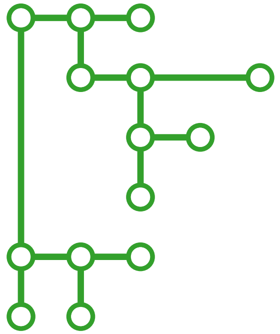
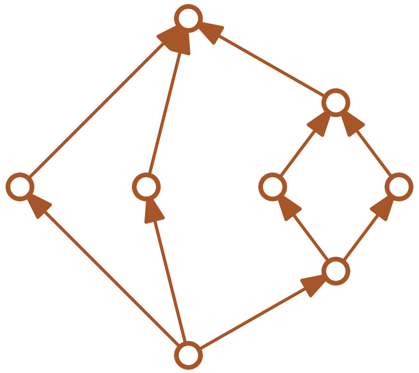
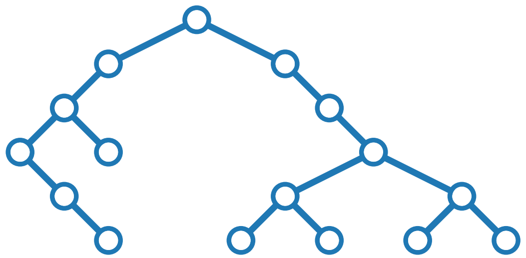
Visualization of Graphs

Lecture 1b:
Drawing Trees and Series-Parallel Graphs

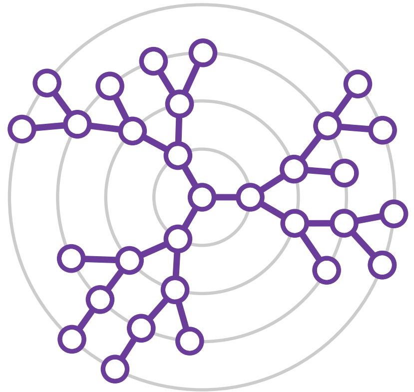




Part I:
Layered Drawings



Jonathan Klawitter



(Rooted) Trees

Leaf: Vertex of degree 1

Rooted tree: tree with designated **root**

Ancestor: Vertex on path to root

Parent: Neighbor on path to root

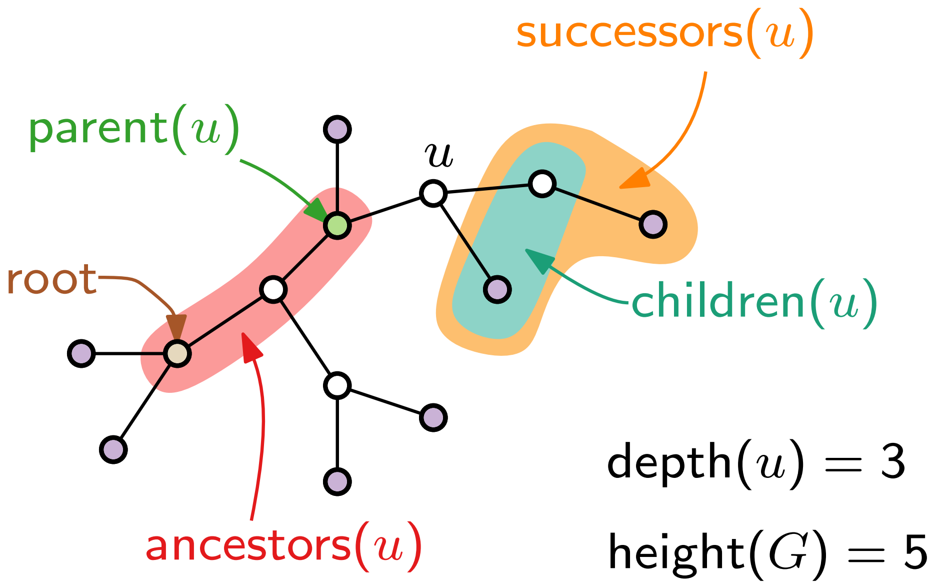
Successor: Vertex on path away from root

Child: Neighbor not on path to root

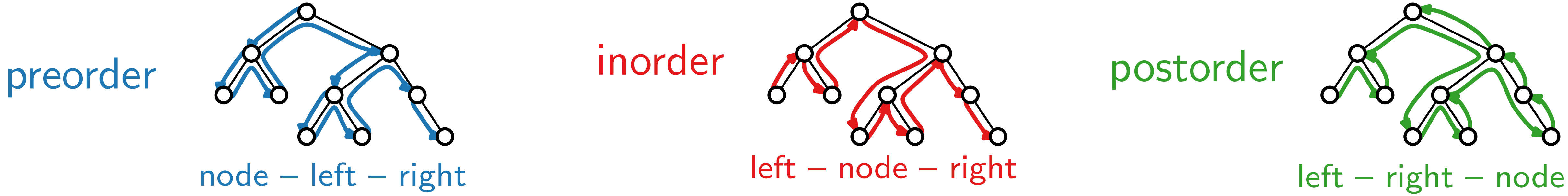
Depth: Length of path to root

Height: Maximum depth of a leaf

Binary Tree: At most two children per vertex (left / right child)

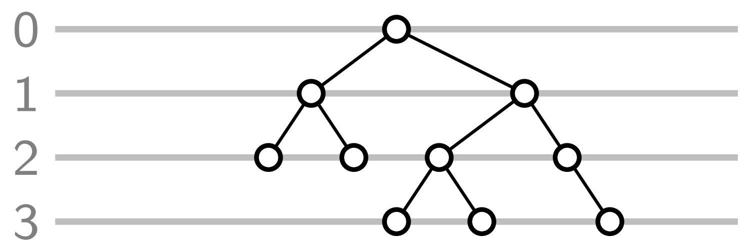


3 traversals:

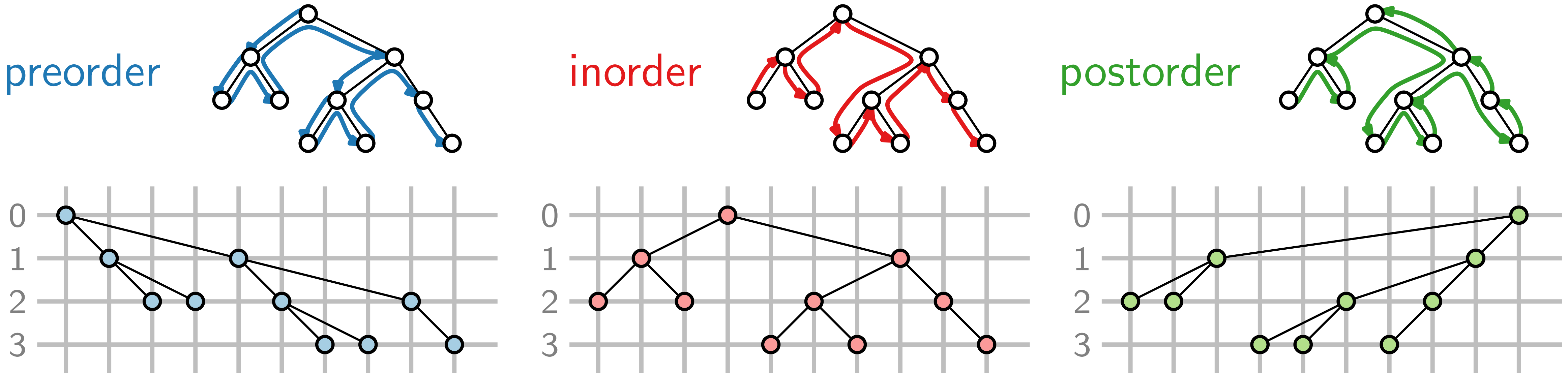


First Grid Layout of Binary Trees

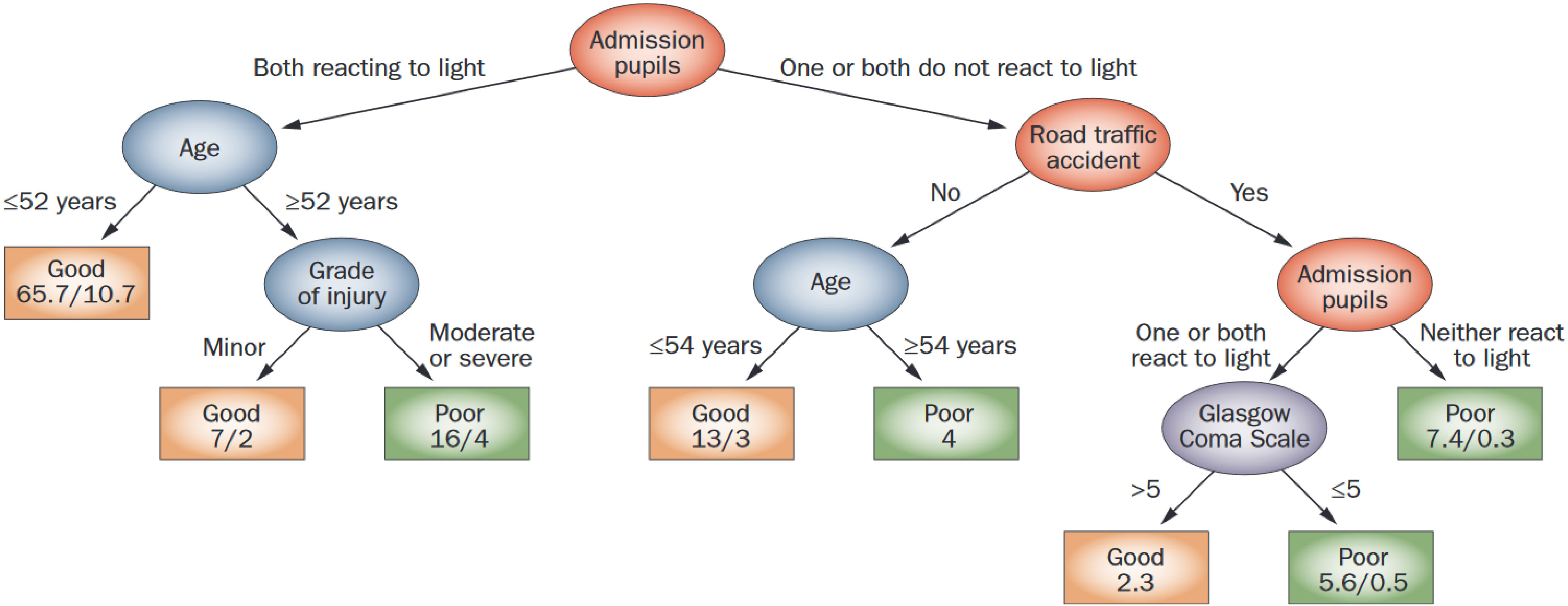
1. Choose y-coordinates: $y(u) =$depth(u)



2. Choose x-coordinates:



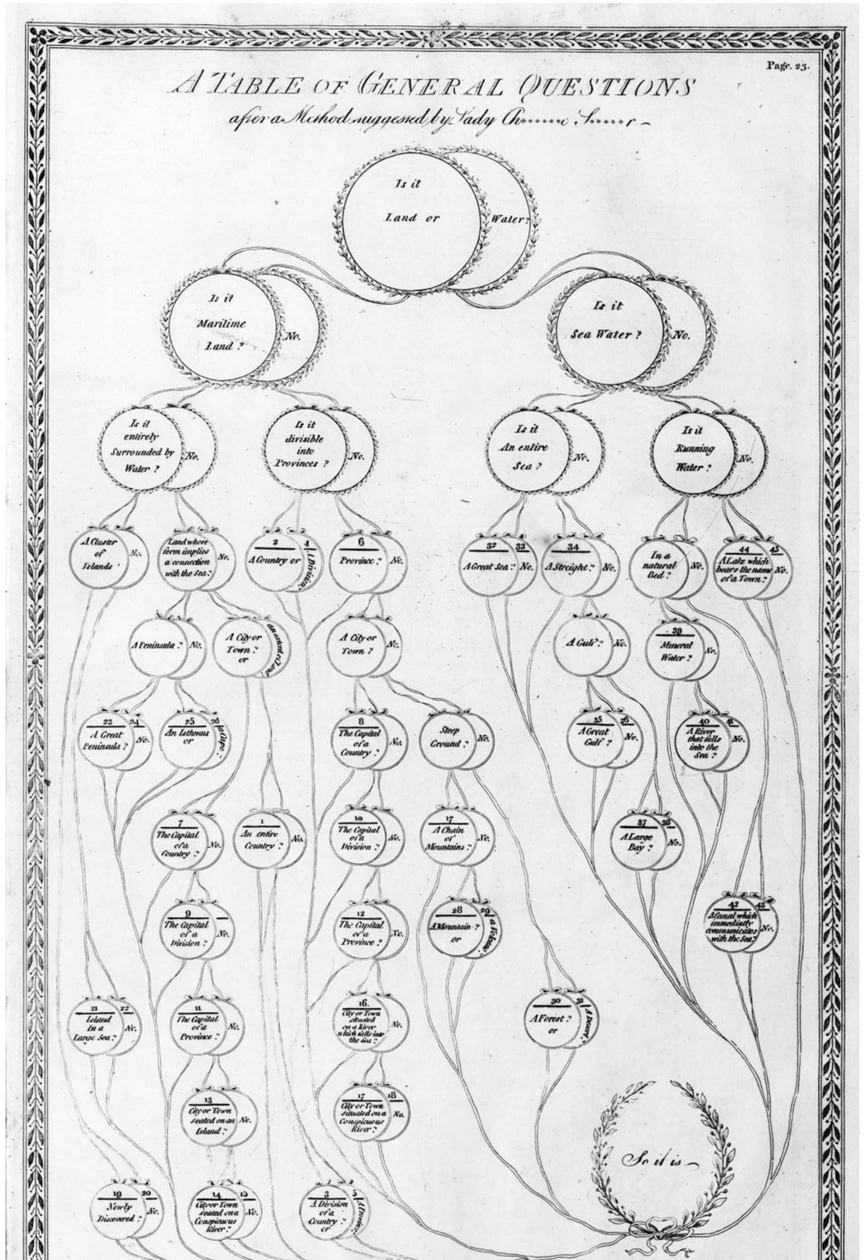
Layered Drawings – Applications



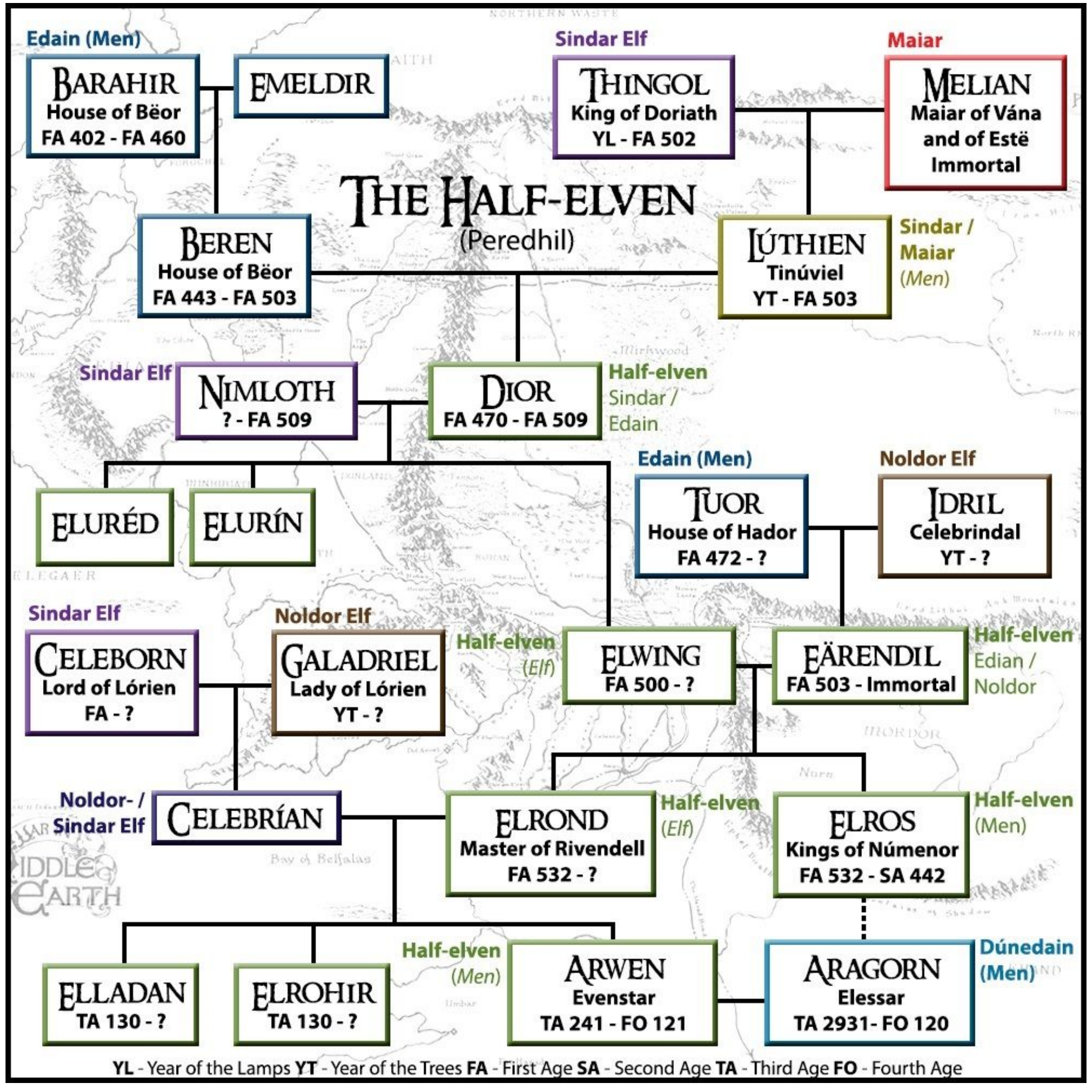
Decision tree for outcome prediction after traumatic brain injury

Source: Nature Reviews Neurology

Layered Drawings – Applications

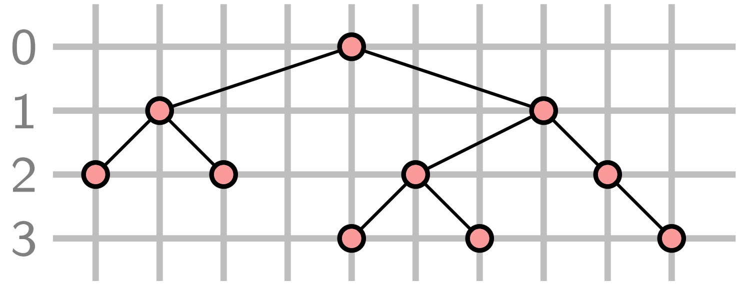


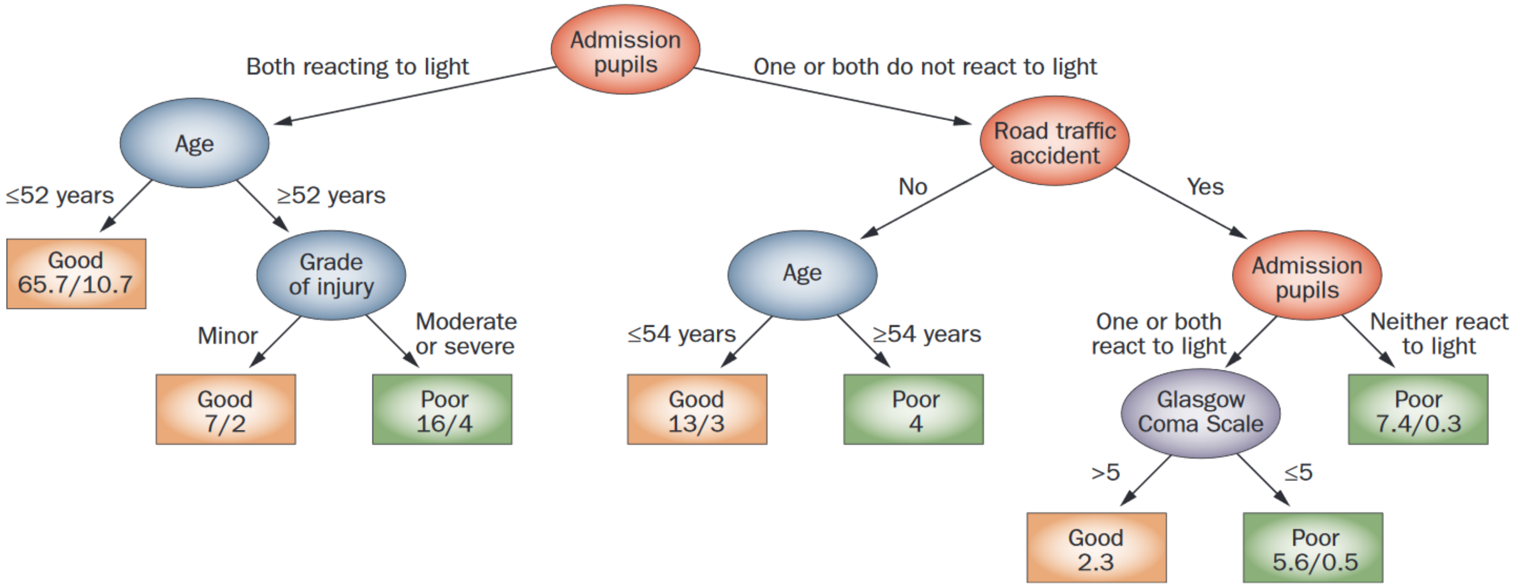
Aloisius Gaultier 1821



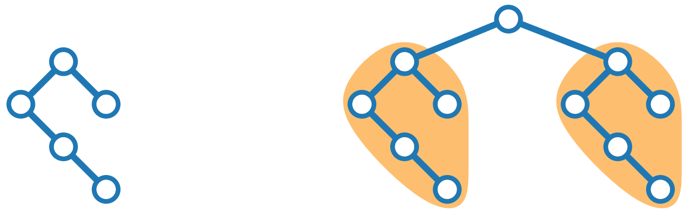
Family tree of LOTR elves and half-elves

Layered Drawings – Drawing Style





- What are properties of the layout?
- What are the drawing conventions?
- What are aesthetics to optimize?



Drawing conventions

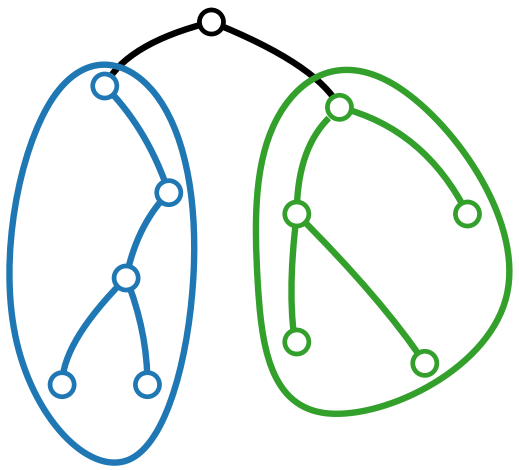
- Vertices lie on layers and have integer coordinates
- Parent centered above children
- Edges are straight-line segments
- Isomorphic subtrees have identical drawings

Drawing aesthetics

- Area
- Symmetries

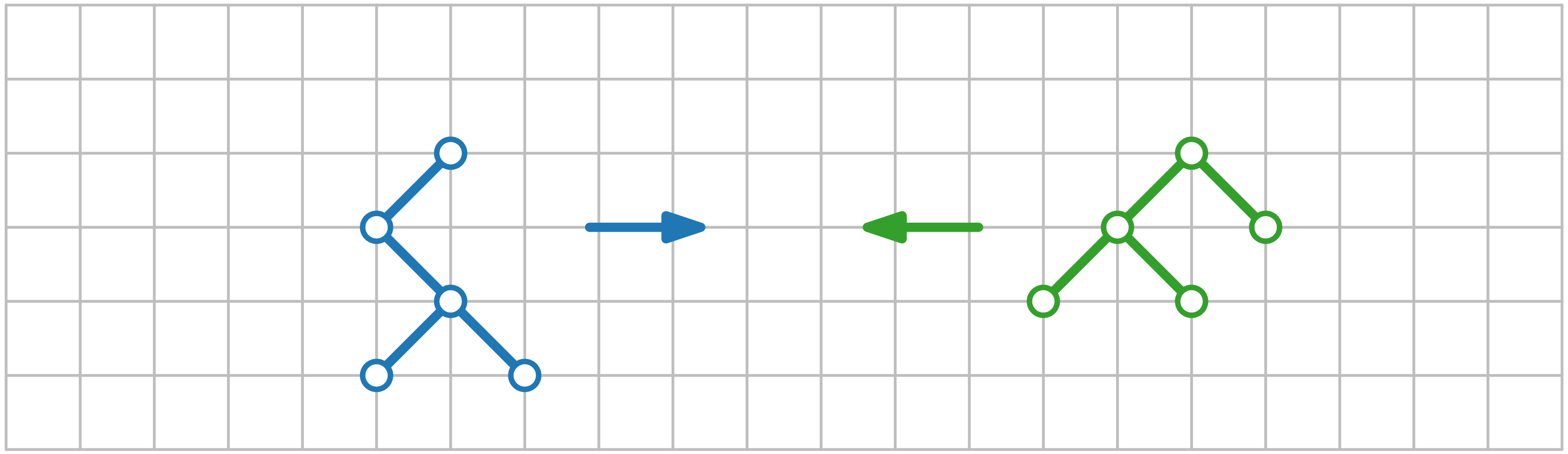
Layered Drawings – Algorithm

Input: A binary tree T
Output: A layered drawing of T



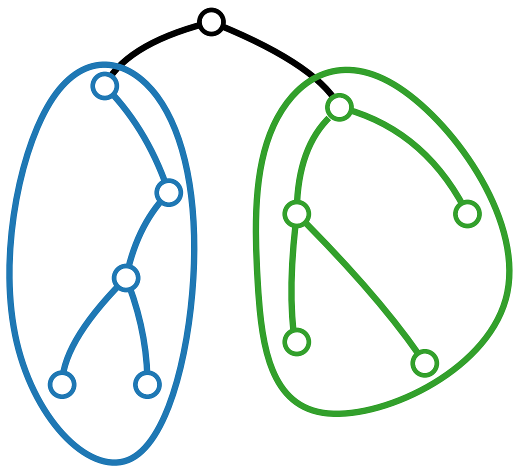
Base case: A single vertex 
Divide: Recursively apply the algorithm to draw the left and right subtrees

Conquer:



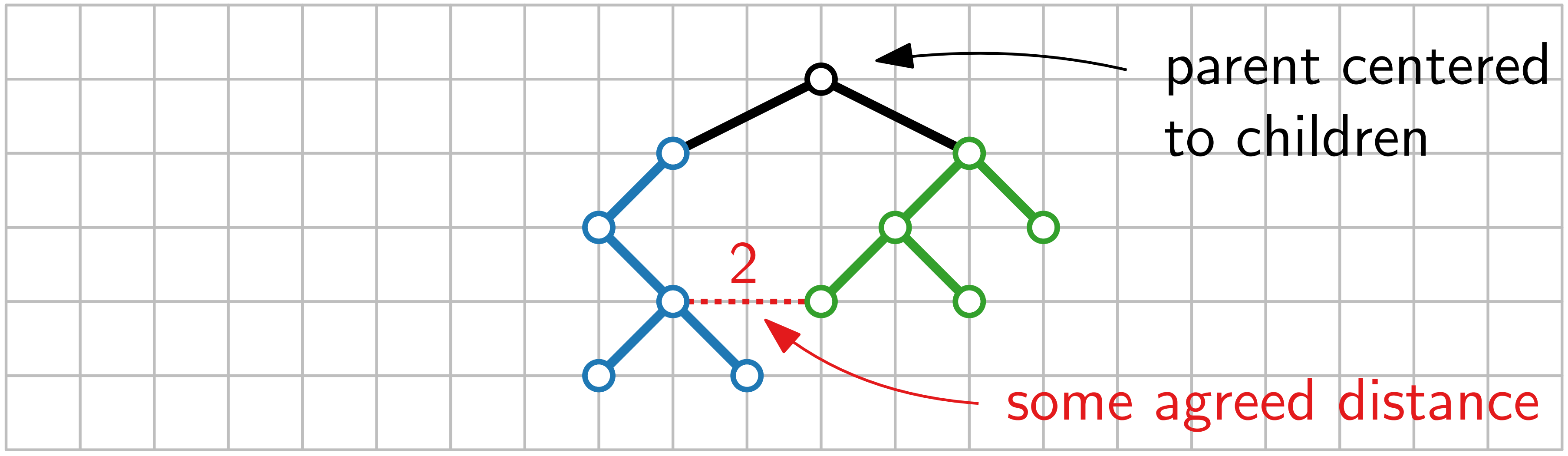
Layered Drawings – Algorithm

Input: A binary tree T
Output: A layered drawing of T



Base case: A single vertex
Divide: Recursively apply the algorithm to draw the left and right subtrees

Conquer:



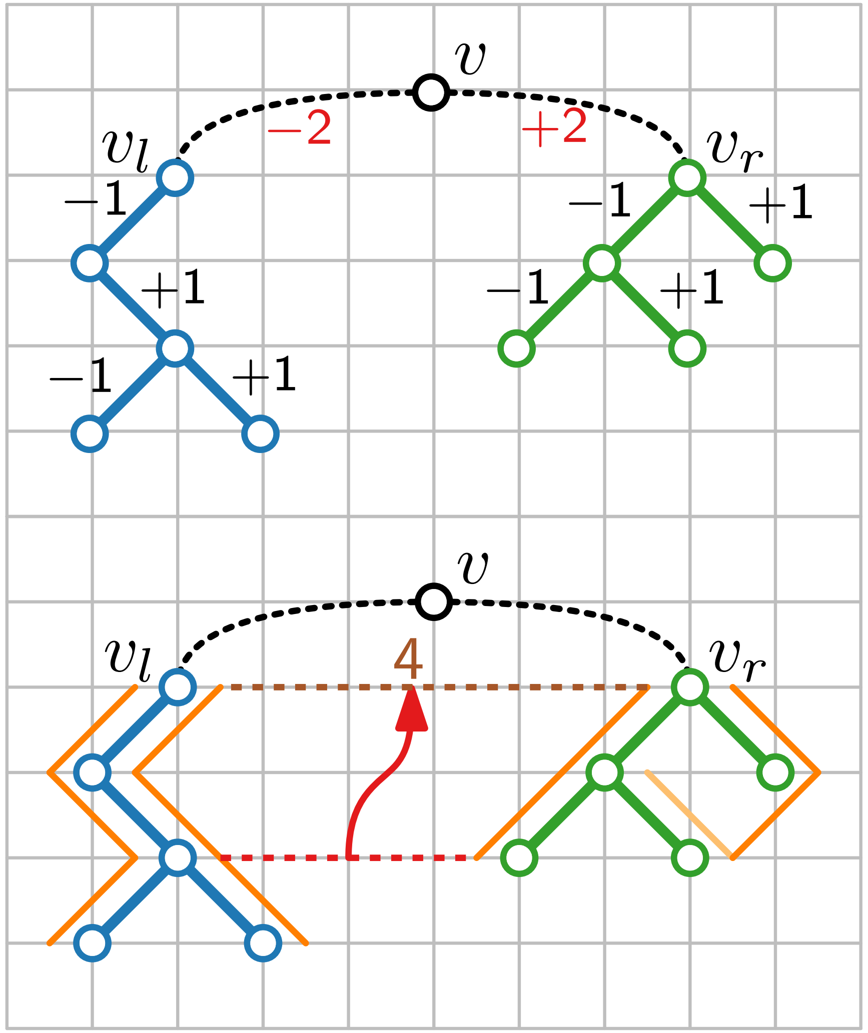
Layered Drawings – Algorithm Details

Phase 1 – postorder traversal:

- For each vertex compute horizontal displacement of left and right child
- x-offset$(v_l) = -\lceil \frac{d_v}{2} \rceil$, x-offset$(v_r) = \lceil \frac{d_v}{2} \rceil$
- At vertex u (below v) store left and right contour of subtree $T(u)$
- Contour is linked list of vertex coordinates/offsets
- Find d_v = min. horiz. distance between v_l and v_r

Phase 2 – preorder traversal:

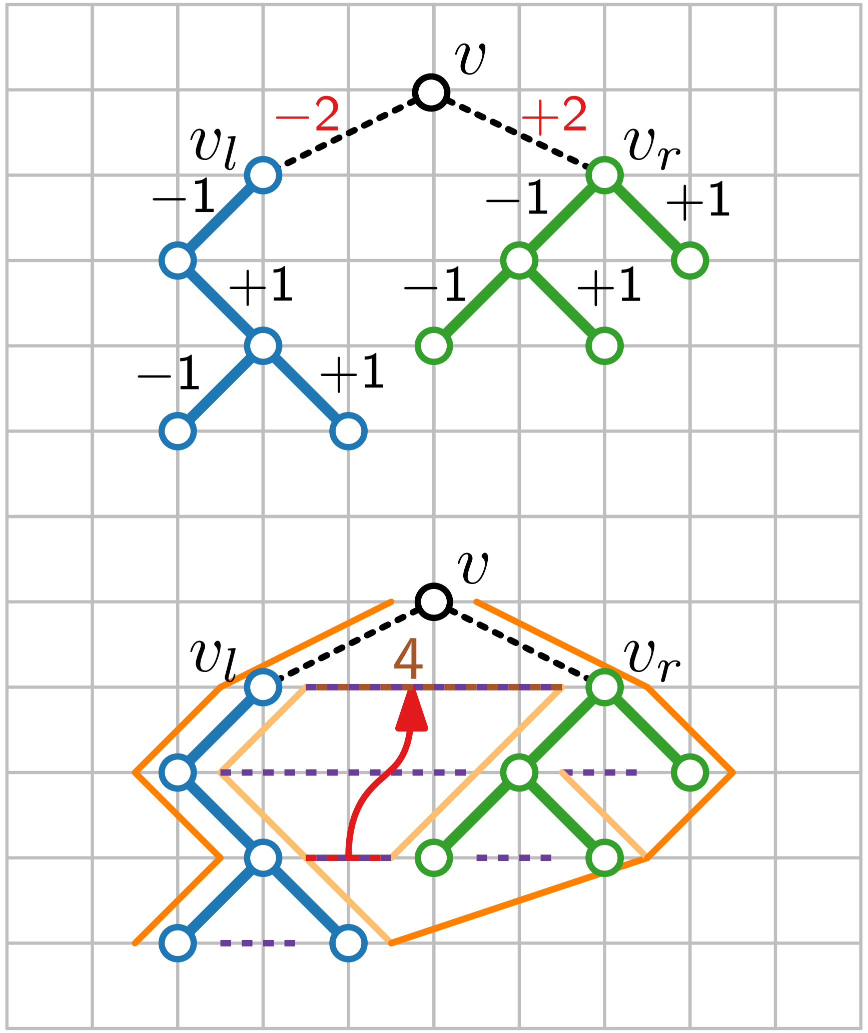
- Compute x- and y-coordinates



Layered Drawings – Algorithm Details

Phase 1 – postorder traversal:

- For each vertex compute horizontal displacement of left and right child
- x-offset$(v_l) = -\lceil \frac{d_v}{2} \rceil$, x-offset$(v_r) = \lceil \frac{d_v}{2} \rceil$
- At vertex u (below v) store left and right contour of subtree $T(u)$
- Contour is linked list of vertex coordinates/offsets
- Find d_v = min. horiz. distance between v_l and v_r



Phase 2 – preorder traversal:

- Compute x- and y-coordinates

Runtime?

- How often do we have to walk along a contour?

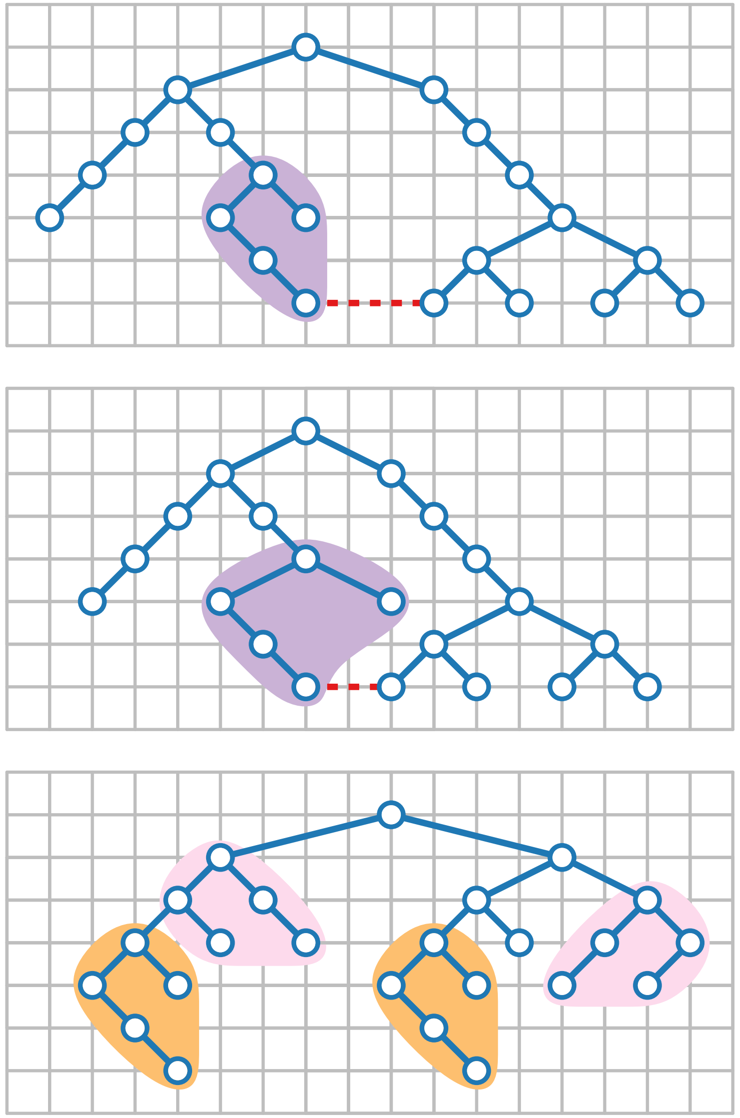
$\Rightarrow \mathcal{O}(n)$

Layered Drawings – Result

Theorem. [Reingold & Tilford '81]
Let T be a binary tree with n vertices. We can construct a drawing Γ of T in $\mathcal{O}(n)$ time, such that:

- Γ is planar, straight-line and strictly downward
- Γ is layered: y-coordinate of vertex v is $-\text{depth}(v)$
- Horizontal and Vertical distances are at least 1
- Each vertex is centred wrt its children
- Area of Γ is in $\mathcal{O}(n^2)$ – but not optimal!
- Simply isomorphic subtrees have congruent drawings, up to translation
- Axially isomorphic subtrees have congruent drawings, up to translation and reflection

NP-hard

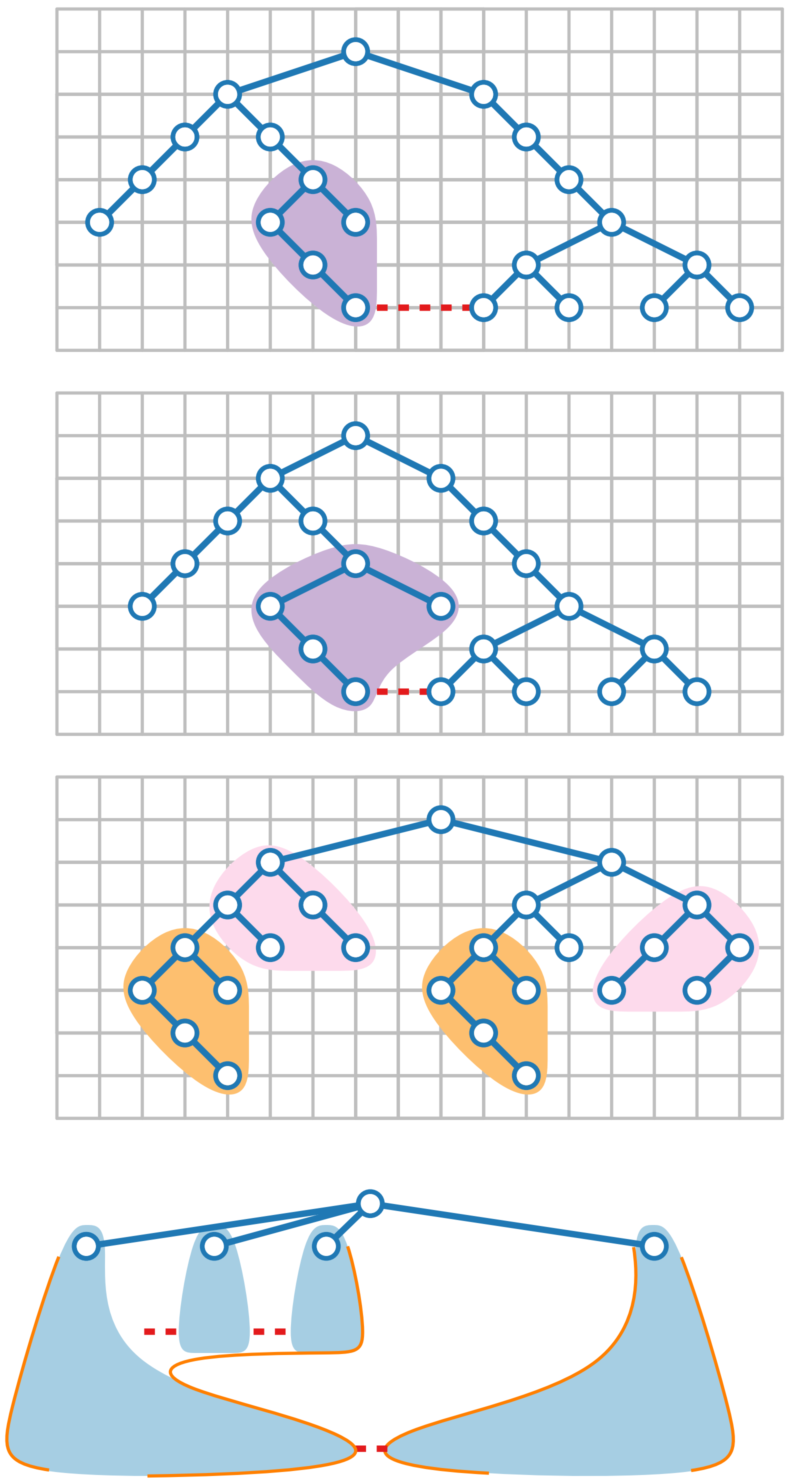


Layered Drawings – Result

Theorem. [Reingold & Tilford '81]

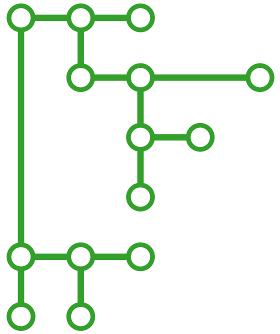
Let T be a ~~binary~~ rooted tree with n vertices. We can construct a drawing Γ of T in $\mathcal{O}(n)$ time, such that:

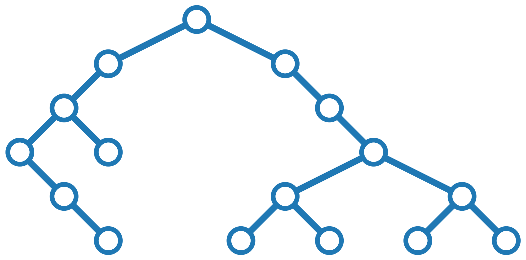
- Γ is planar, straight-line and strictly downward
- Γ is layered: y-coordinate of vertex v is $-$depth(v)
- Horizontal and Vertical distances are at least 1
- Each vertex is centred wrt its children
- Area of Γ is in $\mathcal{O}(n^2)$ – but not optimal! NP-hard
- Simply isomorphic subtrees have congruent drawings, up to translation
- ~~Axially isomorphic subtrees have congruent drawings, up to translation and reflection~~



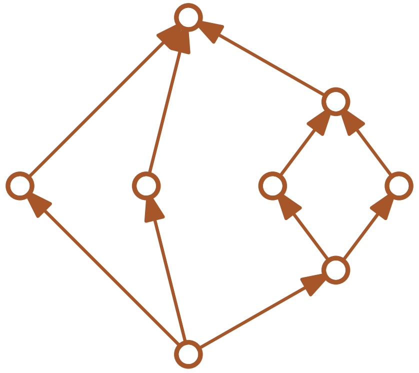
Visualization of Graphs

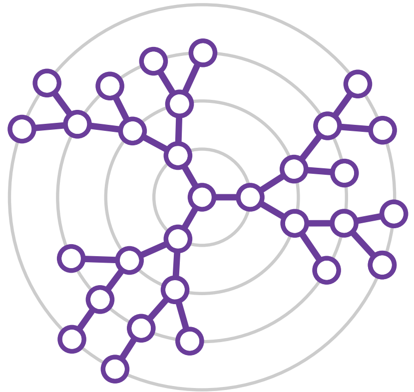
Lecture 1b:
Drawing Trees and Series-Parallel Graphs





Part II:
HV-Drawings



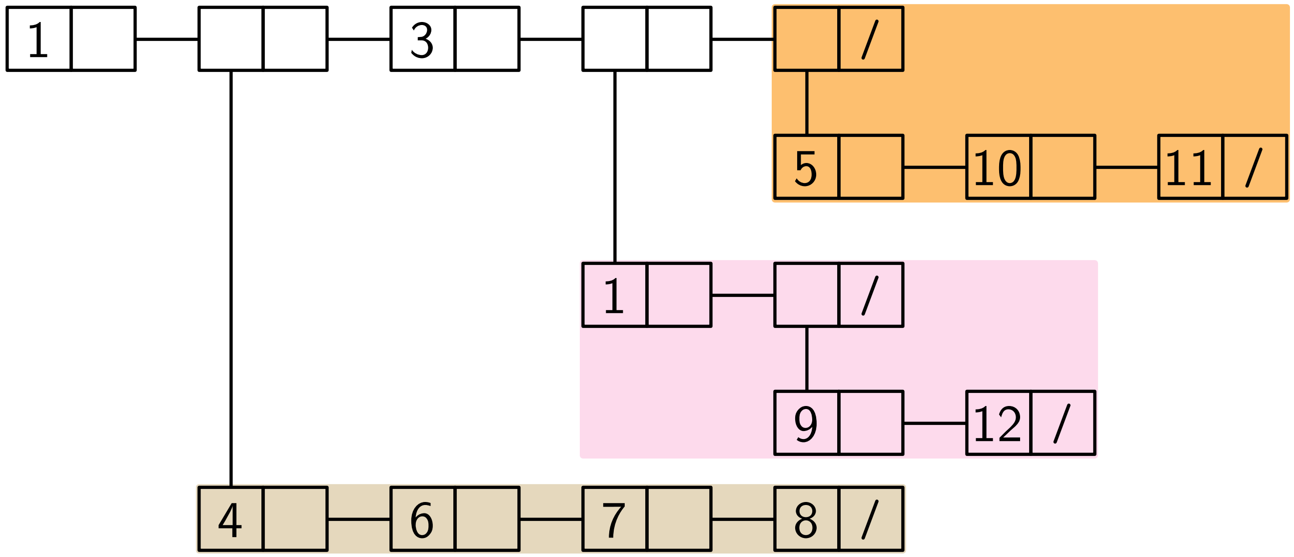


Jonathan Klawitter

HV-Drawings – Drawing Style

Applications

- Cons cell diagram in LISP
- *Cons*(constructs) are memory objects which hold two values or pointers to values



Source: after `gajon.org/trees-linked-lists-common-lisp/`

Drawing conventions

- Children are vertically or horizontally aligned with their parent
- The bounding boxes of the subtrees of the children are disjoint
- Edges are strictly down- or rightwards

Drawing aesthetics

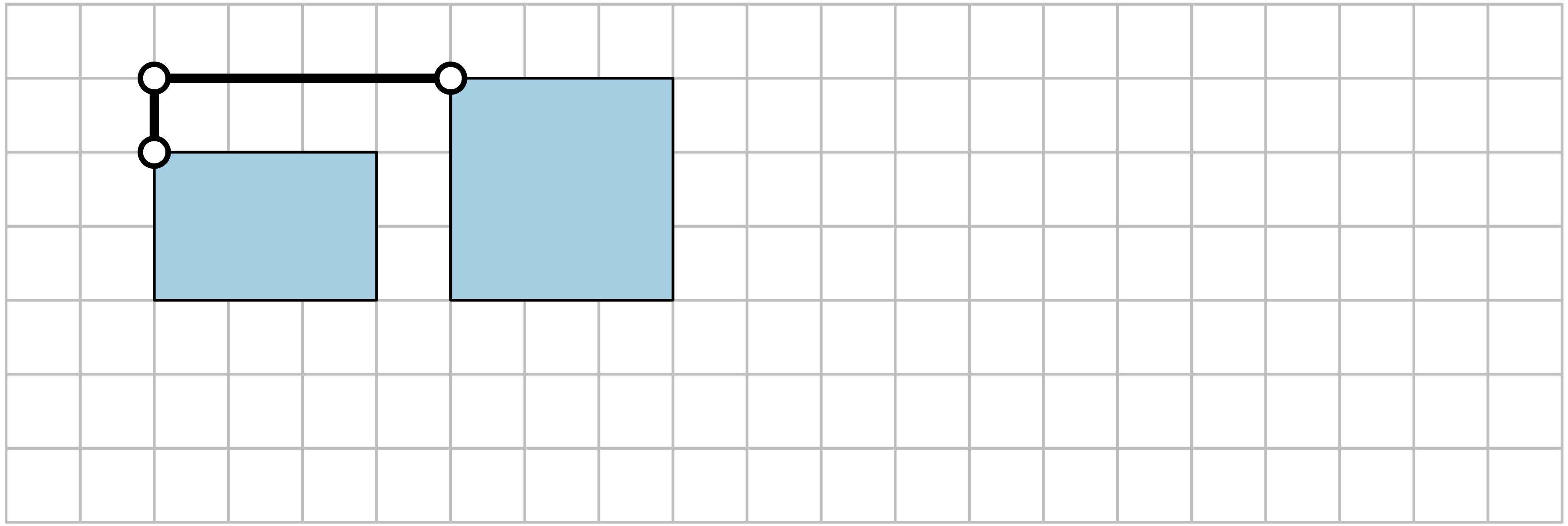
- Height, width, area

HV-Drawings – Algorithm

Input: A binary tree T
Output: An HV-drawing of T

Base case: 
Divide: Recursively apply the algorithm to draw the left and right subtrees

Conquer: horizontal combination



HV-Drawings – Algorithm

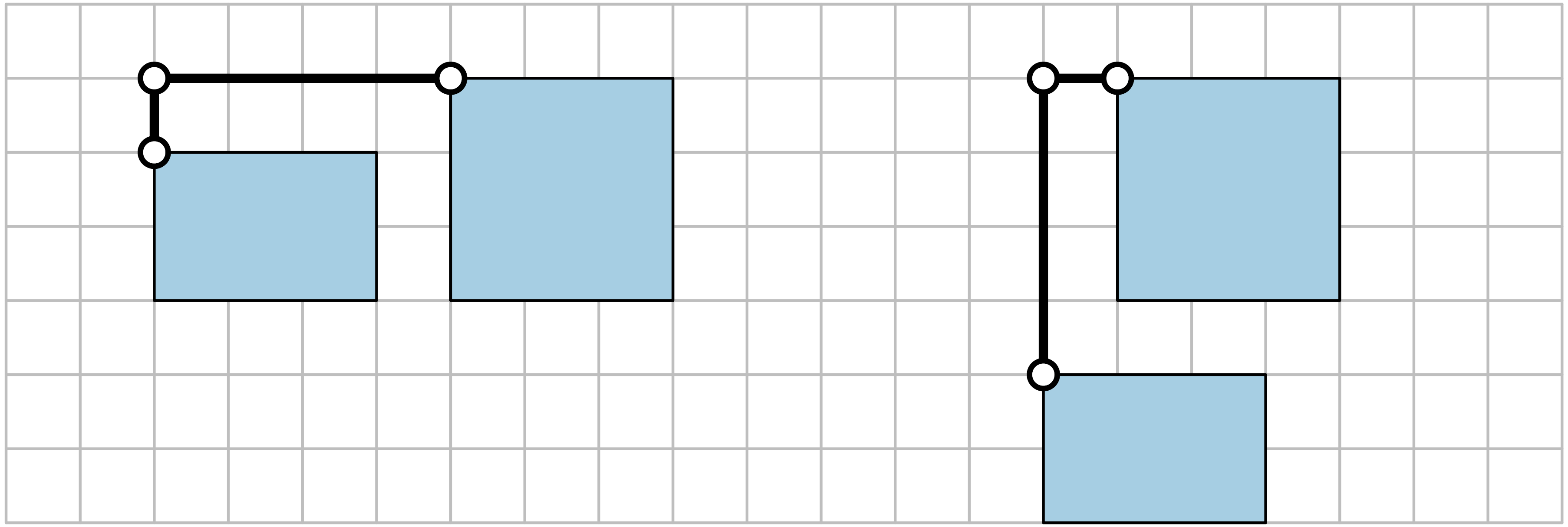
Input: A binary tree T
Output: An HV-drawing of T

Base case: 
Divide: Recursively apply the algorithm to draw the left and right subtrees

Conquer:

horizontal combination

vertical combination



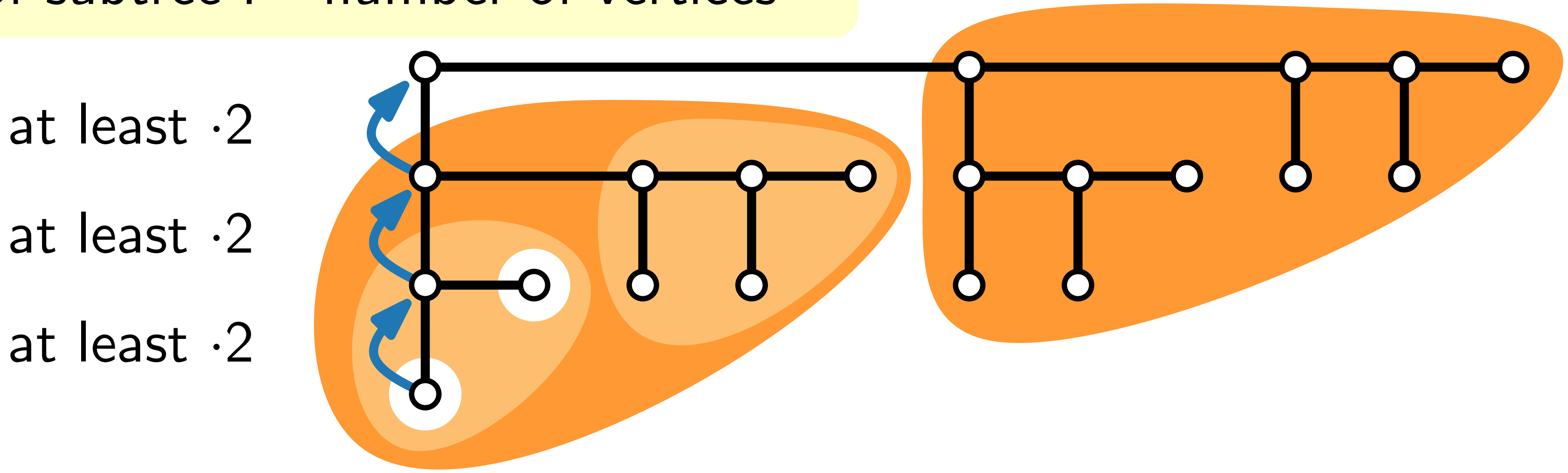
HV-Drawings – Right-Heavy HV-Layout

Right-heavy approach

- Always apply horizontal combination
- Place the larger subtree to the right

Size of subtree := number of vertices

How to implement this in linear time?



Lemma. Let T be a binary tree. The drawing constructed by the right-heavy approach has

- width at most $n-1$ and
- height at most $\log n$.

HV-Drawings – Result

Theorem.
Let T be a binary tree with n vertices. The right-heavy algorithm constructs in $O(n)$ time a drawing Γ of T s.t.:

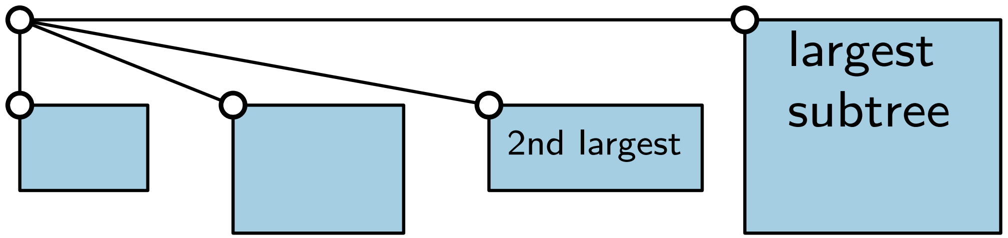
- Γ is an HV-drawing
  (planar, orthogonal, strictly right-/downward)
- Width is at most $n - 1$
- Height is at most $\log n$
- Area is in $\mathcal{O}(n \log n)$
- Simply and axially isomorphic subtrees have congruent drawings up to translation

HV-Drawings – Result

Theorem.
Let T be a ~~binary~~ rooted tree with n vertices. The right-heavy algorithm constructs in $O(n)$ time a drawing Γ of T s.t.:

- Γ is an HV-drawing (planar, ~~orthogonal,~~ strictly right-/downward)
- Width is at most $n-1$
- Height is at most $\log n$
- Area is in $\mathcal{O}(n \log n)$
- Simply and axially isomorphic subtrees have congruent drawings up to translation

General rooted tree



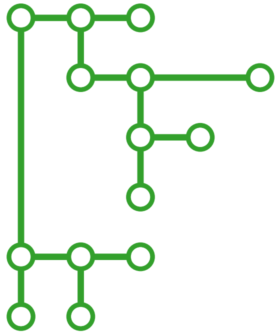
Optimal area?

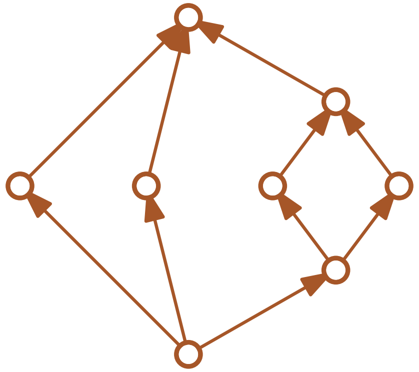
Not with divide & conquer approach, but can be computed with Dynamic Programming.

Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

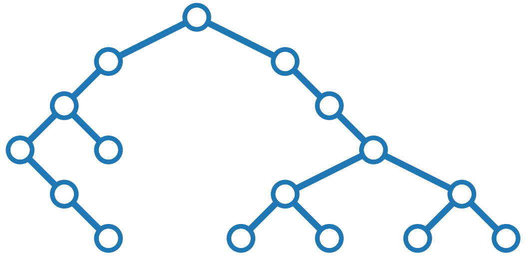
Visualization of Graphs

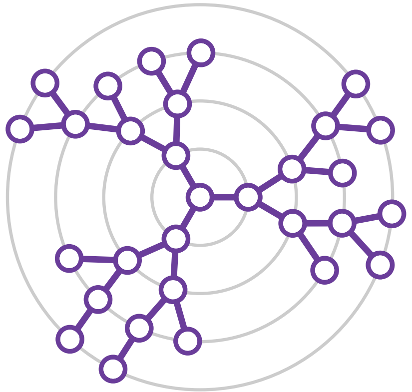
Lecture 1b:
Drawing Trees and Series-Parallel Graphs





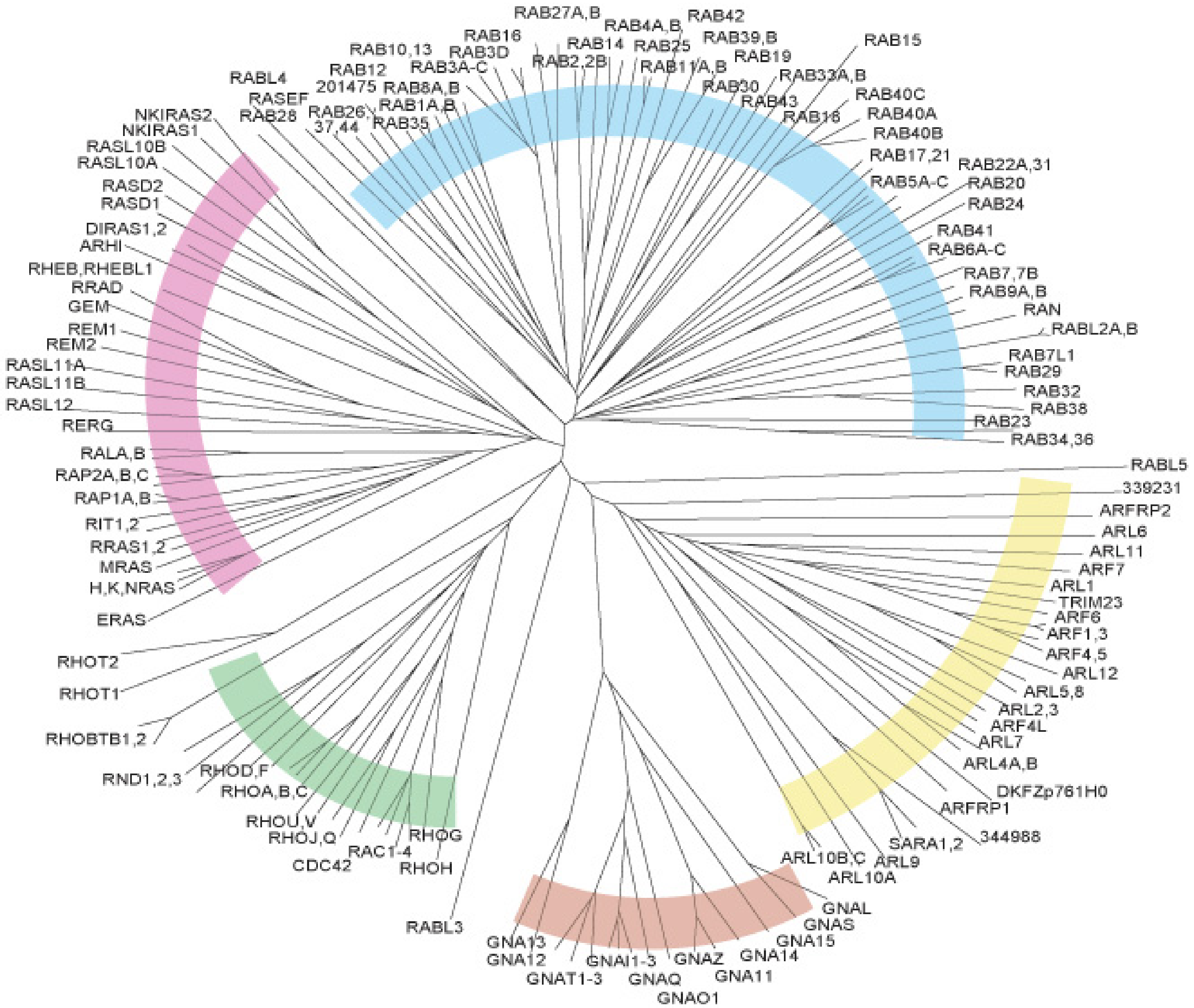
Part III:
Radial Layouts





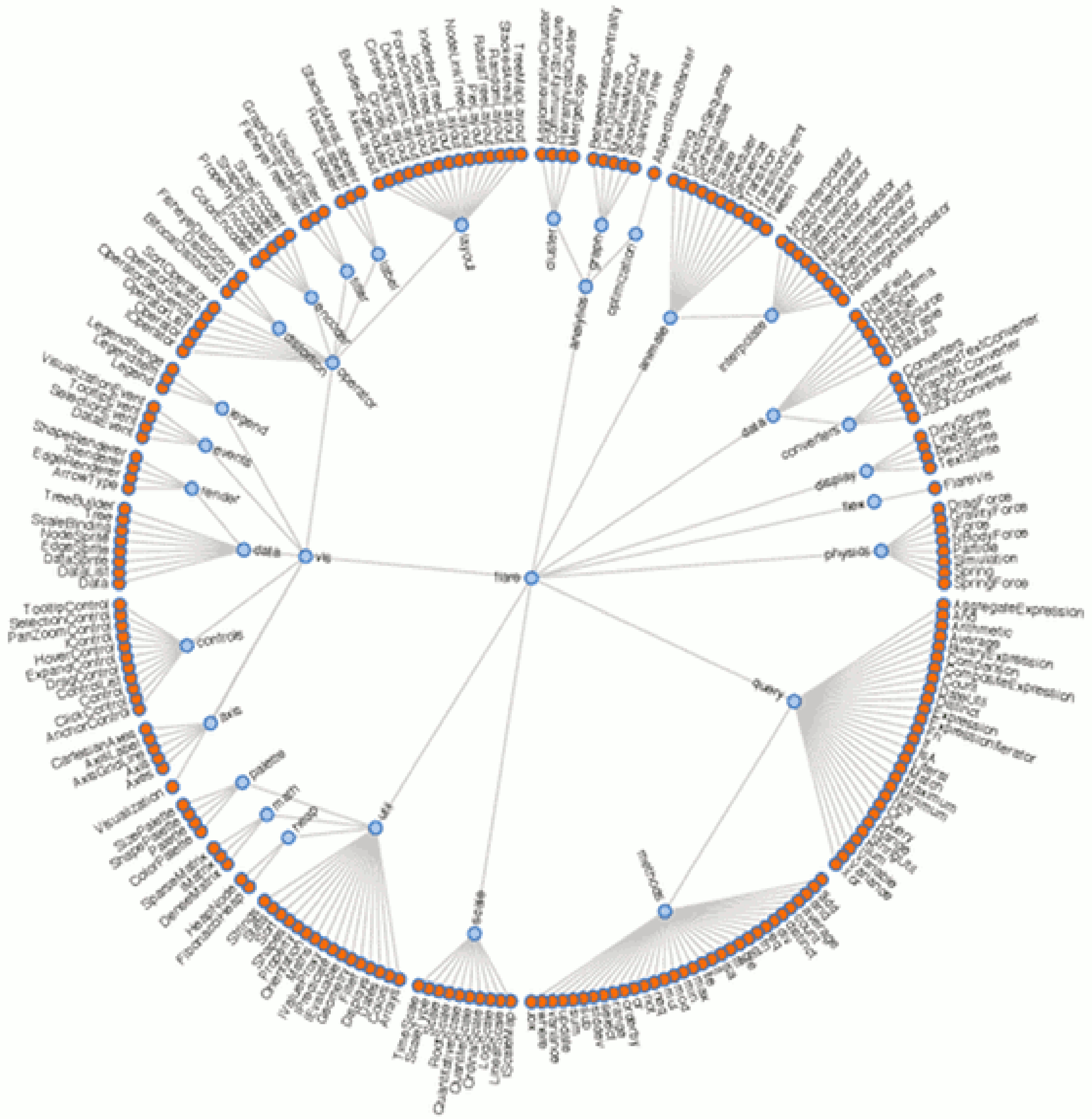
Jonathan Klawitter

Radial Layouts – Applications

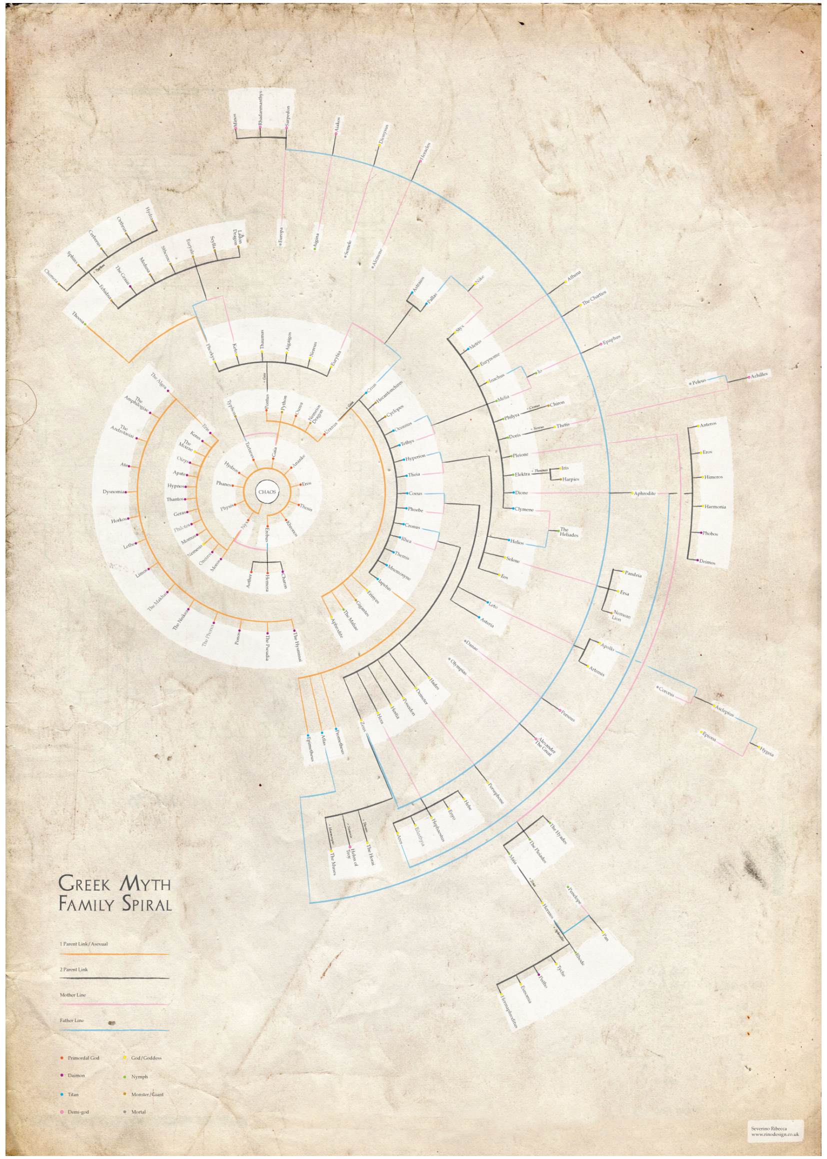


Phylogenetic tree
by Colicelli, ScienceSignaling, 2004

Radial Layouts – Applications

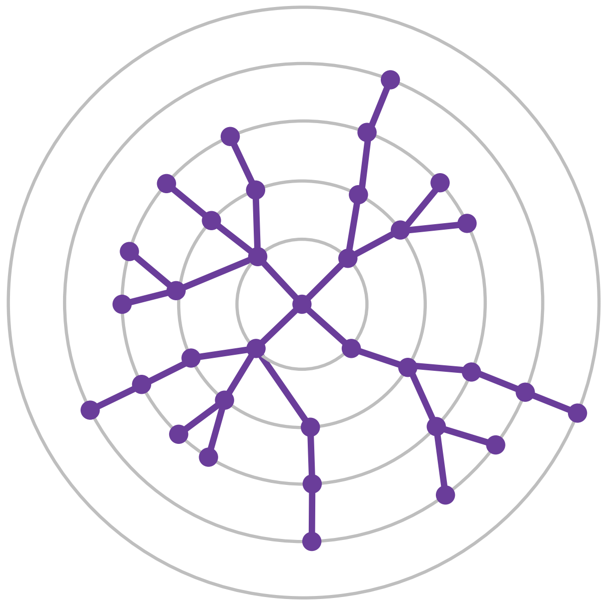


Flare Visualization Toolkit code structure
by Heer, Bostock and Ogievetsky, 2010



Greek Myth Family
by Ribecca, 2011

Radial Layouts – Drawing Style



Drawing conventions

- Vertices lie on circular layers according to their depth
- Drawing is planar

Drawing aesthetics

- Distribution of the vertices

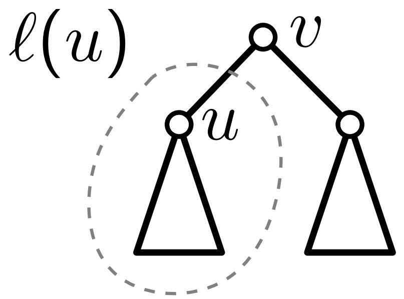
How can an algorithm optimize the distribution of the vertices?

Radial Layouts – Algorithm Attempt

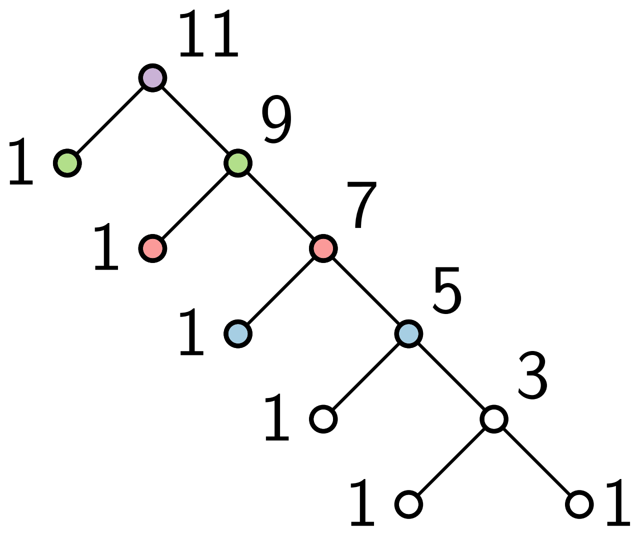
Idea

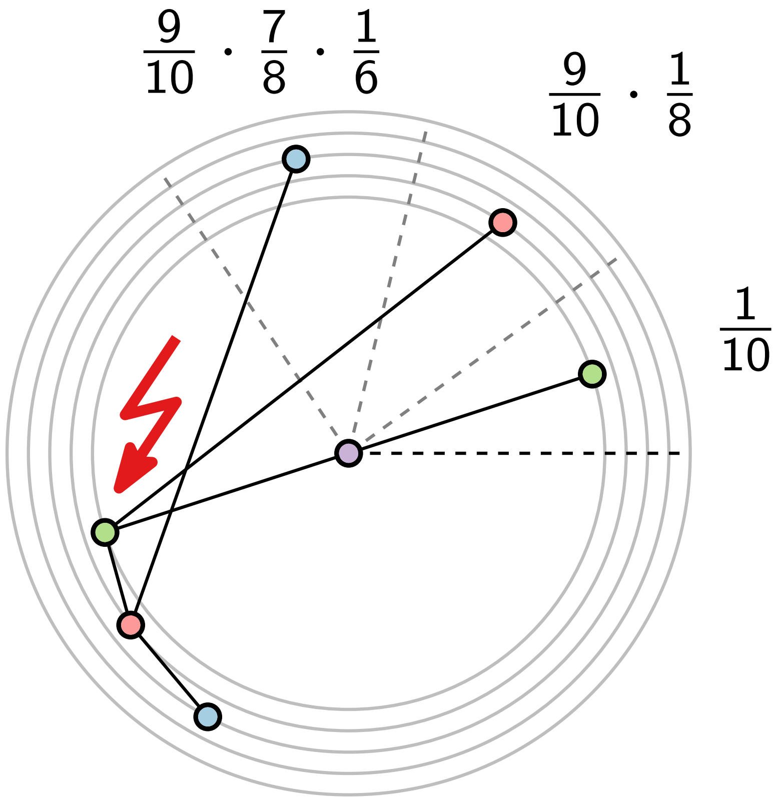
- Reserve area corresponding to size $\ell(u)$ of $T(u)$:

$$\tau_u = \frac{\ell(u)}{\ell(v) - 1}$$

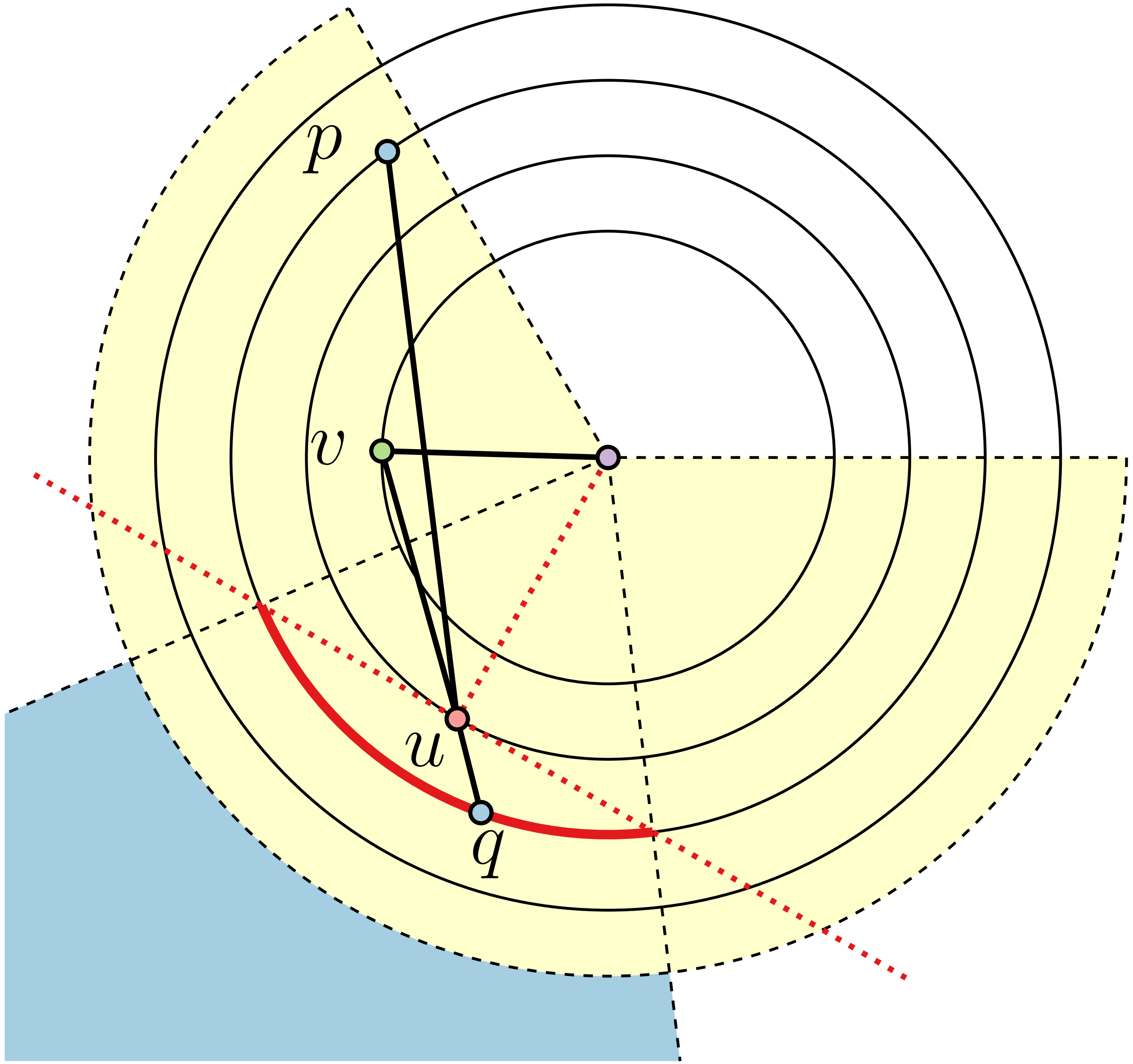


- Place u in middle of area

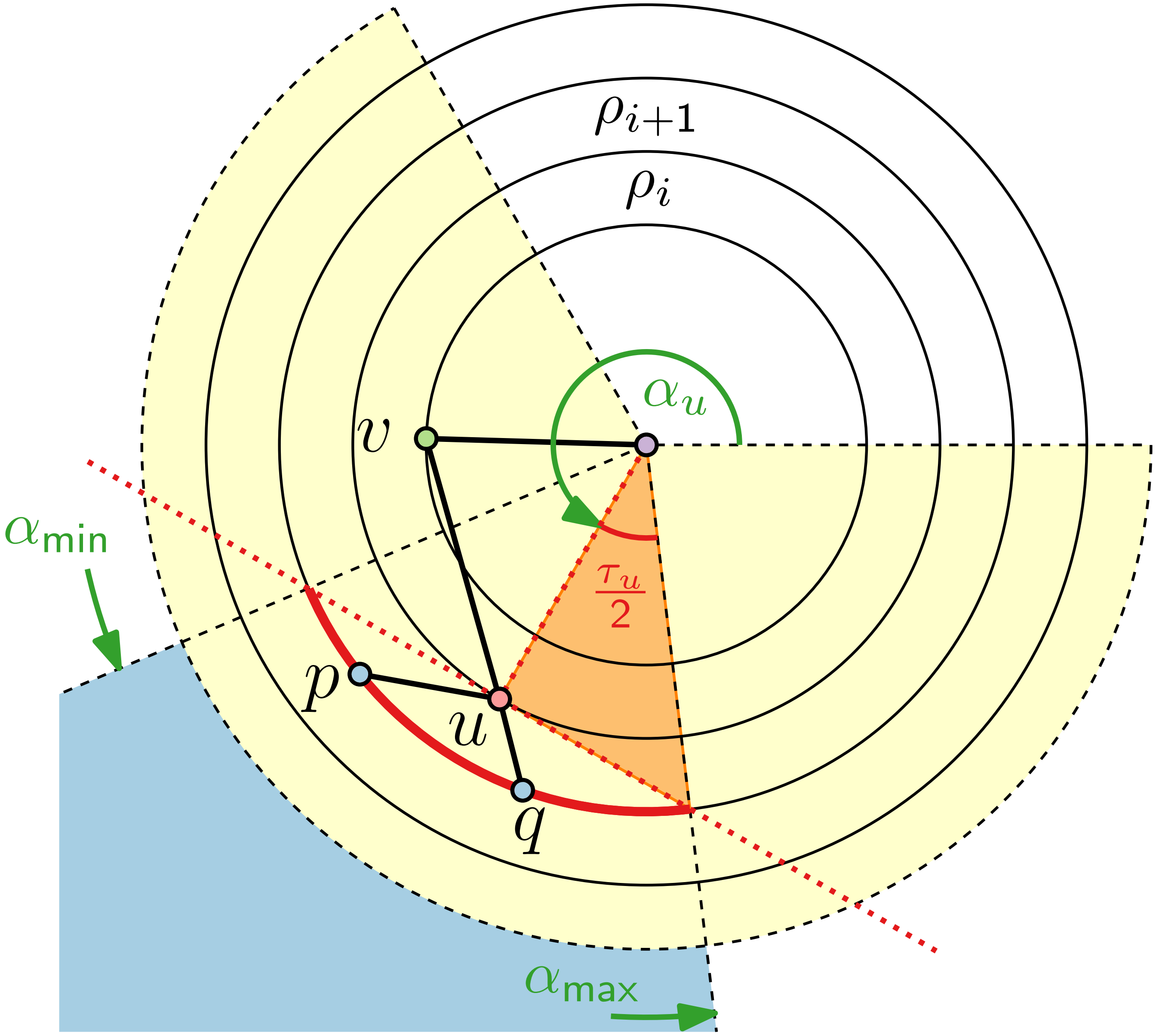




Radial Layouts – How To Avoid Crossings



Radial Layouts – How To Avoid Crossings



- τ_u – angle of the wedge corresponding to vertex u
- $\ell(u)$ – number of nodes in the subtree rooted at u
- ρ_i – radius of layer i
- $\cos \frac{\tau_u}{2} = \frac{\rho_i}{\rho_{i+1}}$
- $\tau_u = \min\{\frac{\ell(u)}{\ell(v)-1}, 2\arccos\frac{\rho_i}{\rho_{i+1}}\}$
- Alternative:

  $\alpha_{\text{min}} = \alpha_u - \arccos\frac{\rho_i}{\rho_{i+1}}$

  $\alpha_{\text{max}} = \alpha_u + \arccos\frac{\rho_i}{\rho_{i+1}}$

Radial Layouts – Pseudocode

RadialTreeLayout(tree T, root $r \in T$, radii $\rho_1 < \cdots < \rho_k$)

begin

  $\mathit{postorder}(r)$

  $\mathit{preorder}(r, 0, 0, 2\pi)$

  return $(d_v, \alpha_v)_{v \in V(T)}$

  // vertex pos./polar coord.

postorder(vertex v)

  $\ell(v) \leftarrow 1$

  foreach child w of v **do**

    $\mathit{postorder}(w)$

    $\ell(v) \leftarrow \ell(v) + \ell(w)$

preorder(vertex v, $t, \alpha_{\mathsf{min}}, \alpha_{\mathsf{max}}$)

  $d_v \leftarrow \rho_t$

  $\alpha_v \leftarrow (\alpha_{\mathsf{min}} + \alpha_{\mathsf{max}})/2$

  //output

  if $t > 0$ **then**

    $\alpha_{\mathsf{min}} \leftarrow \max\{\alpha_{\mathsf{min}}, \alpha_v - \arccos \frac{\rho_t}{\rho_{t+1}}\}$

    $\alpha_{\mathsf{max}} \leftarrow \min\{\alpha_{\mathsf{max}}, \alpha_v + \arccos \frac{\rho_t}{\rho_{t+1}}\}$

  $\mathit{left} \leftarrow \alpha_{\mathsf{min}}$

  foreach child w of v **do**

    $\mathit{right} \leftarrow \mathit{left} + \frac{\ell(w)}{\ell(v)-1} \cdot (\alpha_{\mathsf{max}} - \alpha_{\mathsf{min}})$

    $\mathit{preorder}(w, t+1, \mathit{left}, \mathit{right})$

    $\mathit{left} \leftarrow \mathit{right}$

Runtime? $\mathcal{O}(n)$

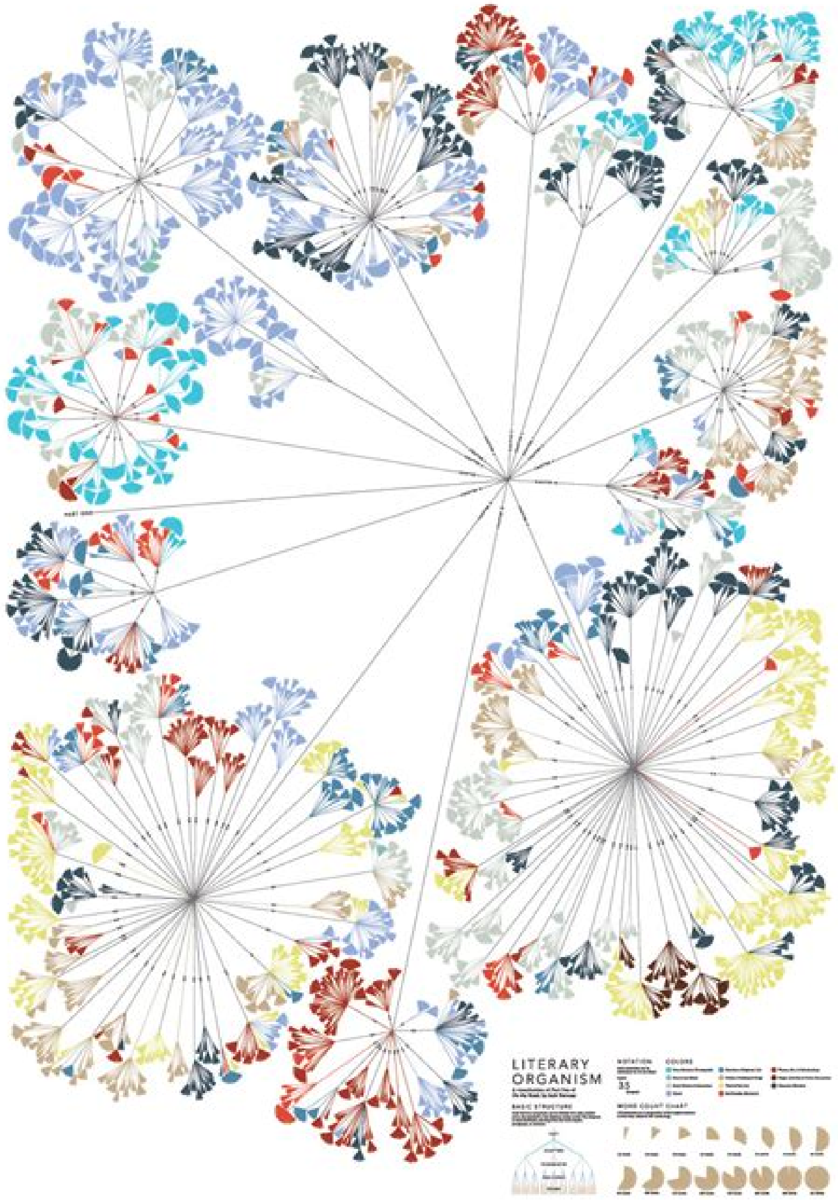
Correctness? ✓

Radial Layouts – Result

Theorem.
Let T be a tree with n vertices. The RadialTreeLayout algorithm constructs in $O(n)$ time a drawing Γ of T s.t.:

- Γ is radial drawing
- Vertices lie on circle according to their depth
- Area quadratic in max degree times height of T
  (see [GD Ch. 3.1.3] if interested)

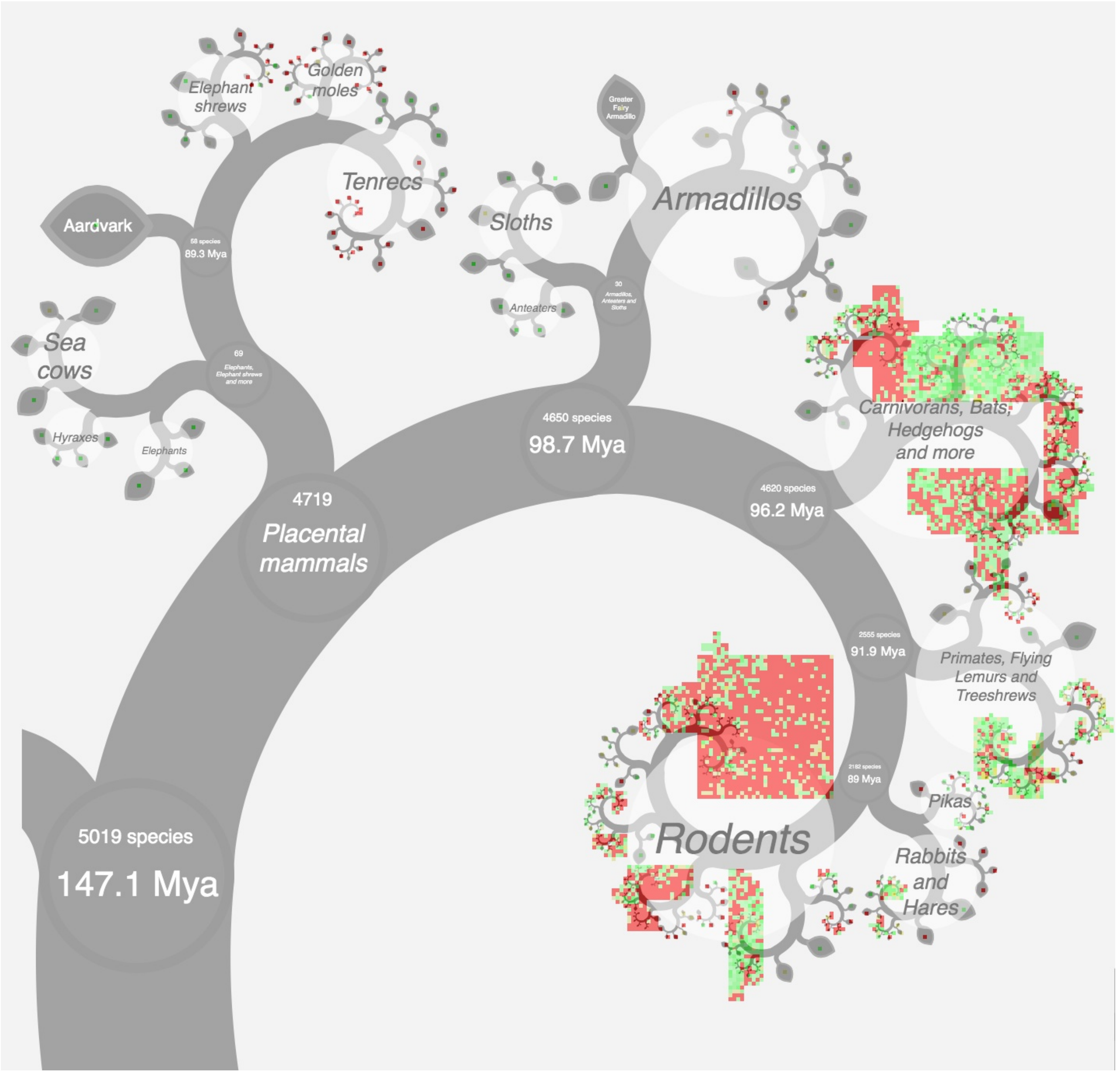
Other tree visualisation styles



Writing Without Words:
The project explores methods to visualises the differences in writing styles of different authors.

Similar to ballon layout

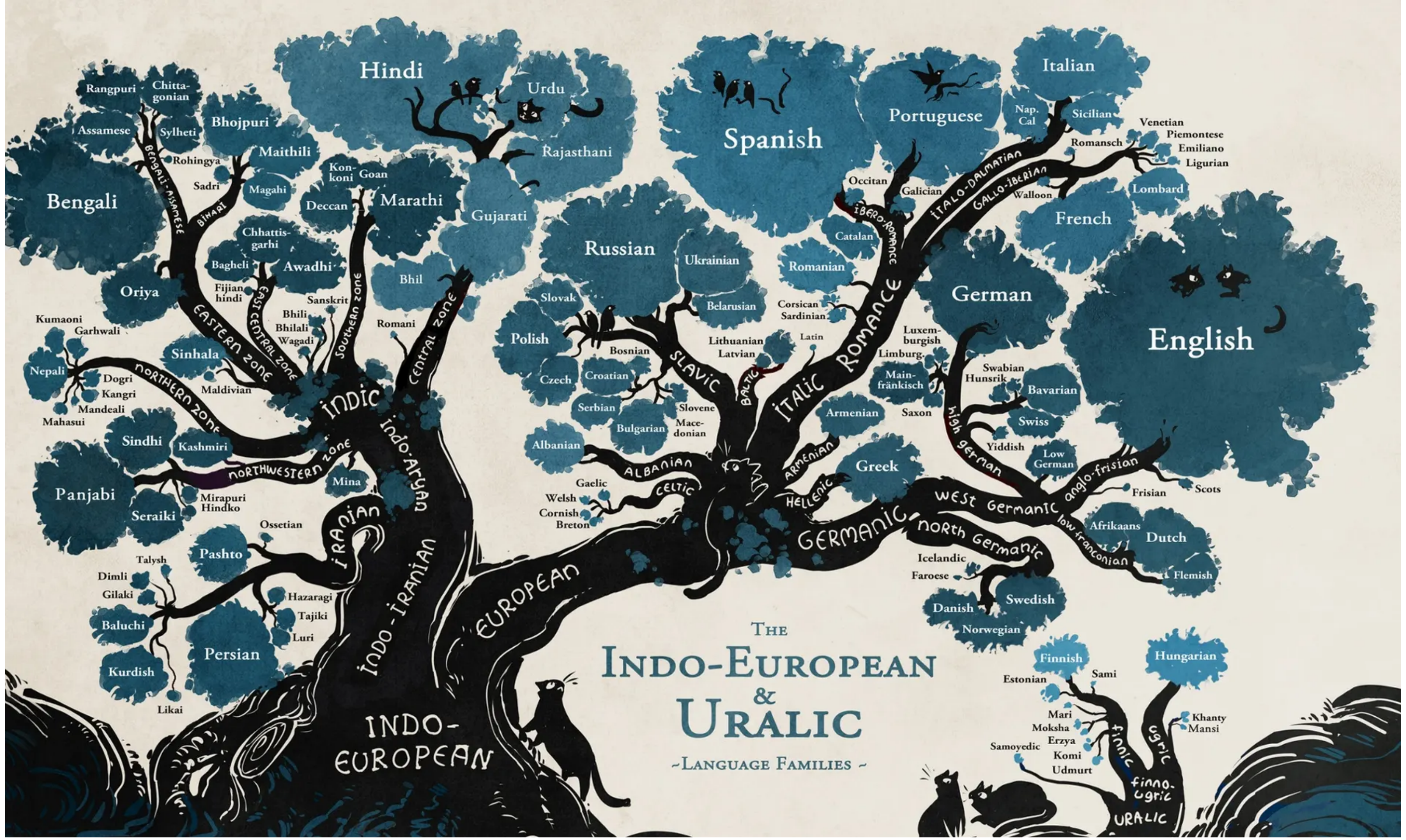
Other tree visualisation styles



A phylogenetically organised display of data for all placental mammal species.

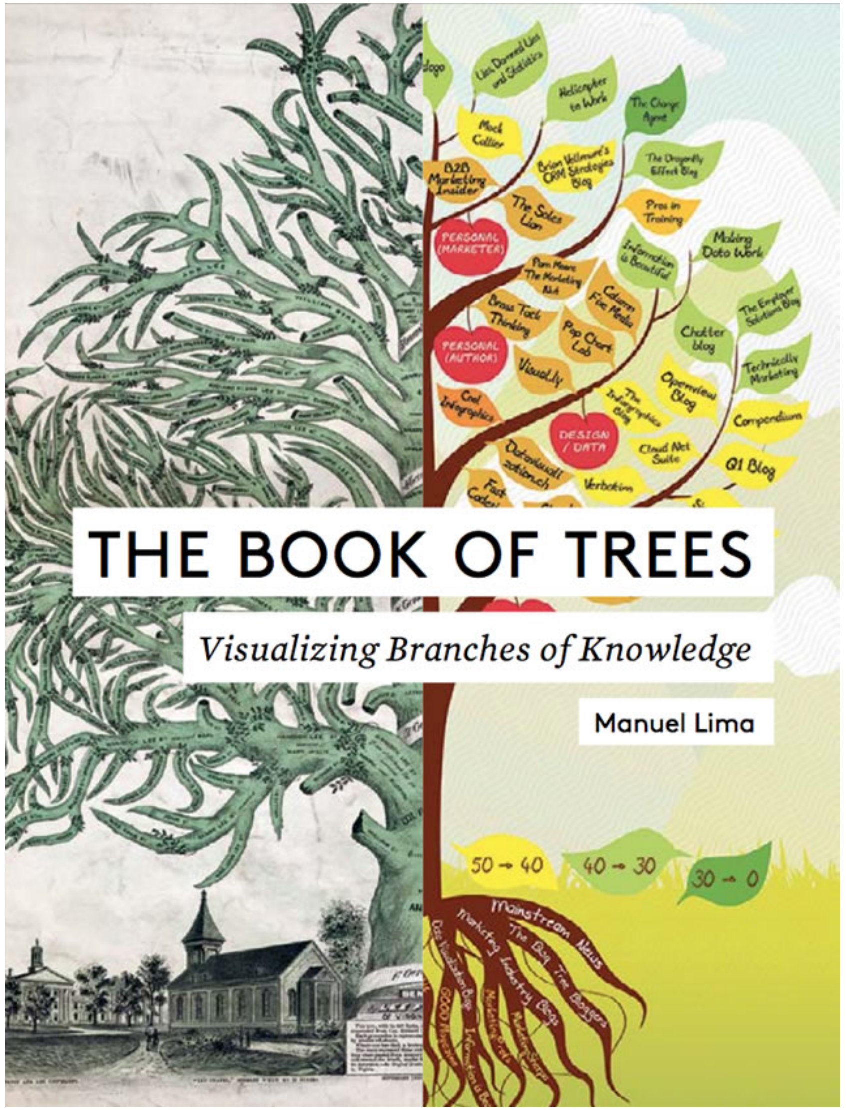
Fractal layout

Other tree visualisation styles



A language family tree – in pictures

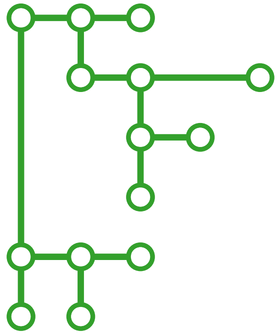
Other tree visualisation styles

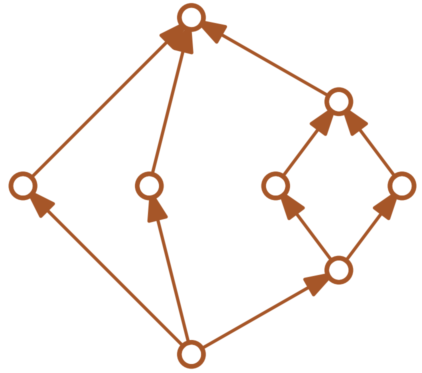


treevis.net

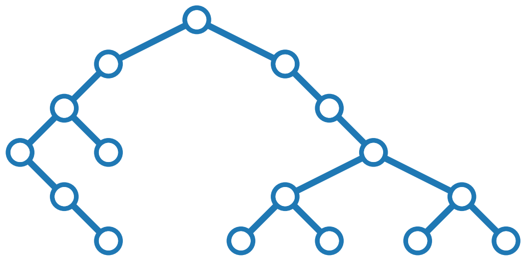
Visualization of Graphs

Lecture 1b:
Drawing Trees and Series-Parallel Graphs

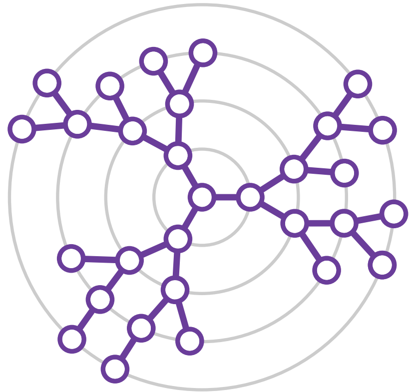




Part IV:
Series-Parallel Graphs



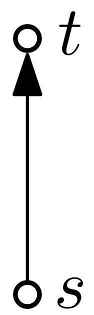
Jonathan Klawitter



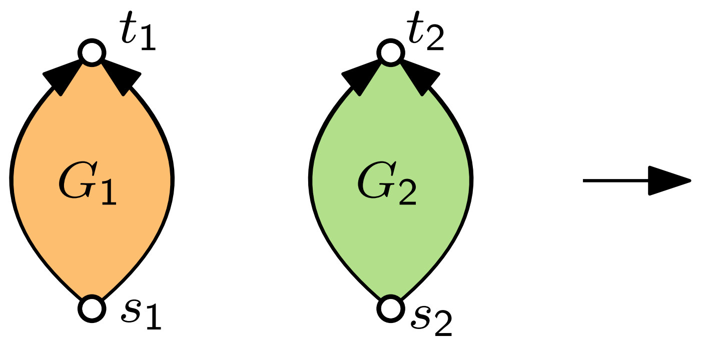
Series-Parallel Graphs

A graph G is **series-parallel**, if

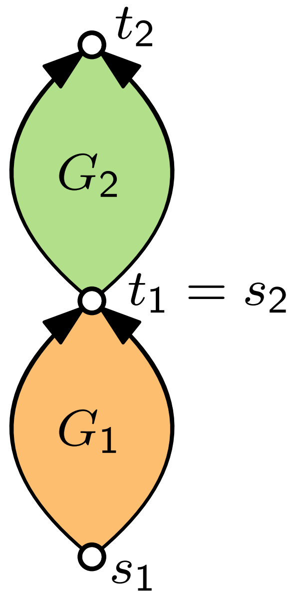
- it contains a single (directed) edge (s, t), or
- it consists of two series-parallel graphs G_1, G_2 with sources s_1, s_2 and sinks t_1, t_2 that are combined using one of the following rules:



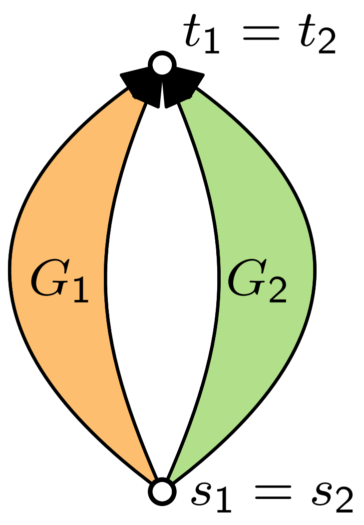
convince yourself that series-parallel graphs are planar



Series composition



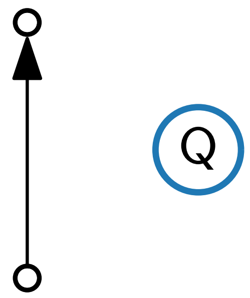
Parallel composition

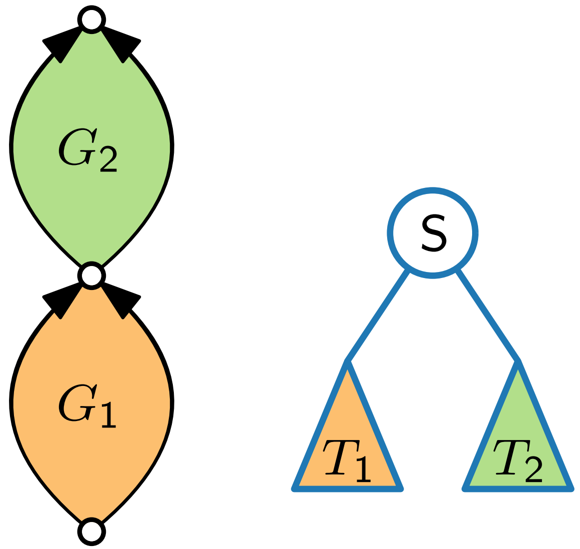


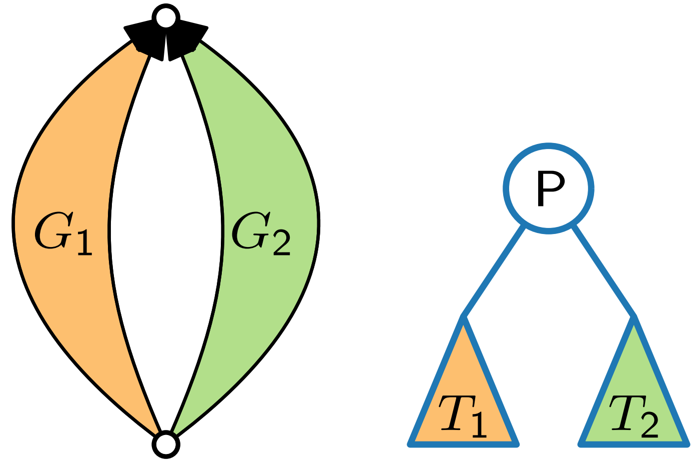
Series-Parallel Graphs – Decomposition Tree

A **decomposition tree** of G is a binary tree T with nodes of three types: **S**, **P** and **Q**-type

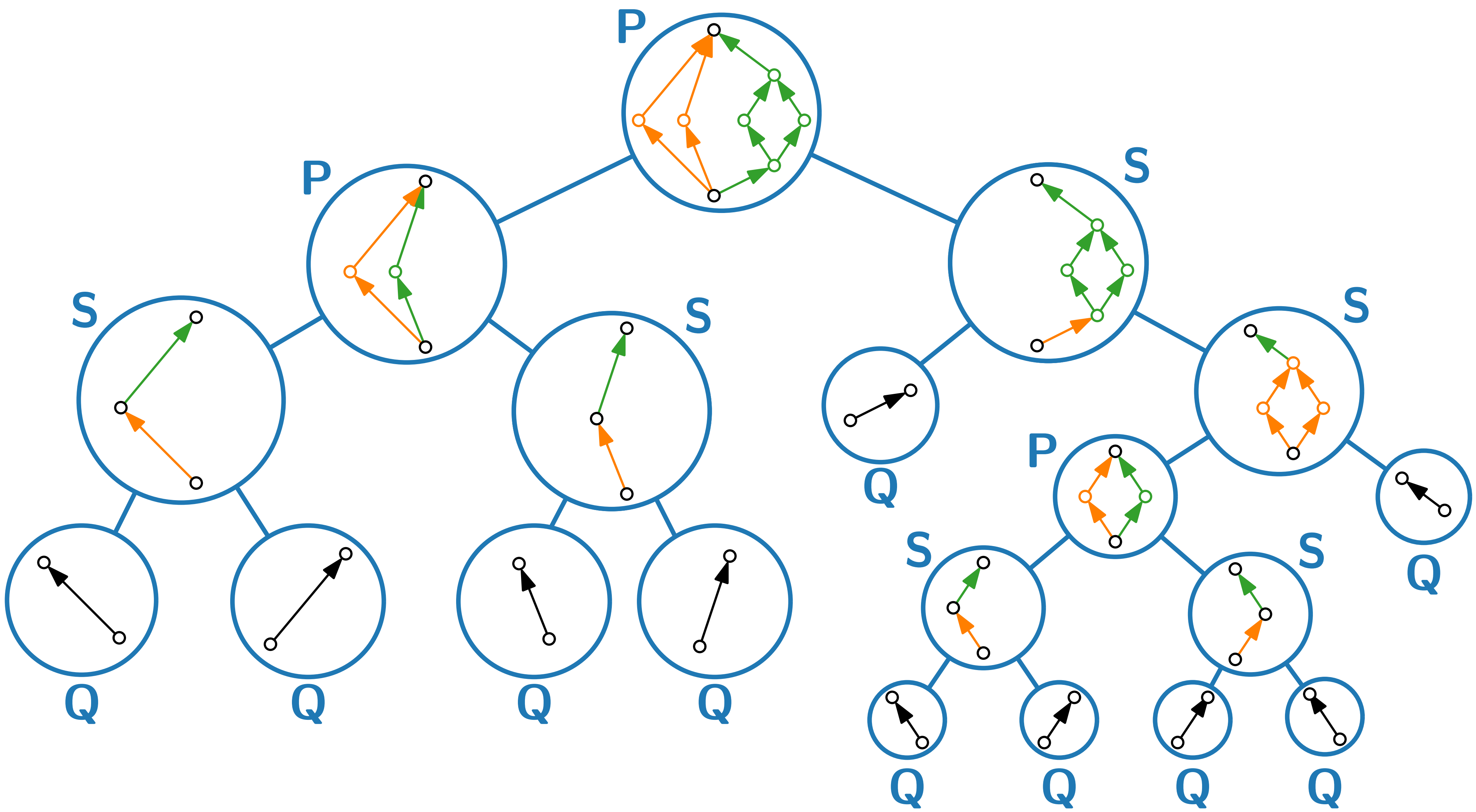
- A **Q**-node represents a single edge
- An **S**-node represents a series composition; its children T_1 and T_2 represent G_1 and G_2
- A **P**-node represents a parallel composition; its children T_1 and T_2 represent G_1 and G_2



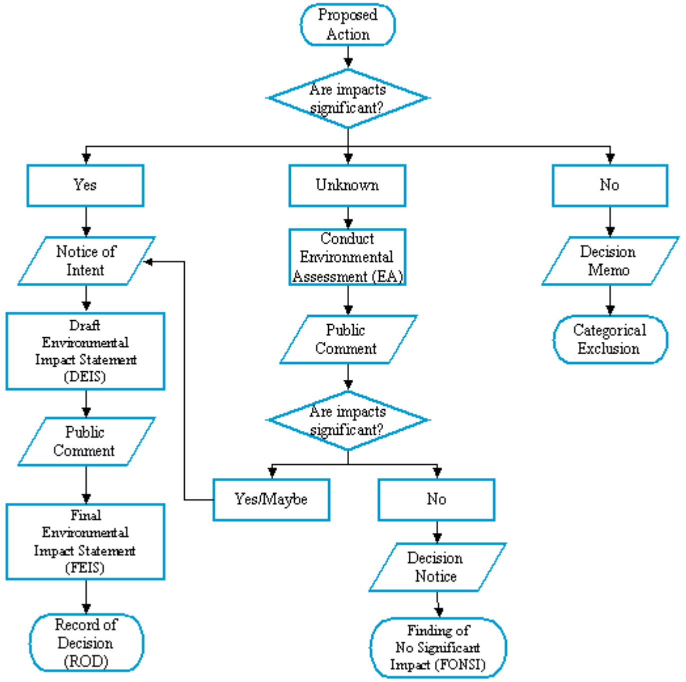




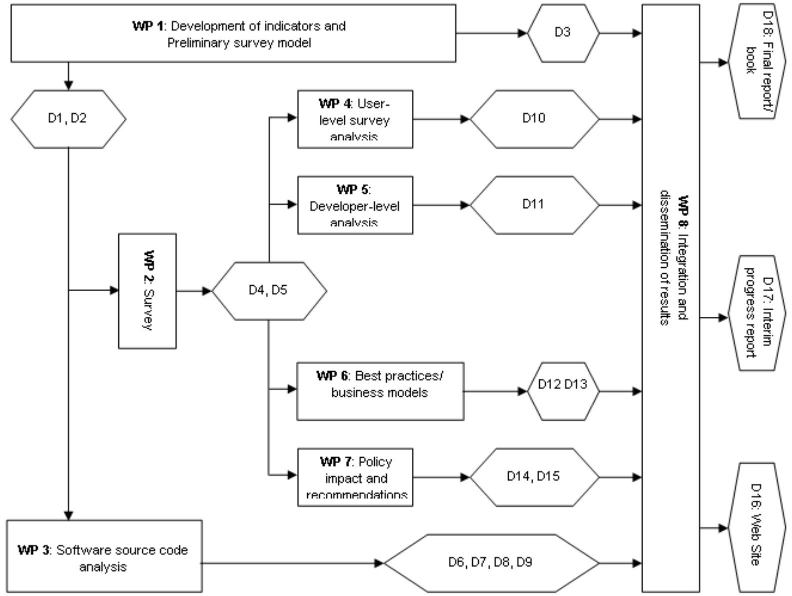
Series-Parallel Graphs – Decomposition Example



Series-Parallel Graphs – Applications



Flowcharts



PERT-Diagrams

(Program Evaluation and Review Technique)

Computational complexity:

Linear time algorithms for $\mathcal{NP}$-hard problems

(e.g. Maximum Matching, MIS, Hamiltonian Completion)

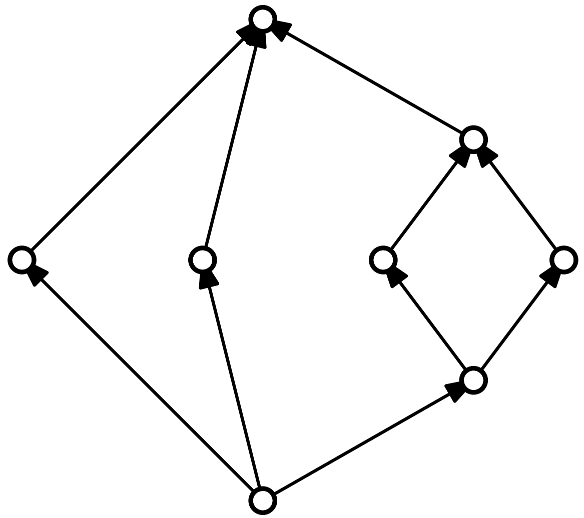
Series-Parallel Graphs – Drawing Style

Drawing conventions

- Planarity
- Straight-line edges
- Upward

Drawing aesthetics

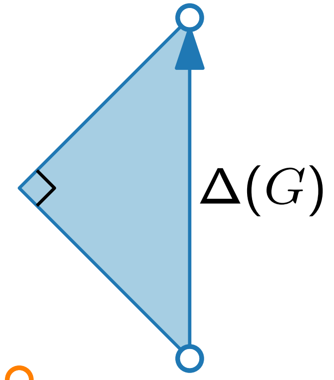
- Area
- Symmetry



Series-Parallel Graphs – Straight-Line Drawings

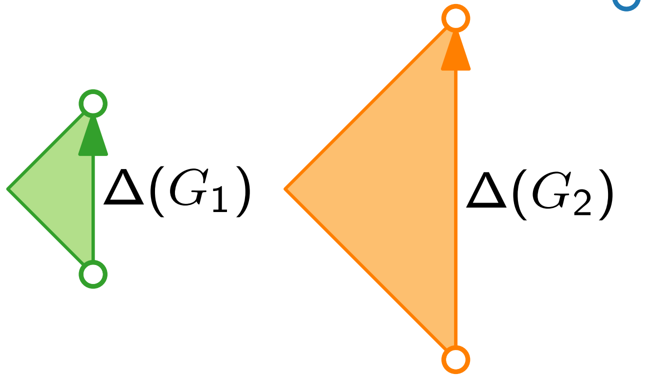
Divide & conquer algorithm using the decomposition tree

- Draw G inside a right-angled isosceles bounding triangle $\Delta(G)$



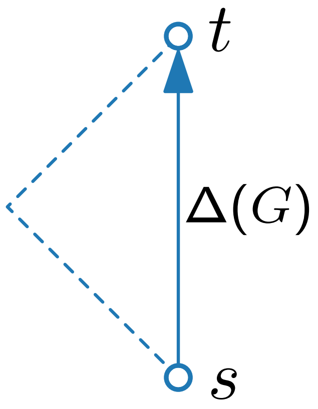
Base case: Q-nodes

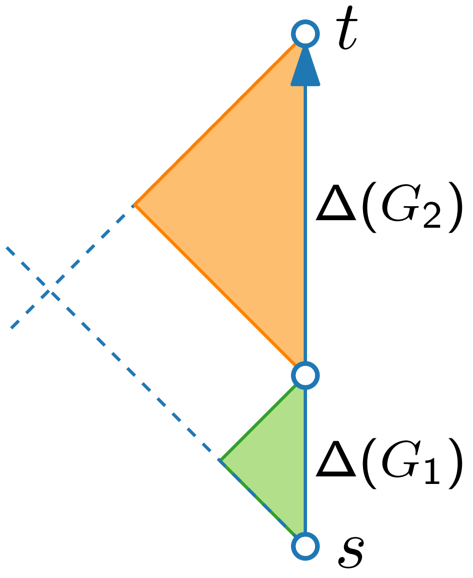
Divide: Draw G_1 and G_2 first

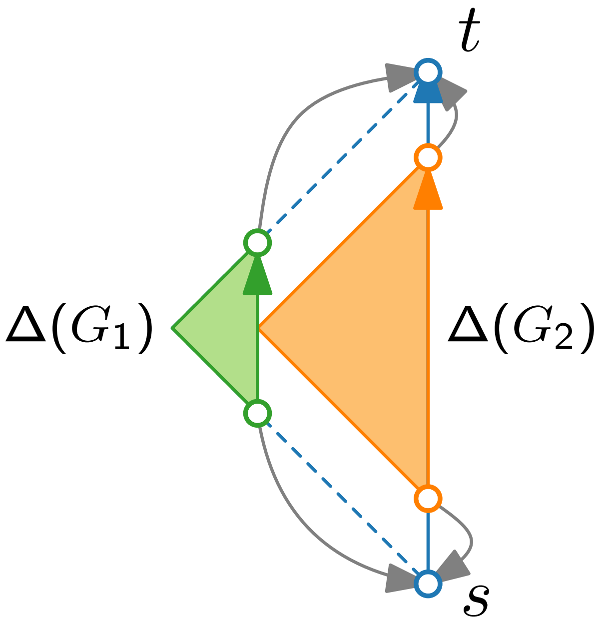


Conquer:

- S-nodes / series composition
- P-nodes / parallel composition





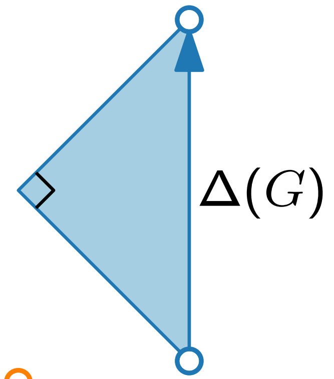


Do you see any problem?

Series-Parallel Graphs – Straight-Line Drawings

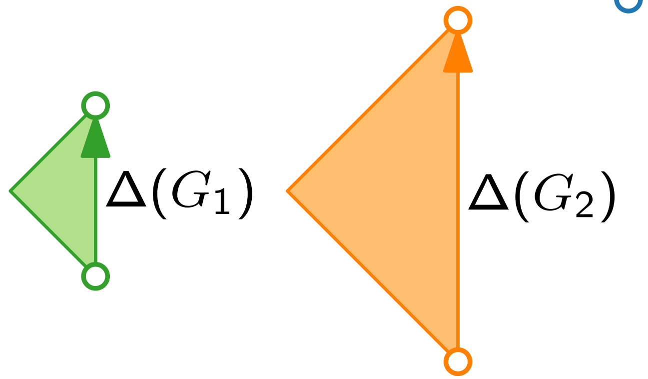
Divide & conquer algorithm using the decomposition tree

- Draw G inside a right-angled isosceles bounding triangle $\Delta(G)$



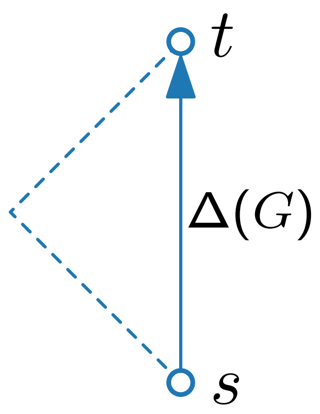
Base case: Q-nodes

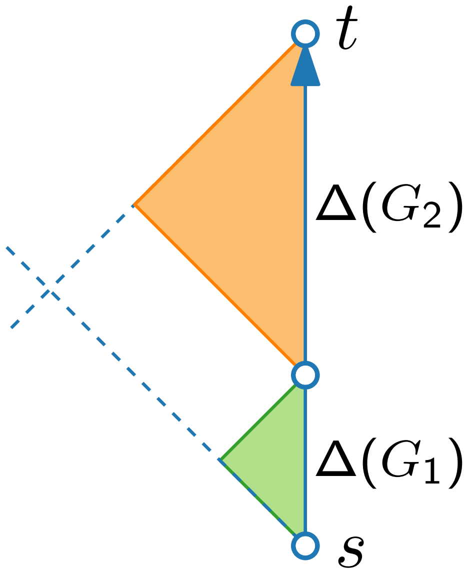
Divide: Draw G_1 and G_2 first

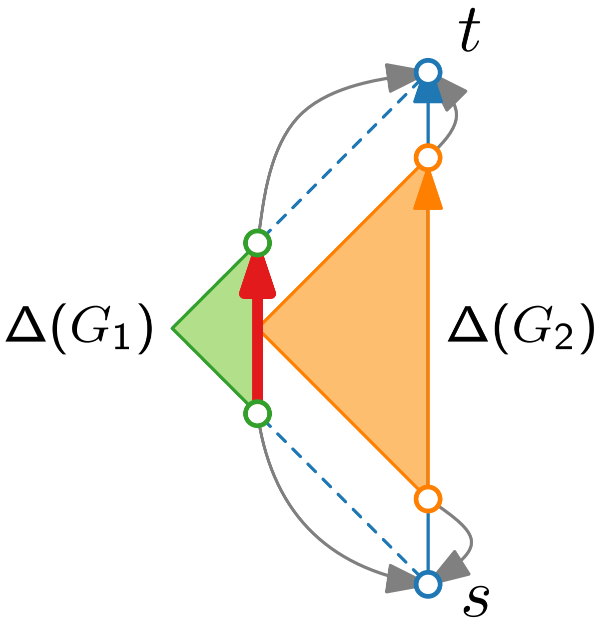


Conquer:

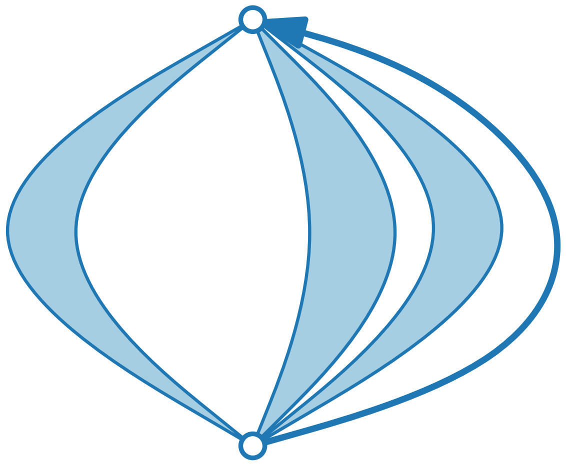
- S-nodes / series composition
- P-nodes / parallel composition





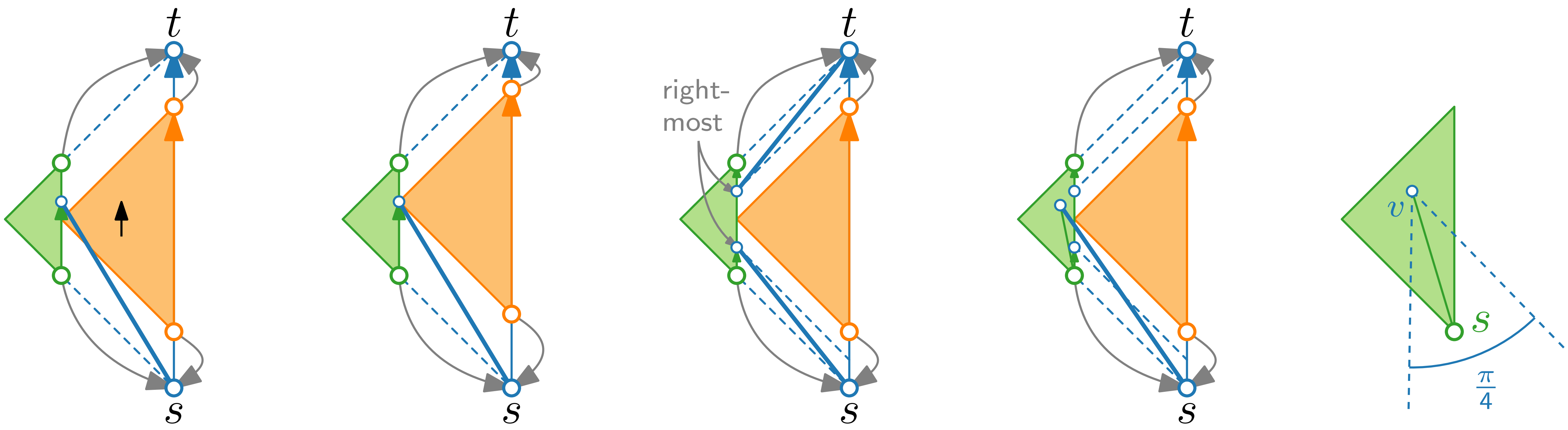


change embedding!



Series-Parallel Graphs – Straight-Line Drawings

- What makes parallel composition possible without creating crossings?



Assume the following holds:
the only vertex in angle(v) is s

- This condition **is** preserved during the induction step.

> **Lemma.**
> The drawing produced by the algorithm is planar.

Series-Parallel Graphs – Result

Theorem.
Let G be a series-parallel graph. Then G (with **variable embedding**) admits a drawing Γ that

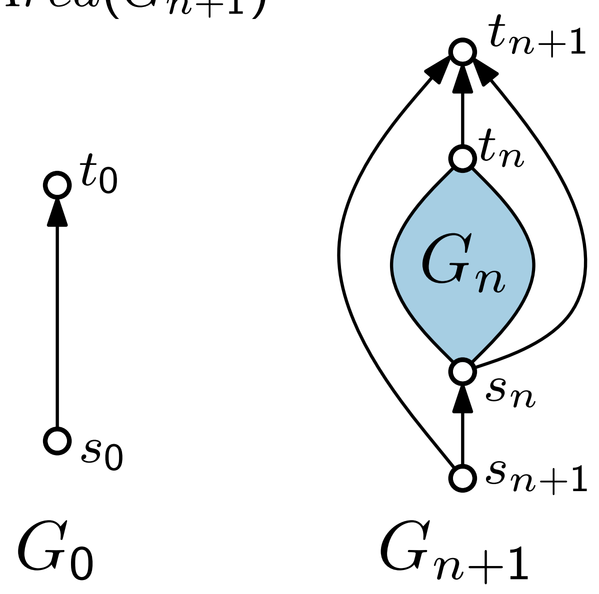
- is upward planar and
- a straight-line drawing
- with area in $\mathcal{O}(n^2)$.
- Isomorphic components of G have congruent drawings up to translation.

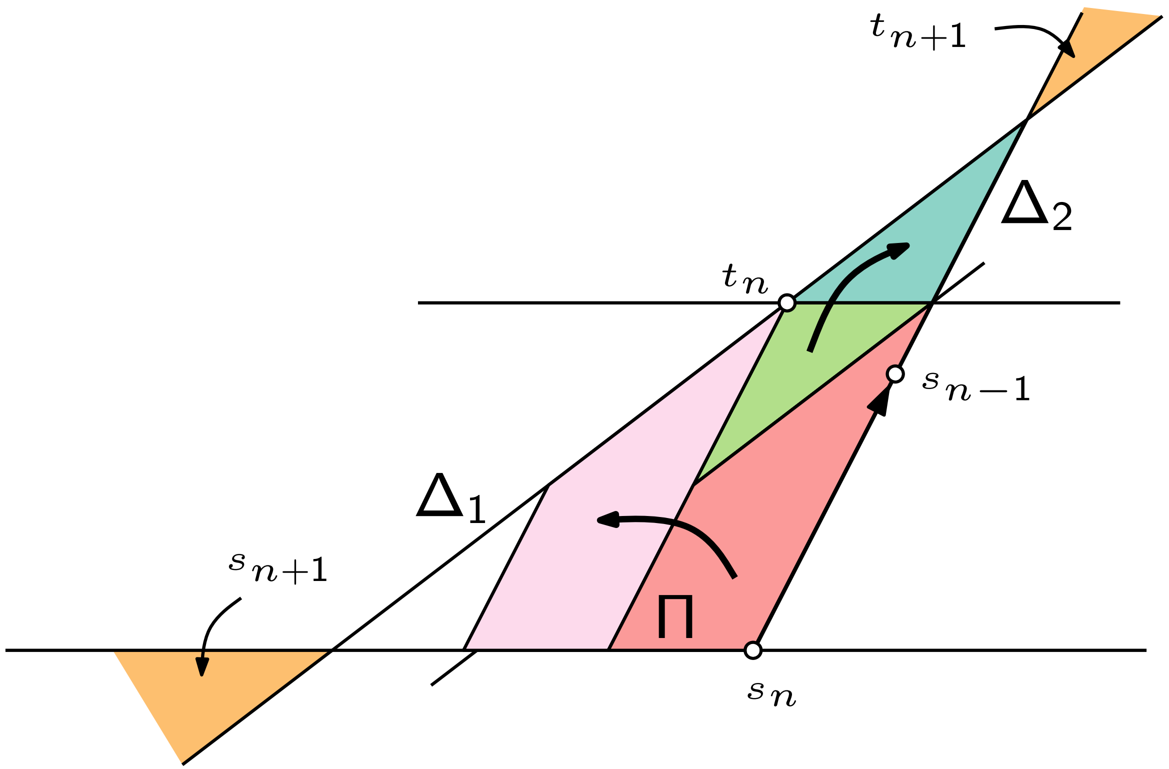
Γ can be computed in $\mathcal{O}(n)$ time.

Series-Parallel Graphs – Fixed Embedding

Theorem. [Bertolazzi et al. 94]
There exists a $2n$-vertex series-parallel graph G_n such that any upward planar drawing of G_n that respects the embedding requires $\Omega(4^n)$ area.

- $2 \cdot Area(G_n) < Area(\Pi)$
- $2 \cdot Area(\Pi) \leq Area(G_{n+1})$
- $4 \cdot Area(G_n) \leq Area(G_{n+1})$





Literature

- [GD Chapter 3] for divide and conquer methods for rooted trees and series-parallel graphs
- [Reingold, Tilford '81] "Tidier Drawings of Trees"
  original paper for level-based layout algo
- [Reingold, Supowit '83] "The complexity of drawing trees nicely"
  NP-hardness proof for area minimisation & LP
- `treevis.net` – compendium of drawing methods for trees