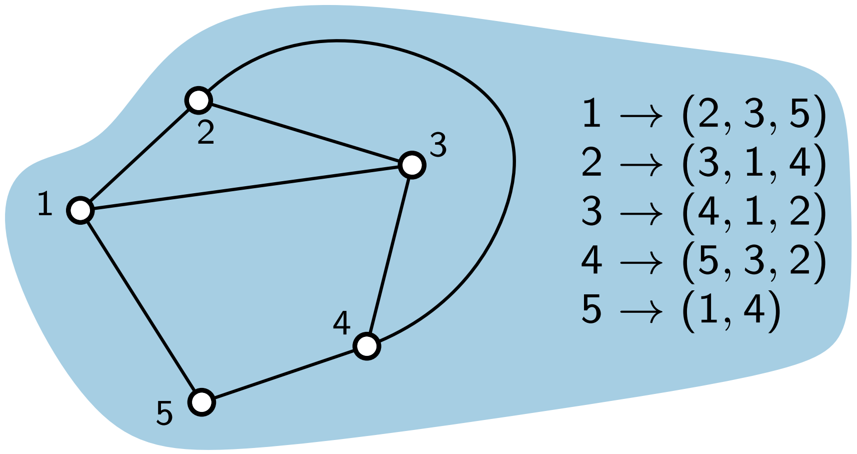

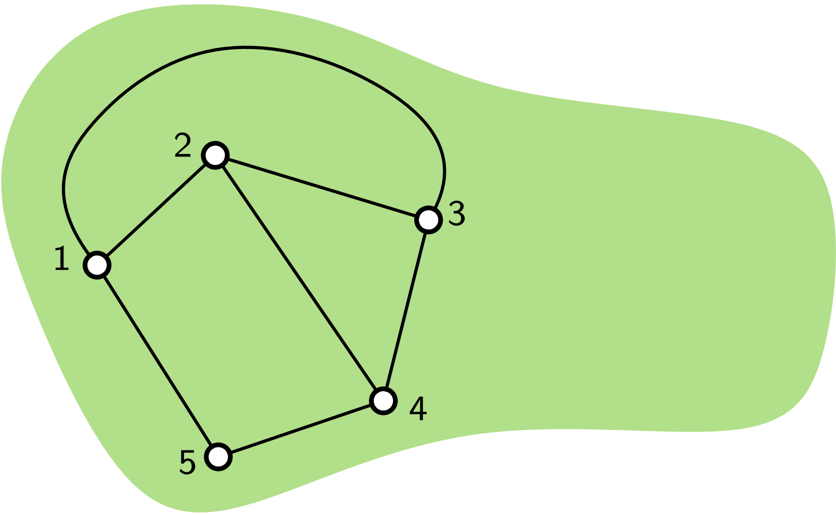

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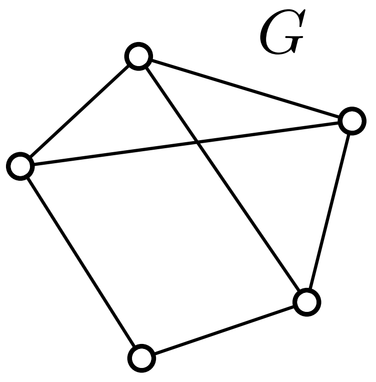
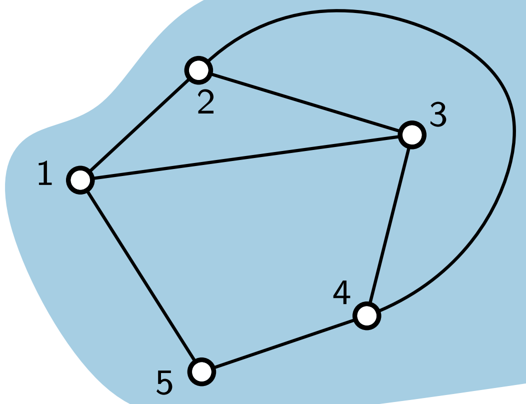
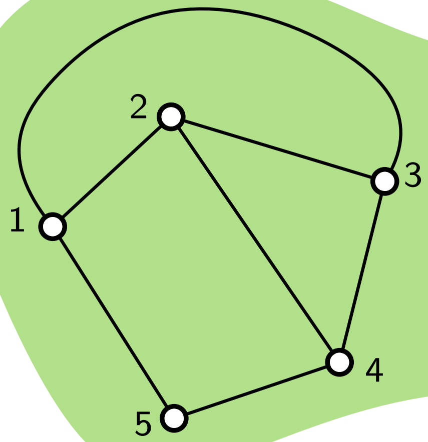




$1 \to (2, 3, 5)$
$2 \to (3, 1, 4)$
$3 \to (4, 1, 2)$
$4 \to (5, 3, 2)$
$5 \to (1, 4)$



$1 \to (2, 5, 3)$
$2 \to (3, 4, 1)$
$3 \to (4, 2, 1)$
$4 \to (5, 2, 3)$
$5 \to (1, 4)$

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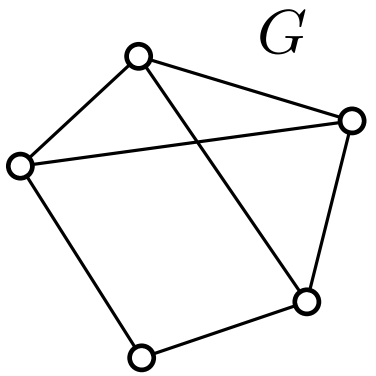

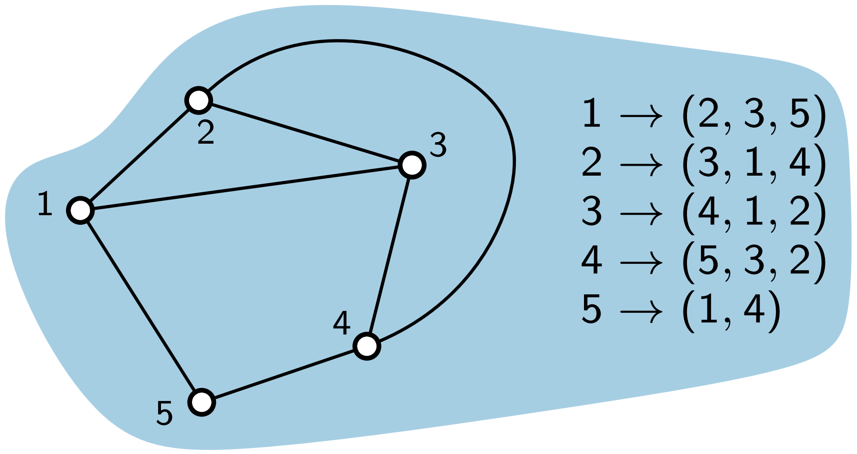

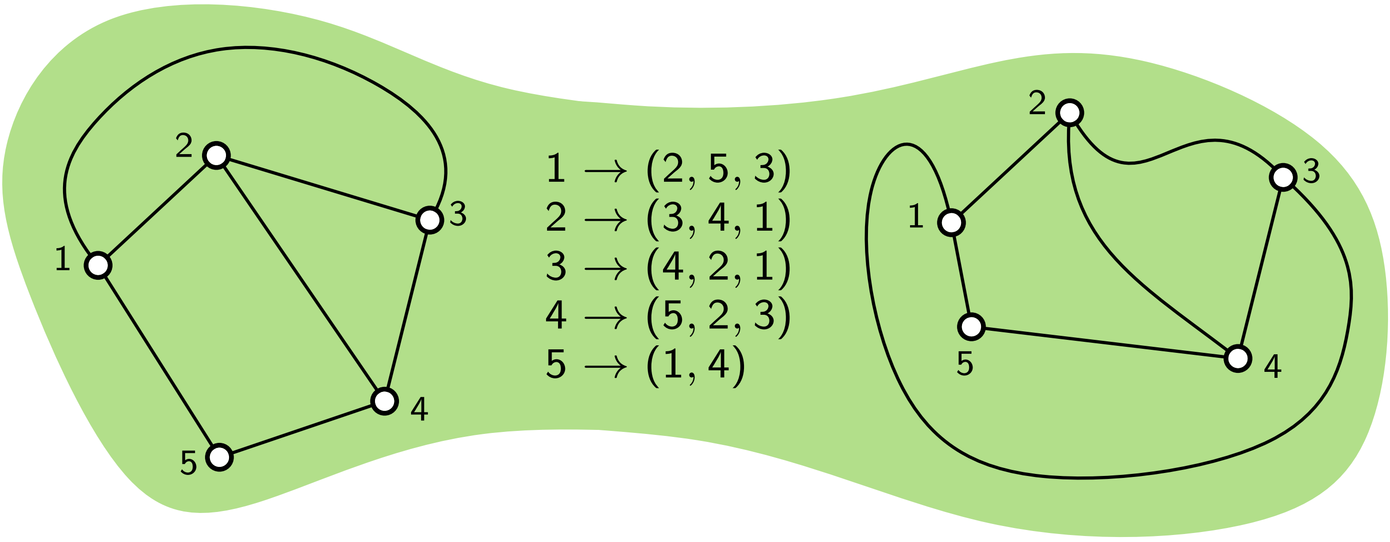

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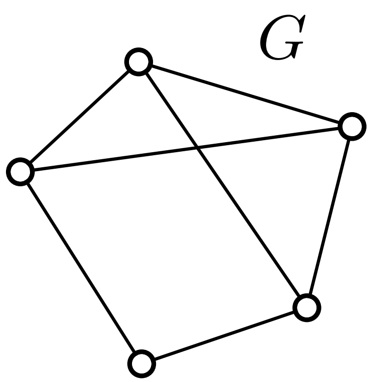

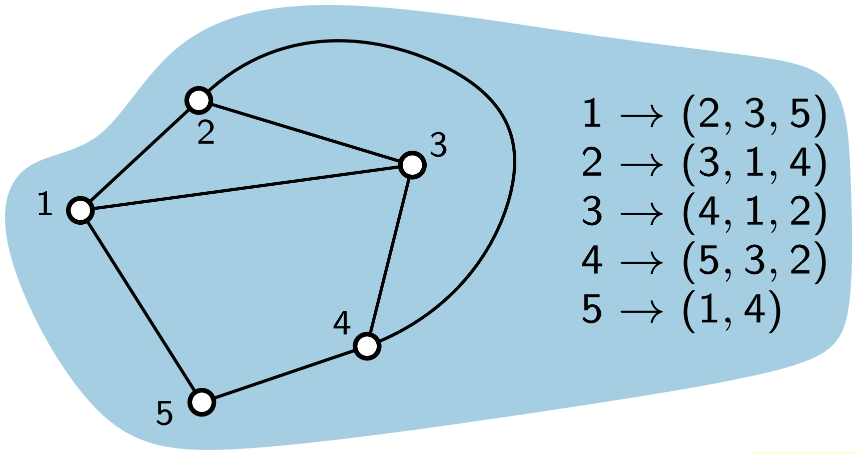

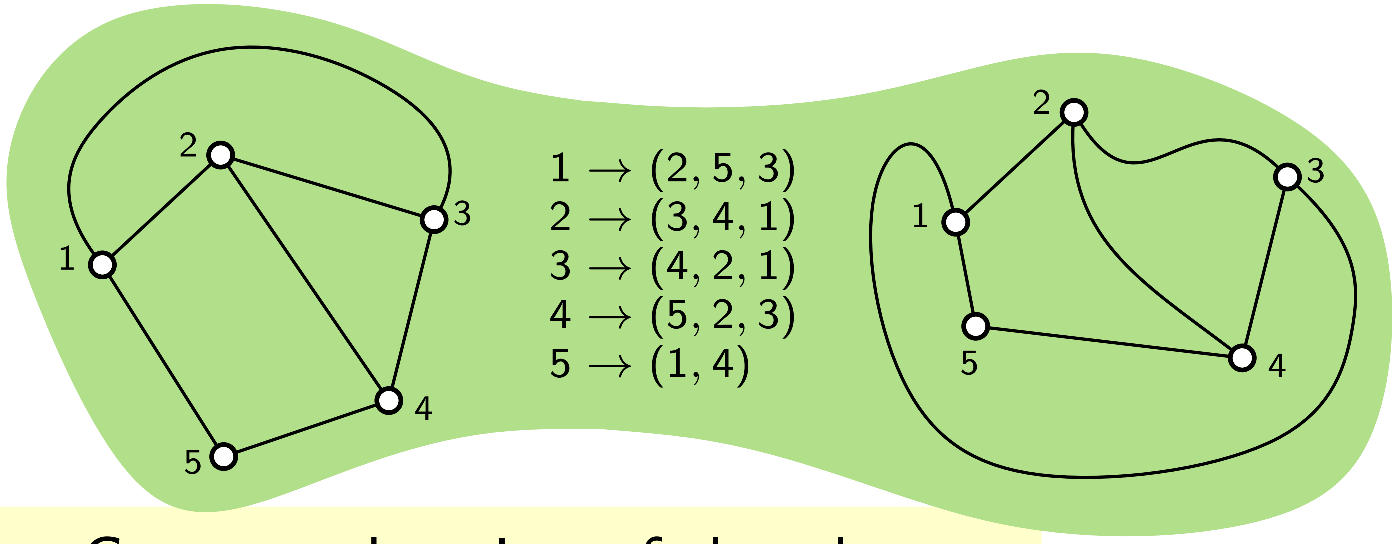

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faces: Connected region of the plane bounded by edges

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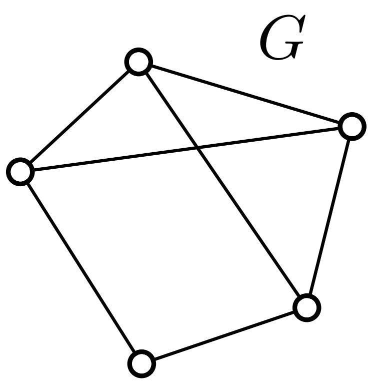

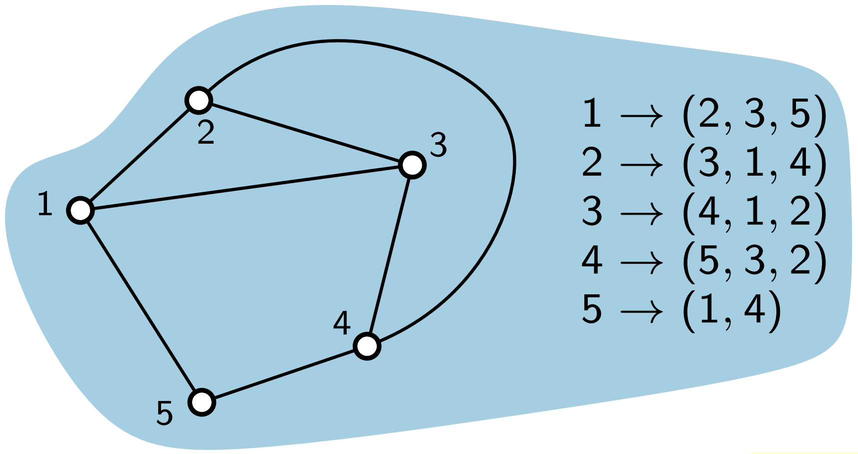

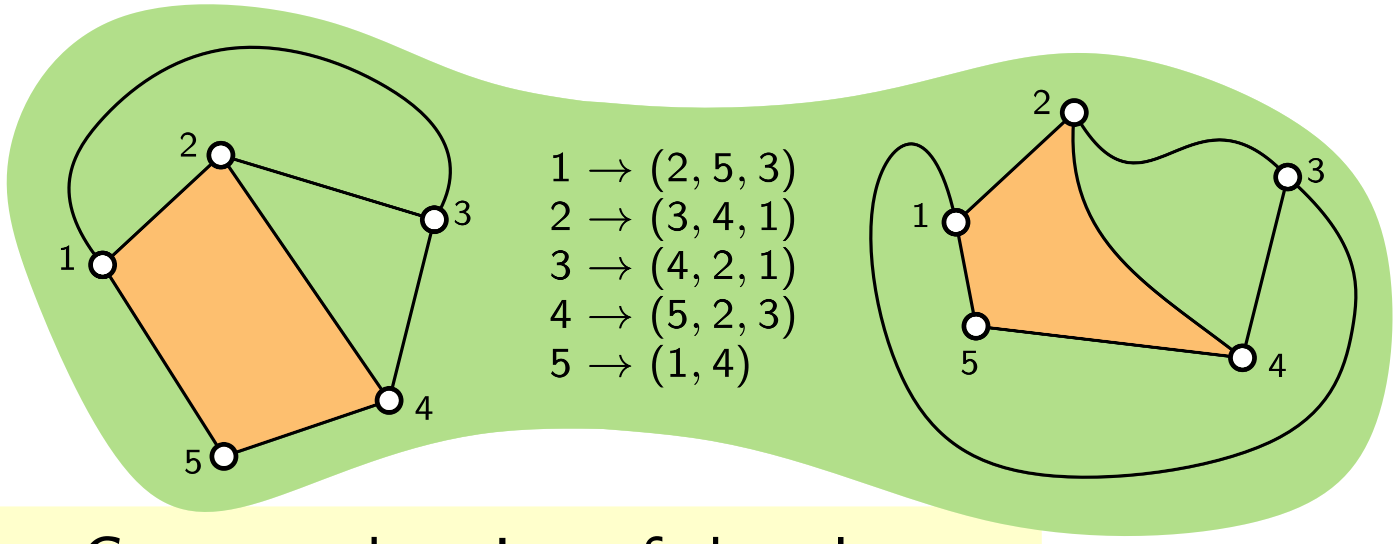

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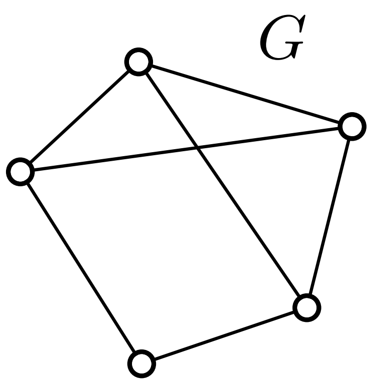

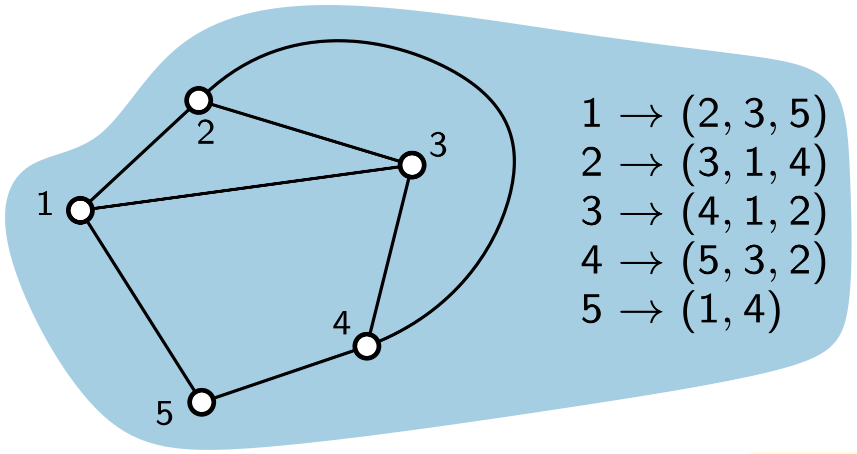

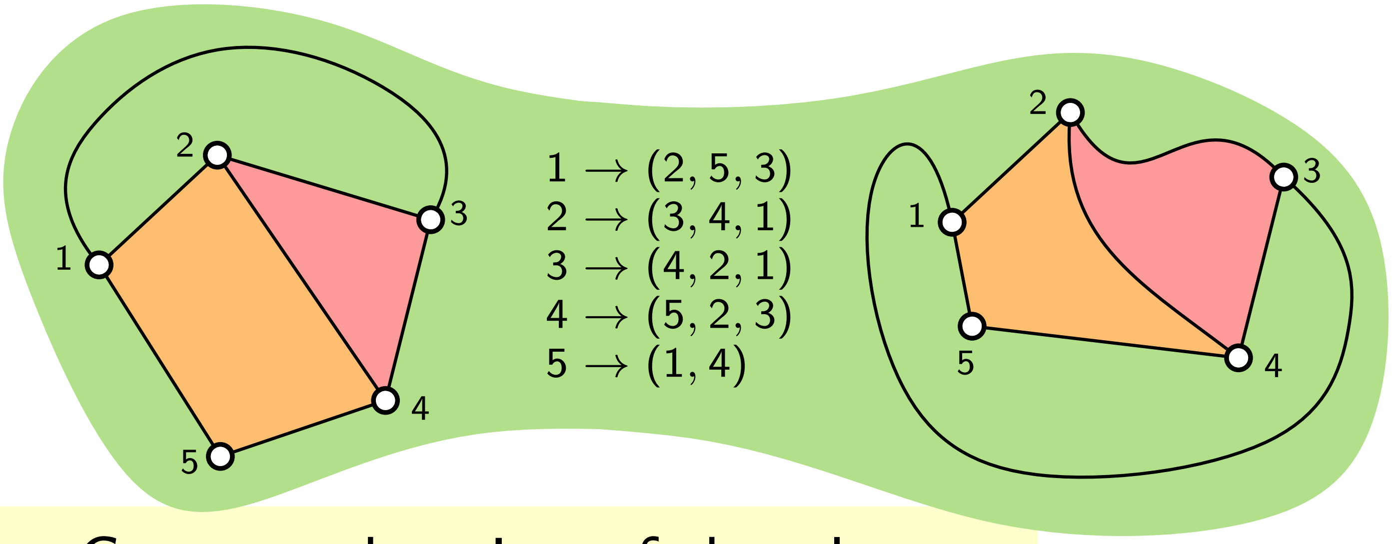

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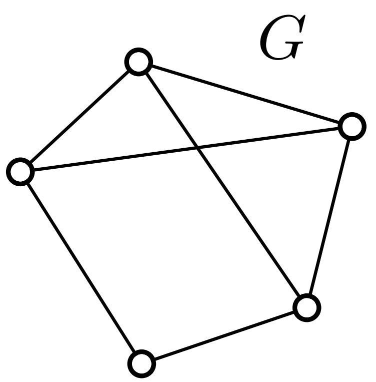

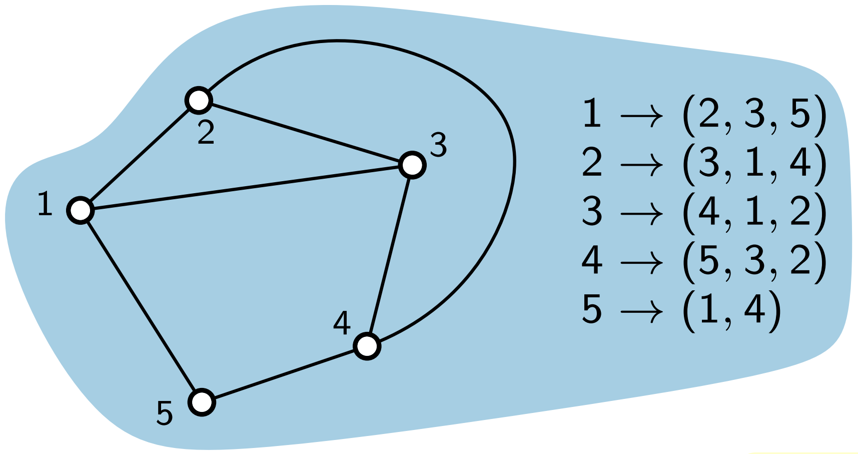

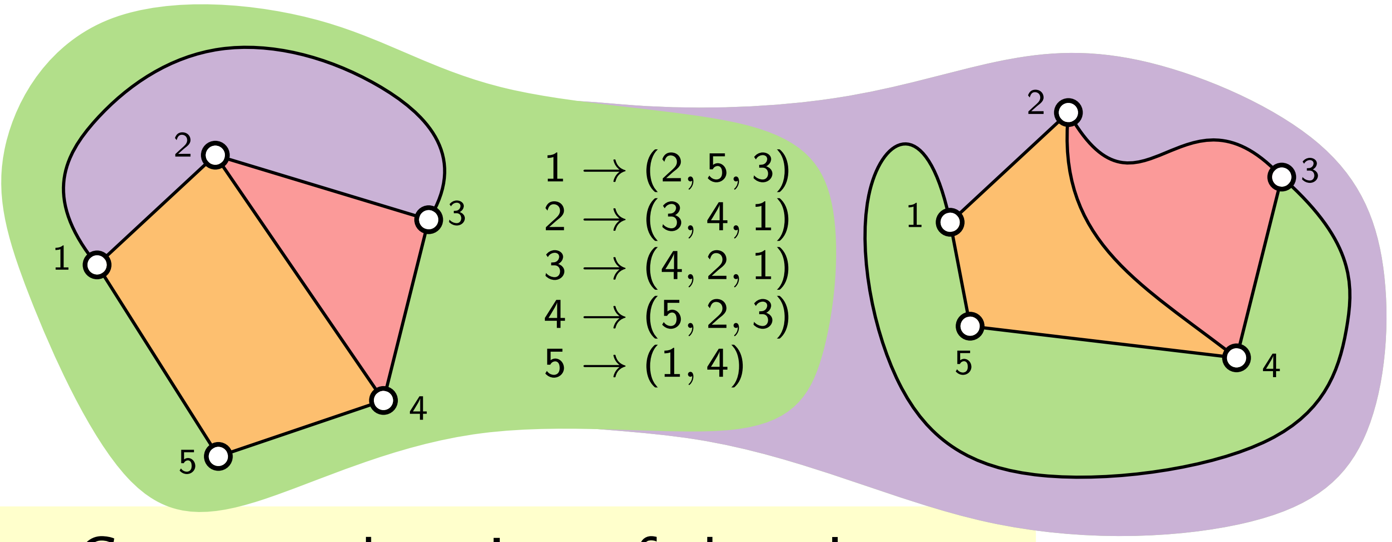

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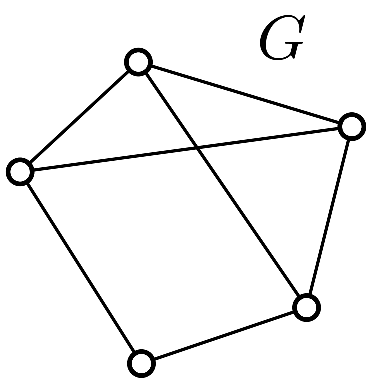

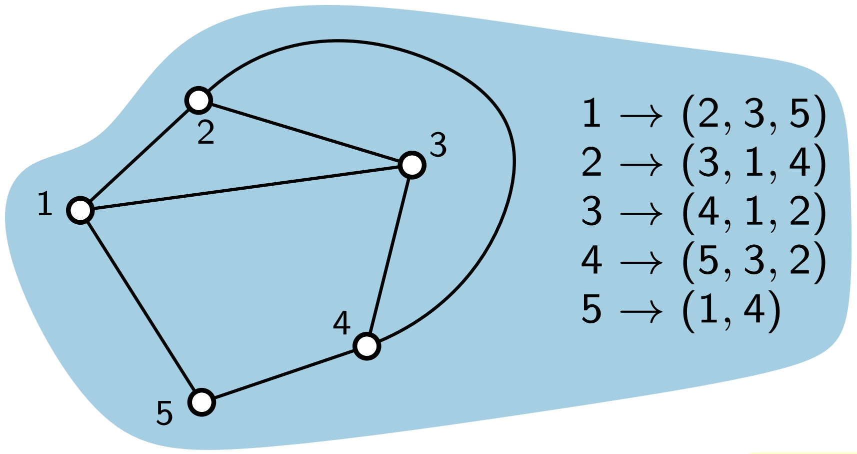

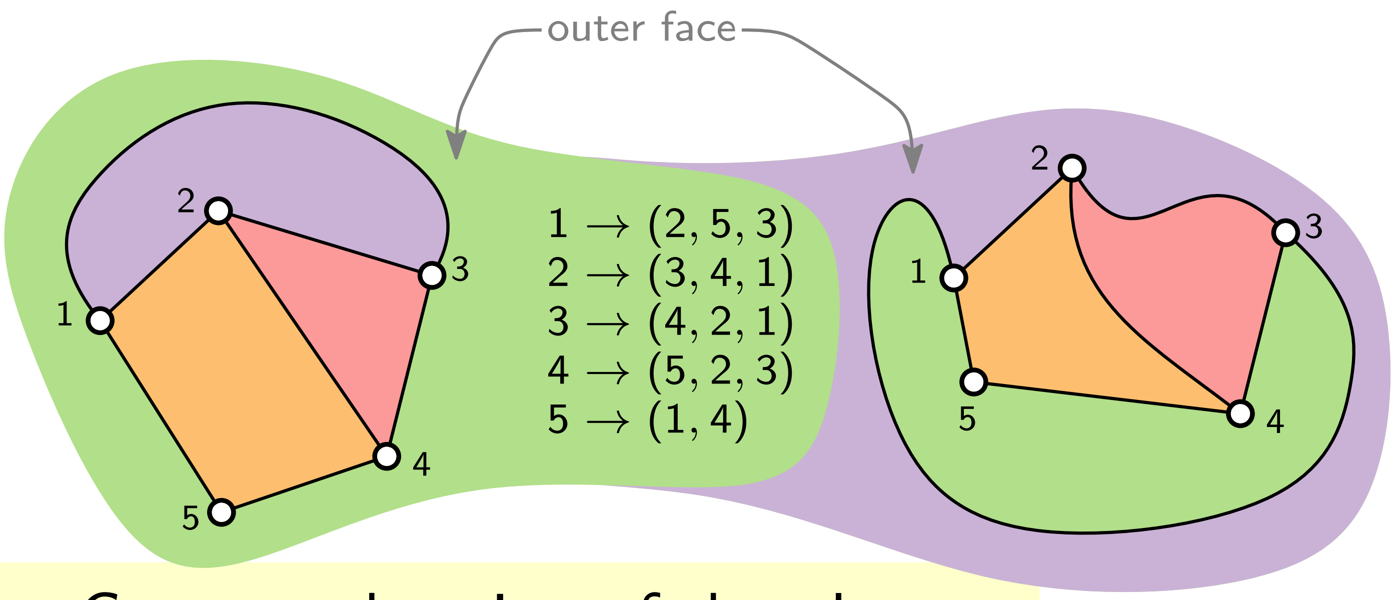

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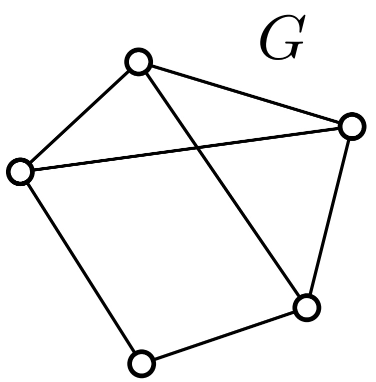

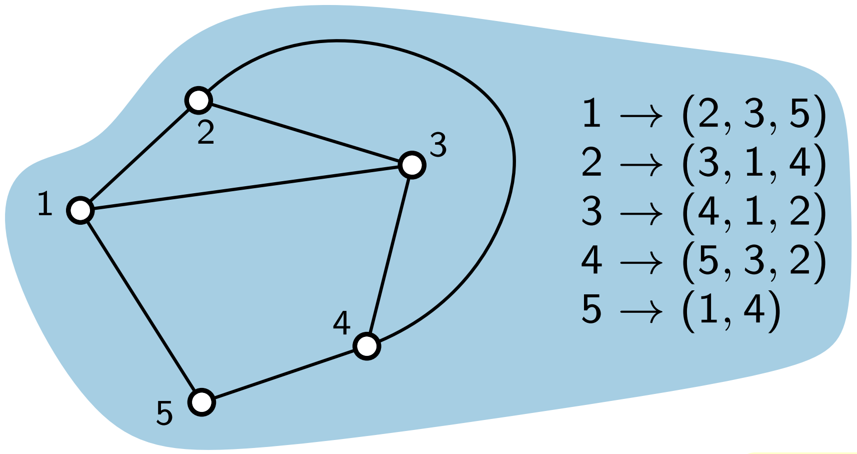

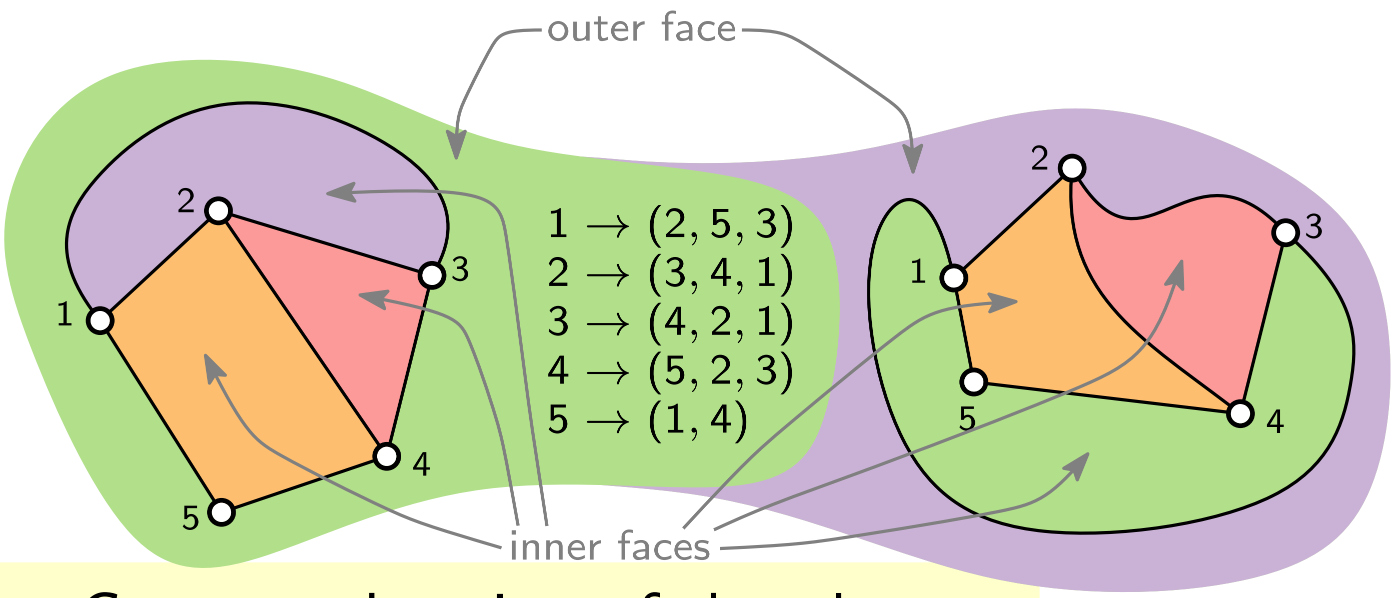

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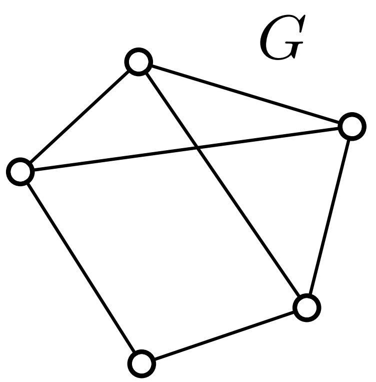

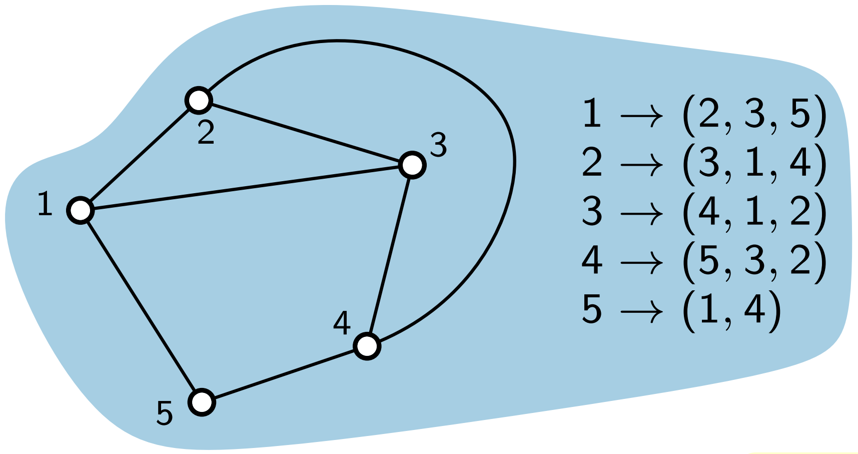

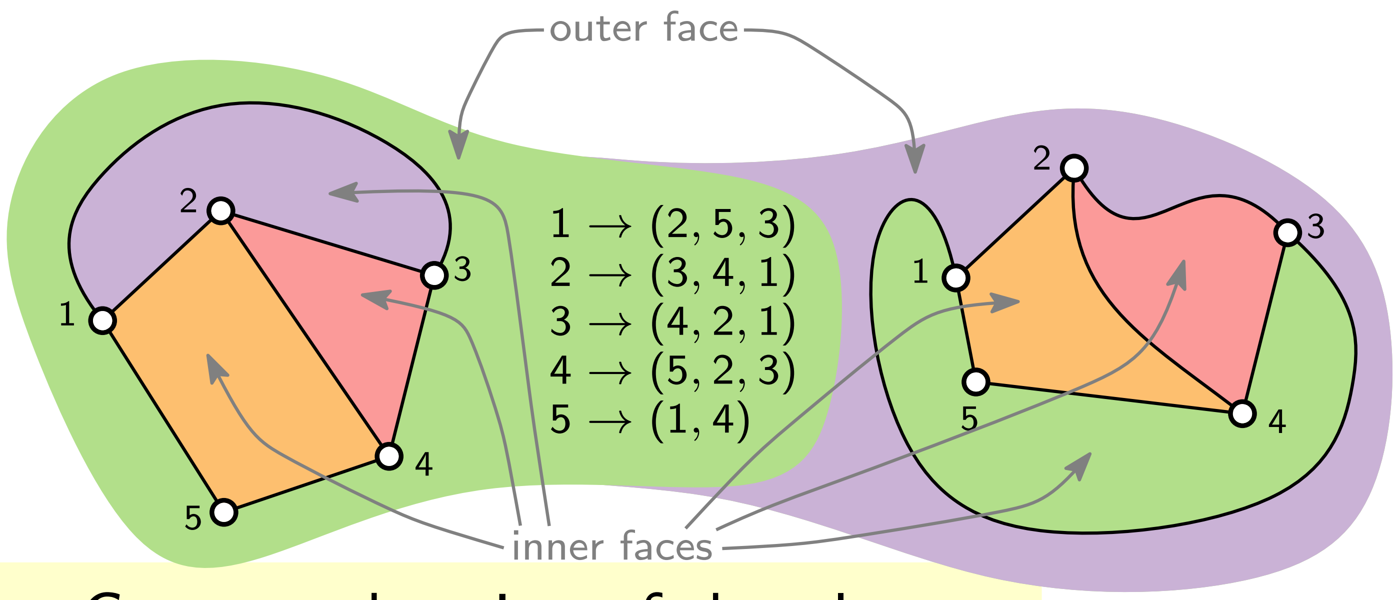

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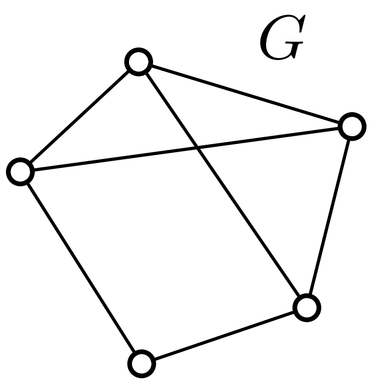
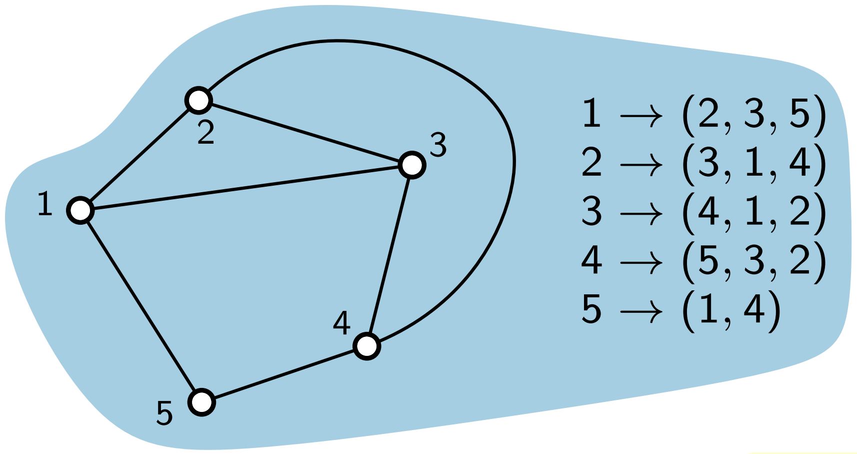
Euler's polyhedra formula.

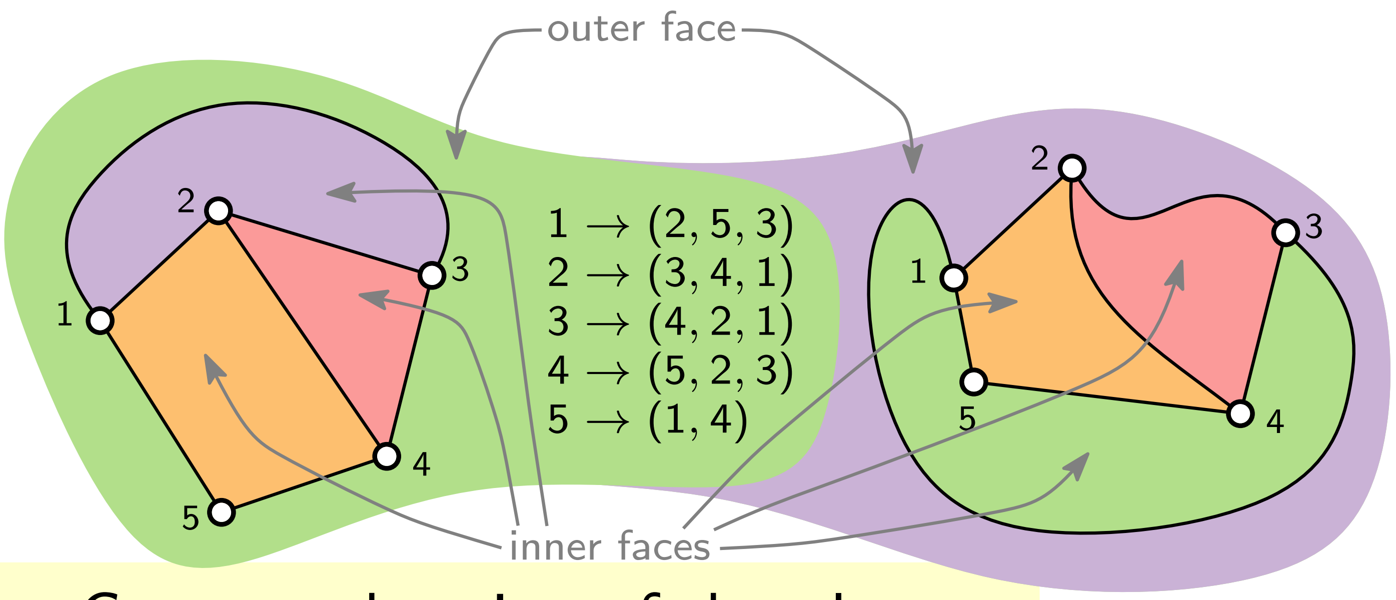
#faces - #edges + #vertices = #conn.comp. + 1

$$f - m + n = c + 1$$

Planar Graphs







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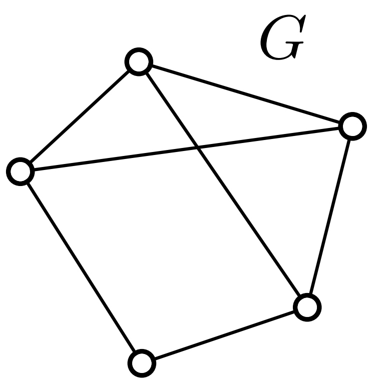
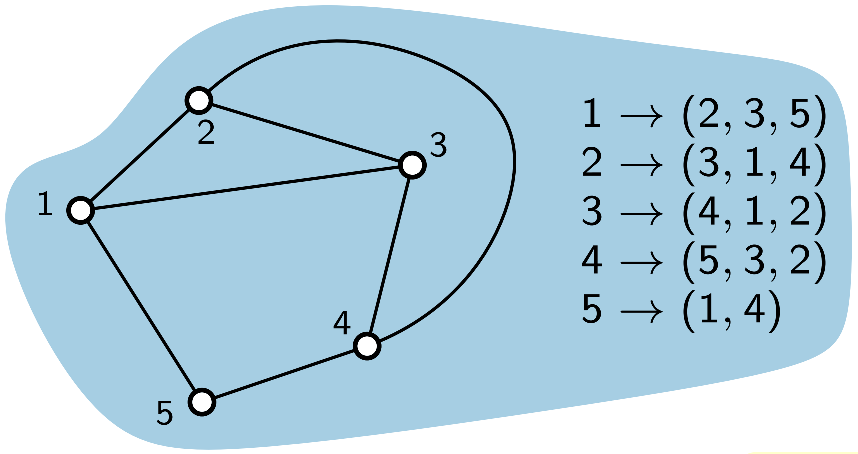
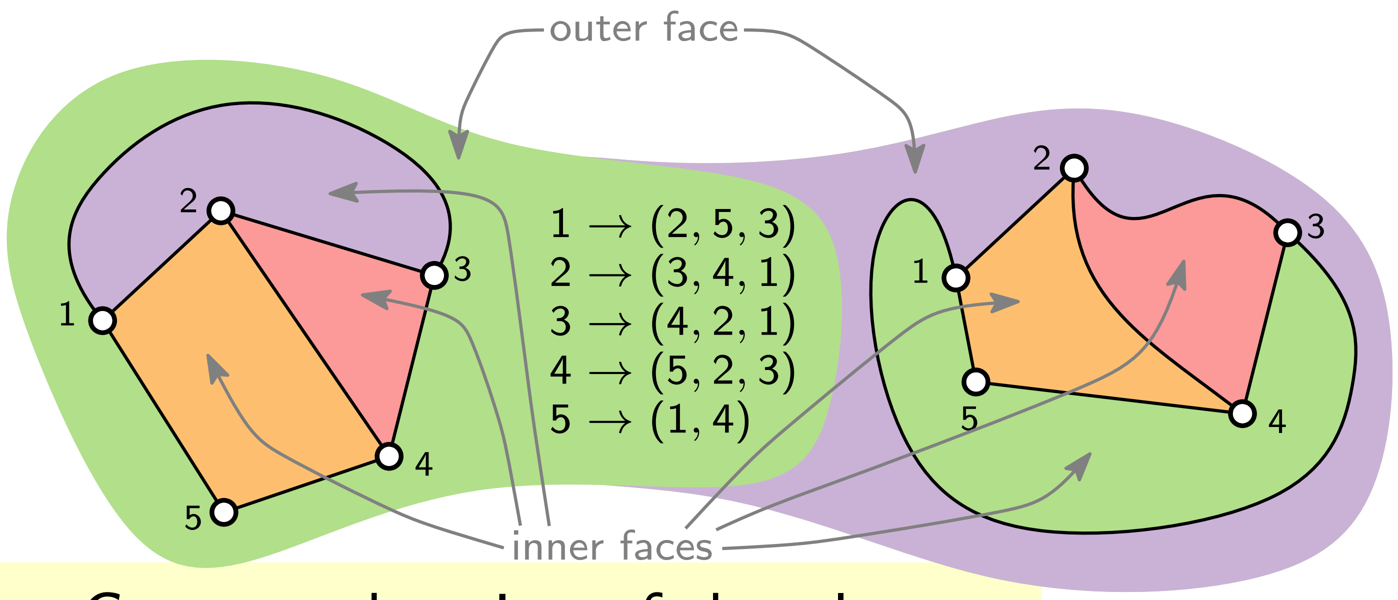
$$f \quad - \quad m \quad + \quad n \quad = \quad c \quad + 1$$

Proof.

Planar Graphs







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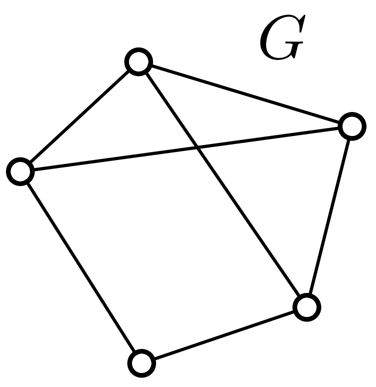
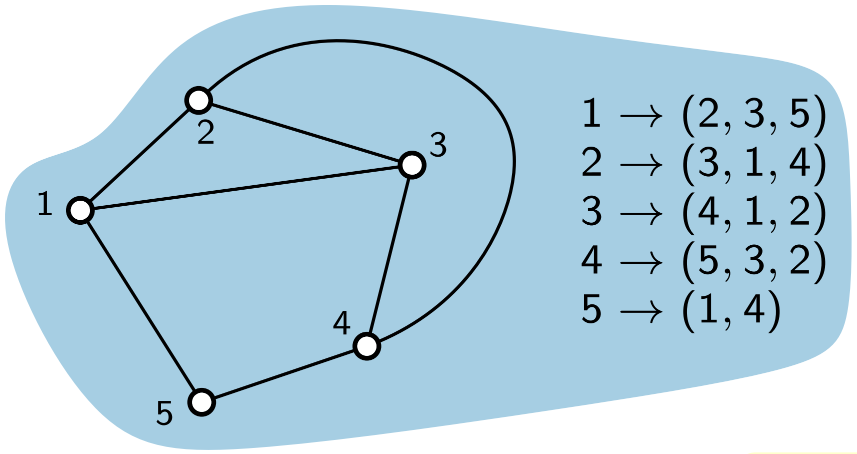
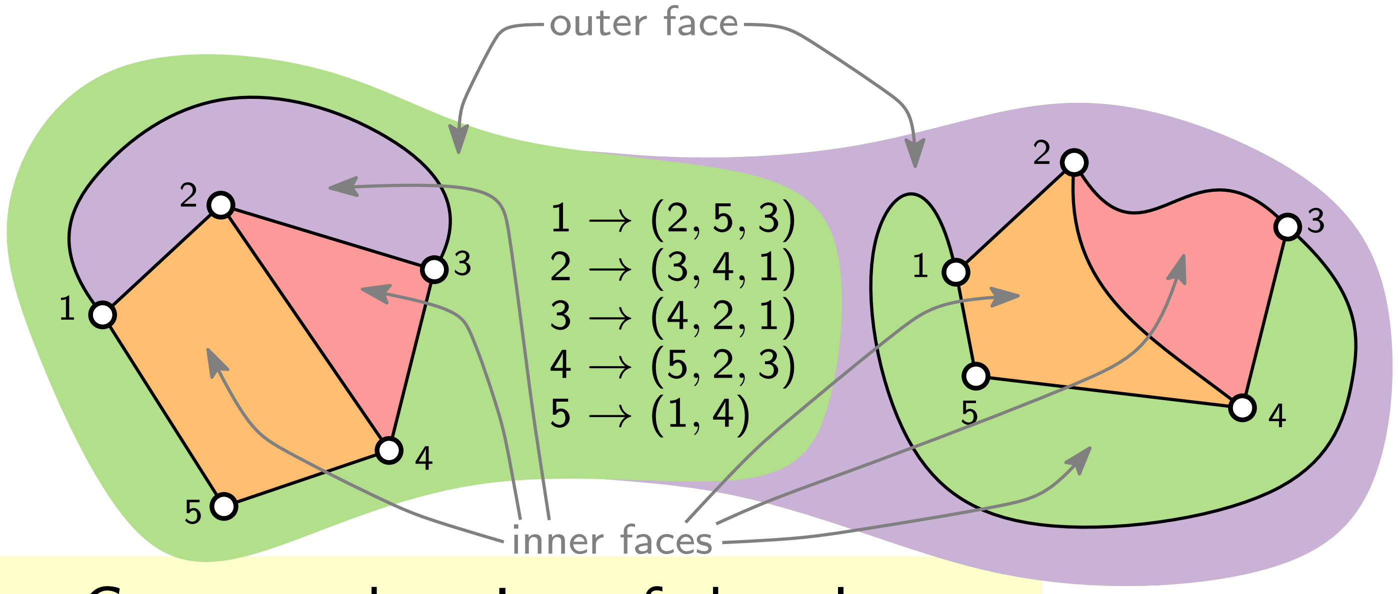
$f - m + n = c + 1$

Proof. By induction on m:

Planar Graphs







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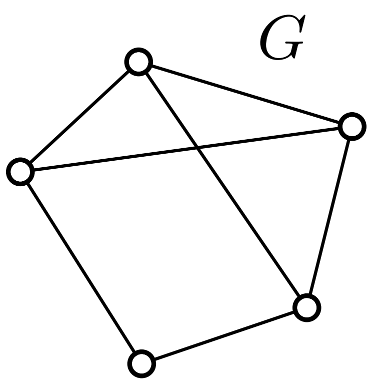
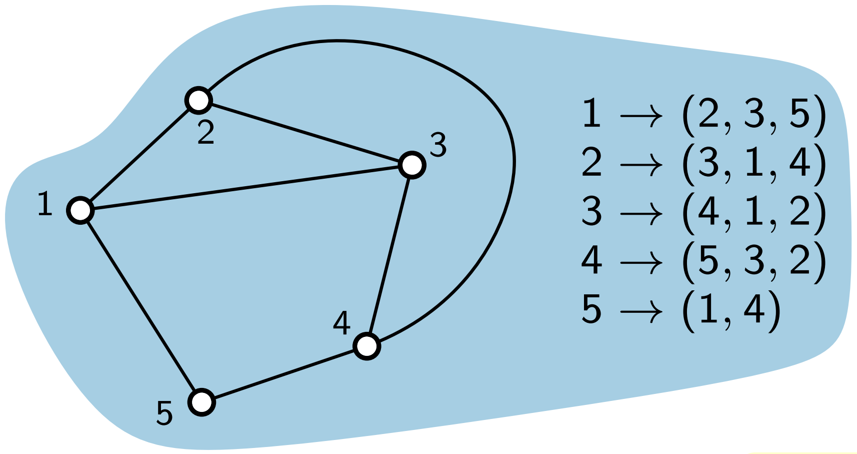
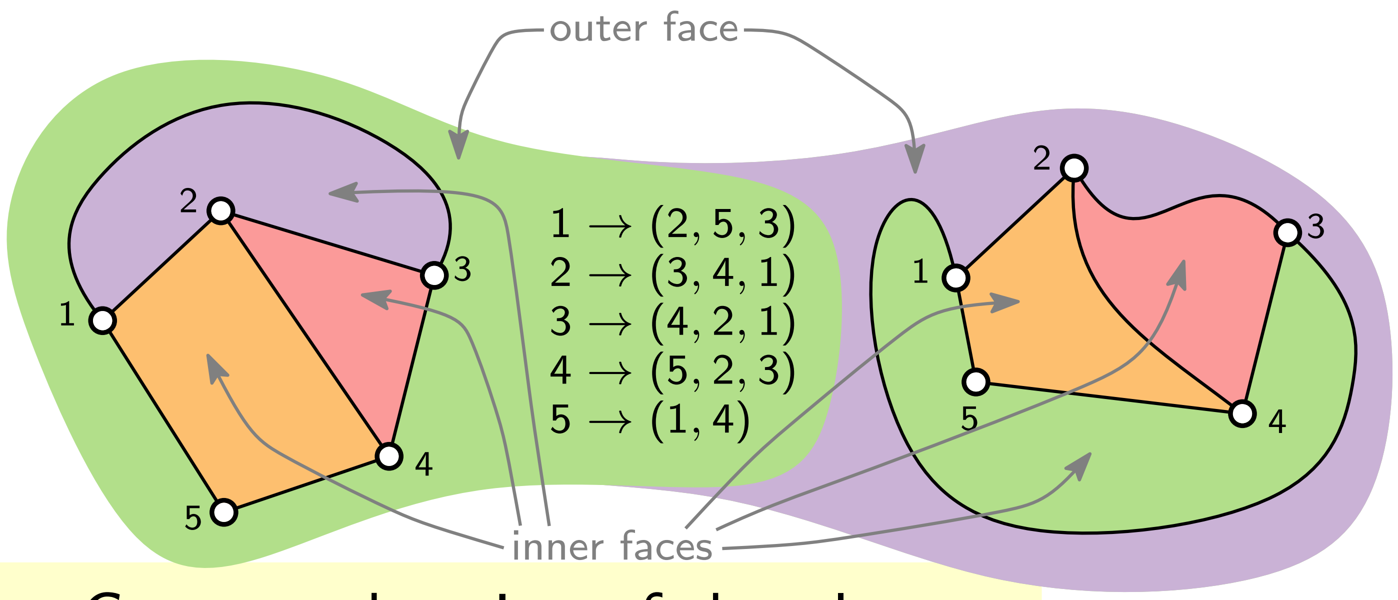
Proof. By induction on m:

$m = 0 \Rightarrow$

Planar Graphs







G is **planar**:
it can be drawn in such a way that no edges cross each other.

planar embedding:
Clockwise orientation of adjacent vertices around each vertex.

A planar graph can have many planar embeddings.

A planar embedding can have many planar drawings!

faces: Connected region of the plane bounded by edges

Euler's polyhedra formula.

#faces - #edges + #vertices = #conn.comp. + 1

$$f - m + n = c + 1$$

Proof. By induction on m:

$m = 0 \Rightarrow f =$? and $c =$?

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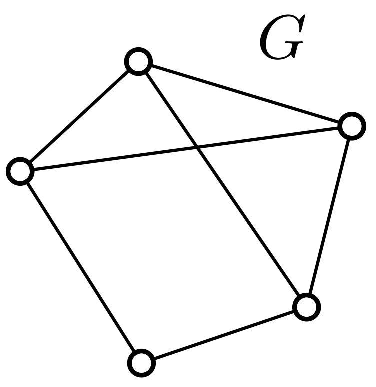

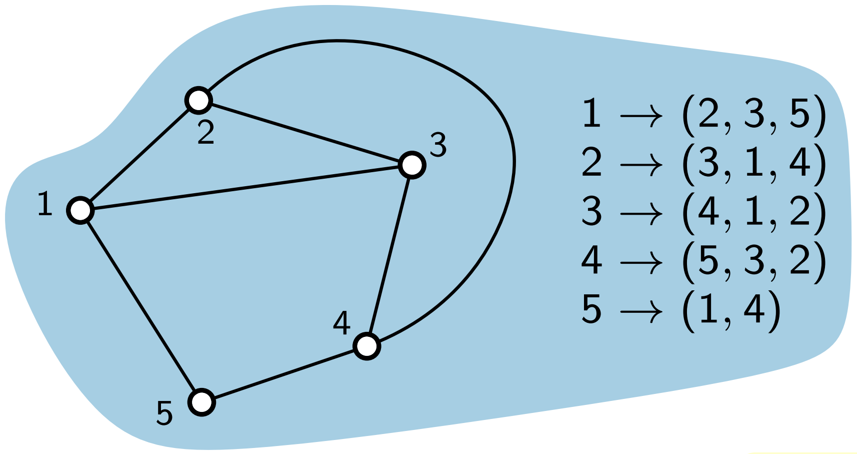

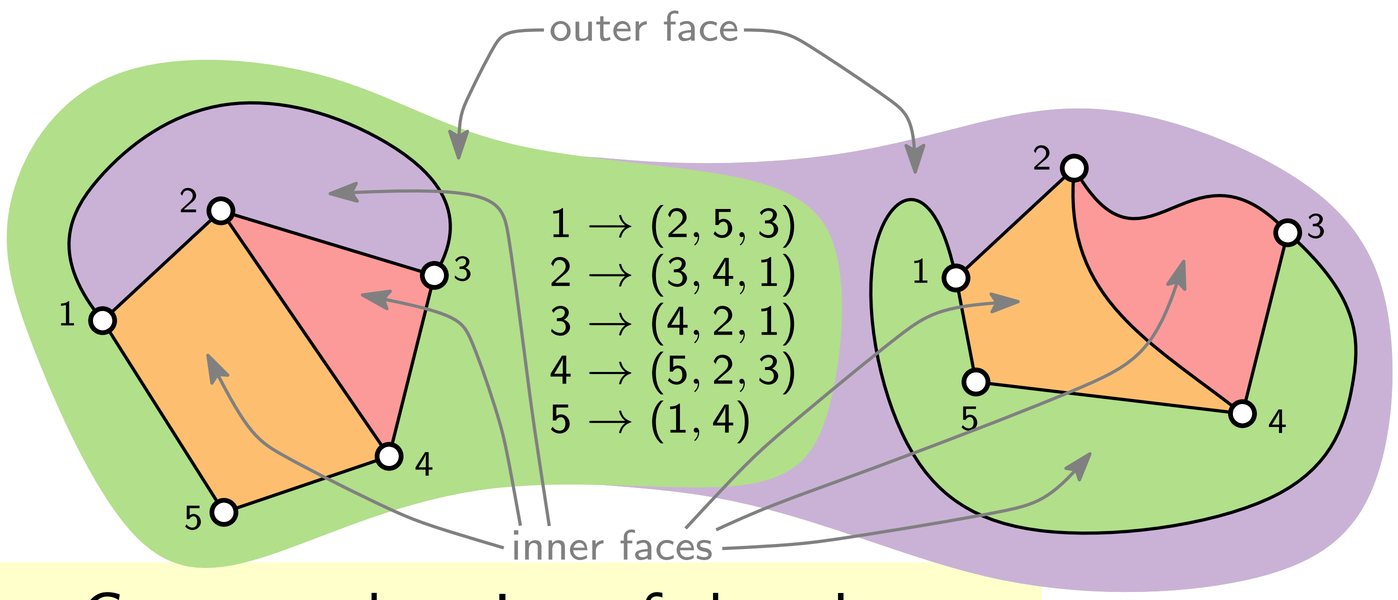

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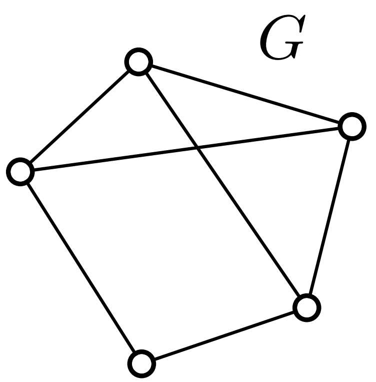

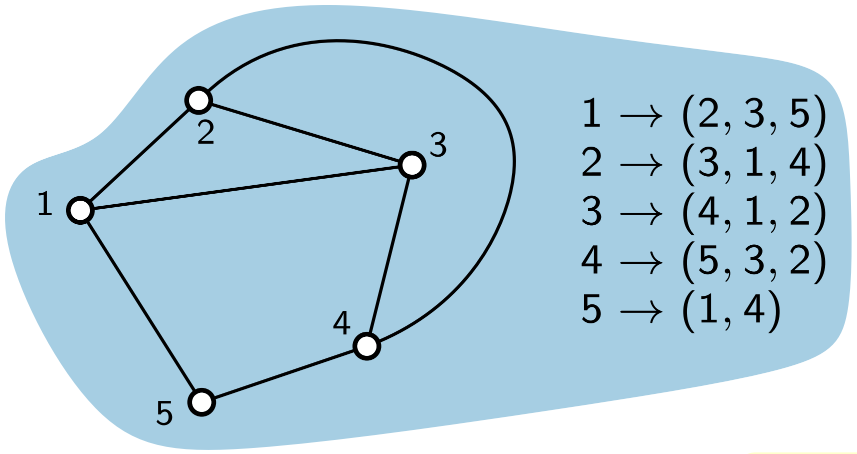

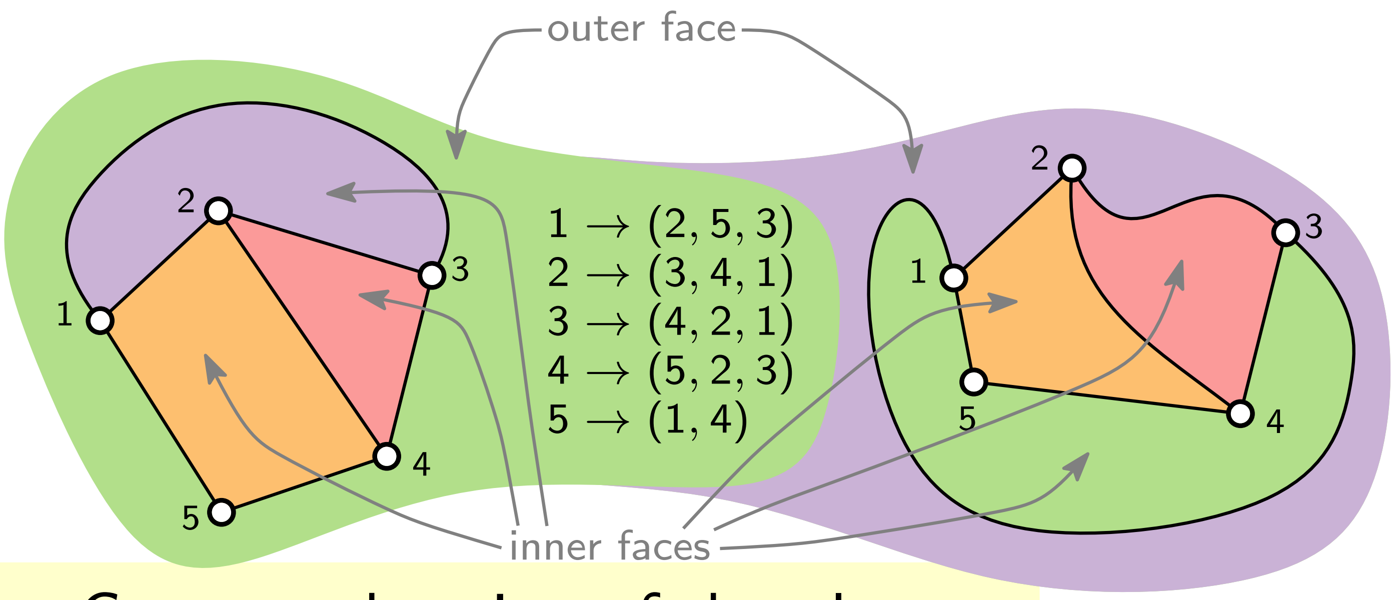

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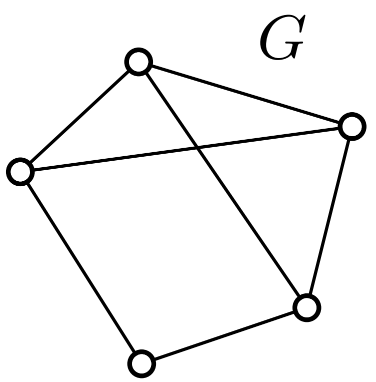

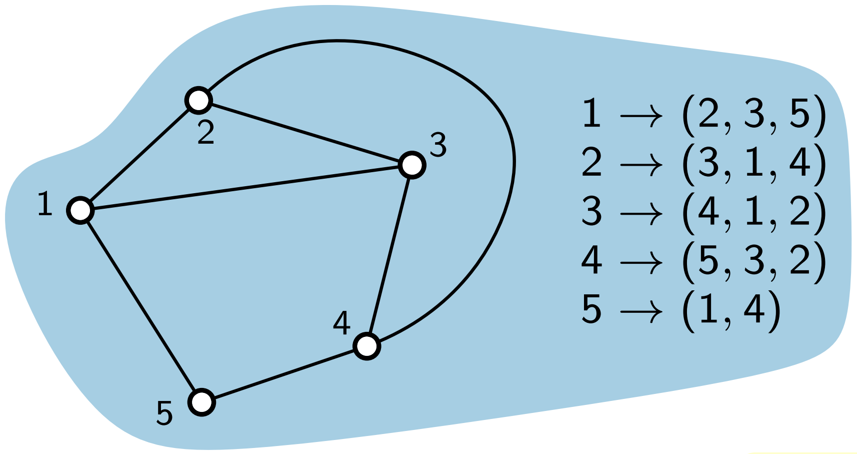

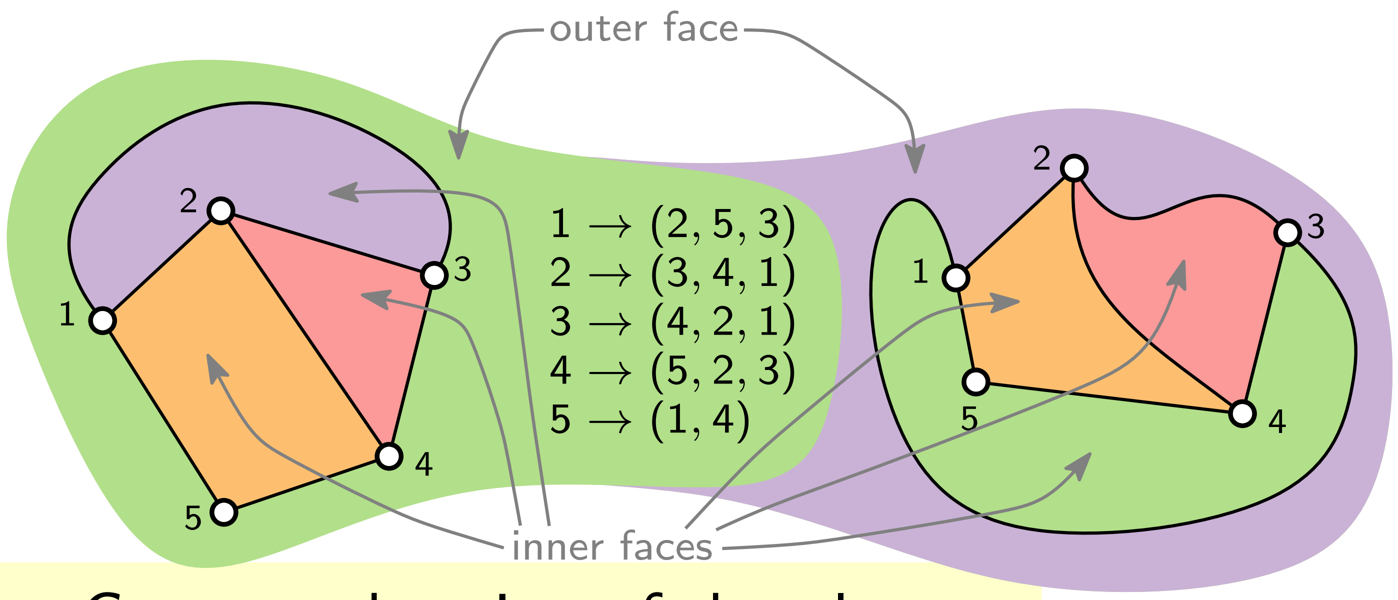

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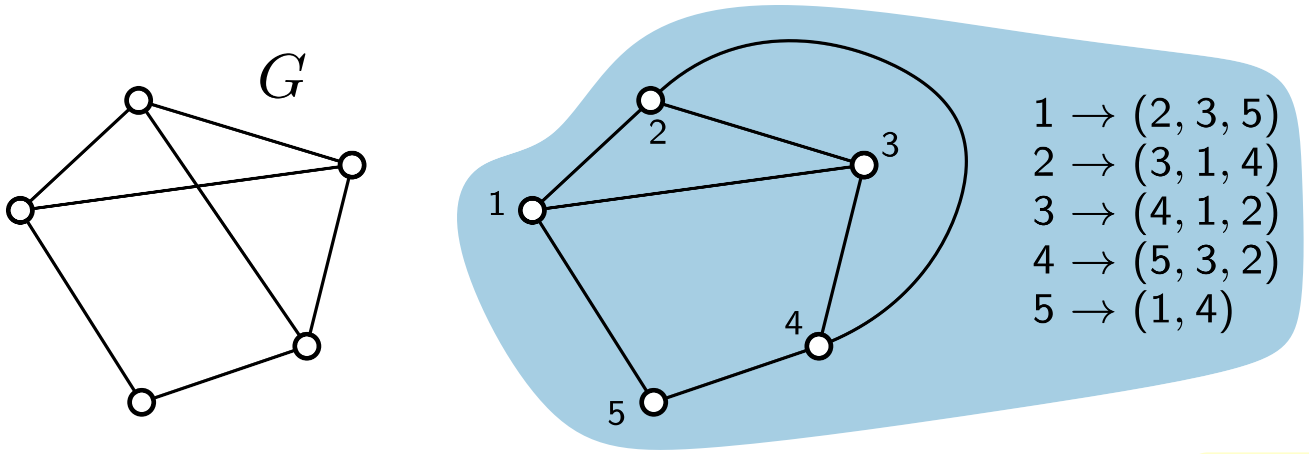

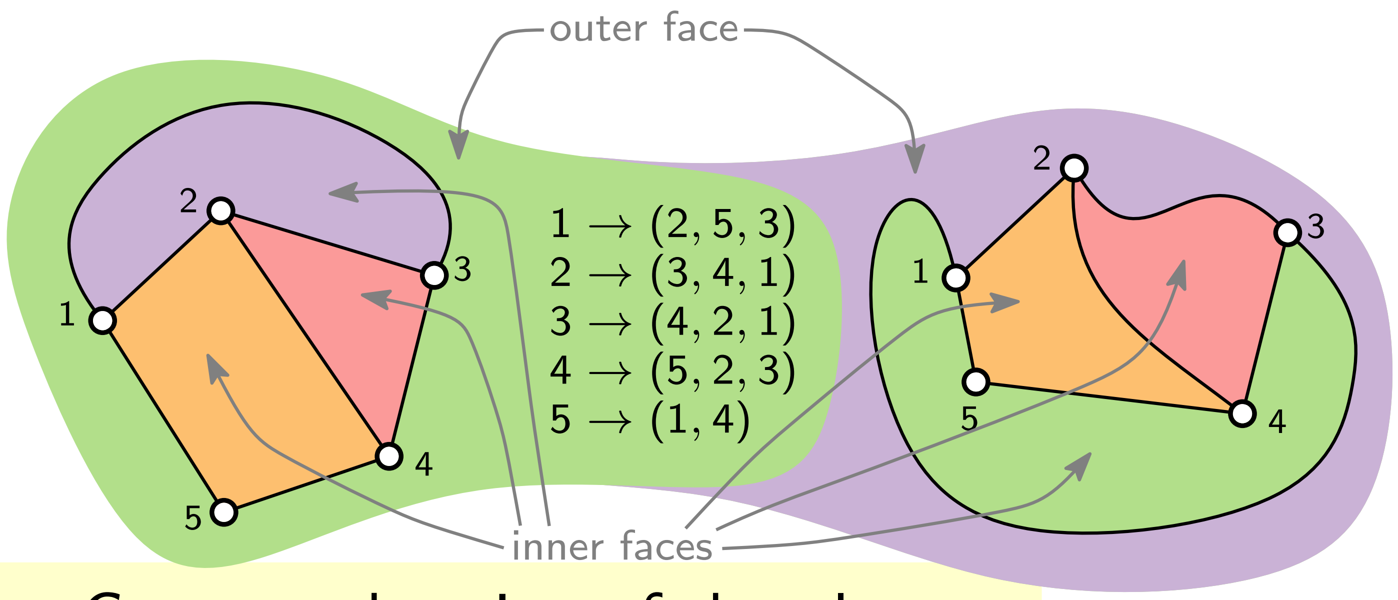

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$m > 1 \Rightarrow$

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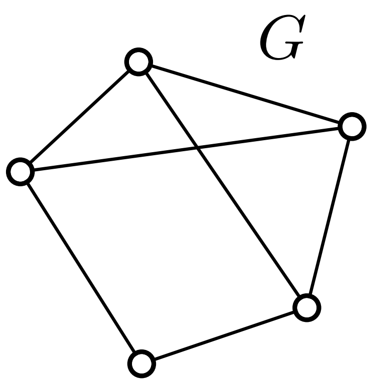

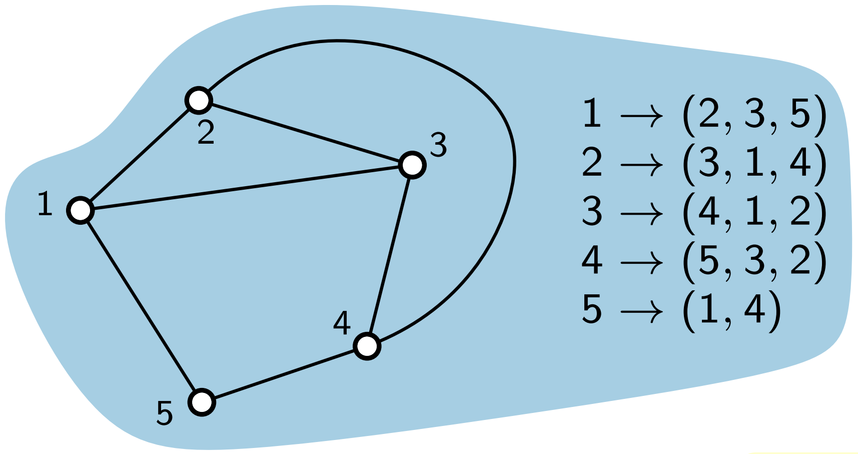

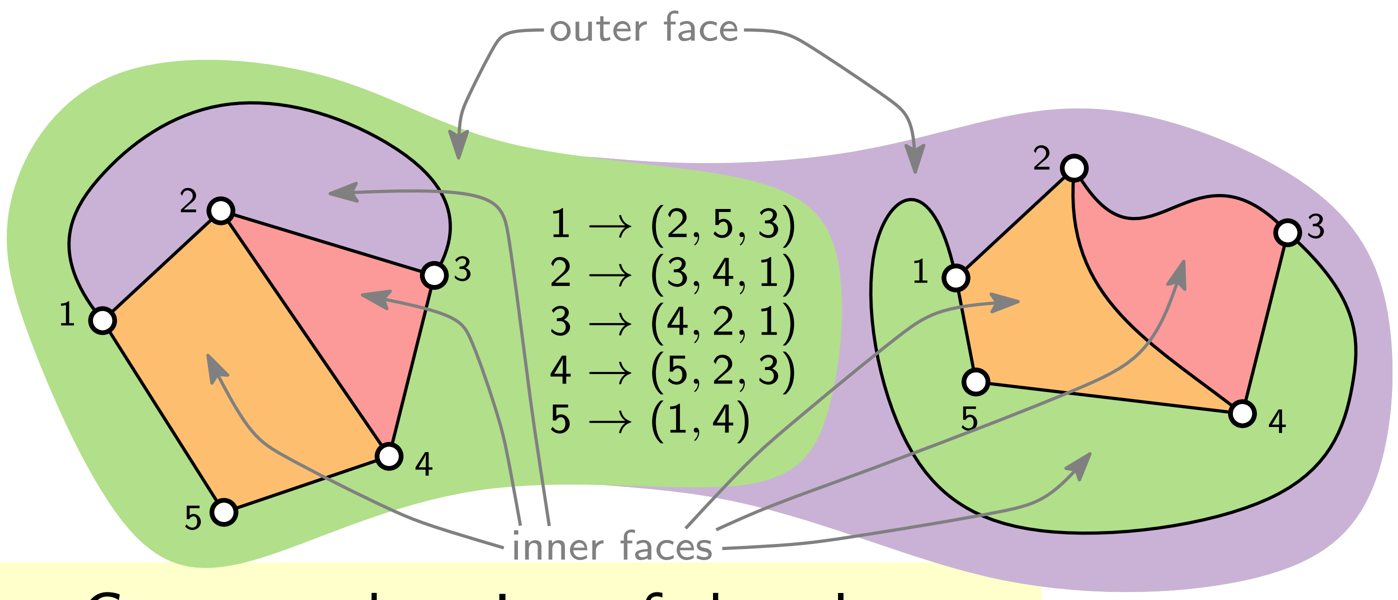

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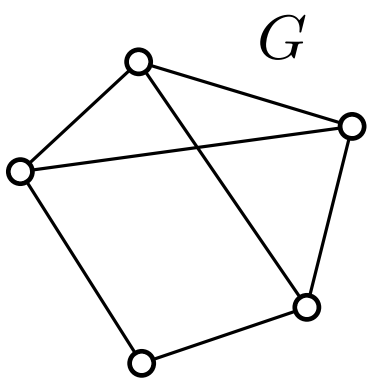
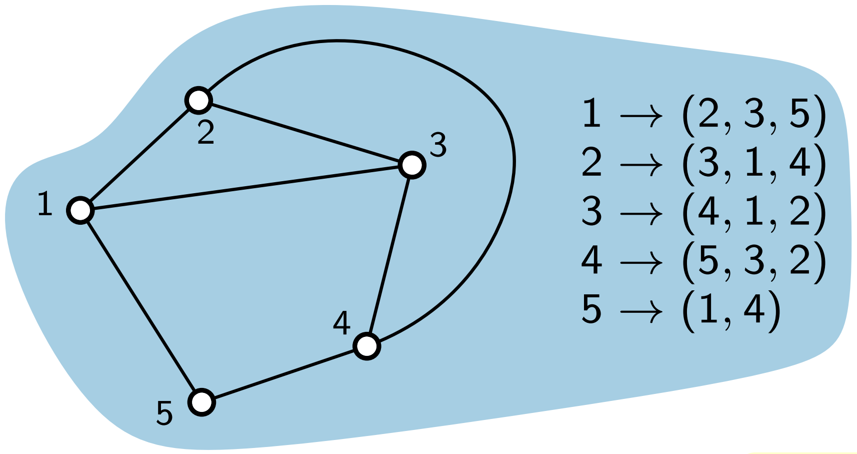
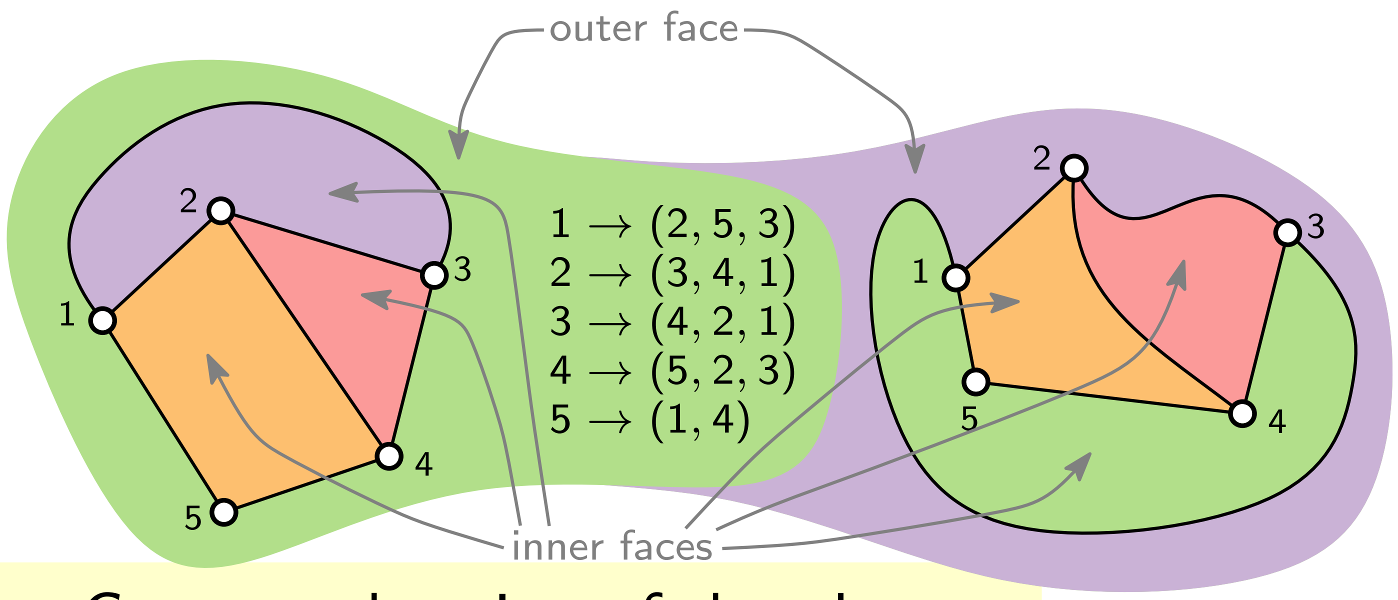
Planar Graphs







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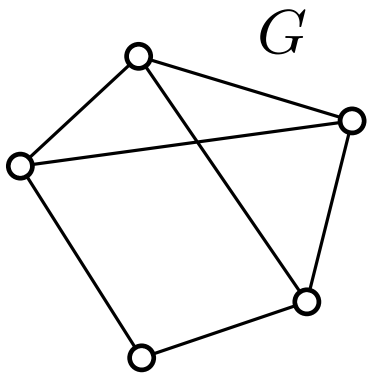

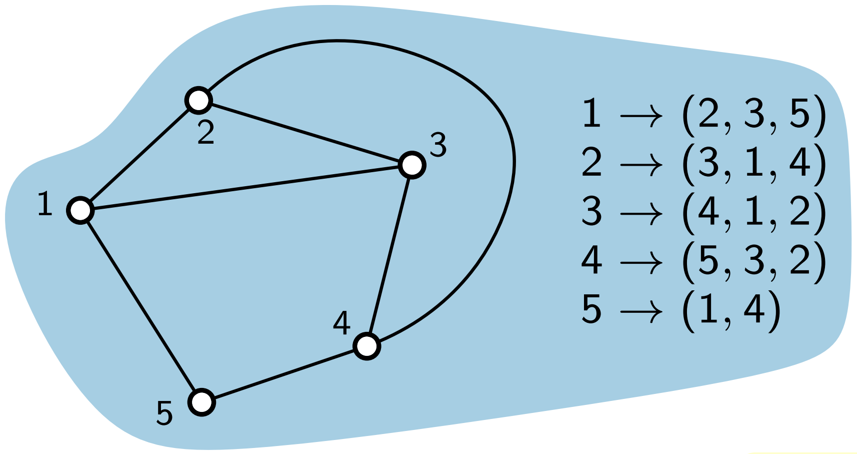

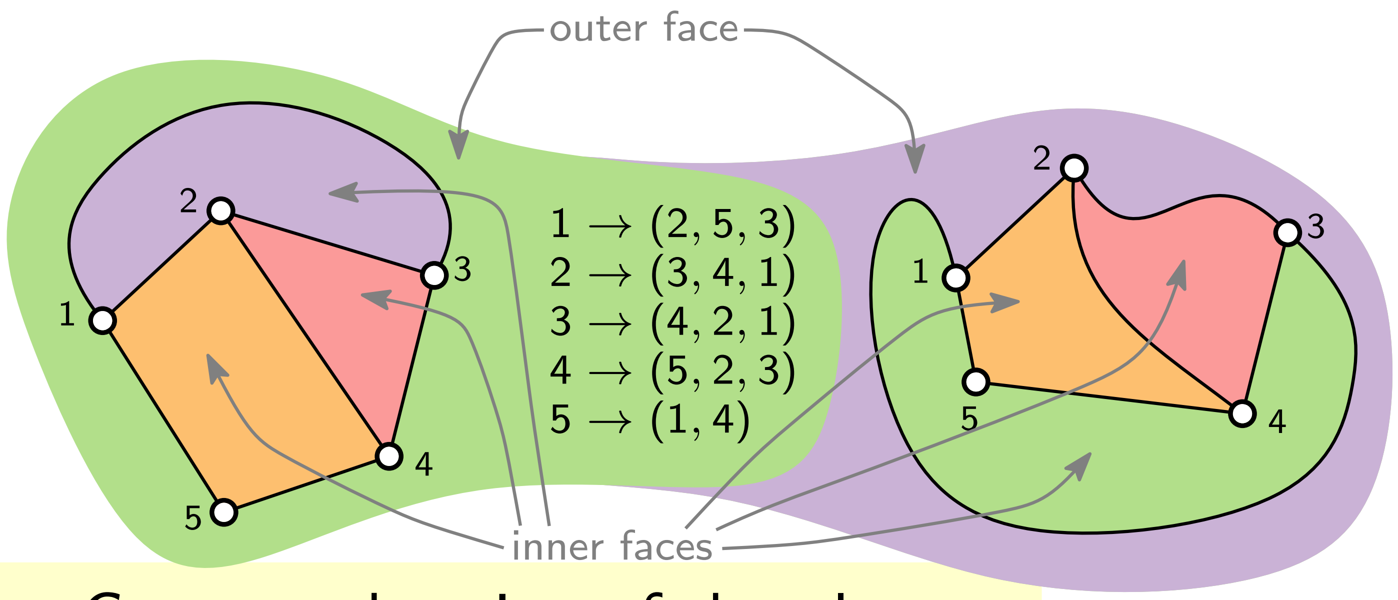

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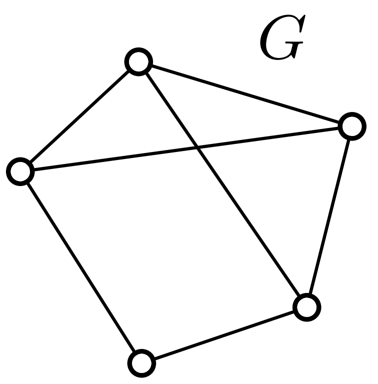

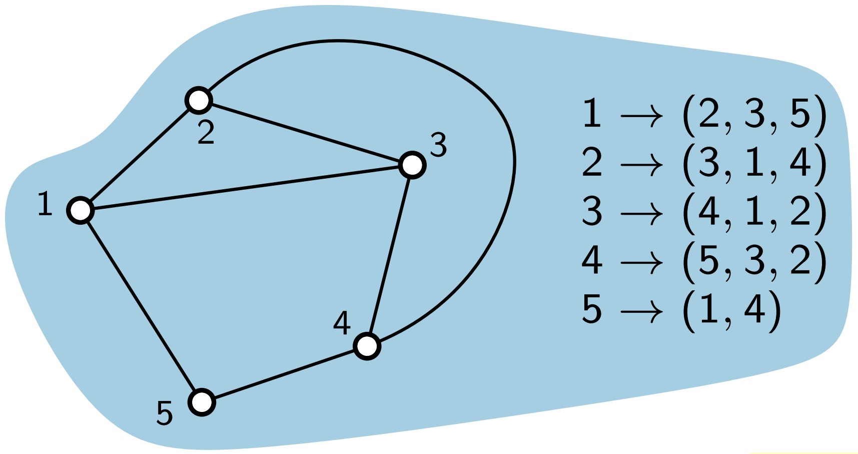

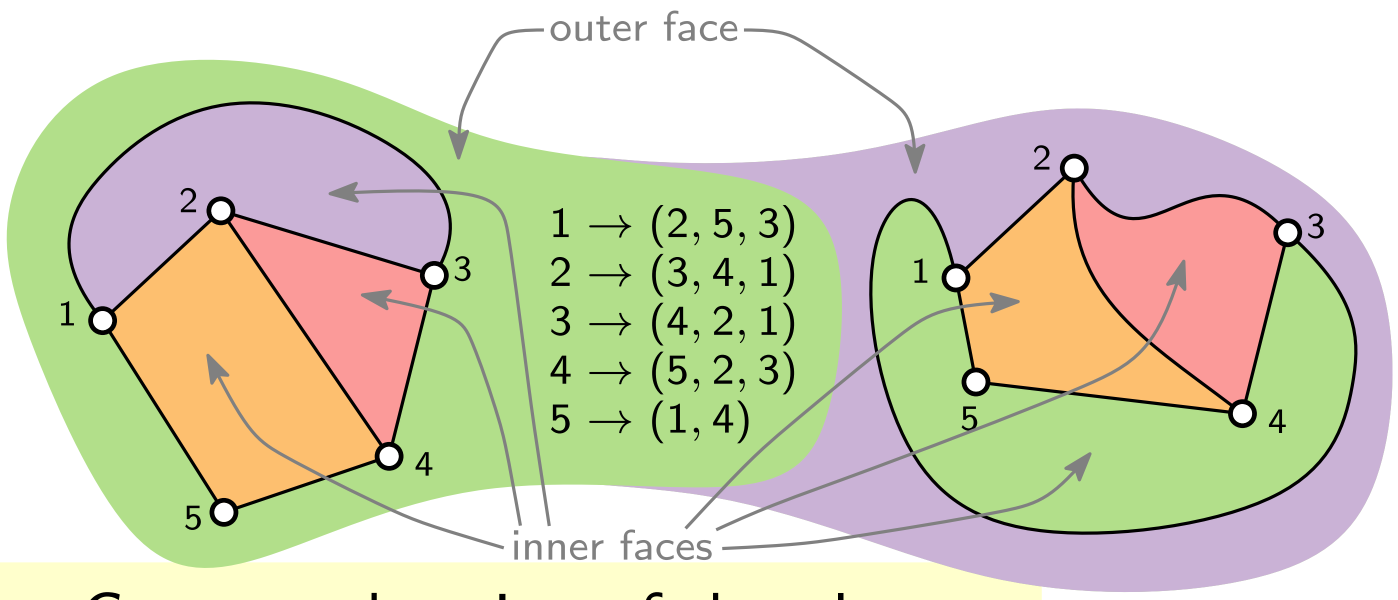

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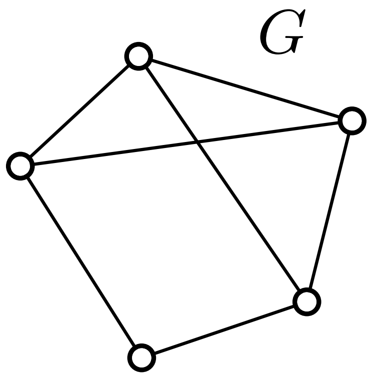

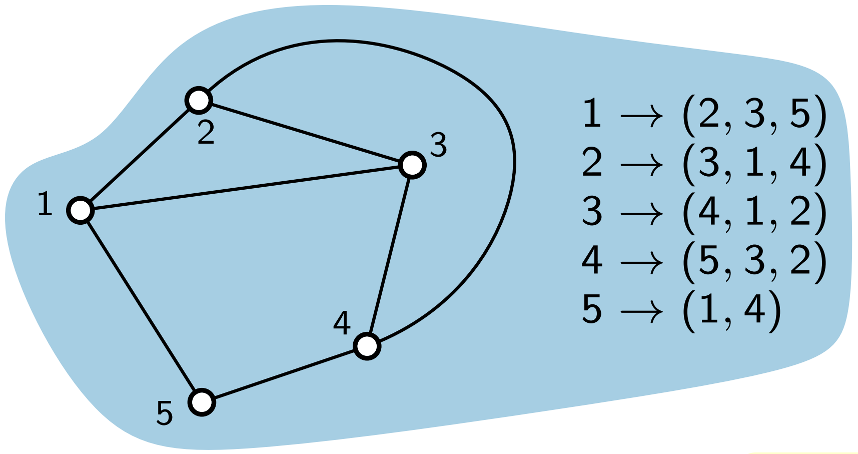

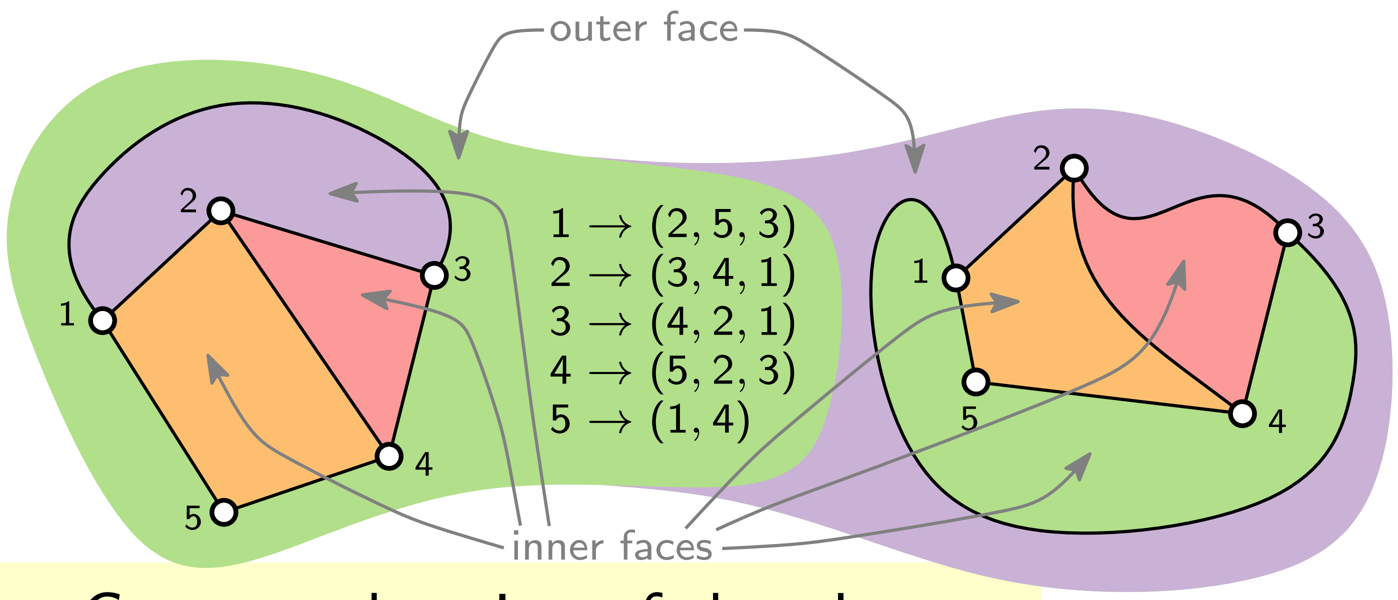

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$\Rightarrow c + 1$

$\Rightarrow f - 1$

Properties of Planar Graphs

Euler's polyhedra formula.

#faces - #edges $+$ #vertices $=$ #conn.comp. $+ 1$

$f - m + n = c + 1$

Properties of Planar Graphs

Euler's polyhedra formula.

#faces - #edges $+$ #vertices $=$ #conn.comp. $+ 1$

f - m $+$ n $=$ c $+ 1$

Theorem. G simple planar graph with $n \geq 3$.

Properties of Planar Graphs

Euler's polyhedra formula.

#faces - #edges + #vertices = #conn.comp. + 1

f - m + n = c + 1

Theorem. G simple planar graph with $n \geq 3$.

1. $m \leq 3n - 6$

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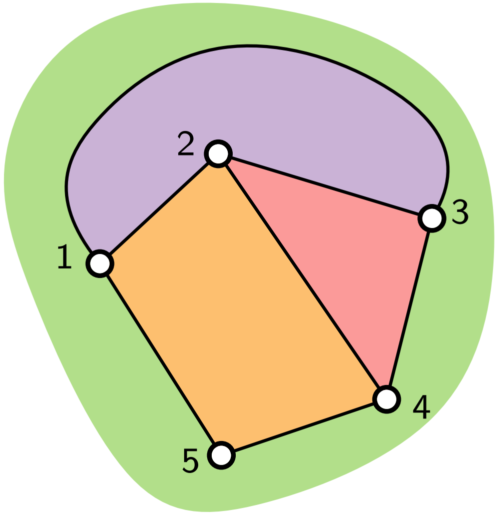

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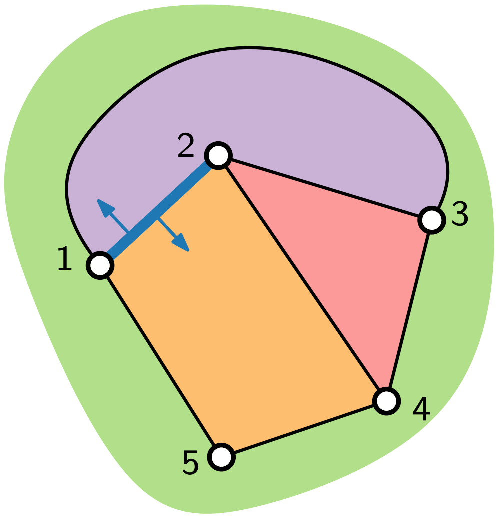

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Properties of Planar Graphs

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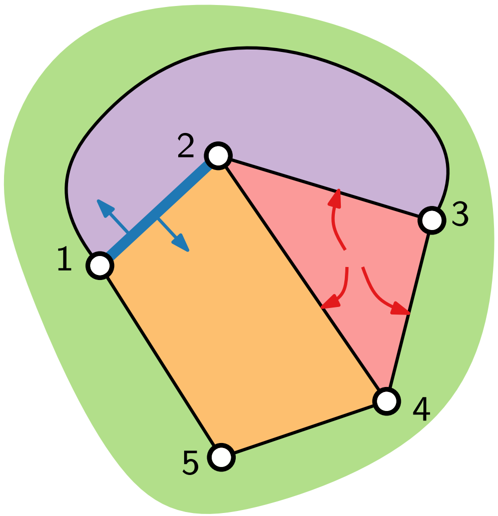

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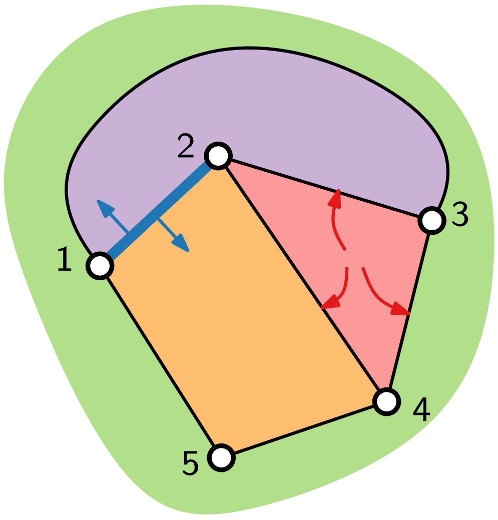
$\Rightarrow 3f \leq 2m$



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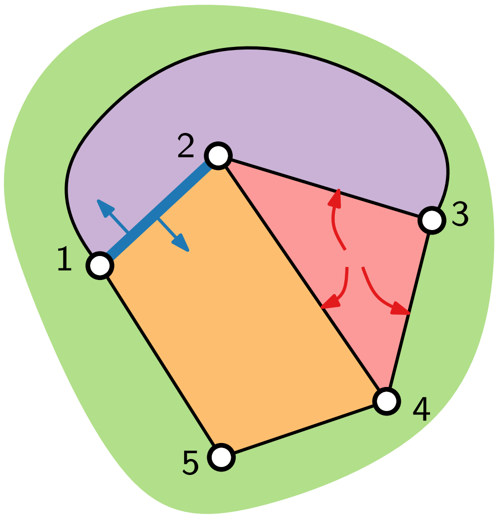

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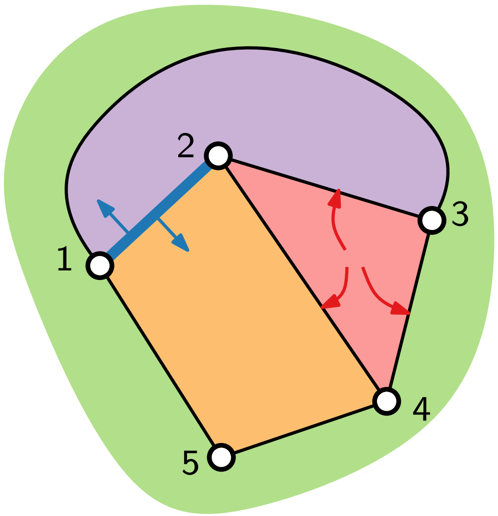

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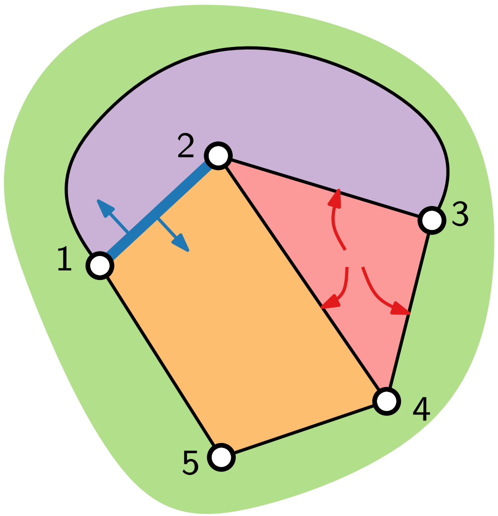

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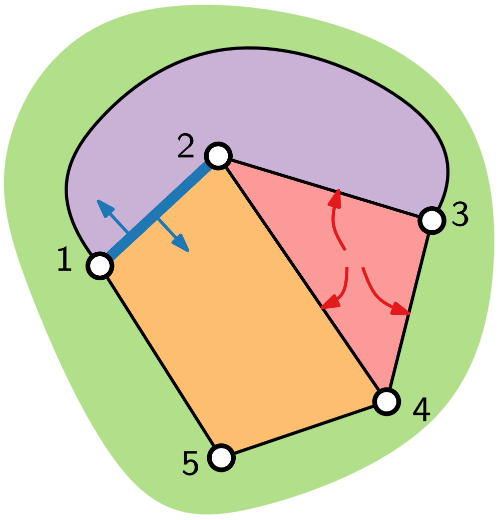

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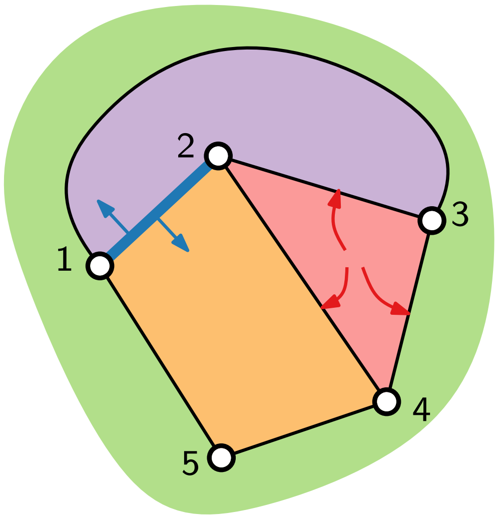

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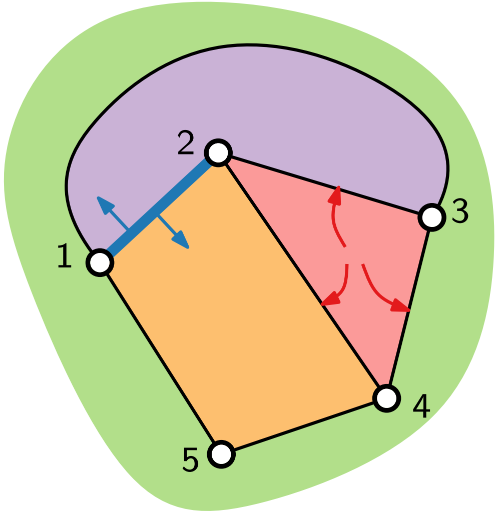

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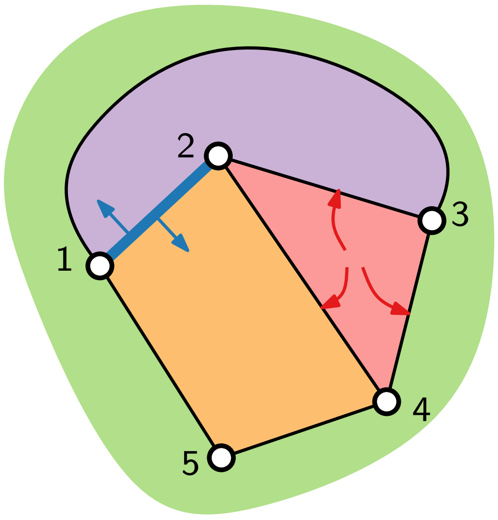

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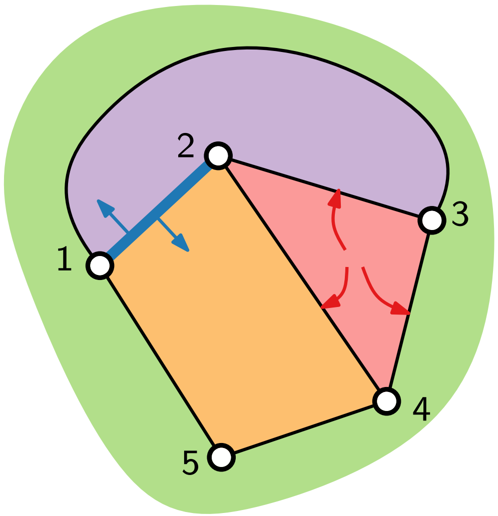

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2. $3f \leq 2m \leq 6n - 12$

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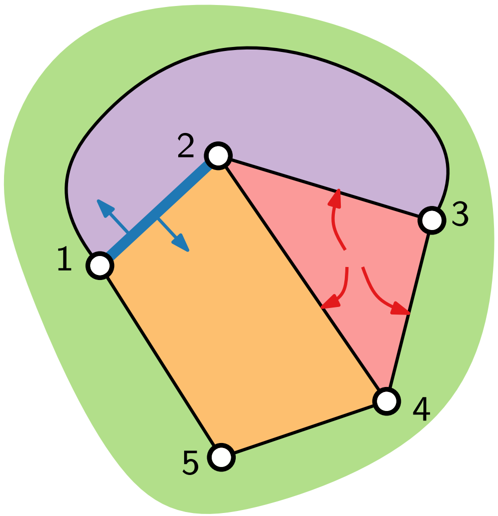

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2. $3f \leq 2m \leq 6n - 12 \Rightarrow f \leq 2n - 4$

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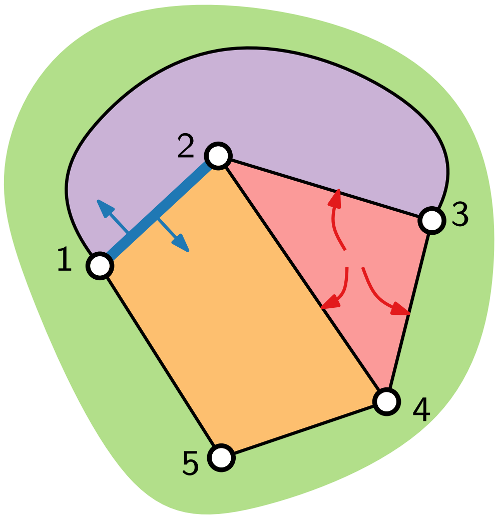
f - m + n = c + 1

Theorem. G simple planar graph with $n \geq 3$.

1. $m \leq 3n - 6$
2. $f \leq 2n - 4$
3. There is a vertex of degree at most five



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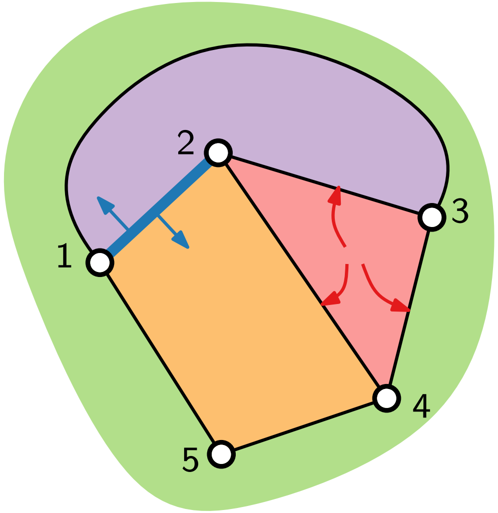

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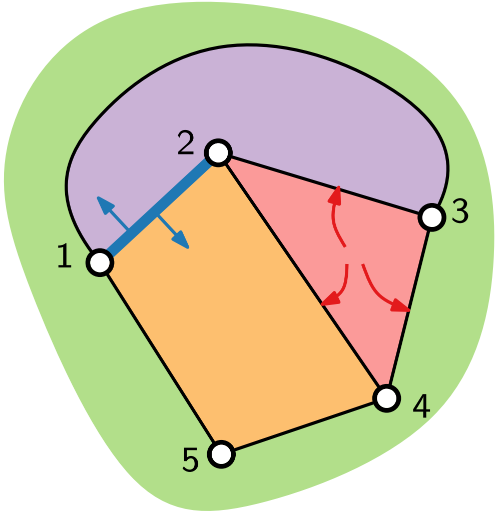

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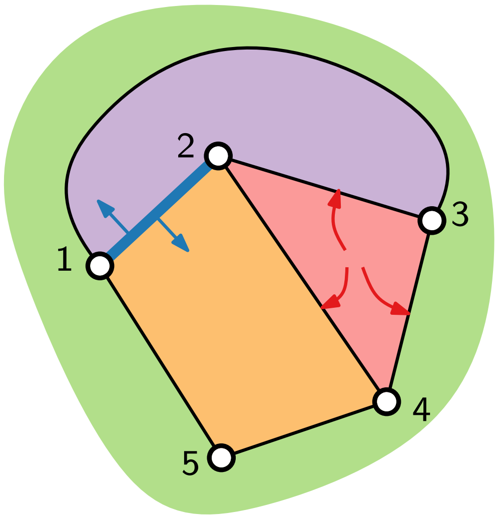

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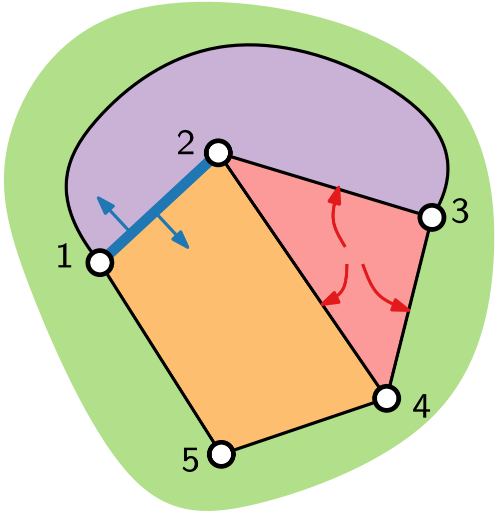

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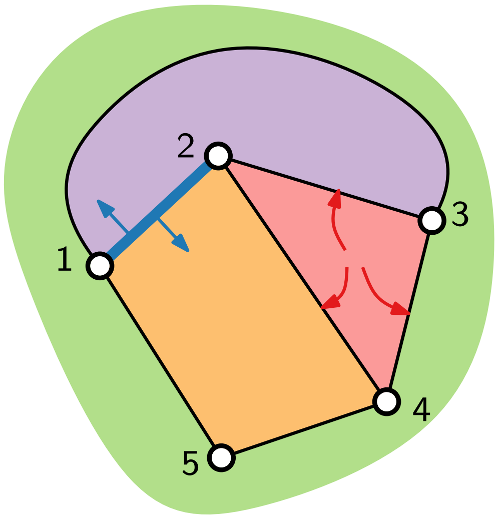

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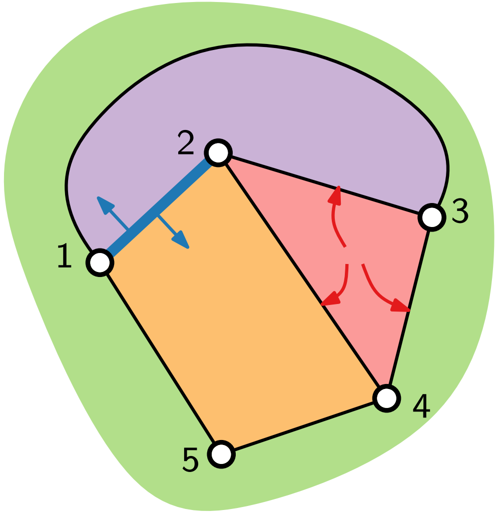

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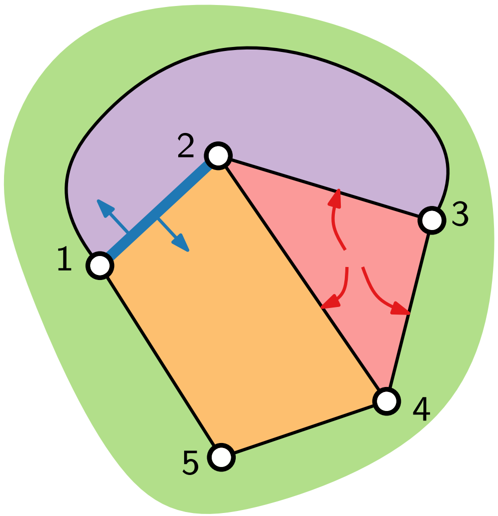

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Triangulations

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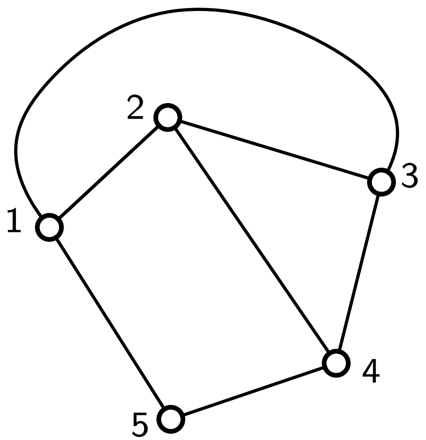

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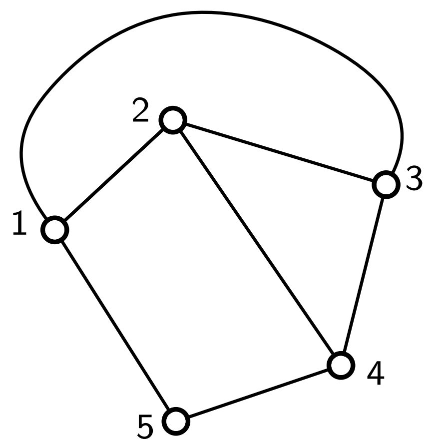

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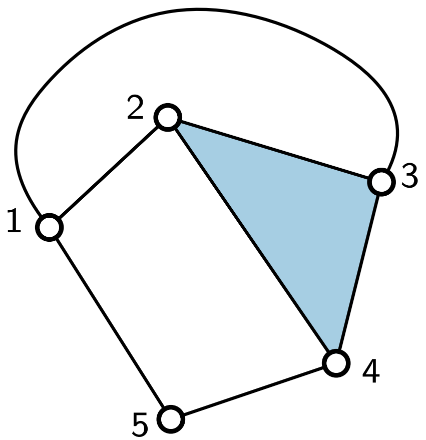

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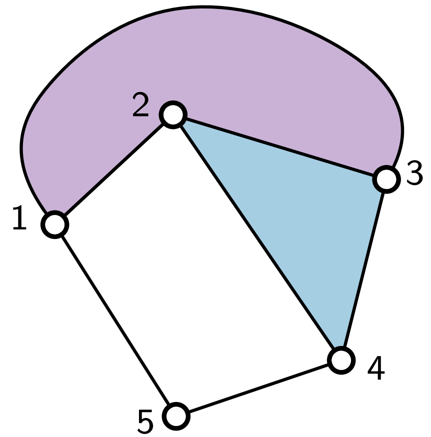

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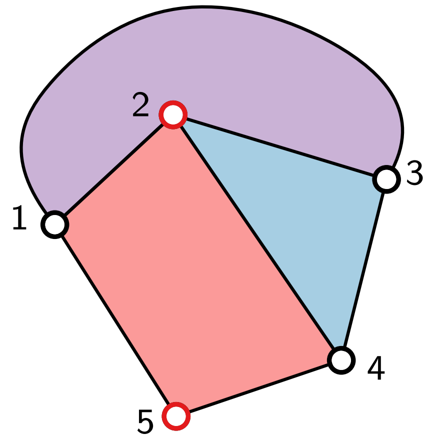

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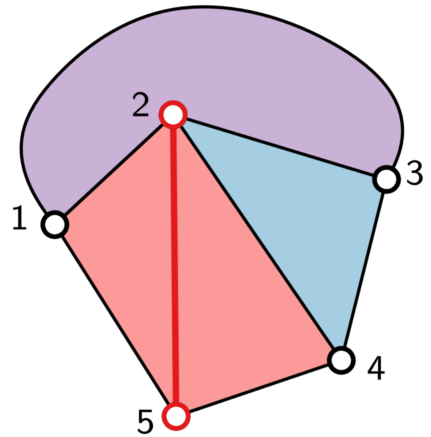

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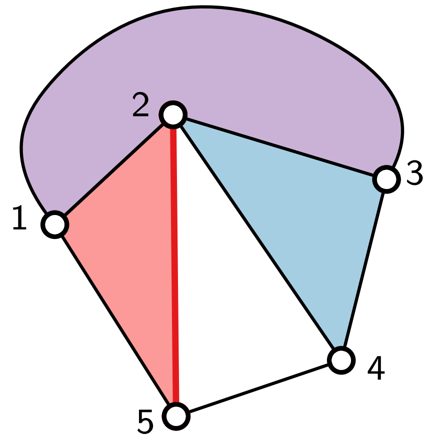

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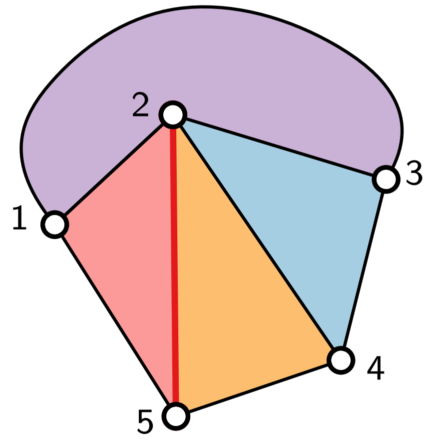

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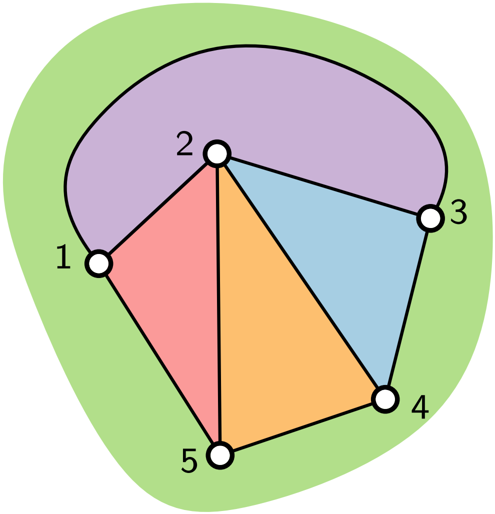

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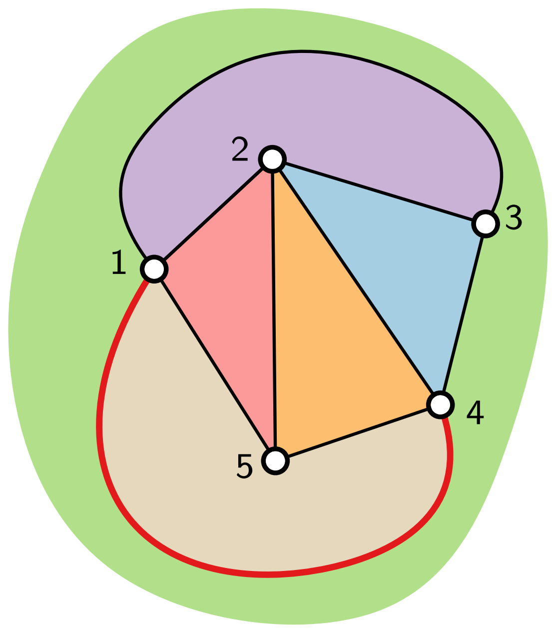

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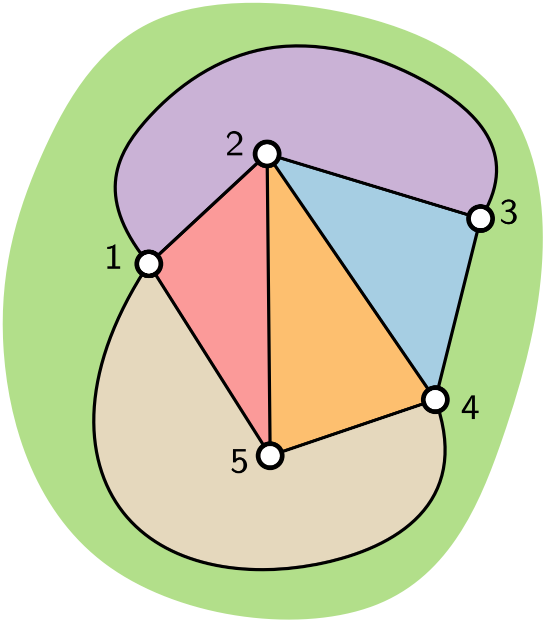

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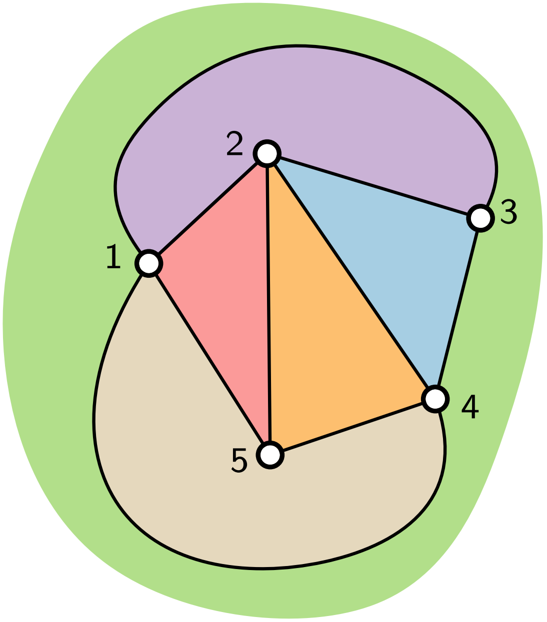

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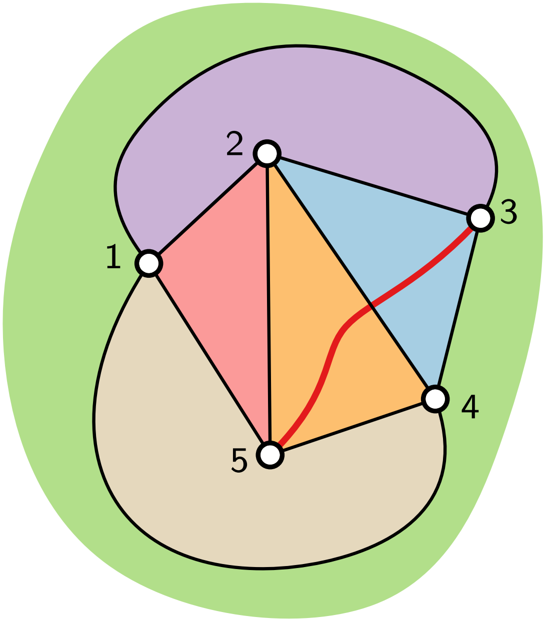

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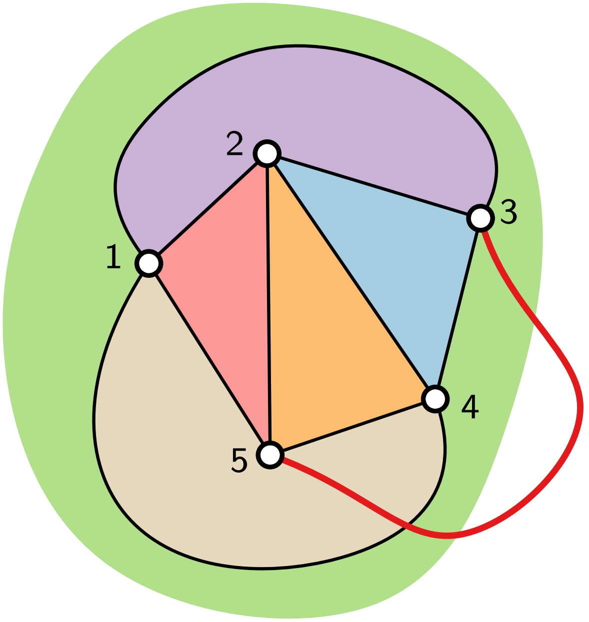

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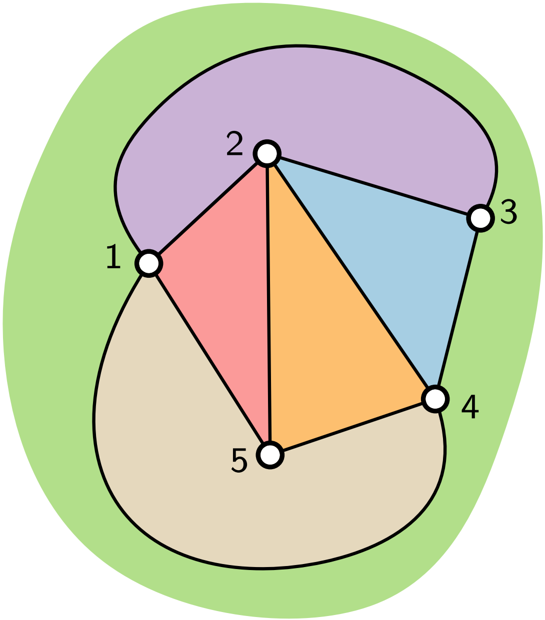

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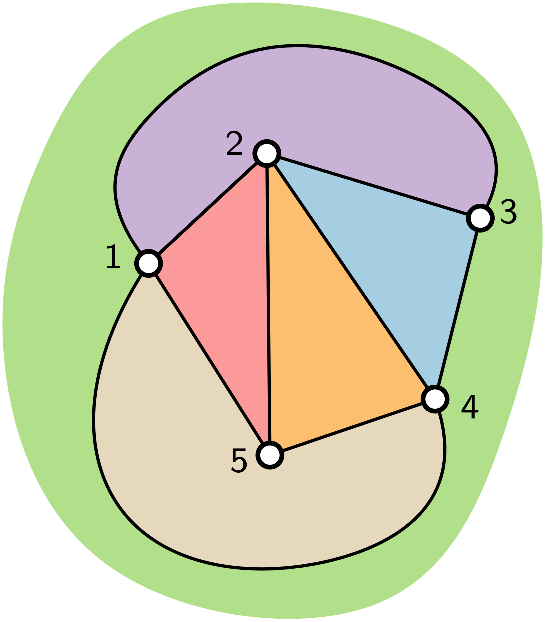

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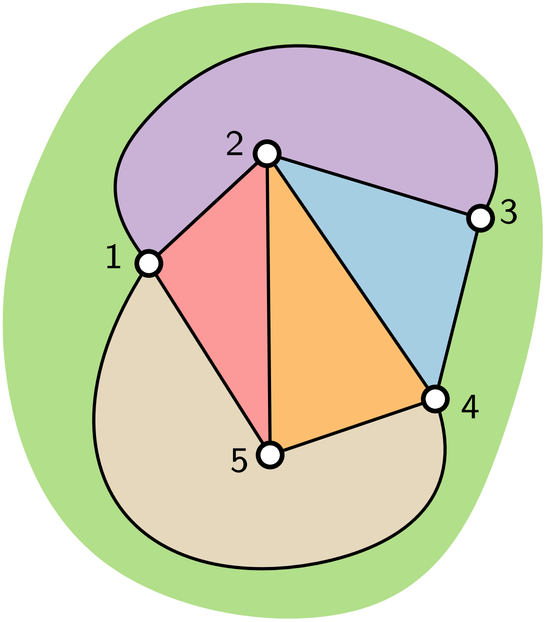

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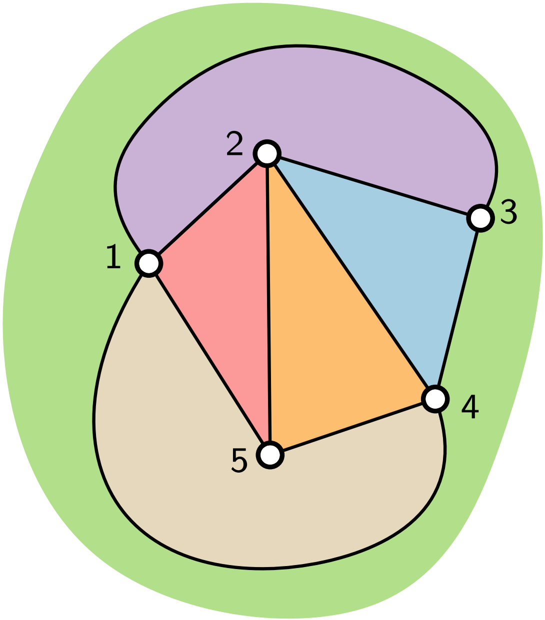

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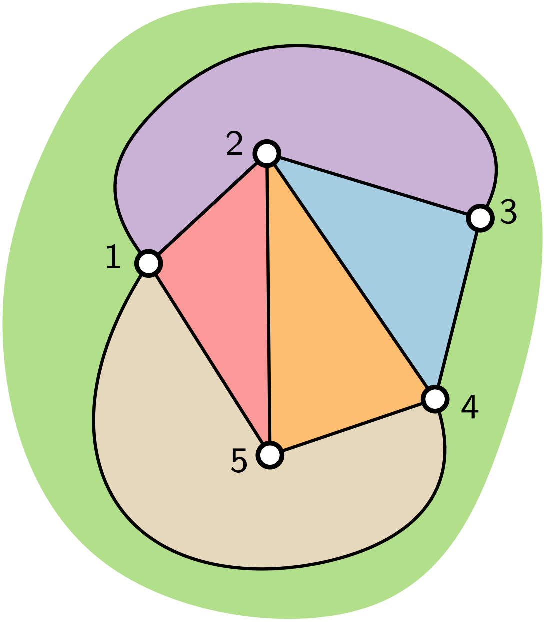

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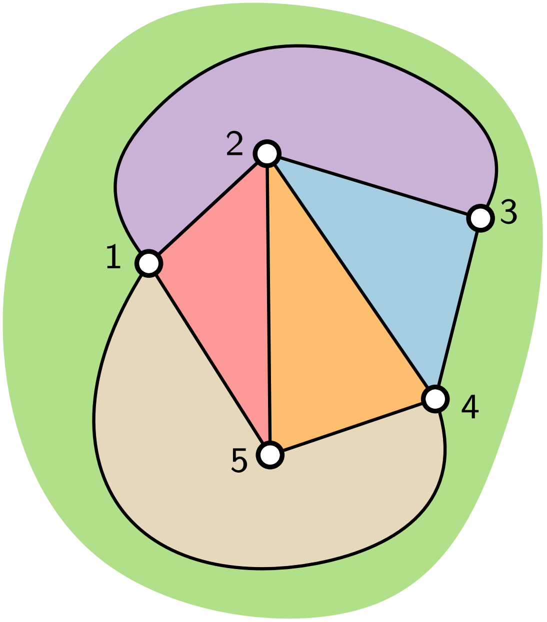

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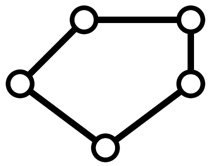

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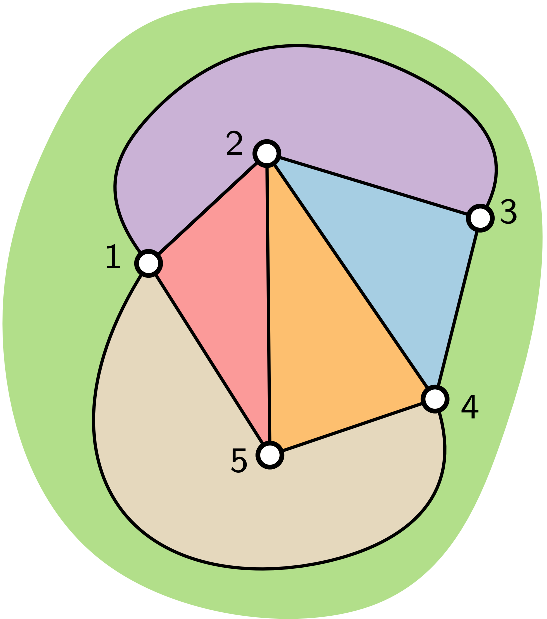

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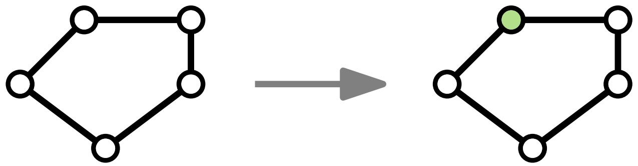

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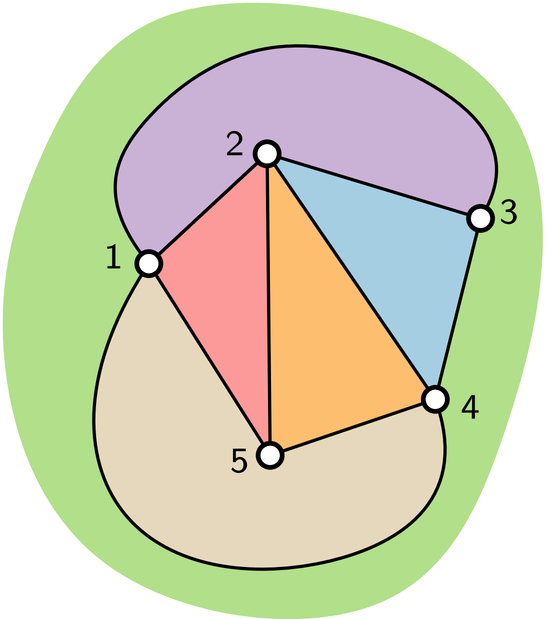

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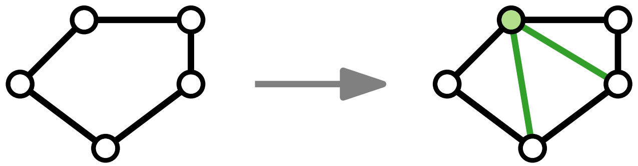

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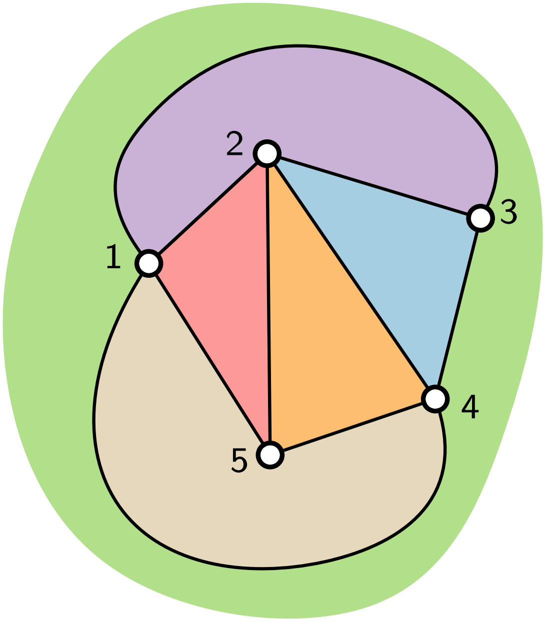

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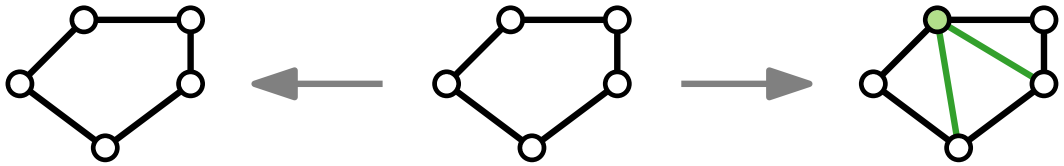

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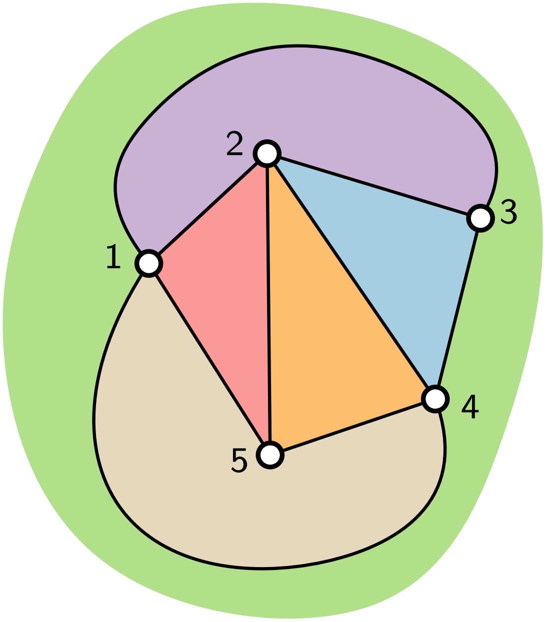

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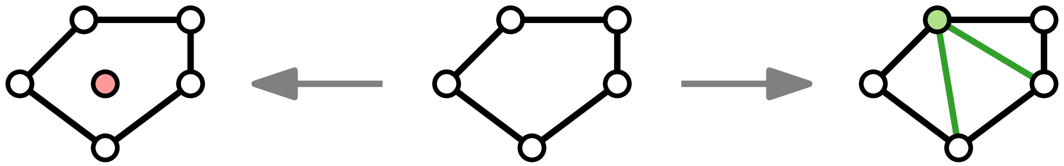

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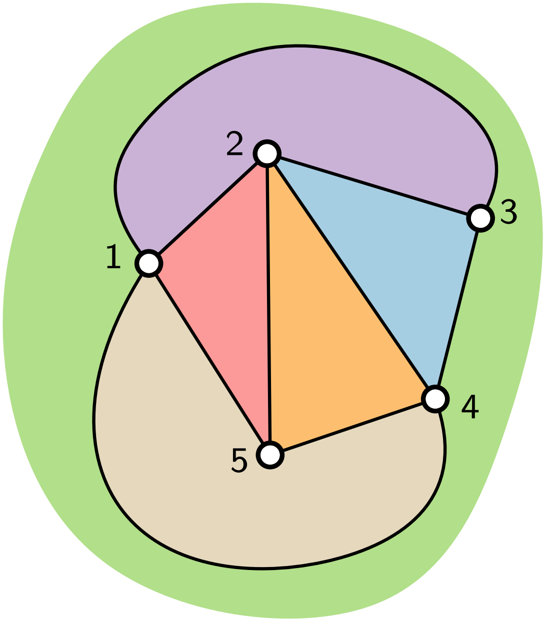

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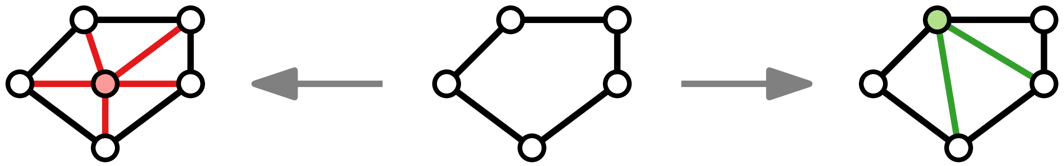

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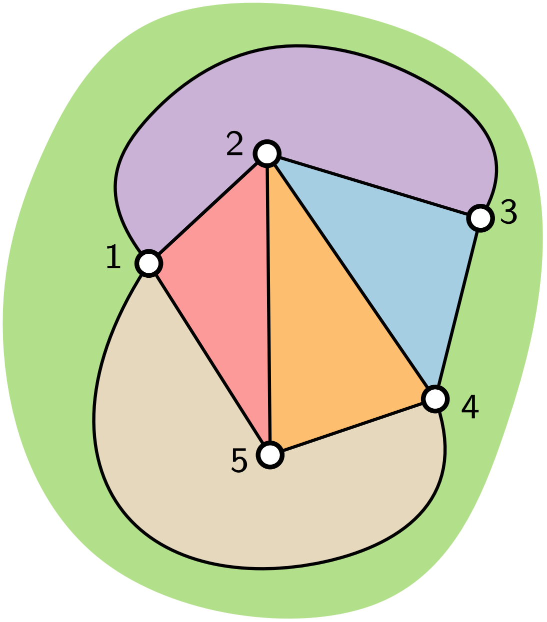

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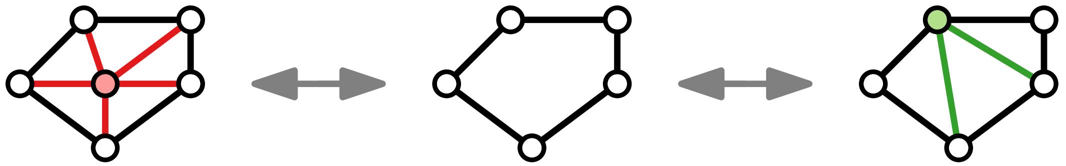

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The Aesthetics of Graph Visualization

3.2. Edge Placement Heuristics

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Drawing conventions

- No crossings $\Rightarrow$ planar
- No bends $\Rightarrow$ straight-line

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Towards Straight-Line Drawings

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Towards Straight-Line Drawings

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Recognition

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Towards Straight-Line Drawings

Theorem. [Kuratowski 1930]
G planar $\Leftrightarrow$
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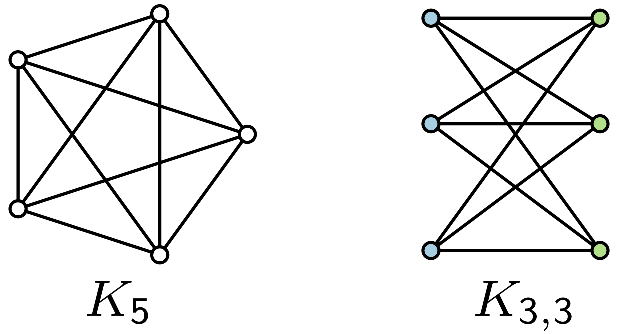

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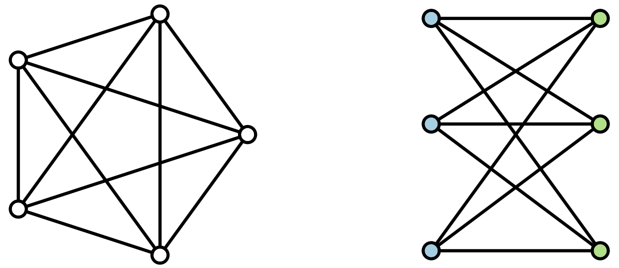

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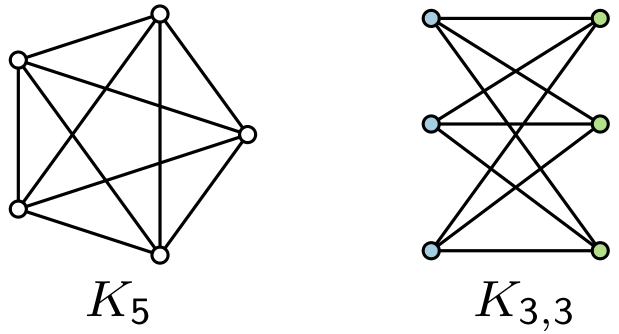

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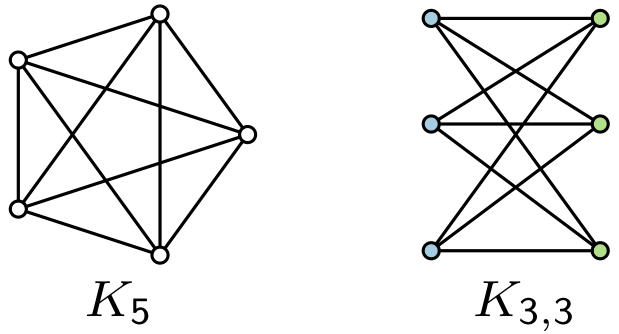

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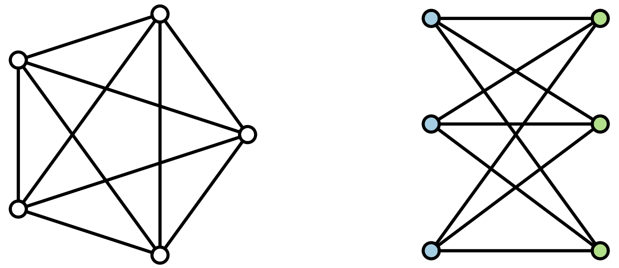

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The algorithms implied by this theory produce drawings with area **not** bounded by any polynomial on n.

Planar straight-line drawings

Theorem. [De Fraysseix, Pach, Pollack '90]
Every n-vertex planar graph has a planar straight-line drawing of size $(2n-4) \times (n-2)$.

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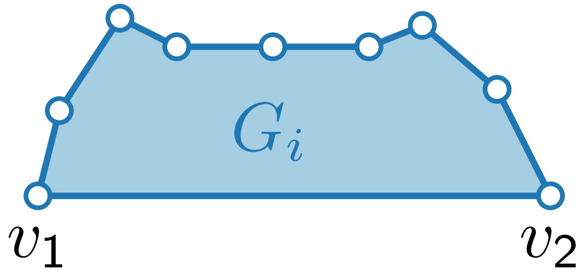

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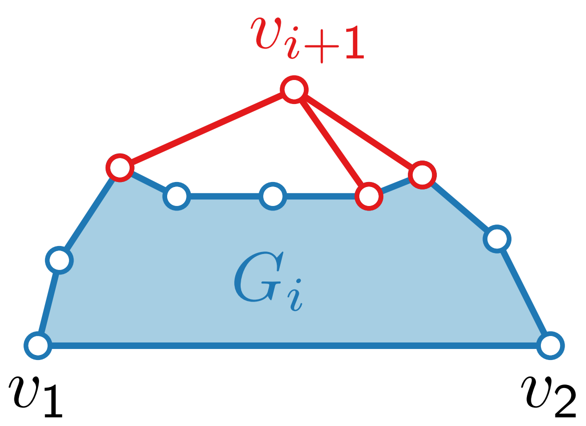

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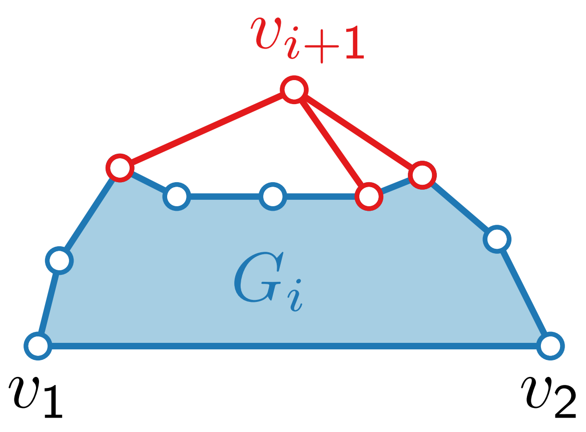

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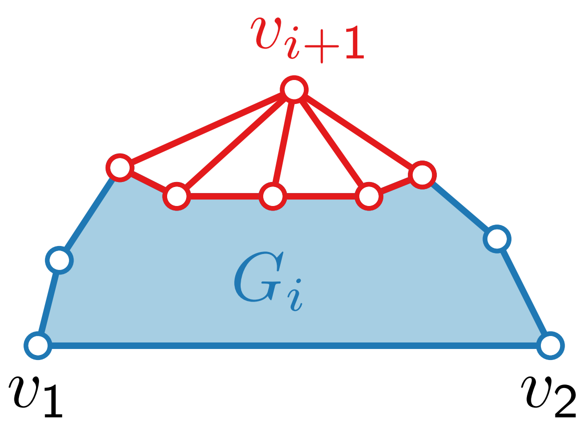

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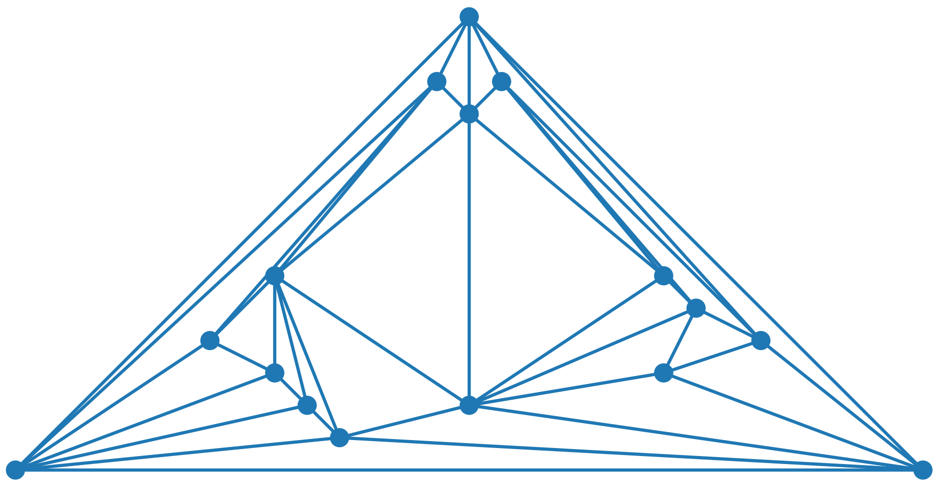
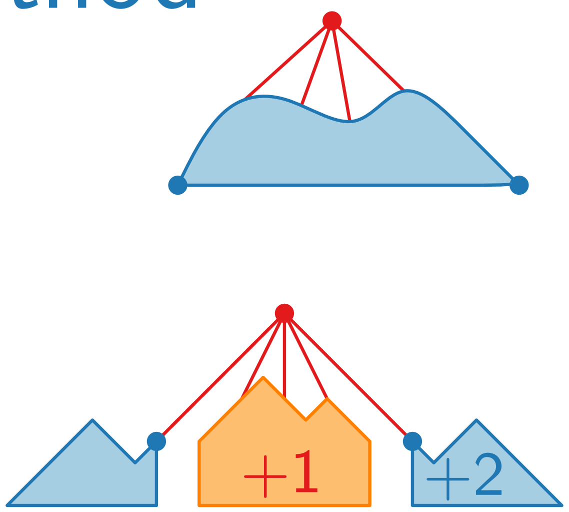
Julius-Maximilians-UNIVERSITÄT WÜRZBURG

Visualization of Graphs

Lecture 3: Straight-Line Drawings of Planar Graphs I: Canonical Ordering and Shift Method



Part II: Canonical Order



Jonathan Klawitter

Canonical Order – Definition

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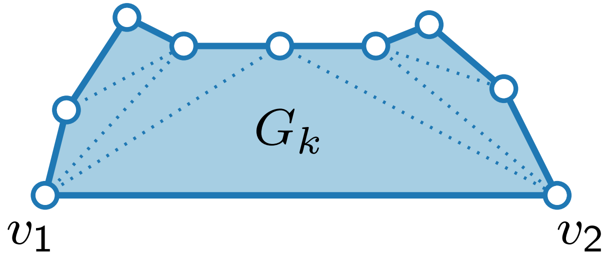

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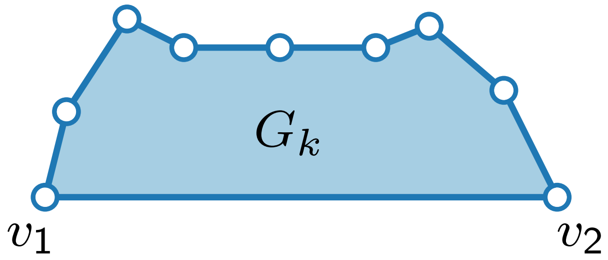

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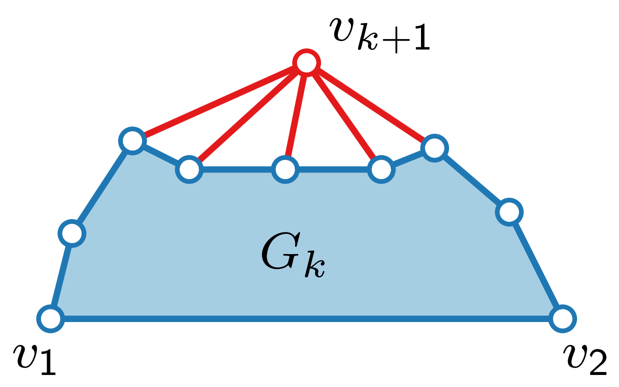

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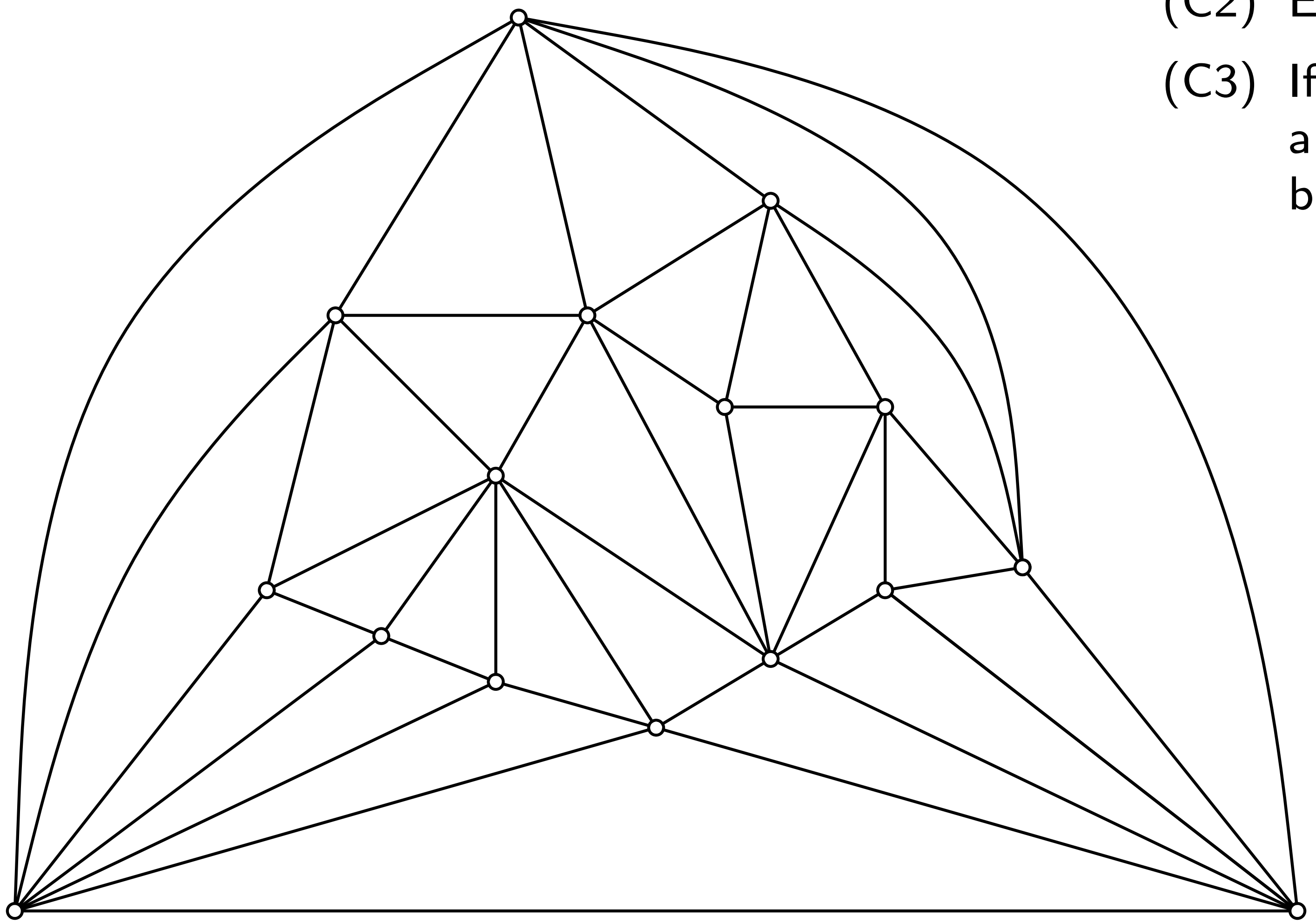

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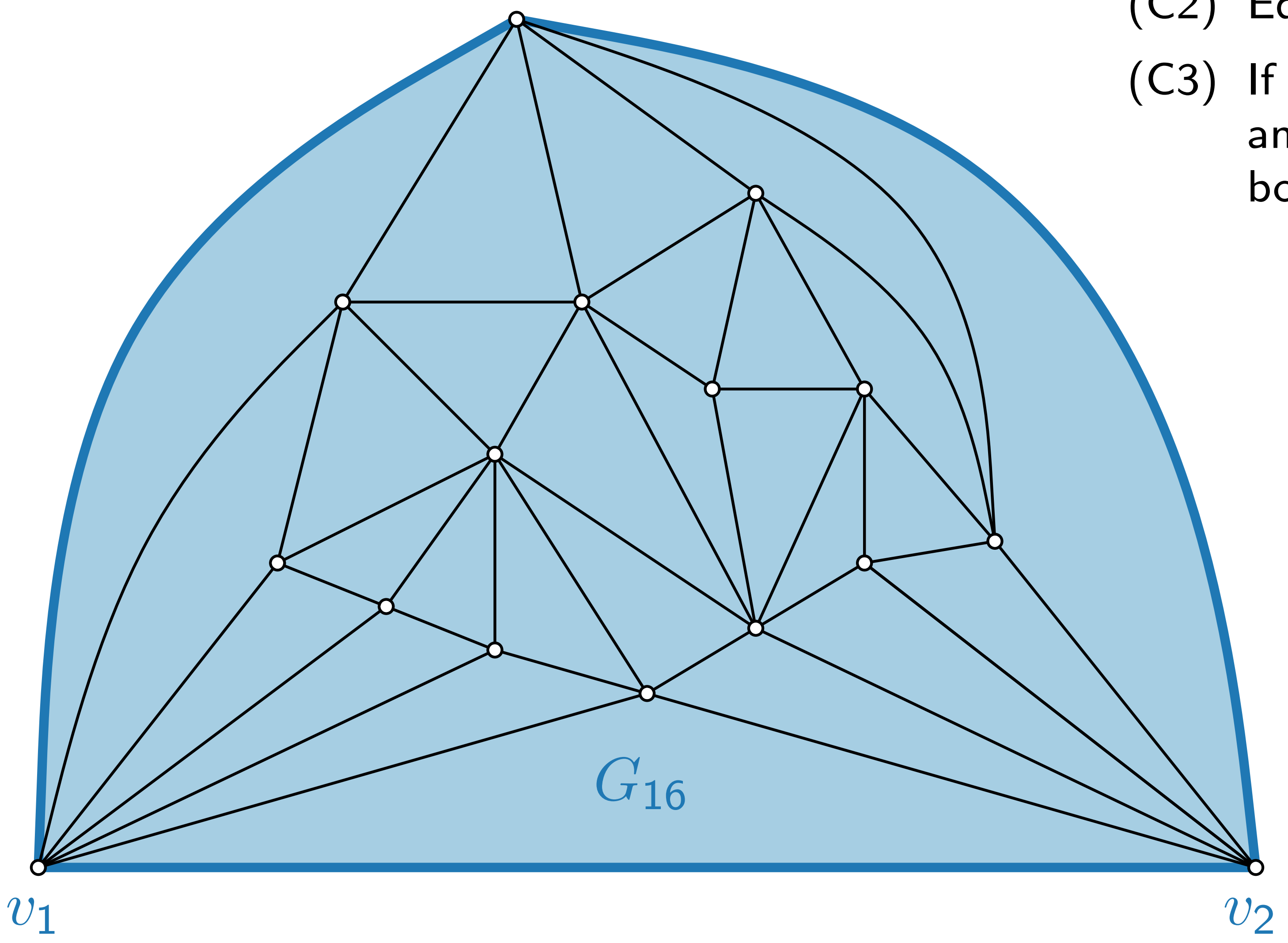

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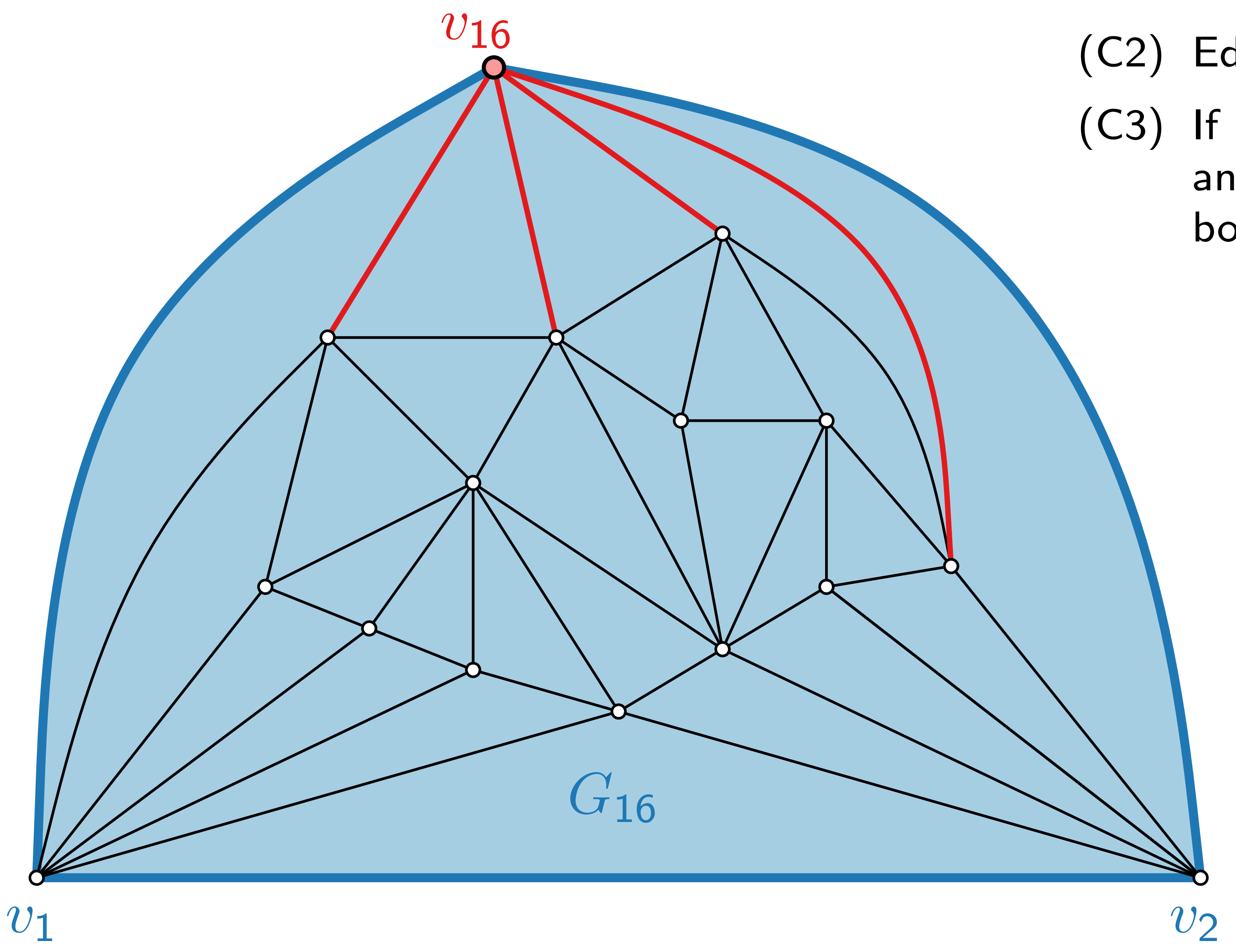

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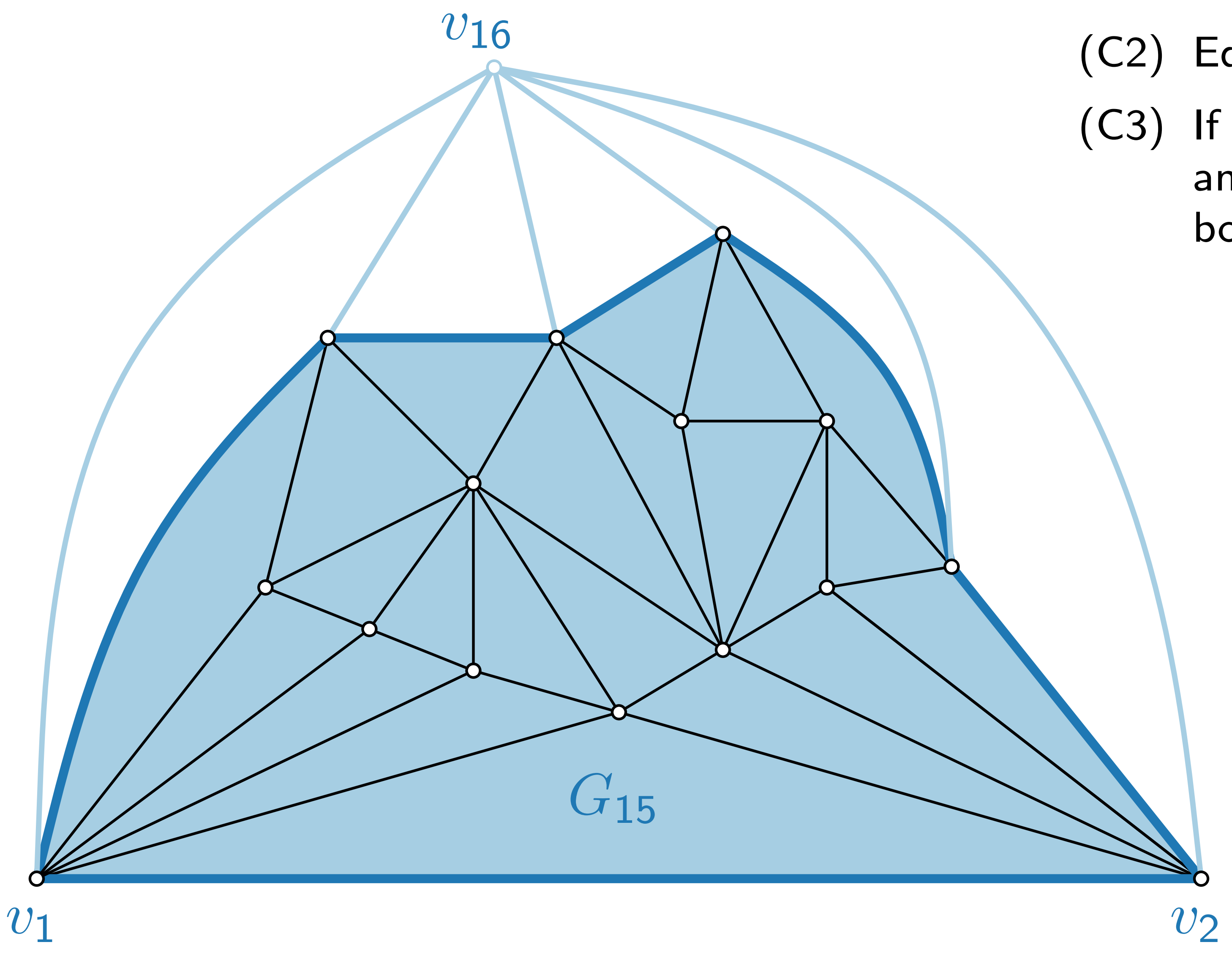

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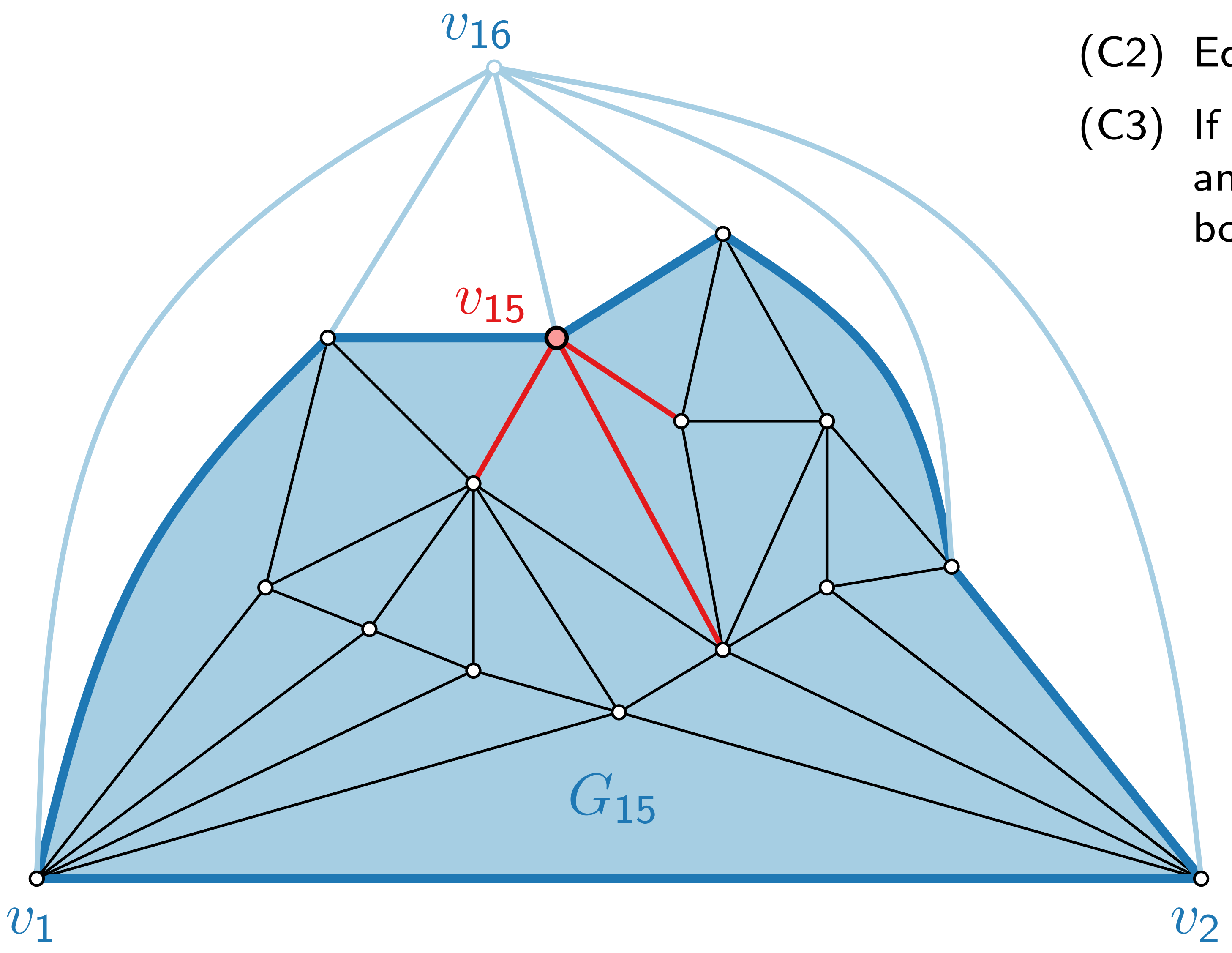

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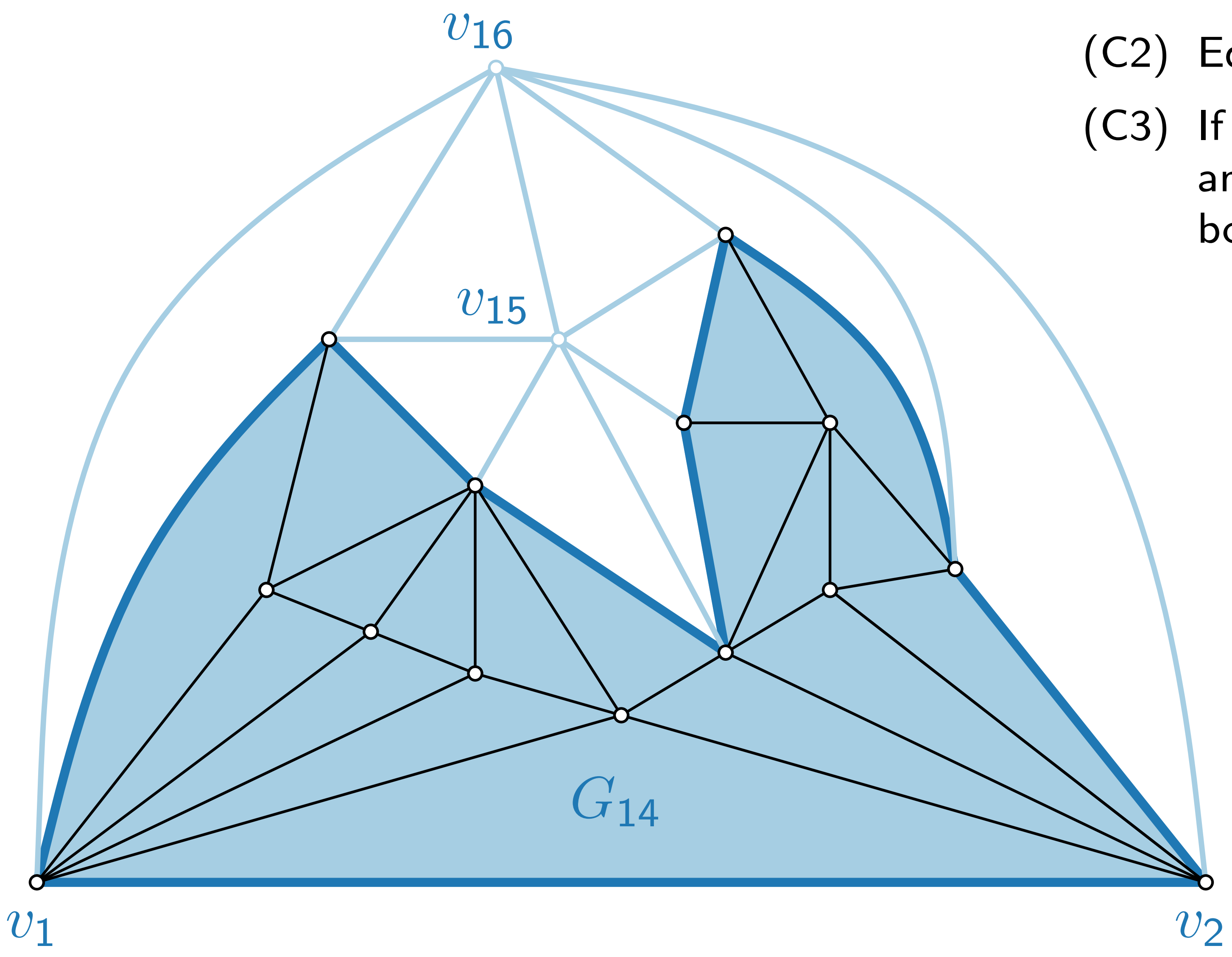

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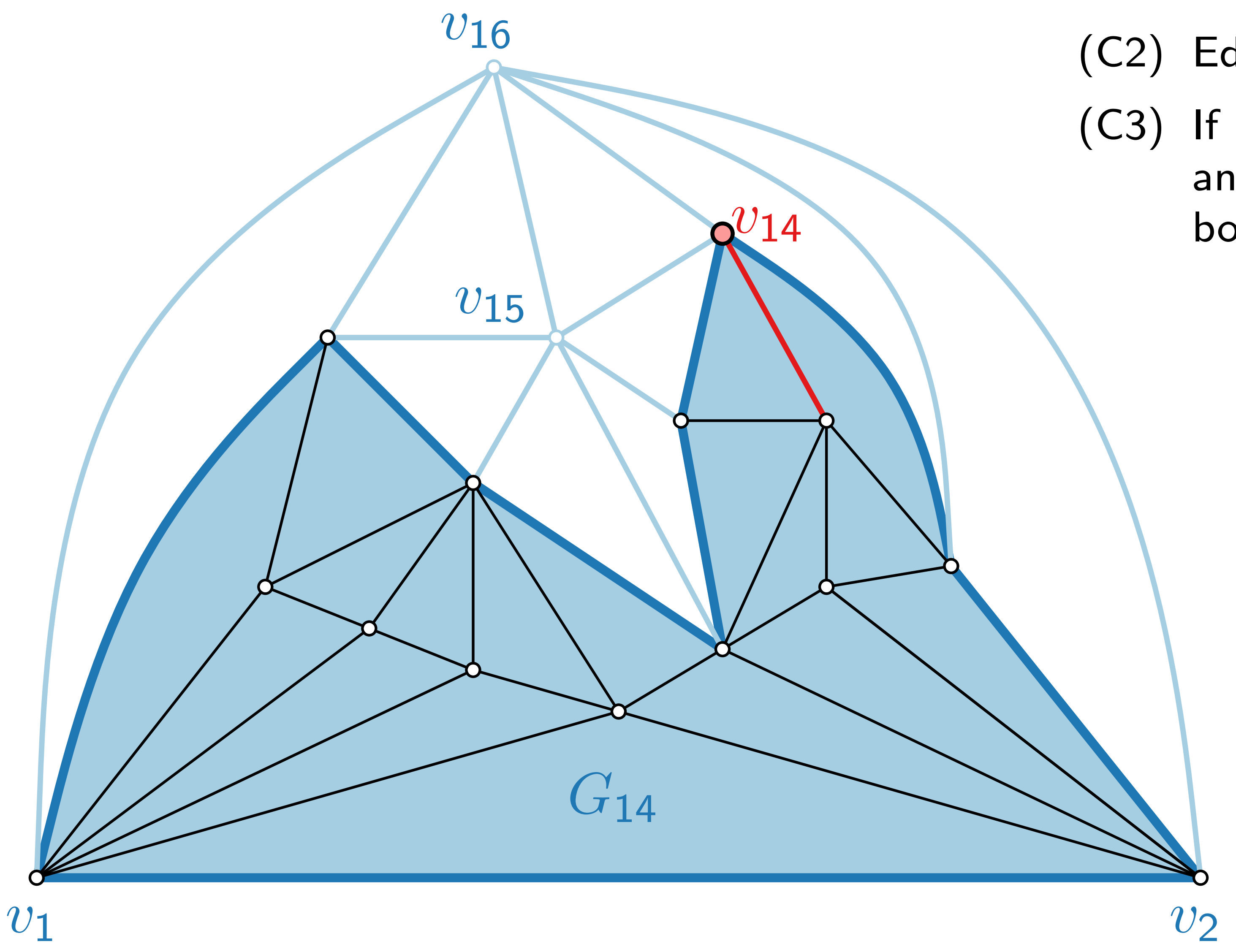

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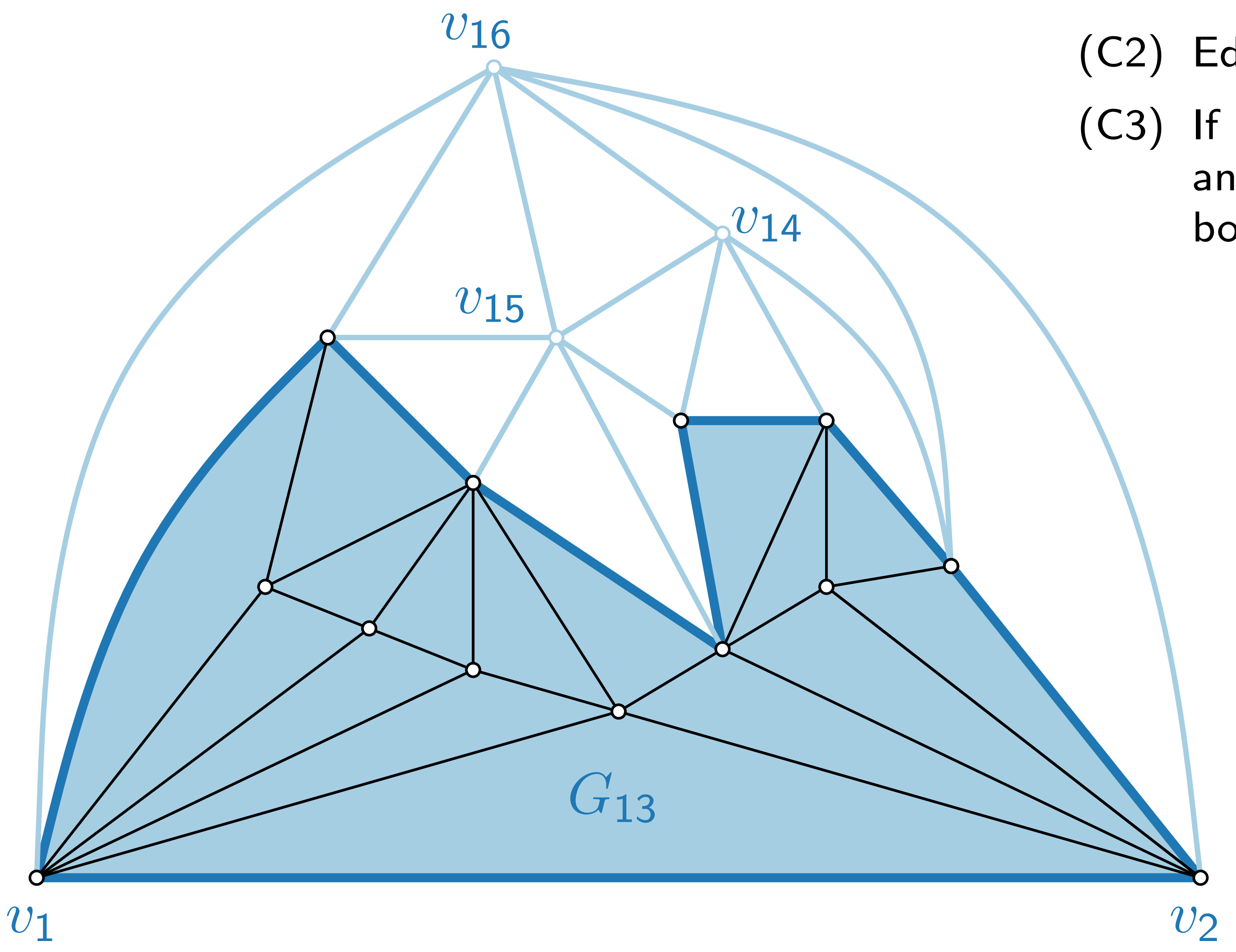

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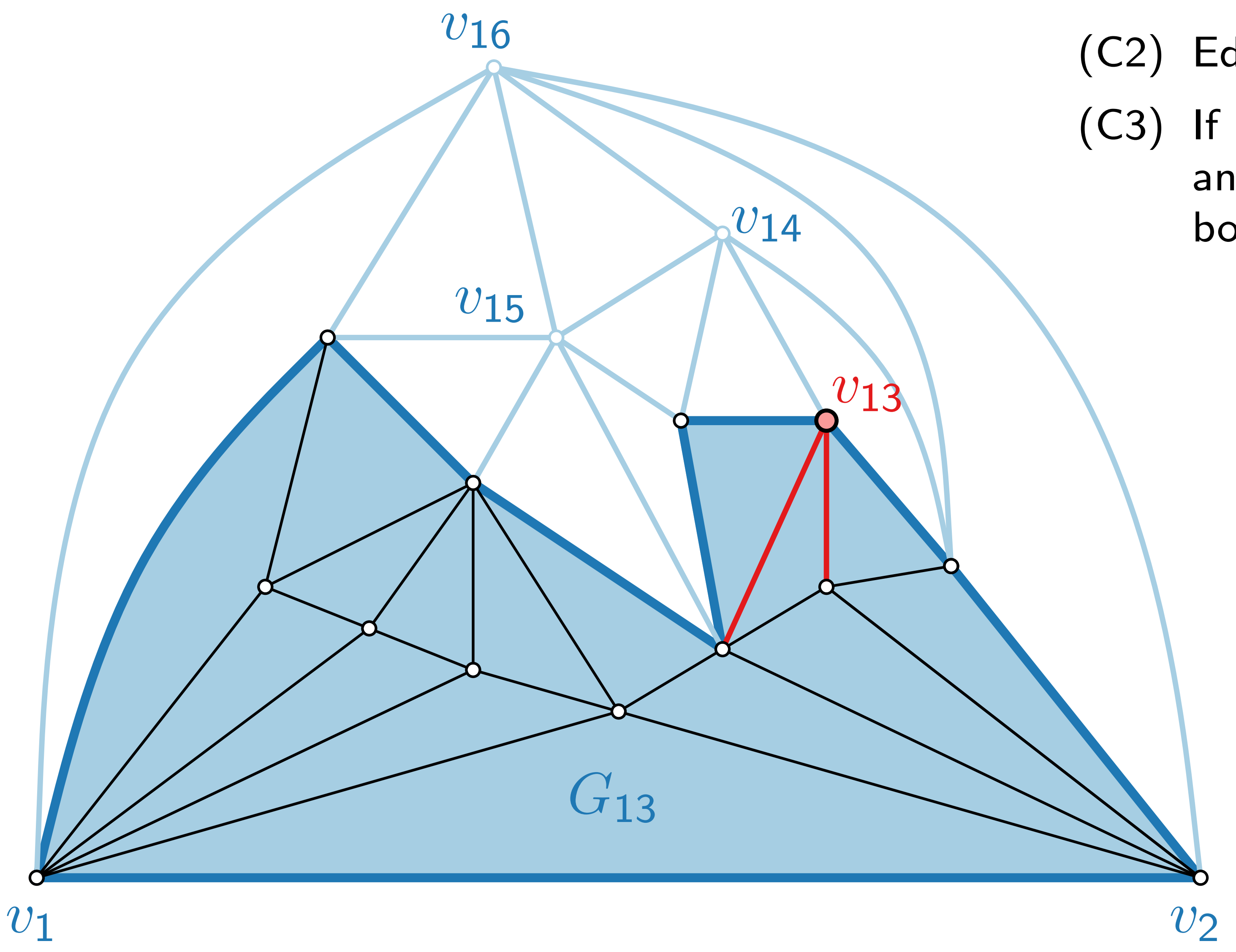

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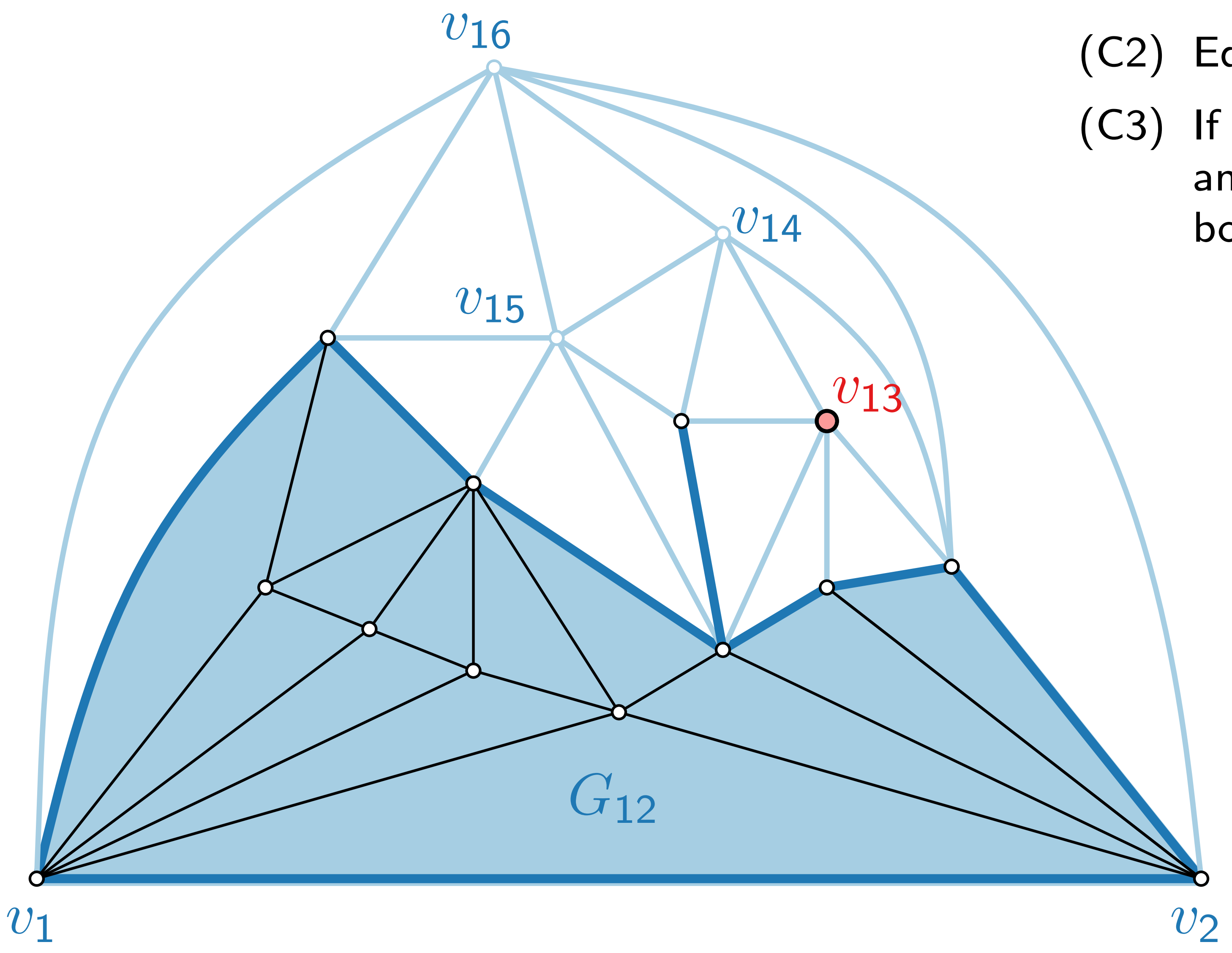

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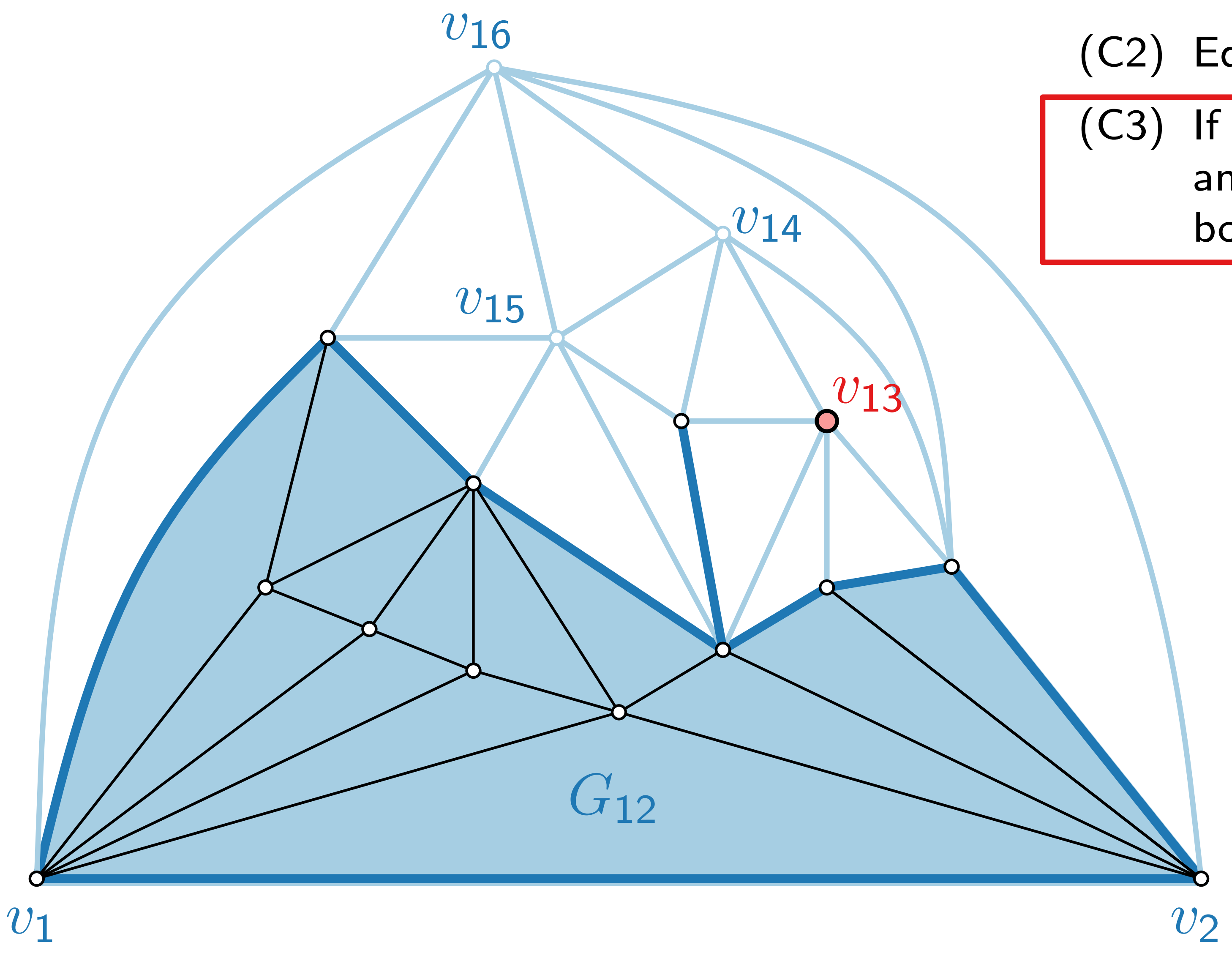

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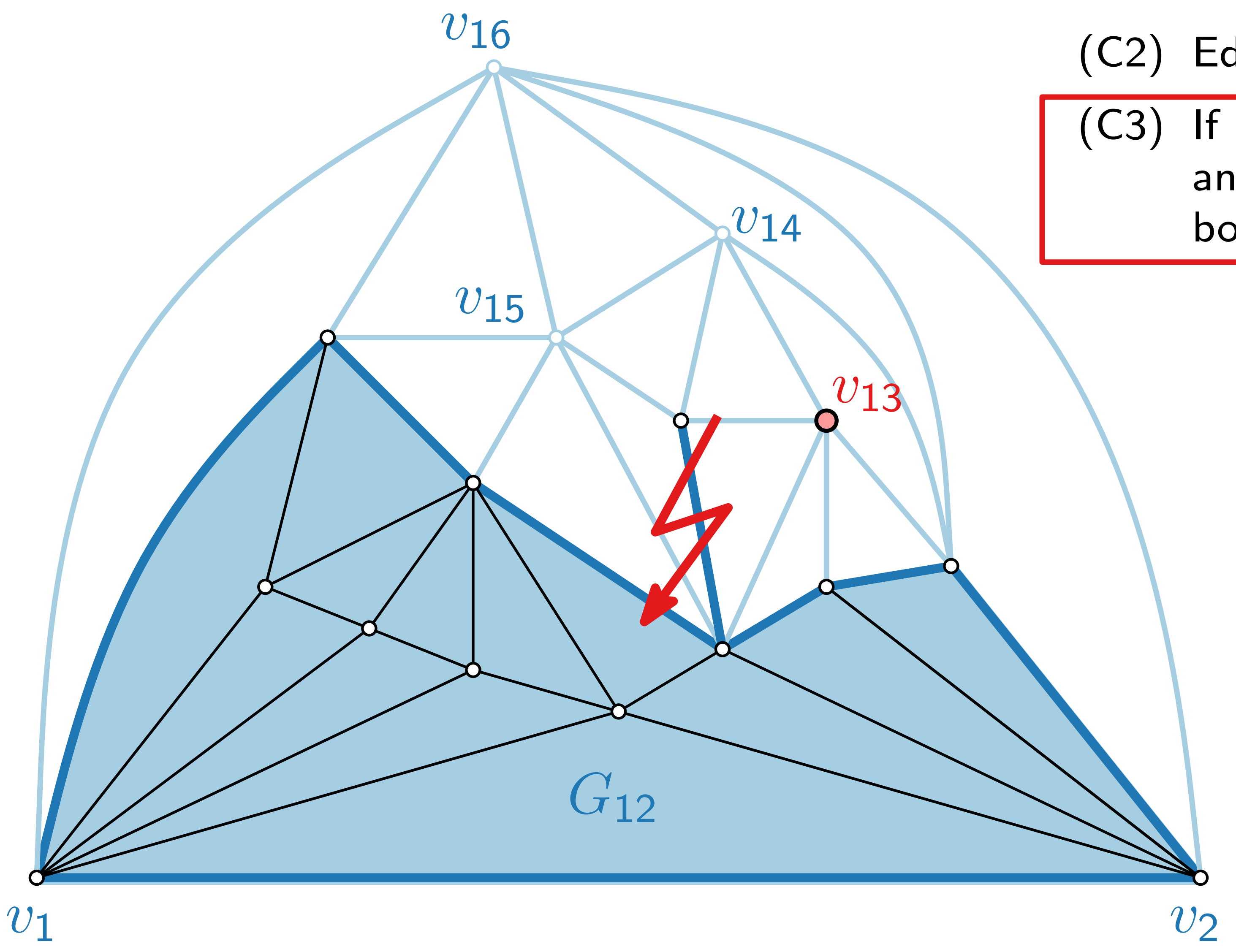

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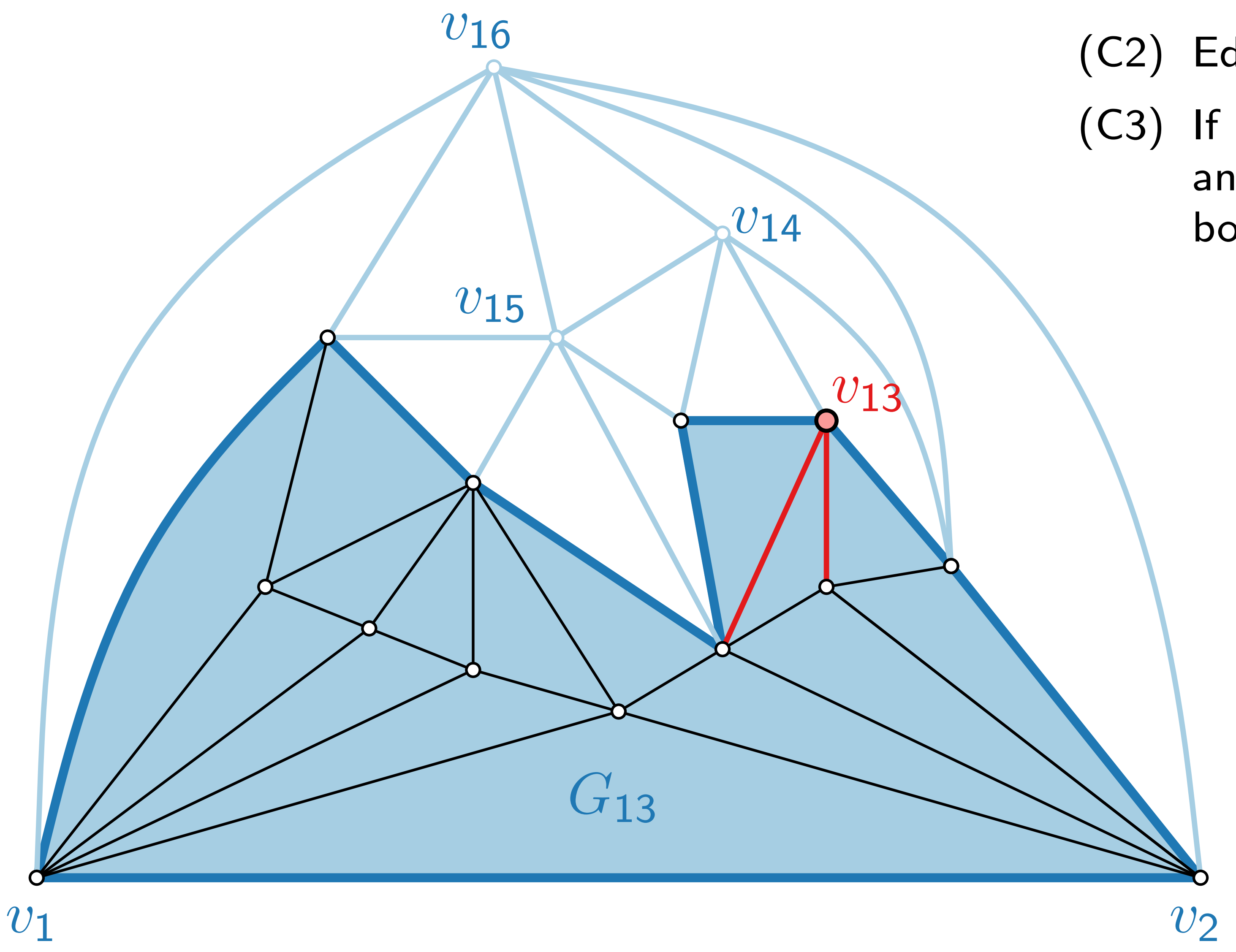

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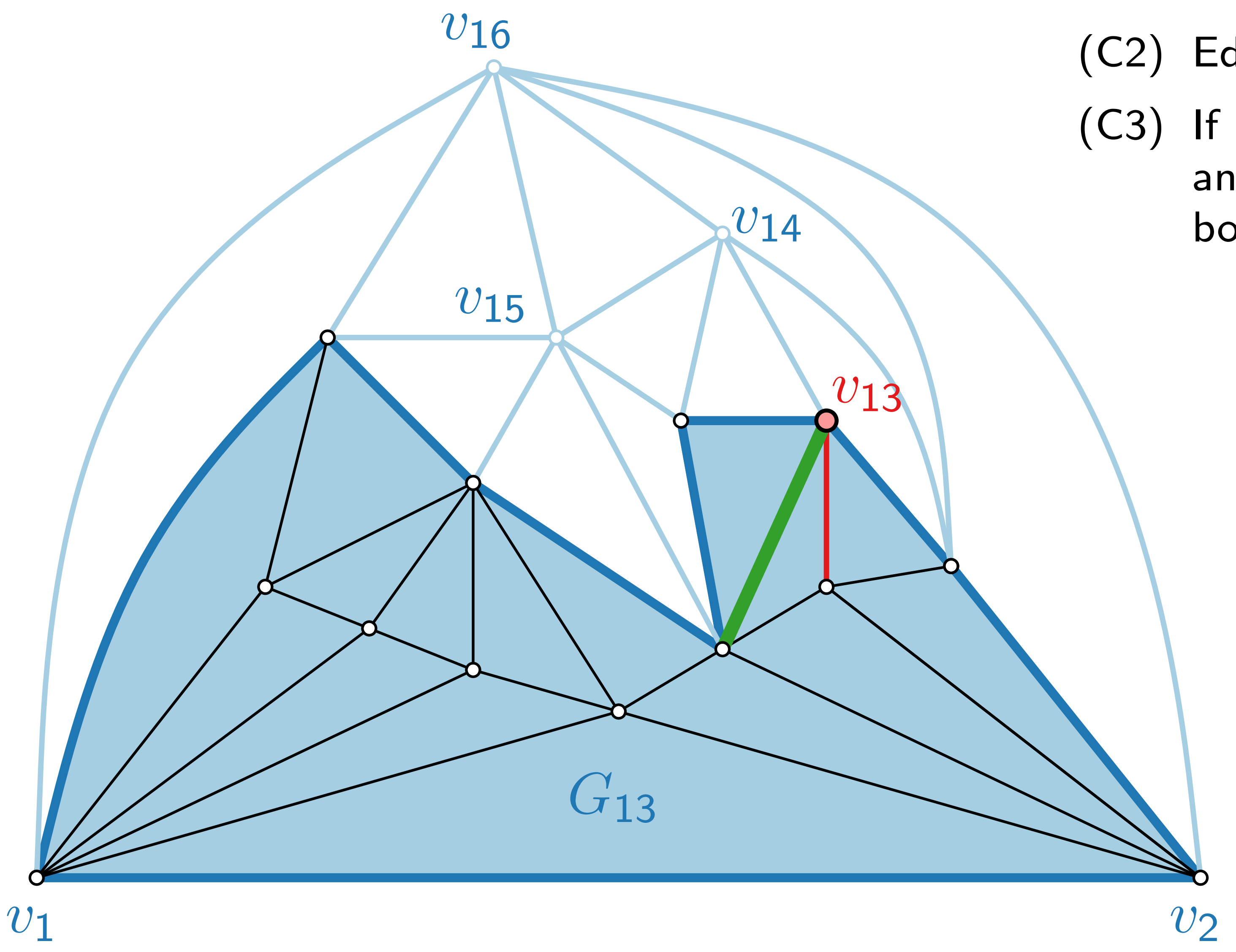

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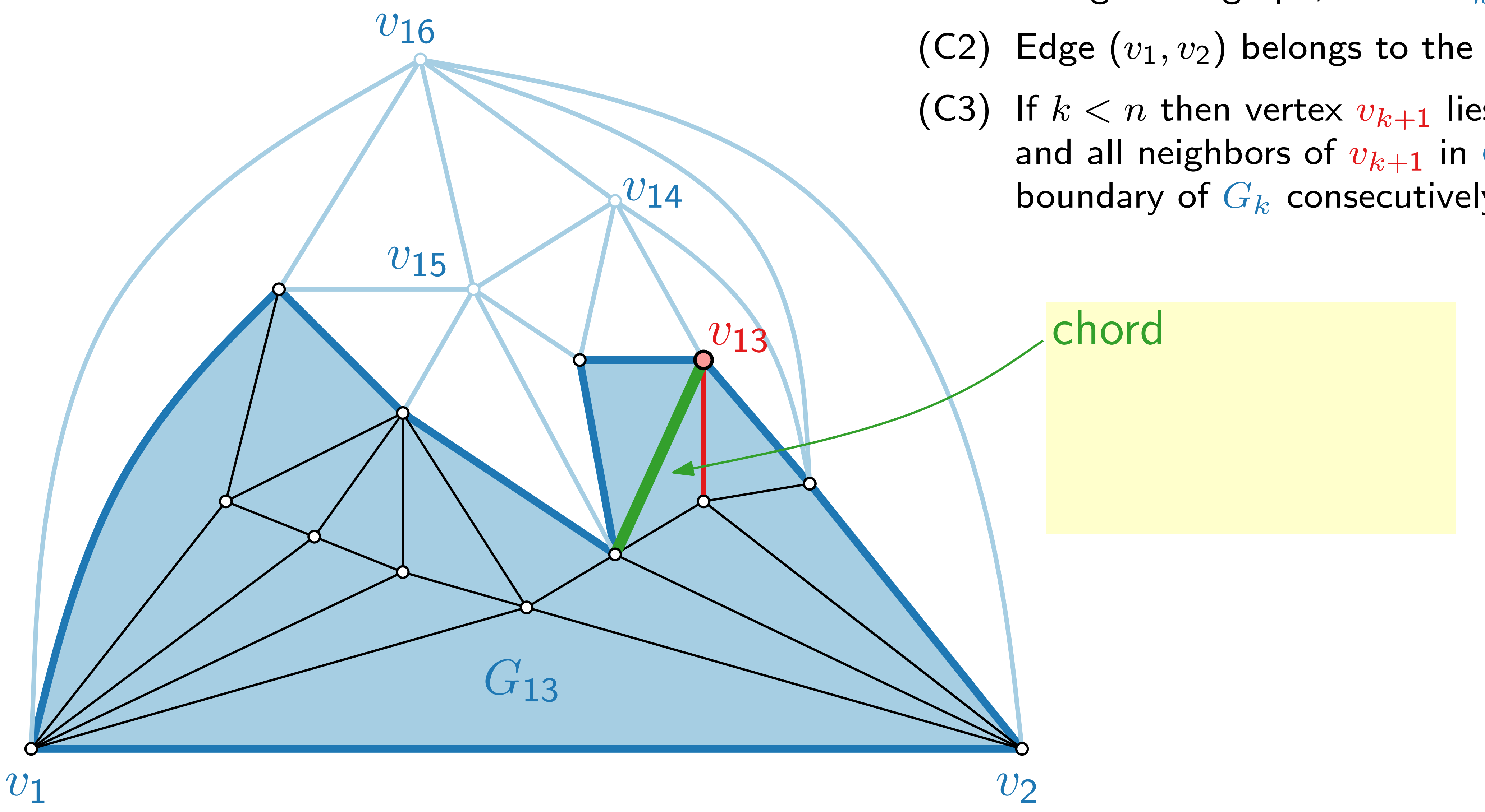

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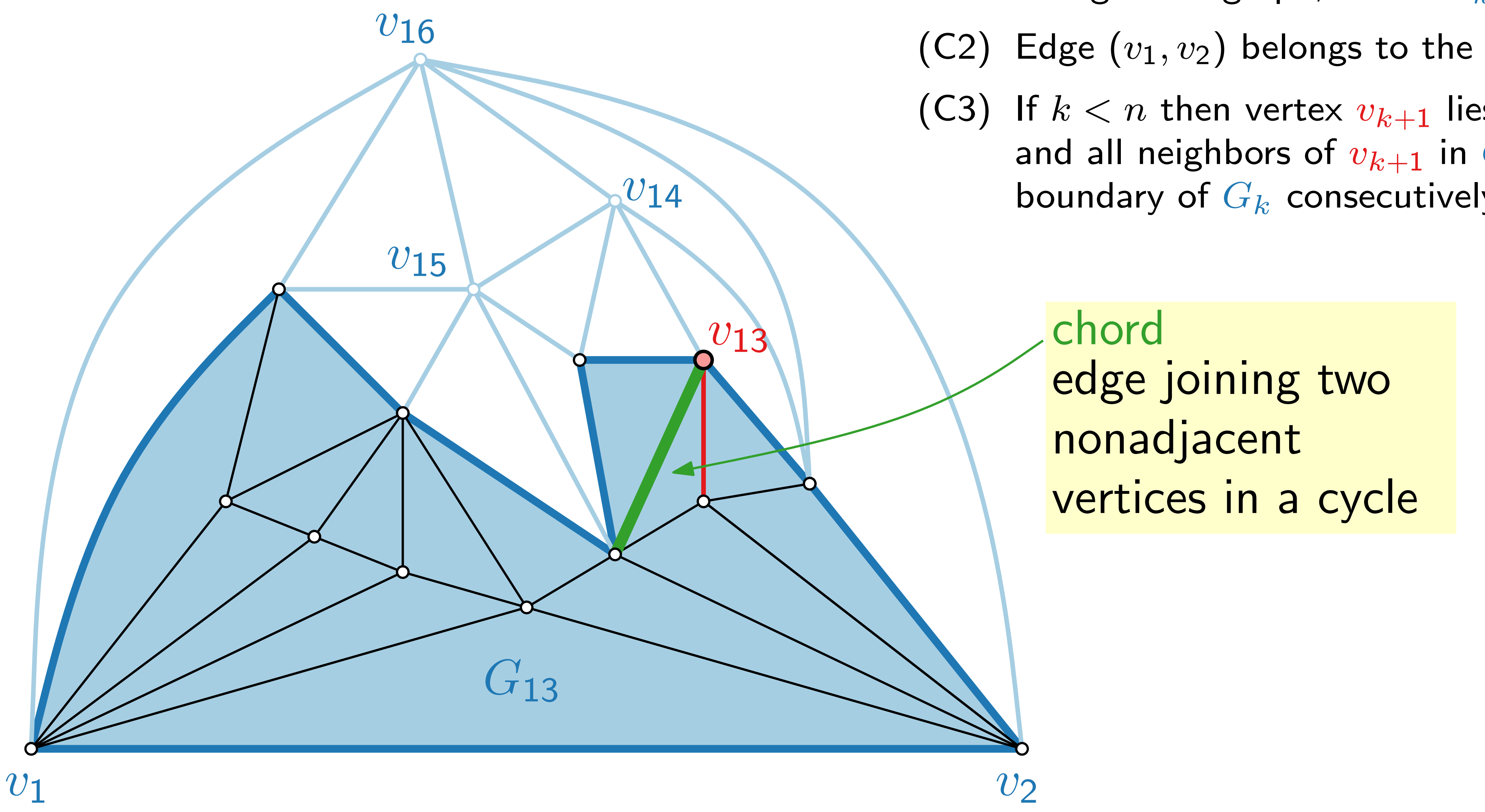

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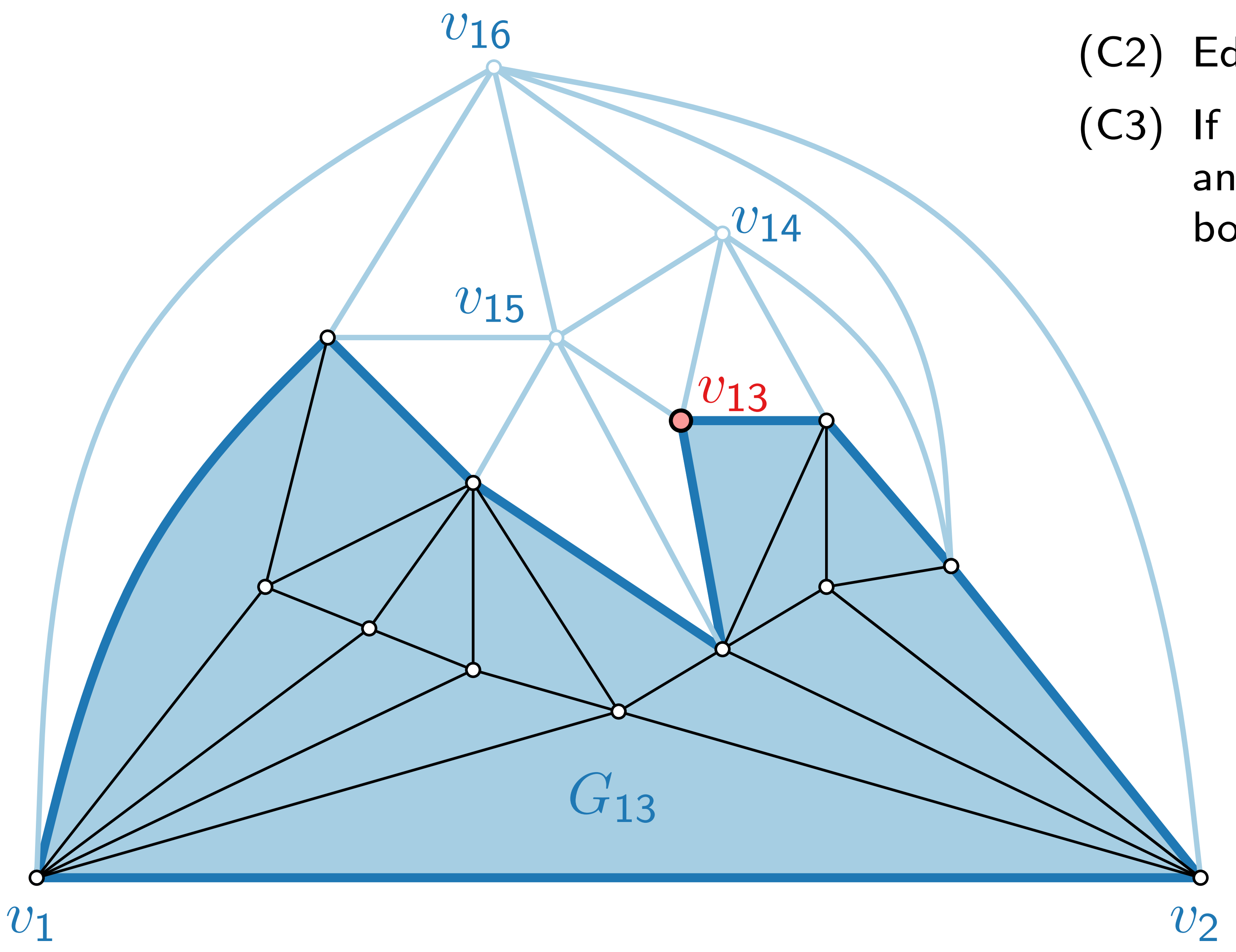

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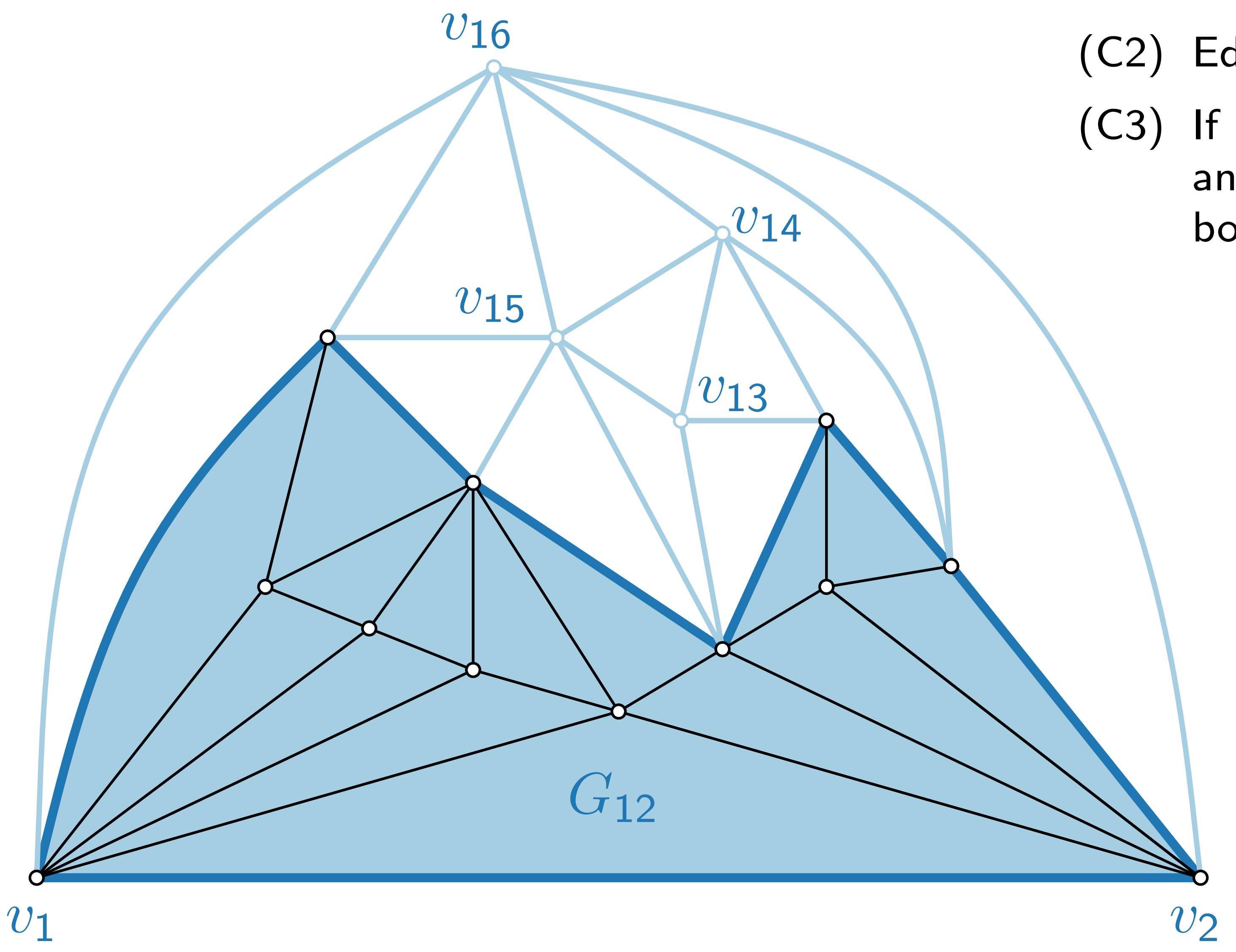

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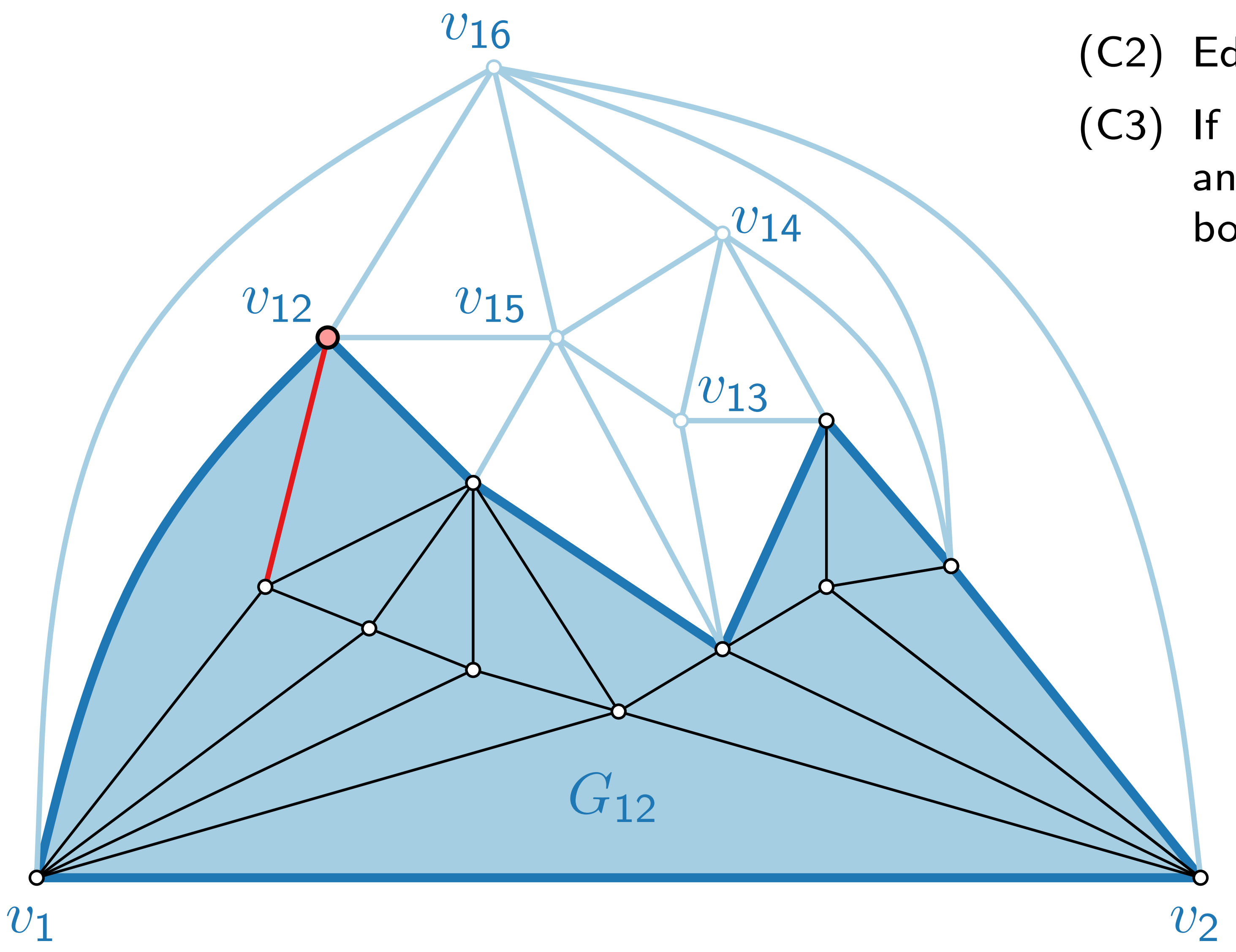

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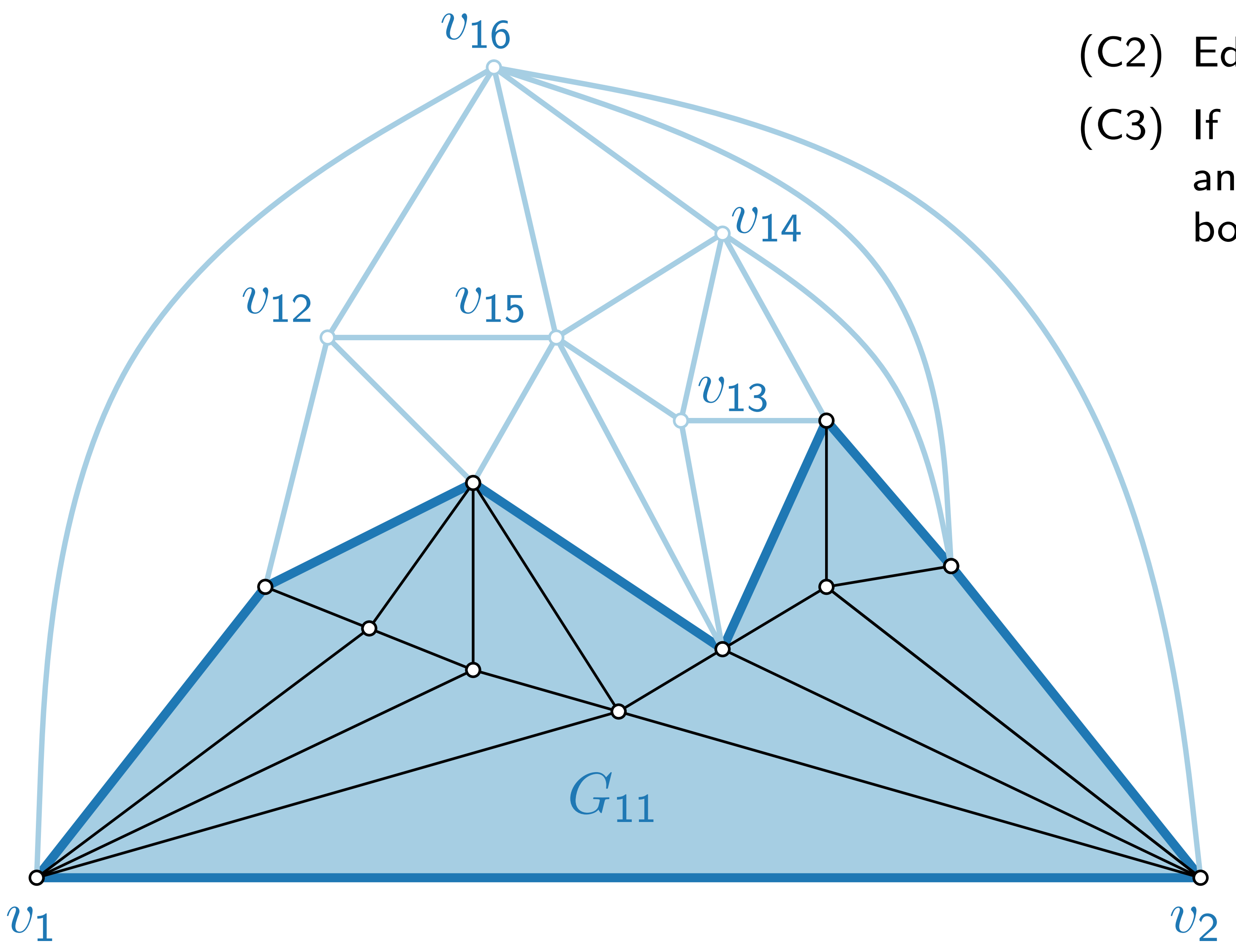

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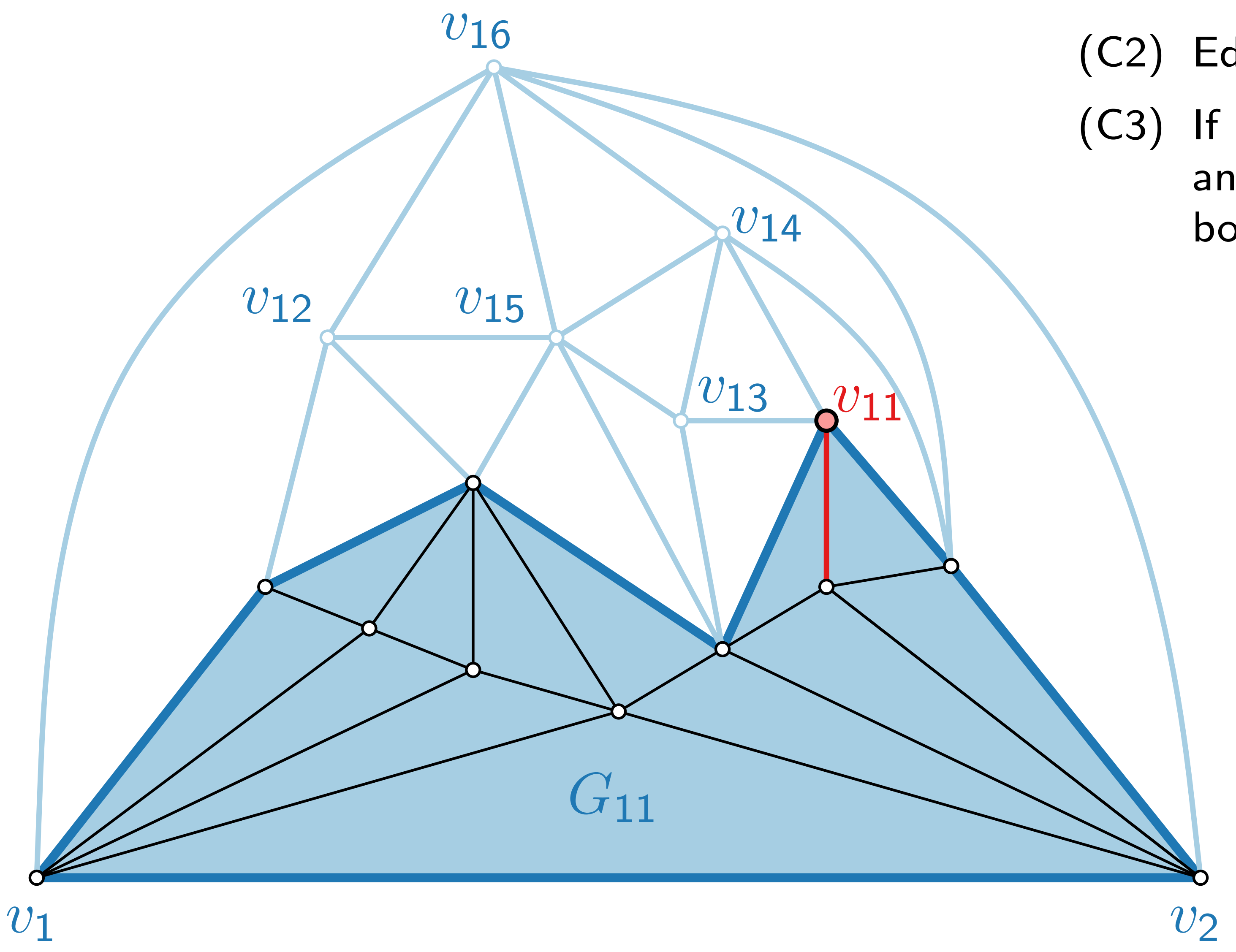

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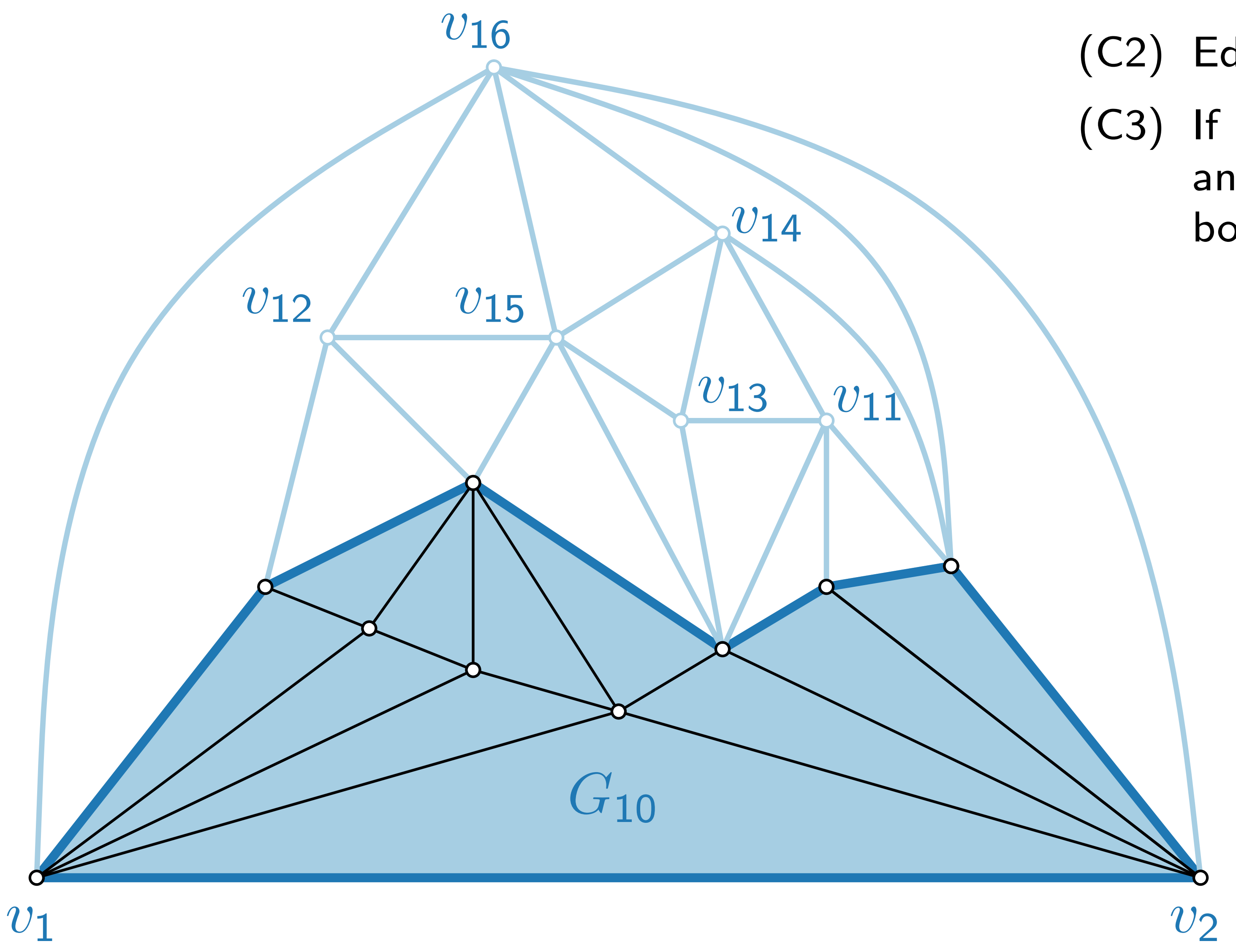

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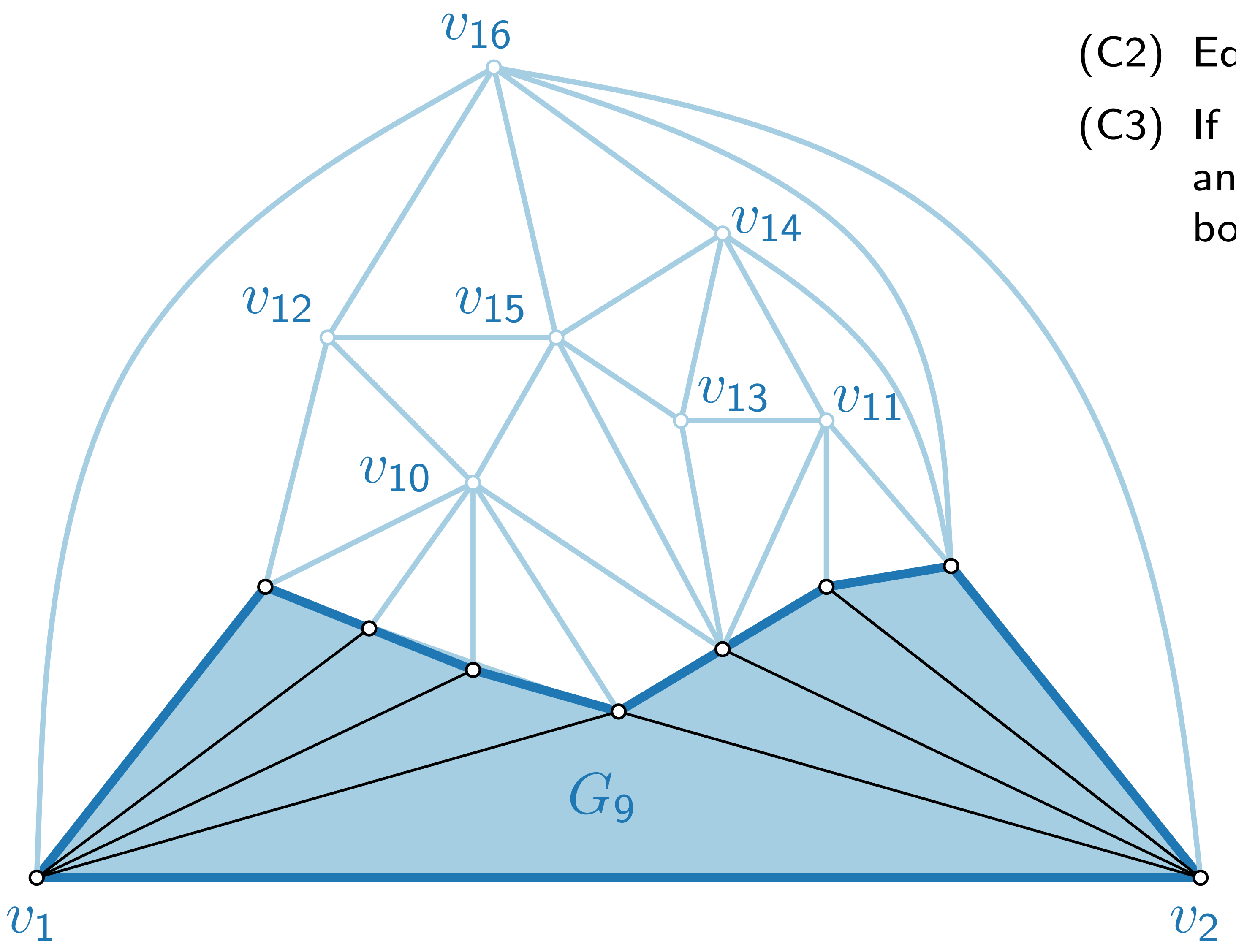

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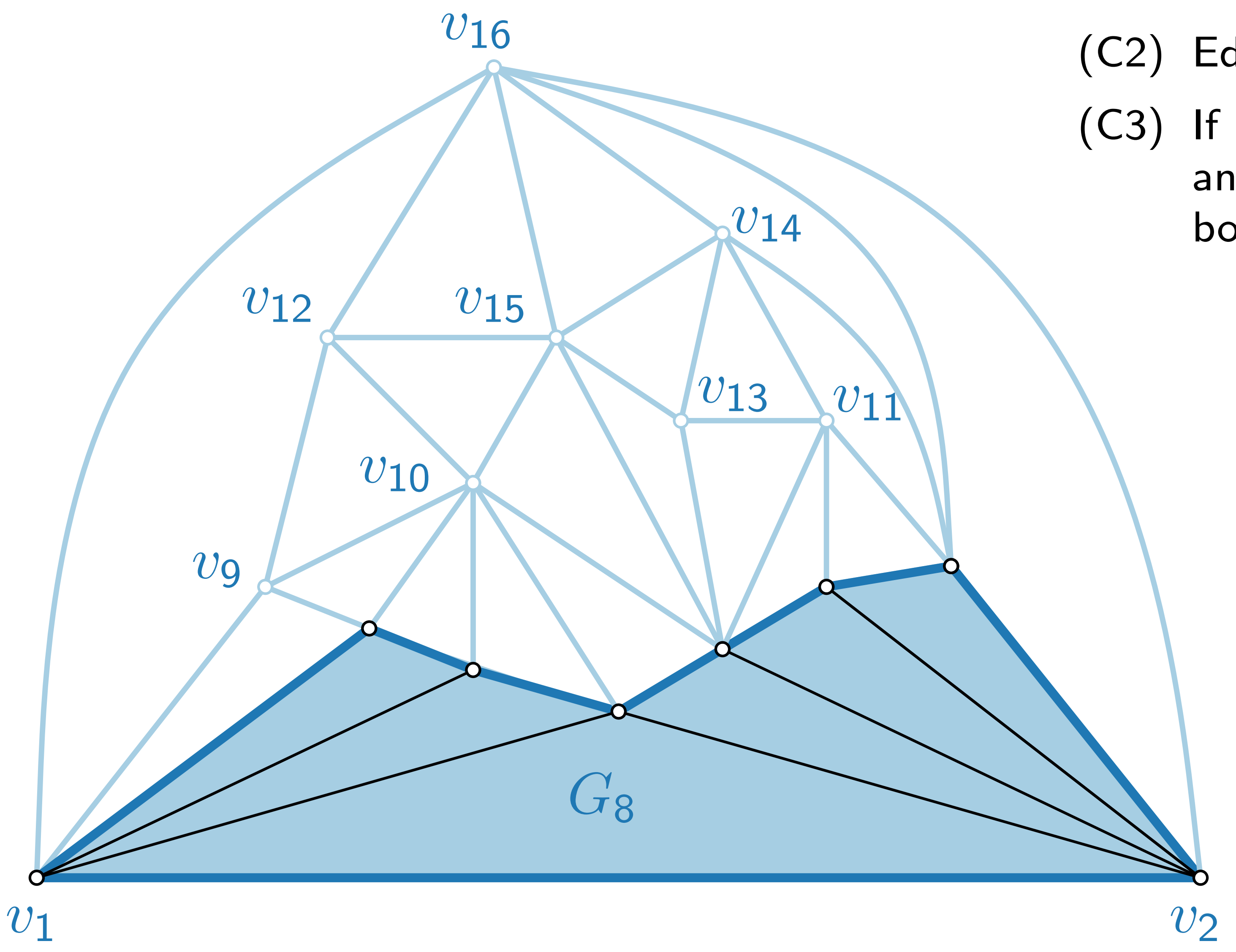

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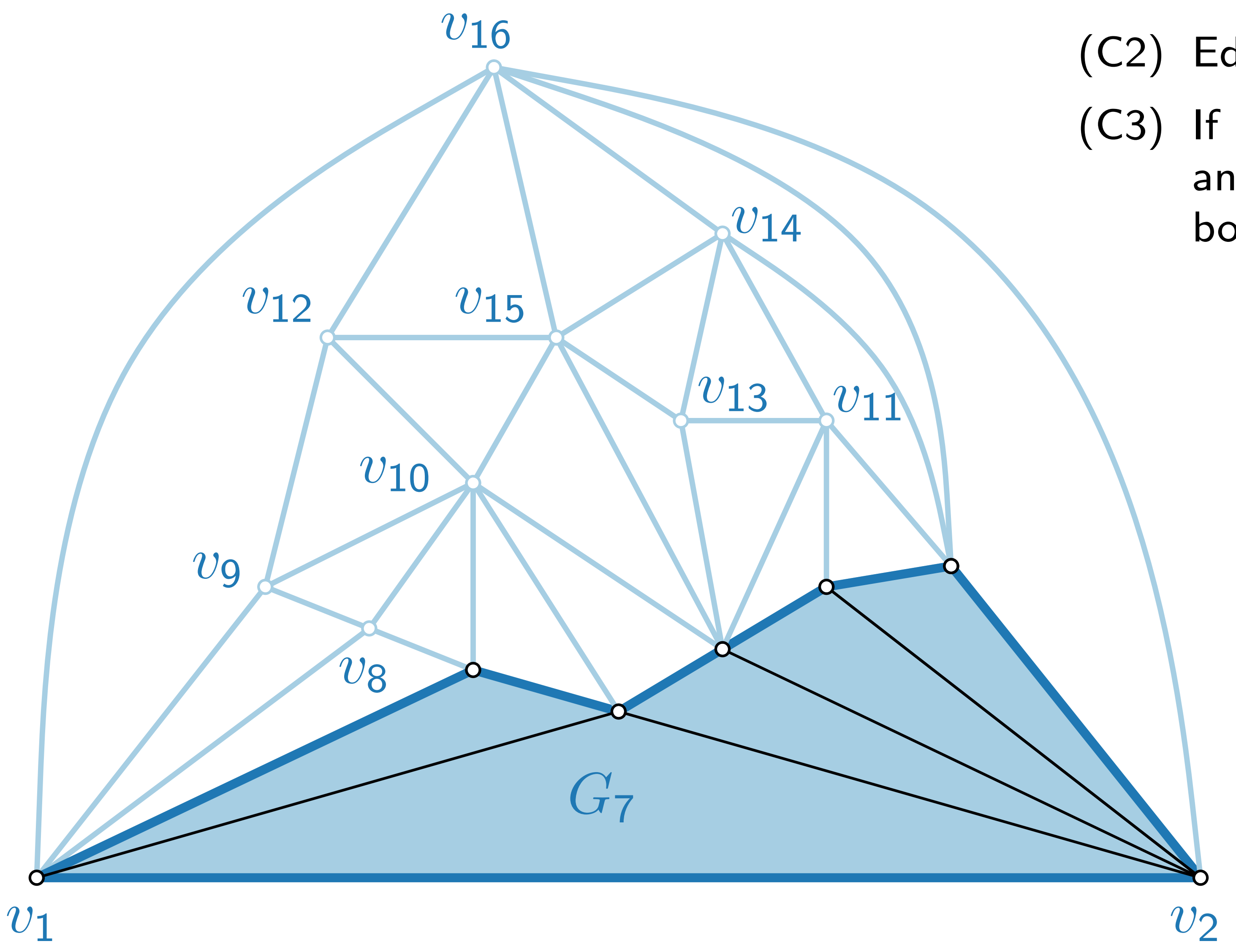

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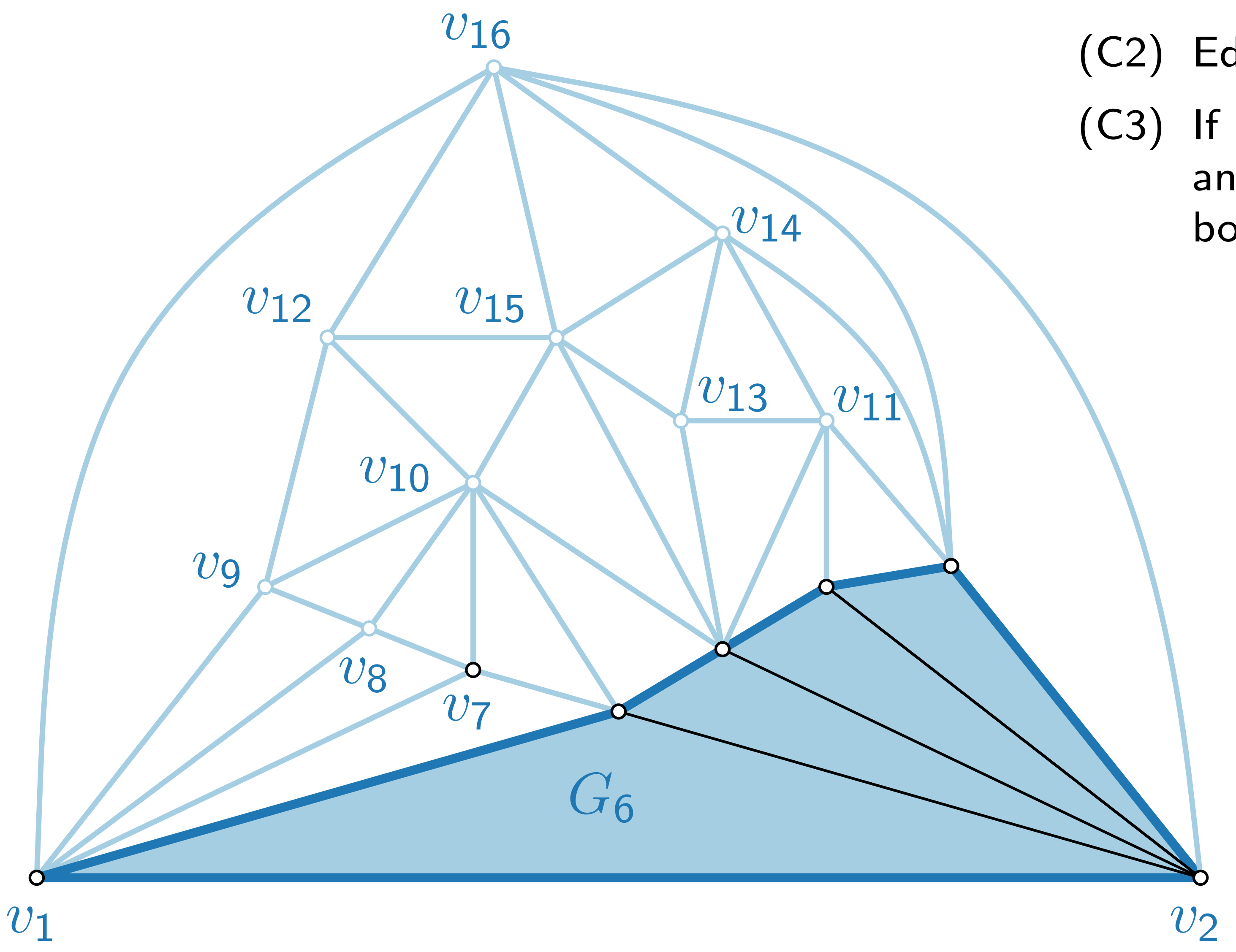

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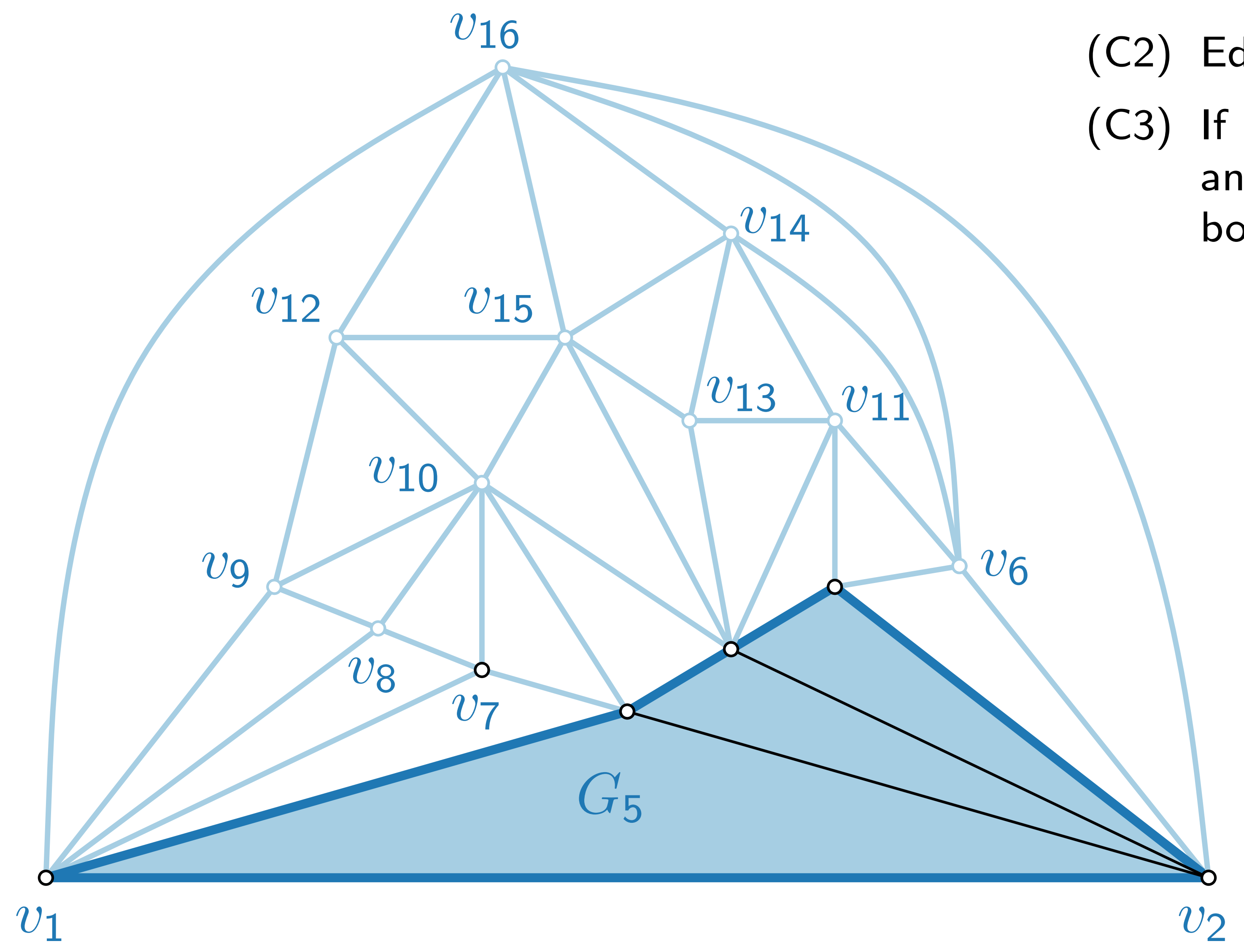

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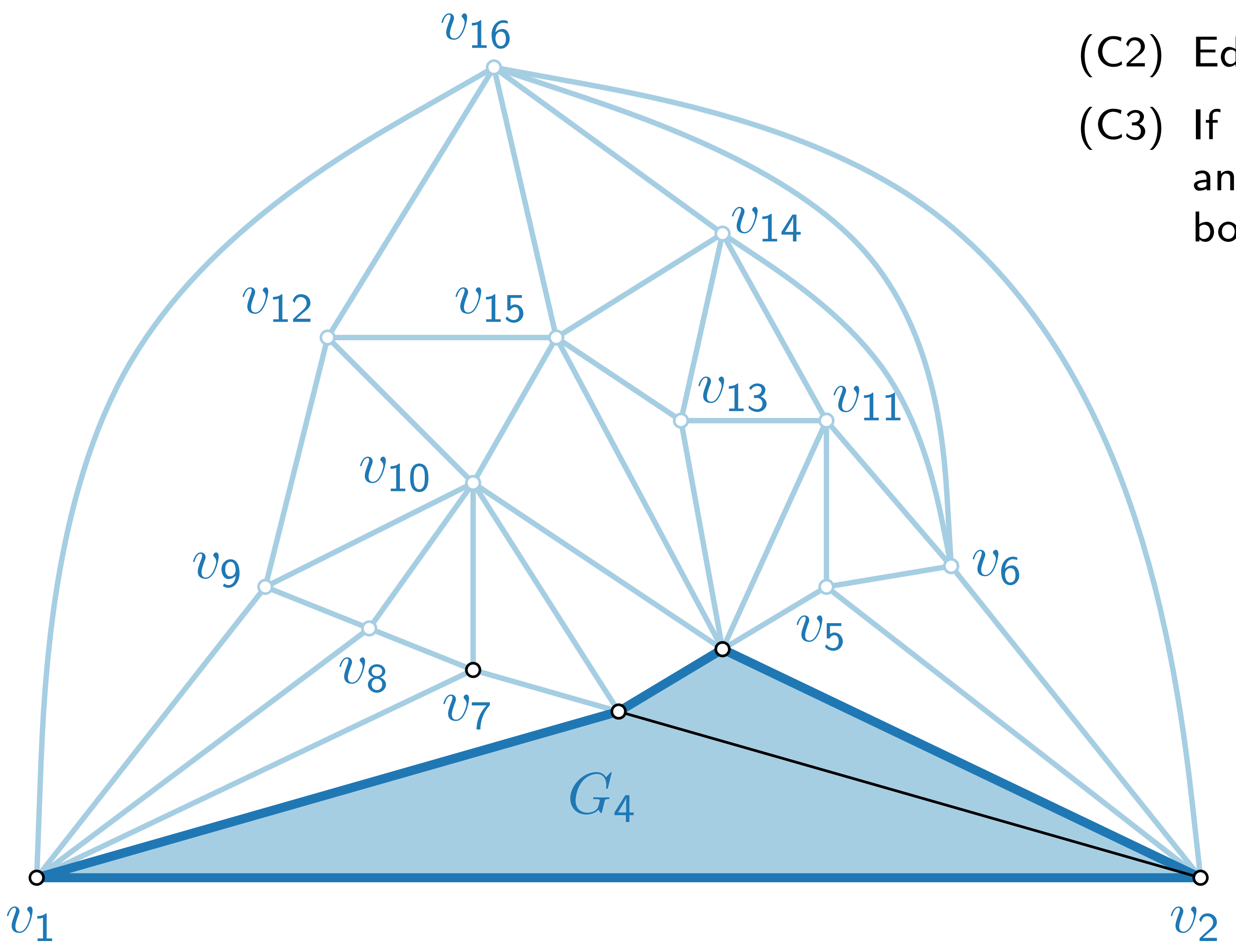

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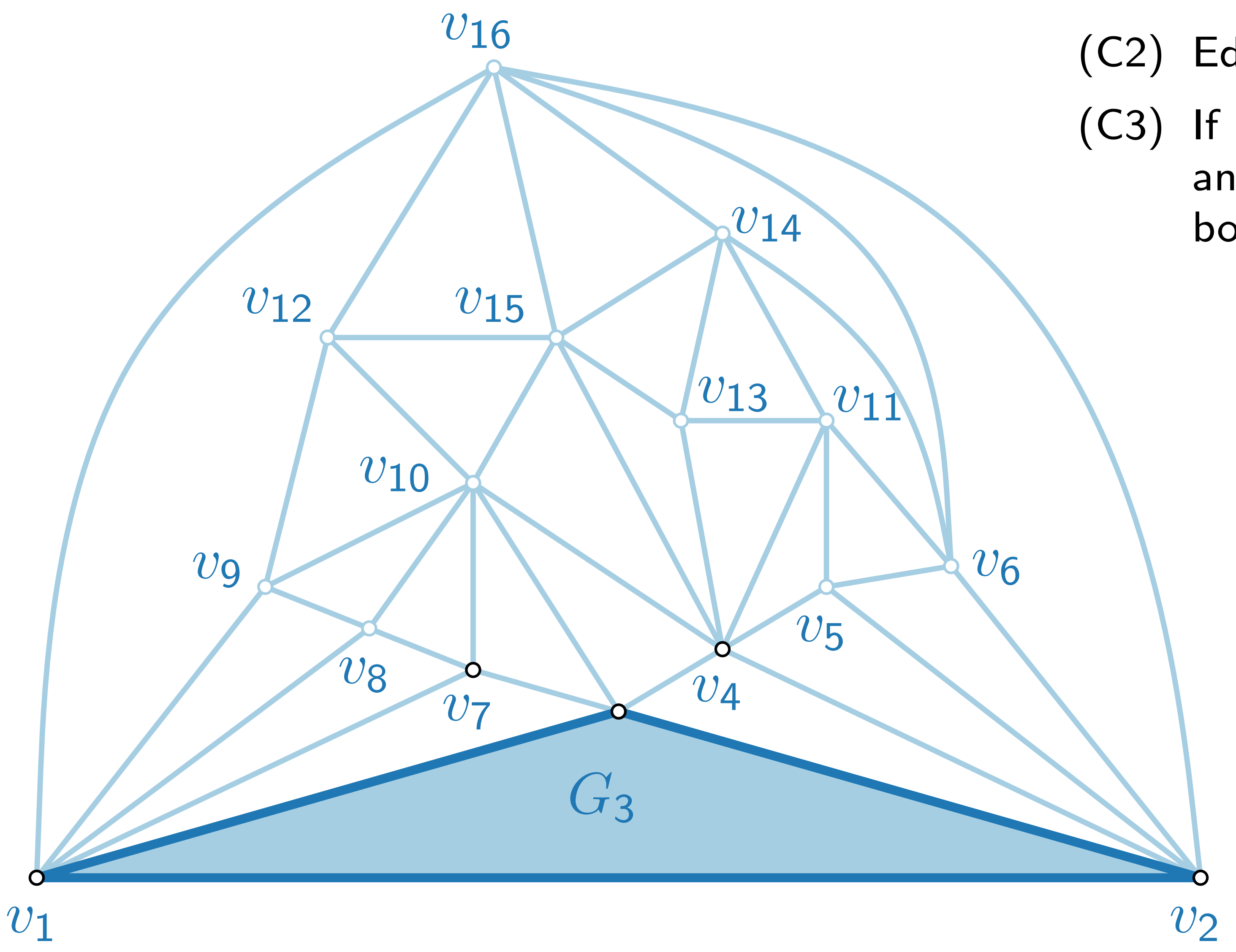

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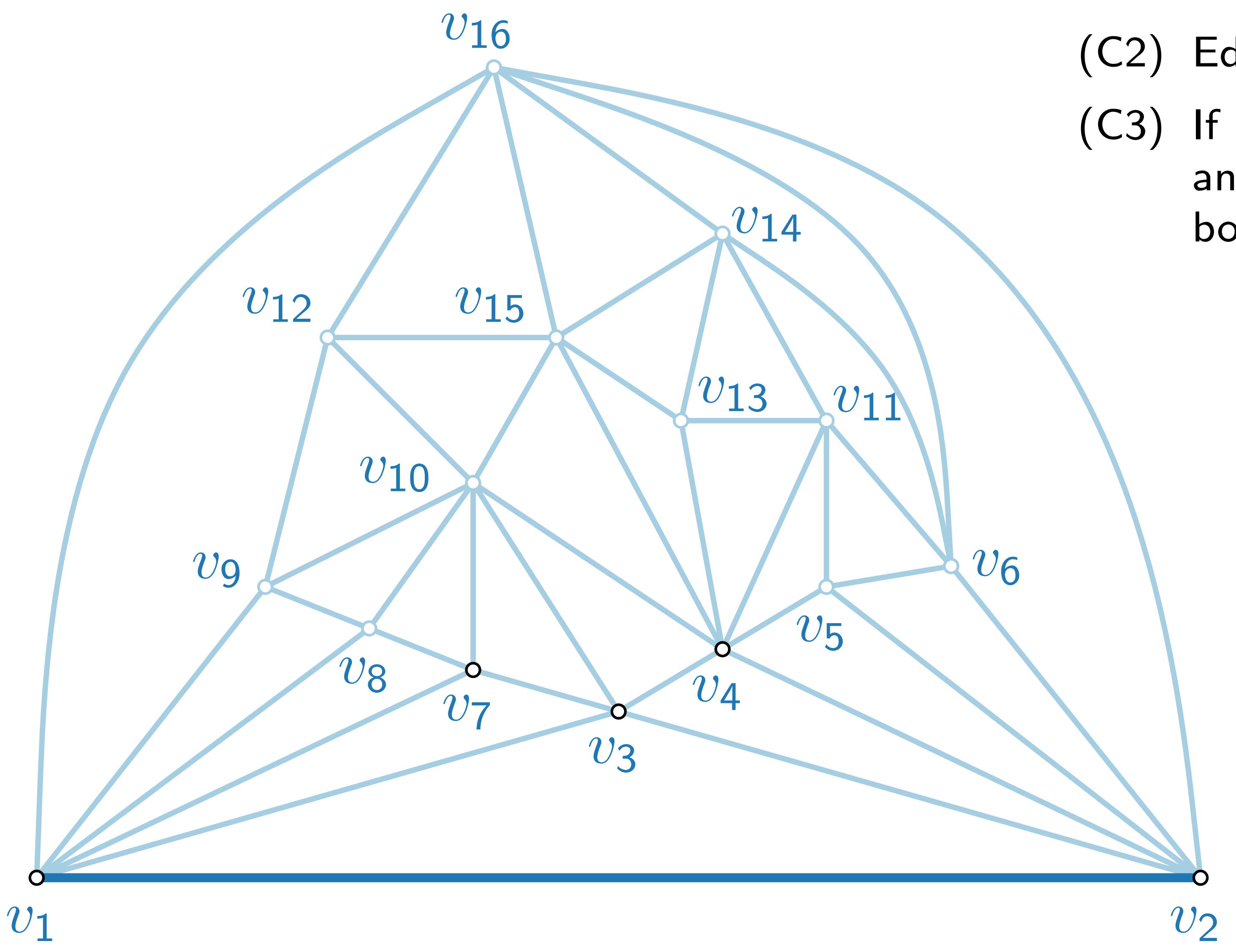

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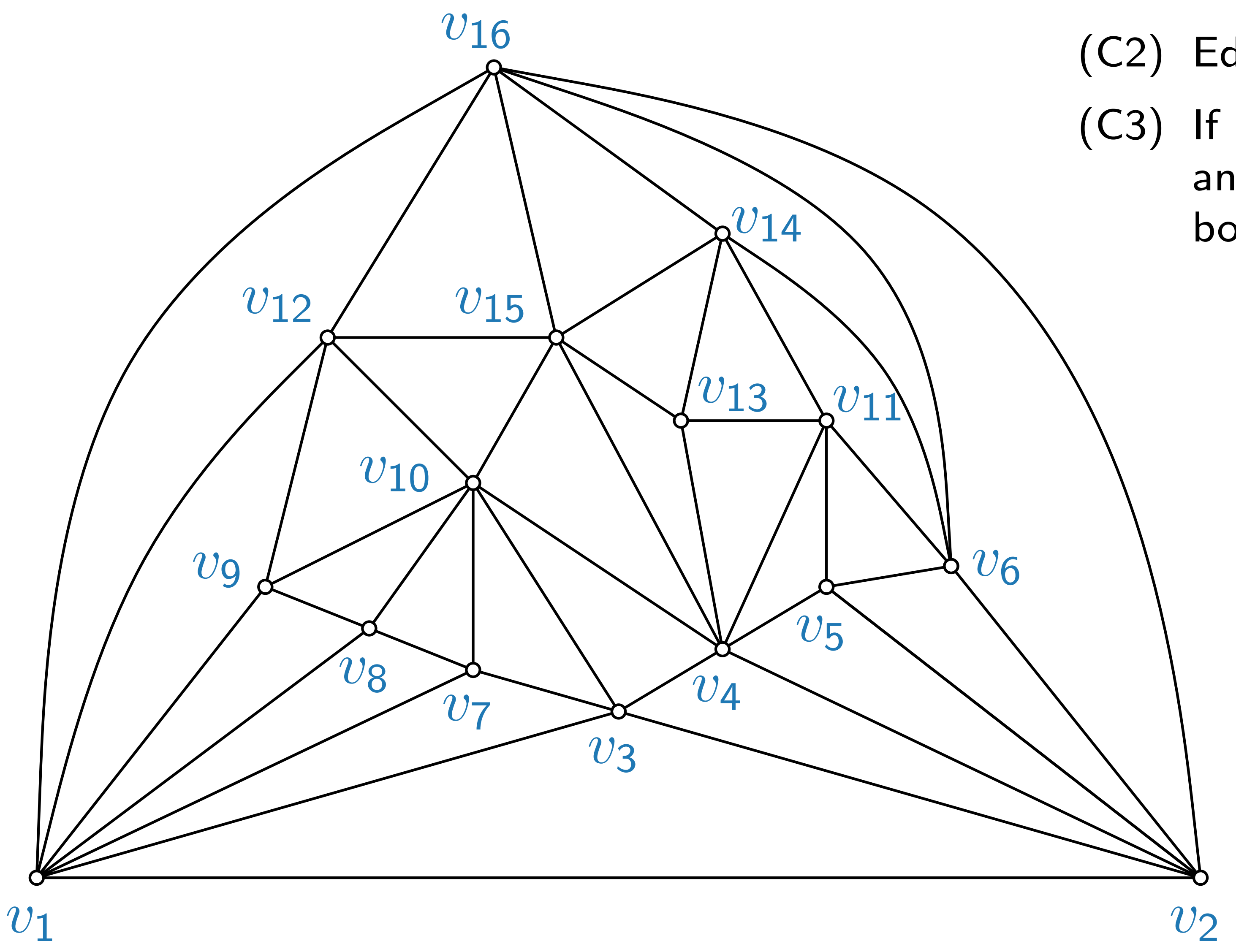

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Lemma.
Every triangulated plane graph has a canonical order.

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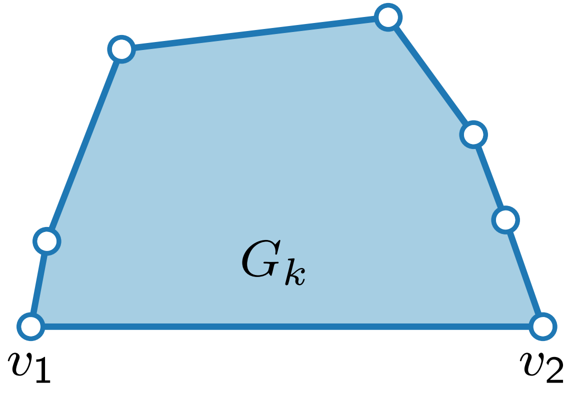

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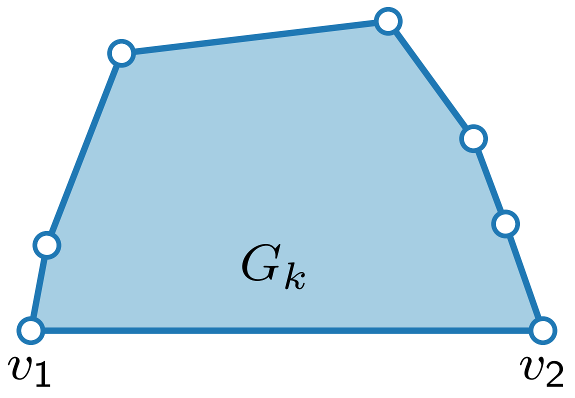

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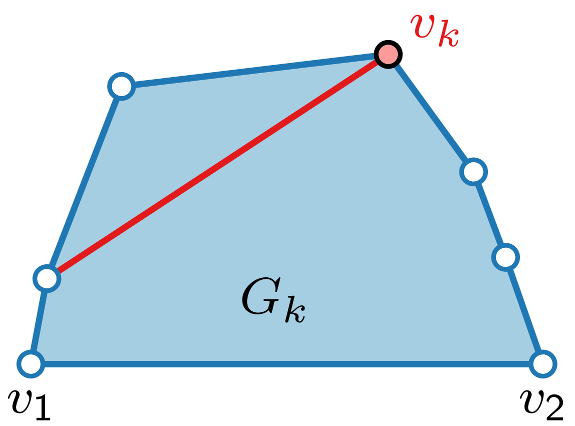

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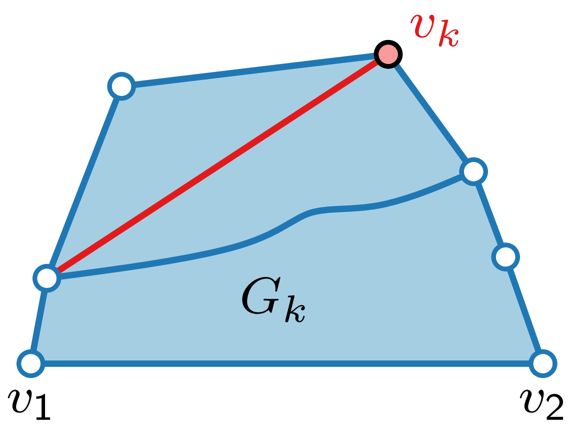

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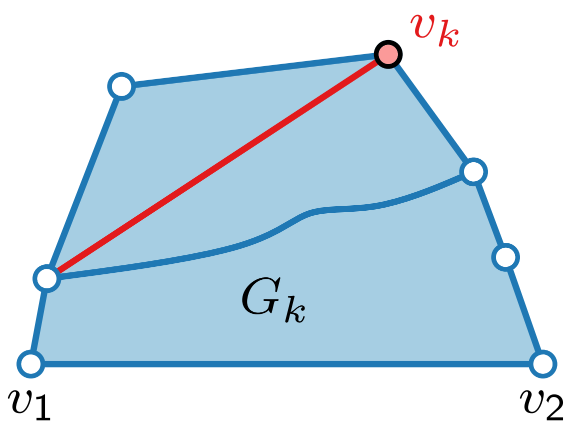

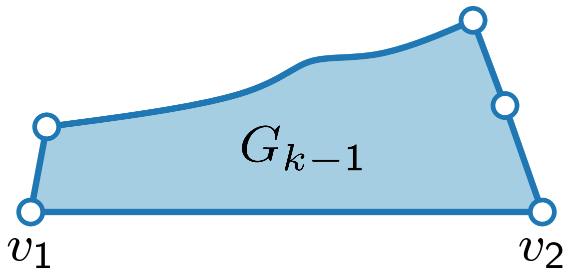

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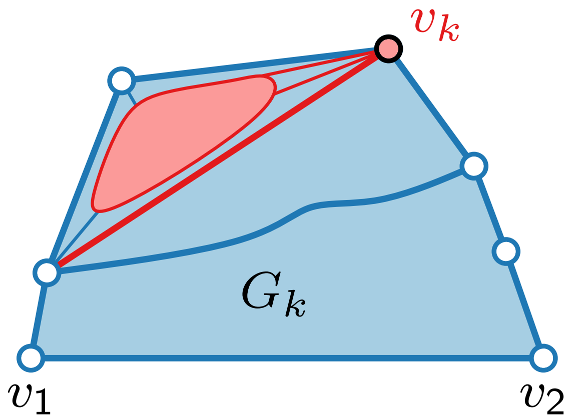

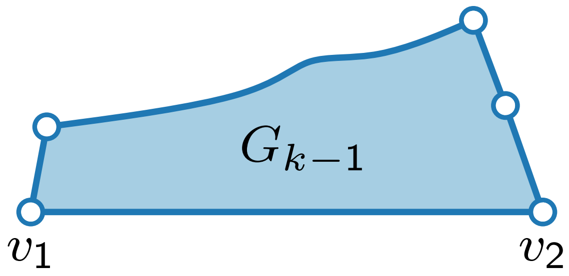

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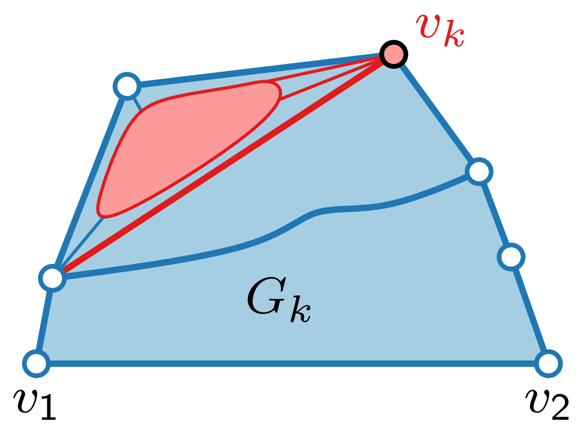

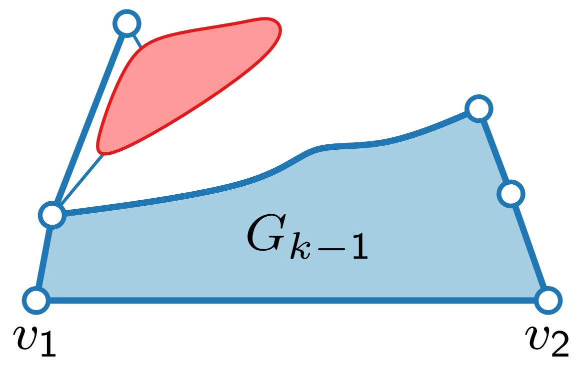

(C1) G_k biconnected and internally triangulated

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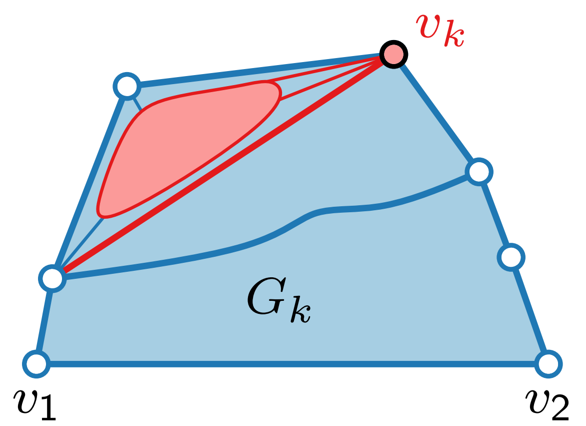

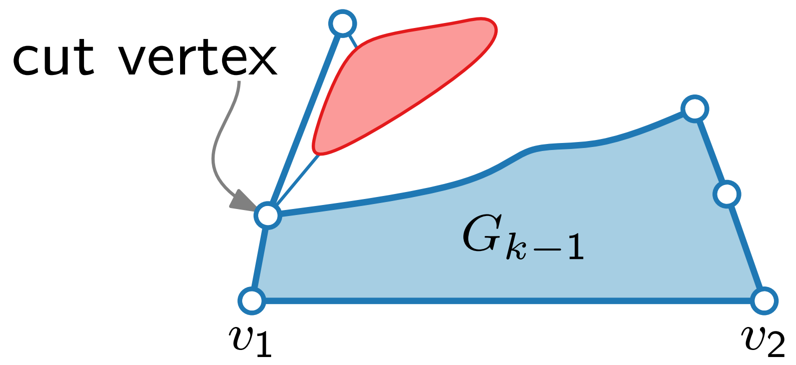

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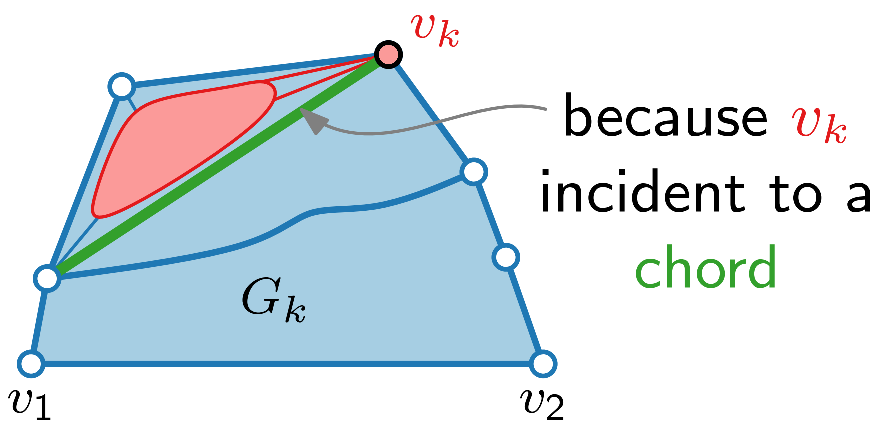

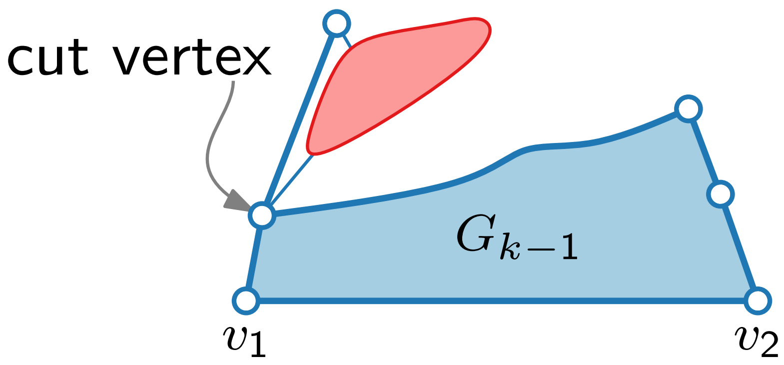

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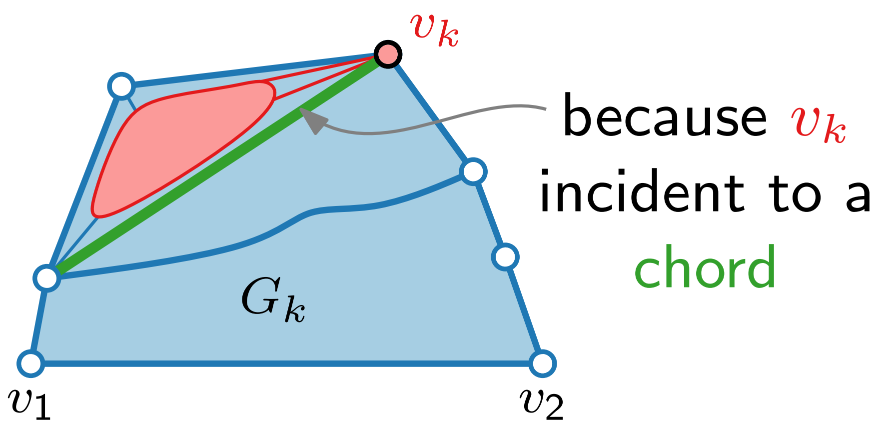

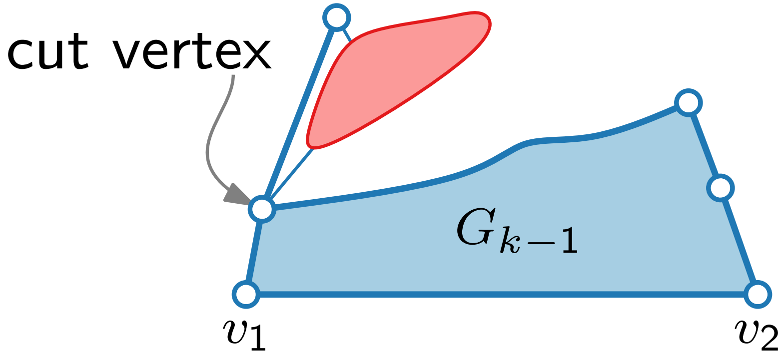

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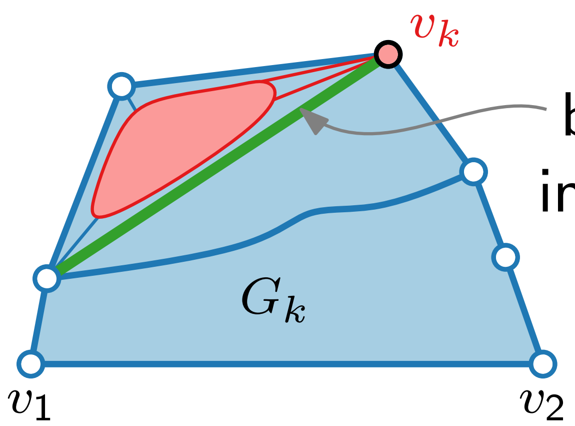
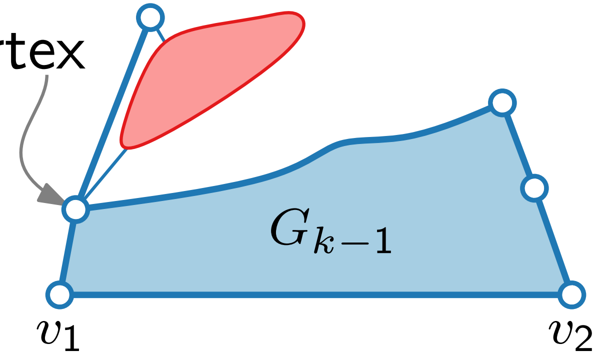


because v_k incident to a chord

cut vertex



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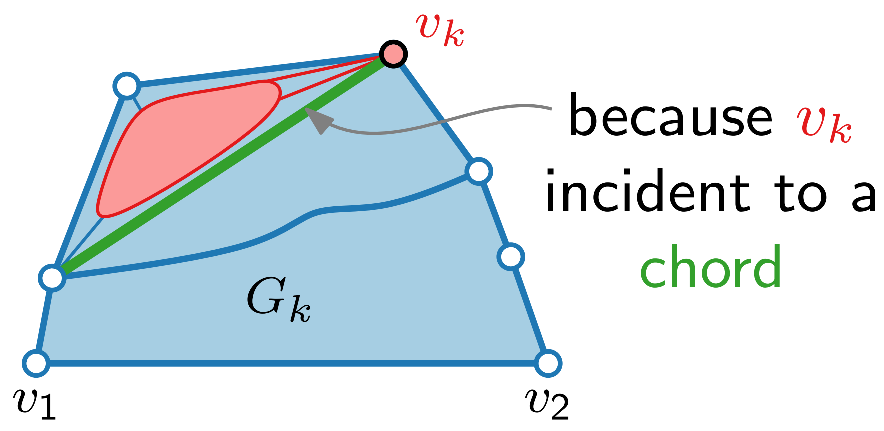
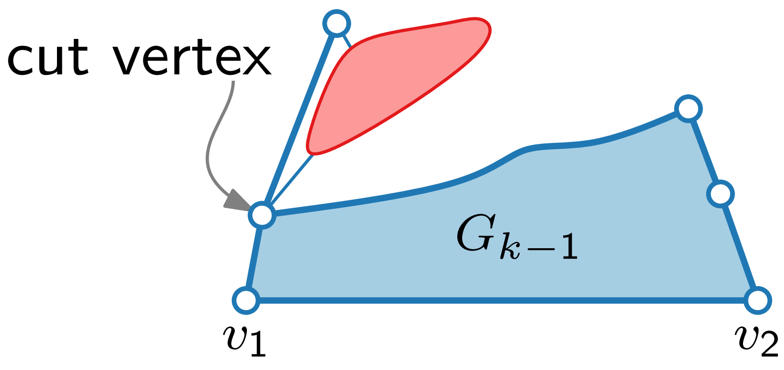
Have to show:
1. v_k not incident to chord is sufficient
2. Such v_k exists

Induction step: Consider G_k. We search for v_k.





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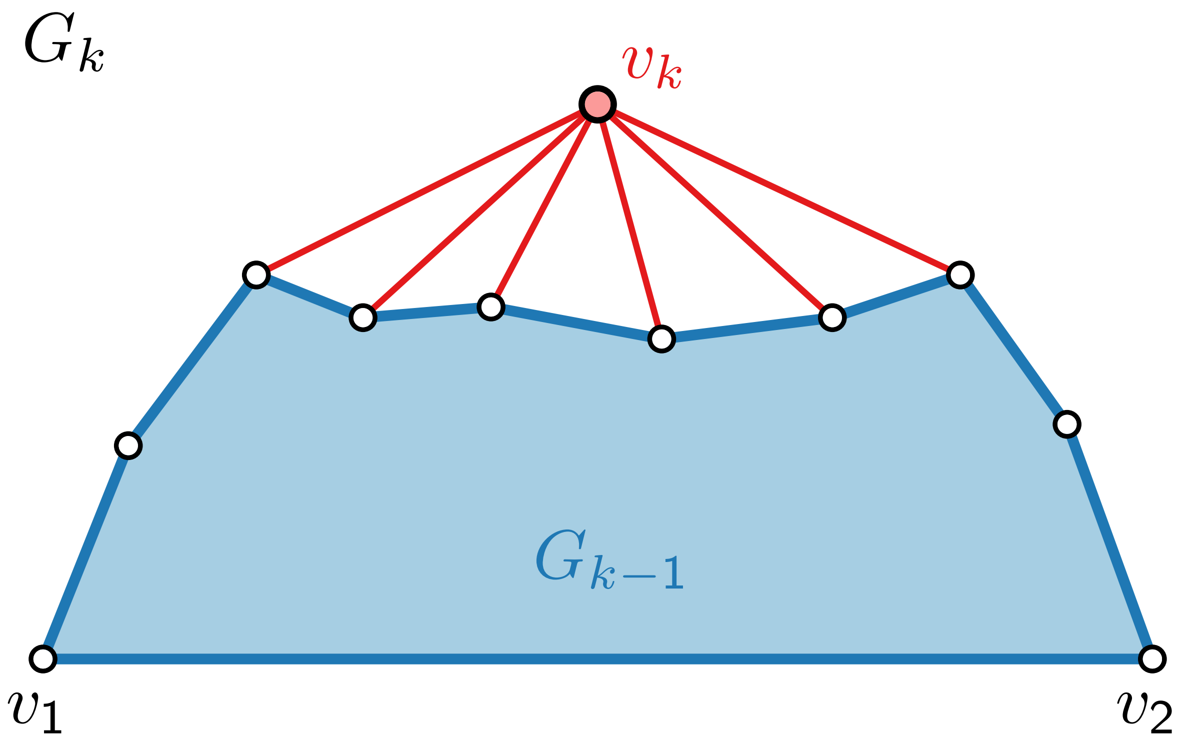

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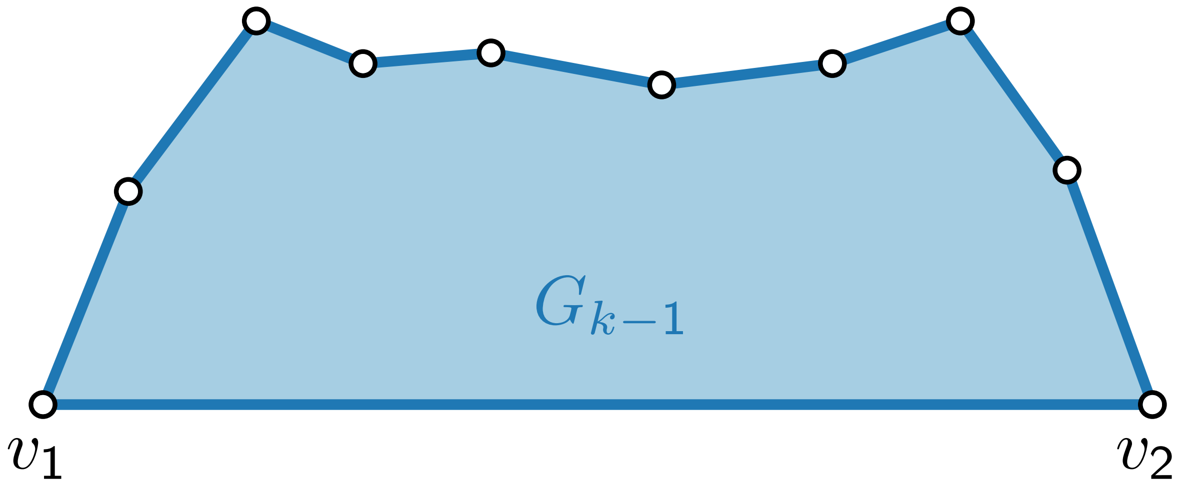

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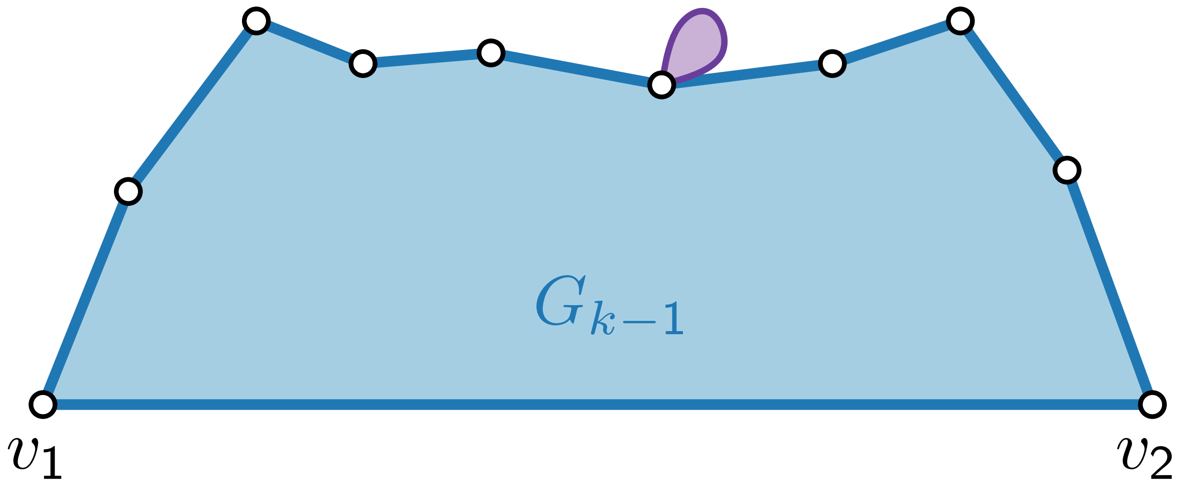

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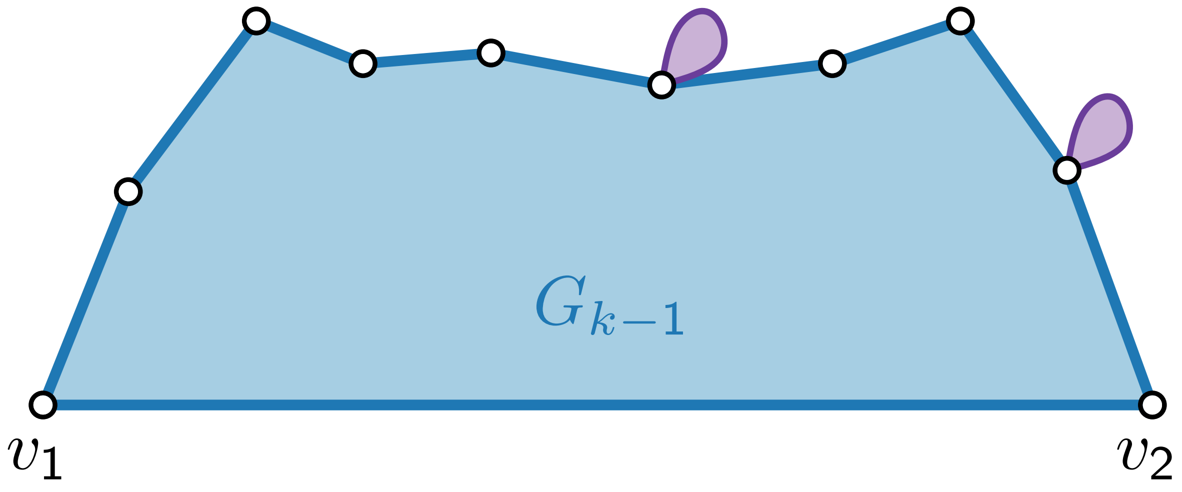

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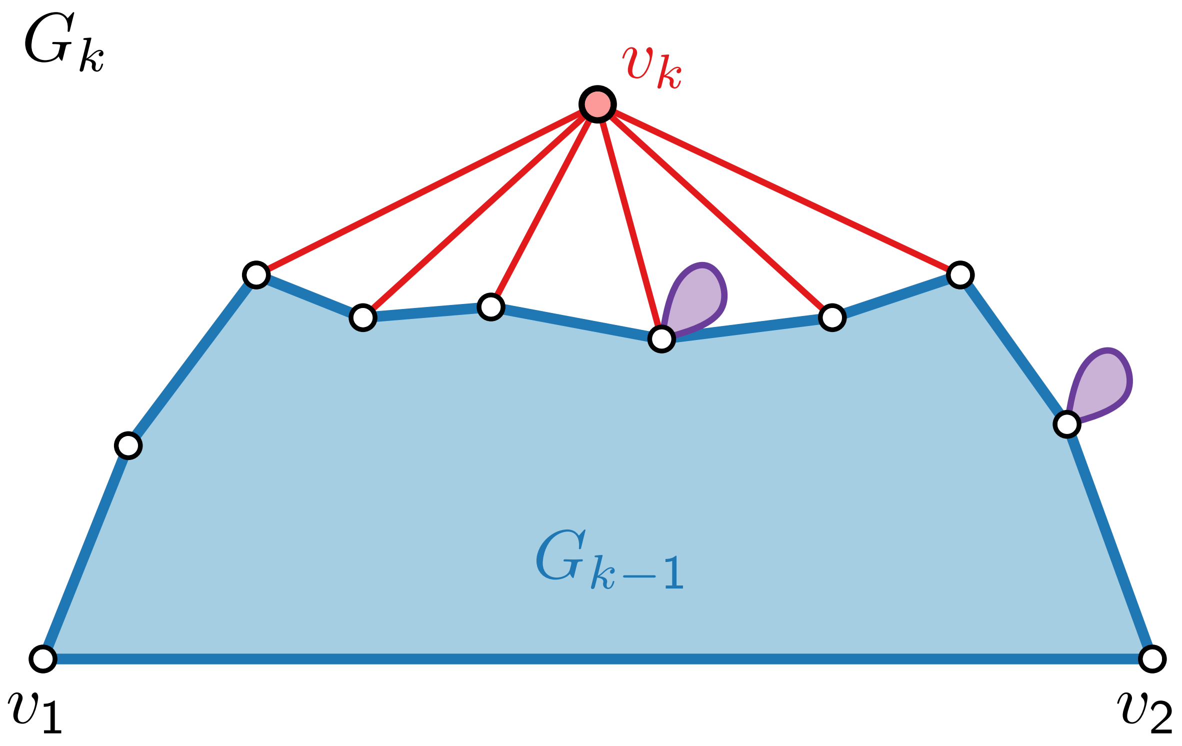

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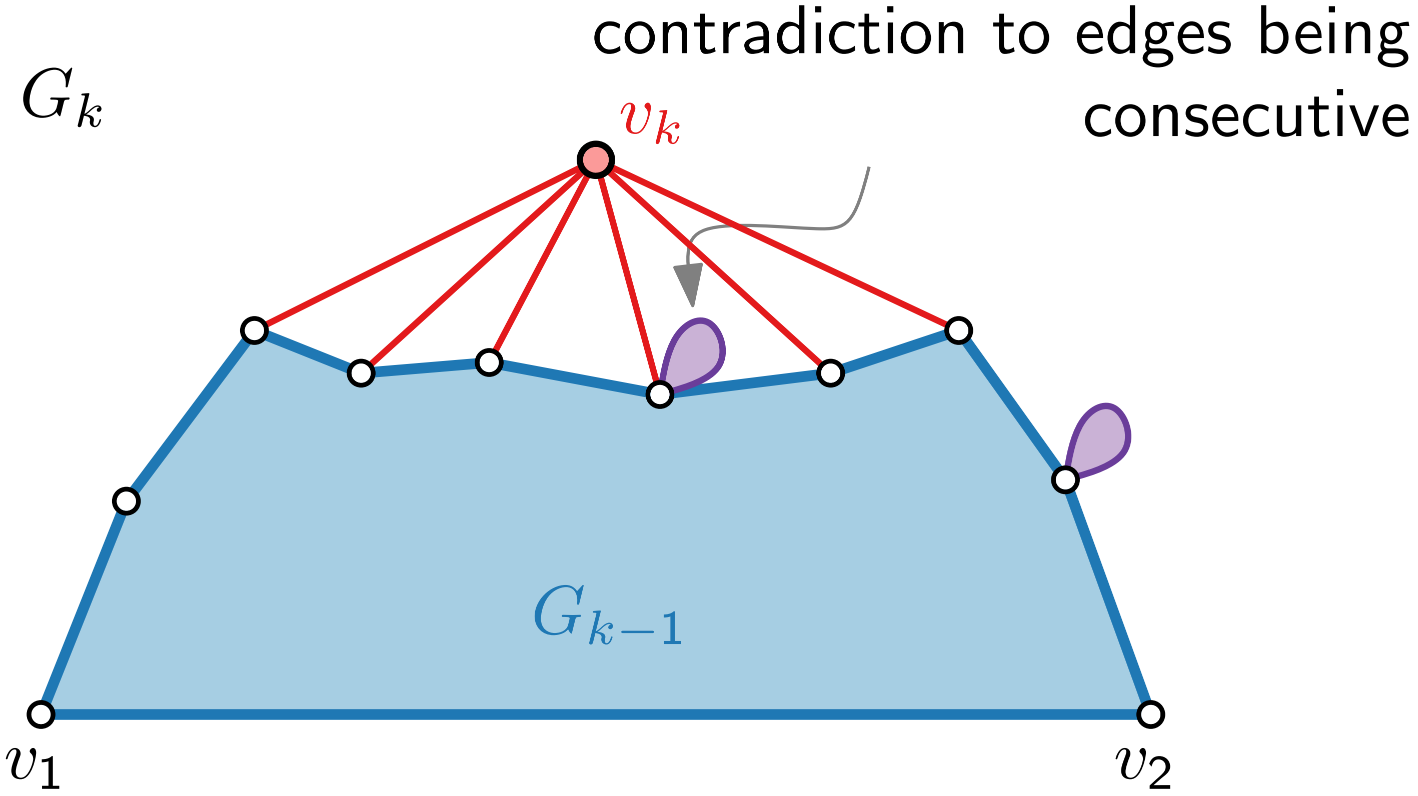

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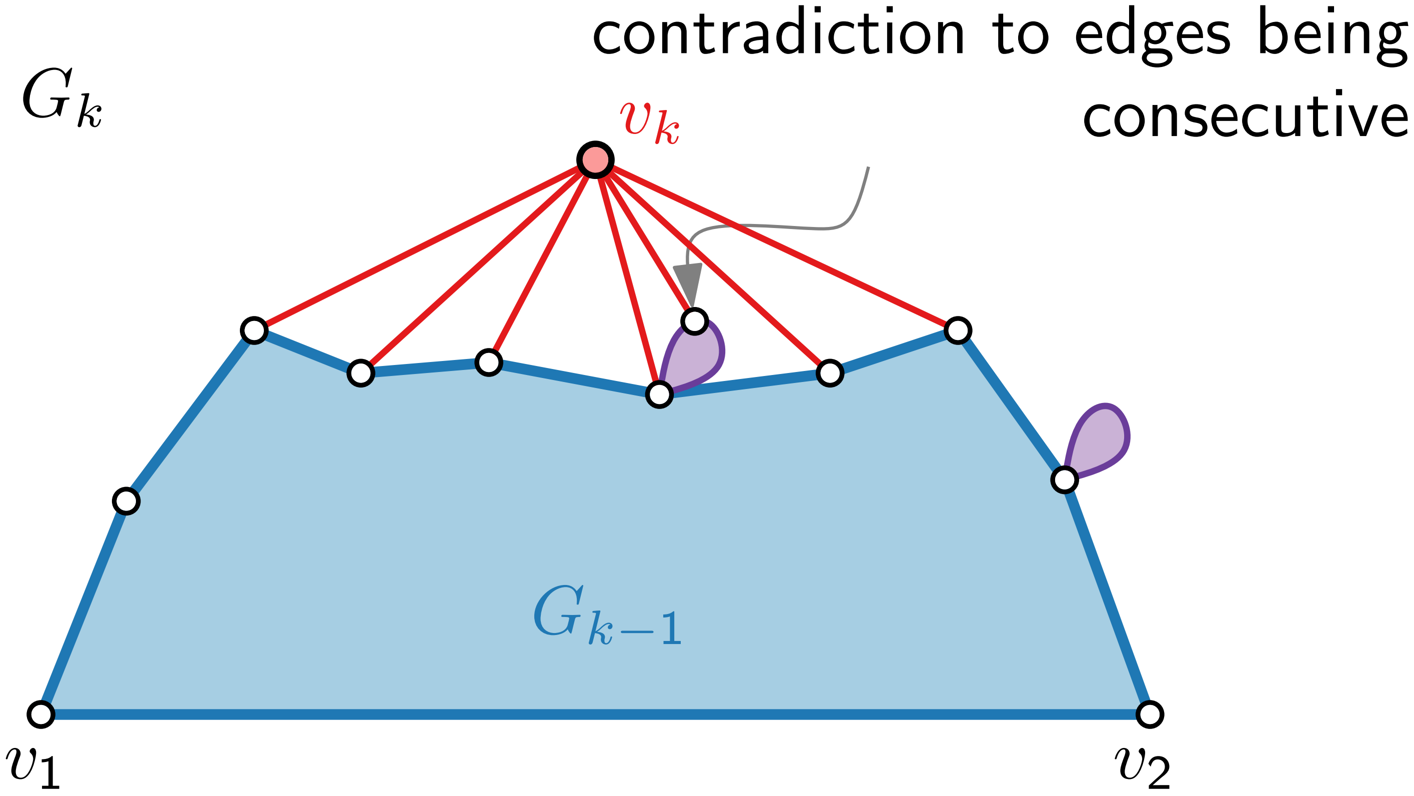

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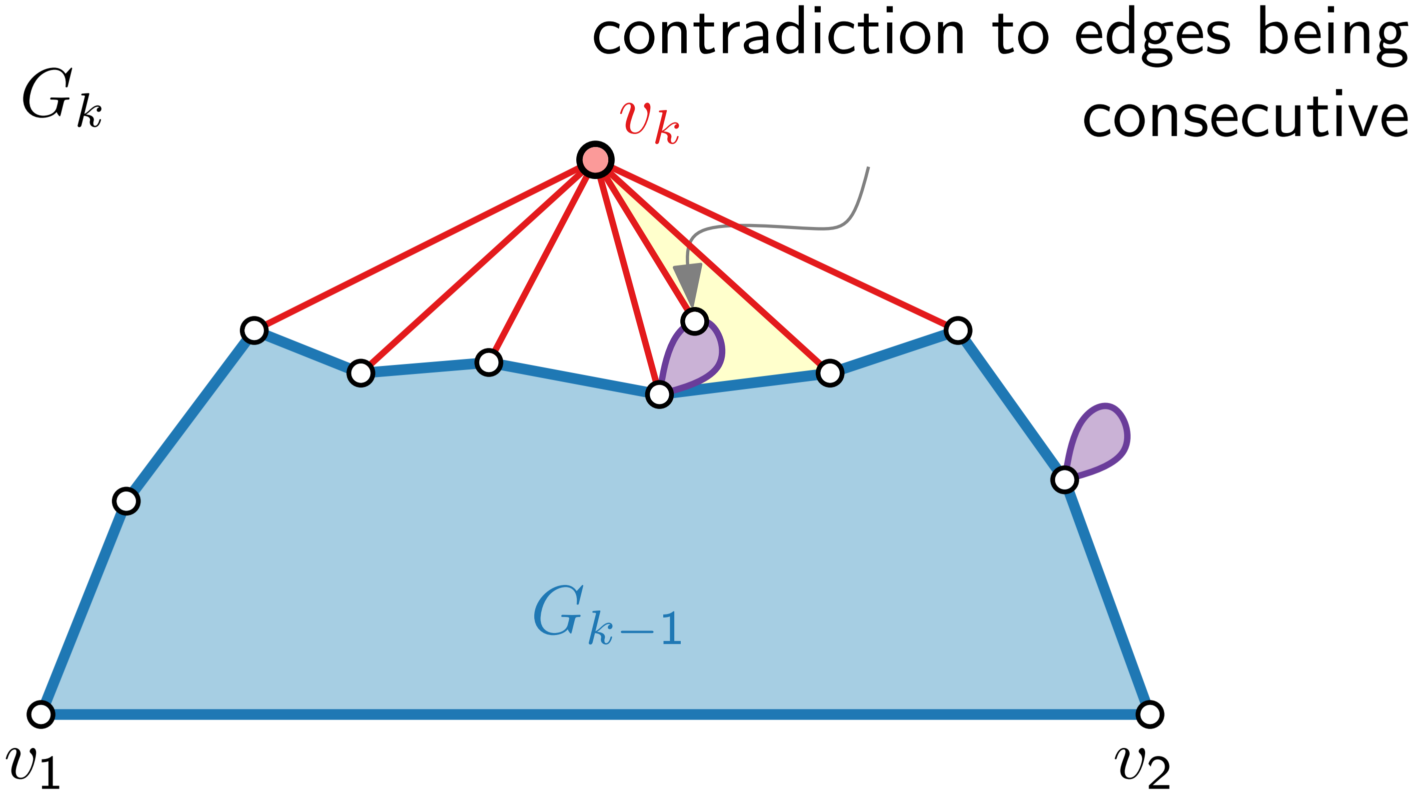

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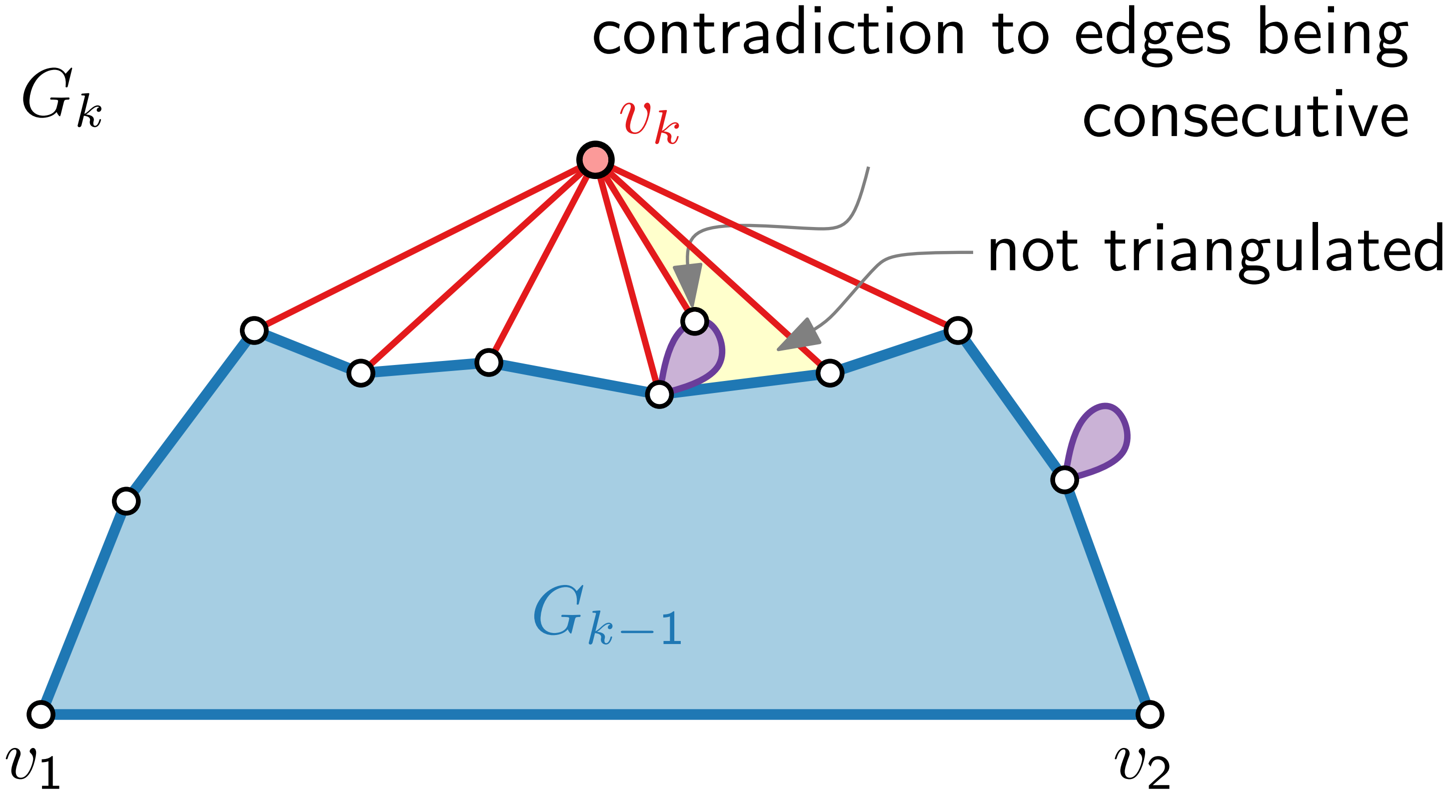

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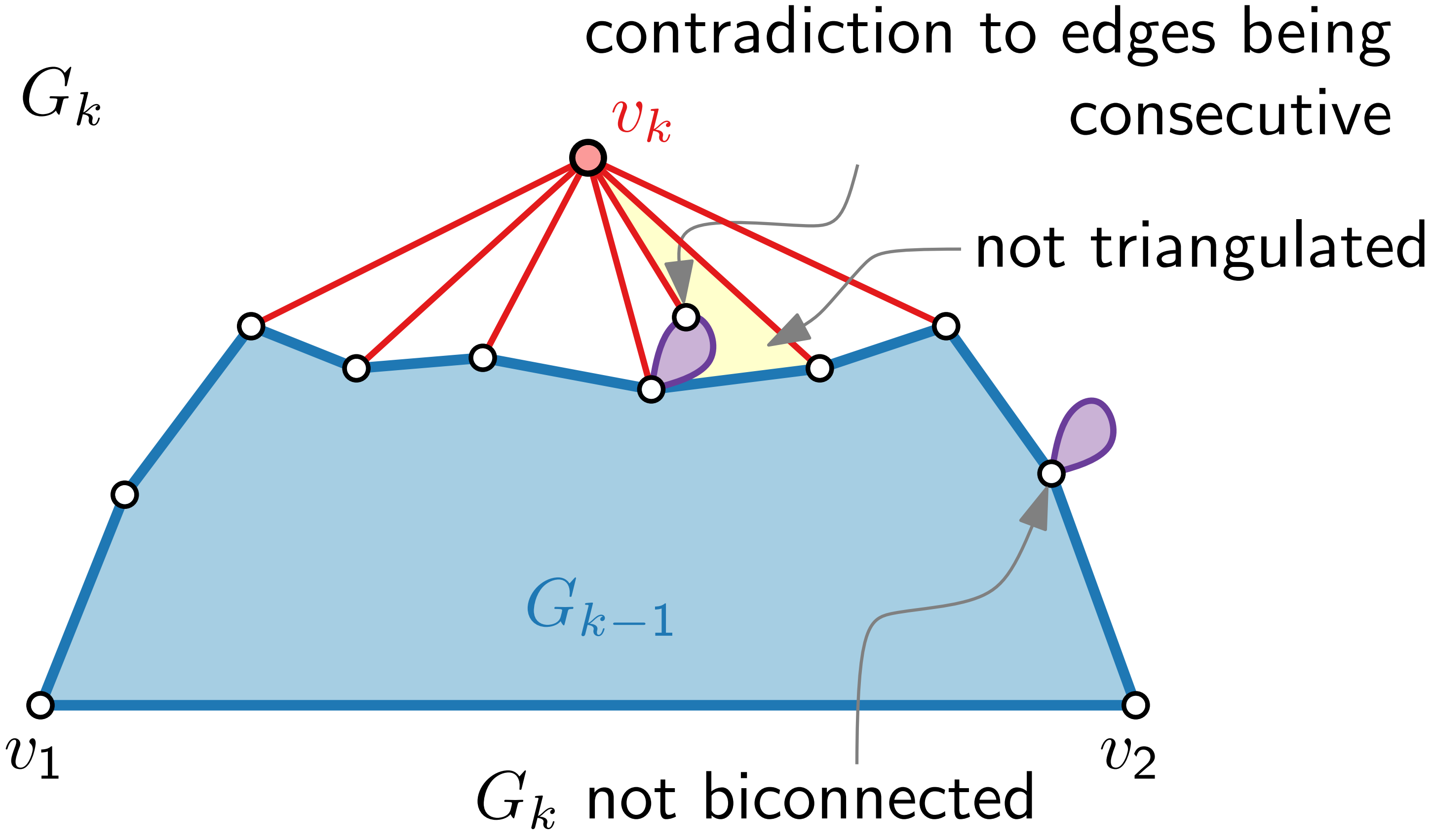

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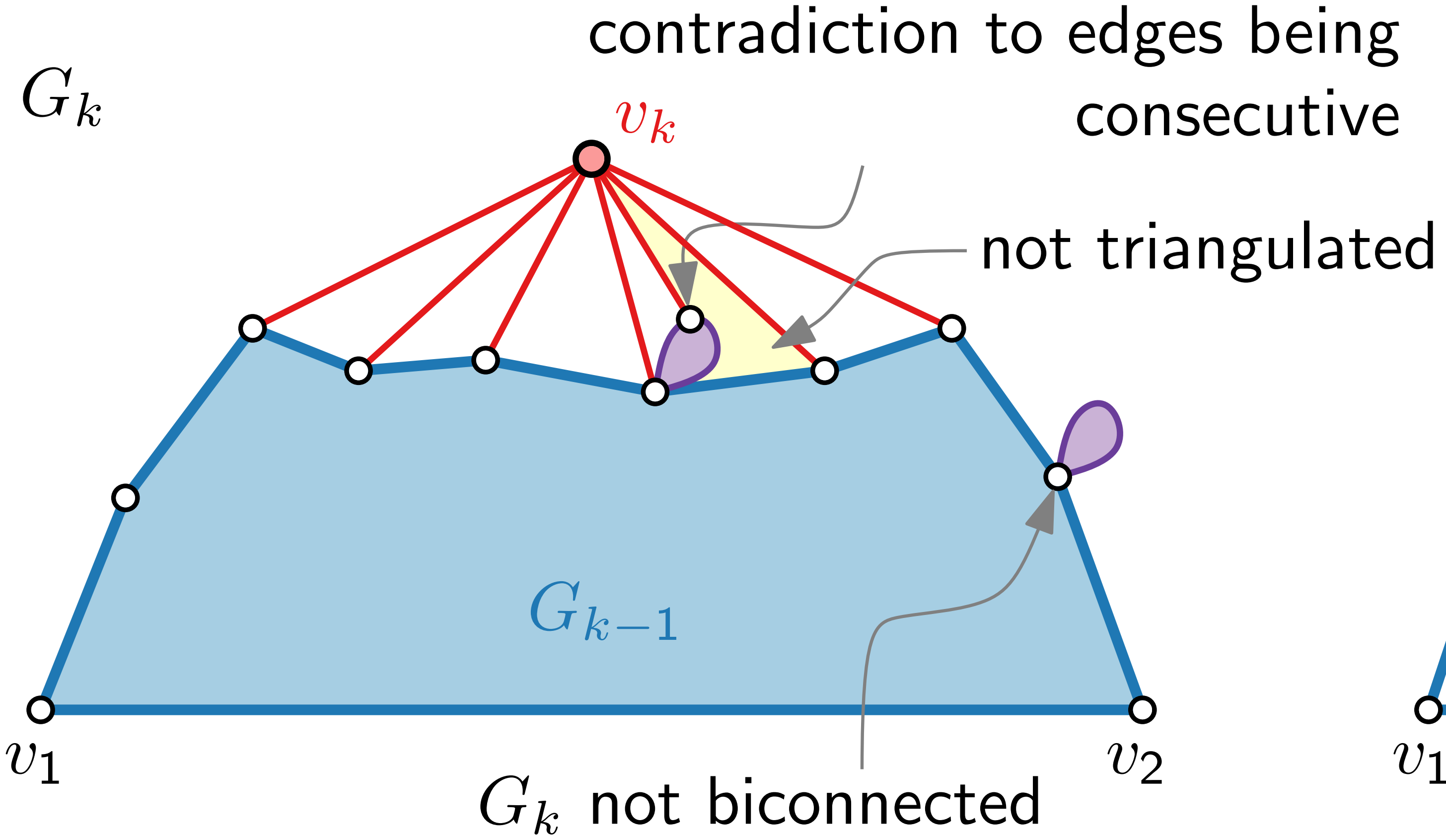

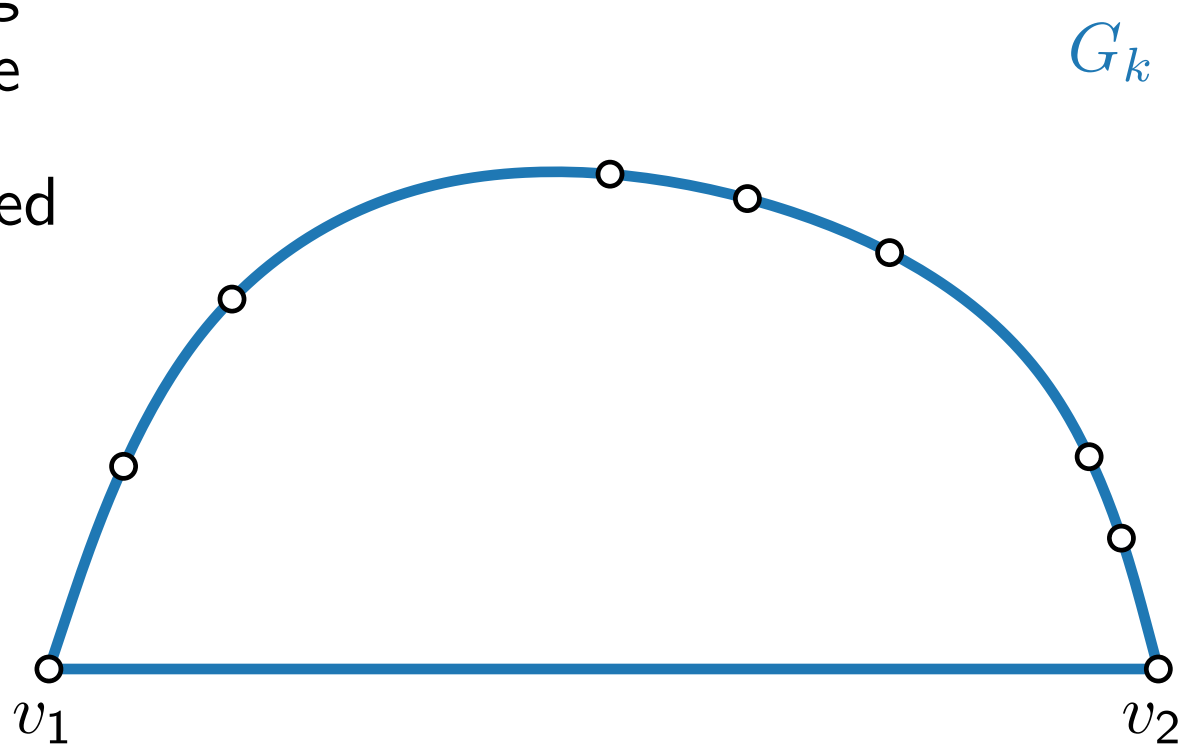

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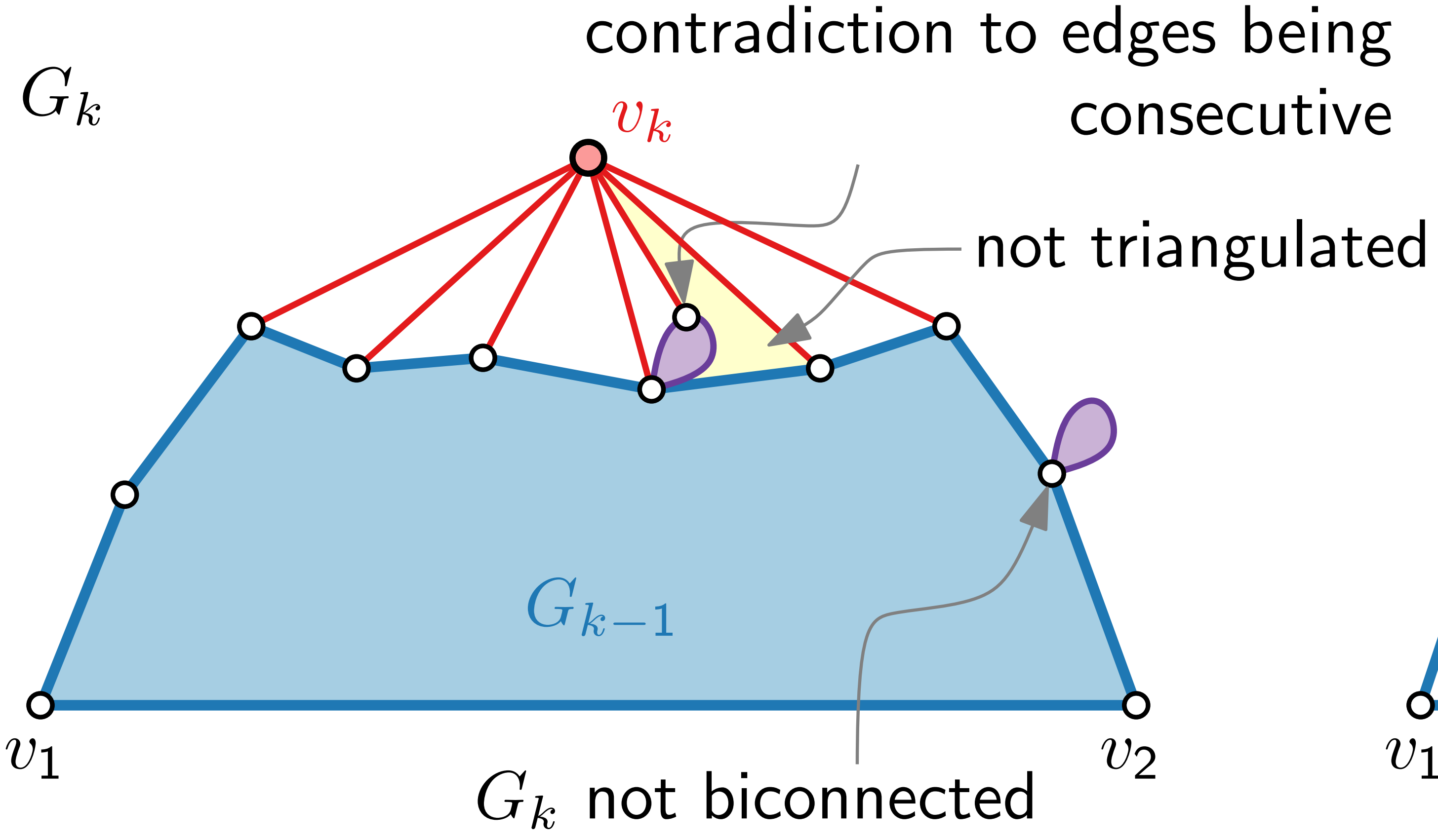

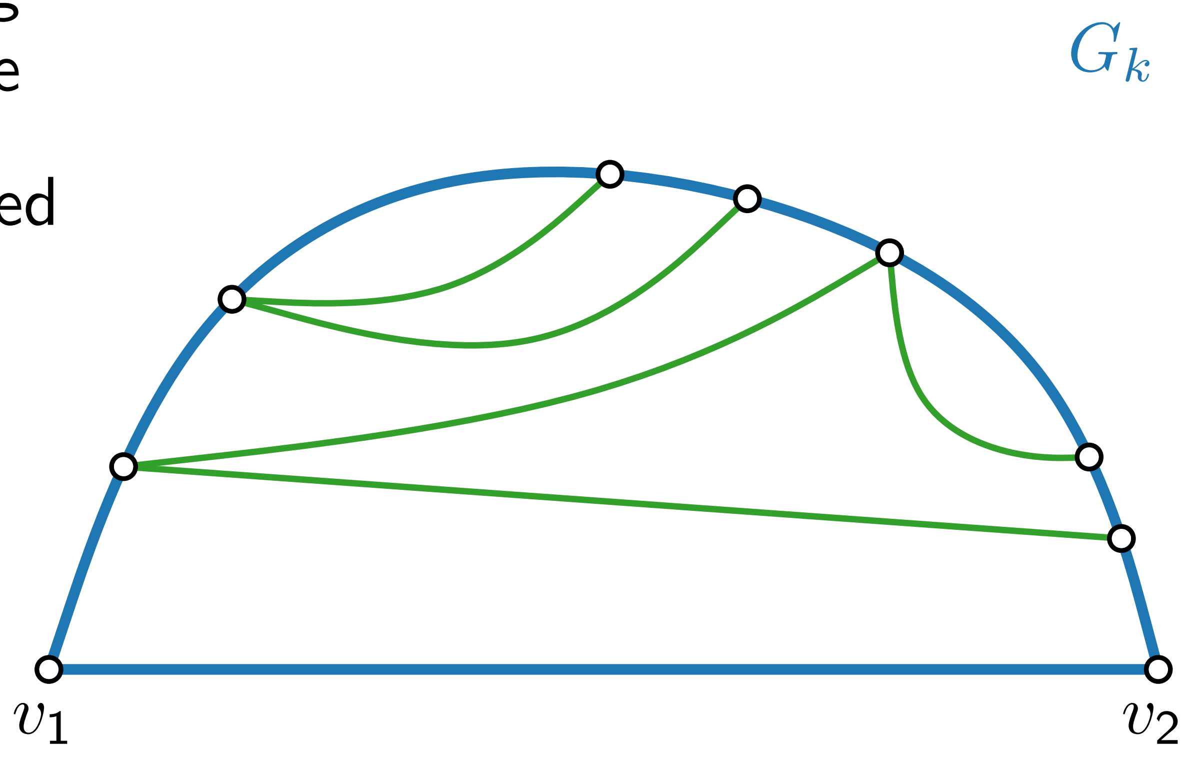

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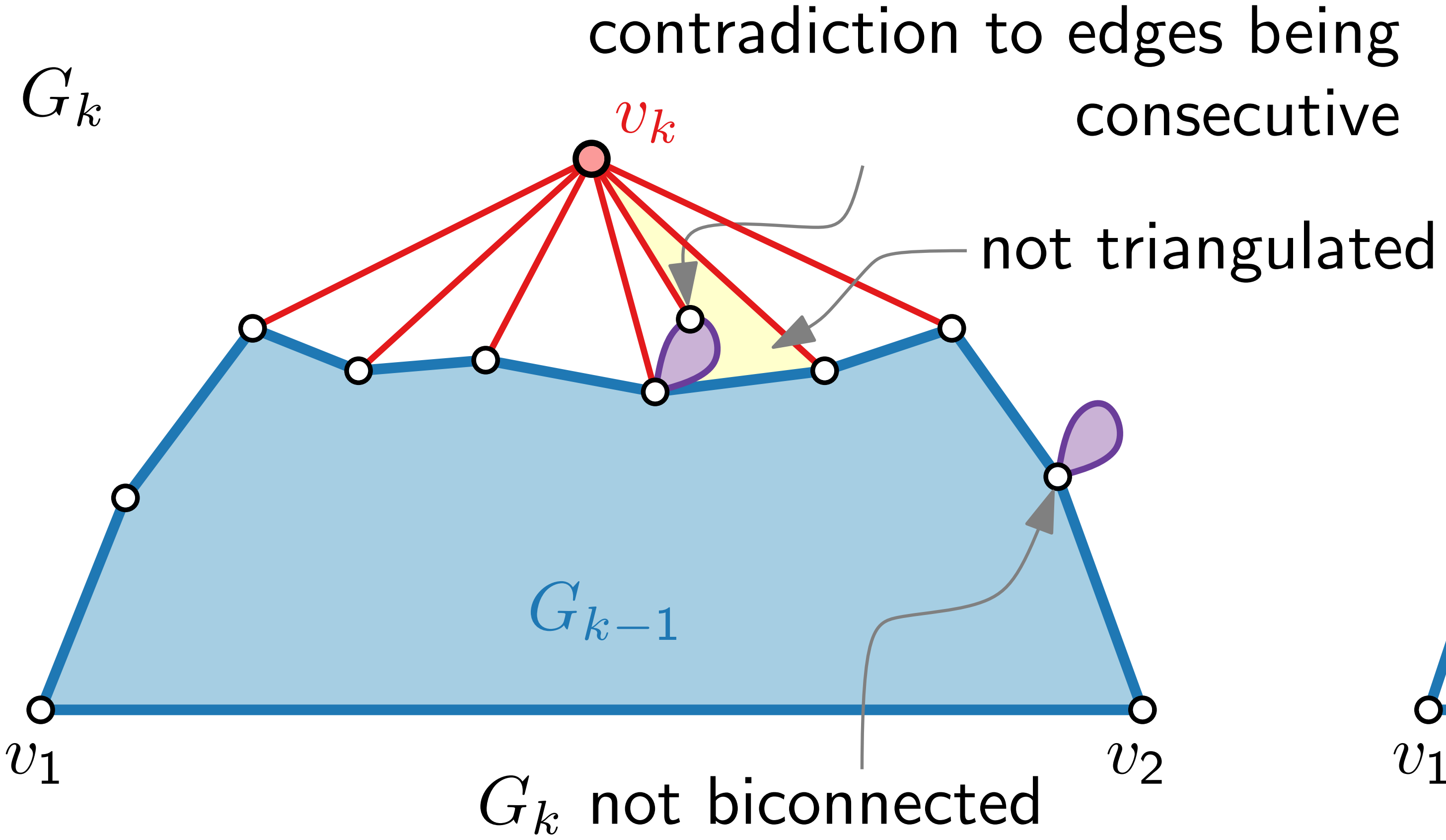

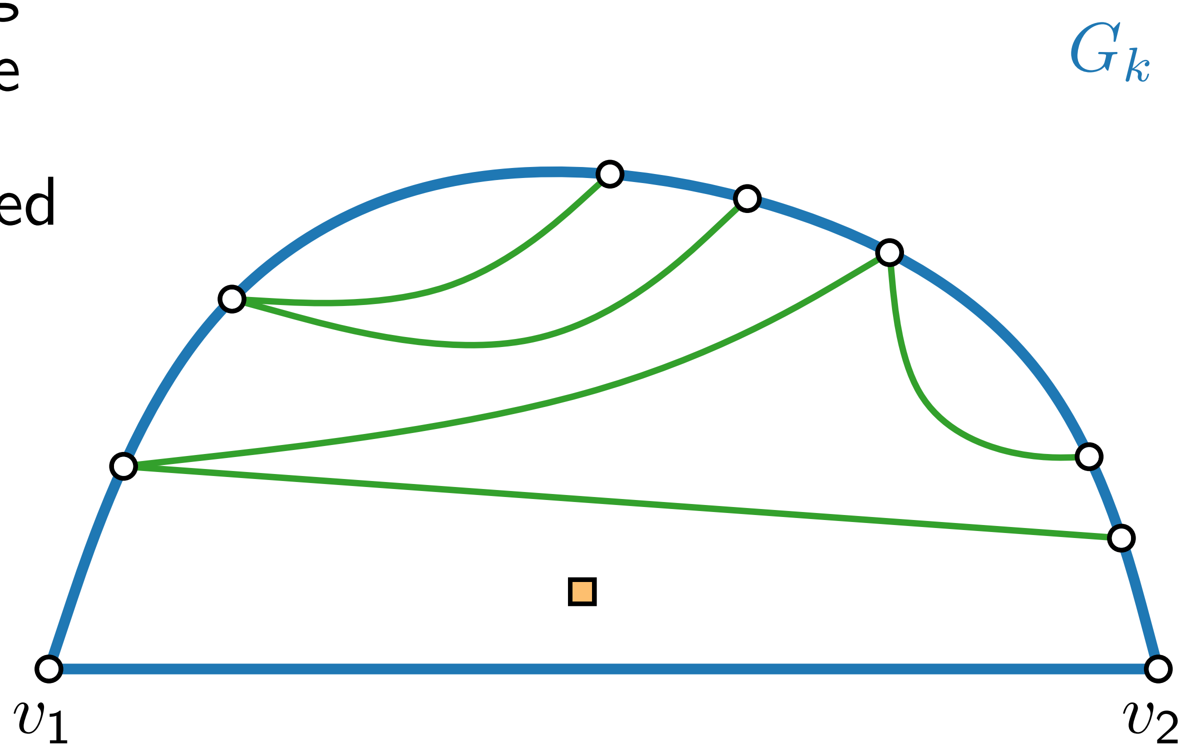

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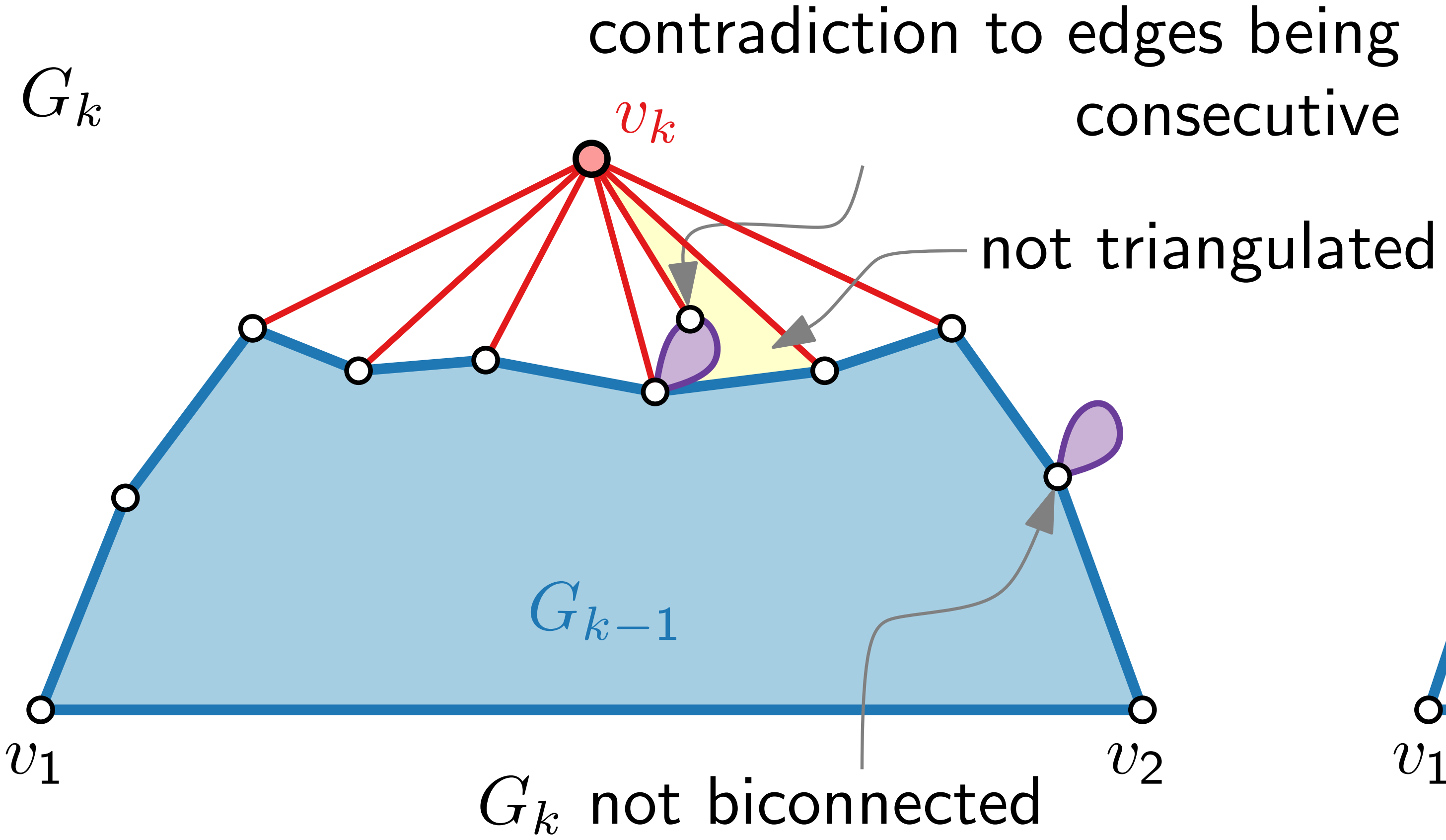

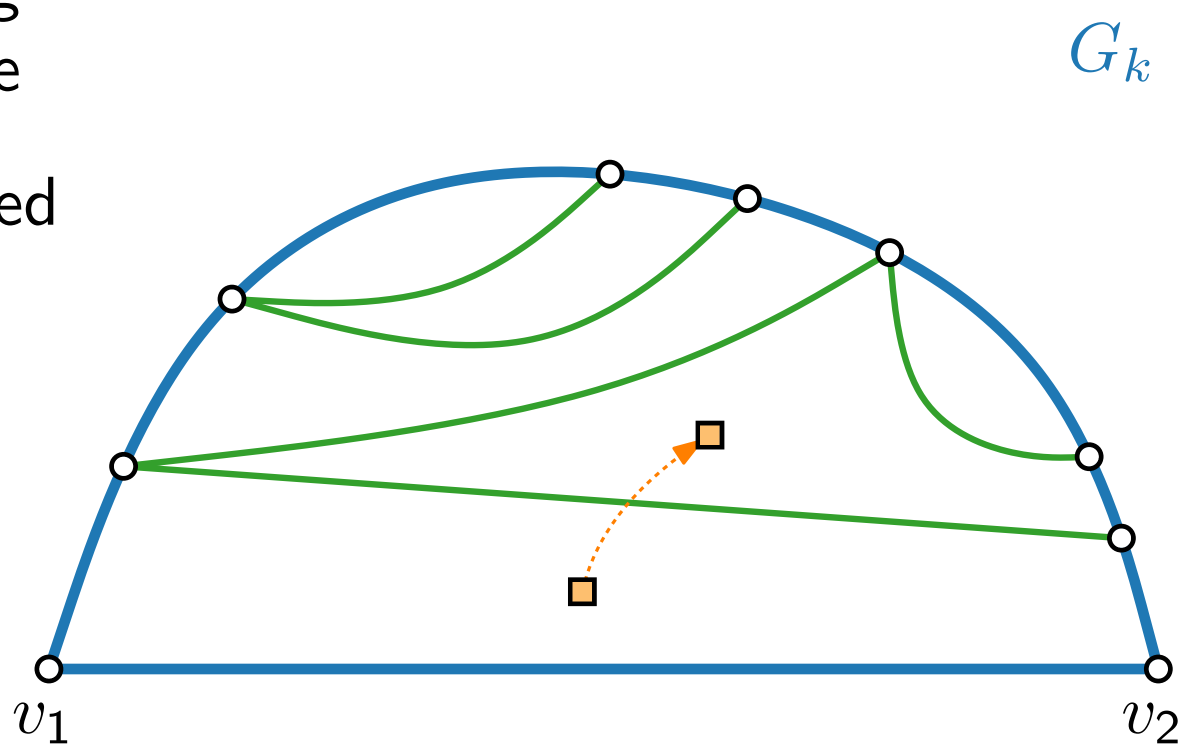

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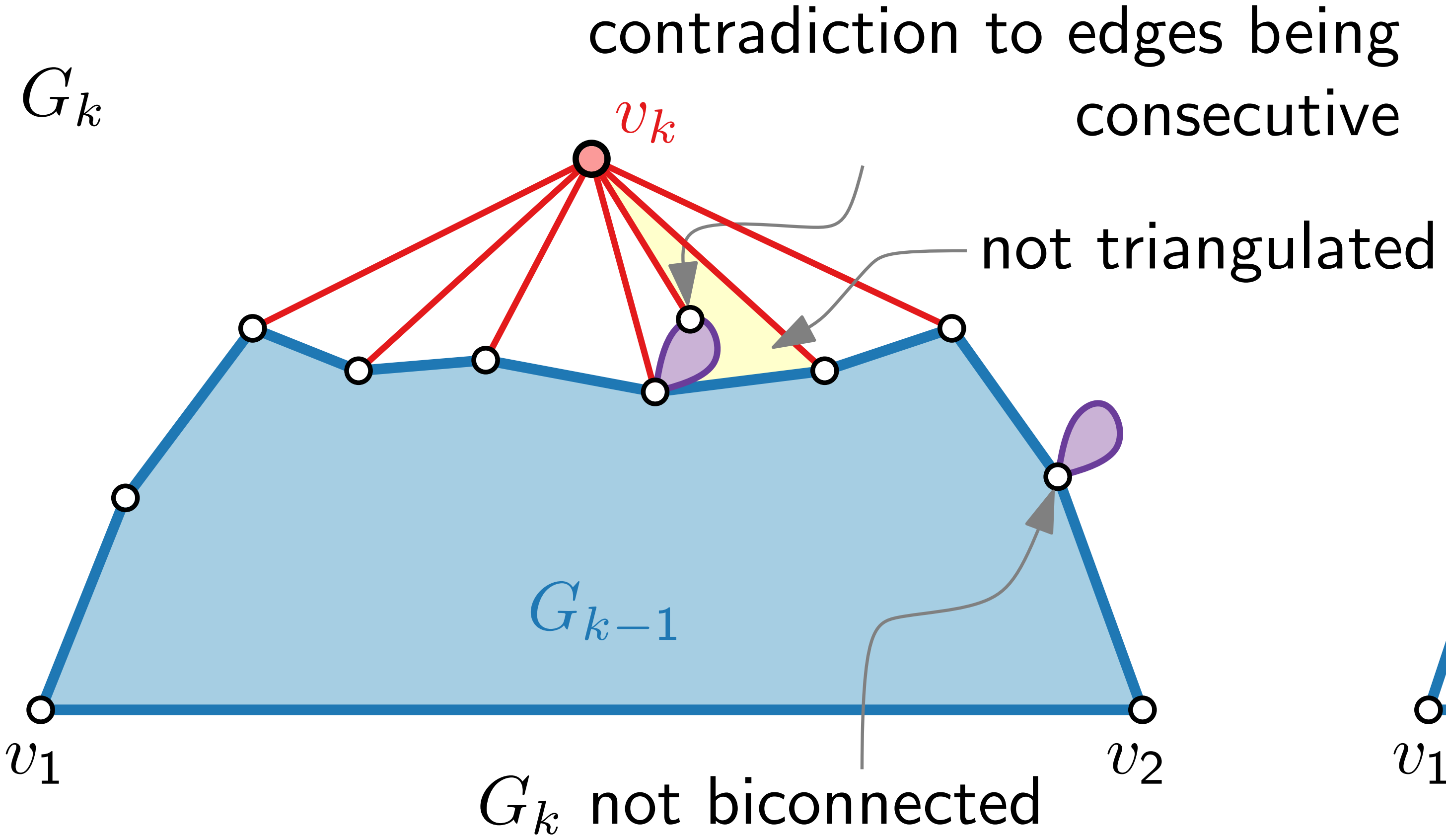

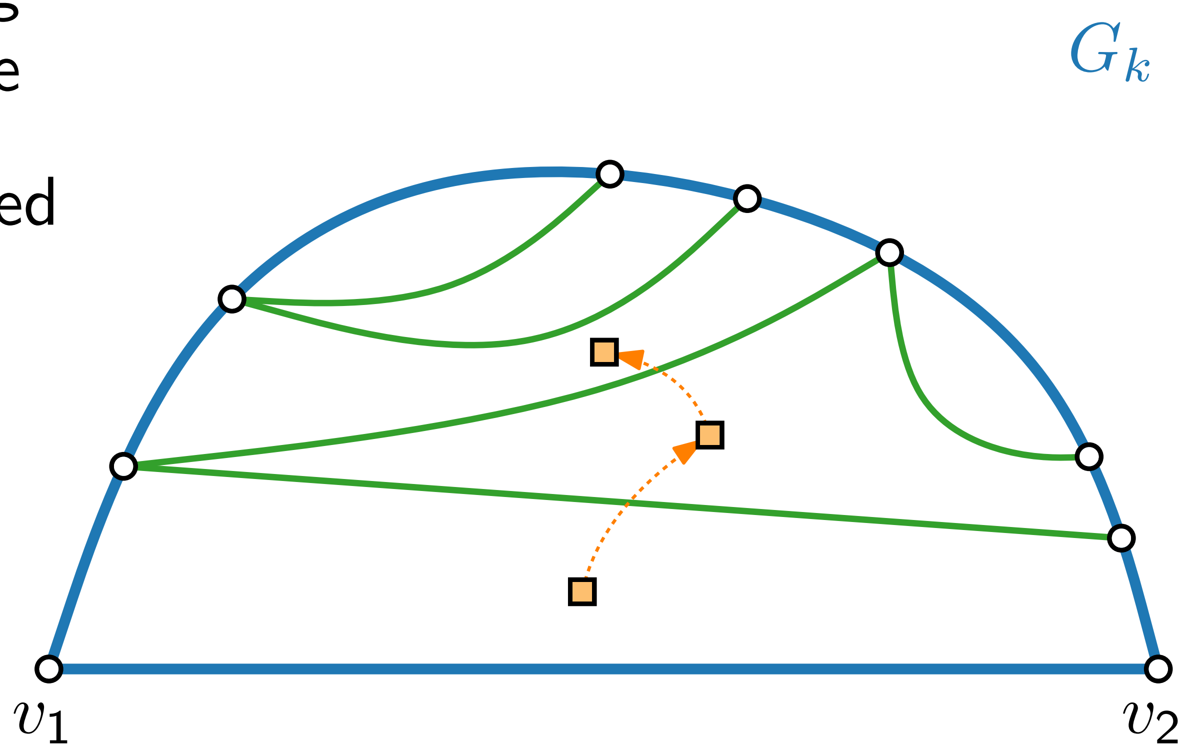

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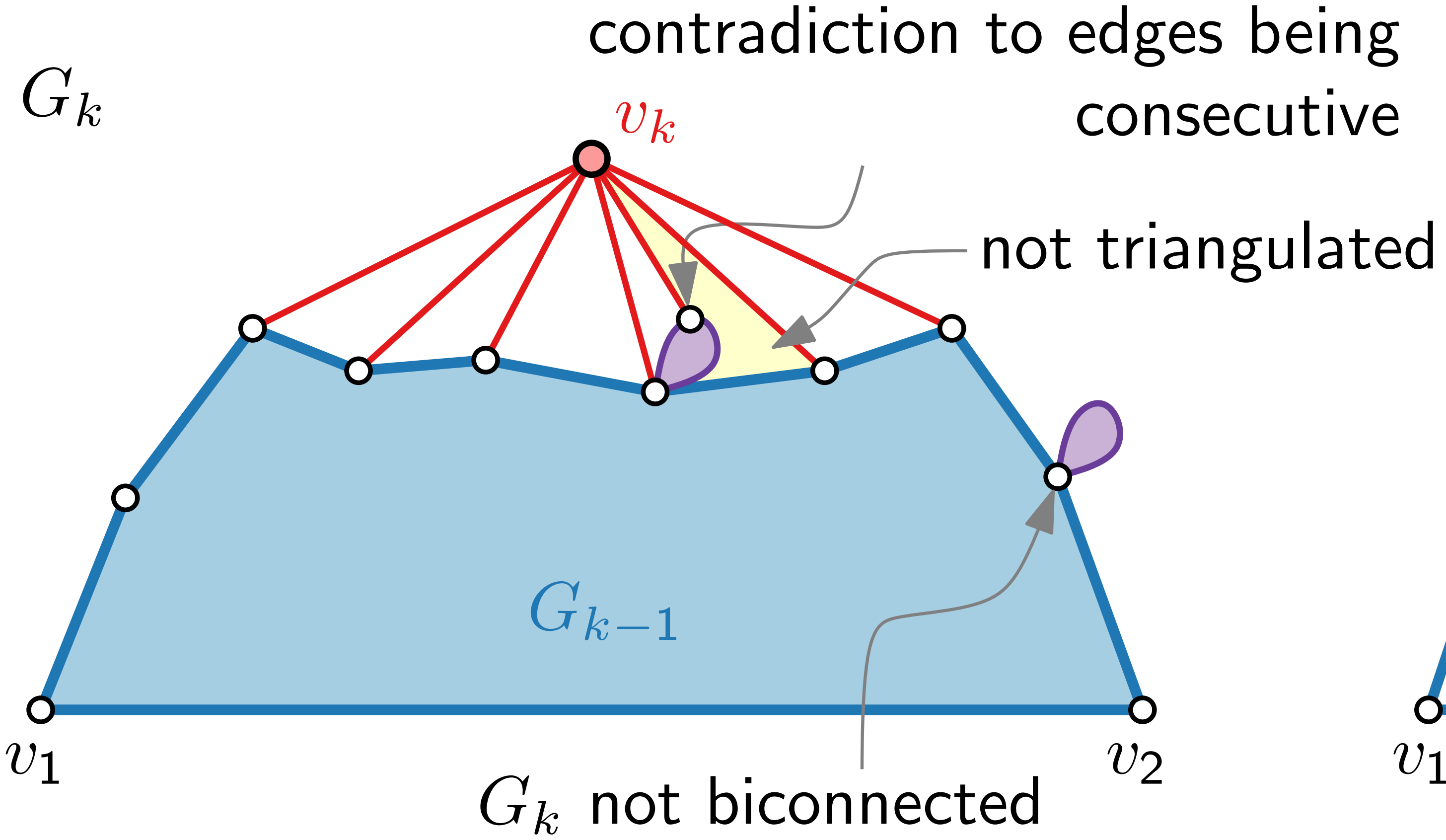

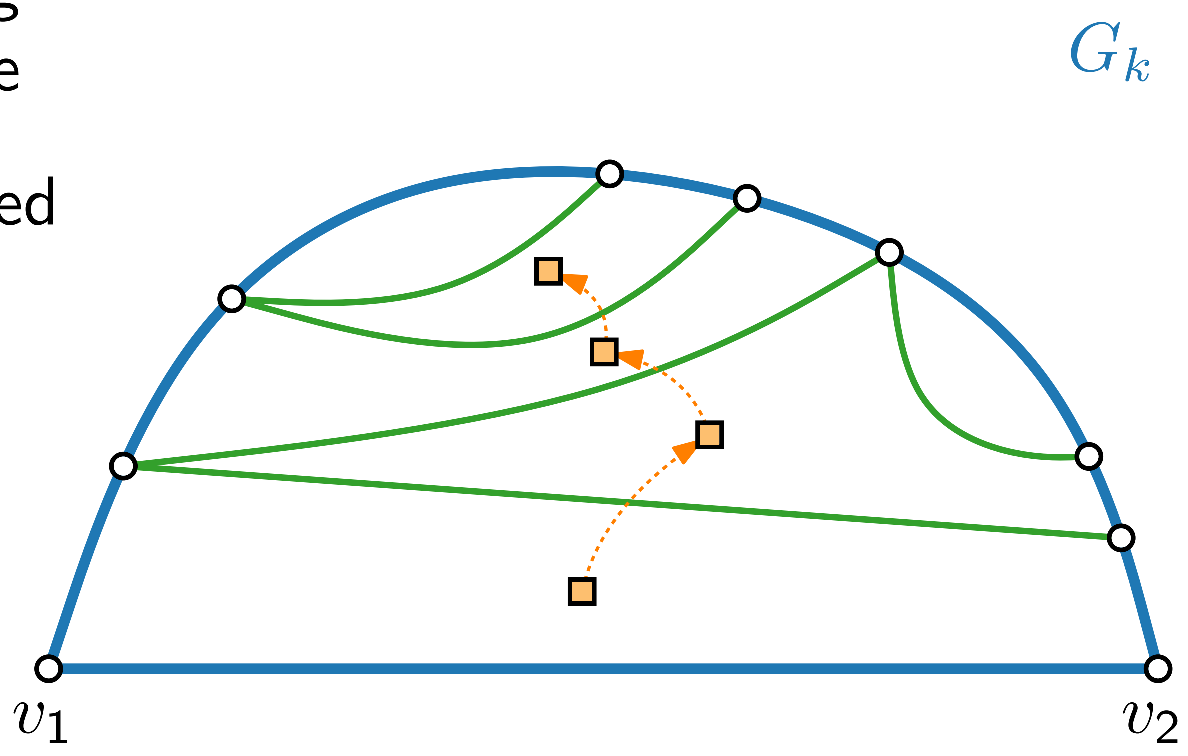

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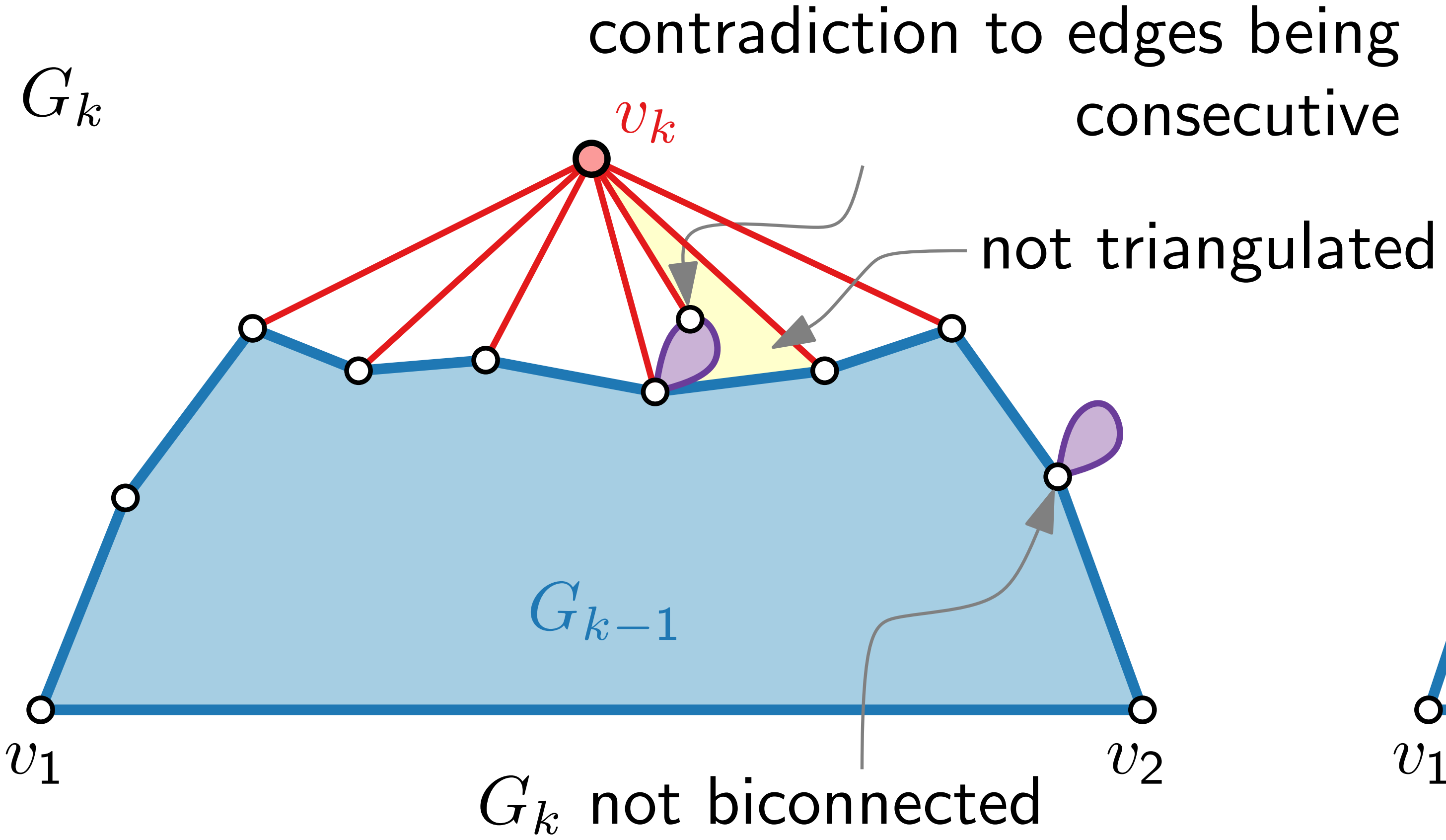

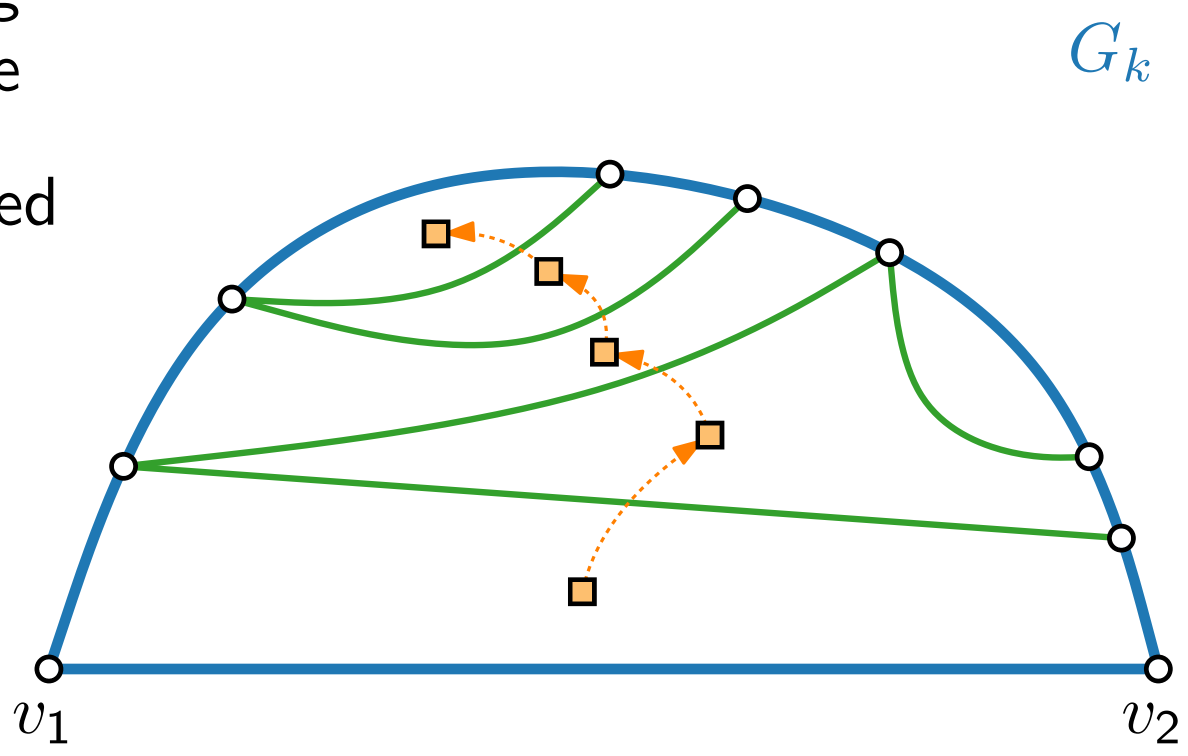

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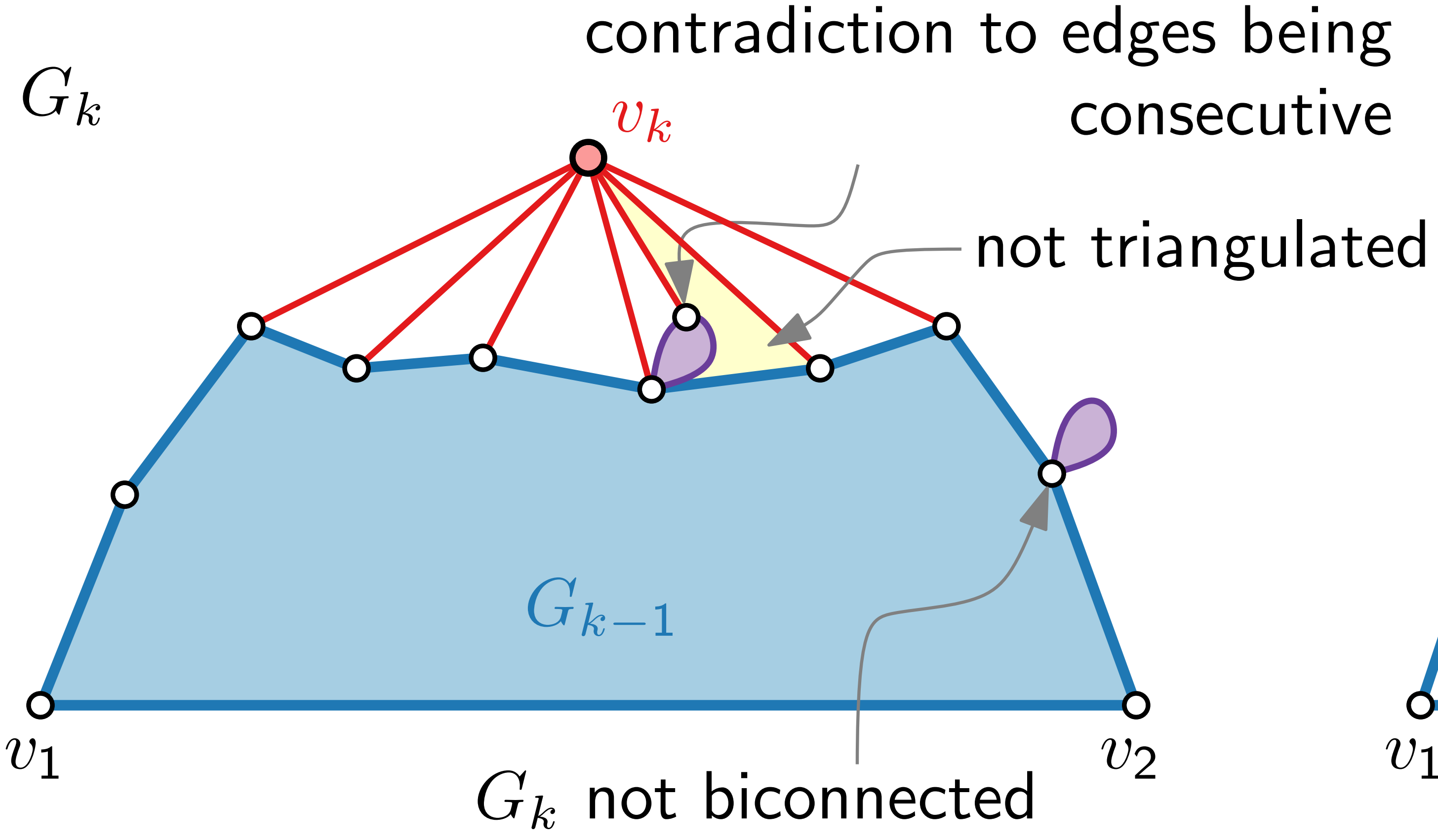

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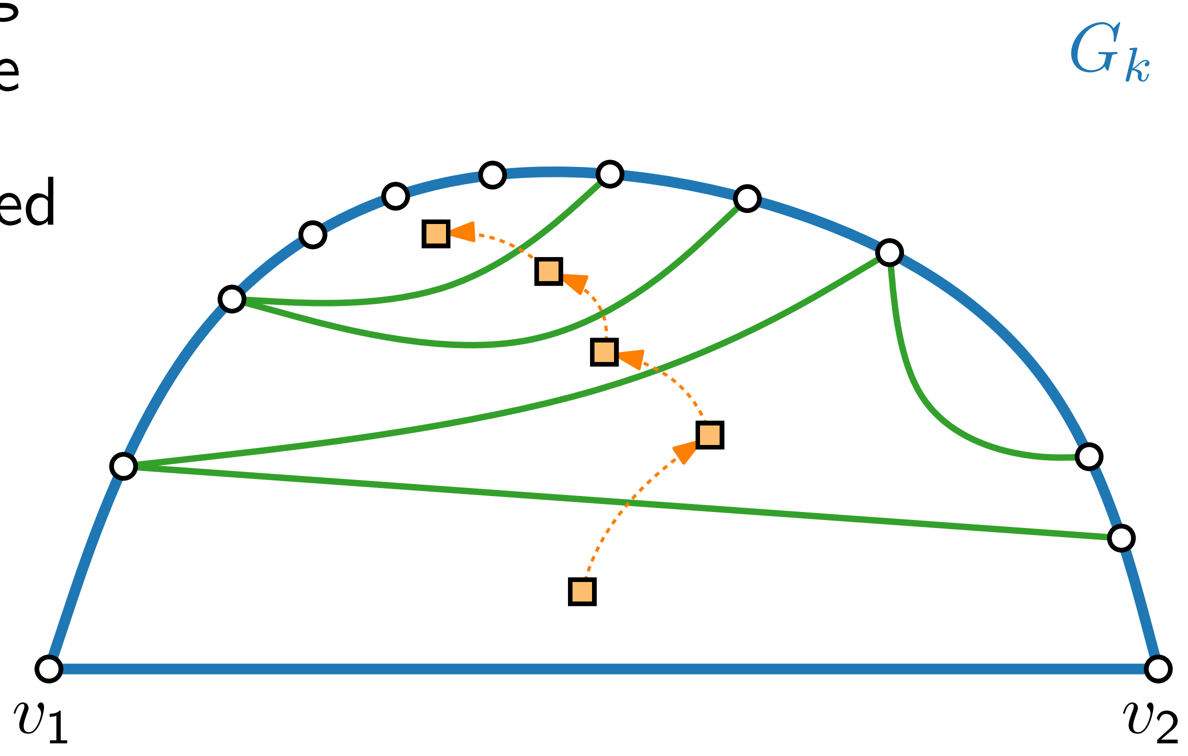

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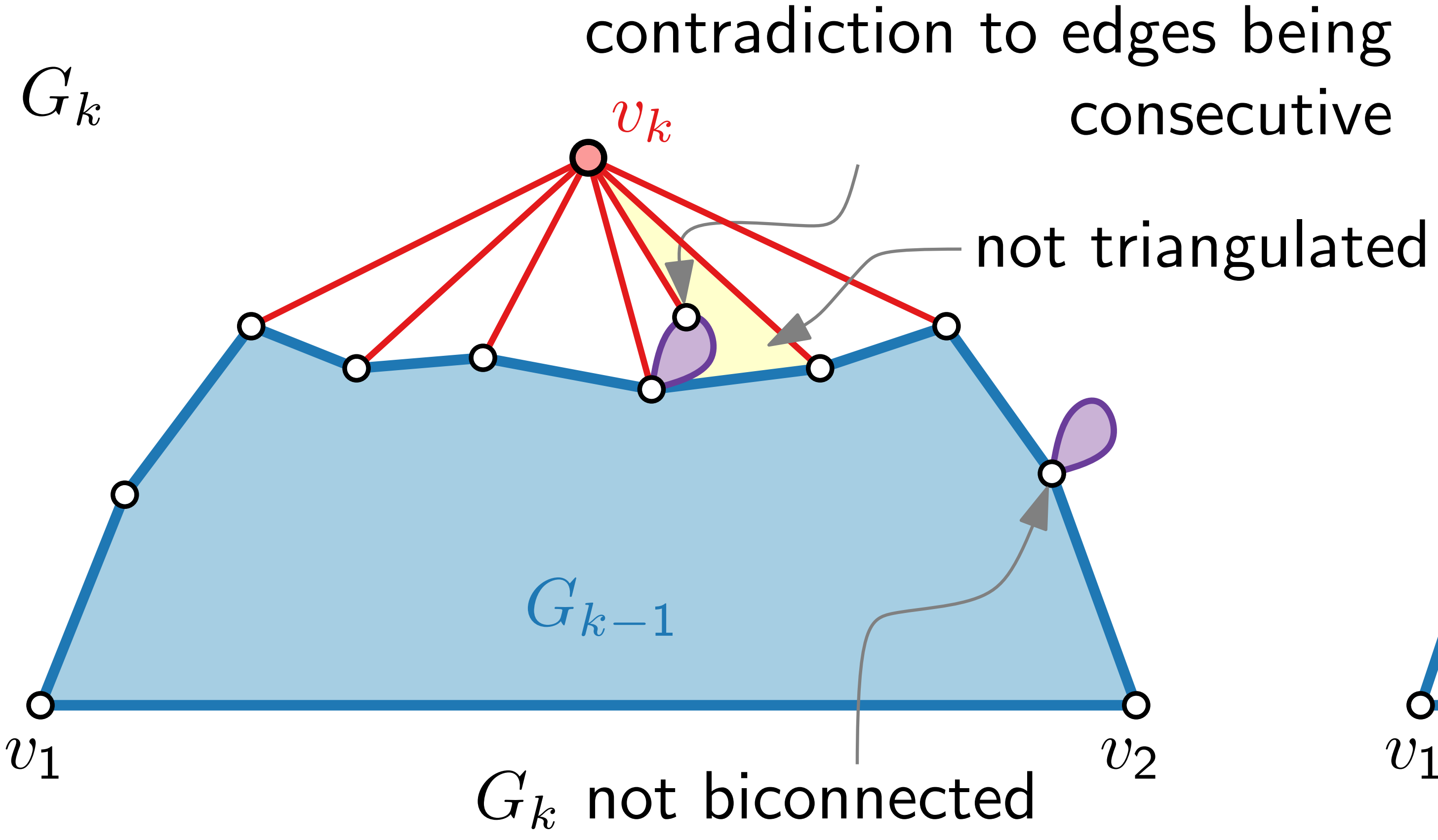

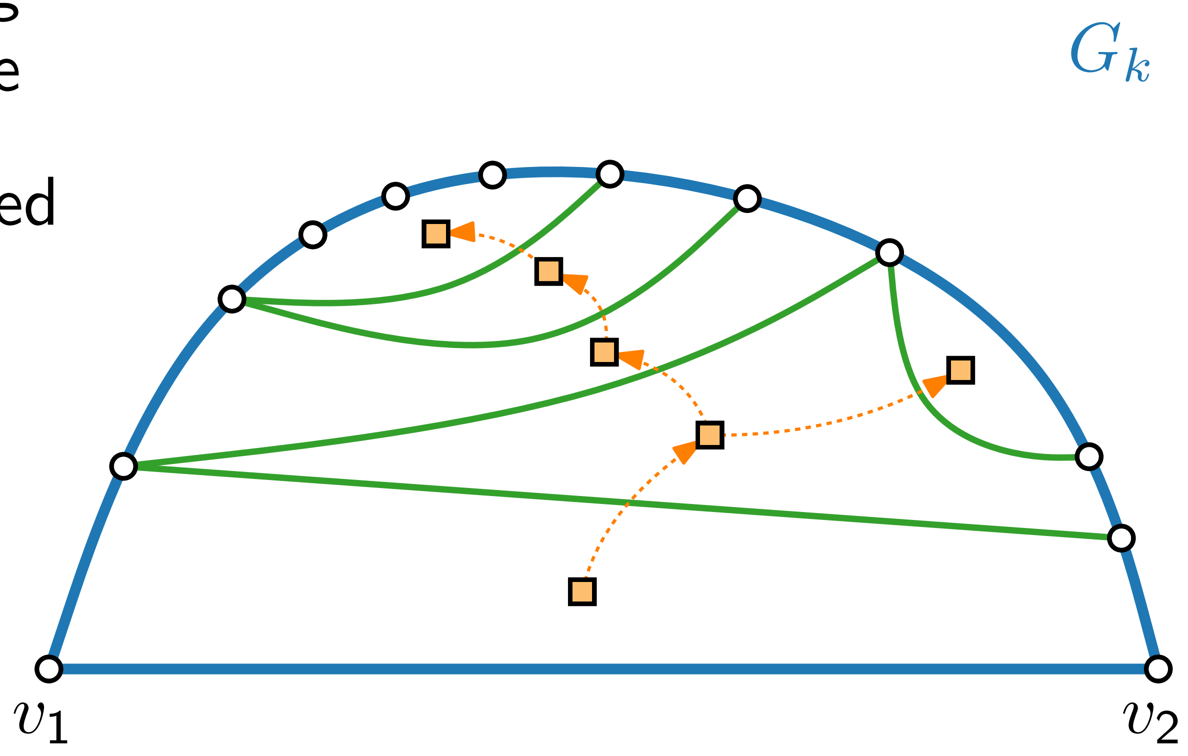

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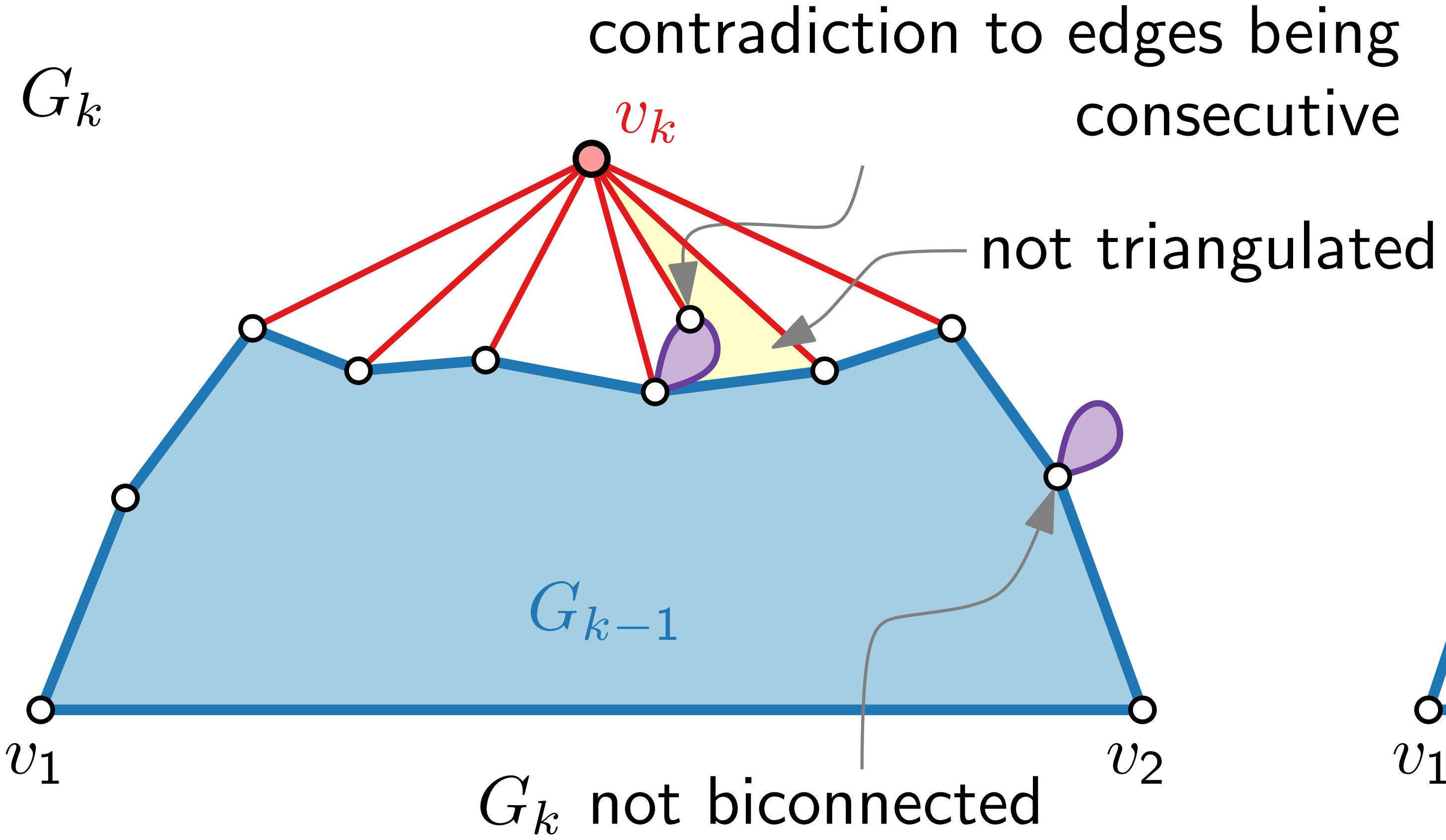

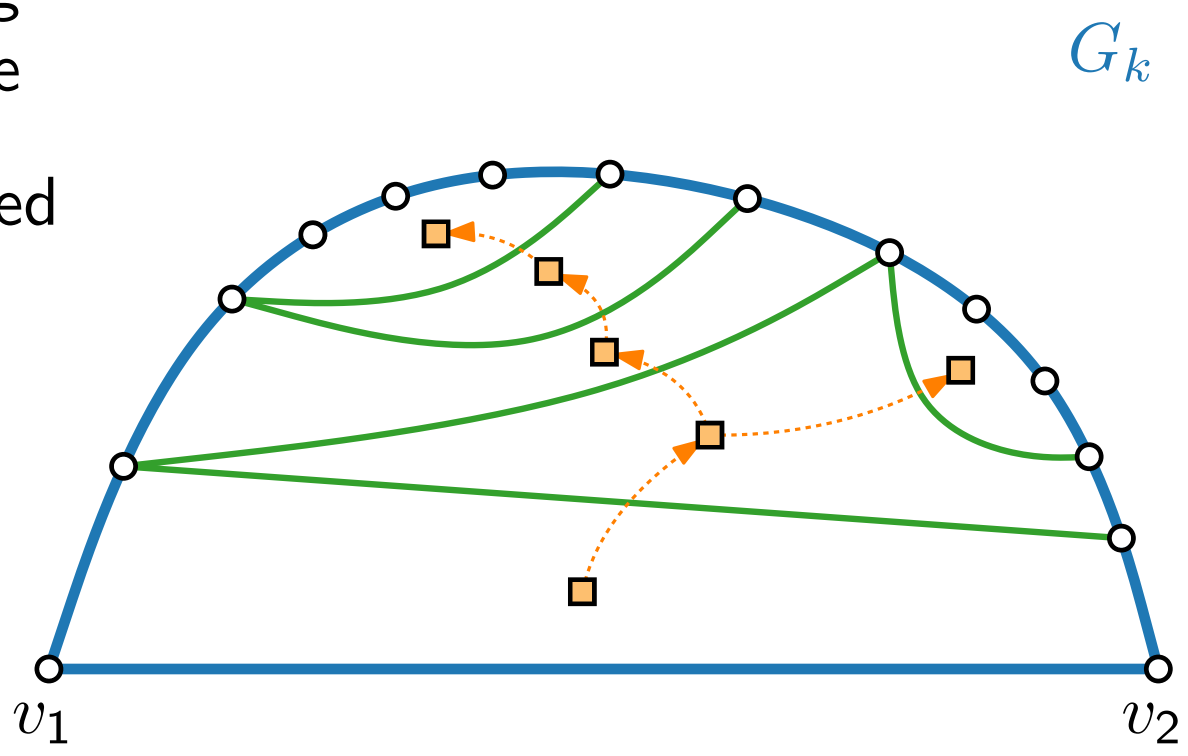

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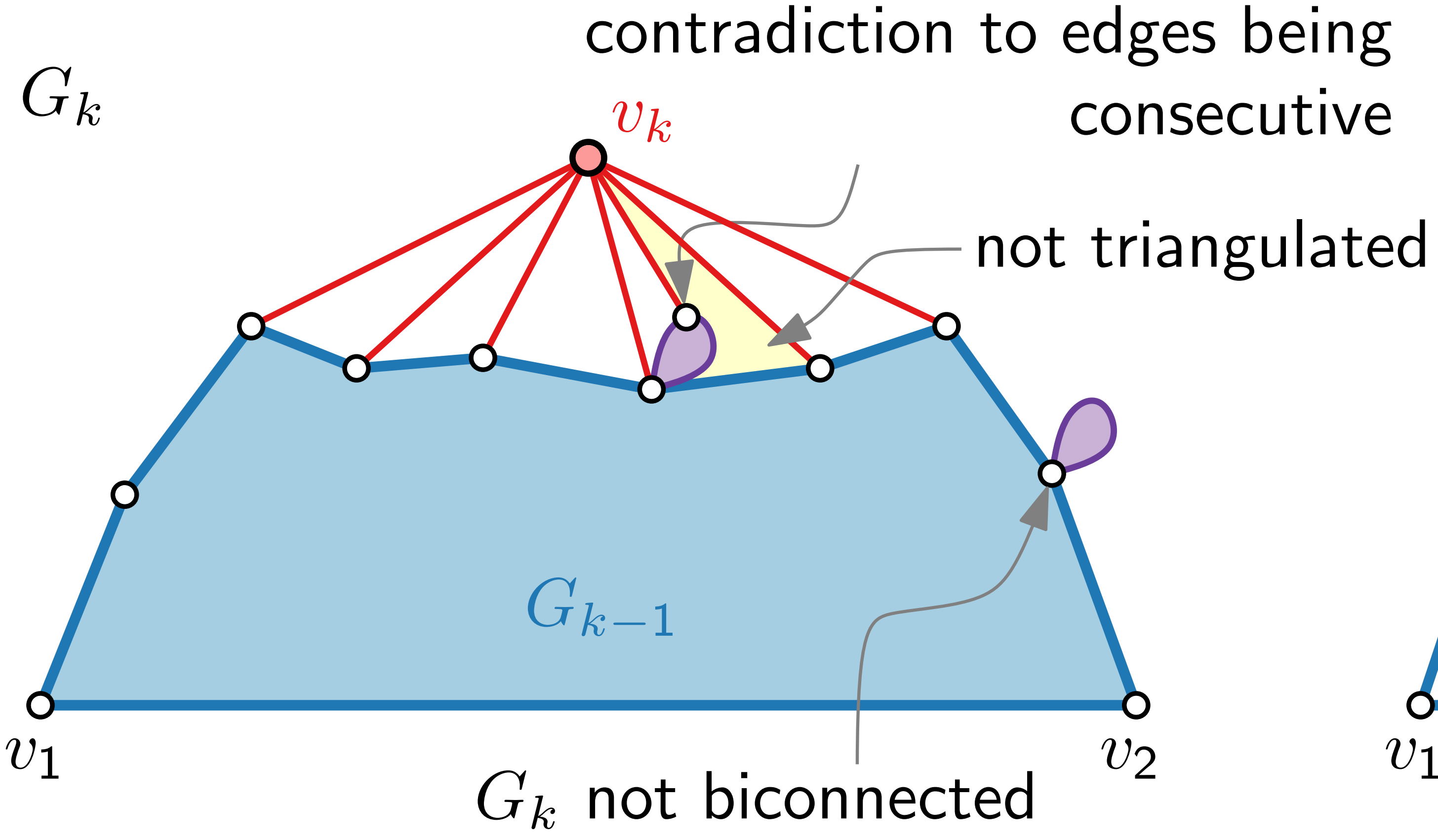

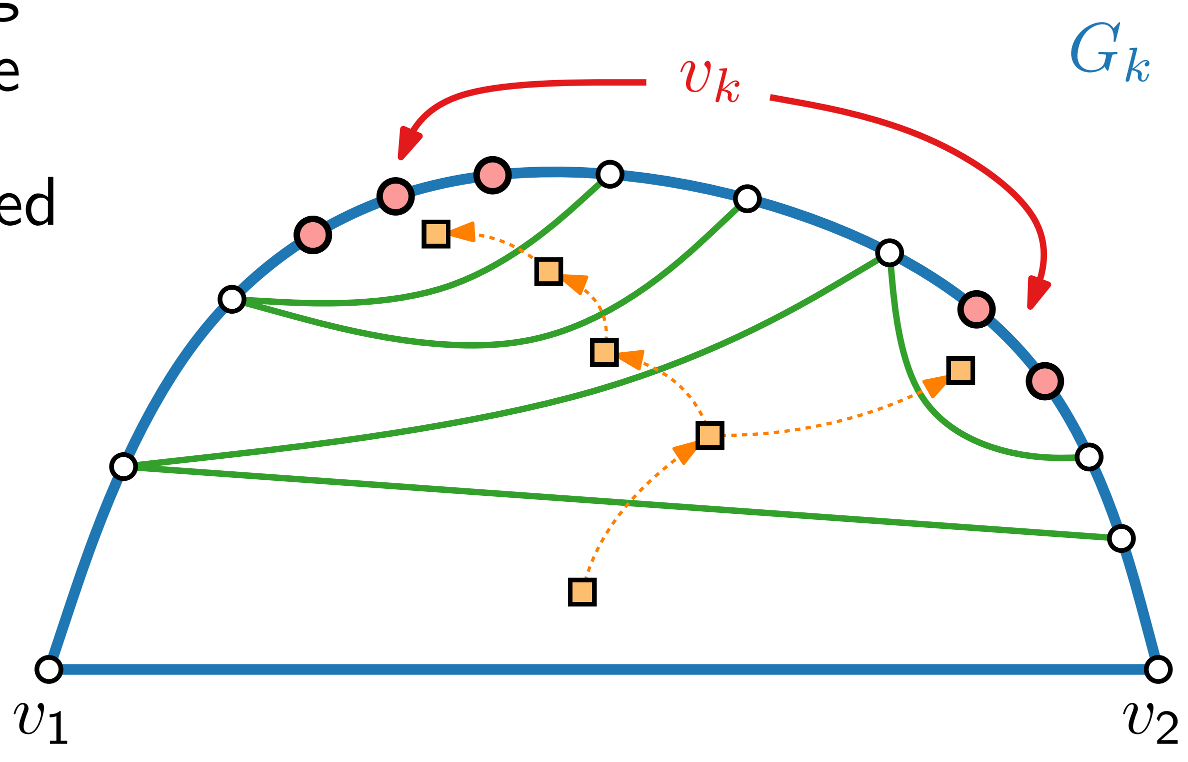

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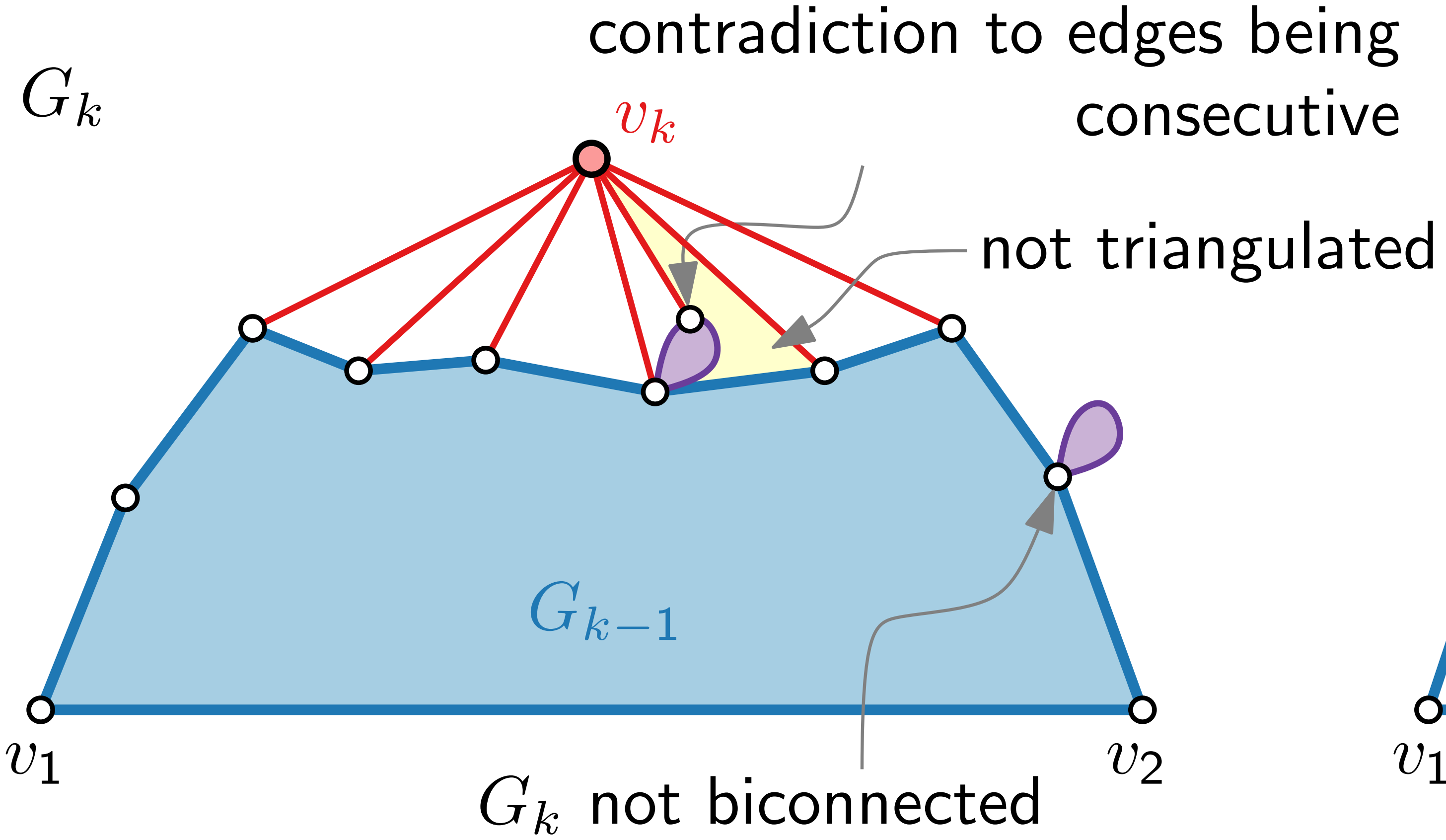

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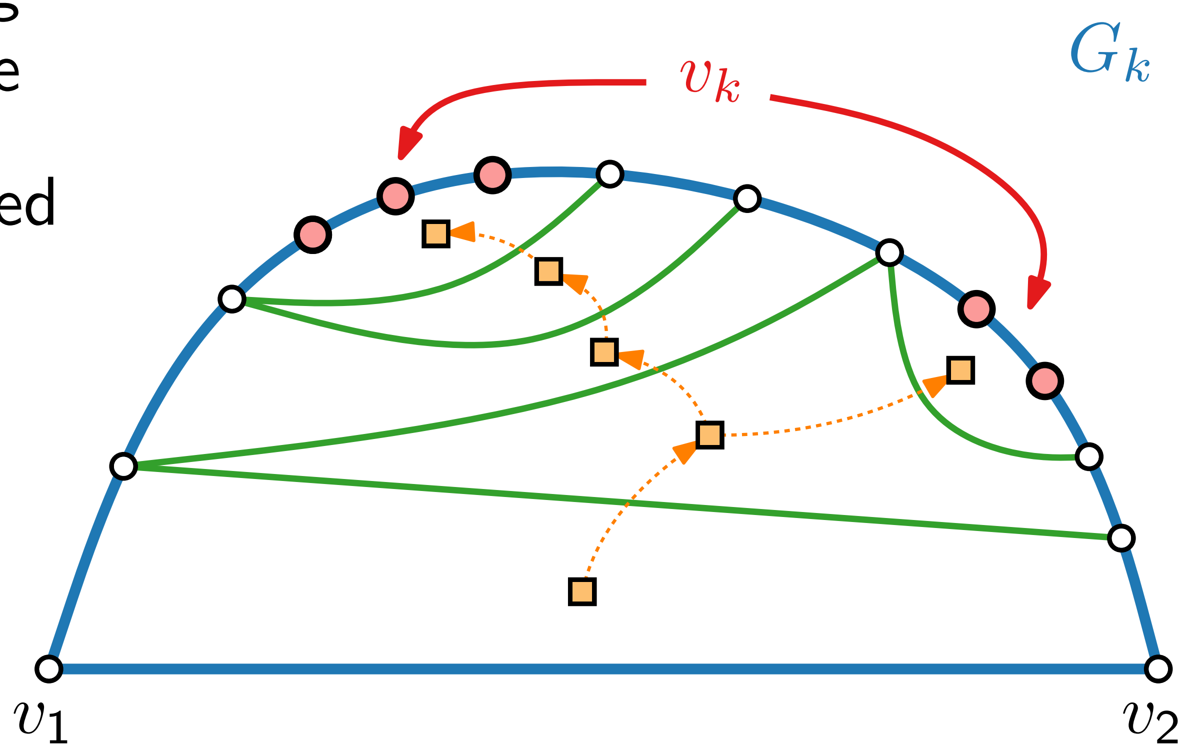

This completes proof of Lemma. □

Canonical Order – Implementation

CanonicalOrder$(G = (V, E), (v_1, v_2, v_n))$

Canonical Order – Implementation

outer face

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CanonicalOrder($G = (V, E), (v_1, v_2, v_n)$)

forall $v \in V$ **do**

Canonical Order – Implementation

outer face

CanonicalOrder($G = (V, E), (v_1, v_2, v_n)$)

forall $v \in V$ **do**

  chords(v) $\leftarrow 0$;

Canonical Order – Implementation

outer face

CanonicalOrder($G = (V, E), (v_1, v_2, v_n)$)

forall $v \in V$ **do**
  chords(v) $\leftarrow$ 0;

- chord(v):
  \# chords adjacent to v

Canonical Order – Implementation

outer face

CanonicalOrder($G = (V, E), (v_1, v_2, v_n)$)

forall $v \in V$ **do**
  chords(v) $\leftarrow$ 0; out(v) $\leftarrow$ false;

- chord(v):
  # chords adjacent to v

Canonical Order – Implementation

outer face

CanonicalOrder($G = (V, E), (v_1, v_2, v_n)$)

forall $v \in V$ **do**
  chords(v) $\leftarrow$ 0; out(v) $\leftarrow$ false;

- chord(v): # chords adjacent to v
- out(v) = true iff v is currently outer vertex

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outer face

CanonicalOrder$(G = (V, E), (v_1, v_2, v_n))$

forall $v \in V$ **do**

  chords$(v) \leftarrow 0$; out$(v) \leftarrow$ false; mark$(v) \leftarrow$ false

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Canonical Order – Implementation

outer face

CanonicalOrder($G = (V, E), (v_1, v_2, v_n)$)

forall $v \in V$ **do**

chords(v) $\leftarrow$ 0; out(v) $\leftarrow$ false; mark(v) $\leftarrow$ false

- chord(v): # chords adjacent to v
- out(v) $=$ true iff v is currently outer vertex
- mark(v) $=$ true iff v has received its number

Canonical Order – Implementation

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Canonical Order – Implementation

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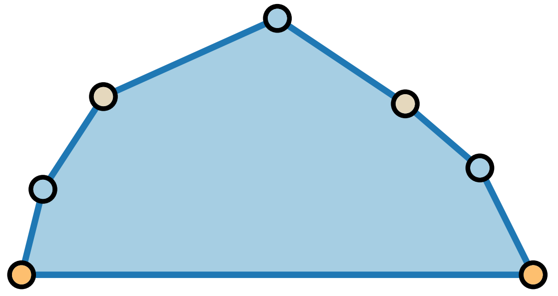

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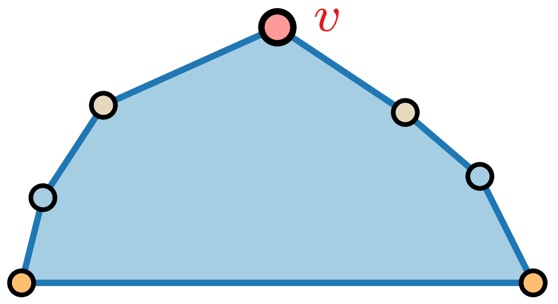

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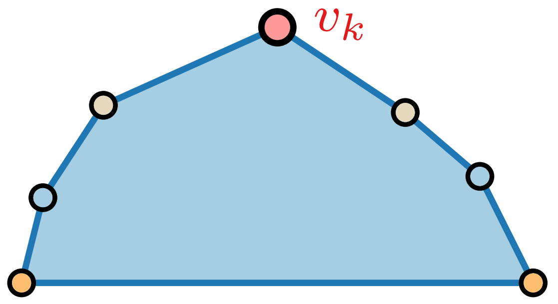

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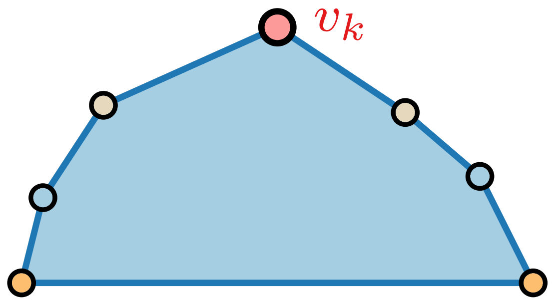

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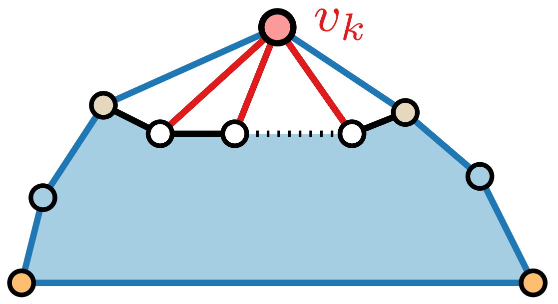

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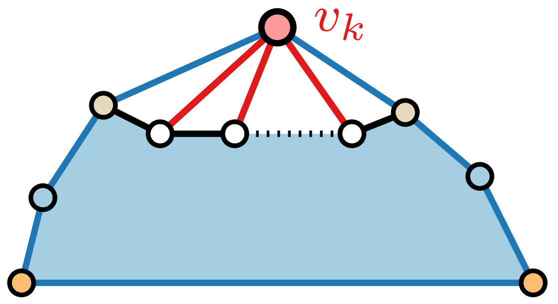

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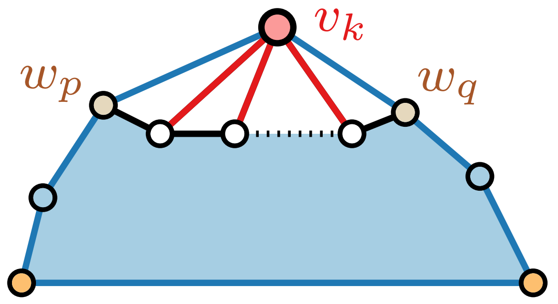

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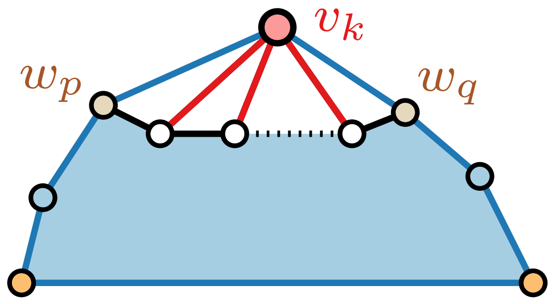

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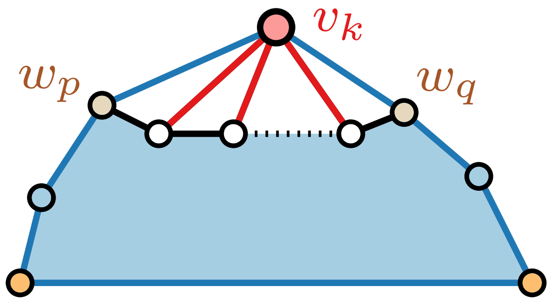

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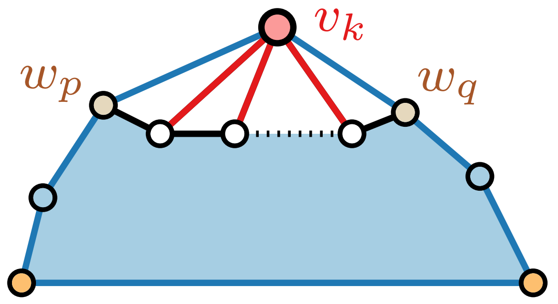

Lemma.
Algorithm CanonicalOrder computes a canonical order of a plane graph in $\mathcal{O}(n)$ time.

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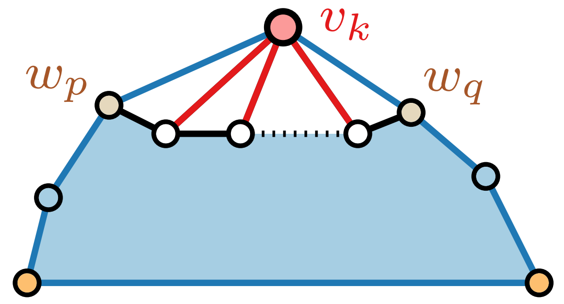

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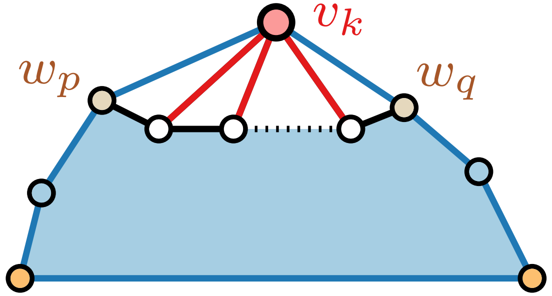

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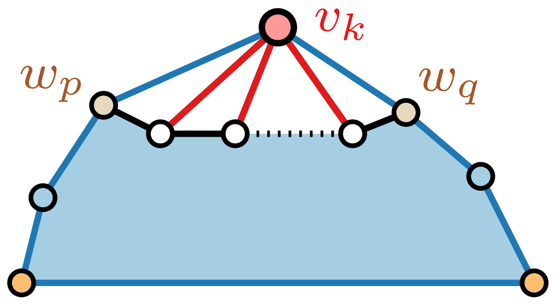

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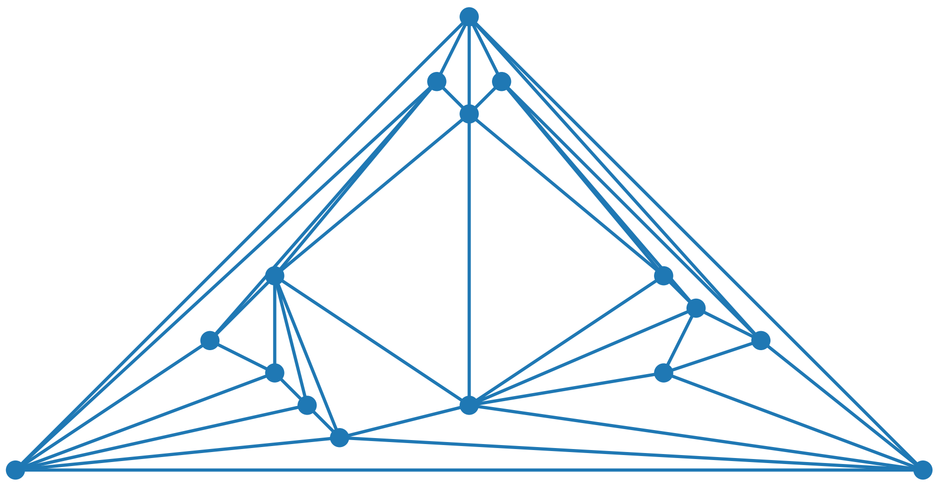
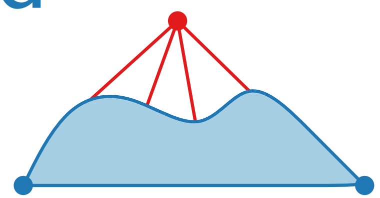
Visualization of Graphs

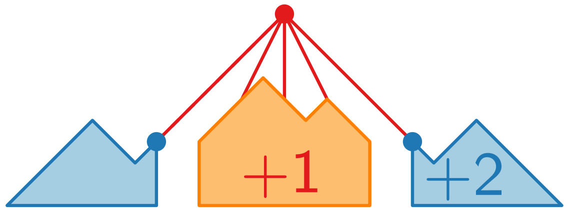
Lecture 3:
Straight-Line Drawings of Planar Graphs I: Canonical Ordering and Shift Method



Part III:
Shift Method

Jonathan Klawitter





Shift Method – Idea

Drawing invariants:

G_{k-1} is drawn such that

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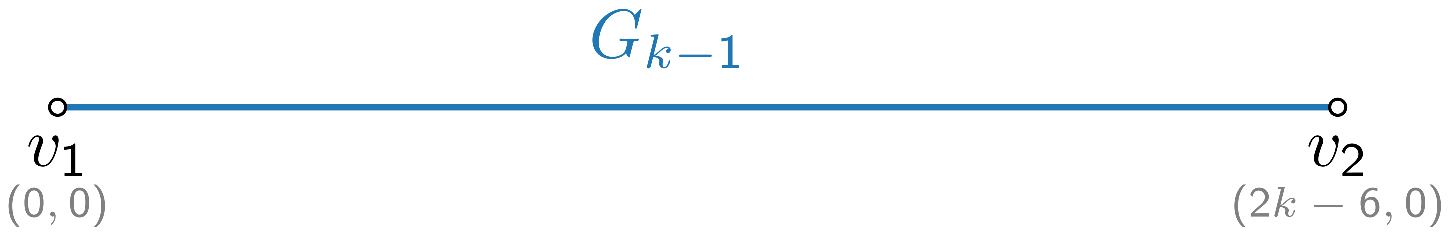

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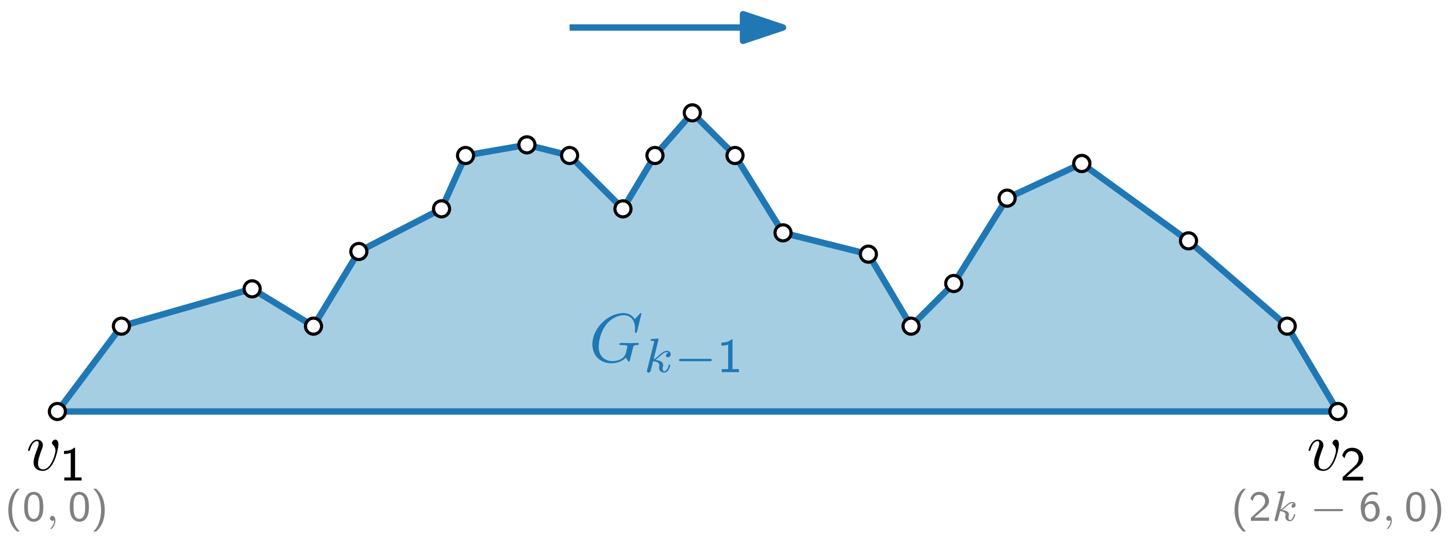

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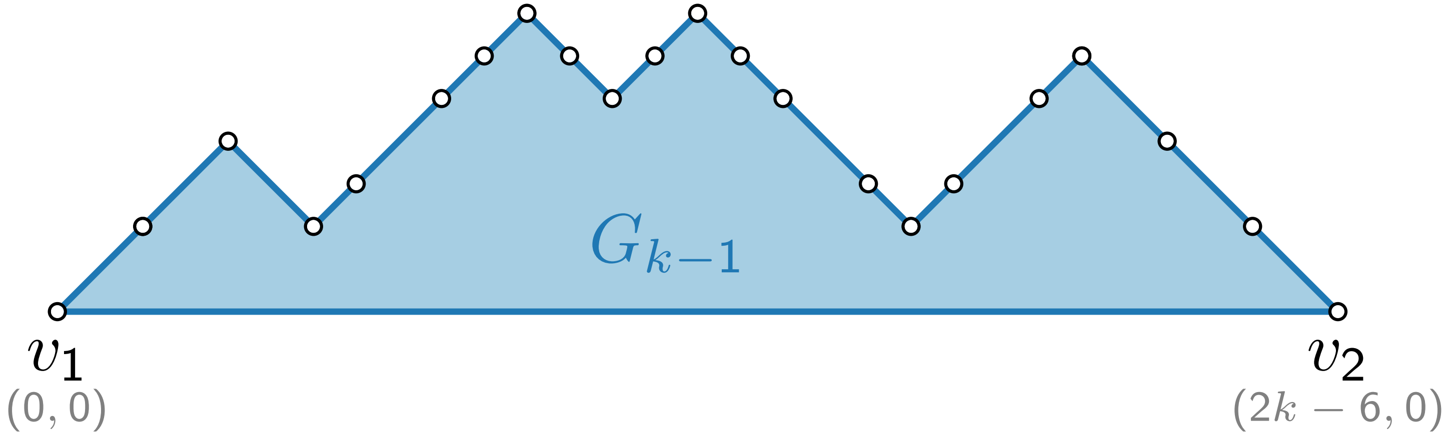

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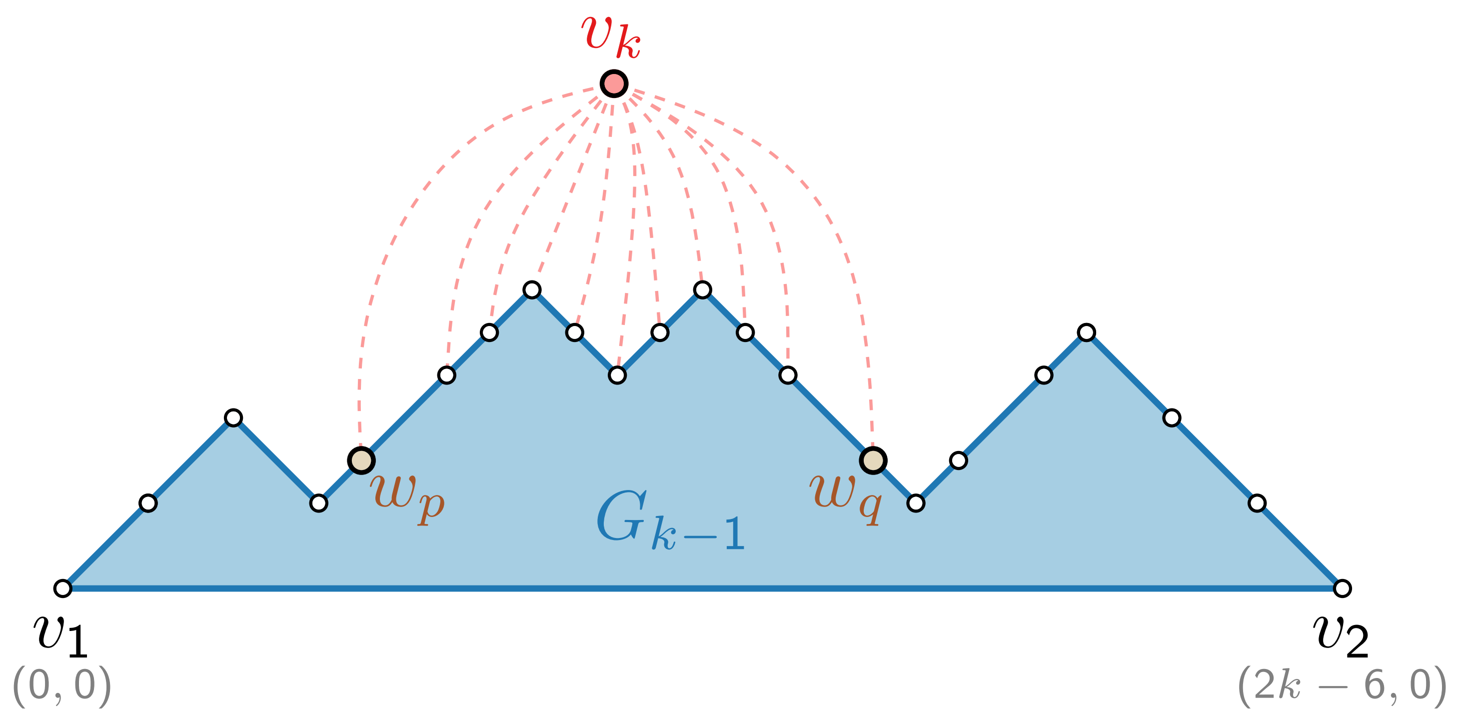

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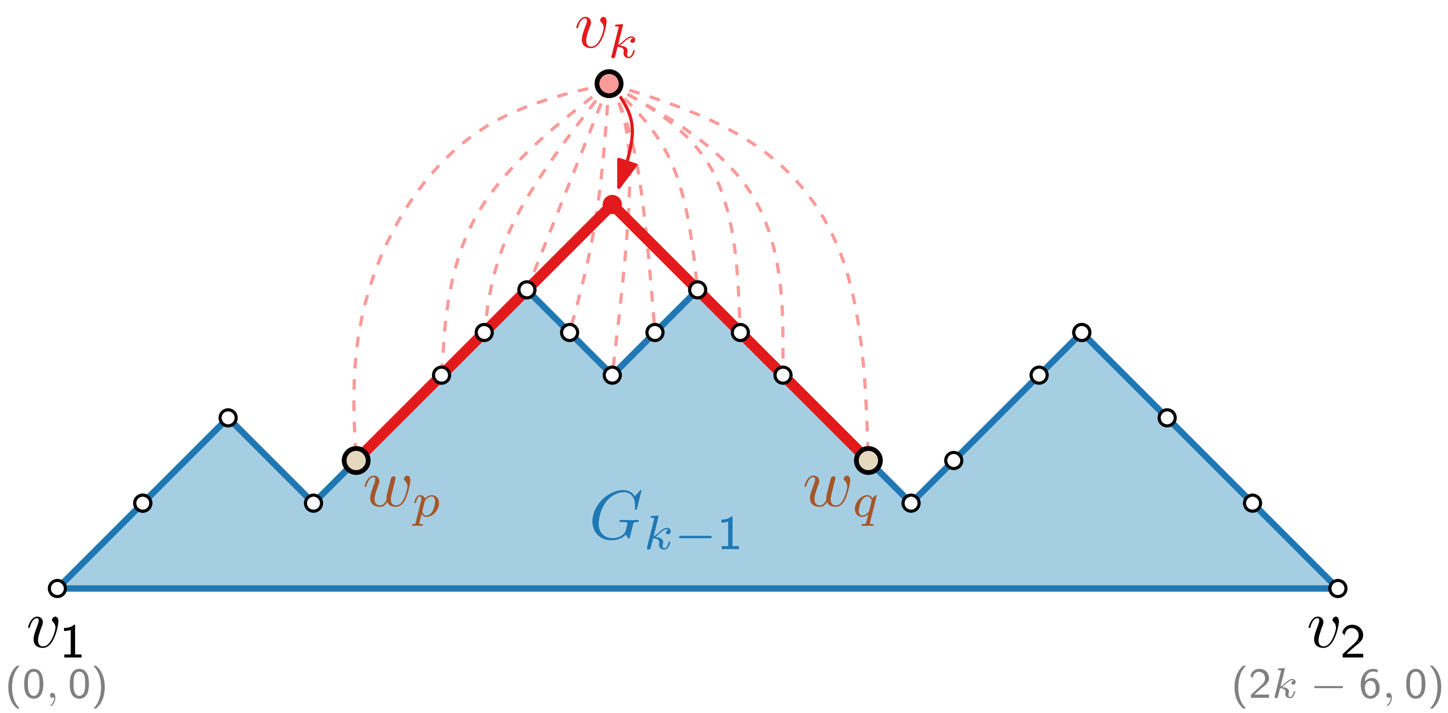

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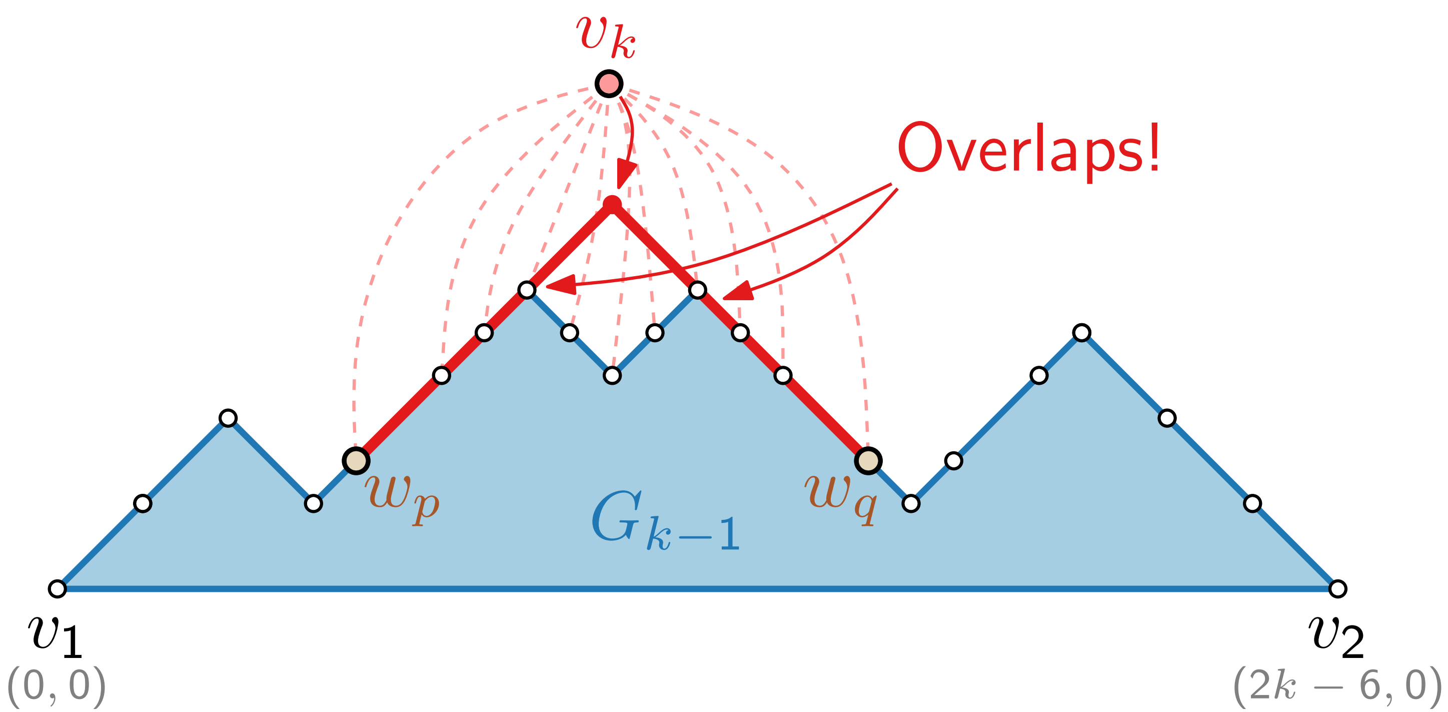

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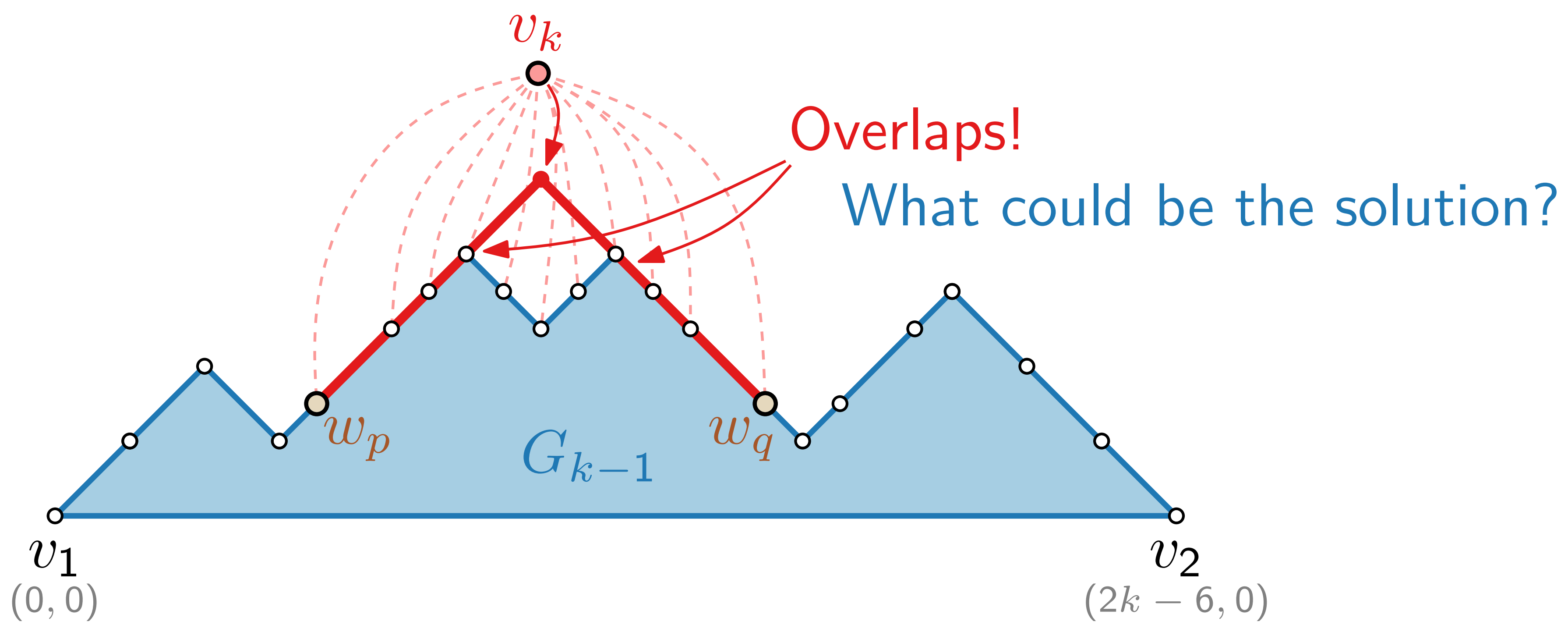

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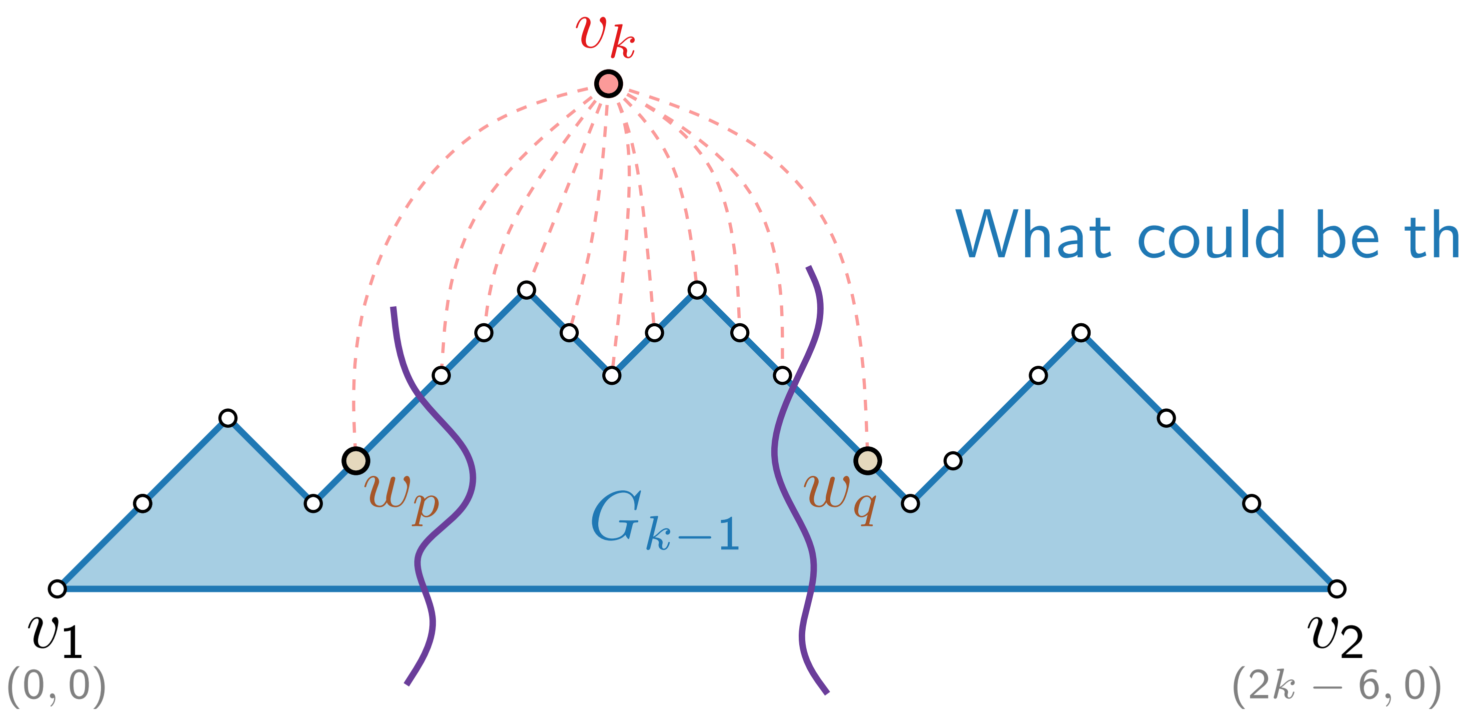

What could be the solution?

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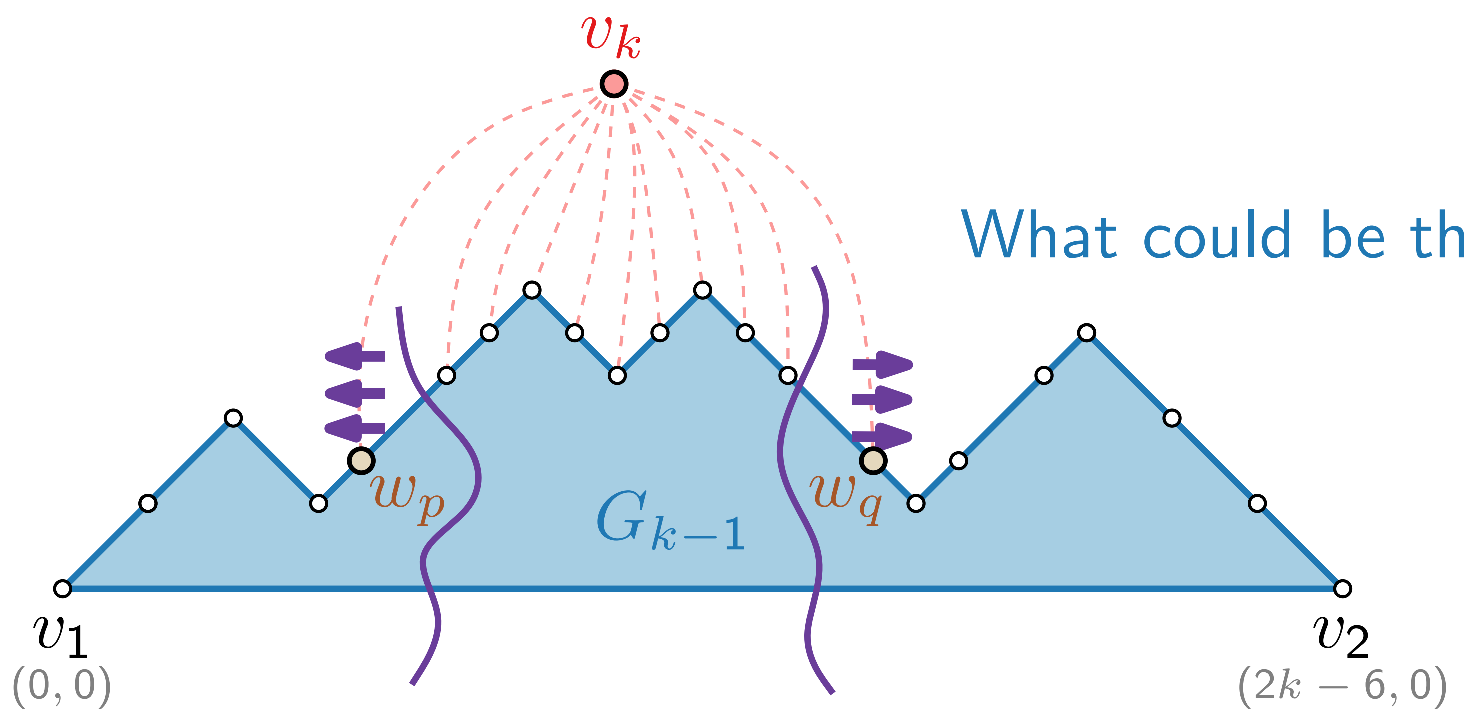

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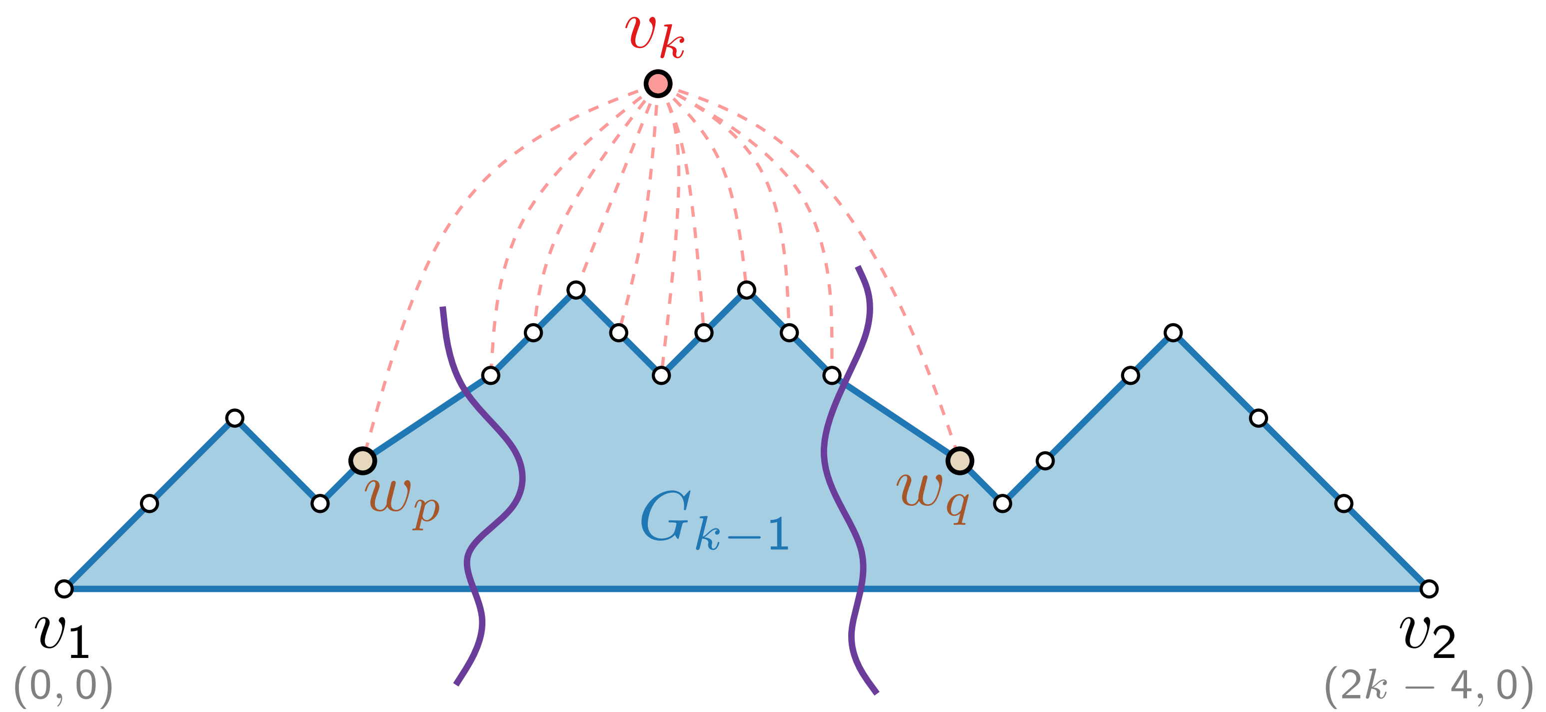

Shift Method – Idea

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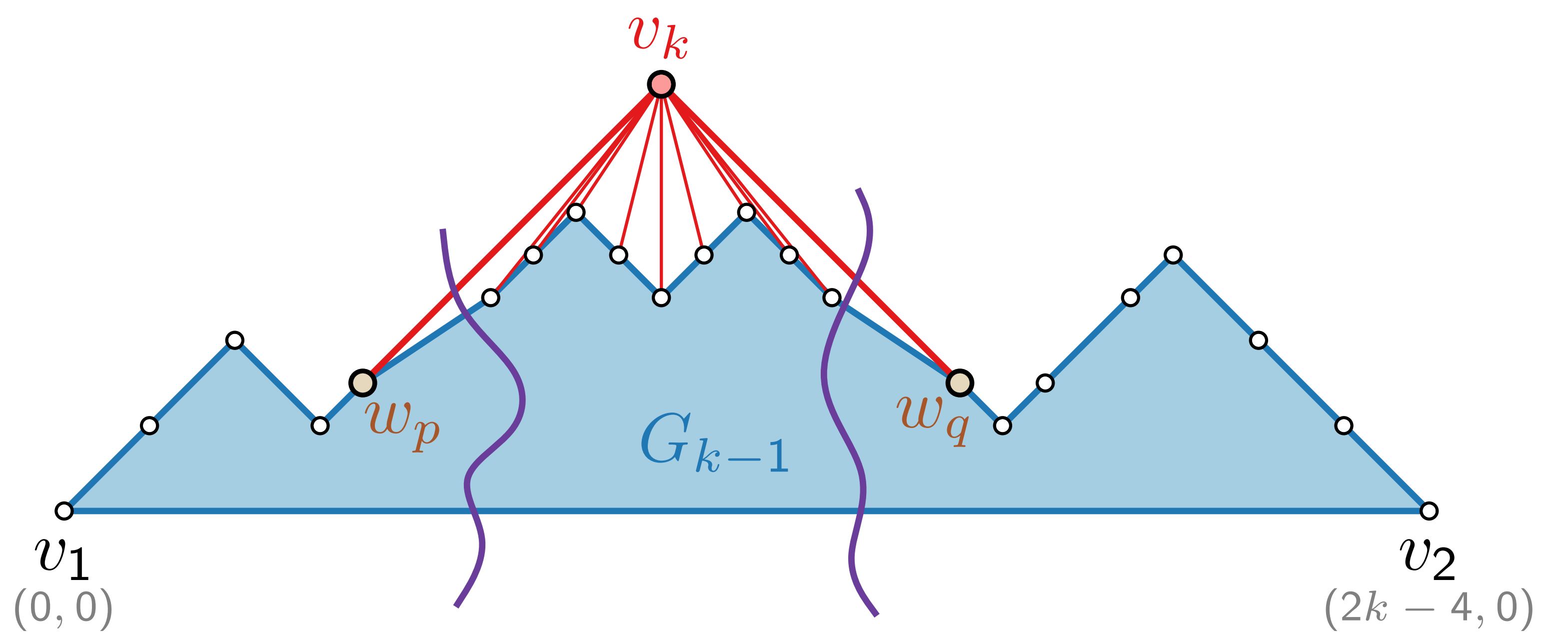

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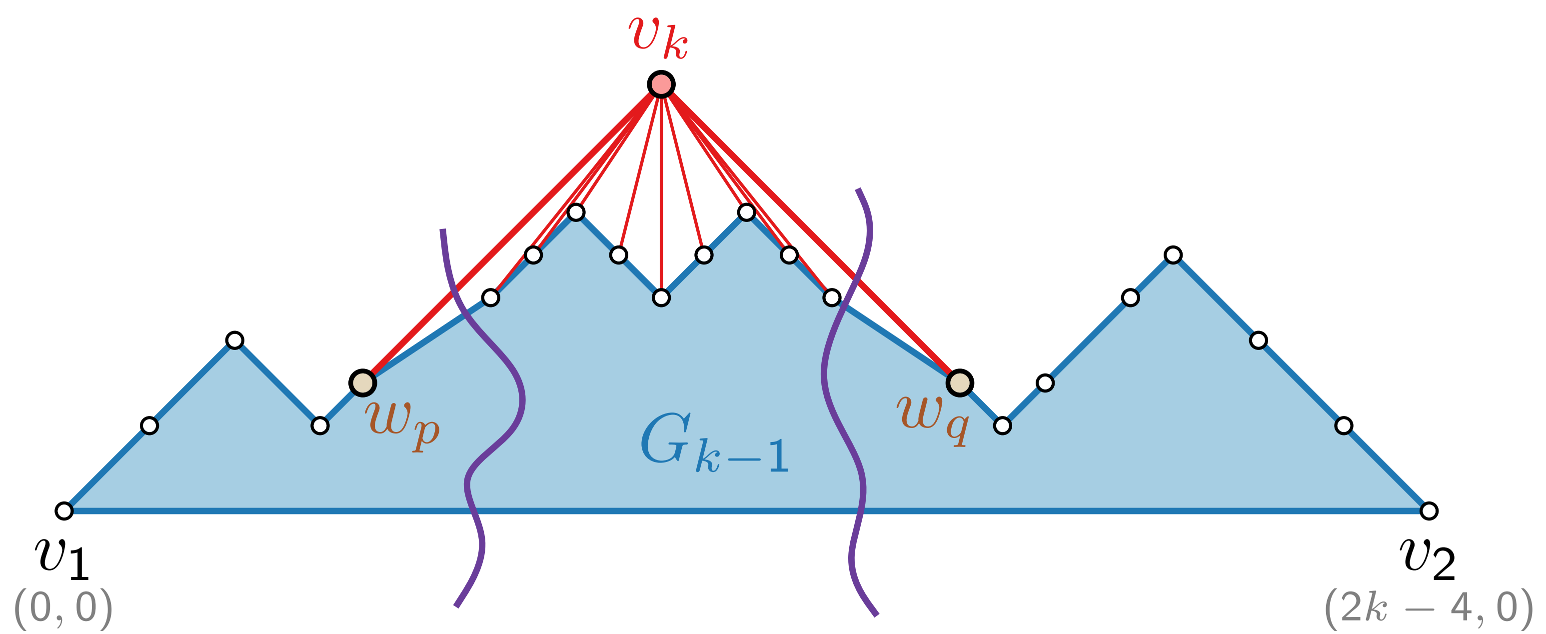

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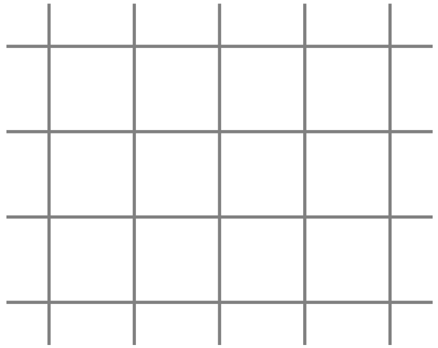

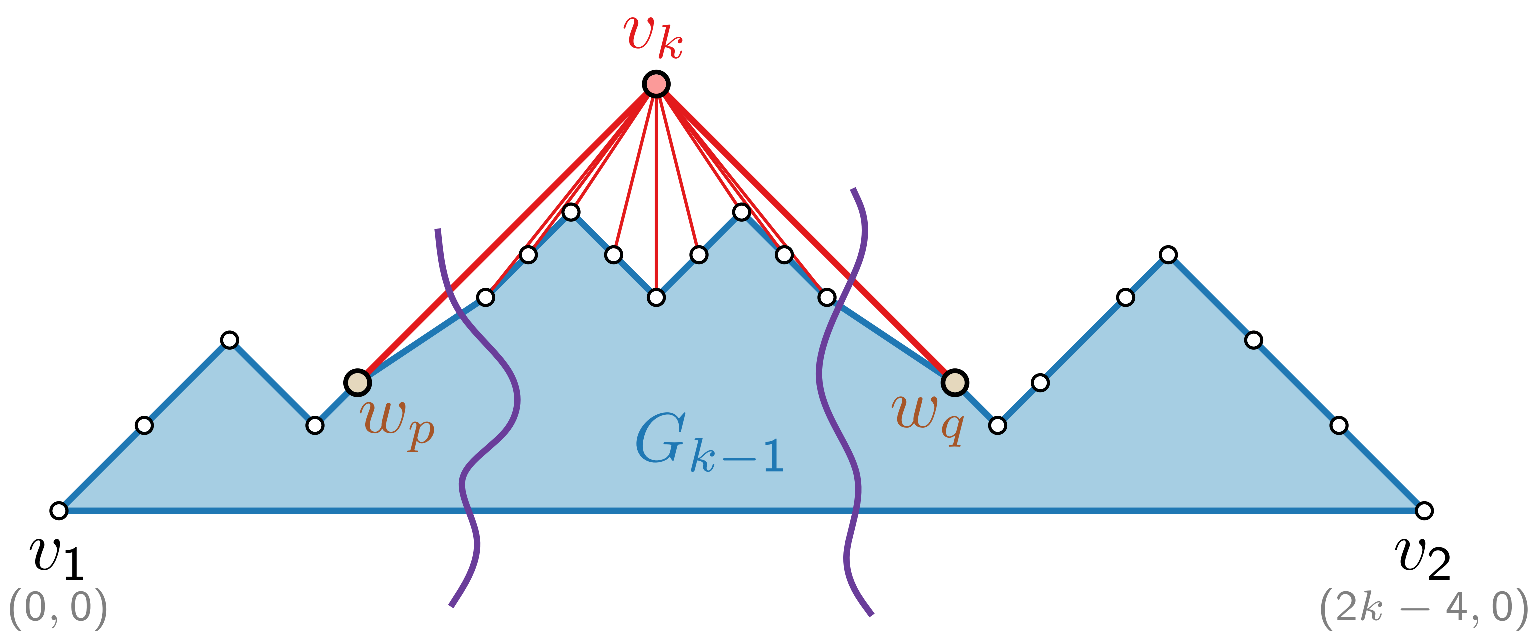

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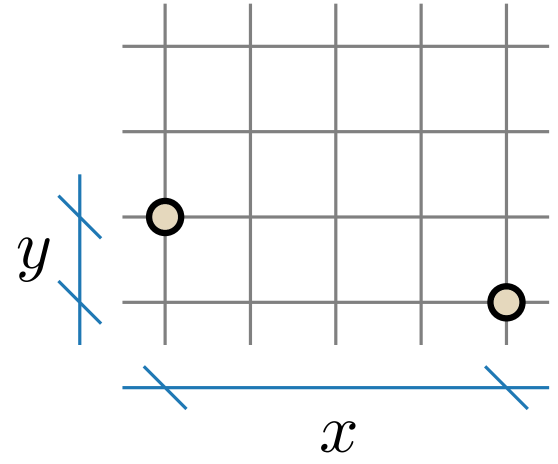

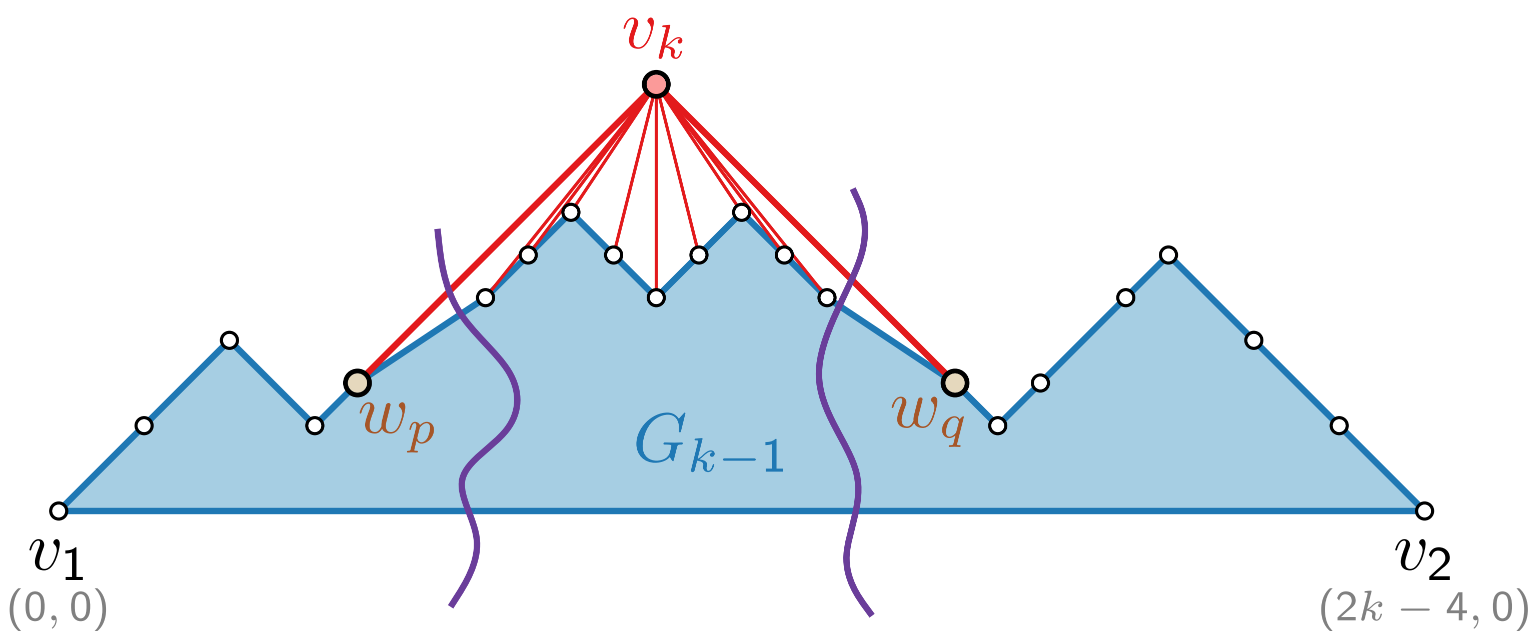

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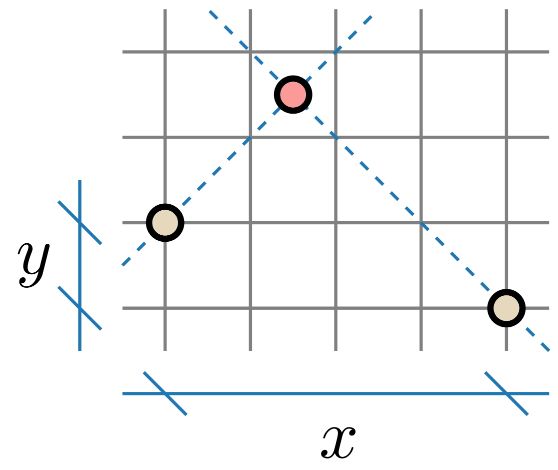

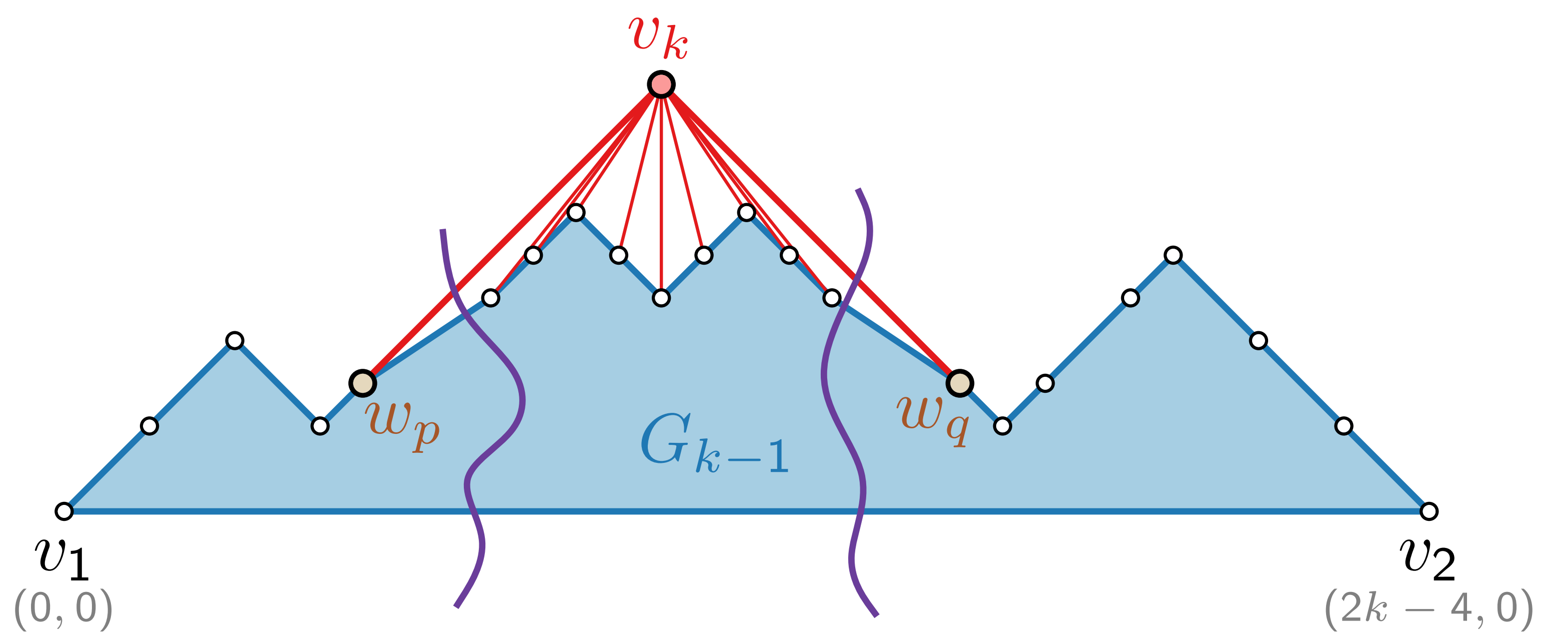

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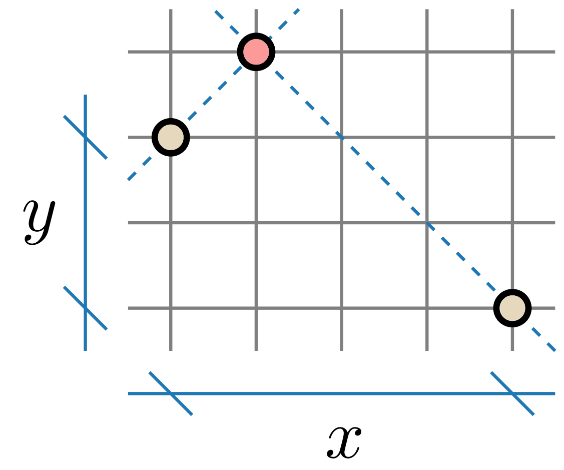

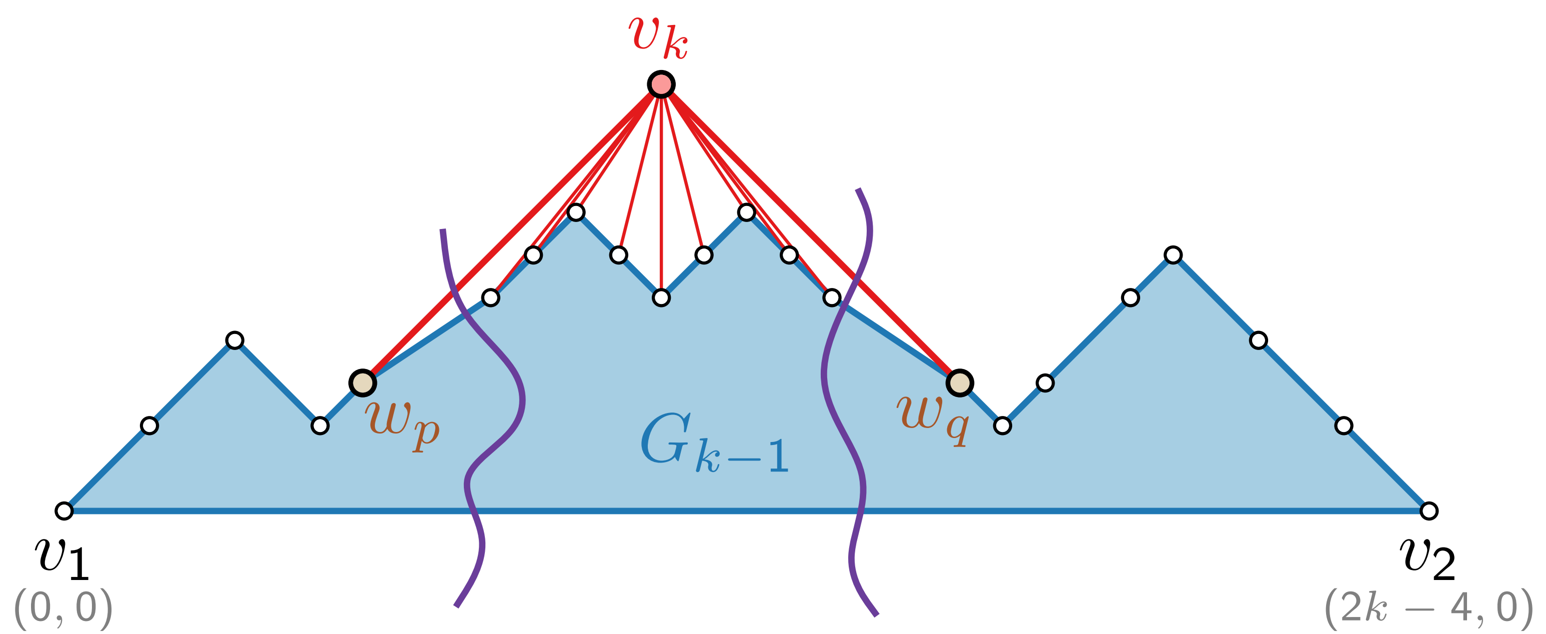

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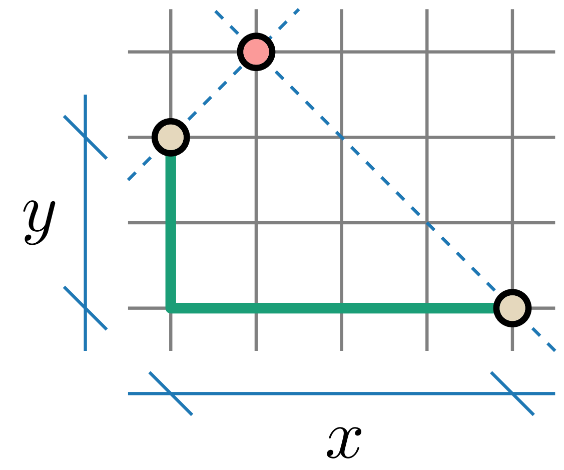

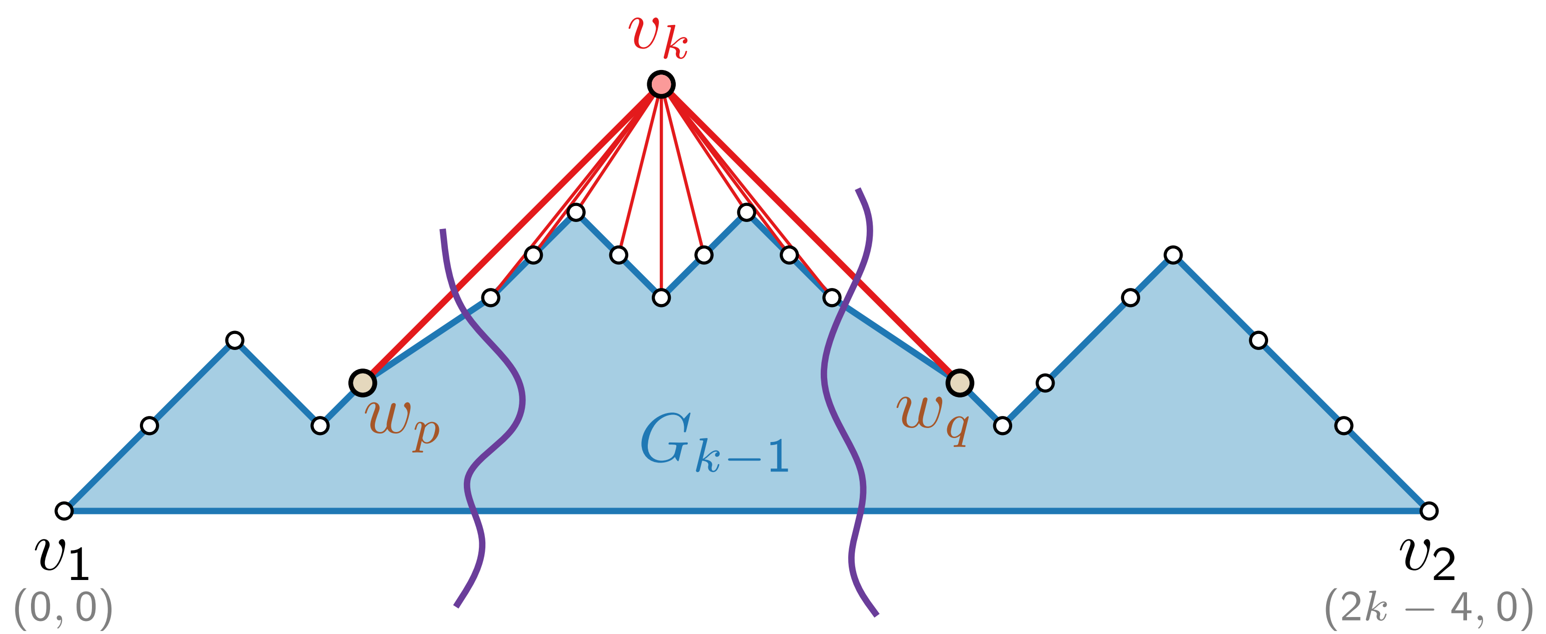

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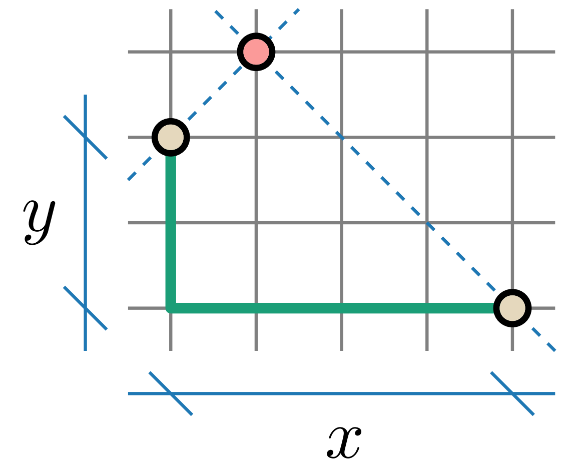
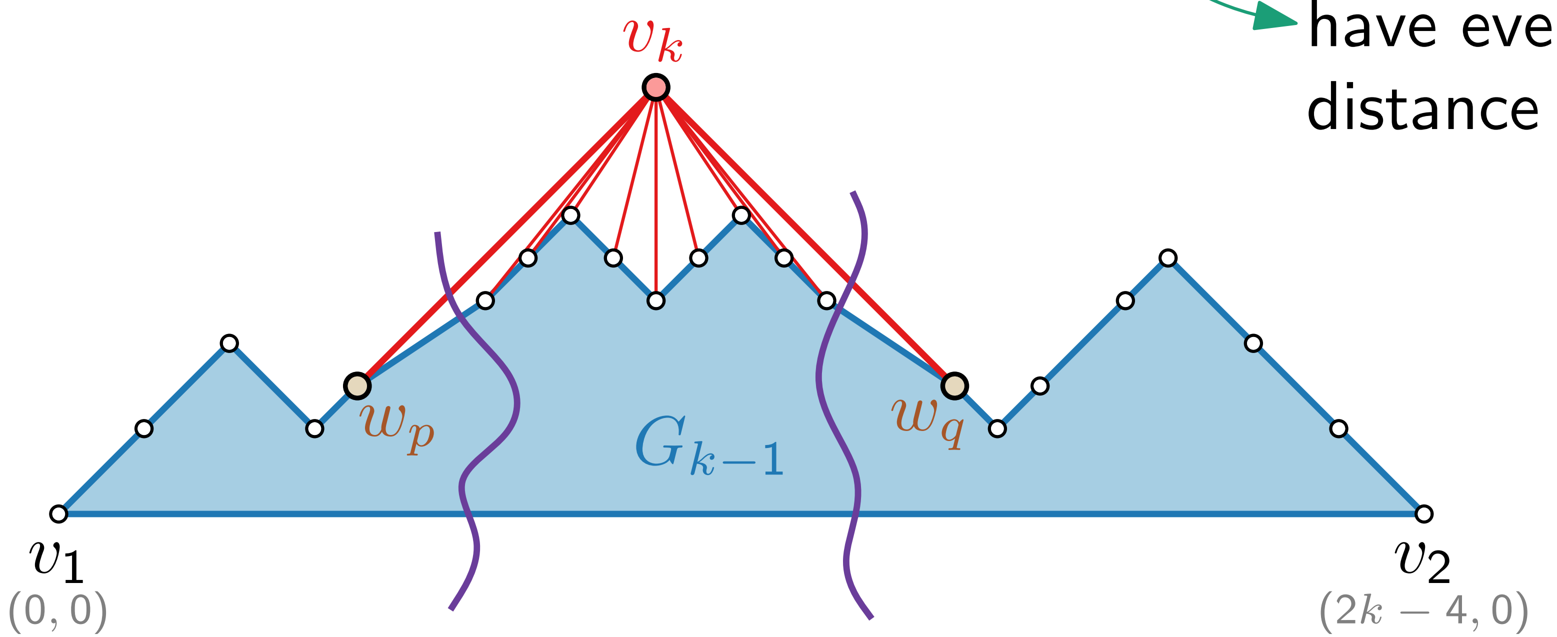


yes, beause w_p and w_q have even Manhattan distance



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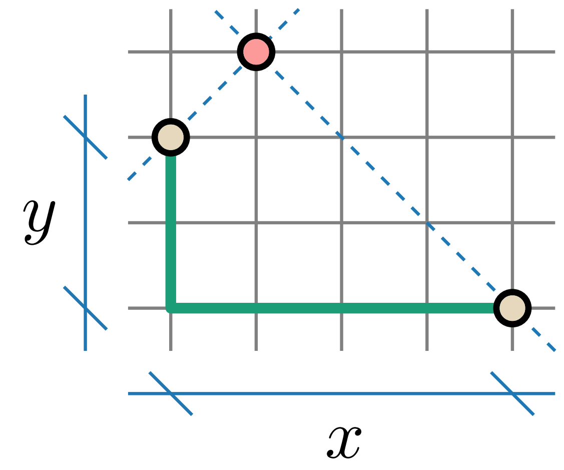

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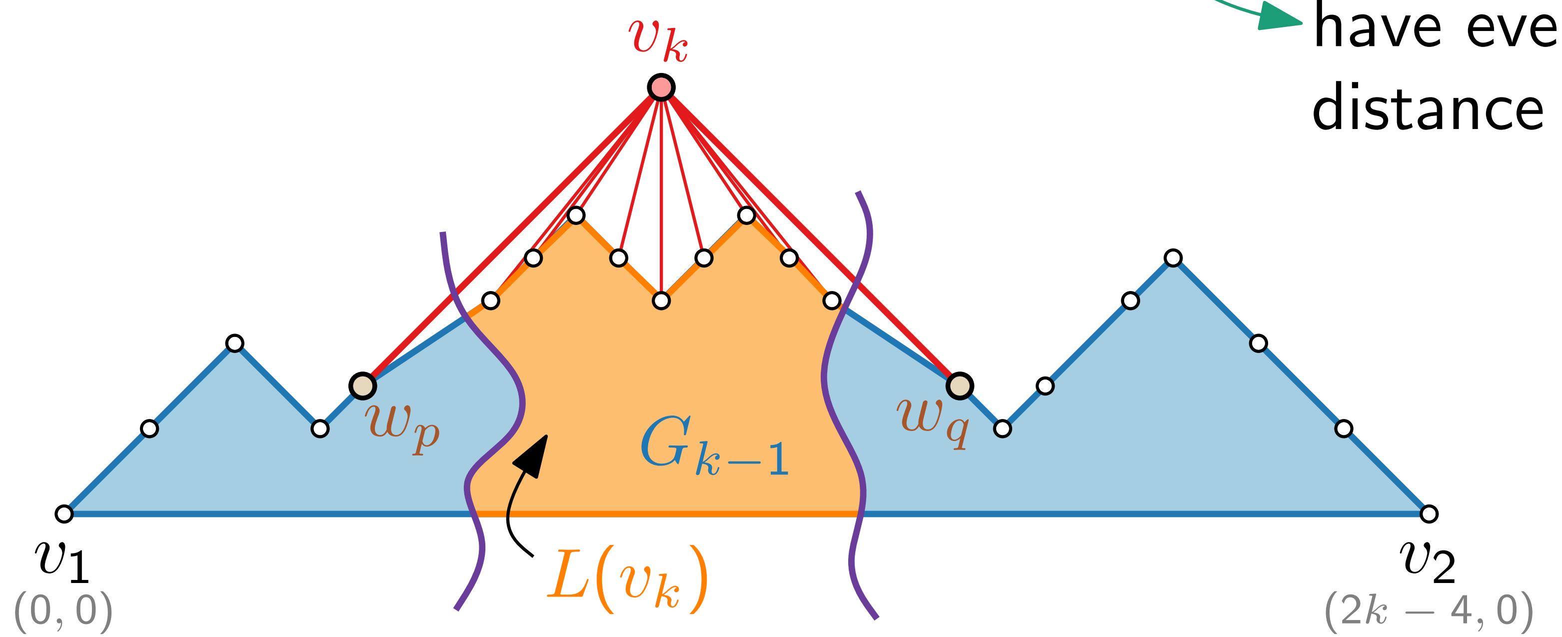

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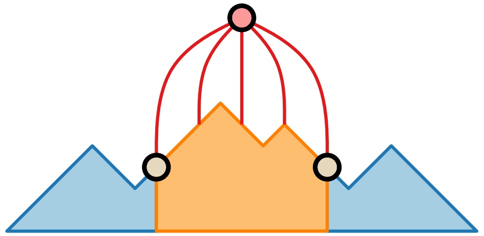

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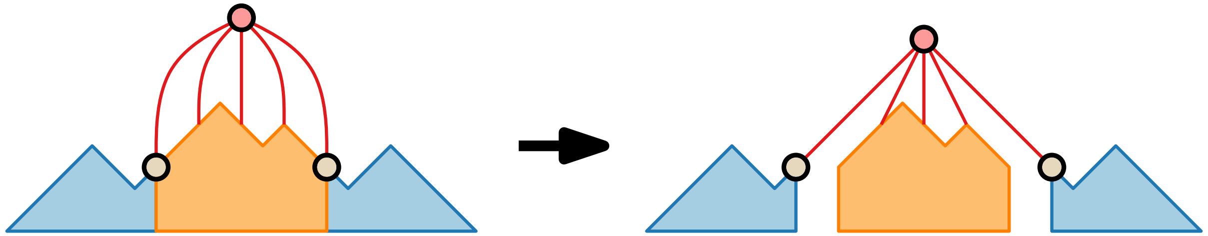

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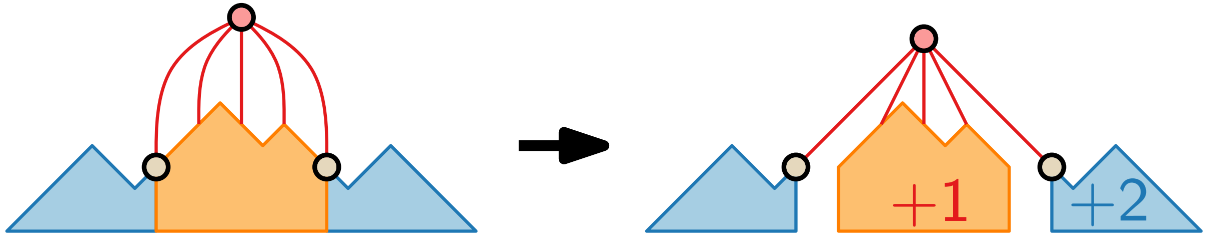

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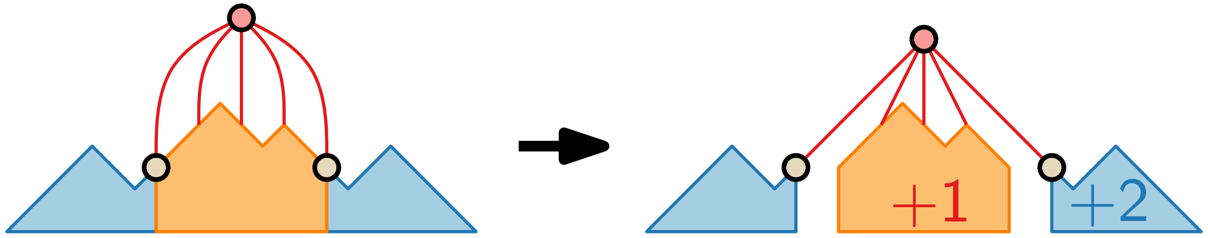

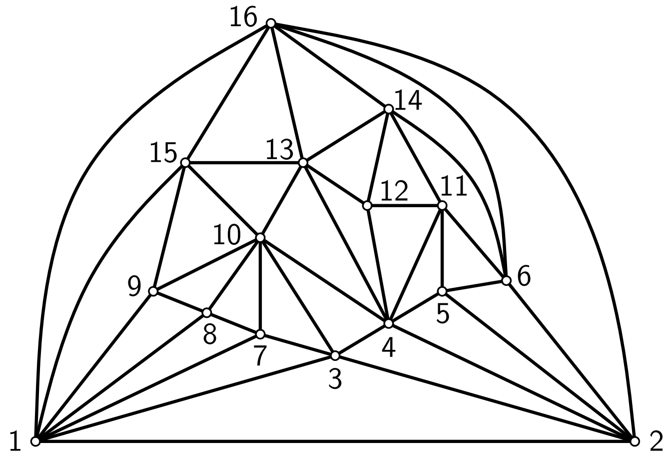

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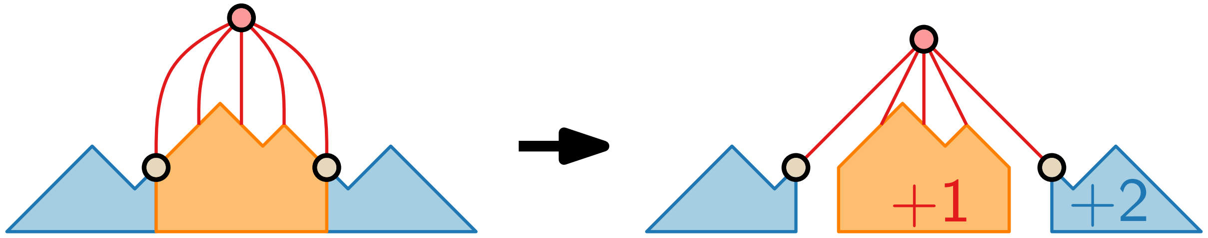

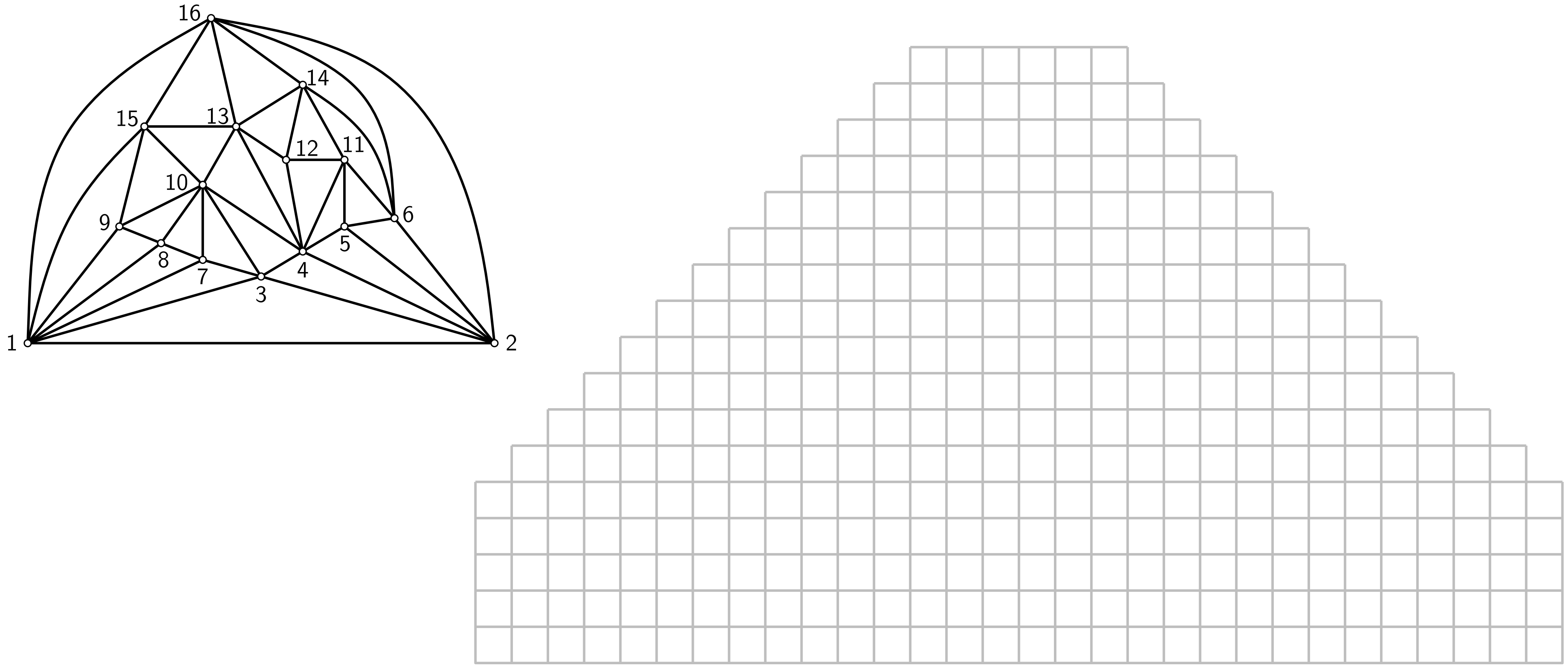

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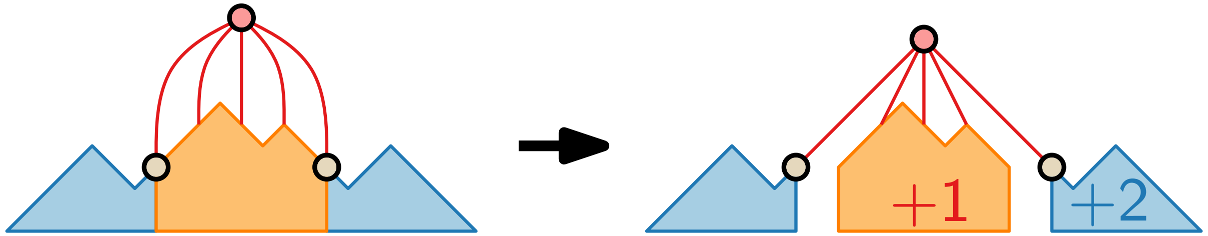

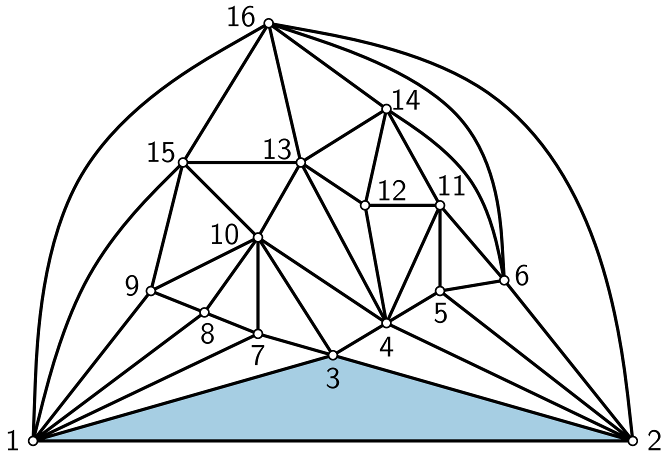

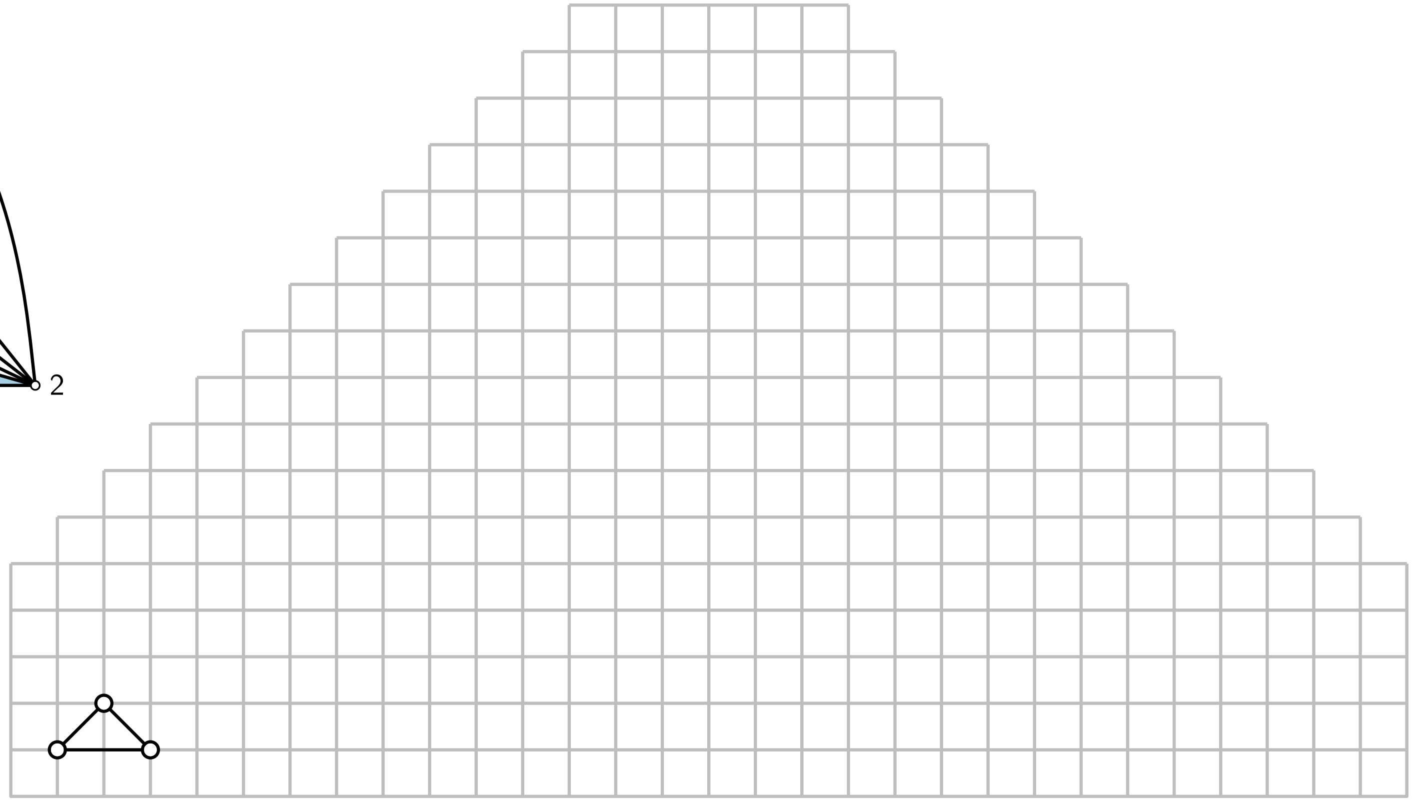

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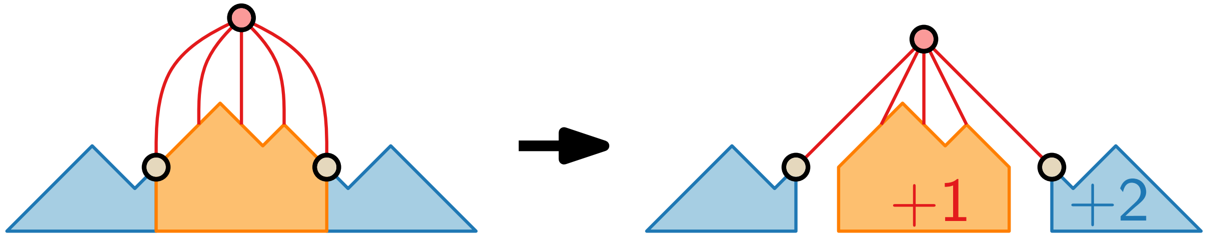

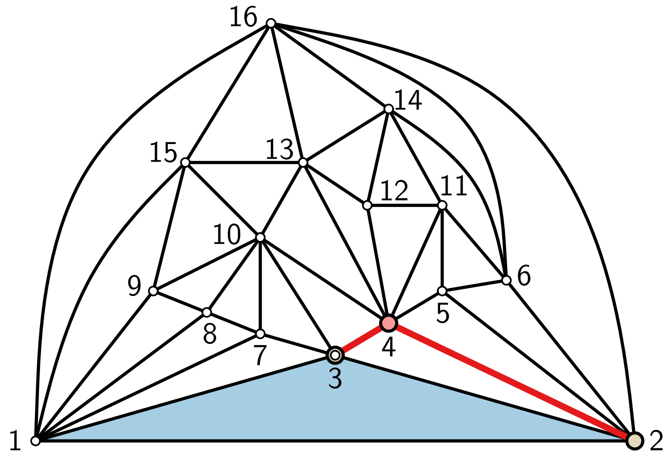

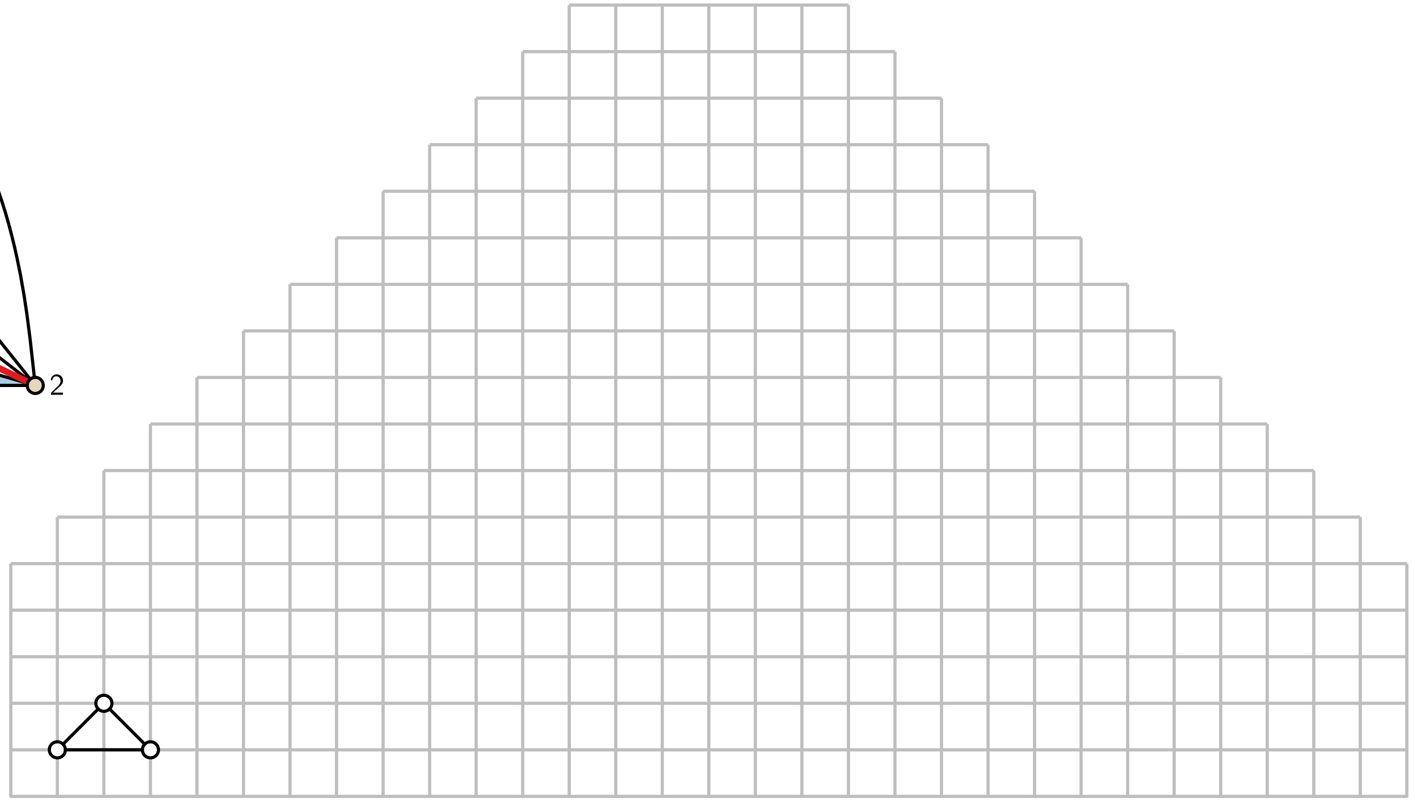

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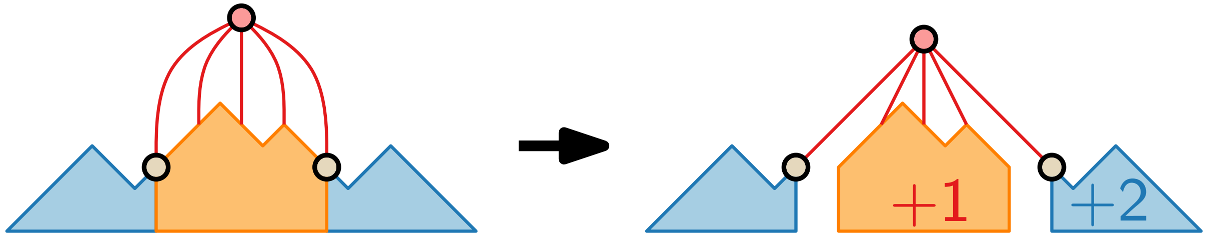

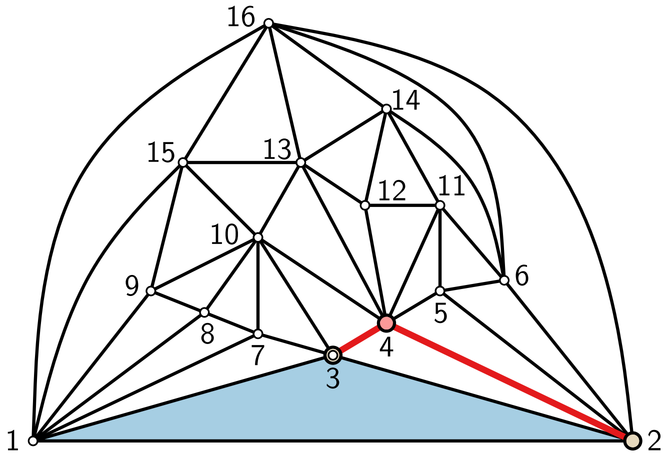

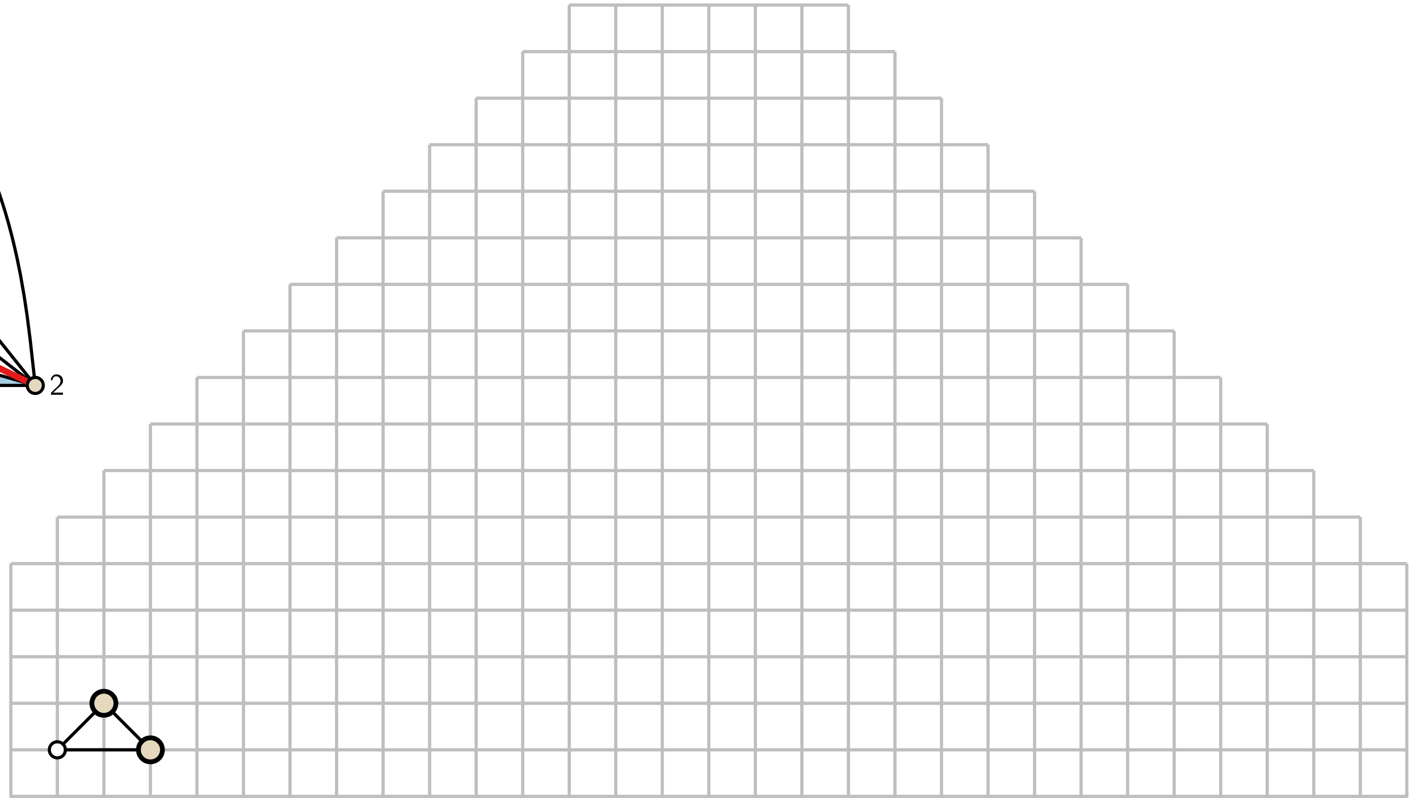

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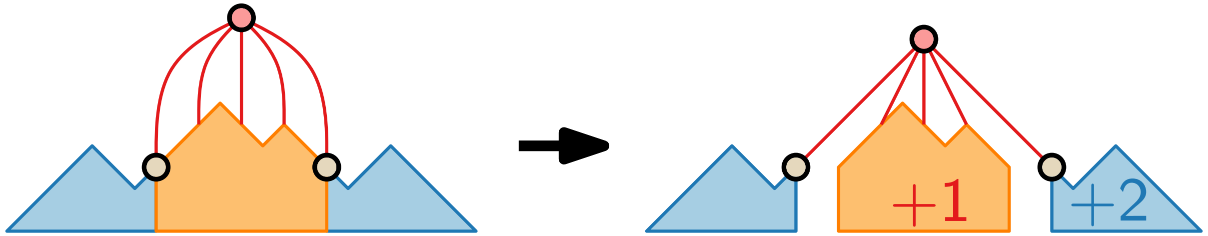

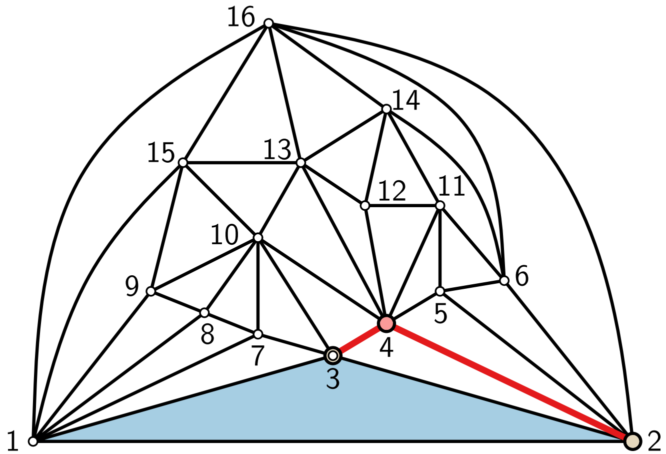

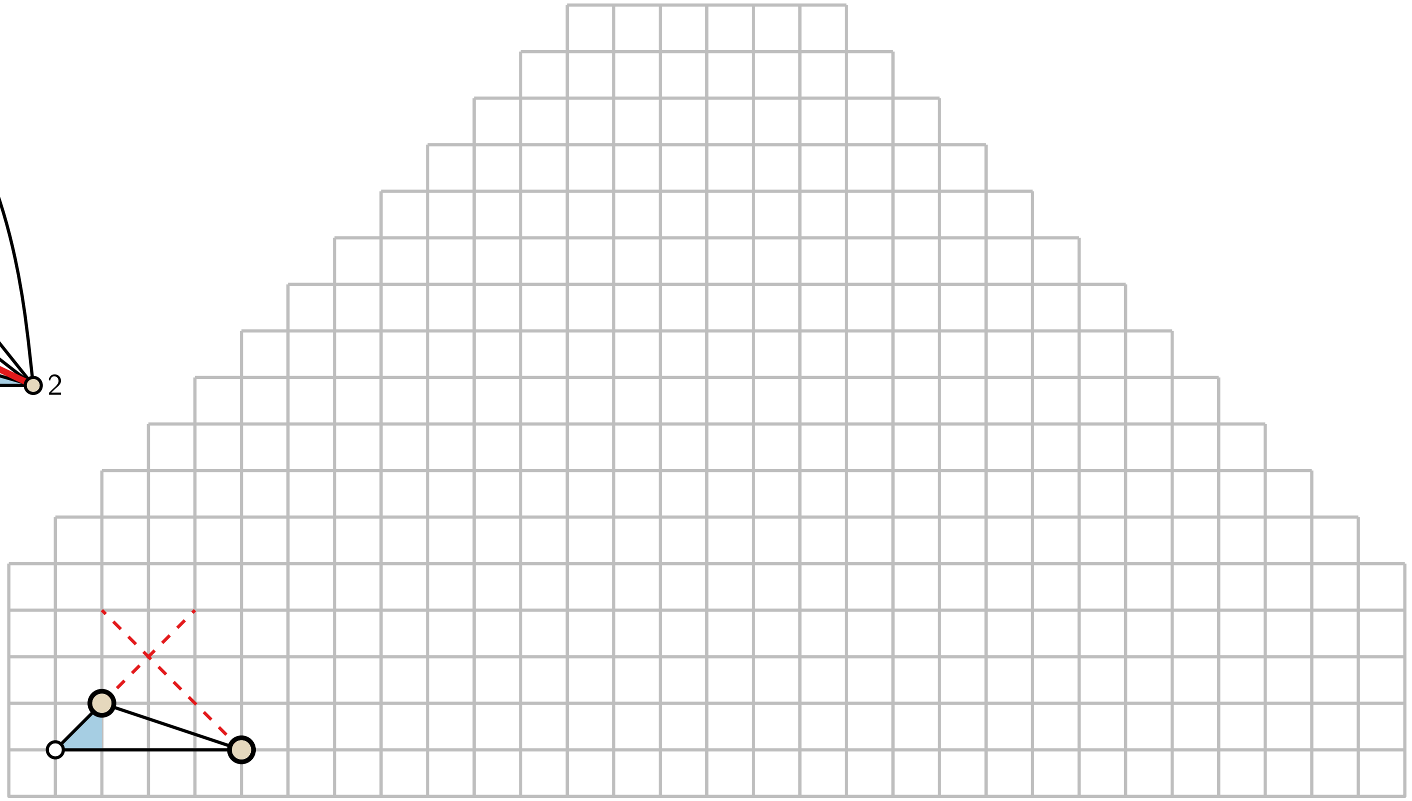

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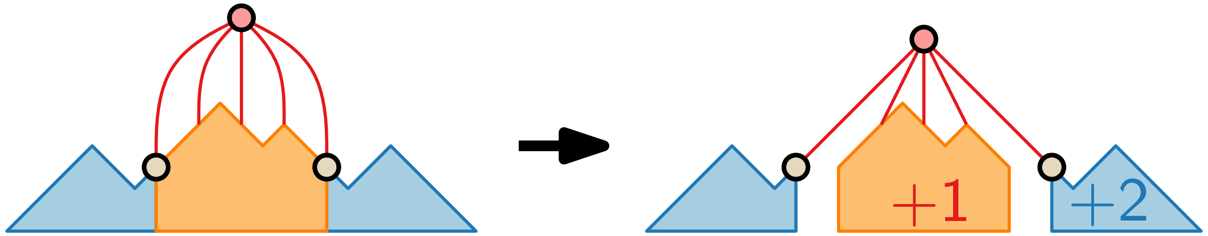

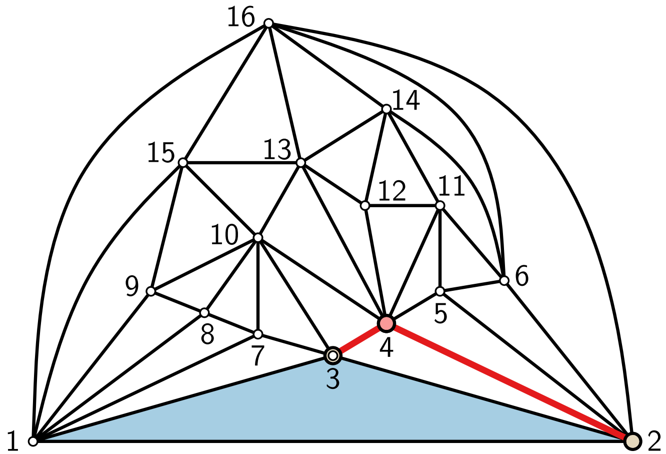

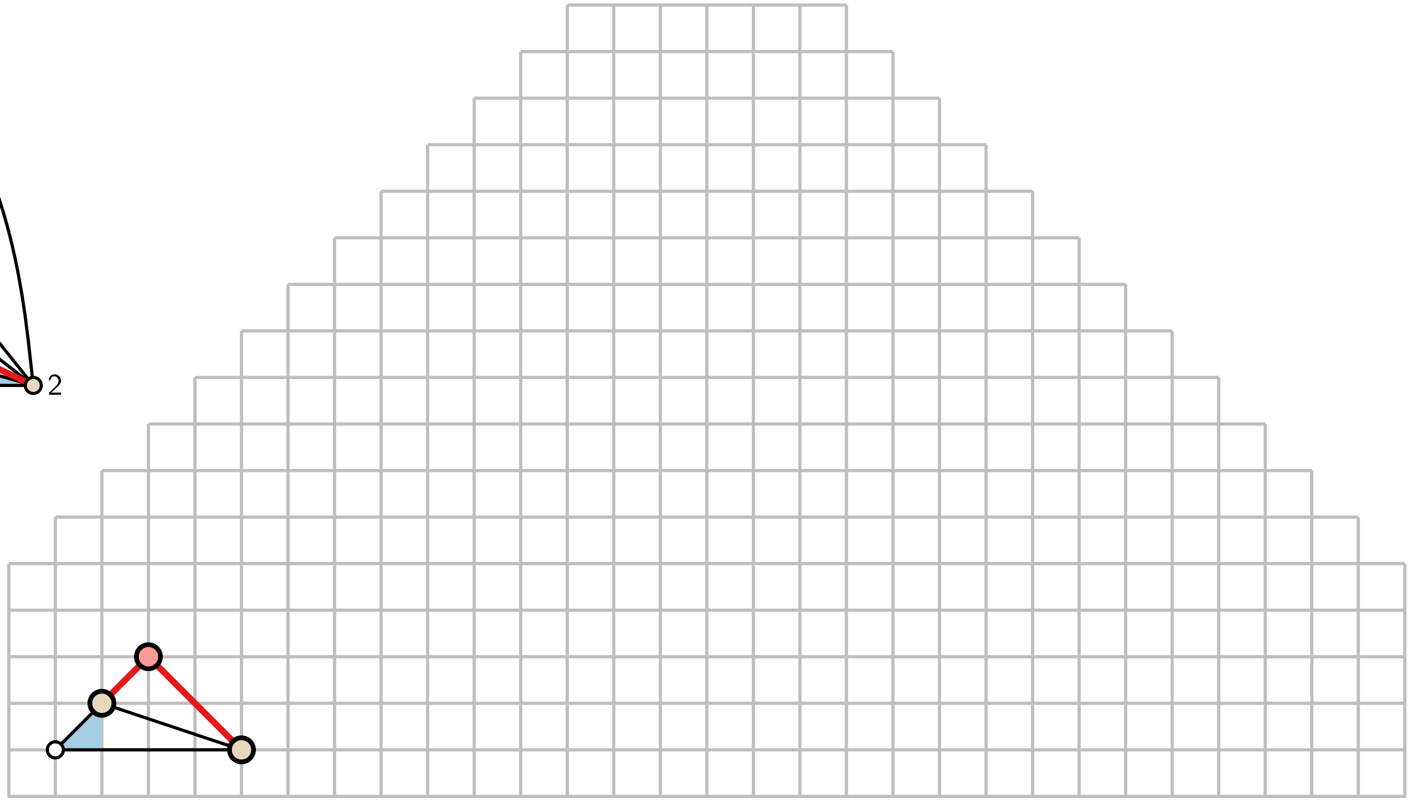

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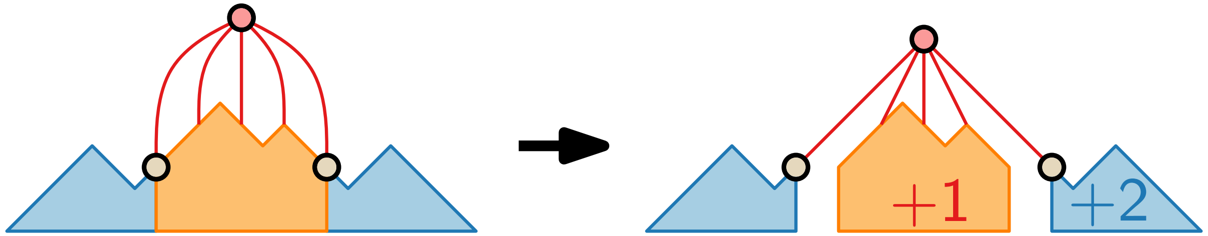

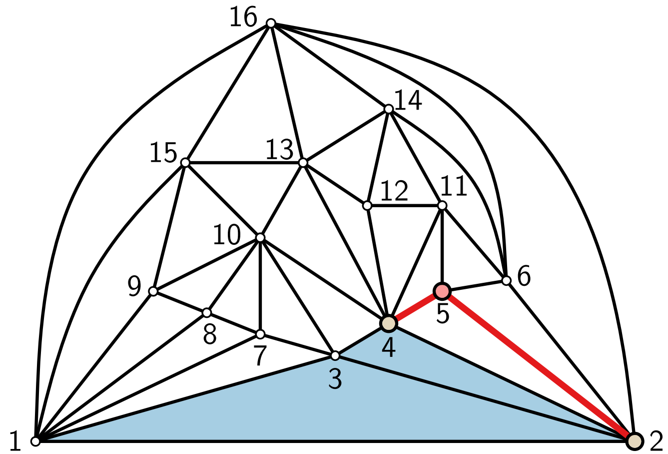

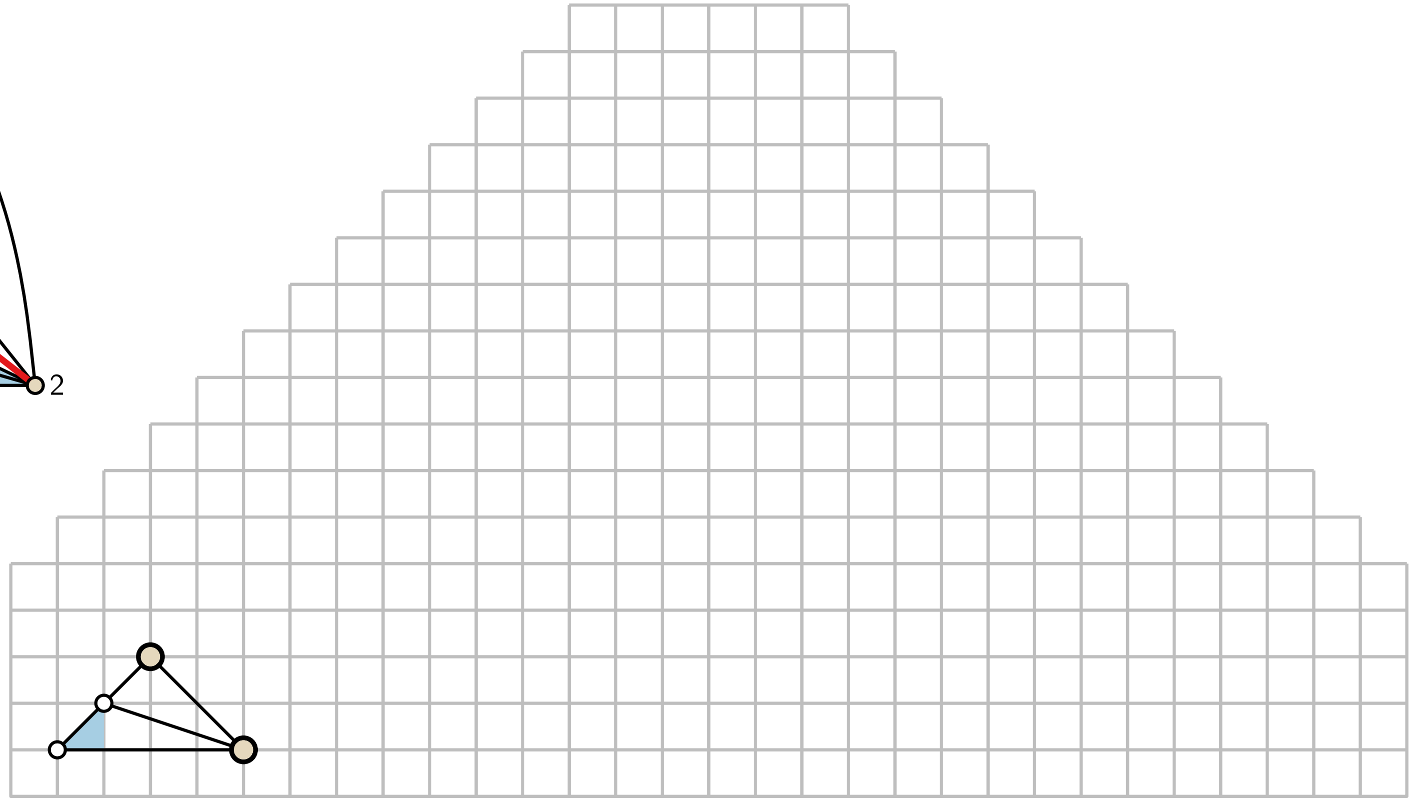

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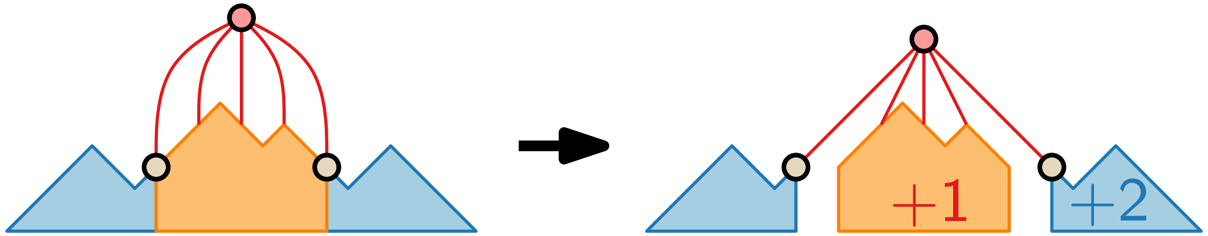

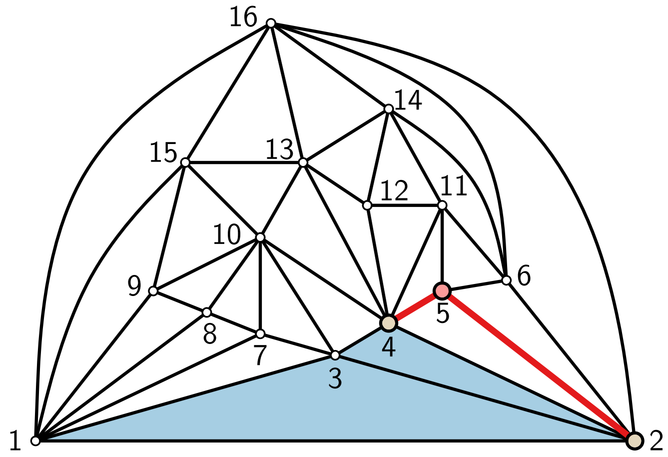

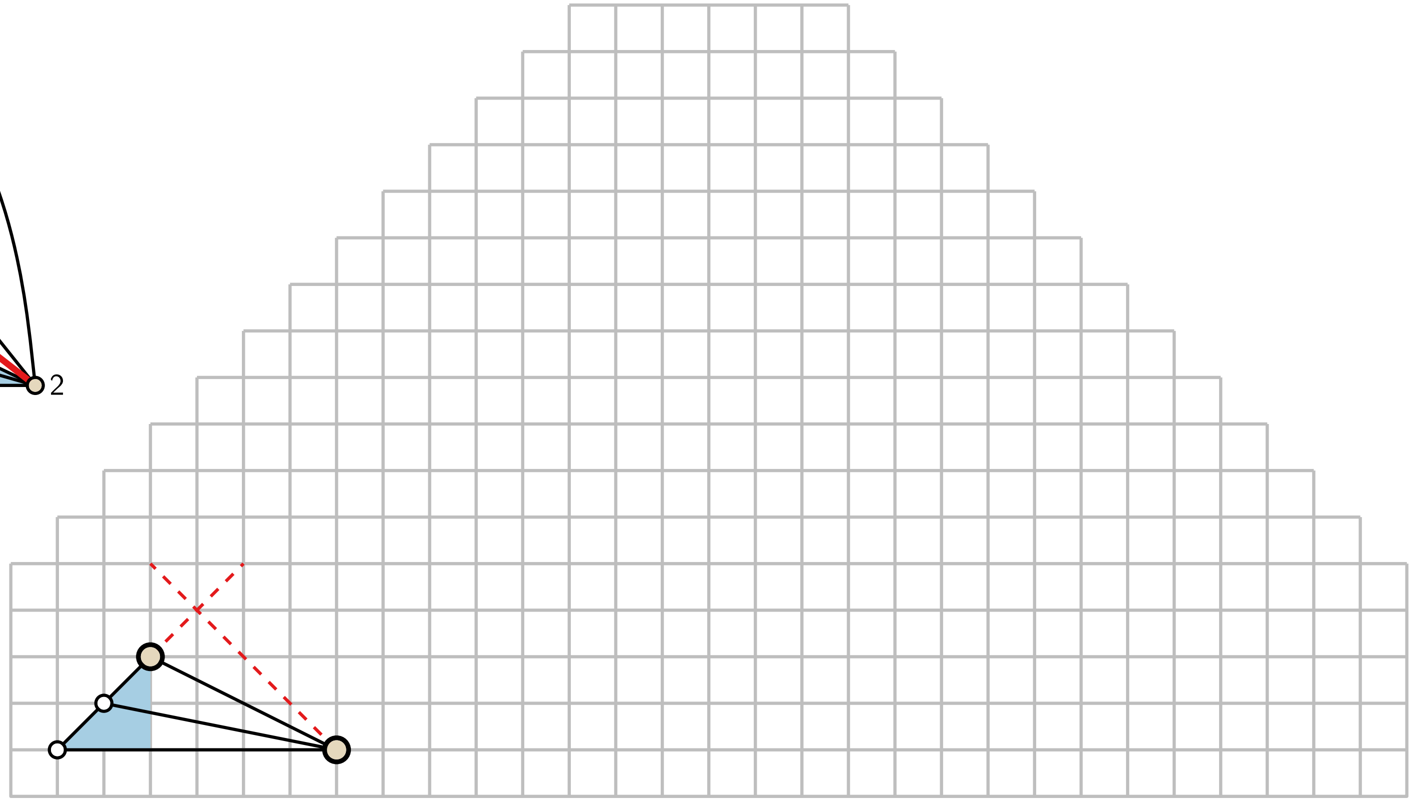

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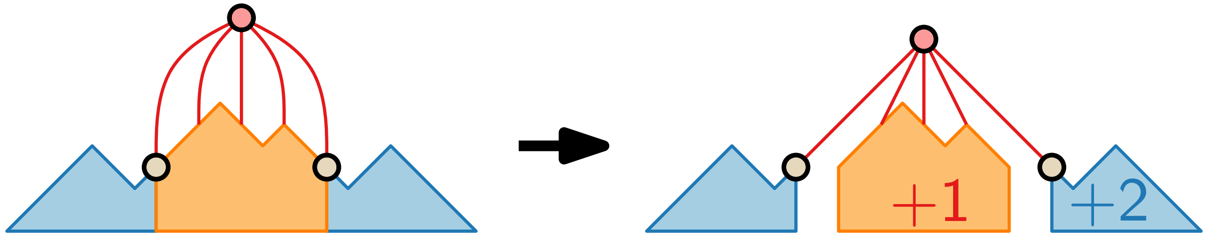

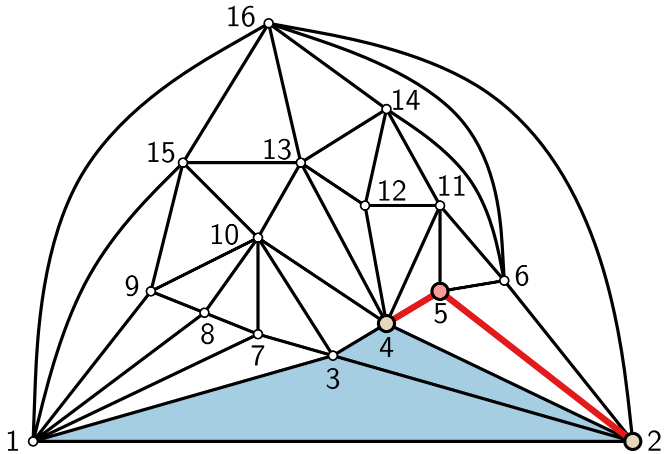

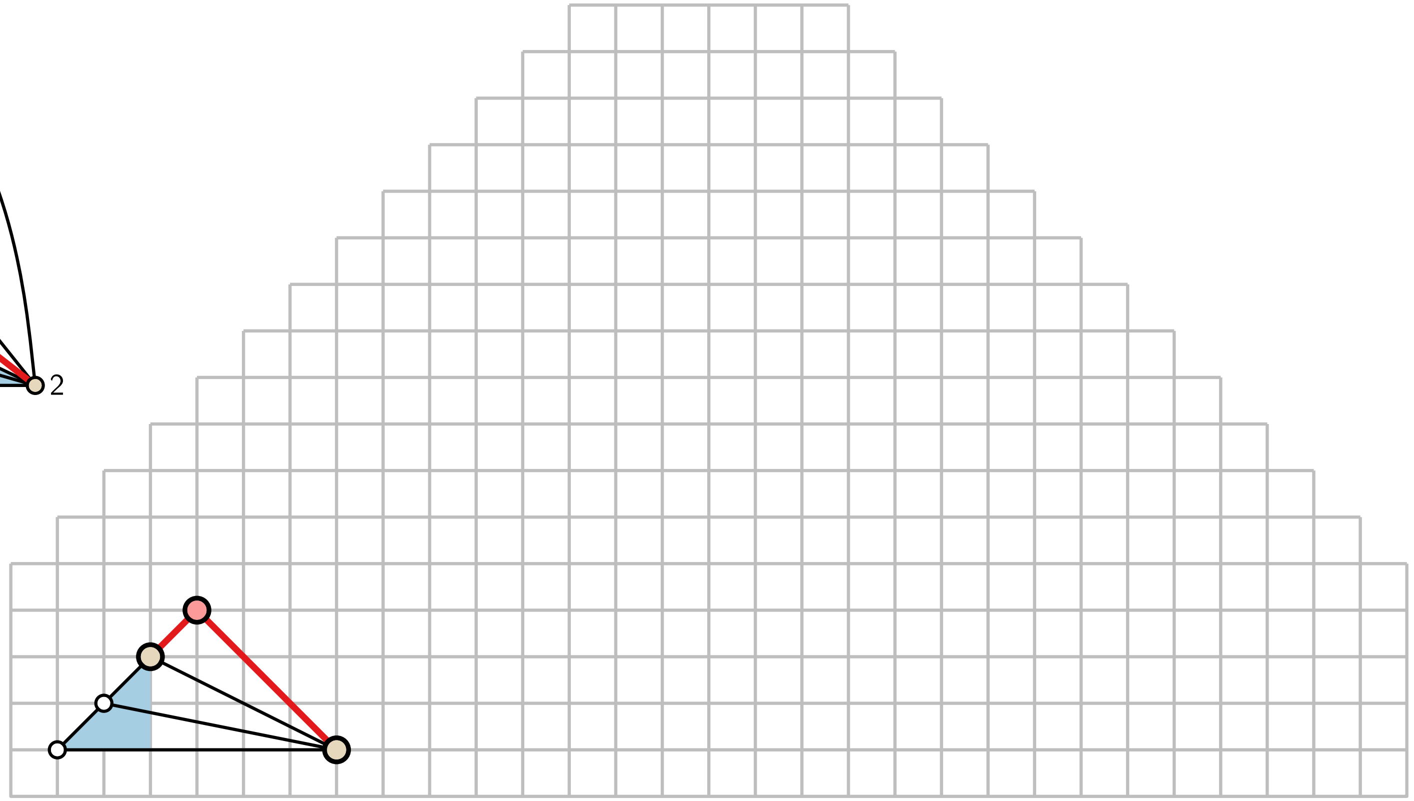

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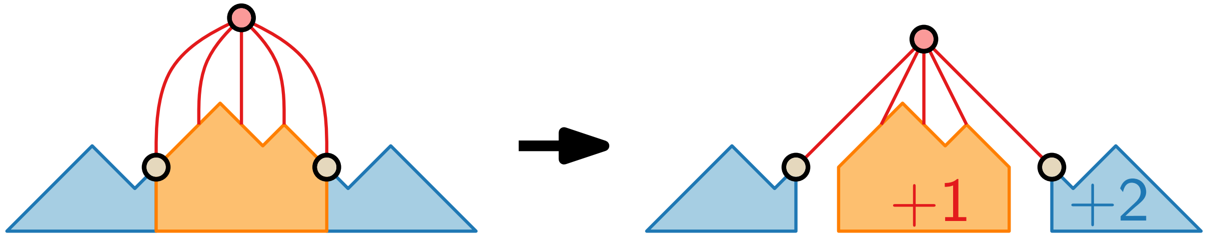

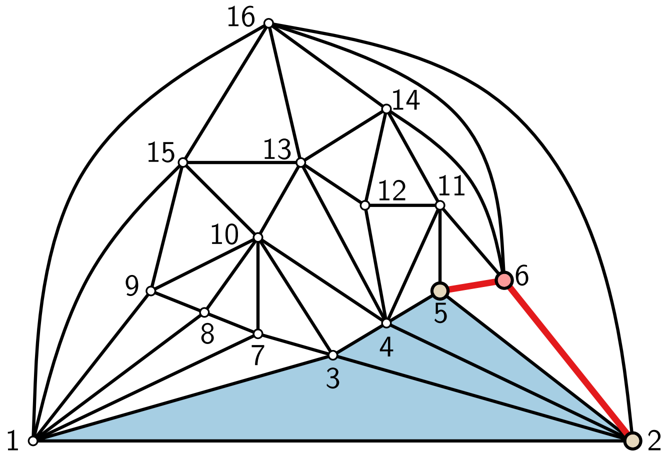

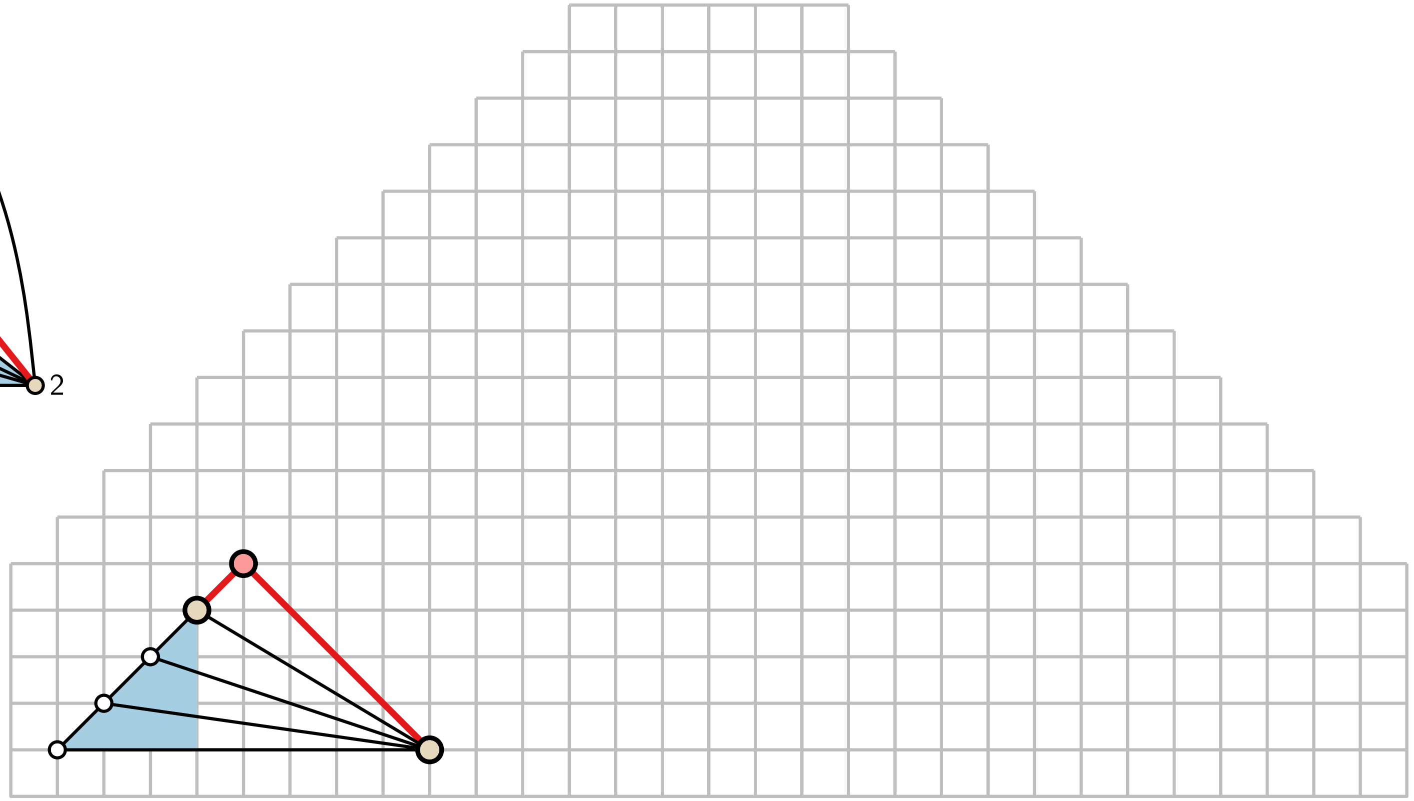

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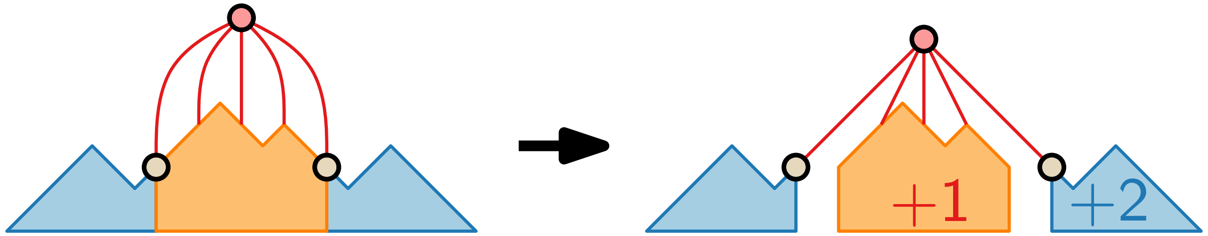

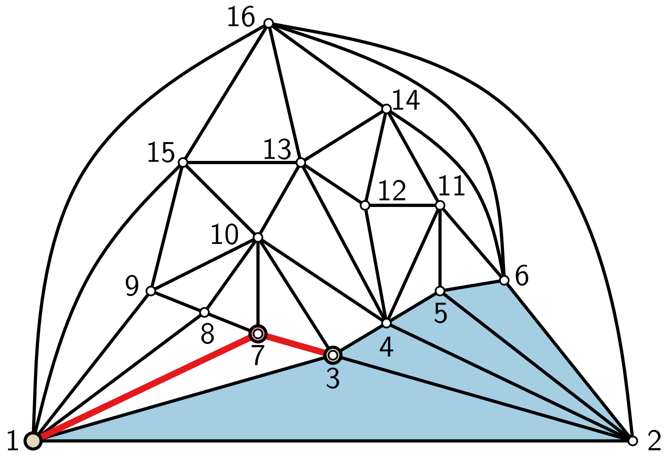

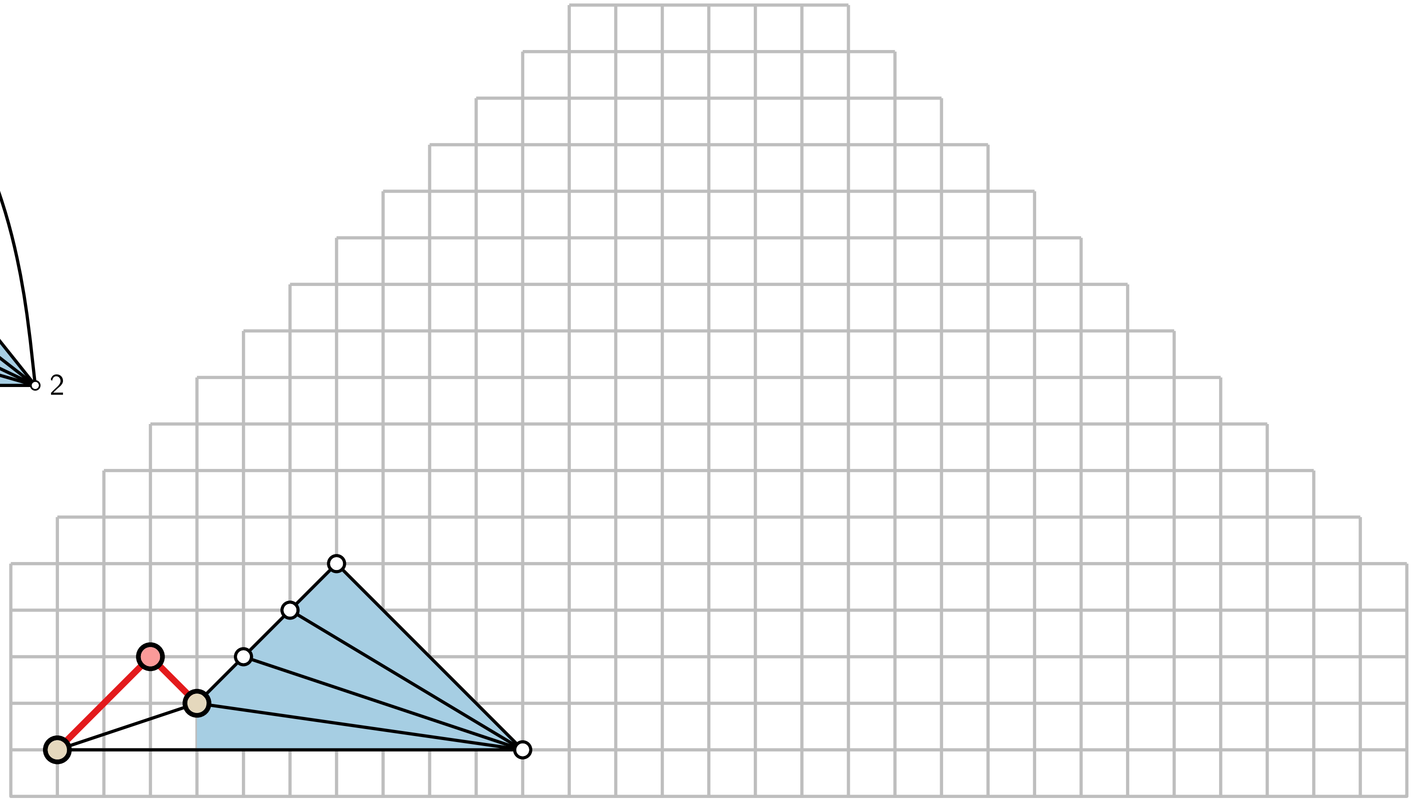

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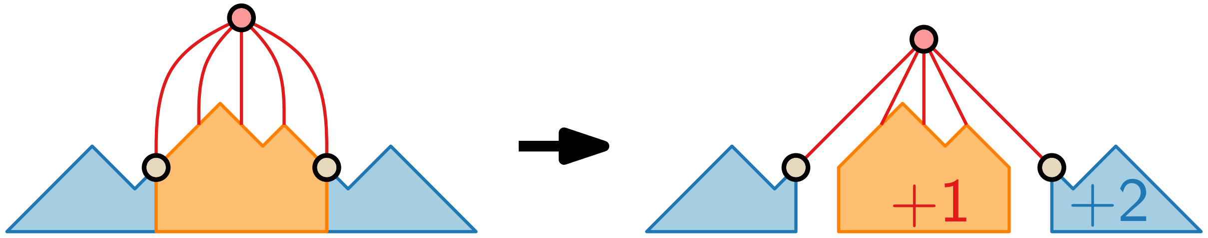

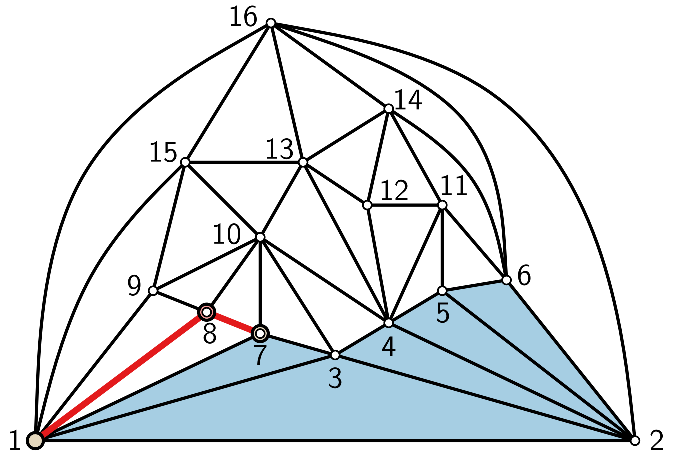

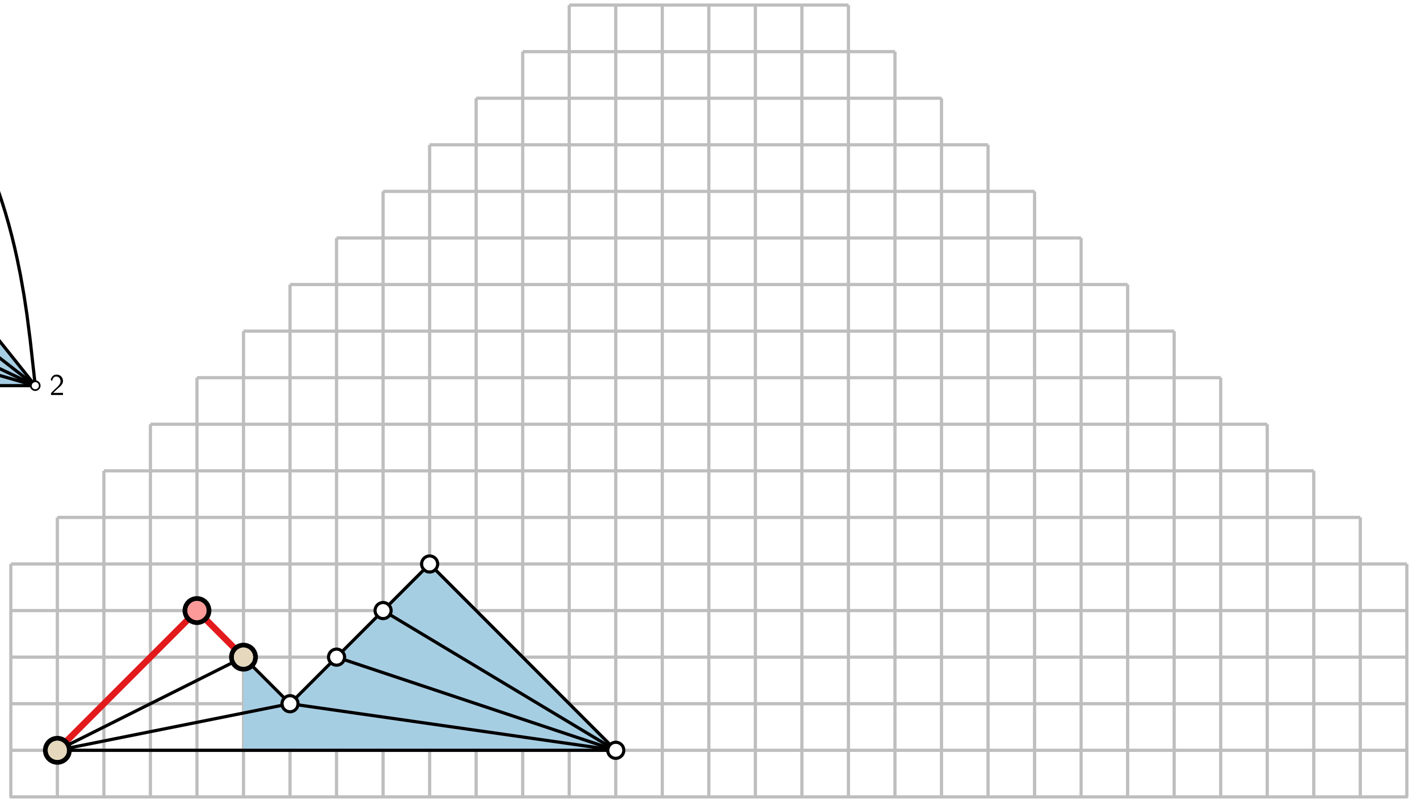

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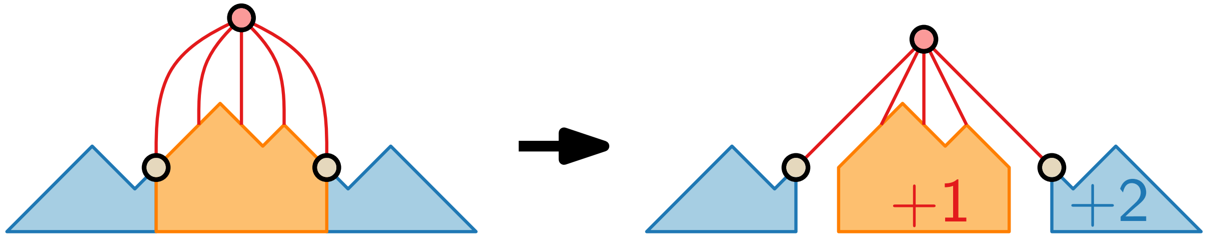

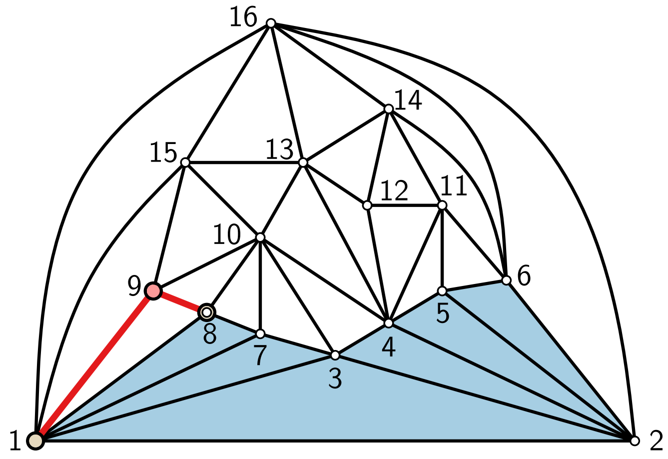

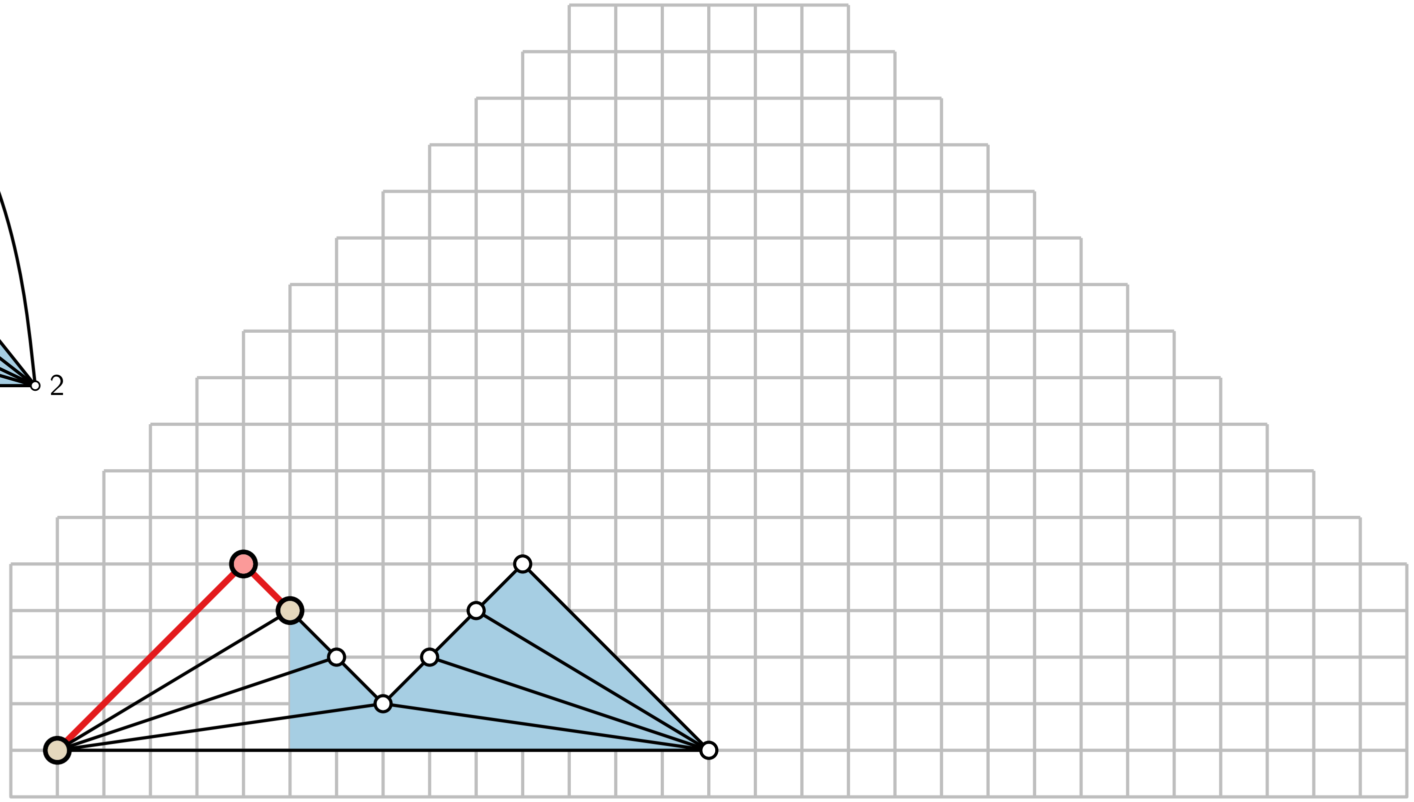

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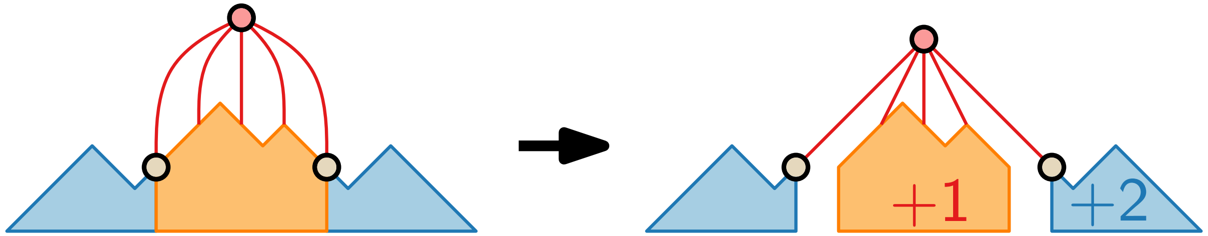

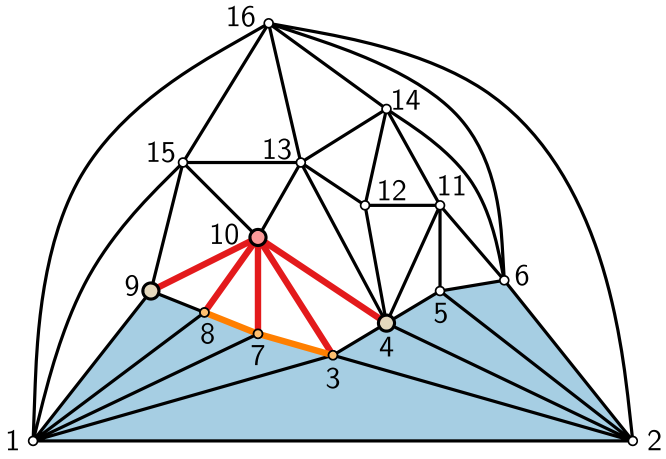

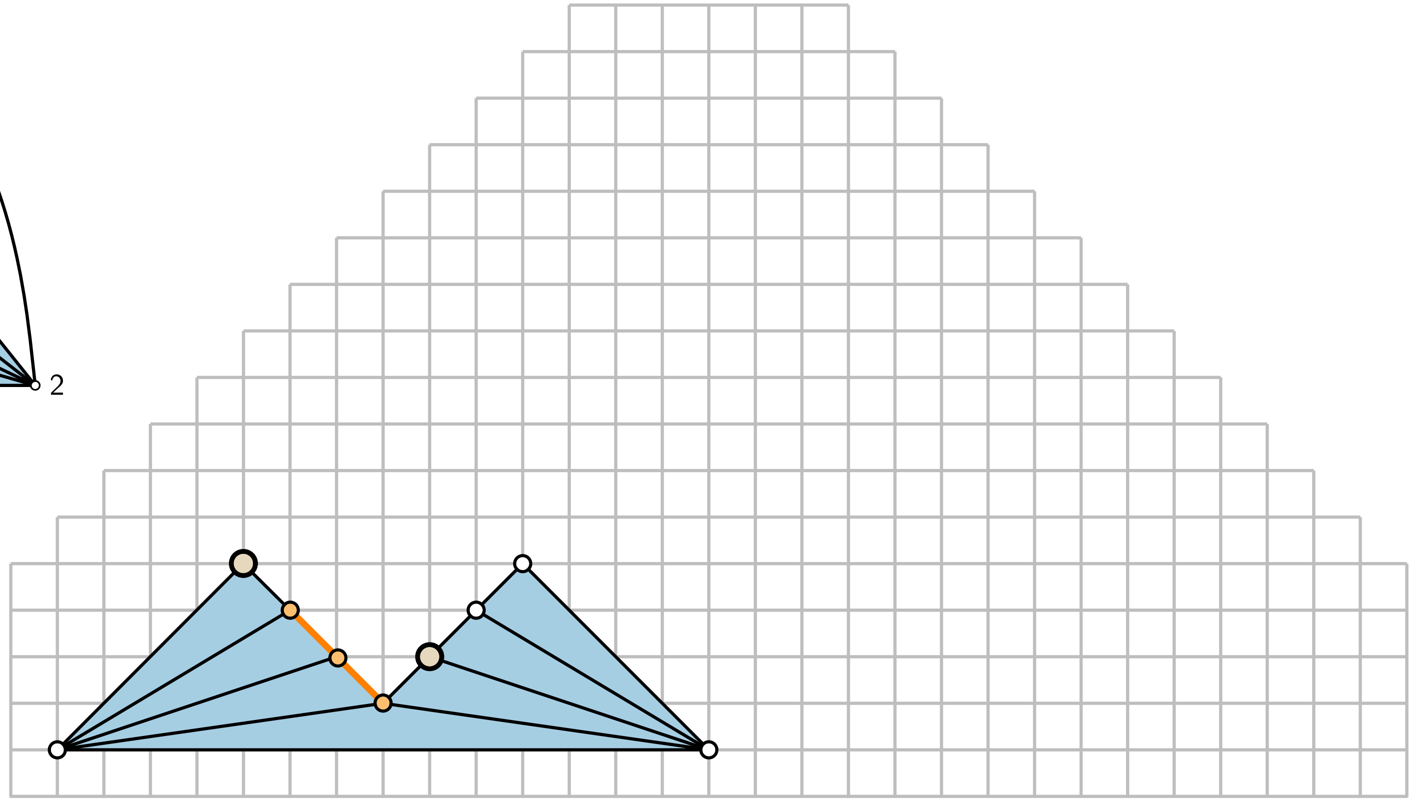

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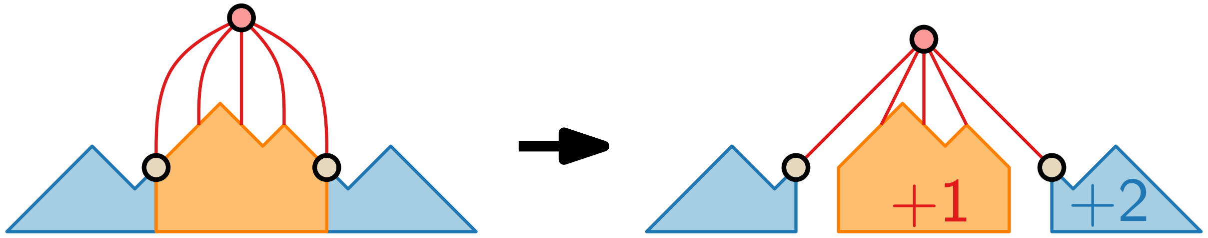
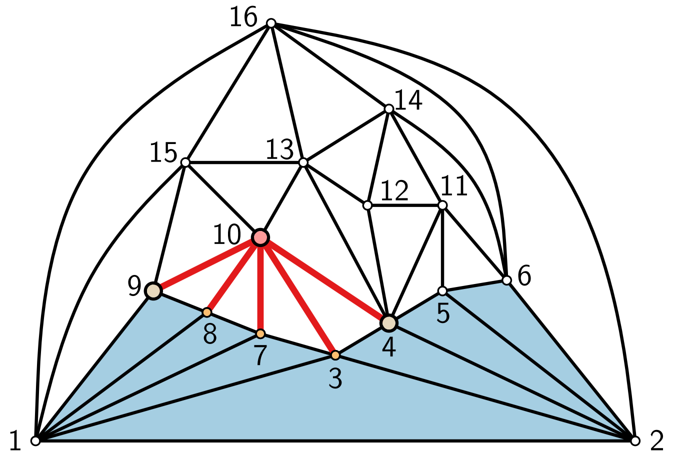
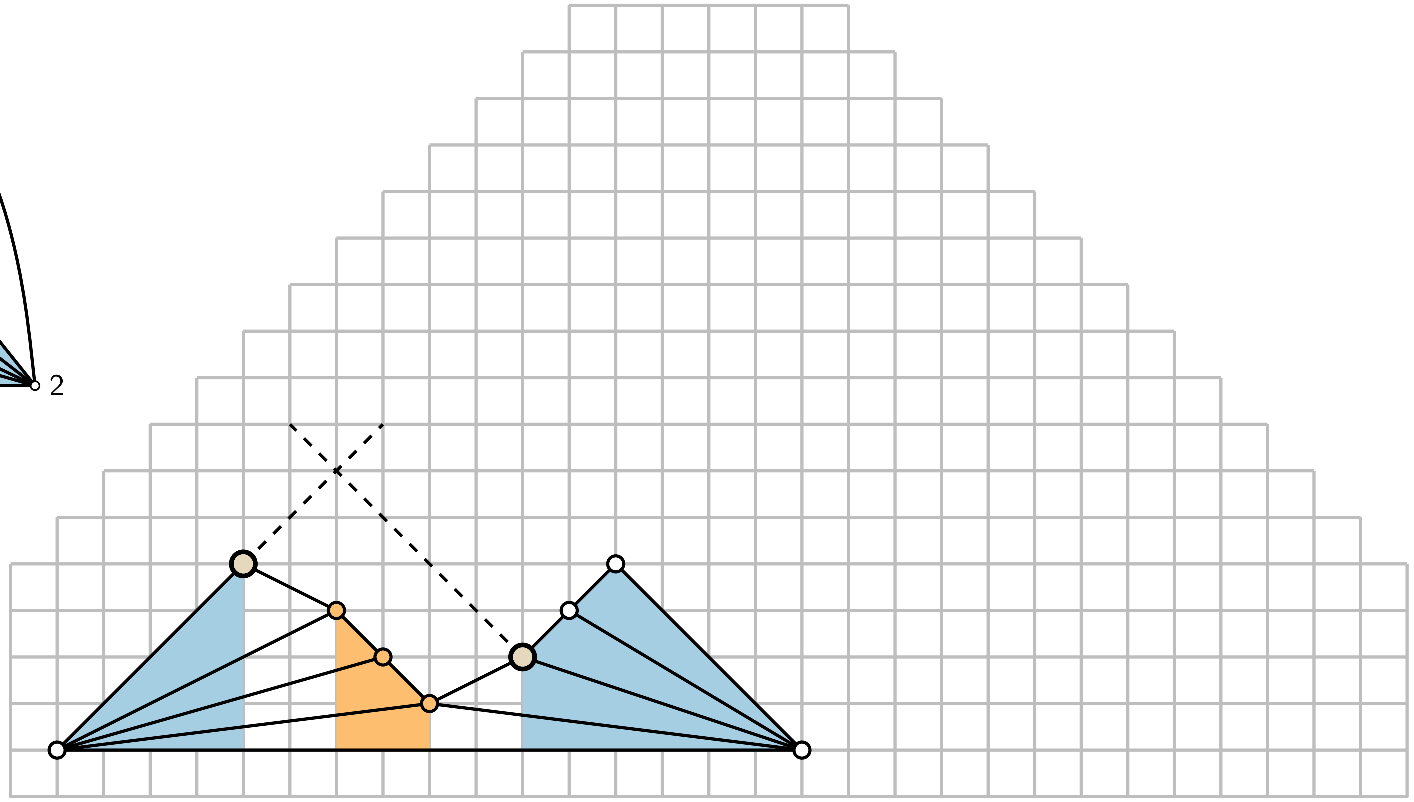
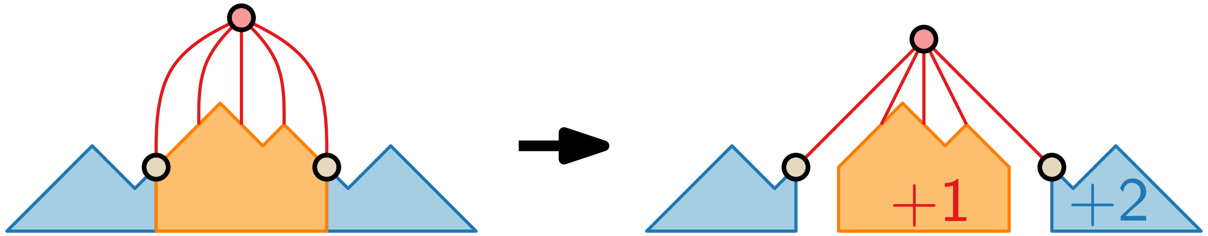
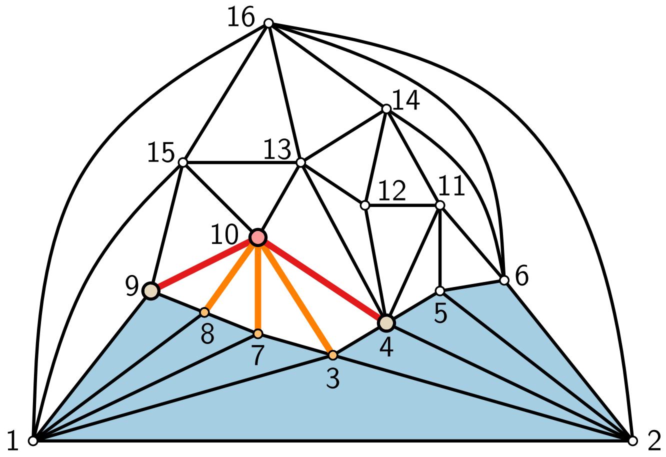


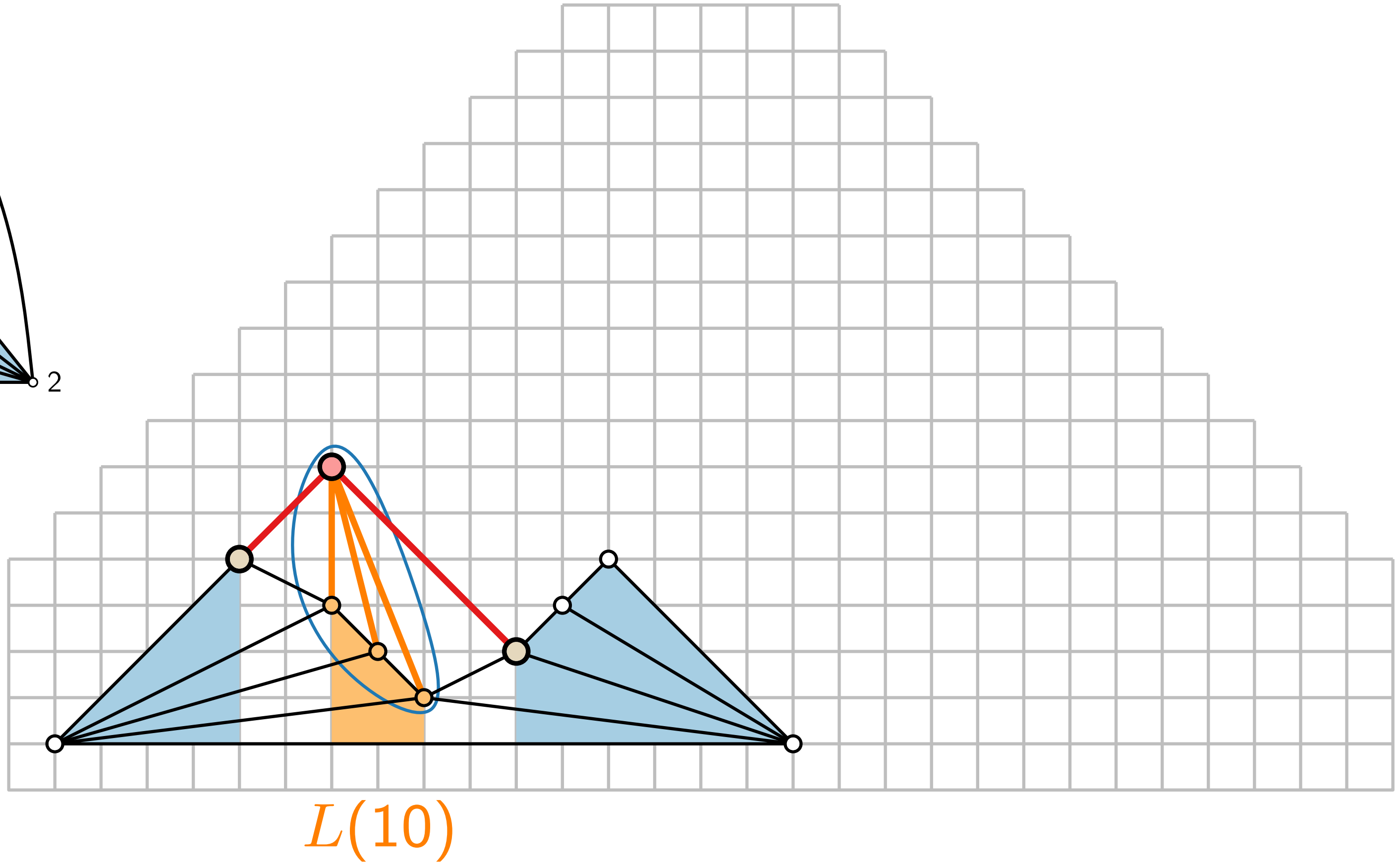




Shift Method – Example







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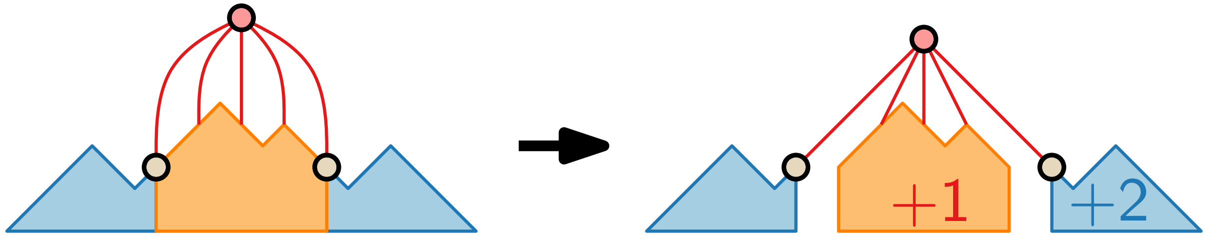

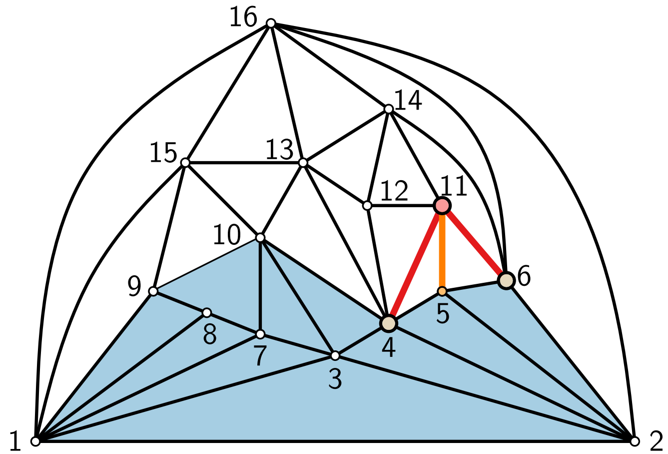

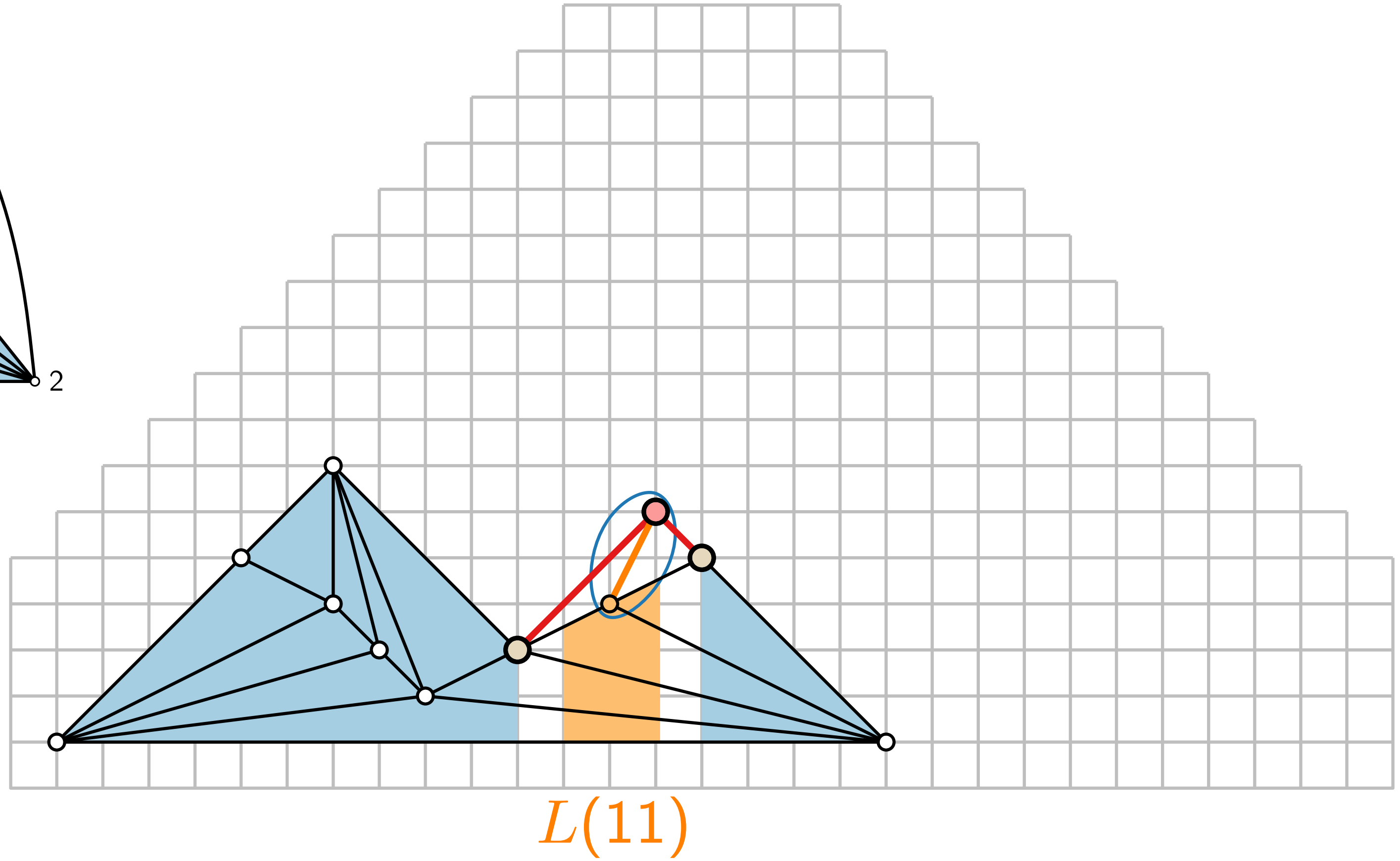

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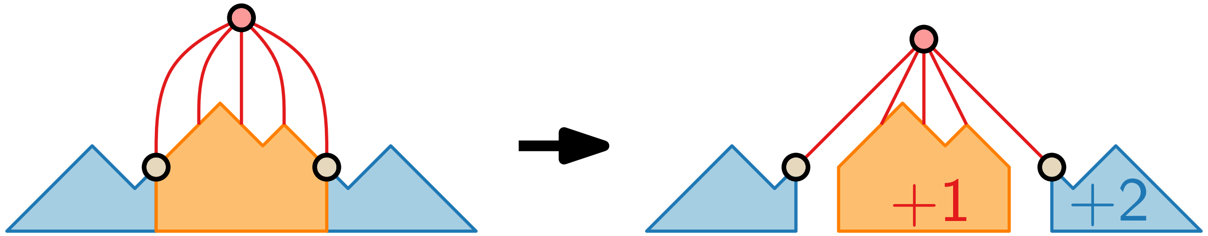

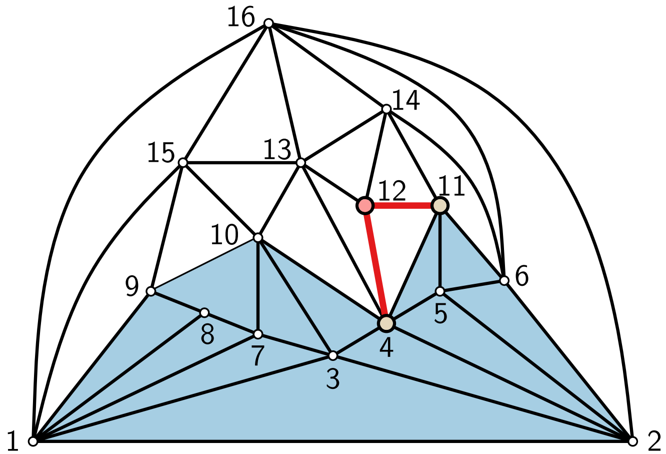

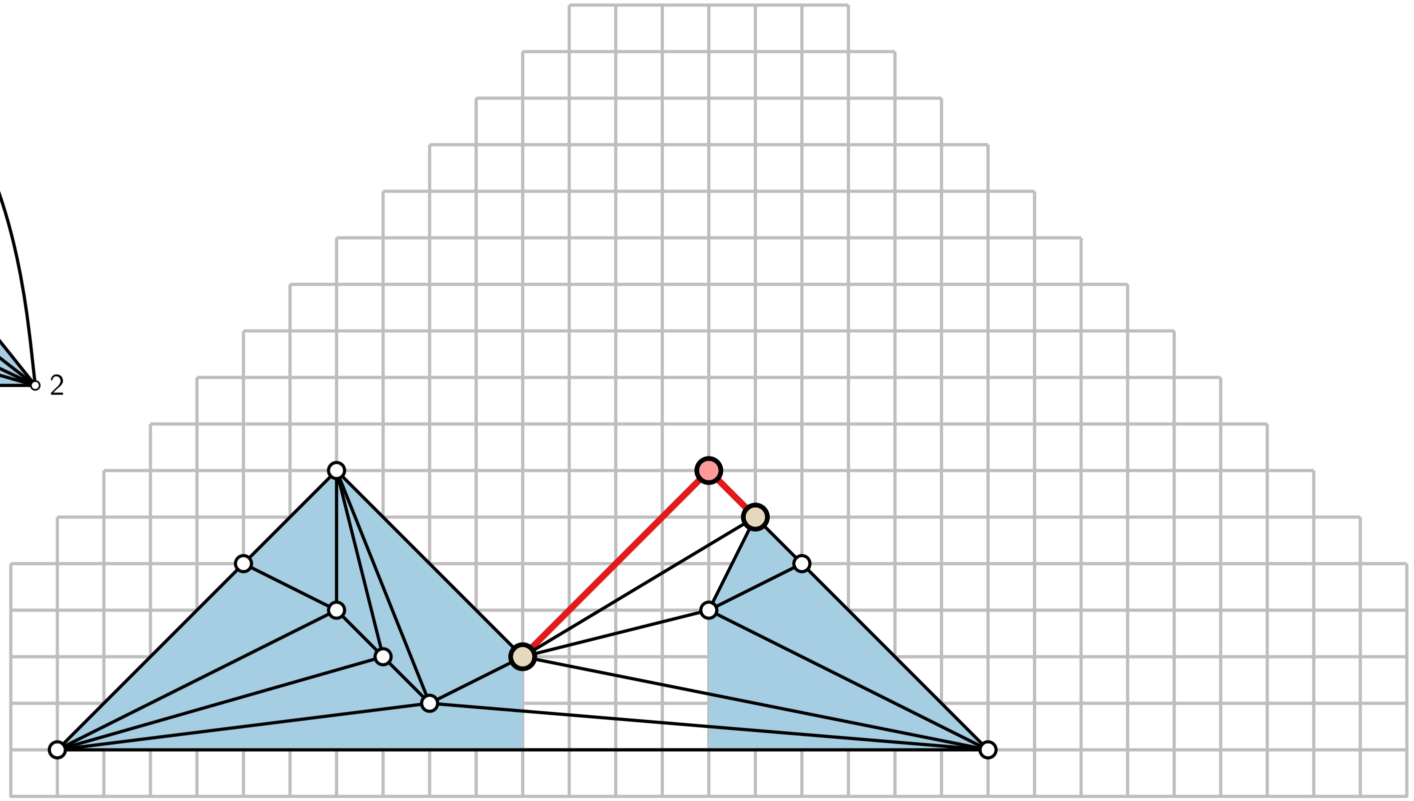

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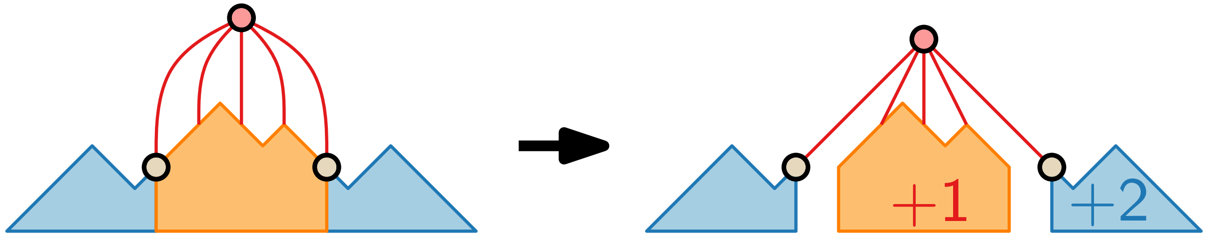

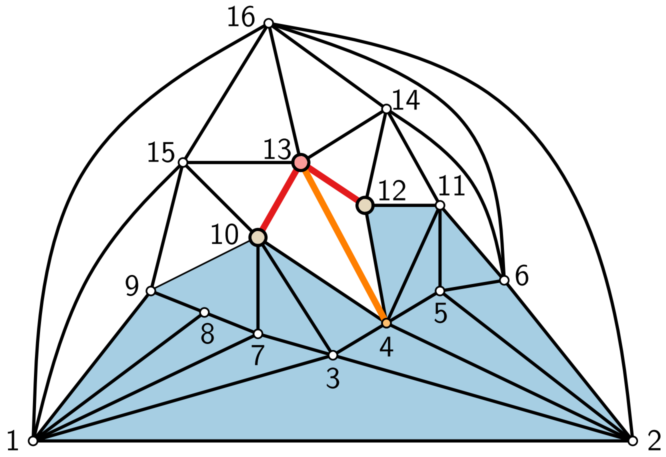

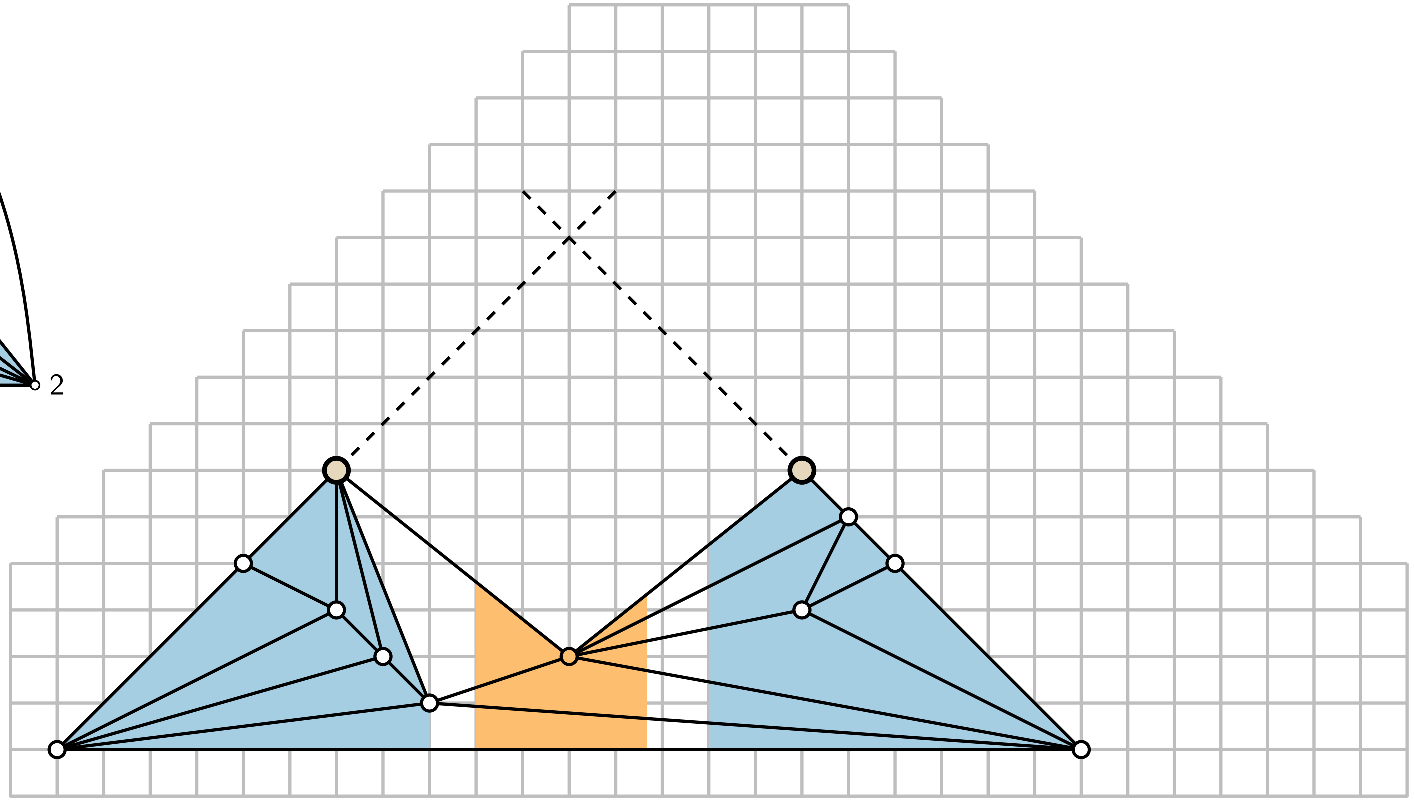

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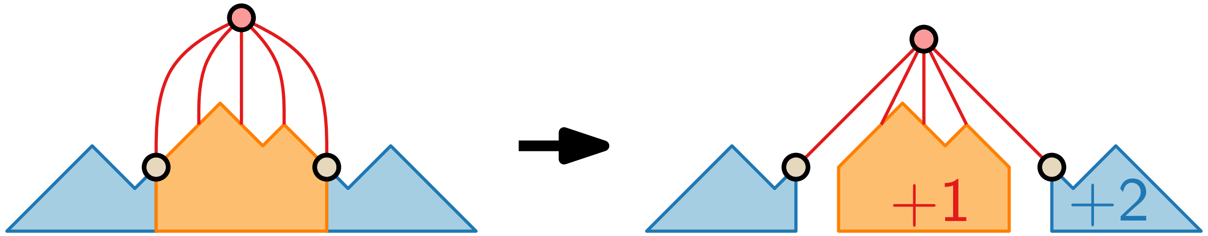

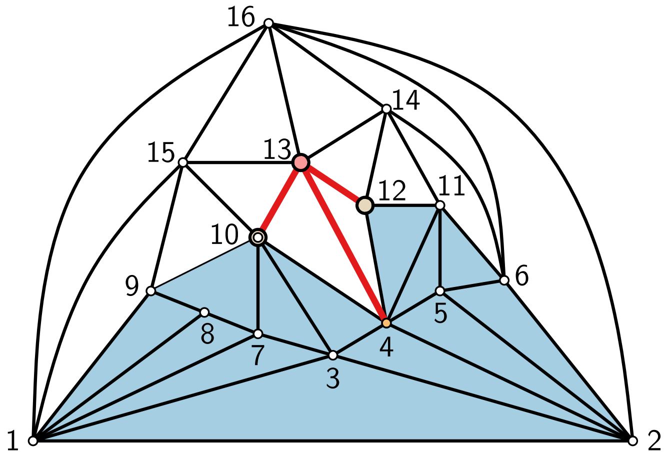

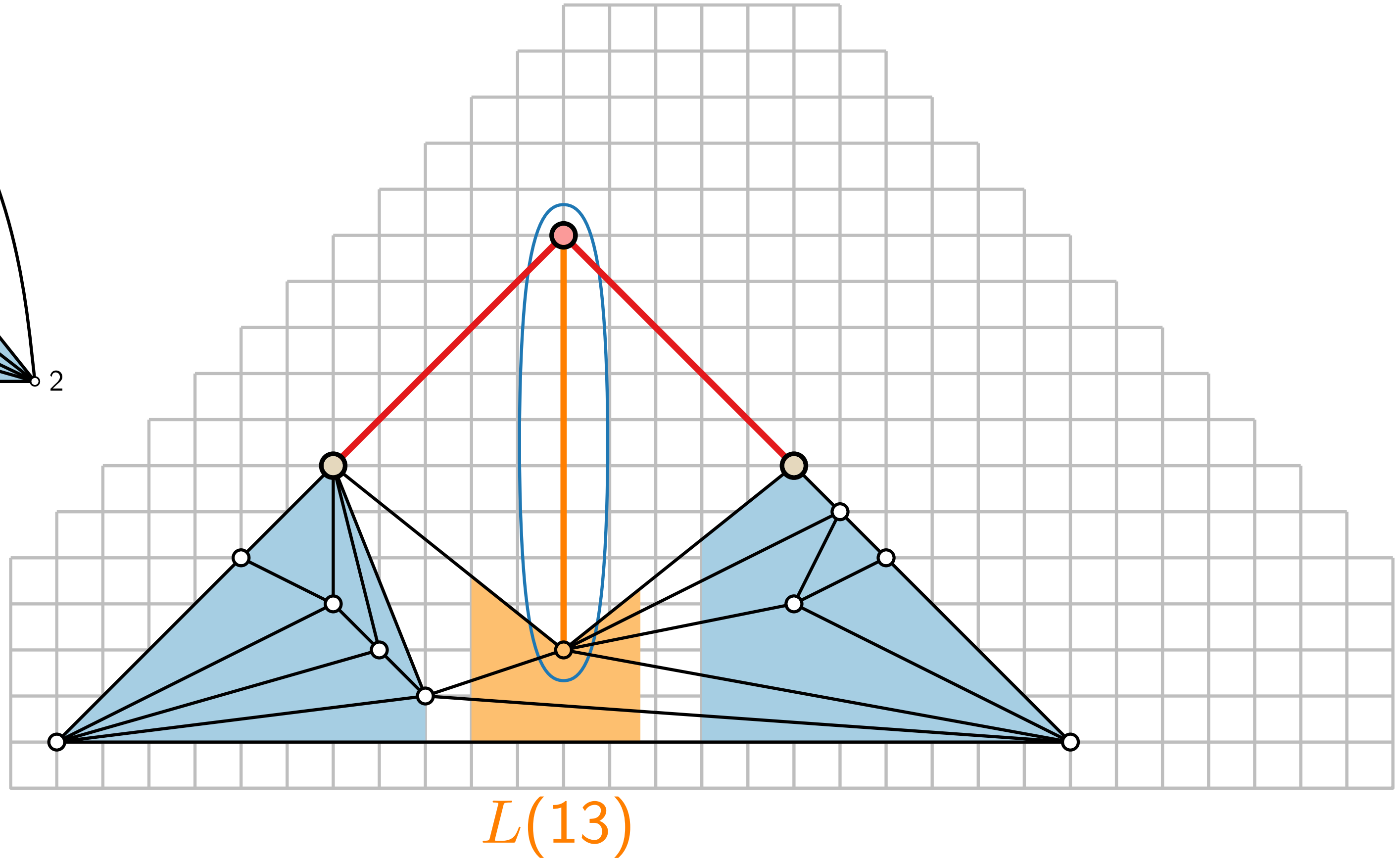

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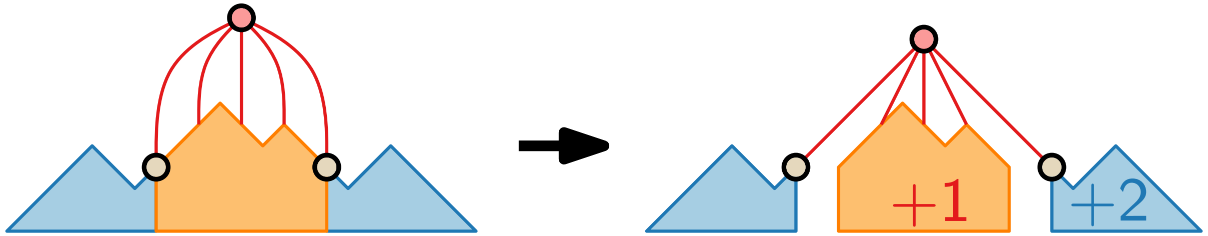

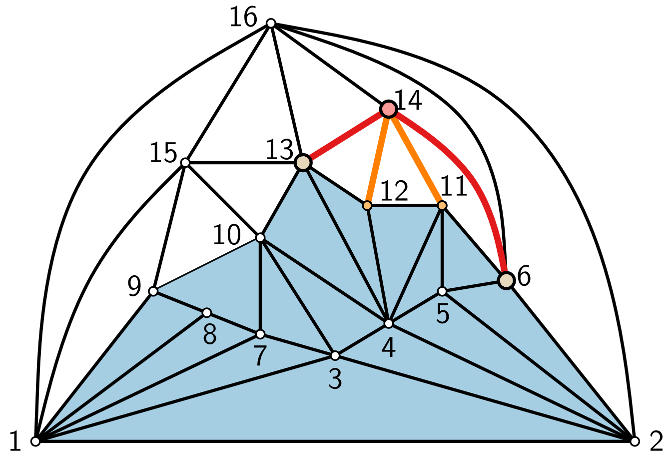

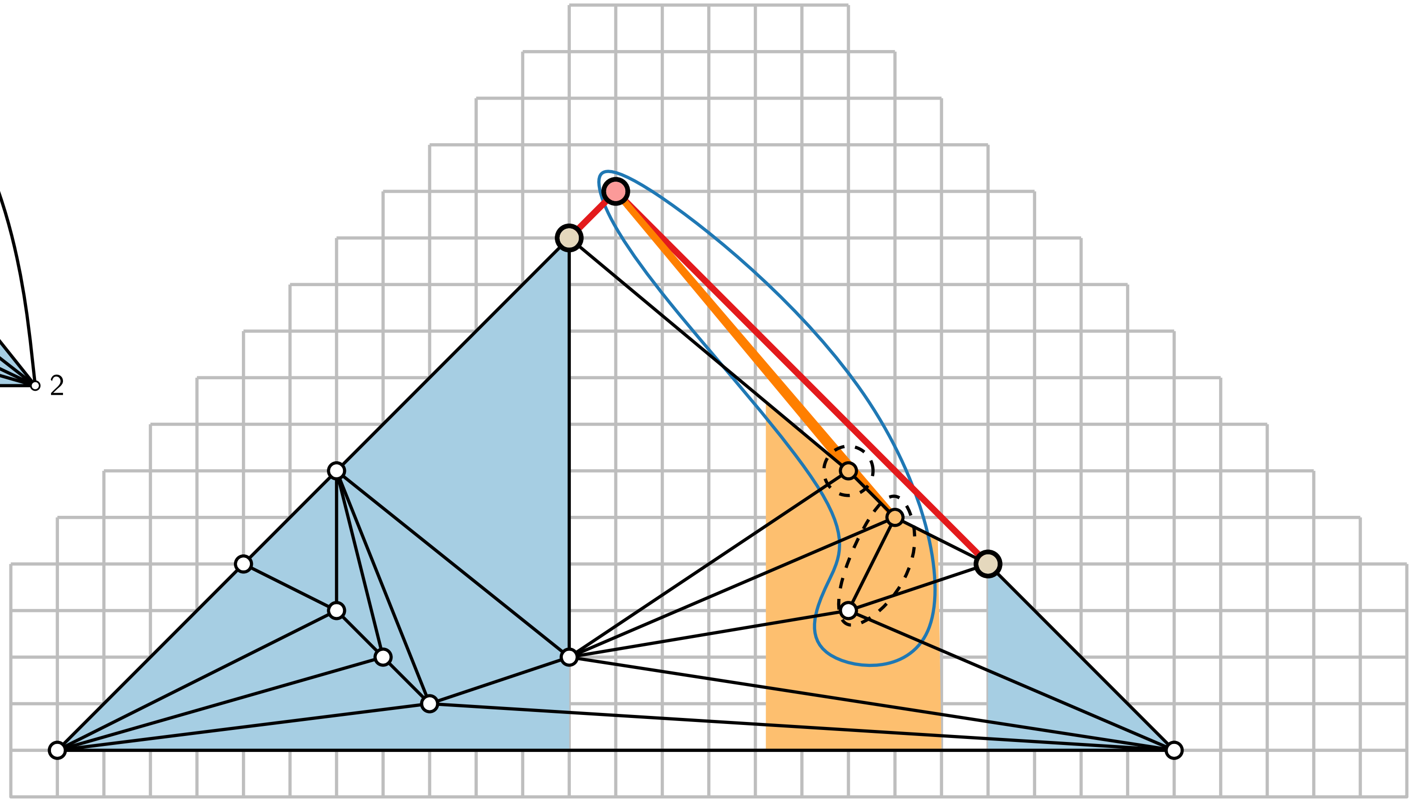

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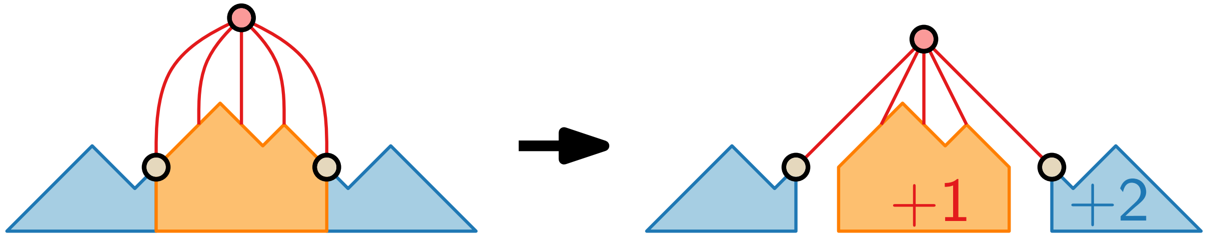

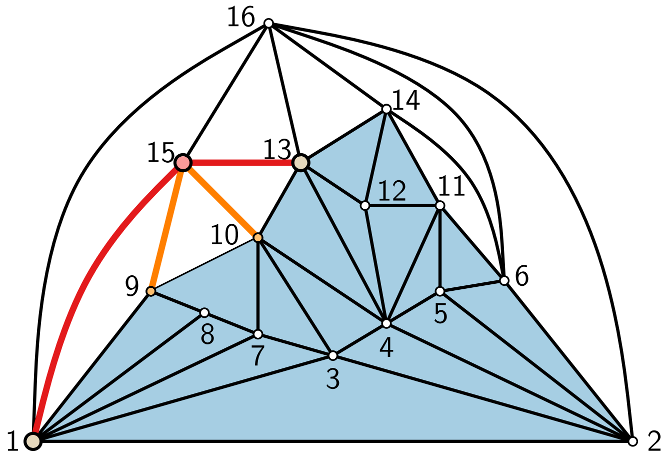

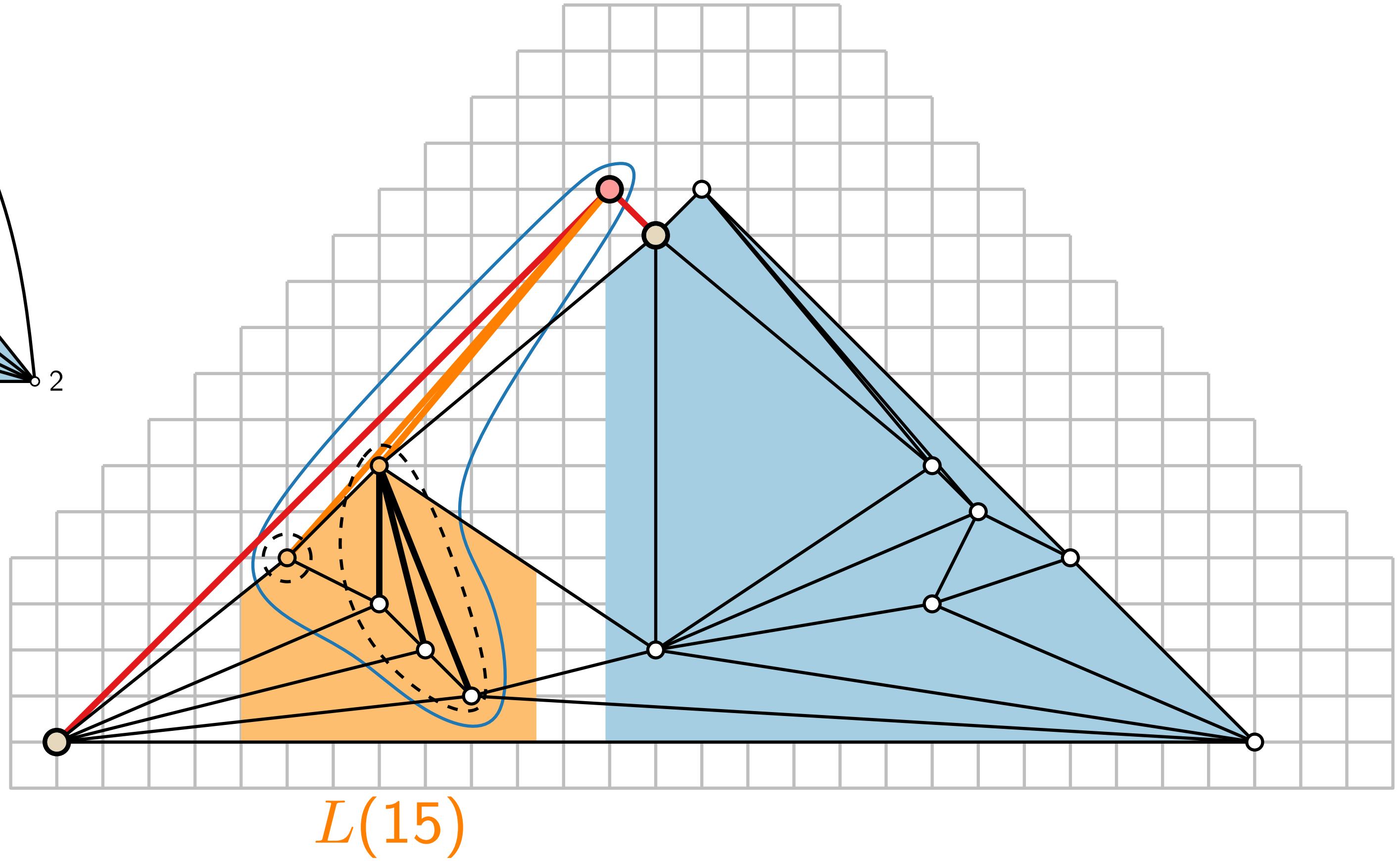

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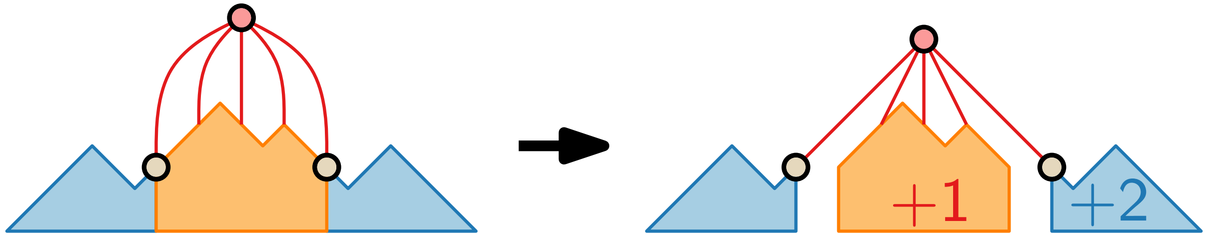

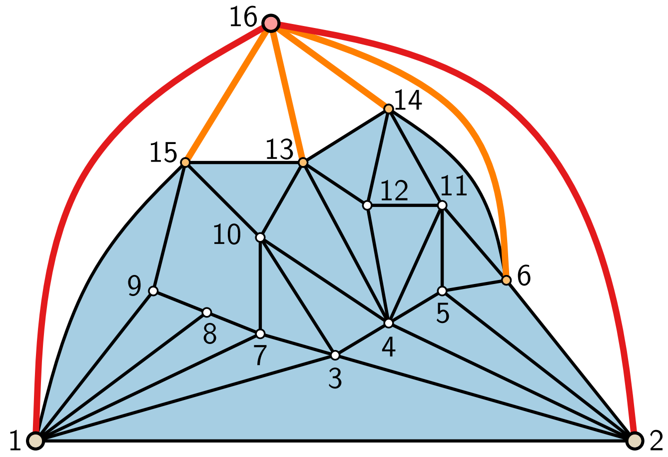

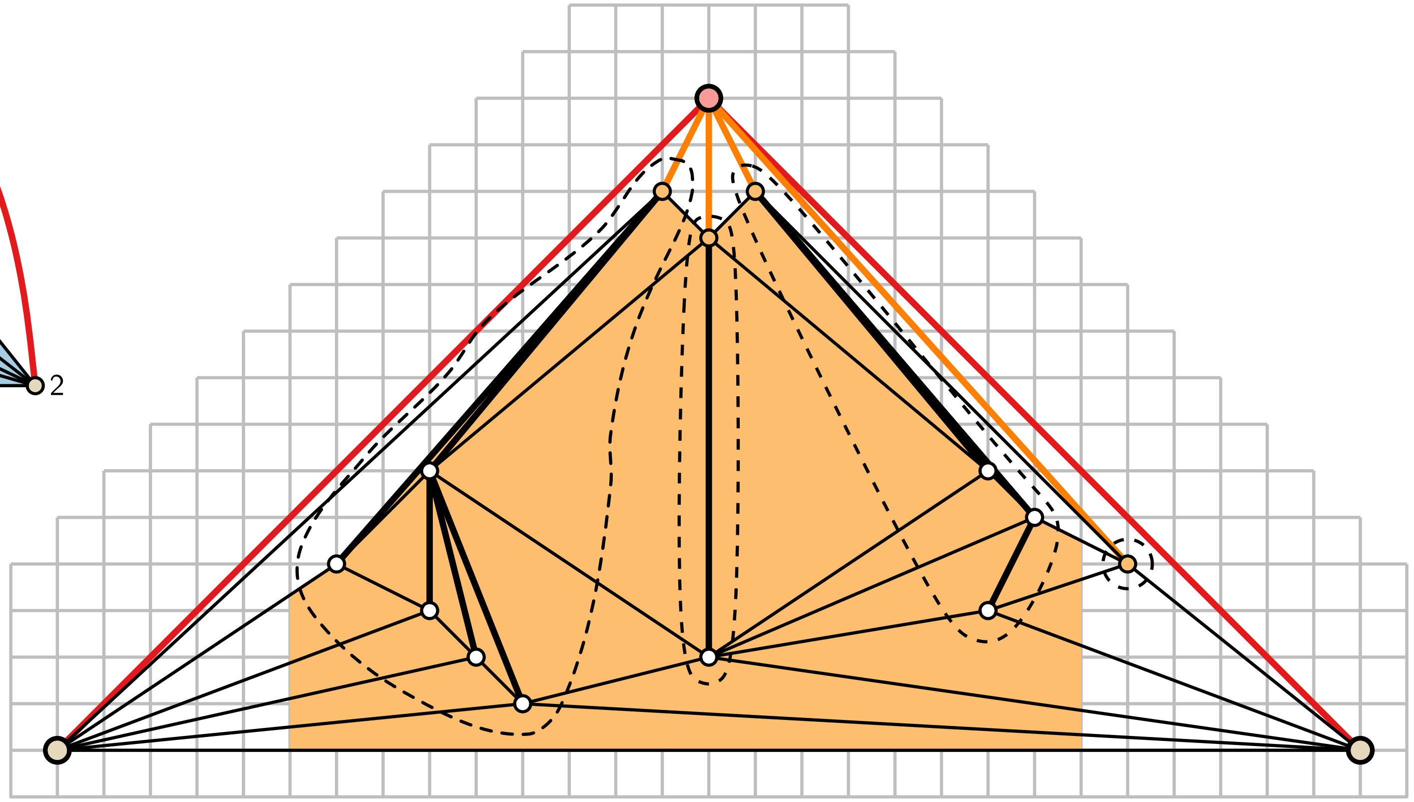

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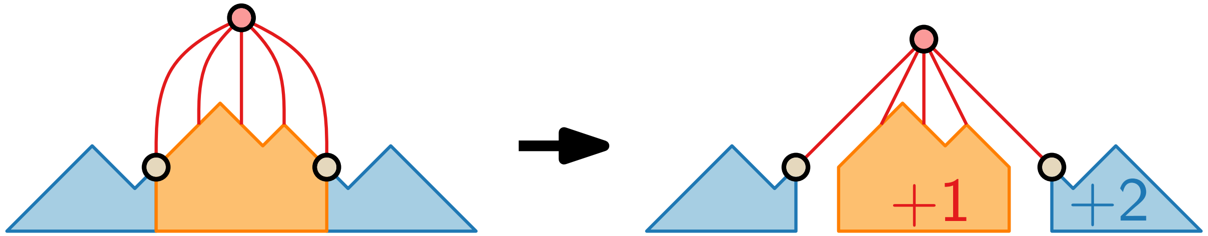
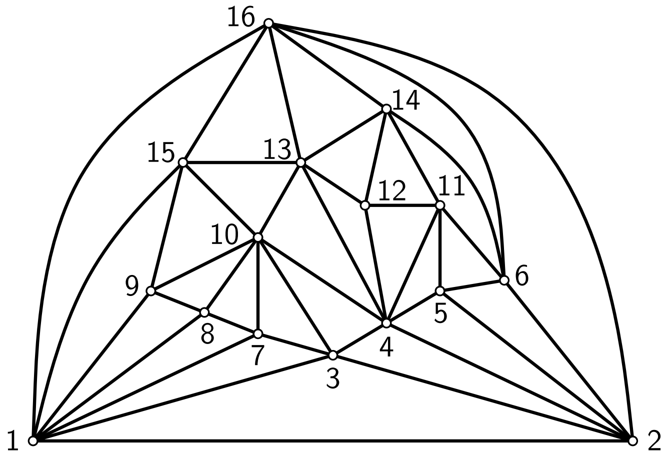
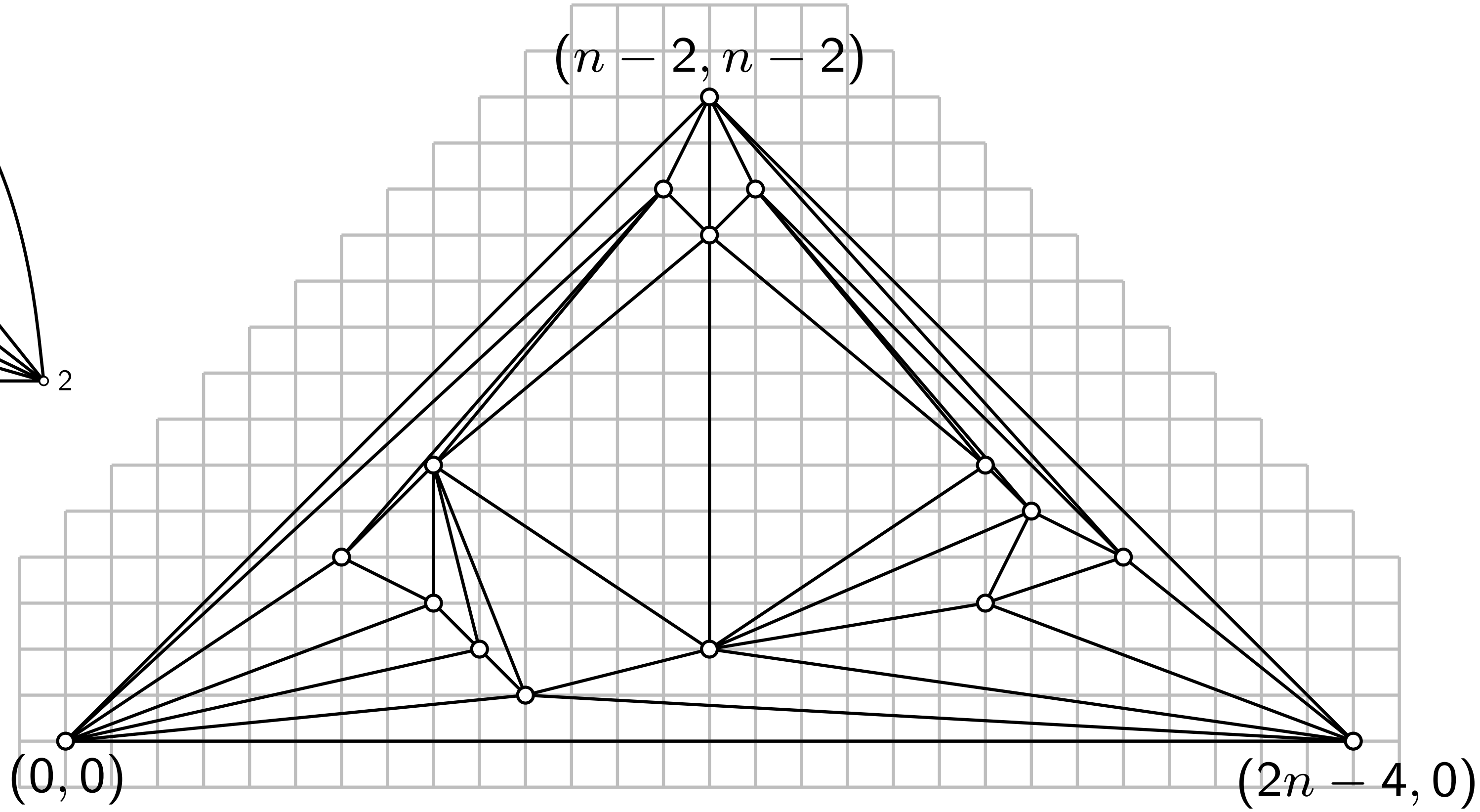
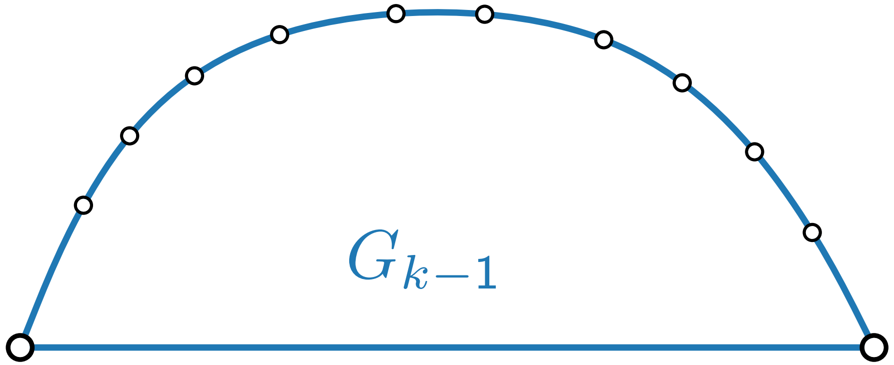






Shift Method – Planarity



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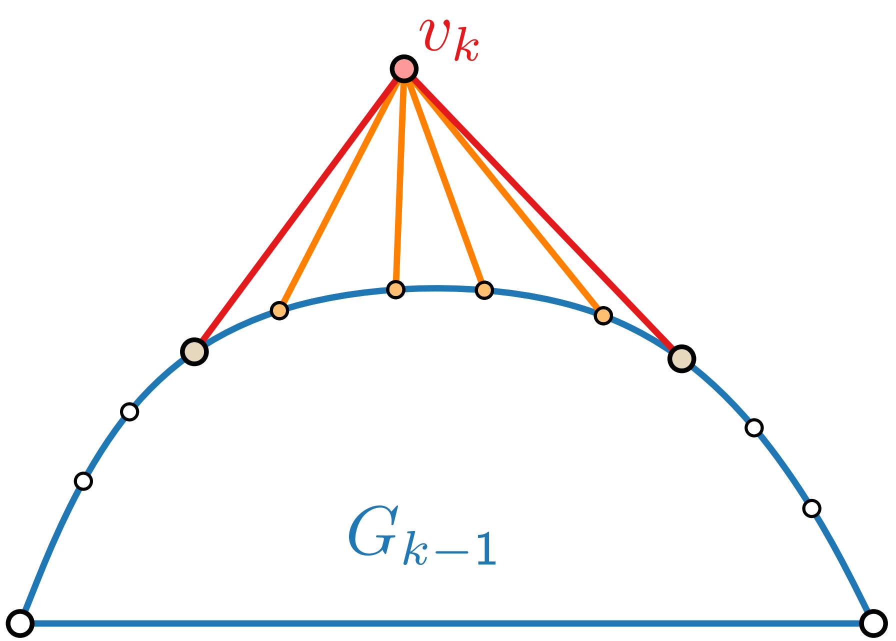

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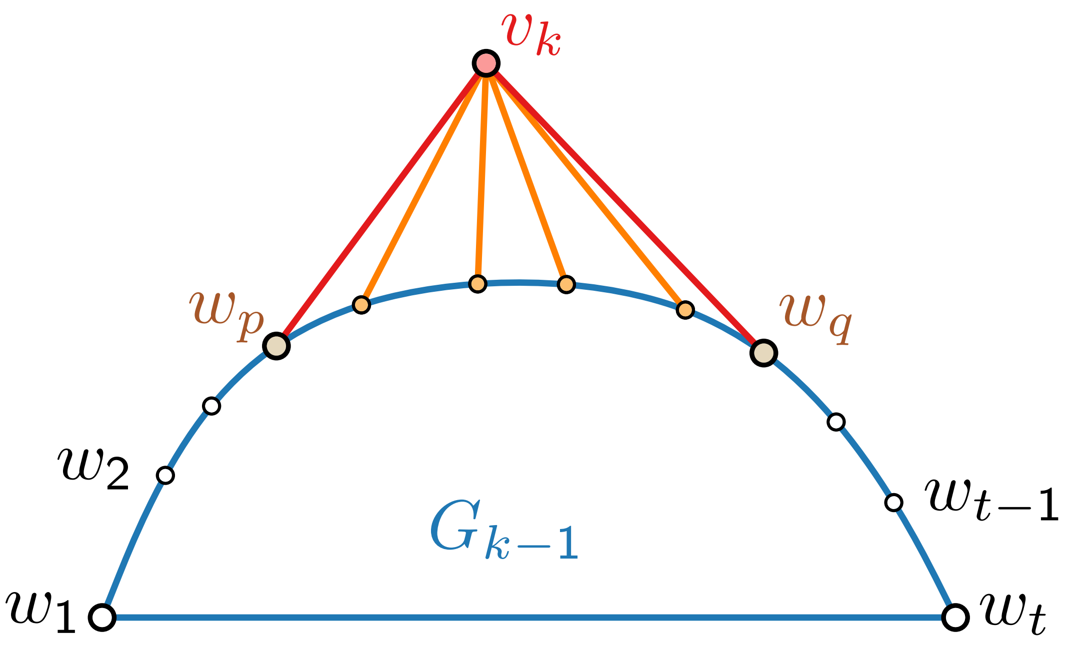

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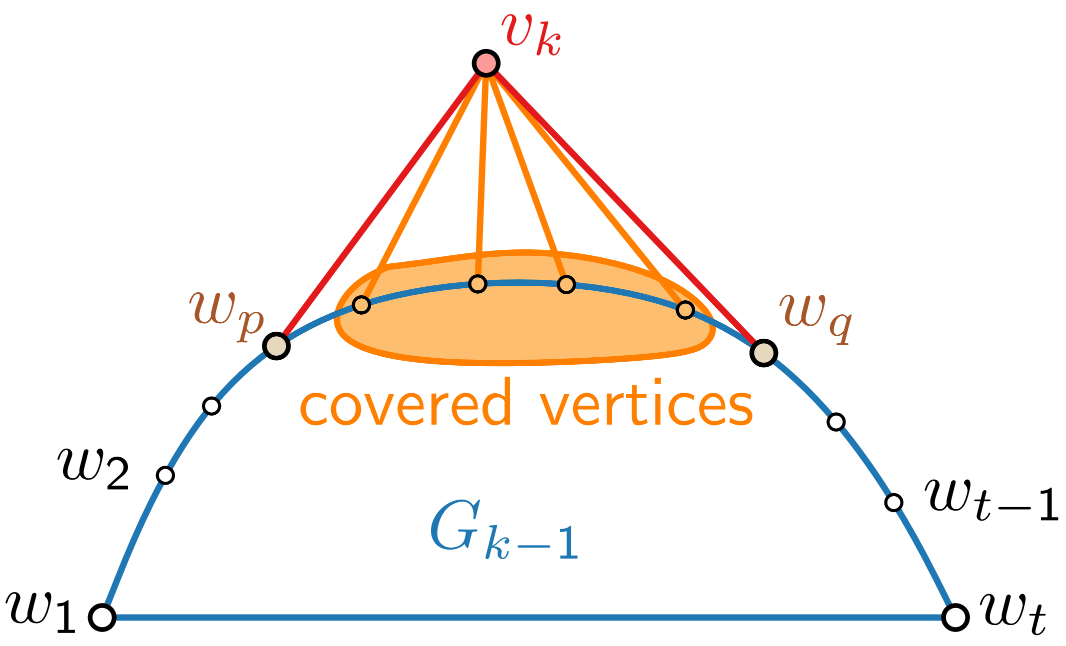
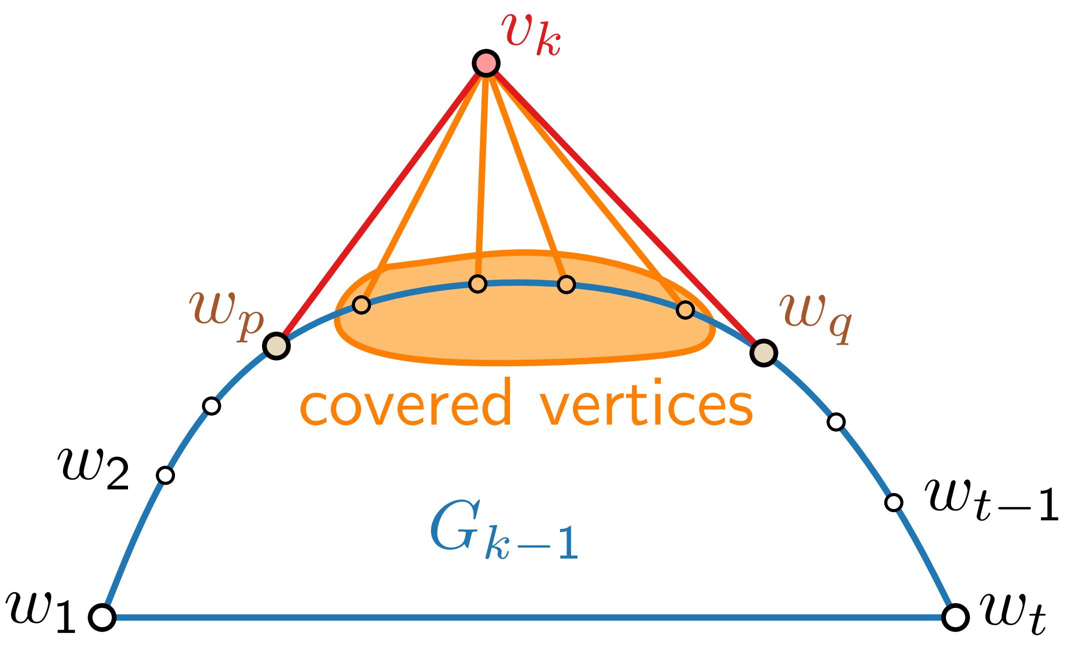


Shift Method – Planarity

Observations.

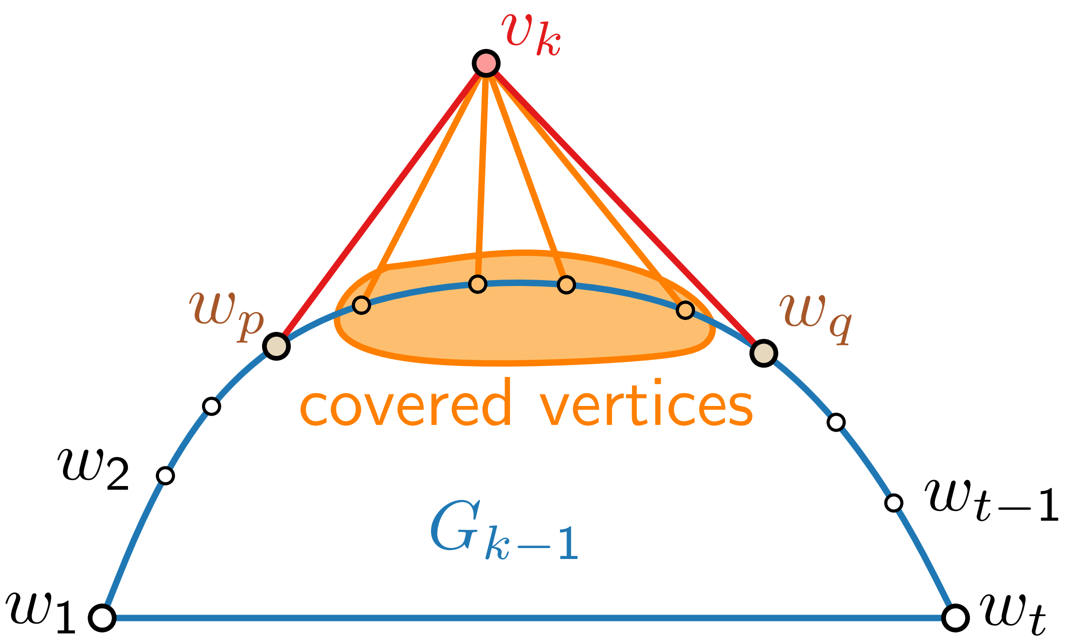
- Each internal vertex is covered exactly once.



Shift Method – Planarity

Observations.

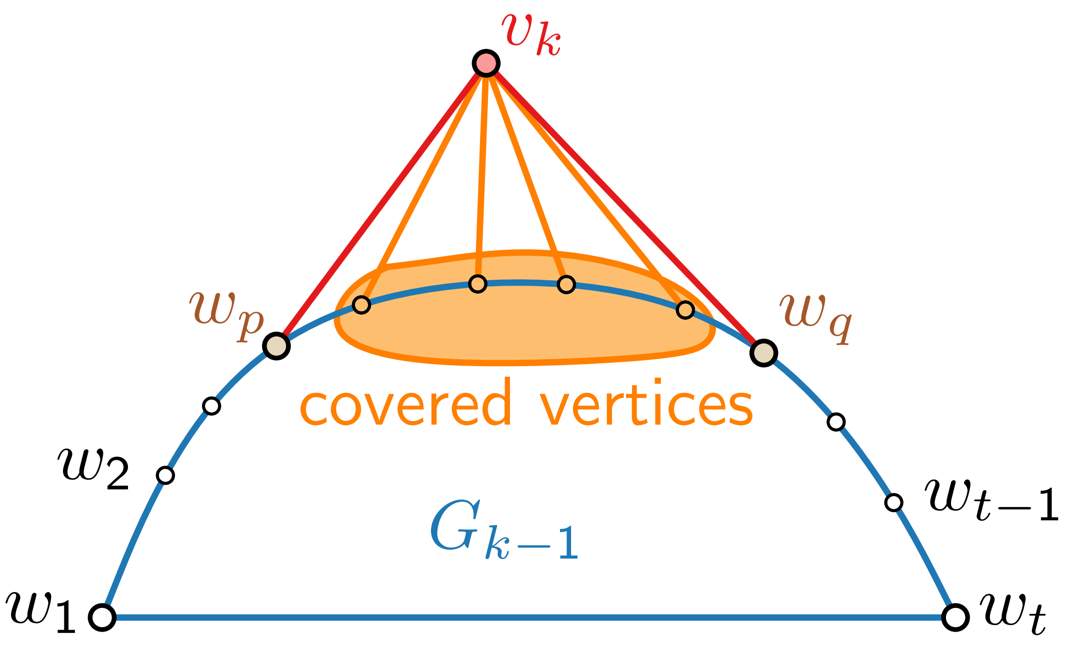
- Each internal vertex is covered exactly once.
- Covering relation defines a tree in G



Shift Method – Planarity

Observations.

- Each internal vertex is covered exactly once.
- Covering relation defines a tree in G
- and a forest in G_i, $1 \leq i \leq n-1$.



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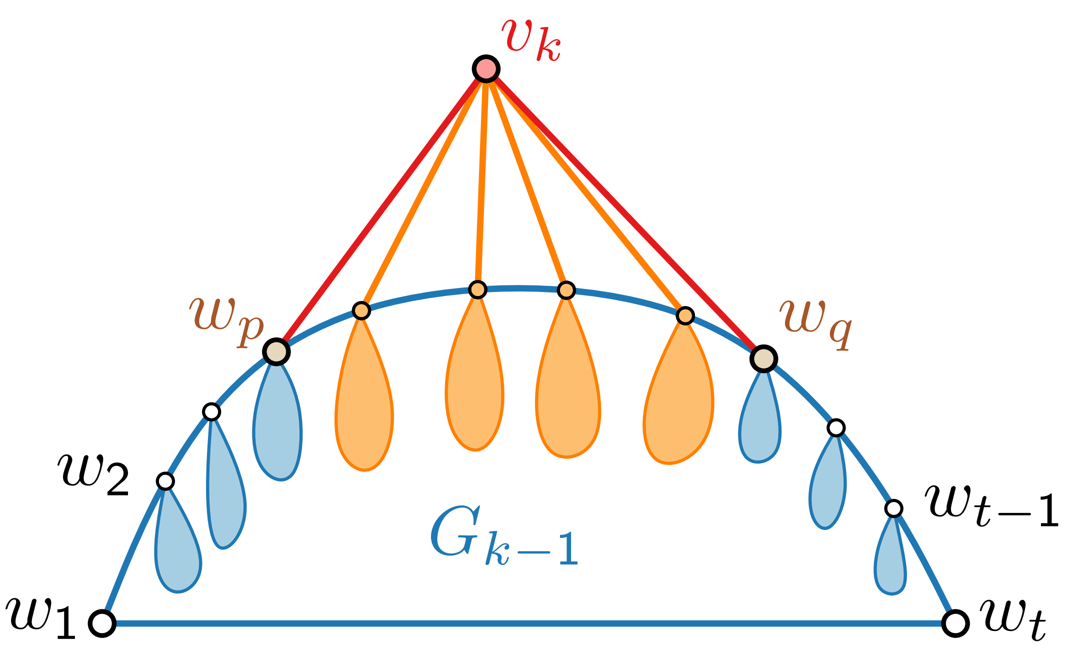

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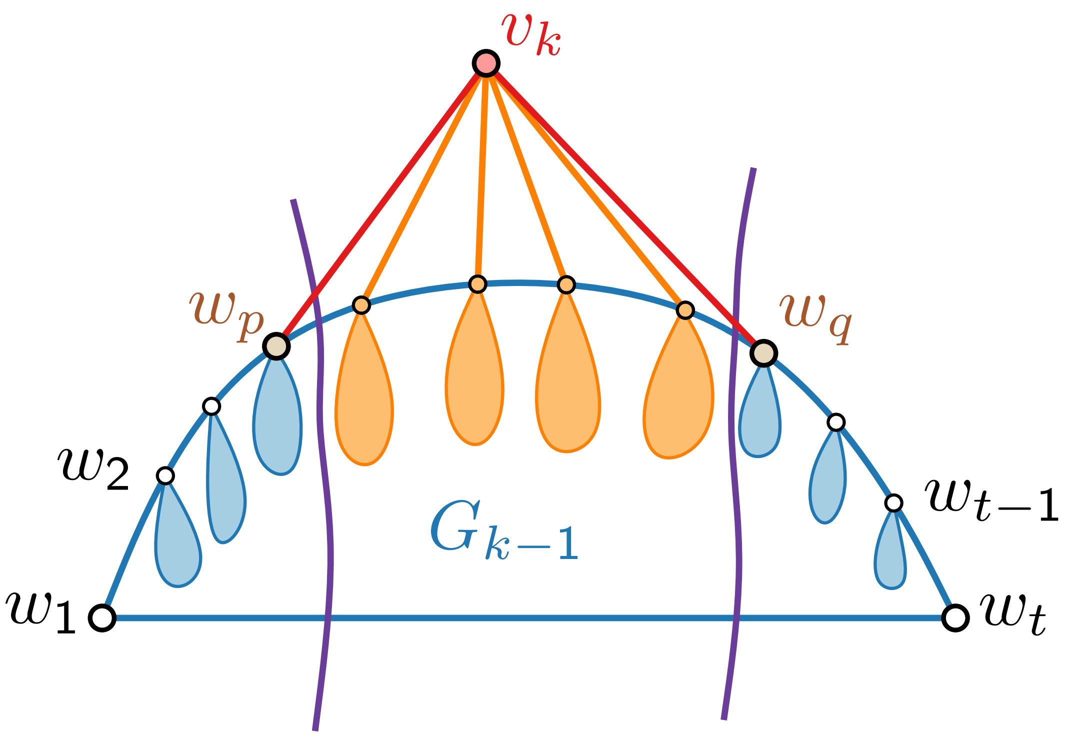

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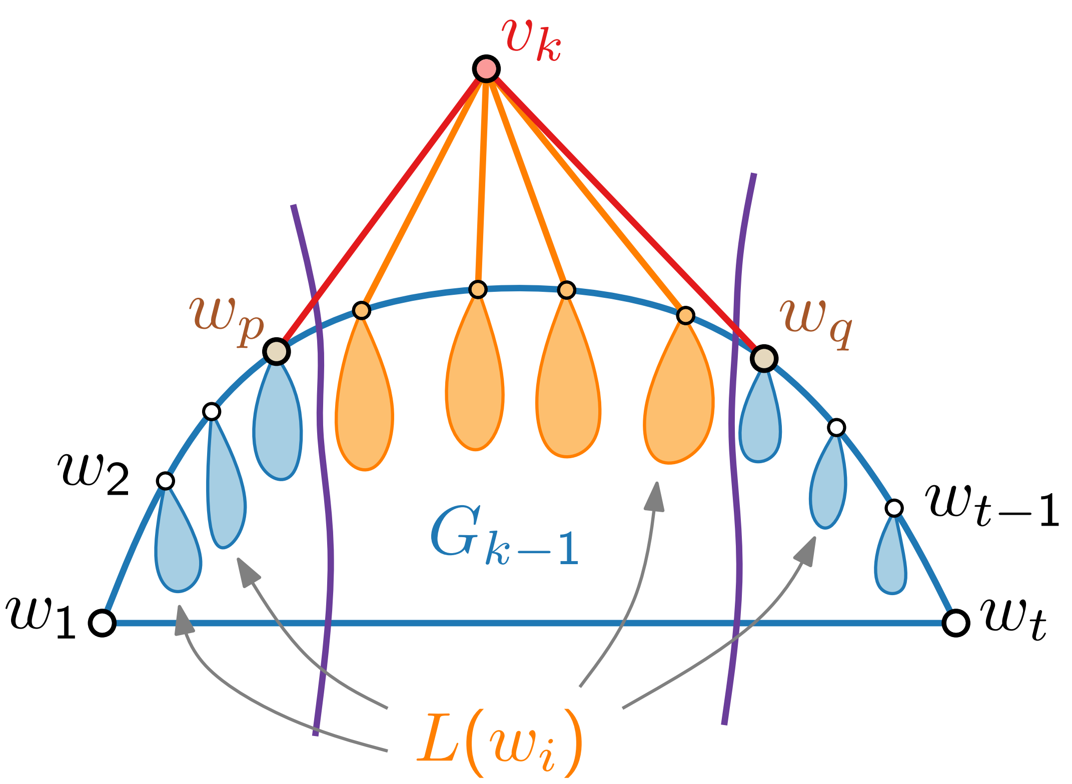

Shift Method – Planarity

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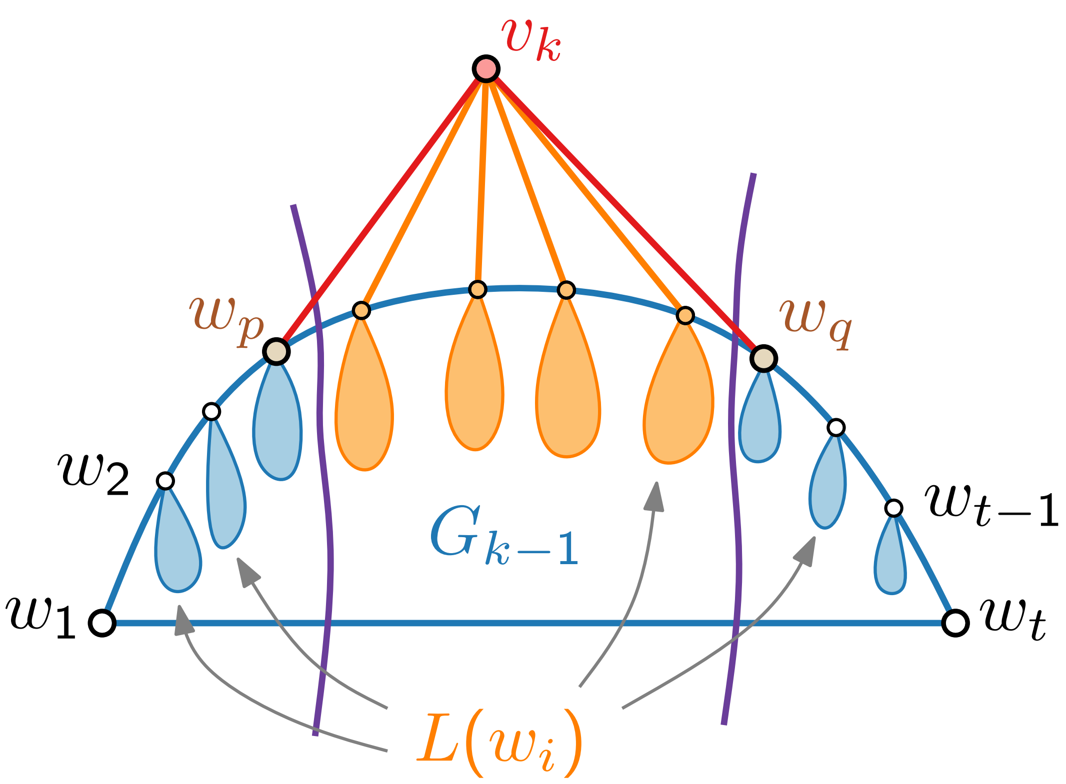
Lemma.
Let $0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_t \in \mathbb{N}$, such that $\delta_q - \delta_p \geq 2$ and even.



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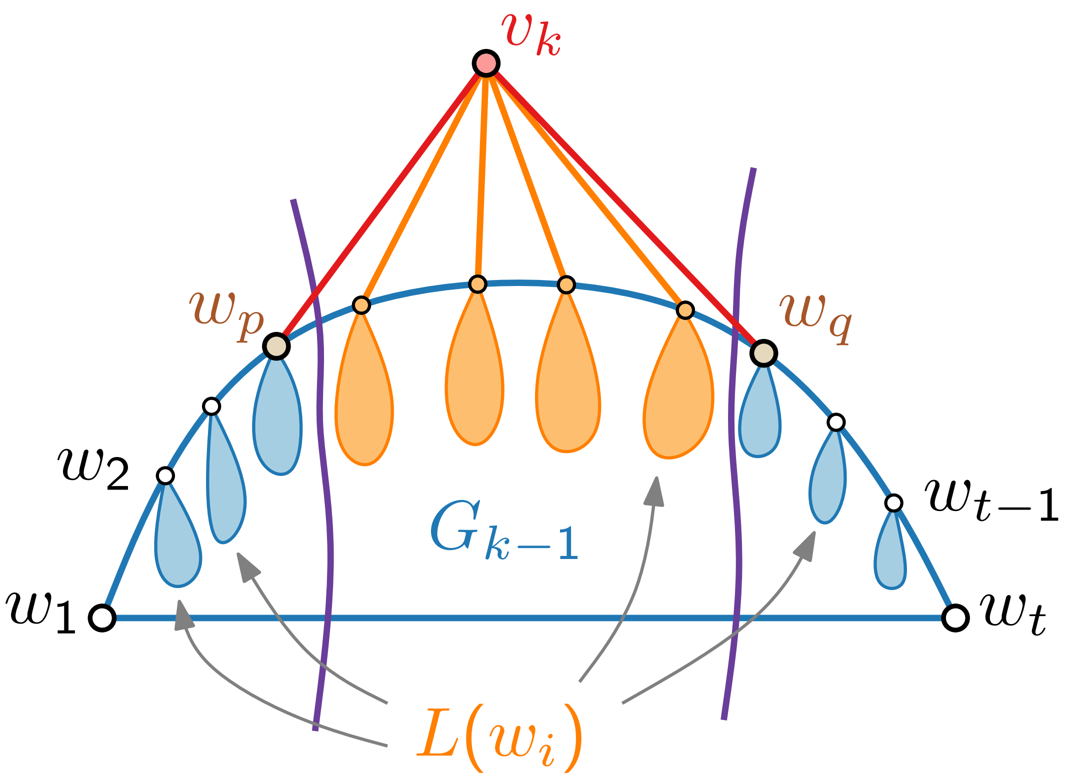
Let $0 < \delta_1 \leq \delta_2 \leq \cdots \leq \delta_t \in \mathbb{N}$, such that $\delta_q - \delta_p \geq 2$ and even.
If we shift $L(w_i)$ by δ_i to the right, then we get a planar straight-line drawing.



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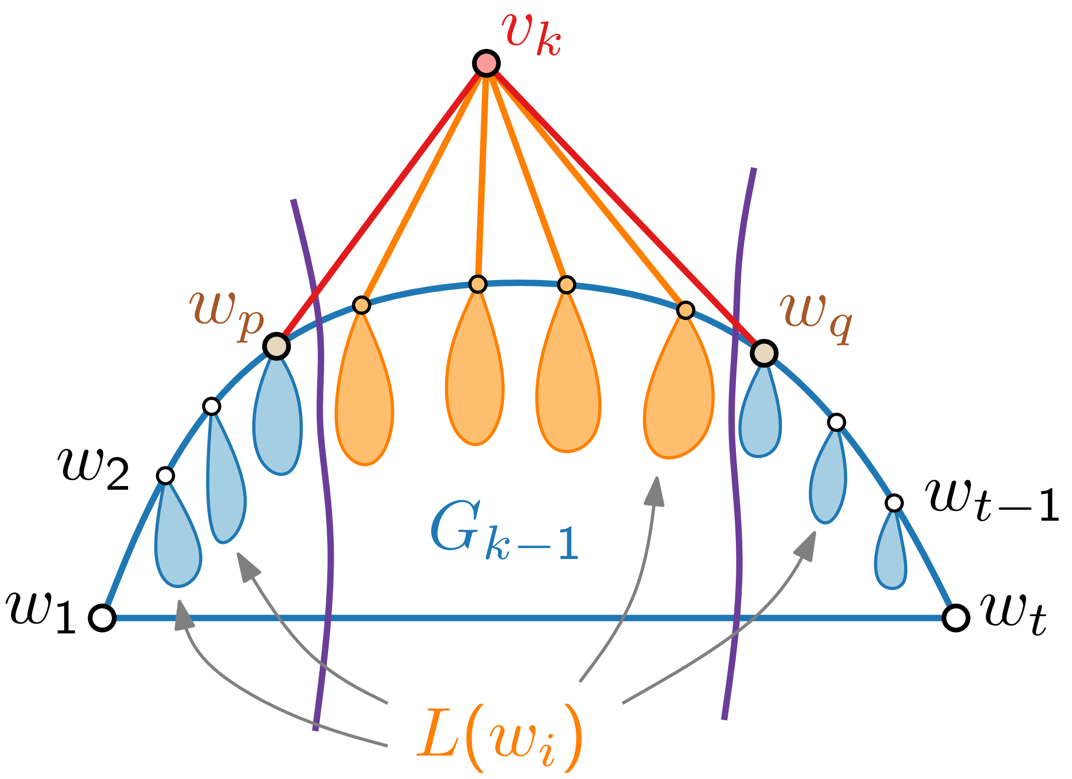

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Proof by induction:
If G_{k-1} is drawn planar and straight-line, then so is G_k.

Shift Method – Pseudocode

Let $v_1, \ldots, v_n$ be a canonical order of G

for $i = 1$ to 3 **do**

for $i = 4$ to n **do**

Shift Method – Pseudocode

Let $v_1, \ldots, v_n$ be a canonical order of G

for $i = 1$ to 3 **do**

  $L(v_i) \leftarrow \{v_i\}$

for $i = 4$ to n **do**

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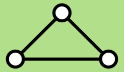
$P(v_1) \leftarrow (0,0); P(v_2) \leftarrow (2,0), P(v_3) \leftarrow (1,1)$ 

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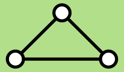
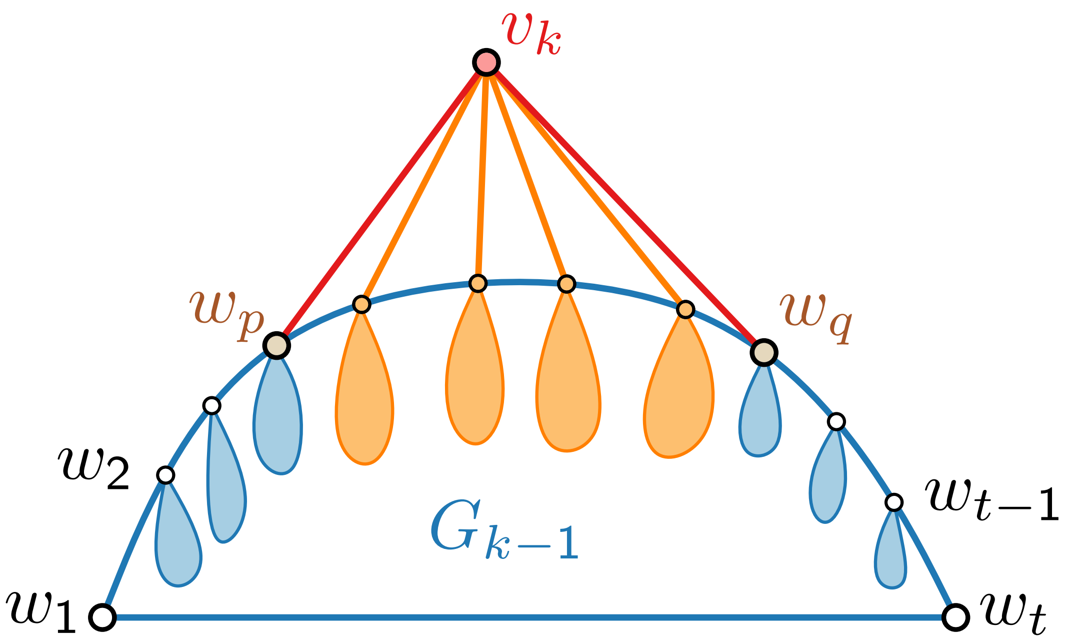


for $i = 4$ to n **do**

  Let $w_1 = v_1, w_2, \ldots, w_{t-1}, w_t = v_2$ denote the boundary of G_{i-1} and let $w_p, \ldots, w_q$ be the neighbours of v_i



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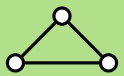

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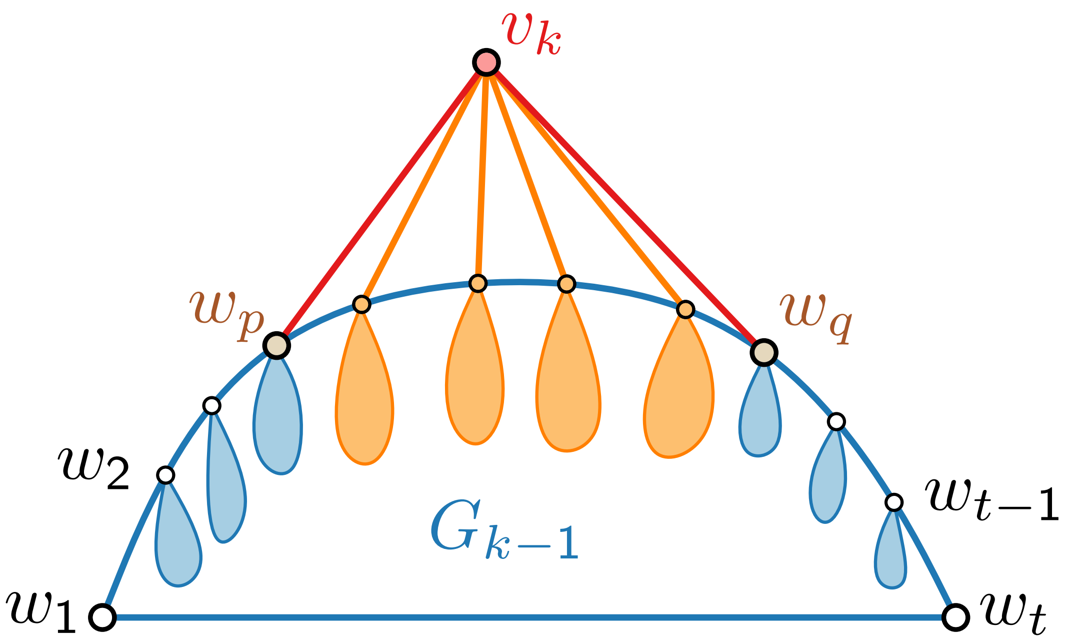

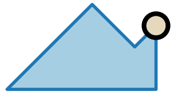

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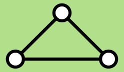

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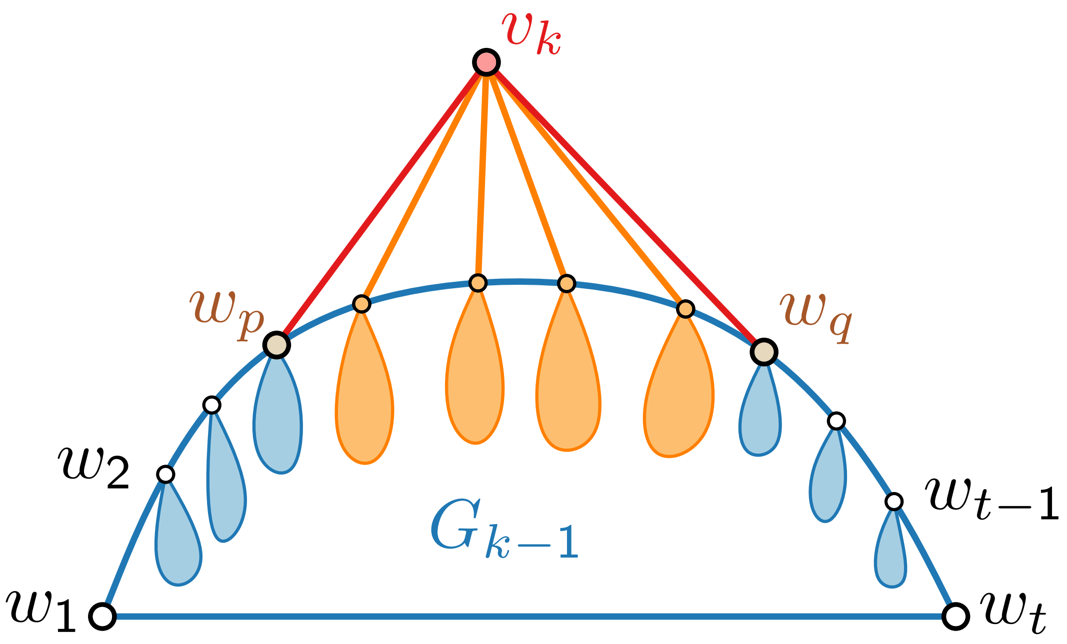
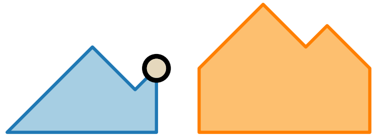
  for $\forall v \in \cup_{j=p+1}^{q-1} L(w_j)$ **do**





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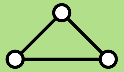

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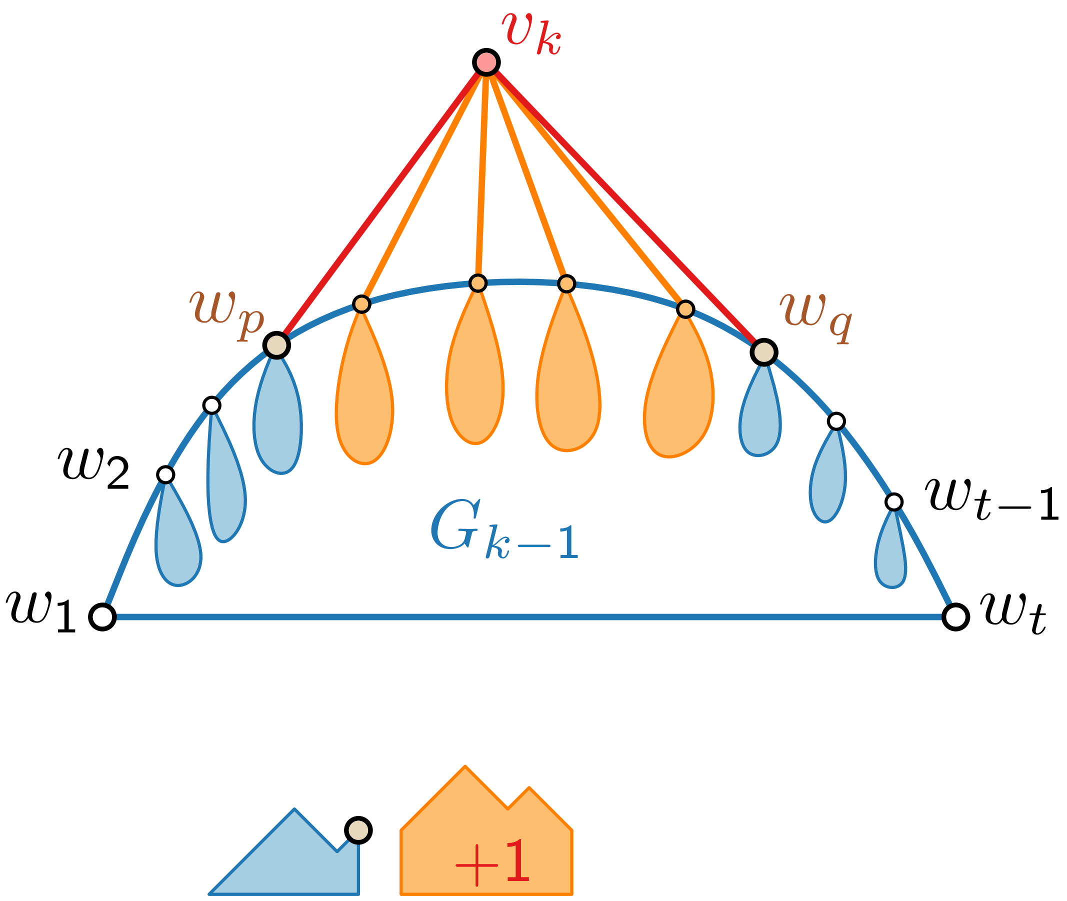
    $x(v) \leftarrow x(v) + 1$



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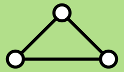

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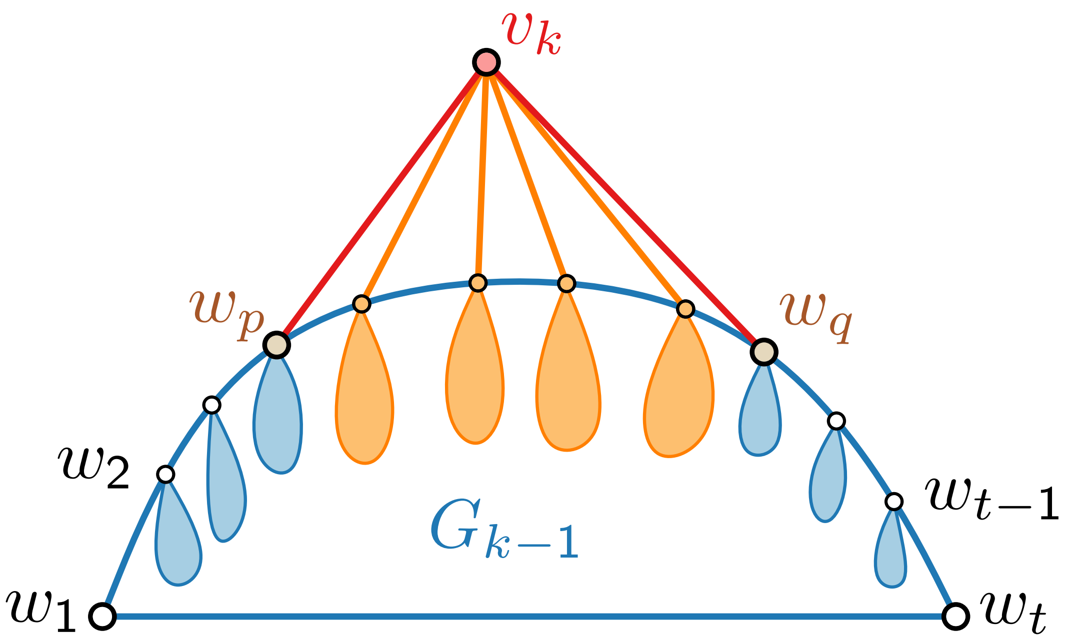
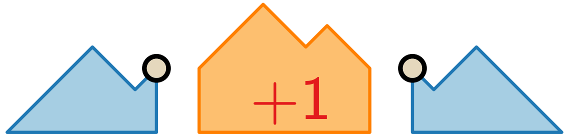
  for $\forall v \in \cup_{j=q}^{t} L(w_j)$ **do**





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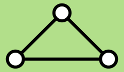


for $i = 4$ to n **do**

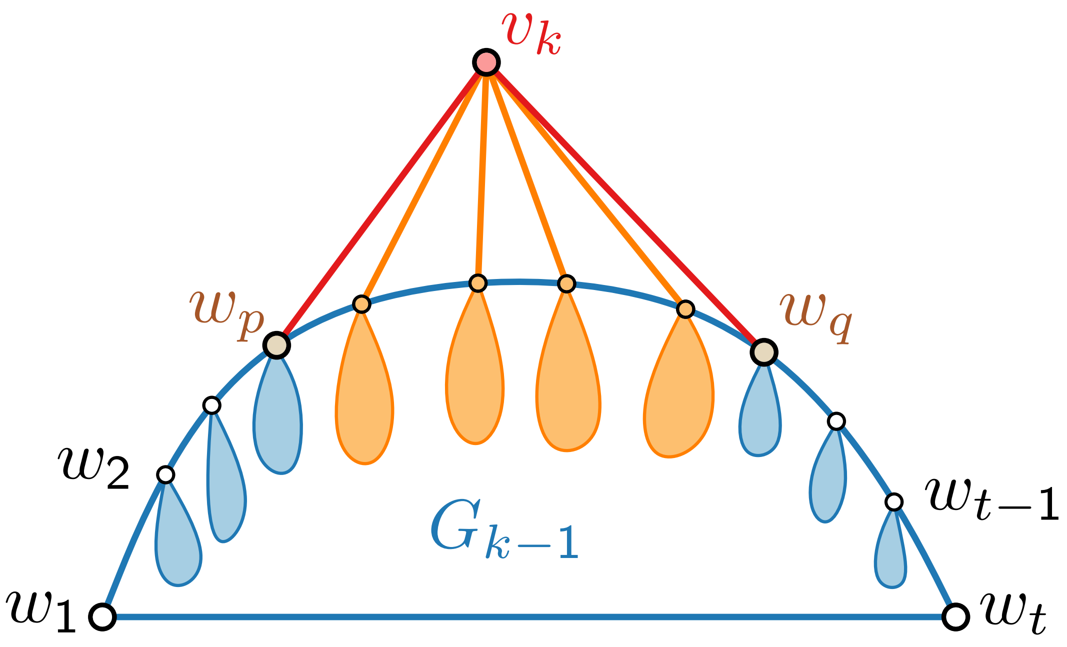
  Let $w_1 = v_1, w_2, \ldots, w_{t-1}, w_t = v_2$ denote the boundary of G_{i-1} and let $w_p, \ldots, w_q$ be the neighbours of v_i

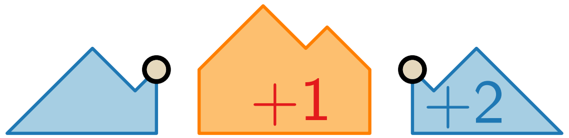
  for $\forall v \in \cup_{j=p+1}^{q-1} L(w_j)$ **do**

    $x(v) \leftarrow x(v) + 1$

  for $\forall v \in \cup_{j=q}^{t} L(w_j)$ **do**

    $x(v) \leftarrow x(v) + 2$





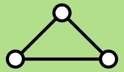
Shift Method – Pseudocode

Let $v_1, \ldots, v_n$ be a canonical order of G

for $i = 1$ to 3 **do**

  $L(v_i) \leftarrow \{v_i\}$

$P(v_1) \leftarrow (0,0); P(v_2) \leftarrow (2,0), P(v_3) \leftarrow (1,1)$



for $i = 4$ to n **do**

  Let $w_1 = v_1, w_2, \ldots, w_{t-1}, w_t = v_2$ denote the boundary of G_{i-1} and let $w_p, \ldots, w_q$ be the neighbours of v_i

  for $\forall v \in \cup_{j=p+1}^{q-1} L(w_j)$ **do**

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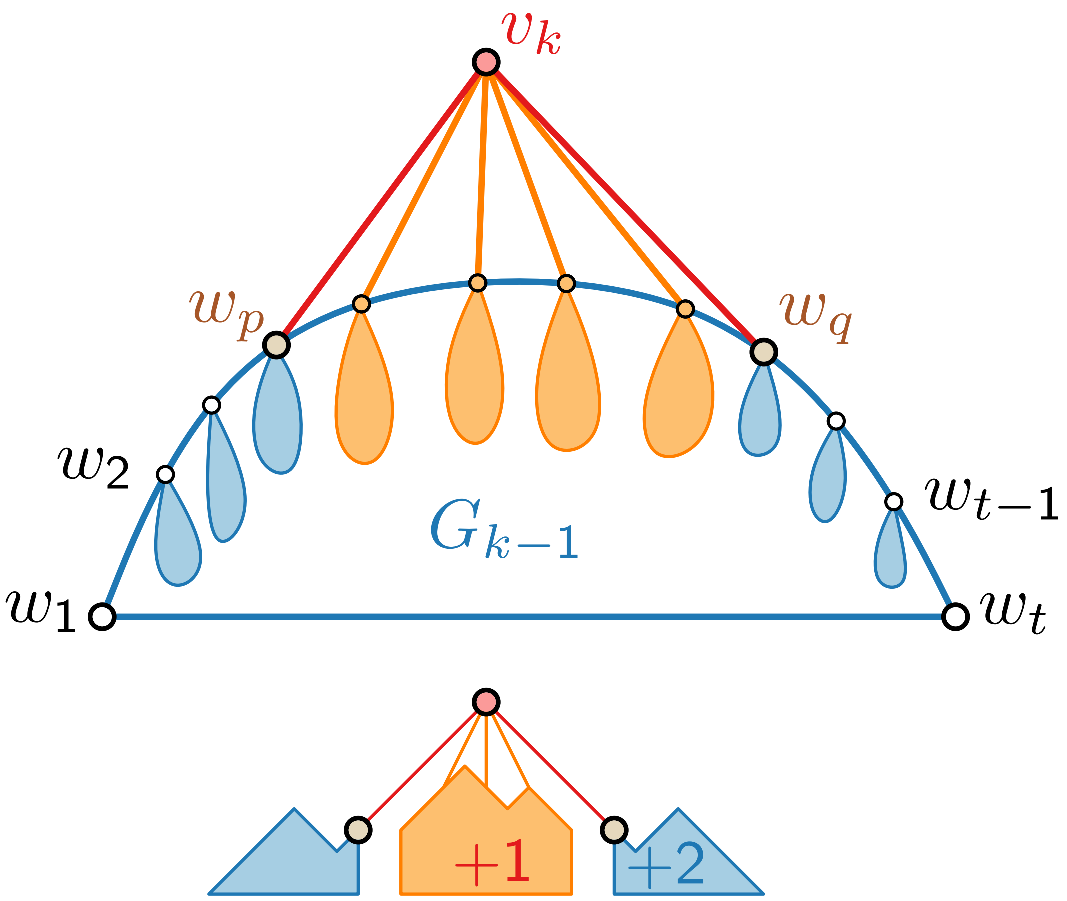
  $P(v_i) \leftarrow$ intersection of $+1/-1$ diagonals through $P(w_p)$ and $P(w_q)$



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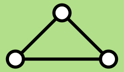

for $i = 4$ to n **do**

  Let $w_1 = v_1, w_2, \ldots, w_{t-1}, w_t = v_2$
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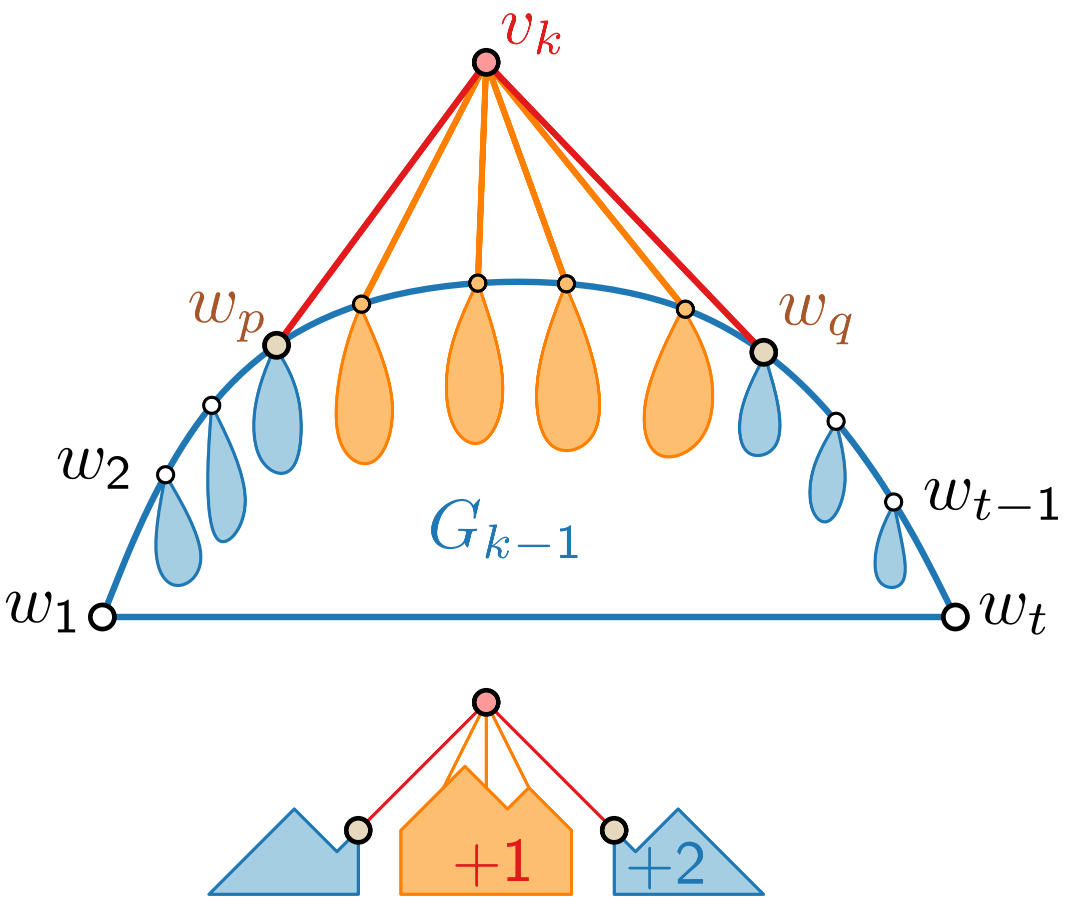
  $L(v_i) \leftarrow \cup_{j=p+1}^{q-1} L(w_j) \cup \{v_i\}$



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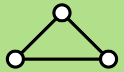
$P(v_1) \leftarrow (0,0); P(v_2) \leftarrow (2,0), P(v_3) \leftarrow (1,1)$ 

for $i = 4$ to n **do**
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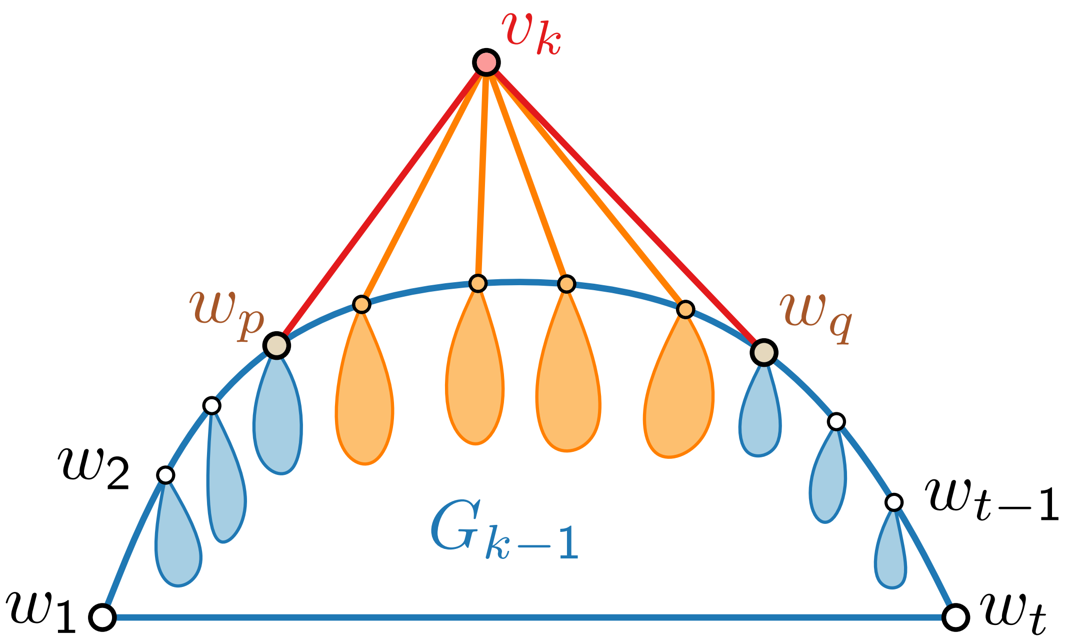
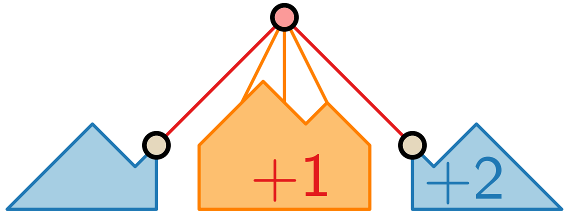




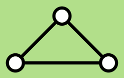
Running Time?

Shift Method – Pseudocode

Let $v_1, \ldots, v_n$ be a canonical order of G

for $i = 1$ to 3 **do**

  $L(v_i) \leftarrow \{v_i\}$

$P(v_1) \leftarrow (0,0)$; $P(v_2) \leftarrow (2,0)$, $P(v_3) \leftarrow (1,1)$ 

for $i = 4$ to n **do**

  Let $w_1 = v_1, w_2, \ldots, w_{t-1}, w_t = v_2$
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  for $\forall v \in \cup_{j=p+1}^{q-1} L(w_j)$ **do** // $\mathcal{O}(n^2)$ *in total*

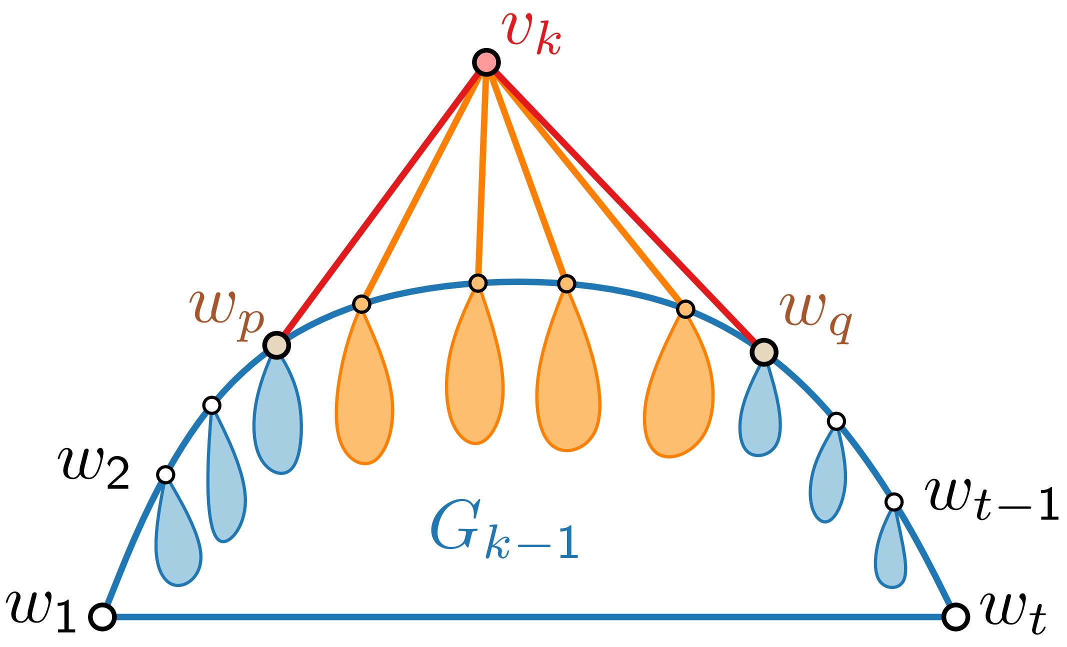
    $x(v) \leftarrow x(v) + 1$

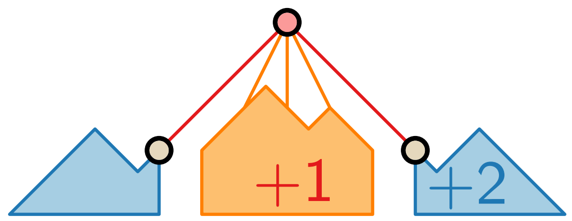
  for $\forall v \in \cup_{j=q}^{t} L(w_j)$ **do** // $\mathcal{O}(n^2)$ *in total*

    $x(v) \leftarrow x(v) + 2$

  $P(v_i) \leftarrow$ intersection of $+1/-1$ diagonals
  through $P(w_p)$ and $P(w_q)$

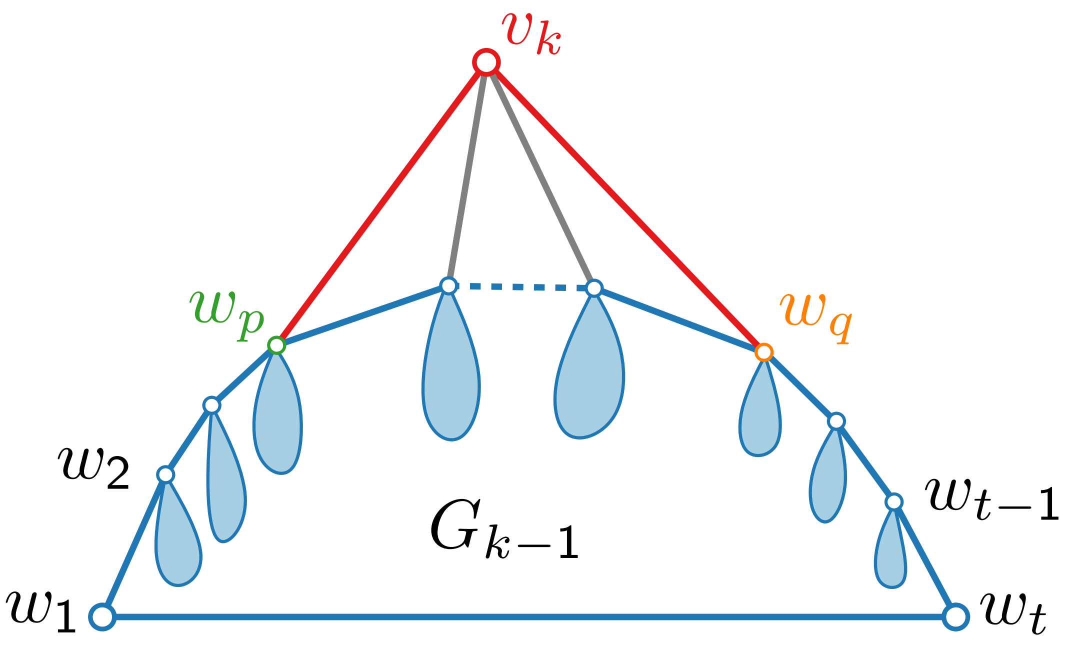
  $L(v_i) \leftarrow \cup_{j=p+1}^{q-1} L(w_j) \cup \{v_i\}$





Running Time?

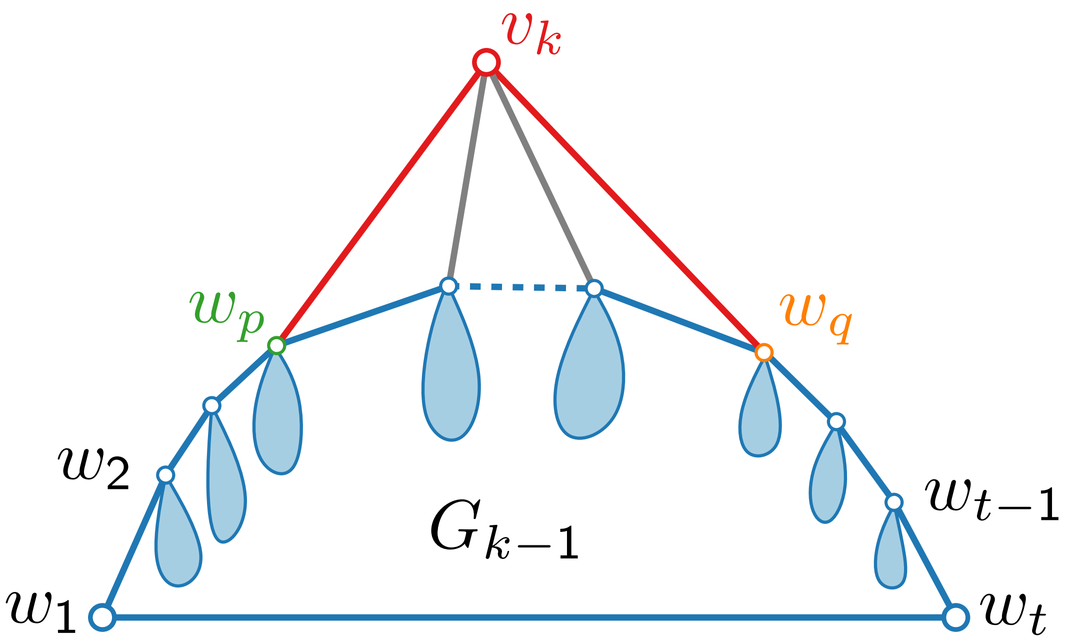
Shift Method – Linear Time Implementation



Shift Method – Linear Time Implementation

Idea 1.

To compute $x(v_k)$ & $y(v_k)$,
we only need $y(w_p)$ and $y(w_q)$ and $x(w_q) - x(w_p)$



Shift Method – Linear Time Implementation

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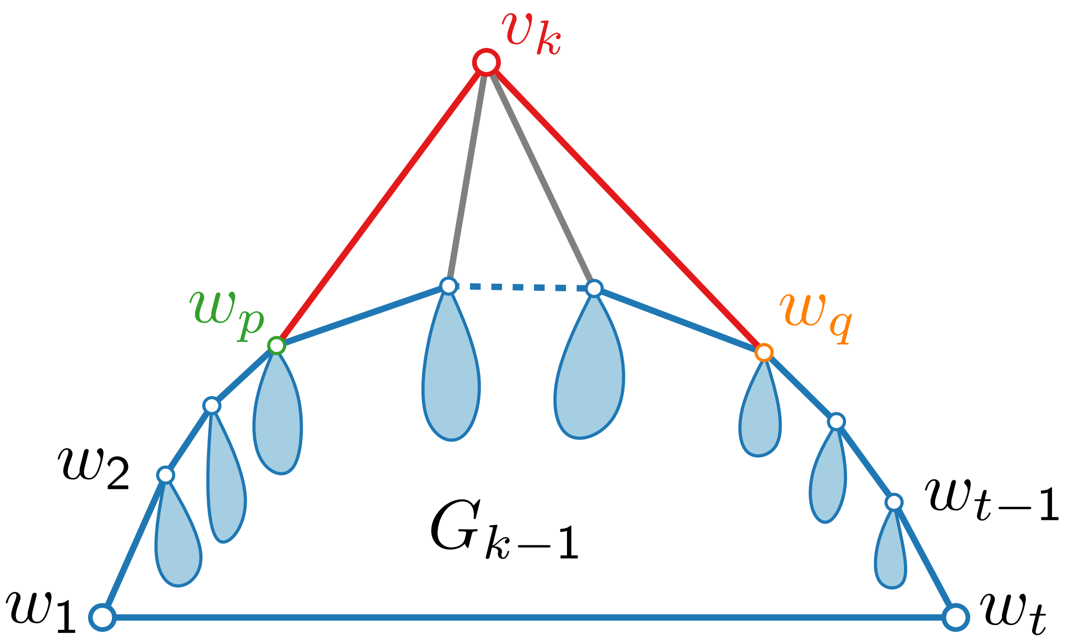
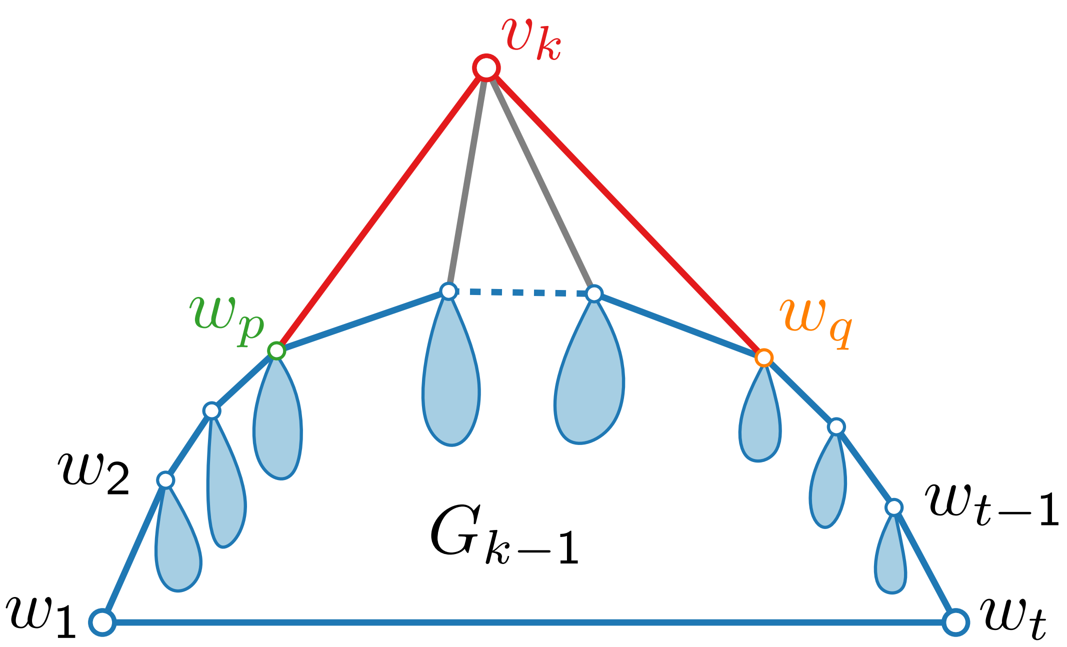


(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

Shift Method – Linear Time Implementation



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To compute $x(v_k)$ & $y(v_k)$,
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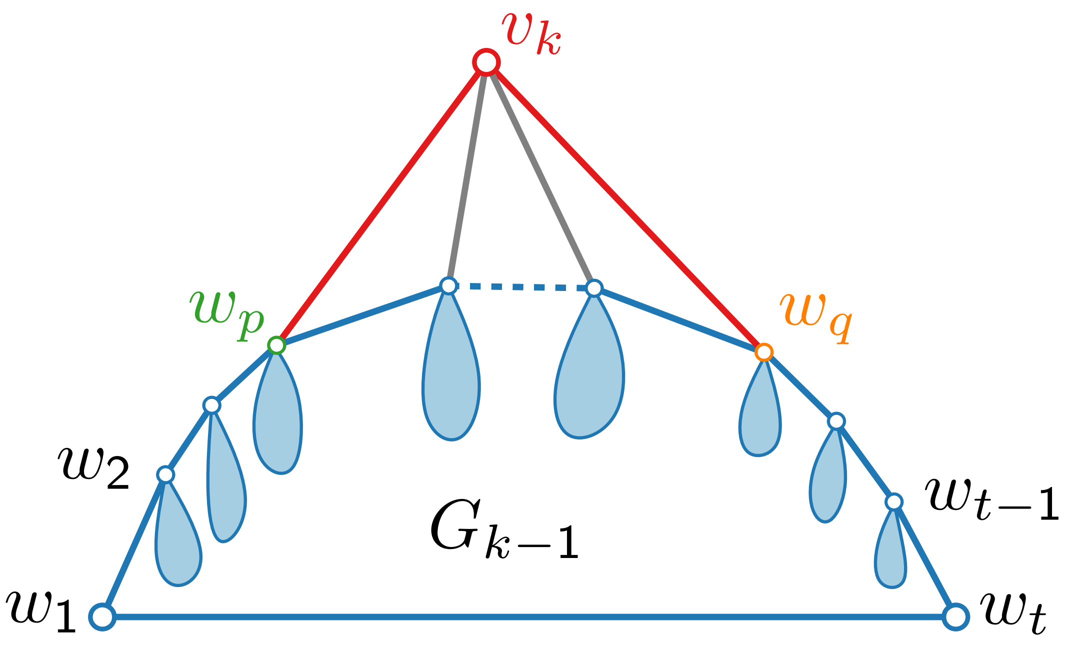
(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

(2) $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$

Shift Method – Linear Time Implementation

Idea 1.
To compute $x(v_k)$ & $y(v_k)$,
we only need $y(w_p)$ and $y(w_q)$ and $x(w_q) - x(w_p)$

Idea 2.
Instead of storing explicit x-coordinates,
we store x distances.



(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

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Shift Method – Linear Time Implementation

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To compute $x(v_k)$ & $y(v_k)$,
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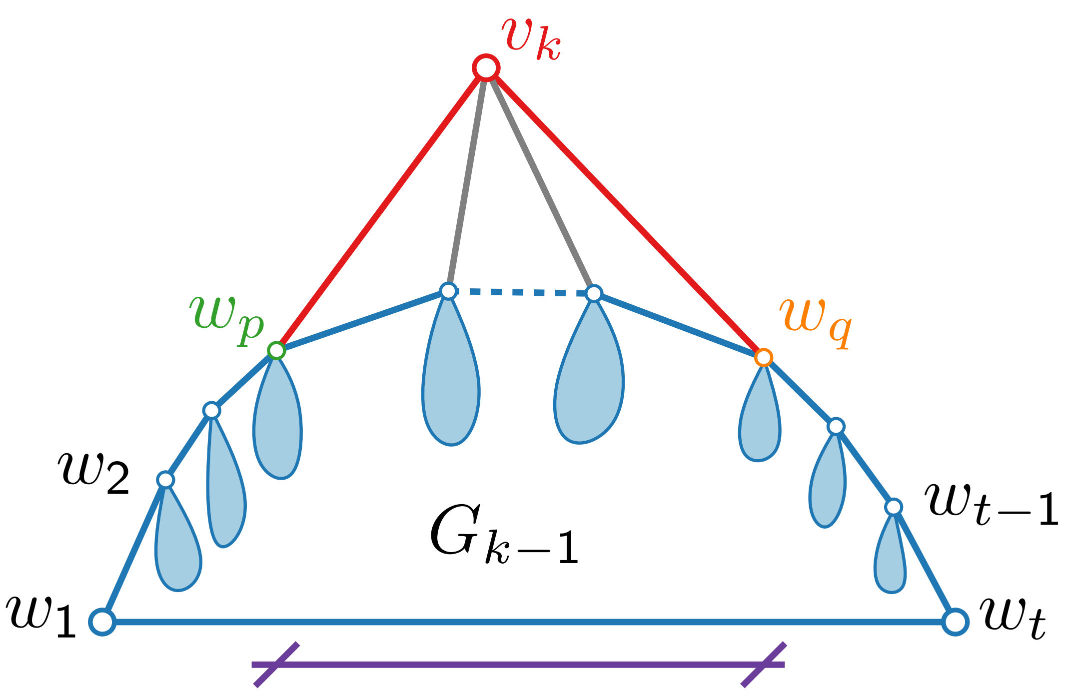

(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

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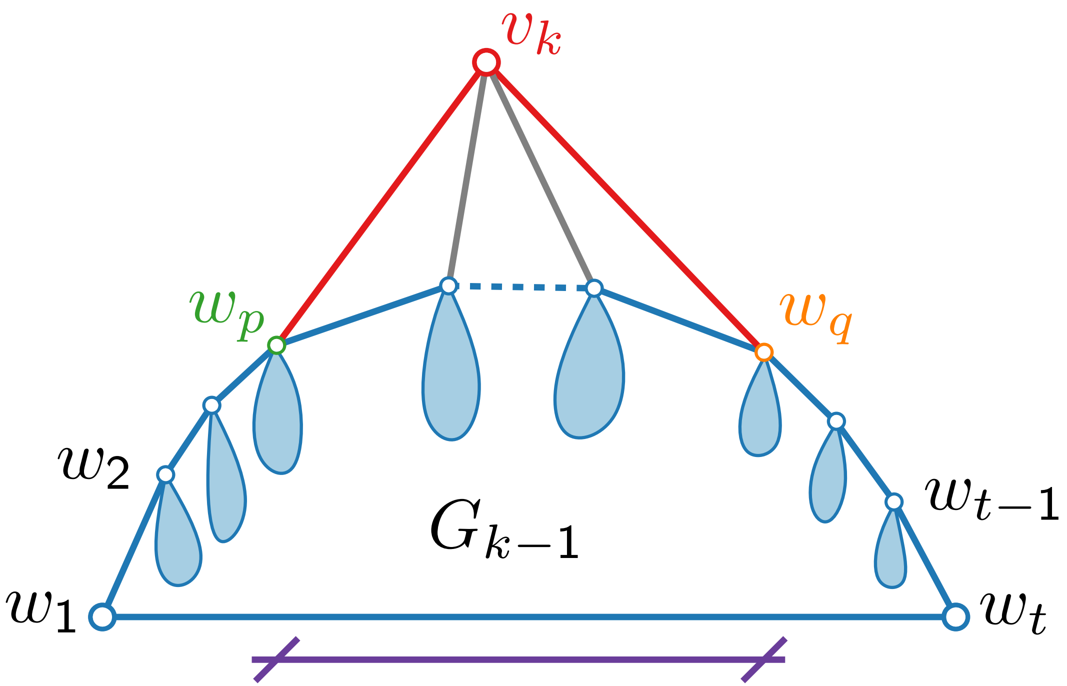
(3) $x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$

Shift Method – Linear Time Implementation

Idea 1.
To compute $x(v_k)$ & $y(v_k)$,
we only need $y(w_p)$ and $y(w_q)$ and $x(w_q) - x(w_p)$

Idea 2.
Instead of storing explicit x-coordinates,
we store x distances.

After x distance for v_n computed, use preorder traversal to compute all x-coordinates.



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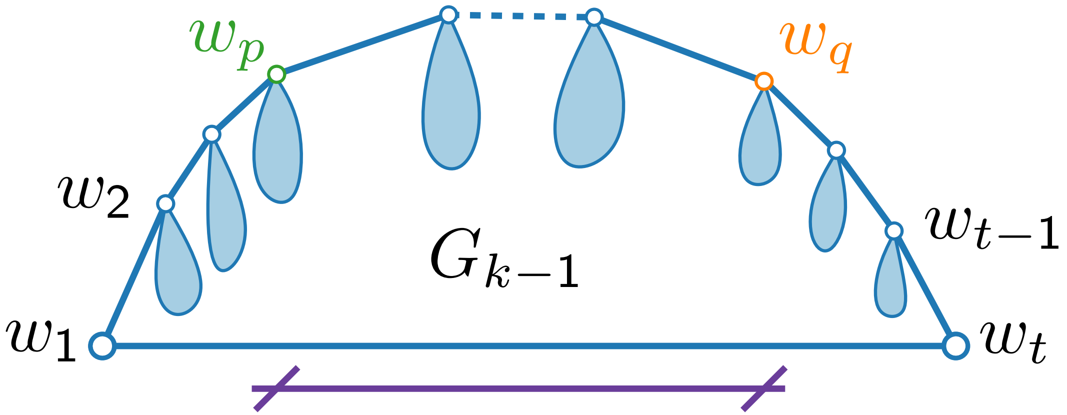
(3) $x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$

Shift Method – Linear Time Implementation

Relative x distance tree.

For each vertex v store

- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$



(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

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Shift Method – Linear Time Implementation

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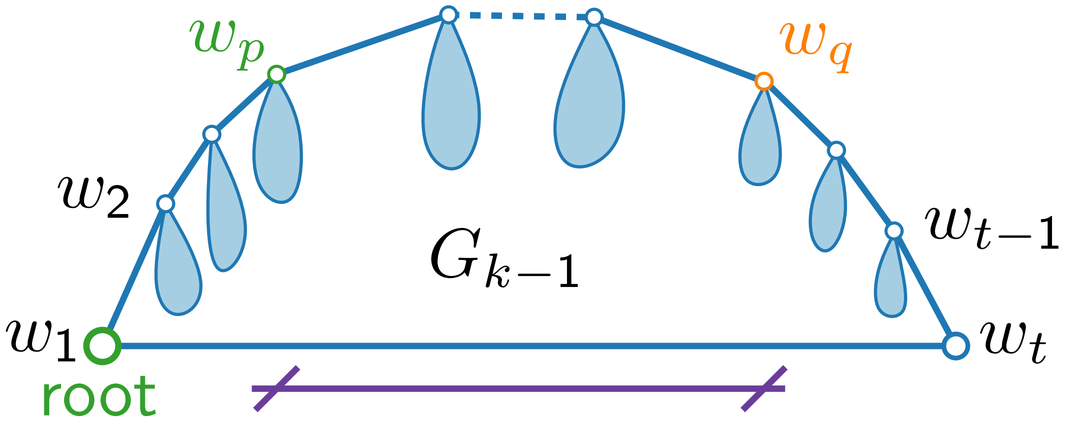

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Shift Method – Linear Time Implementation

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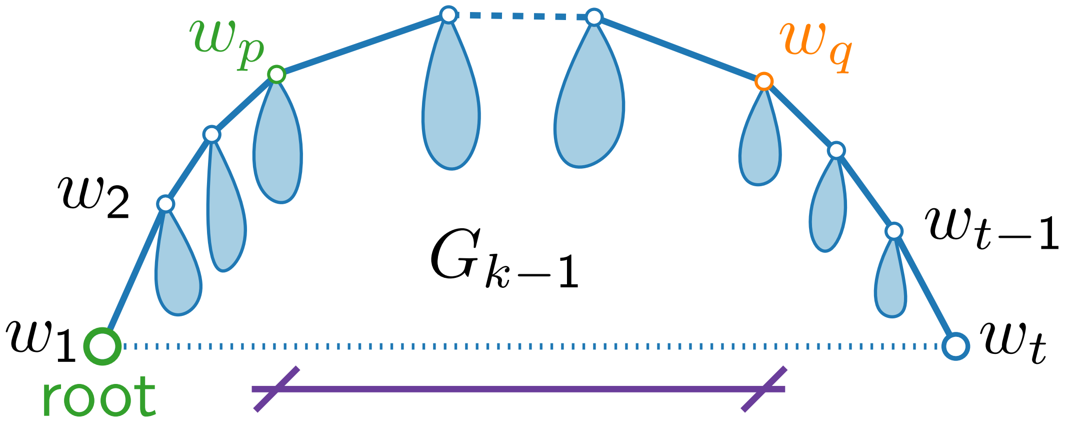

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Shift Method – Linear Time Implementation

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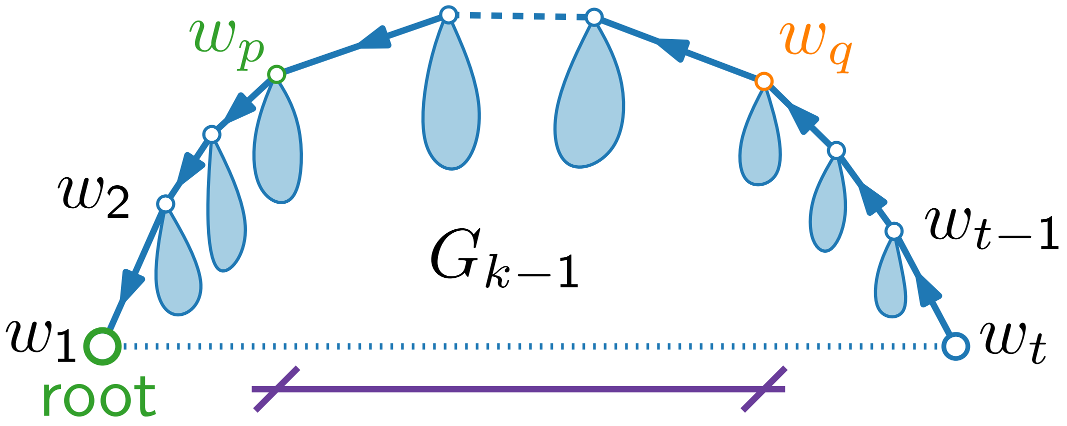

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Shift Method – Linear Time Implementation

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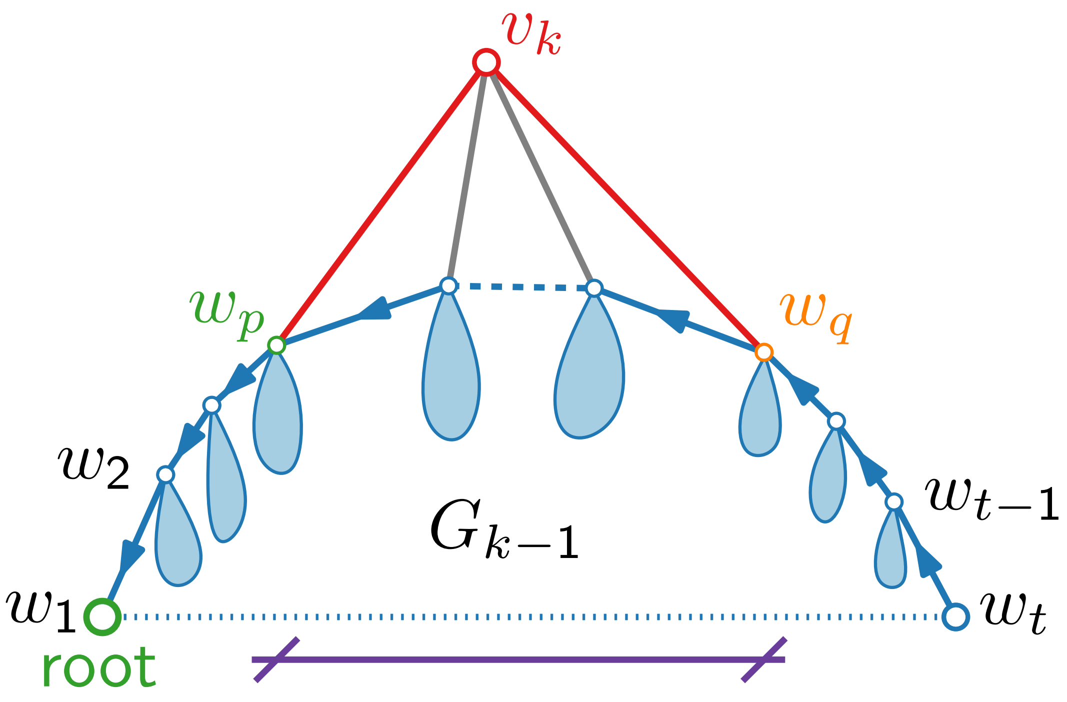

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Shift Method – Linear Time Implementation

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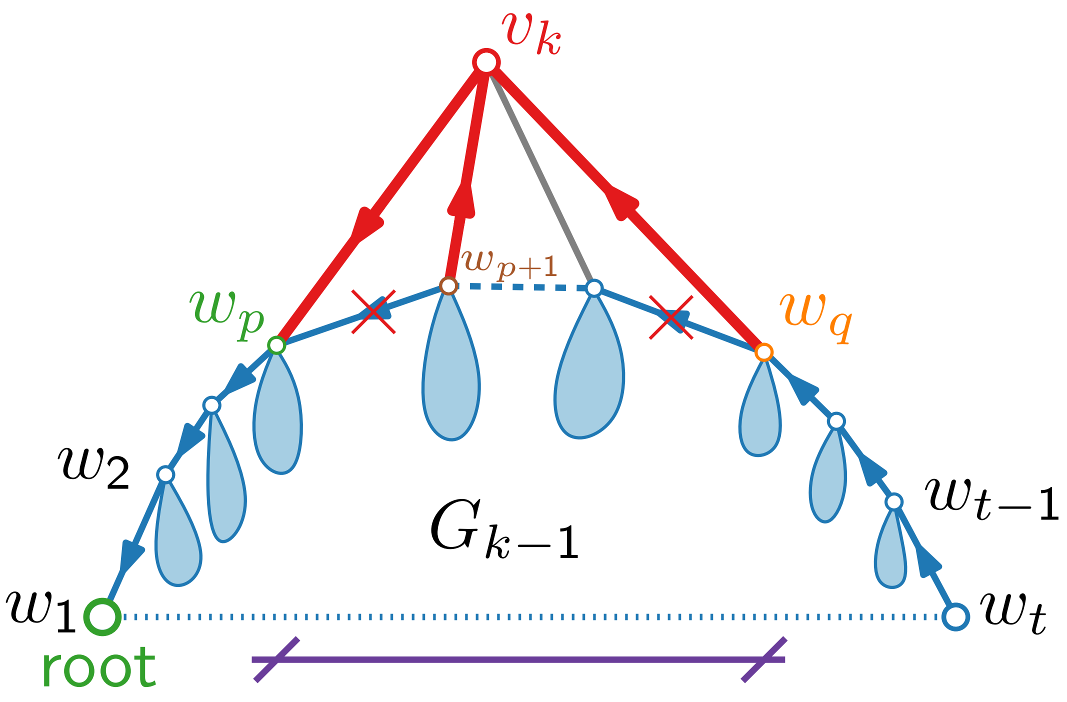

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Shift Method – Linear Time Implementation

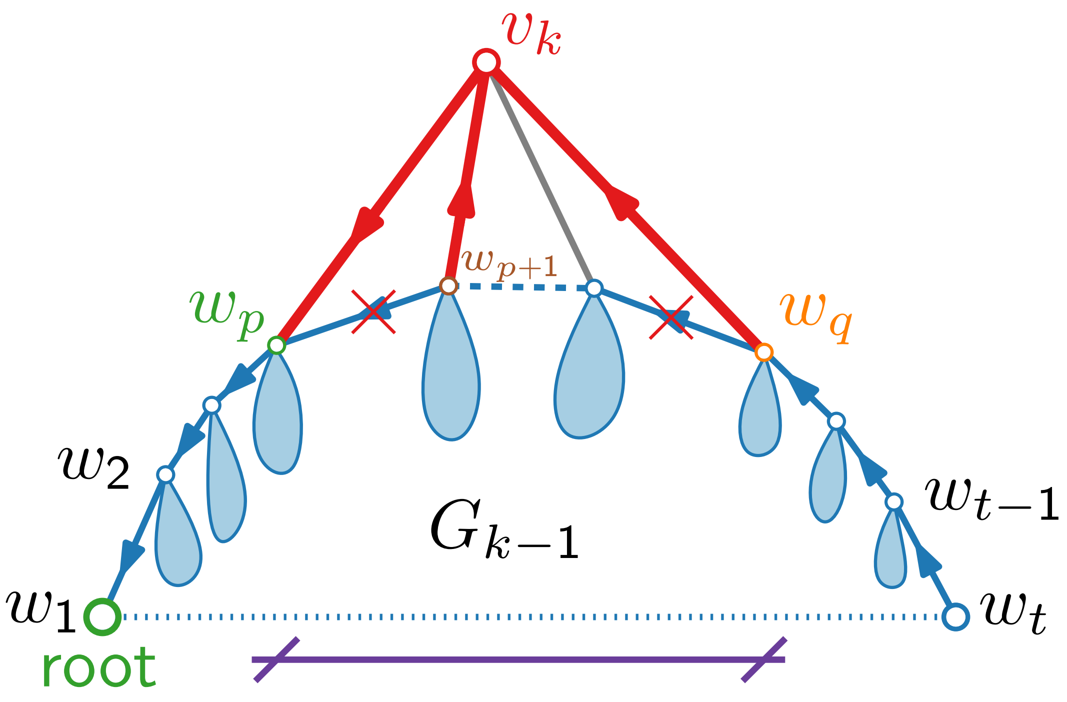
Relative x distance tree.

For each vertex v store

- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$

Calculations.

- $\Delta_x(w_{p+1})$++, $\Delta_x(w_q)$++



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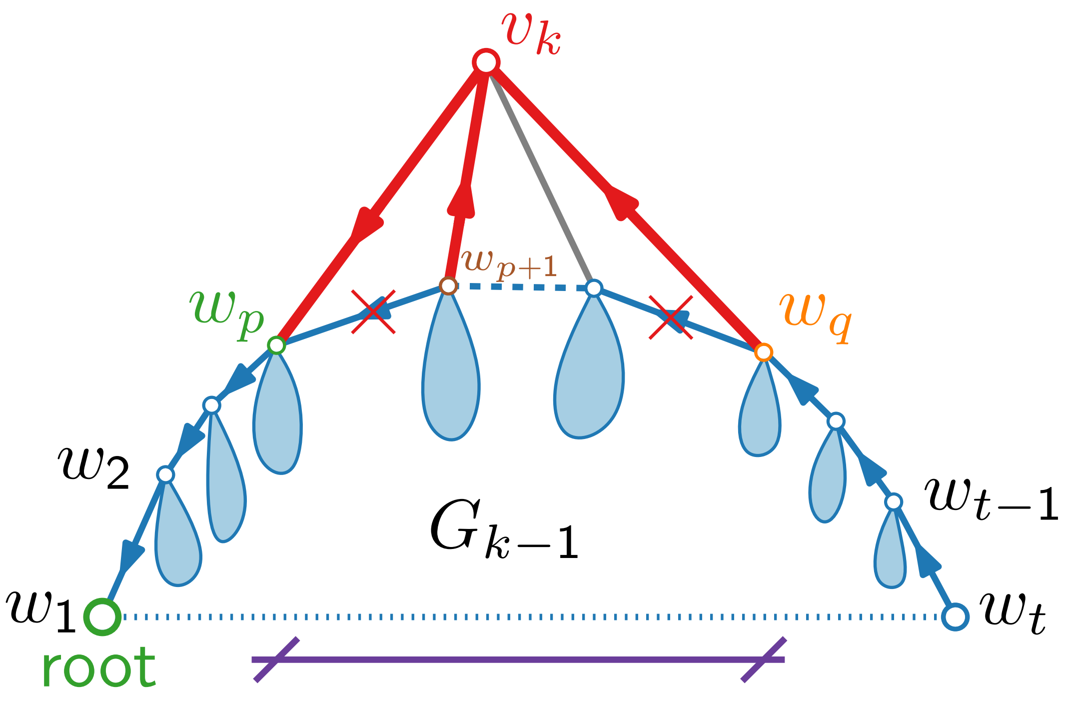
Shift Method – Linear Time Implementation



Relative x distance tree.

For each vertex v store

- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$

Calculations.

- $\Delta_x(w_{p+1})$++, $\Delta_x(w_q)$++
- $\Delta_x(w_p, w_q) = \Delta_x(w_{p+1}) + \ldots + \Delta_x(w_q)$

(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

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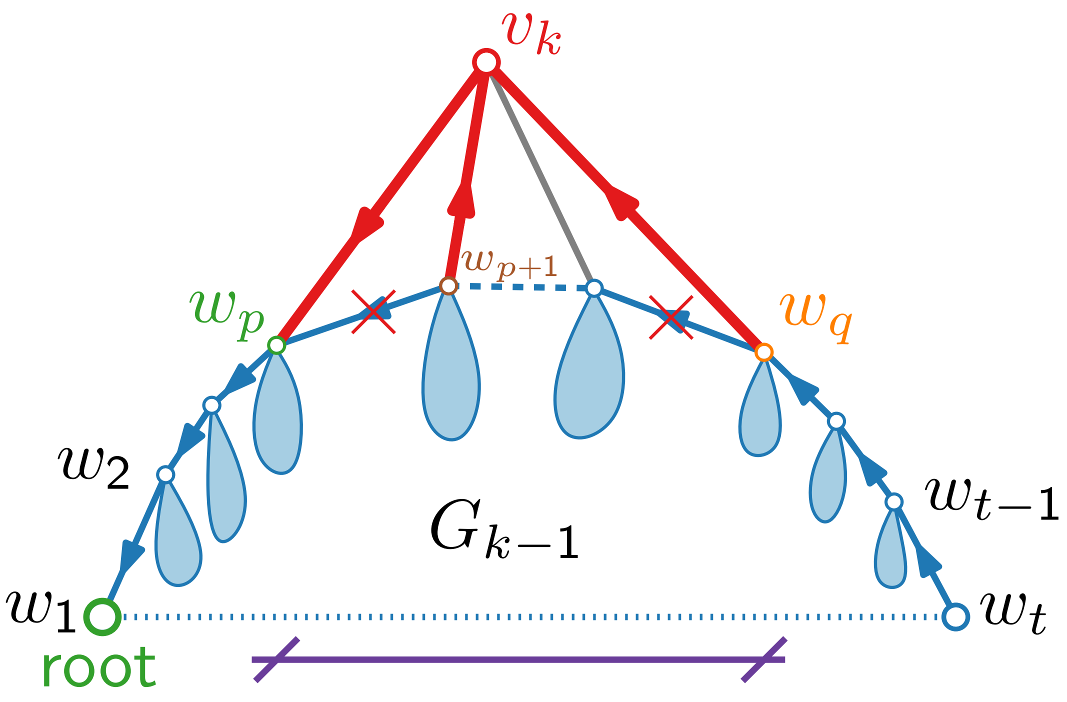
Shift Method – Linear Time Implementation



Relative x distance tree.

For each vertex v store

- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$

Calculations.

- $\Delta_x(w_{p+1})$++, $\Delta_x(w_q)$++
- $\Delta_x(w_p, w_q) = \Delta_x(w_{p+1}) + \ldots + \Delta_x(w_q)$
- $\Delta_x(v_k)$ by (3)

(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

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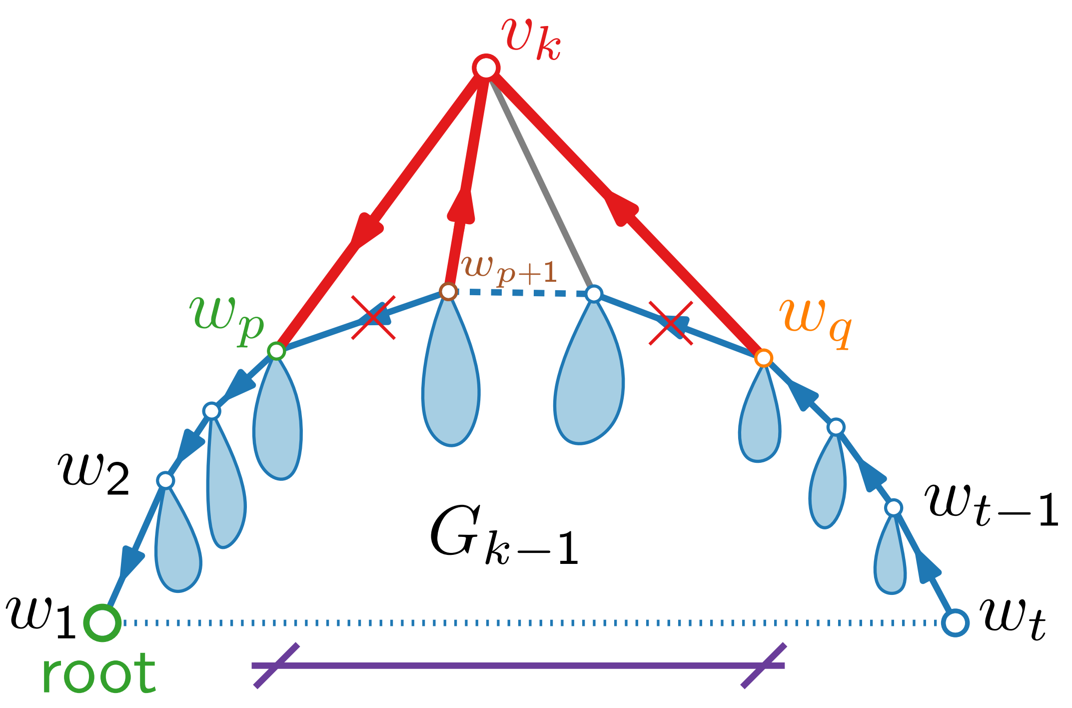
Shift Method – Linear Time Implementation



Relative x distance tree.

For each vertex v store

- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$

Calculations.

- $\Delta_x(w_{p+1})$++, $\Delta_x(w_q)$++
- $\Delta_x(w_p, w_q) = \Delta_x(w_{p+1}) + \ldots + \Delta_x(w_q)$
- $\Delta_x(v_k)$ by (3)
- $y(v_k)$ by (2)

(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

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Shift Method – Linear Time Implementation

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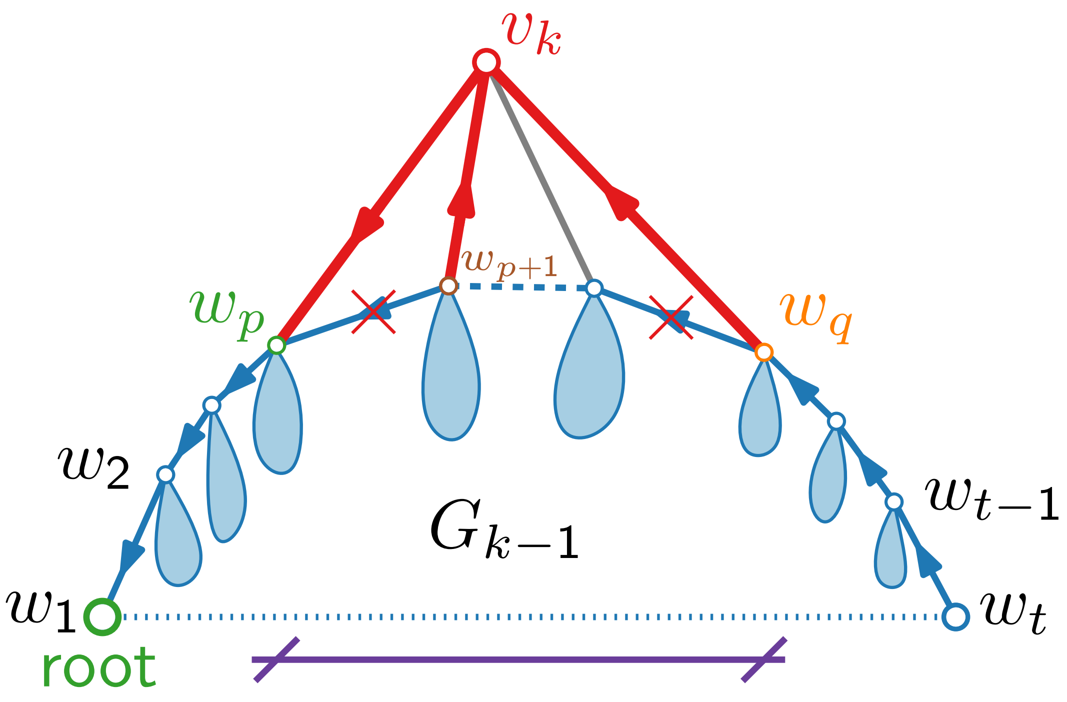
- $\Delta_x(w_{p+1})$++, $\Delta_x(w_q)$++
- $\Delta_x(w_p, w_q) = \Delta_x(w_{p+1}) + \ldots + \Delta_x(w_q)$
- $\Delta_x(v_k)$ by (3)
- $y(v_k)$ by (2)
- $\Delta_x(w_q) = \Delta_x(w_p, w_q) - \Delta_x(v_k)$



(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

(2) $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$

(3) $x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$

Shift Method – Linear Time Implementation

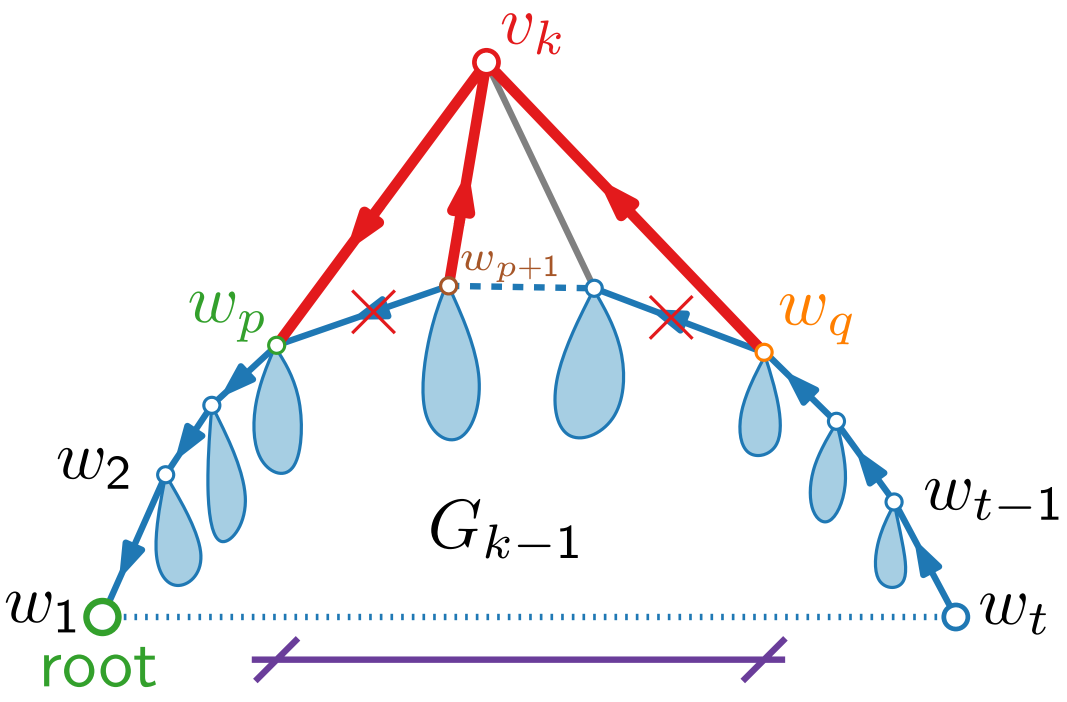
Relative x distance tree.

For each vertex v store

- x-offset $\Delta_x(v)$ from parent
- y-coordinate $y(v)$

Calculations.

- $\Delta_x(w_{p+1})$++, $\Delta_x(w_q)$++
- $\Delta_x(w_p, w_q) = \Delta_x(w_{p+1}) + \ldots + \Delta_x(w_q)$
- $\Delta_x(v_k)$ by (3)
- $y(v_k)$ by (2)
- $\Delta_x(w_q) = \Delta_x(w_p, w_q) - \Delta_x(v_k)$
- $\Delta_x(w_{p+1}) = \Delta_x(w_{p+1}) - \Delta_x(v_k)$



(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

(2) $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$

(3) $x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$

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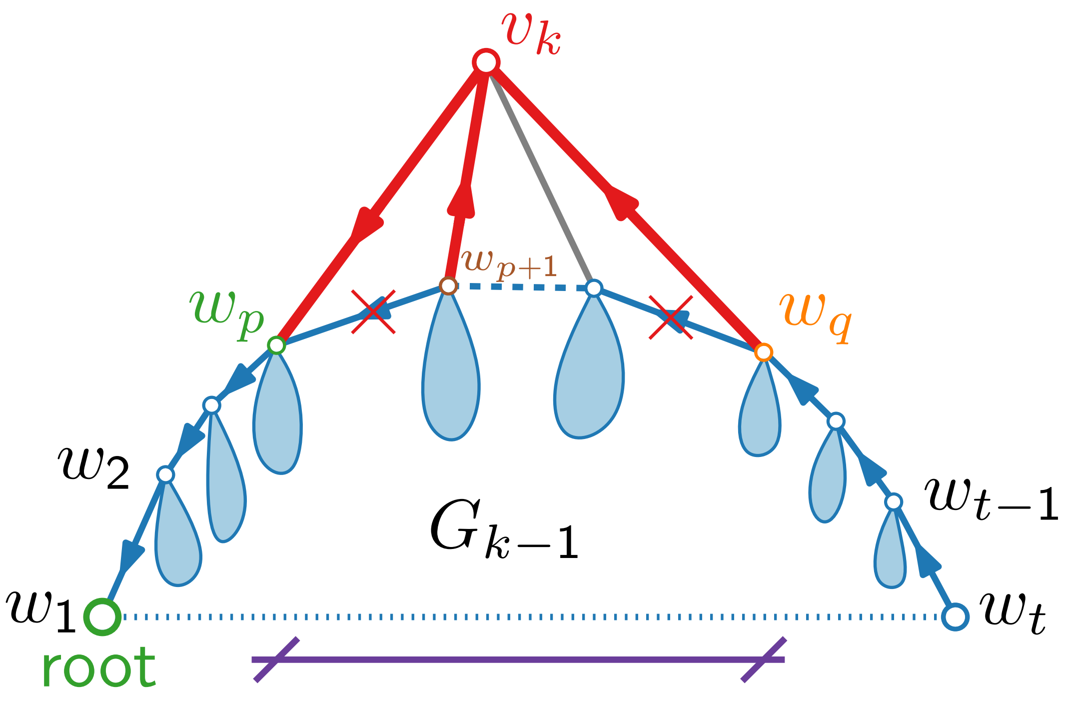
- $\Delta_x(v_k)$ by (3)
- $y(v_k)$ by (2)
- $\Delta_x(w_q) = \Delta_x(w_p, w_q) - \Delta_x(v_k)$
- $\Delta_x(w_{p+1}) = \Delta_x(w_{p+1}) - \Delta_x(v_k)$

$\mathcal{O}(n)$ in total



(1) $x(v_k) = \frac{1}{2}(x(w_q) + x(w_p) + y(w_q) - y(w_p))$

(2) $y(v_k) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) + y(w_p))$

(3) $x(v_k) - x(w_p) = \frac{1}{2}(x(w_q) - x(w_p) + y(w_q) - y(w_p))$

Literature

- [PGD Ch. 4.2] for detailed explanation of shift method
- [de Fraysseix, Pach, Pollack 1990] “How to draw a planar graph on a grid”
  – original paper on shift method