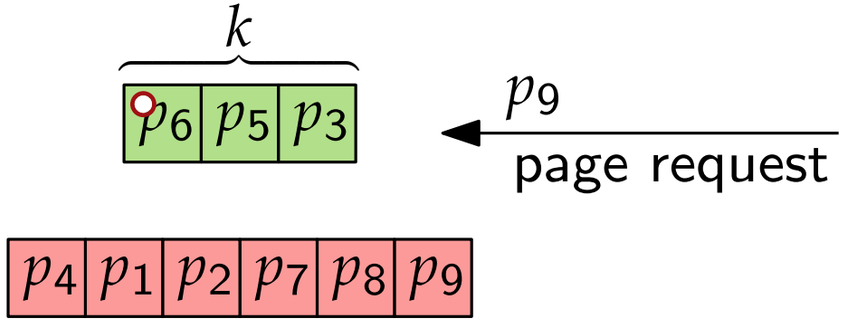
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

Advanced Algorithms

Online Algorithms

Ski-Rental Problem and Paging

Johannes Zink · WS20



Introduction

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Ski-Rental Problem

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- Is it worth **buying** new skis?
- Or should we rather **rent** them?

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Ski-Rental Problem

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. . . this means the ski season is back!*

- But what if there is not always enough snow?
- Is it worth **buying** new skis?
- Or should we rather **rent** them?
- We don't know the weather (much) in advance.

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Ski-Rental Problem – definition

Behavior.

- Every day when there is "good" weather, you go skiing.
  - We call this is a **good** day.

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Plan.

- Not knowing T,
- devise a strategy if and when to buy skis.

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Buying costs M
T good days

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competitive ratio

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not better than the deterministic Strategy III

Ski-Rental Problem – Strategy IV

Renting costs 1/day
Buying costs M
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Can we get below this bound using randomization? – Let's try!

Strategy IV: throw a coin; **HEAD:** buy on the $\mathbf{M}$-th good day
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- With a more sophisticated probability distribution for the time we buy skis, we can even get a competitive ratio of $\frac{e}{e-1} \approx 1.58$.

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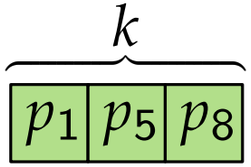

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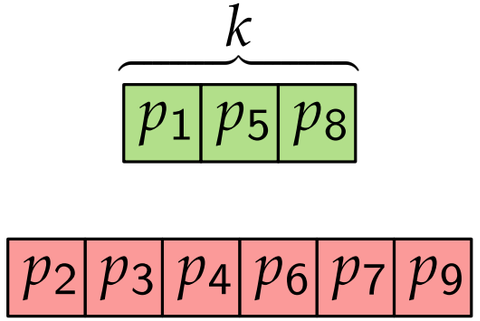

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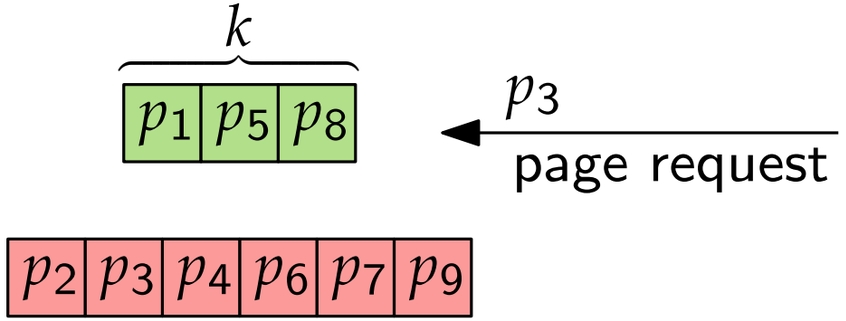

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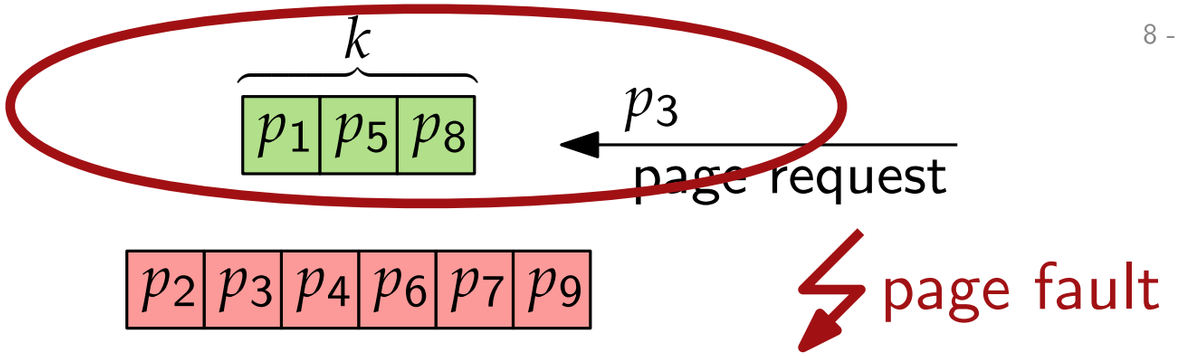

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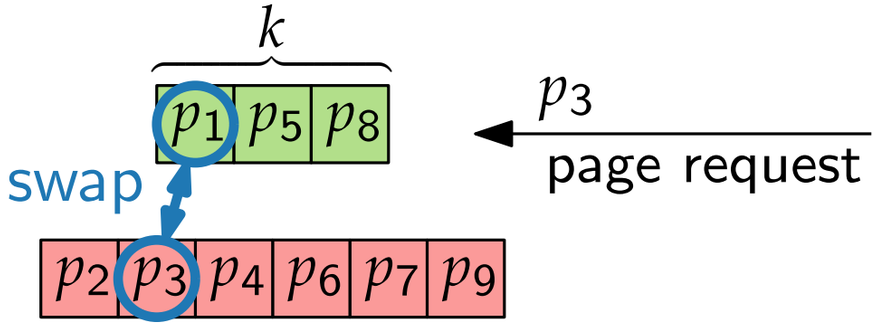

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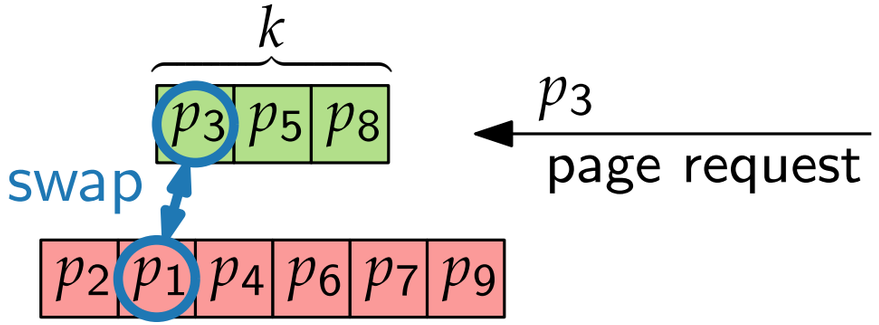

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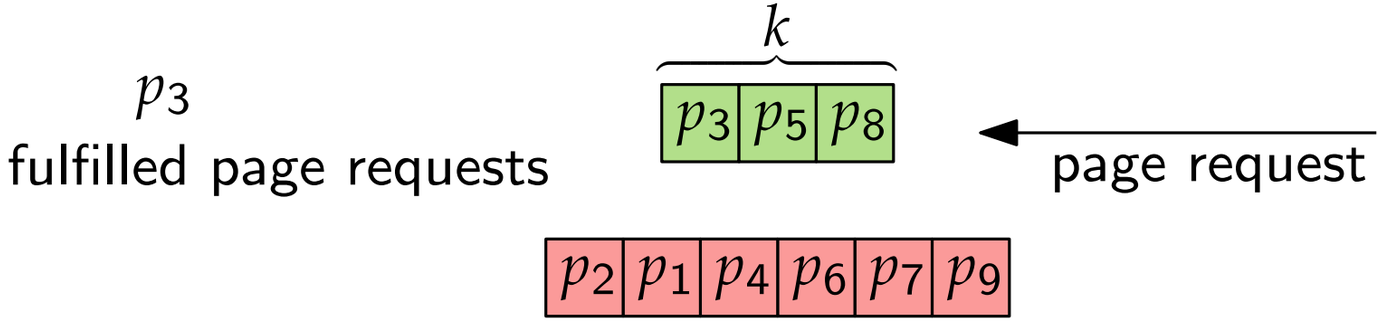

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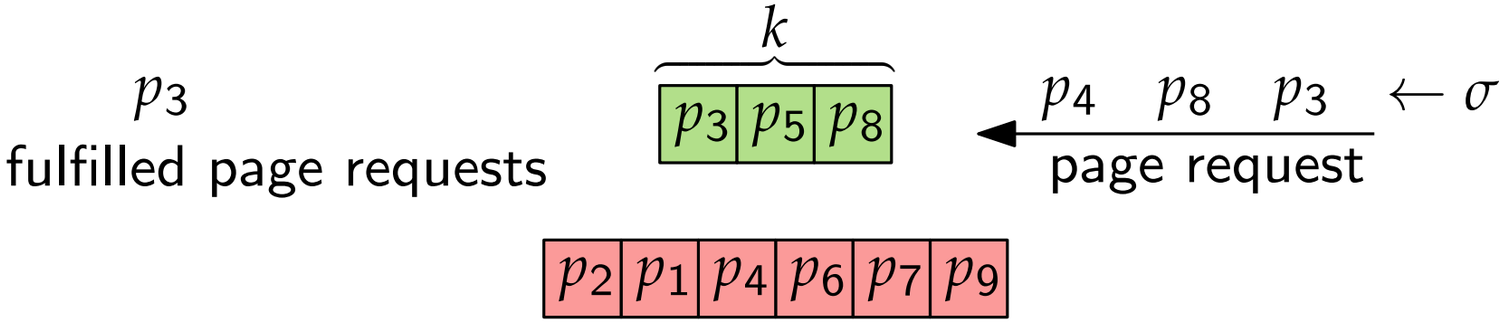

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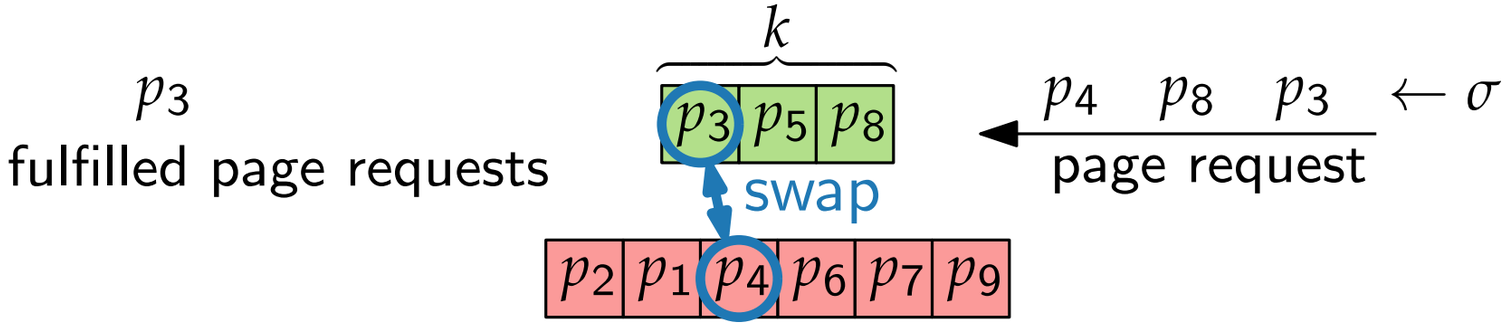

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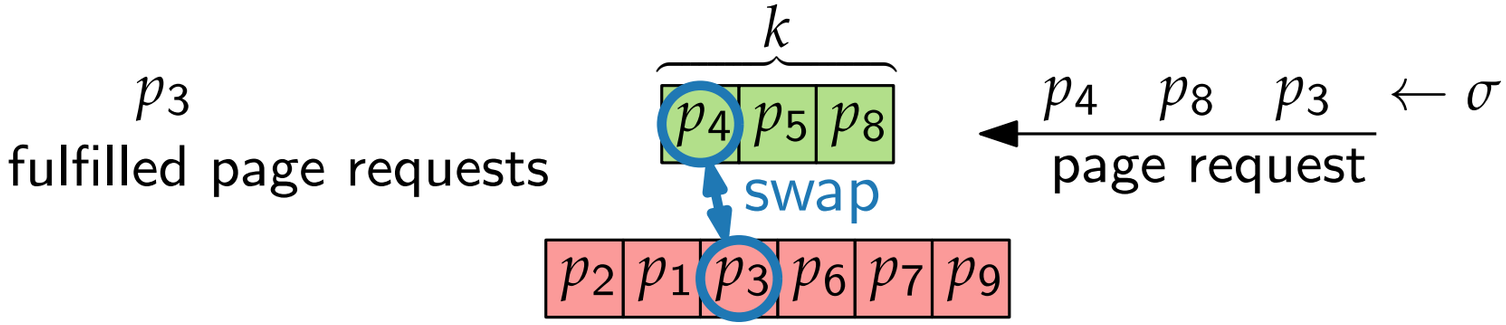

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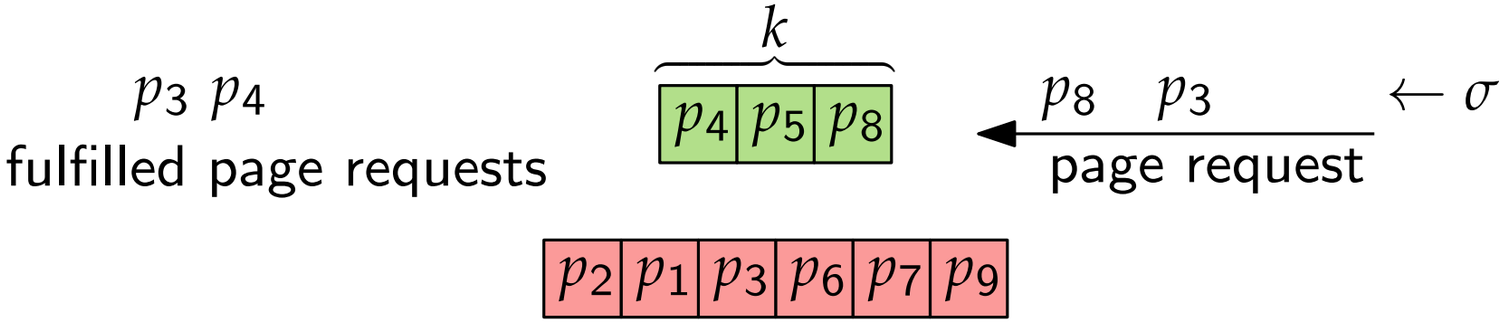

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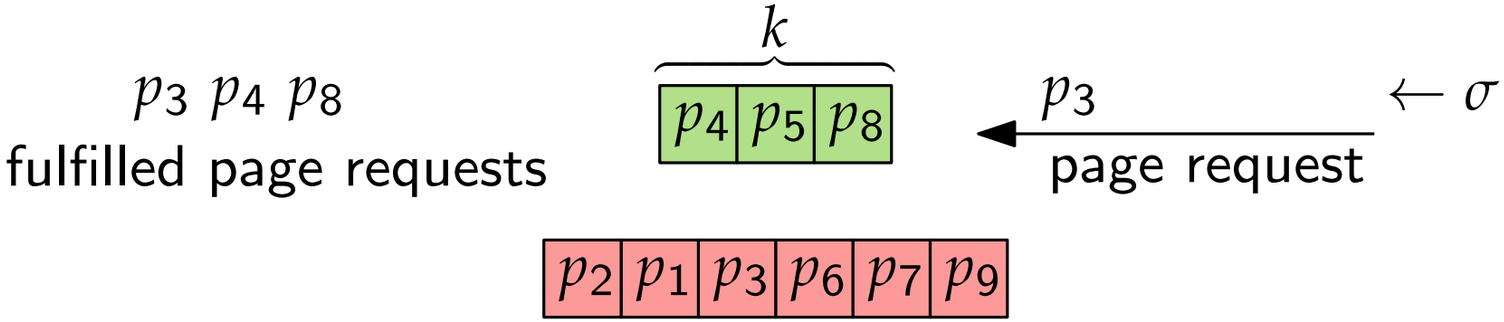

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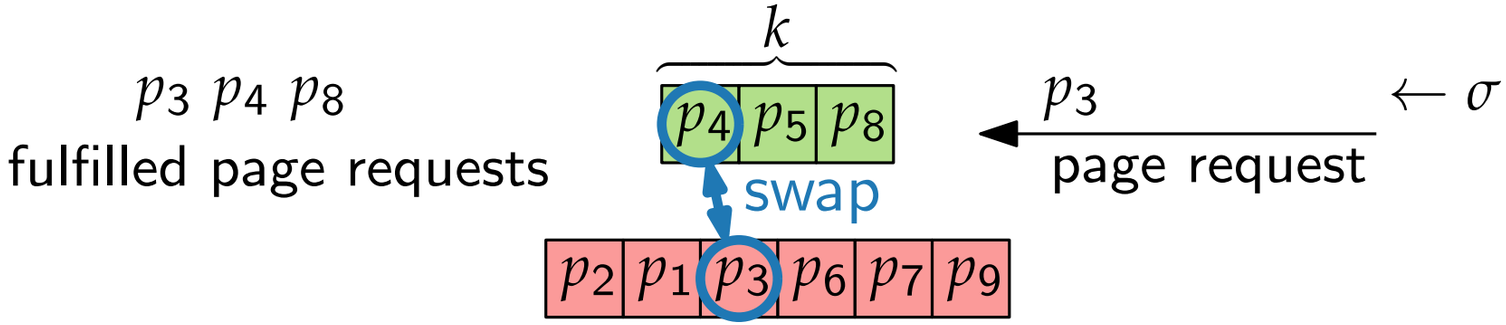

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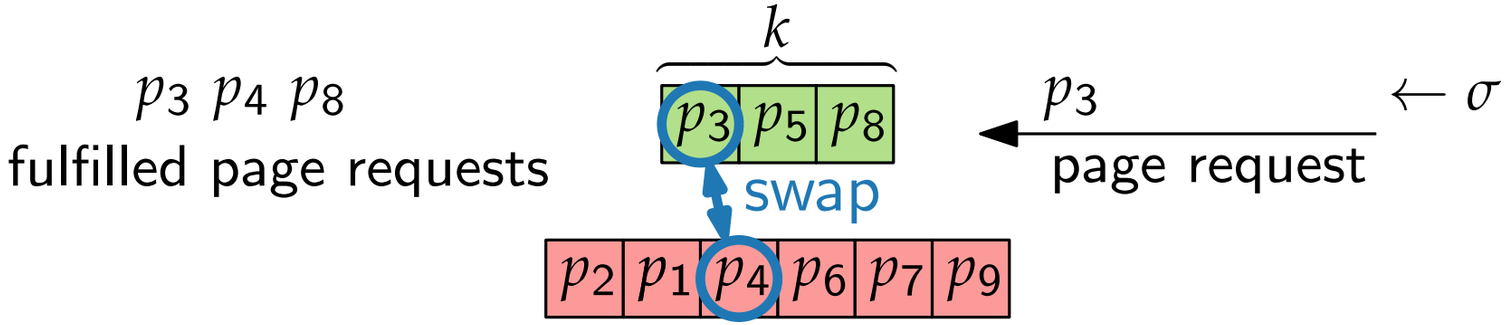

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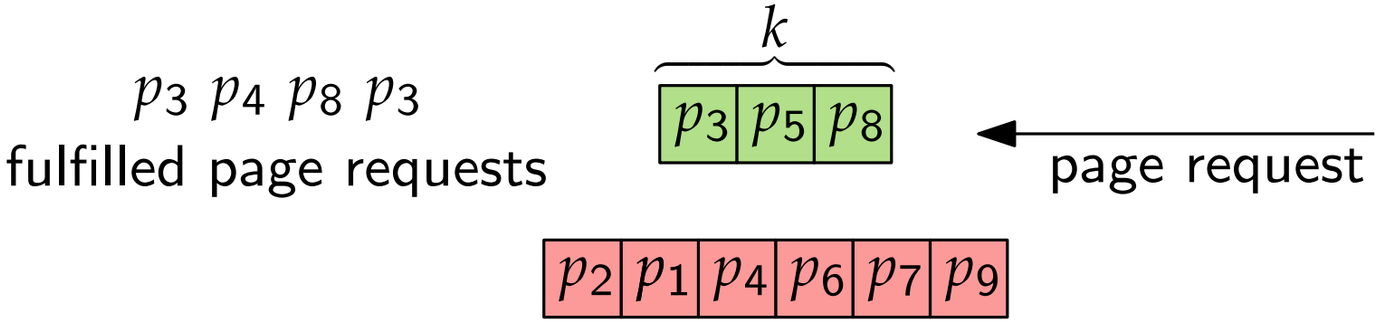

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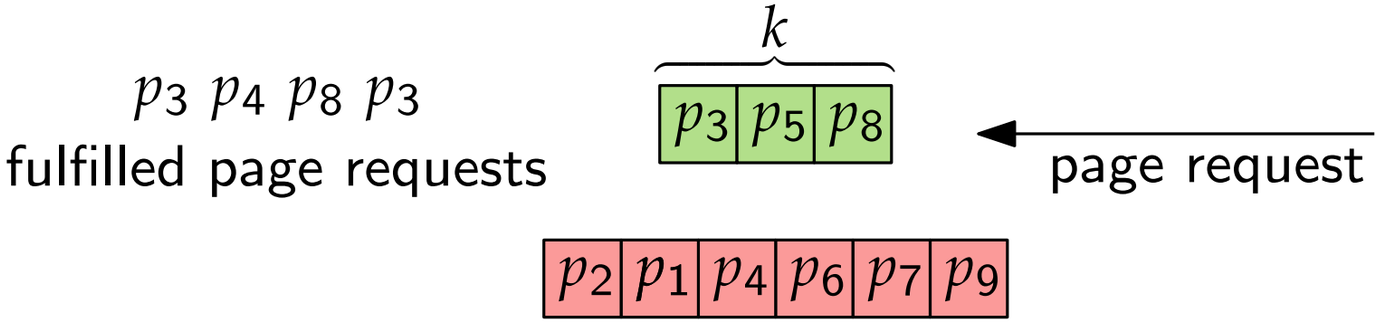

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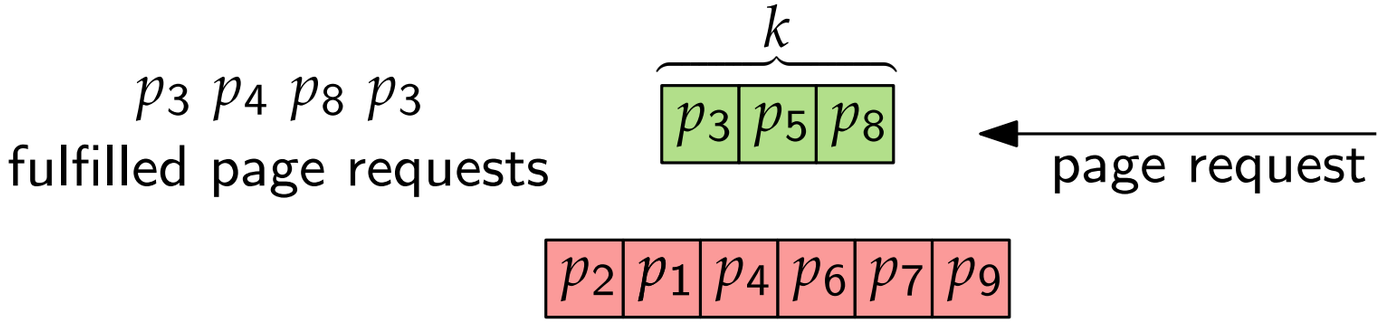

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- Minimize the number of page faults while fulfilling σ.

Paging – det. strat.

- On a page fault, a Paging algorithm chooses which page to evict from the cache.

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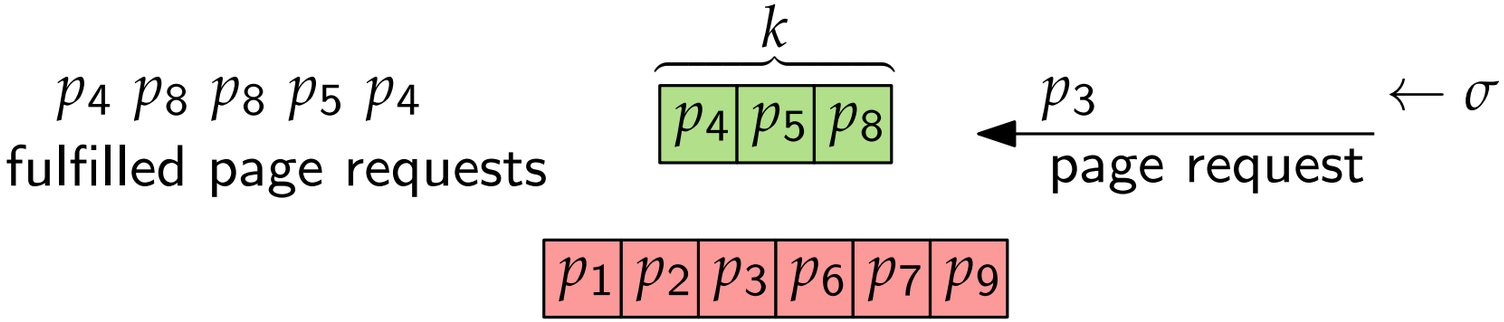

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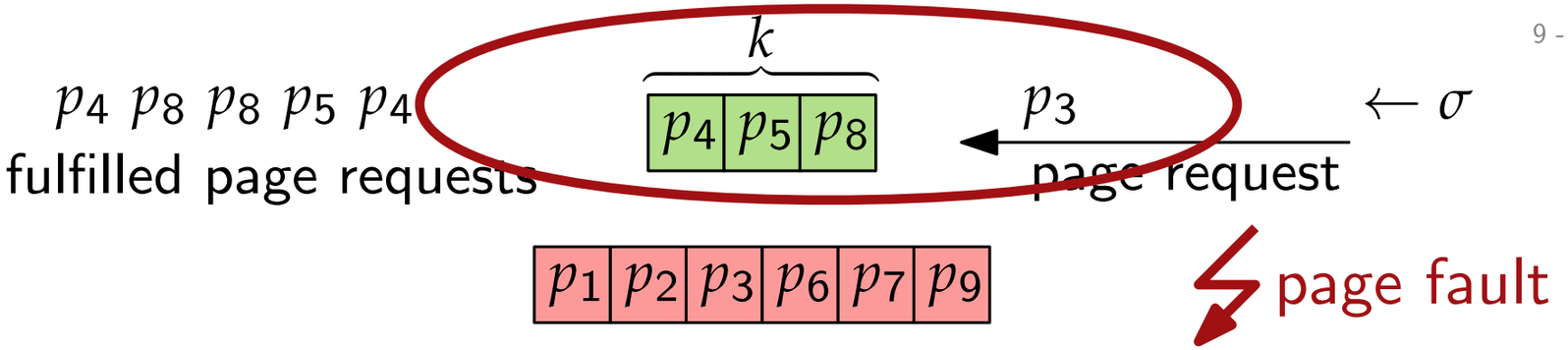

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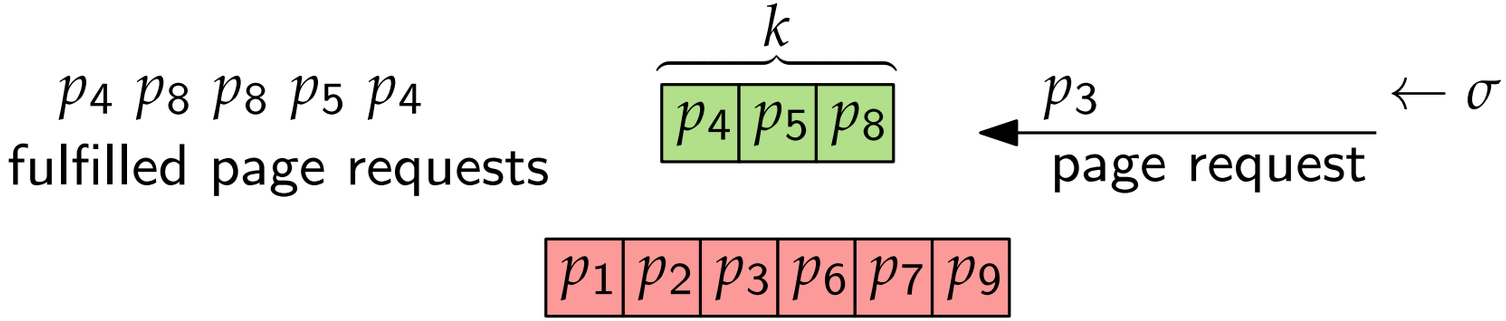

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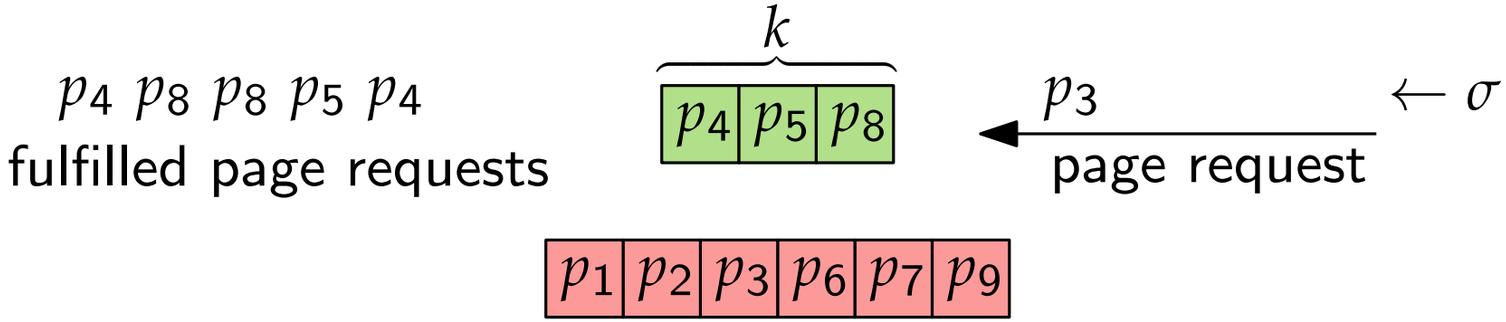

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- Least Frequently Used (LFU): . . . the lowest number of accesses since it was loaded.

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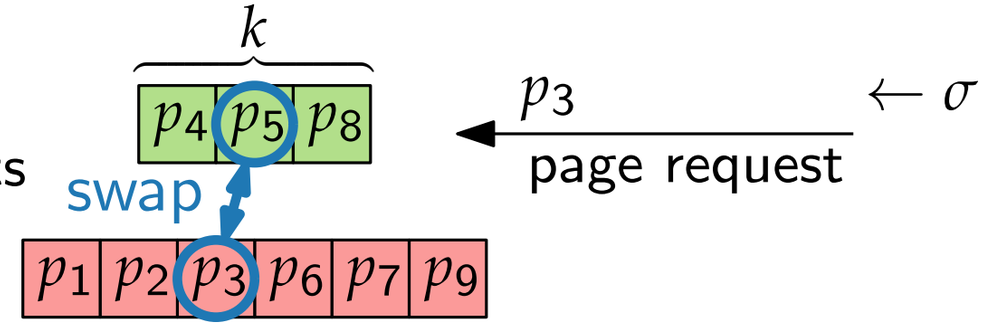
$p_4\ p_8\ p_8\ p_5\ p_4$
fulfilled page requests



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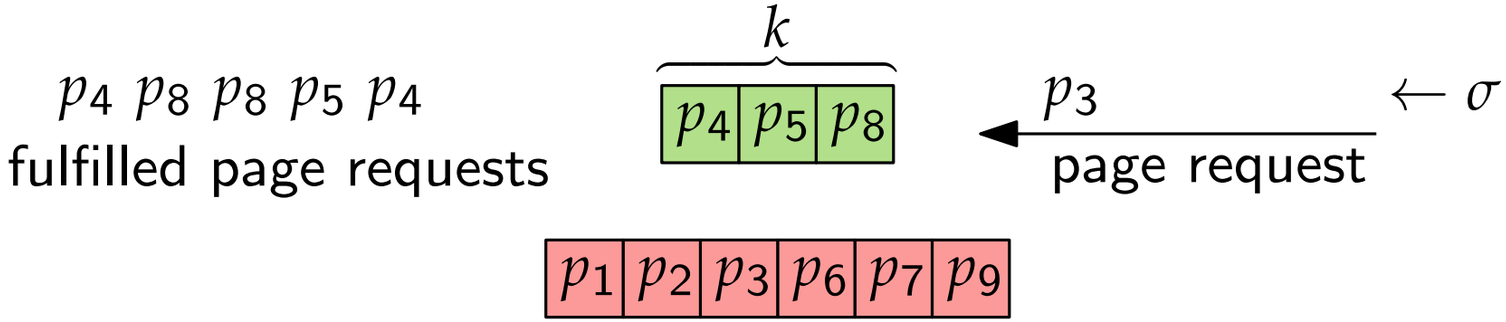

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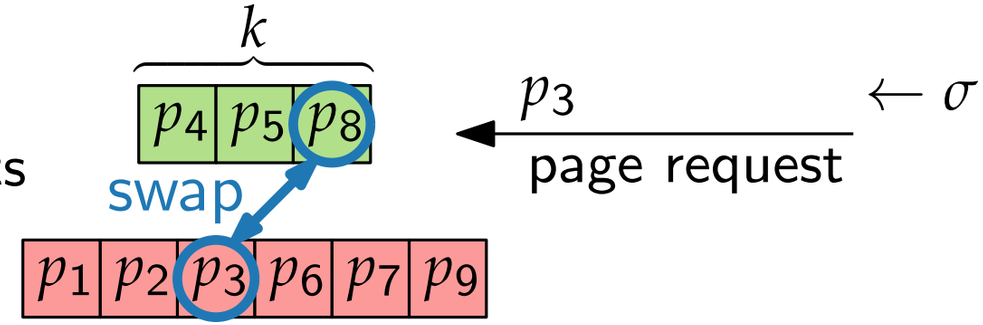

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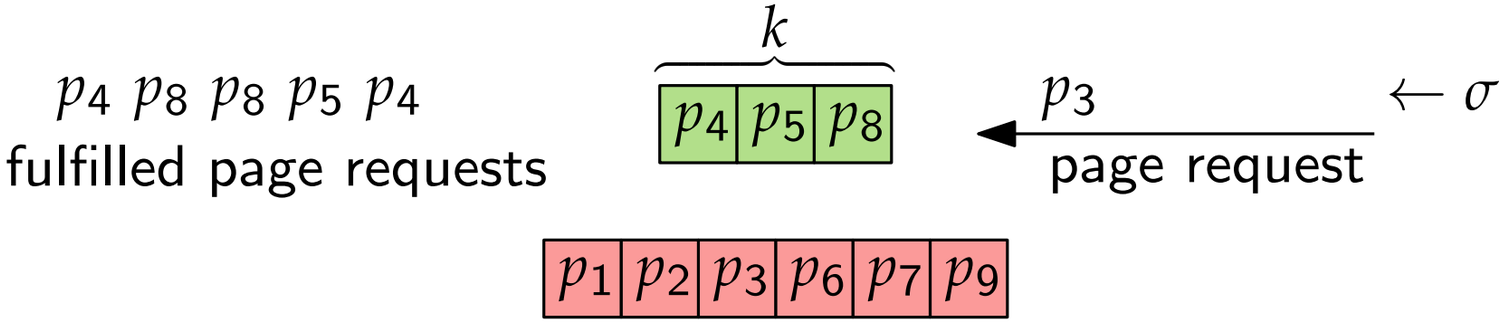

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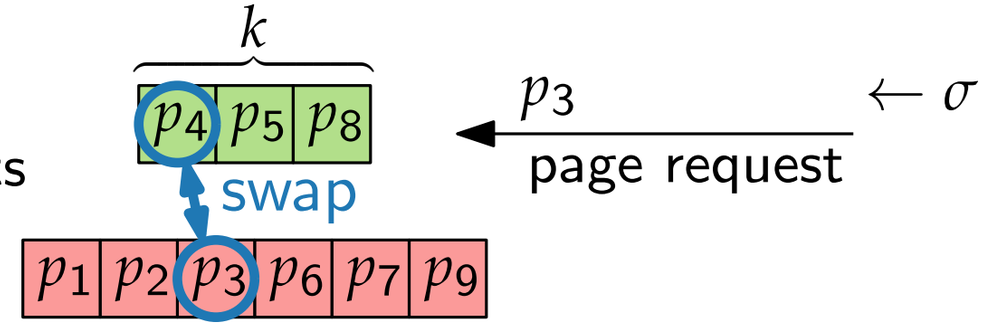

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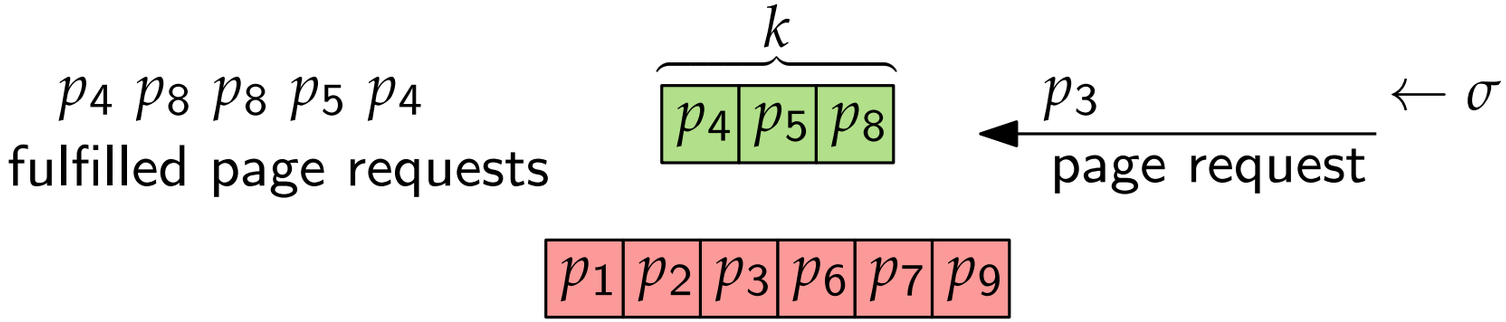

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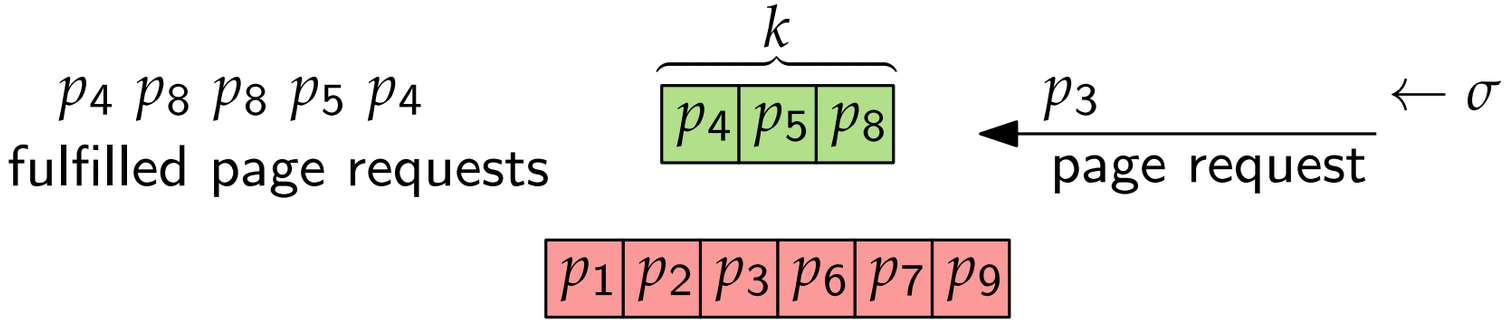

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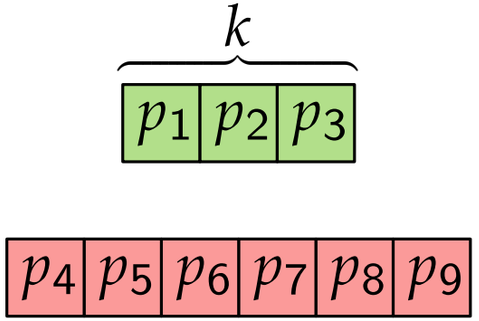

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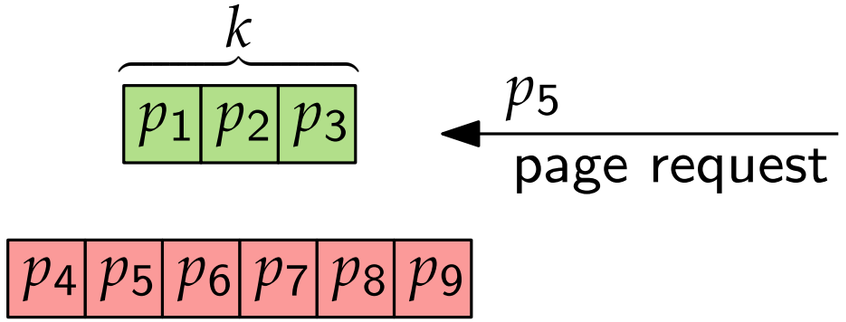

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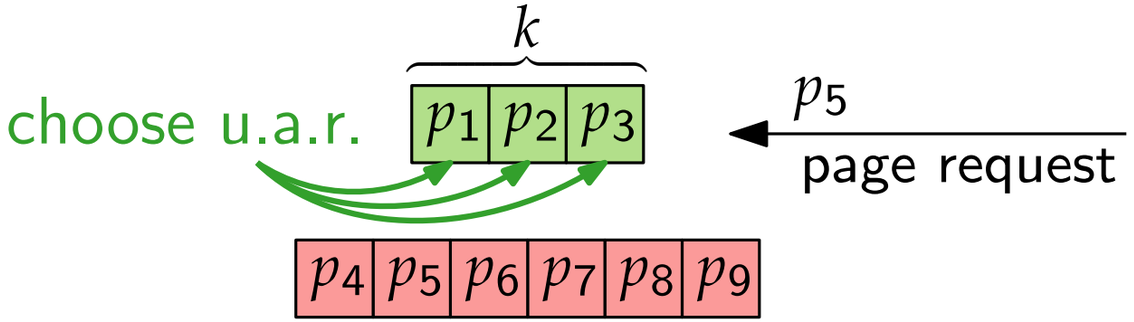

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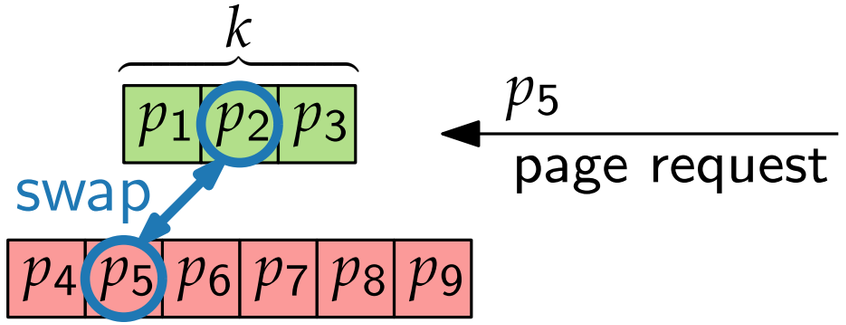

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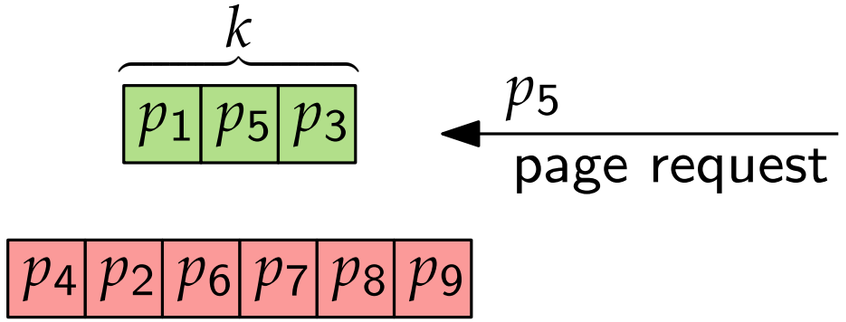

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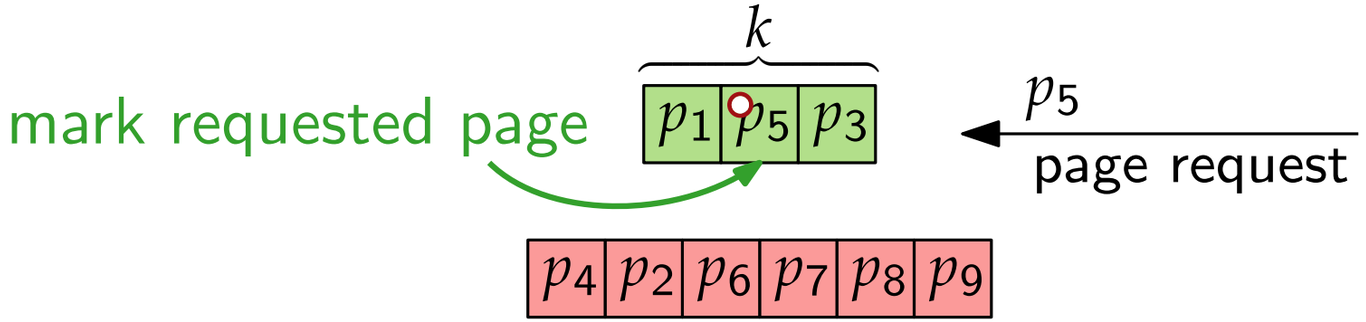

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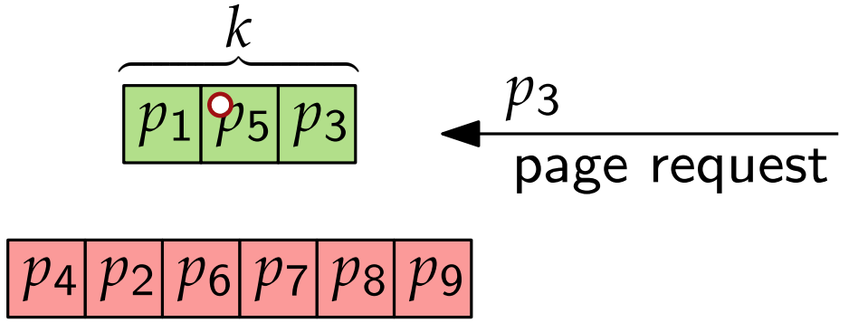

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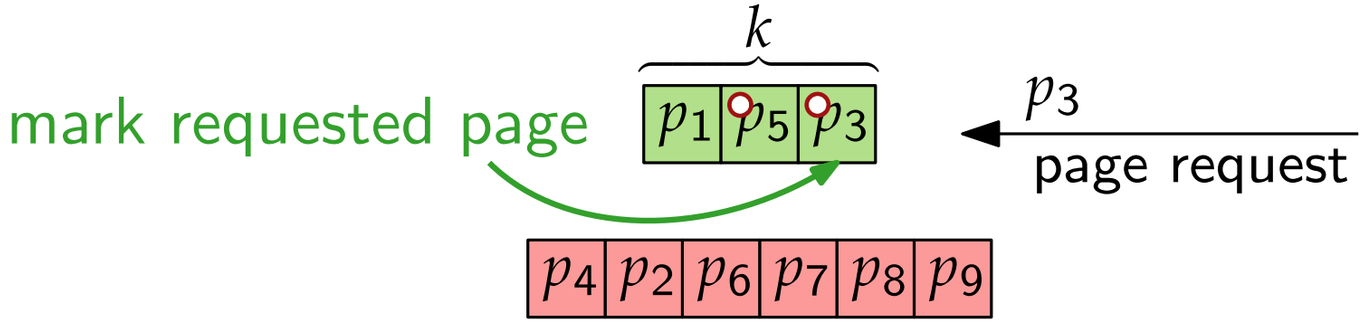

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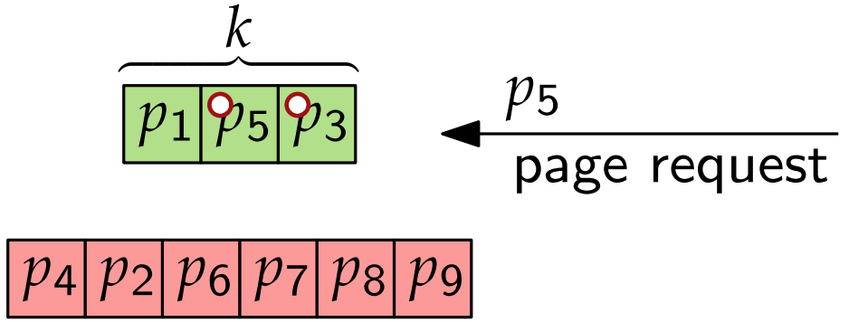

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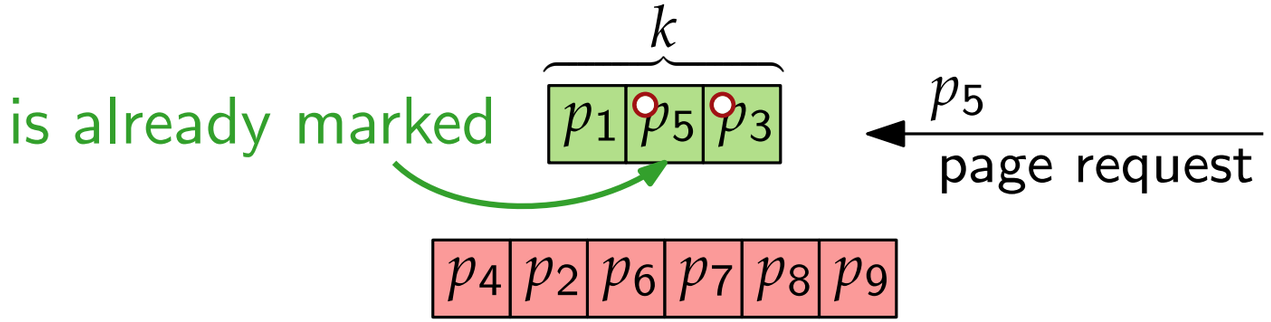

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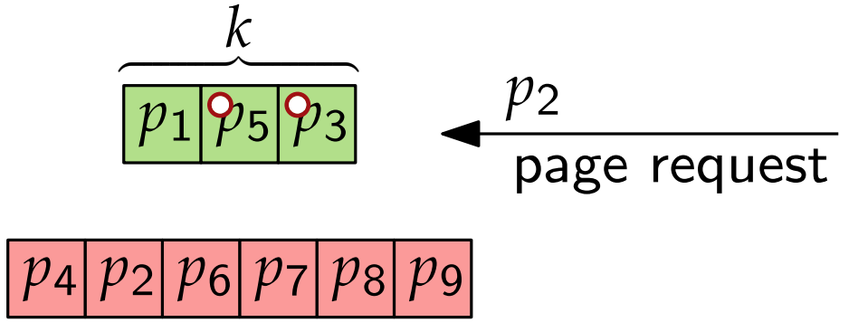

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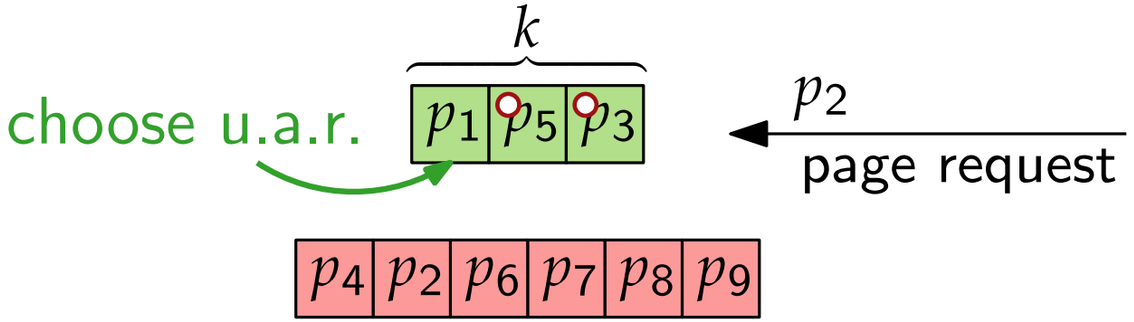

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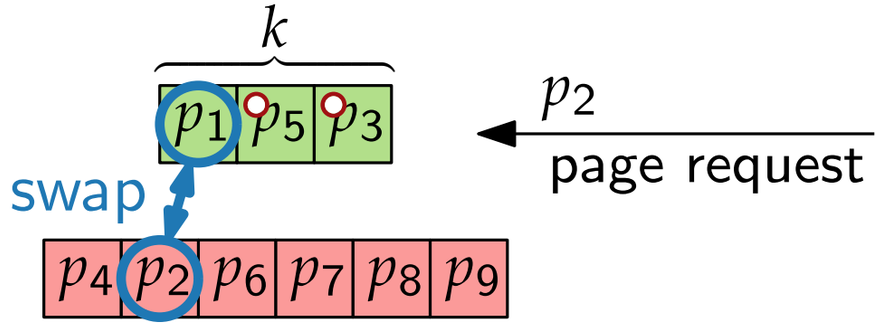

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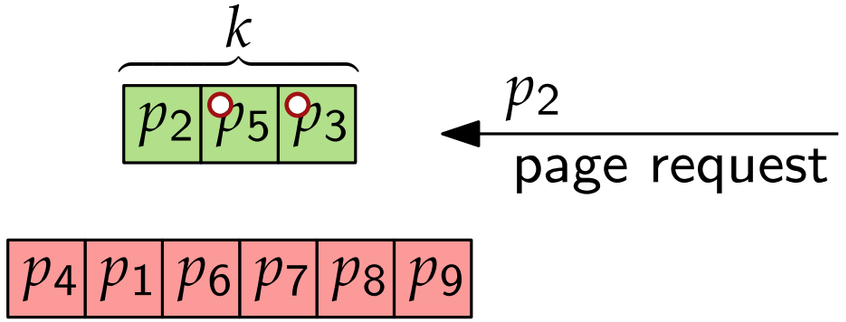

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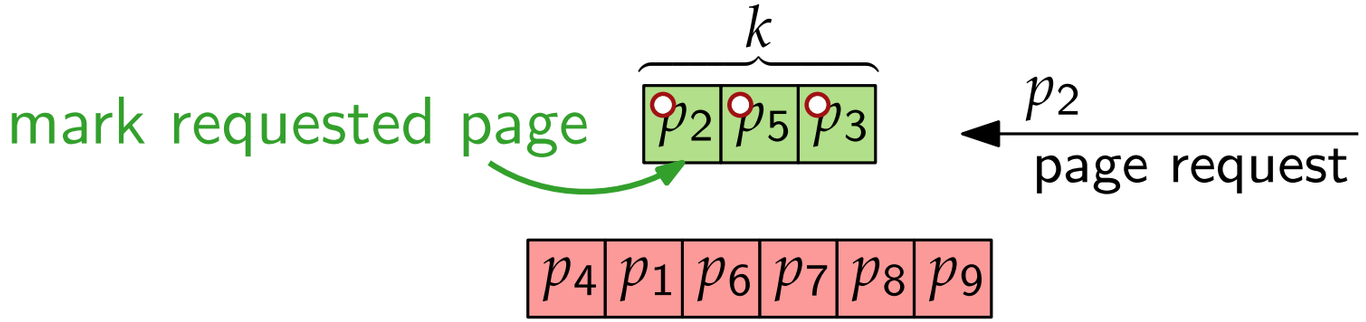

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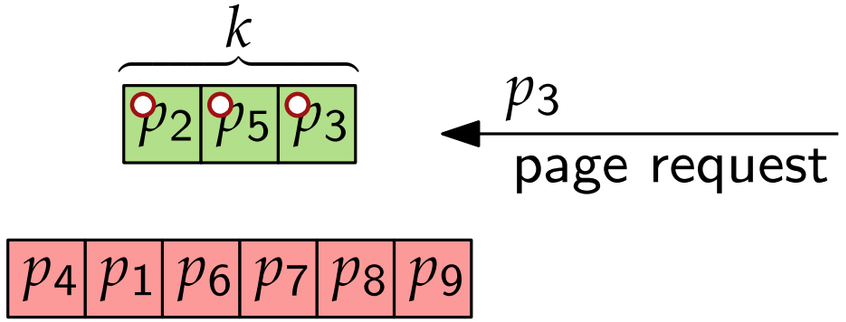
Paging – rand. strat.



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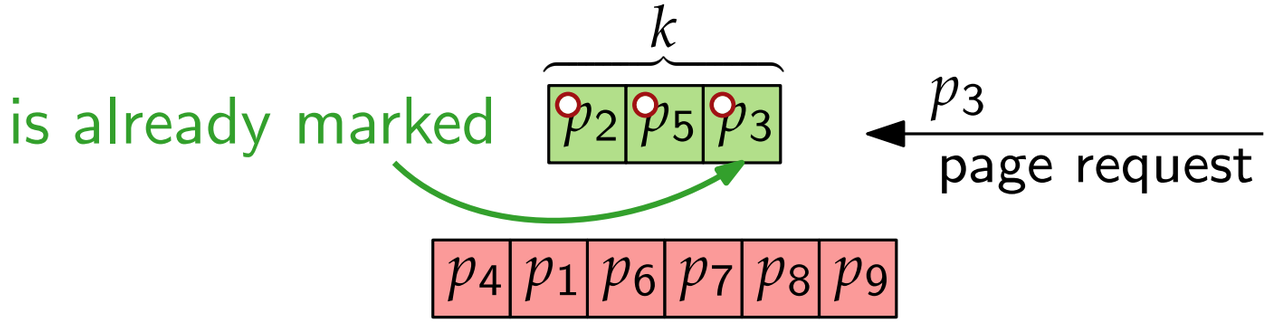

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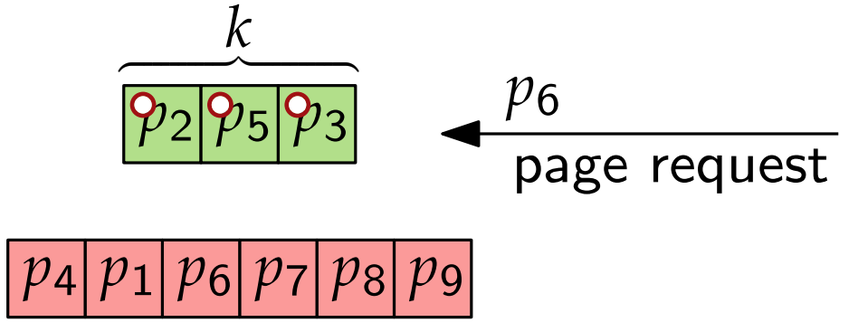

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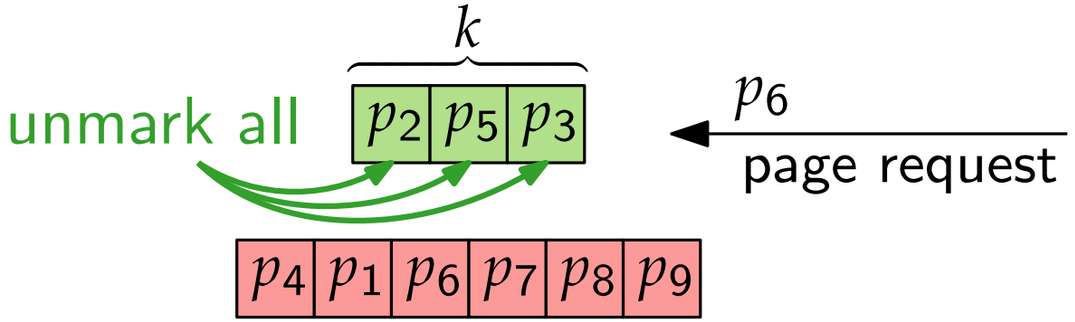

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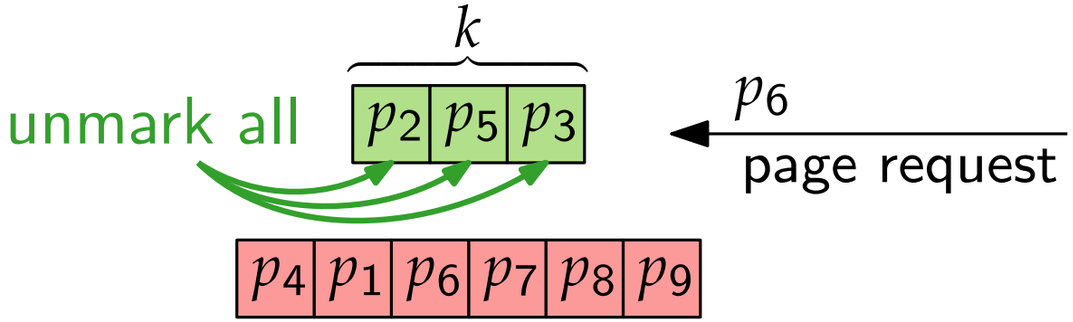

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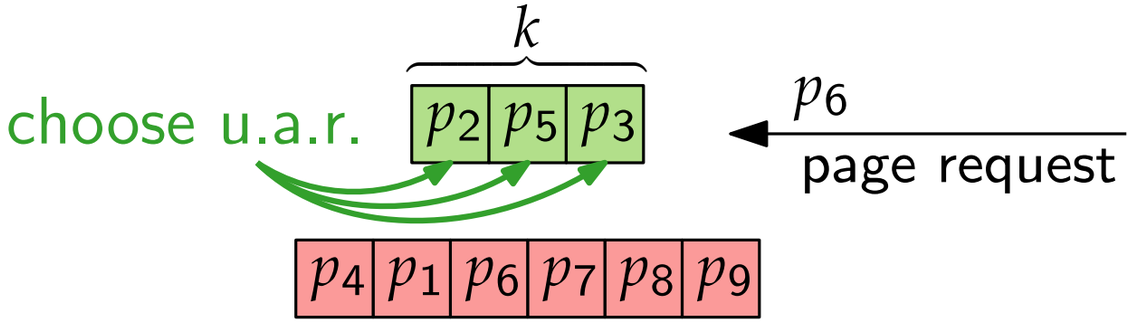

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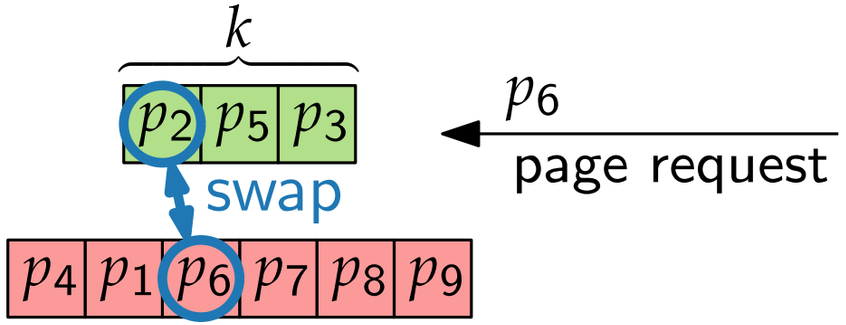

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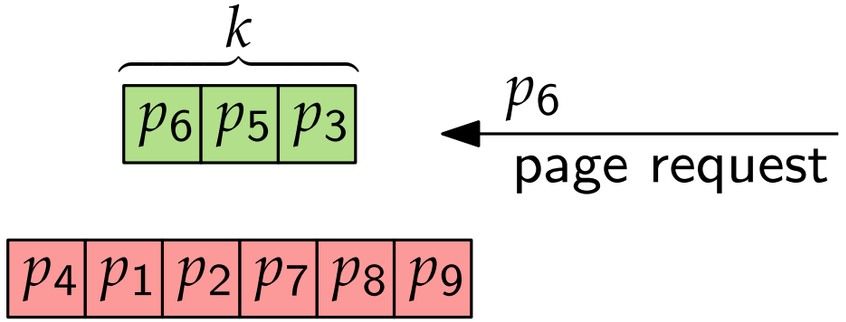

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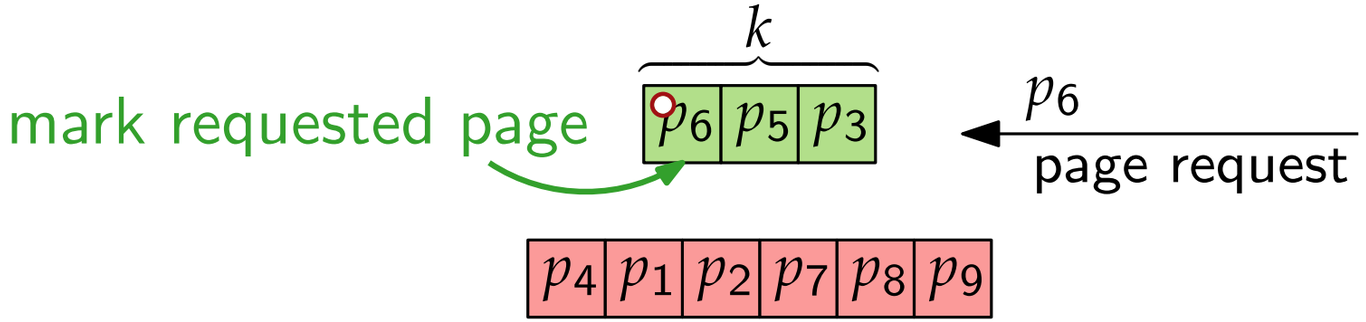

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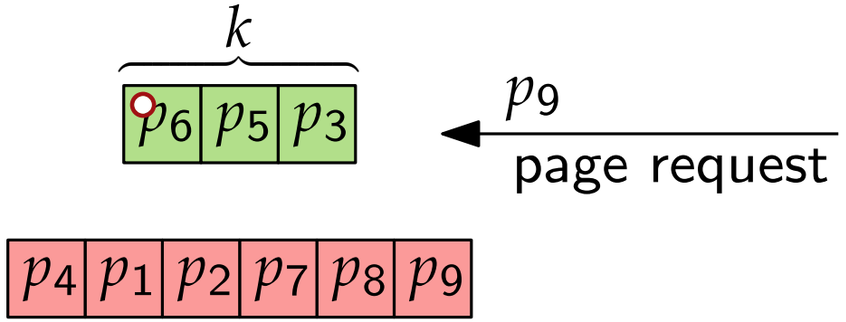

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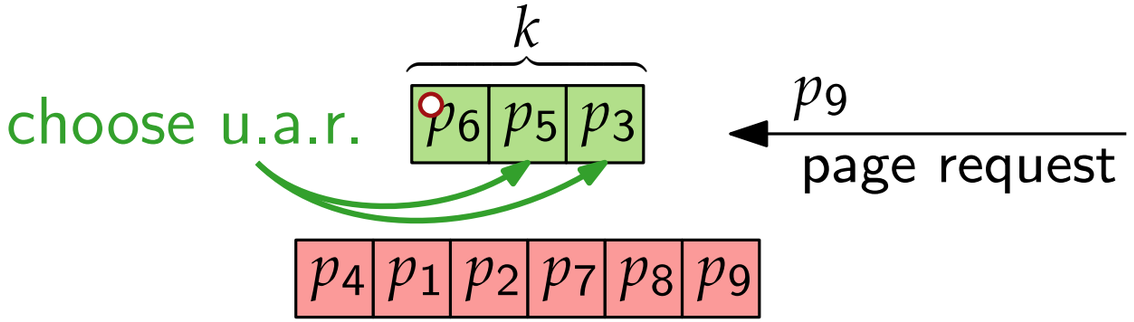

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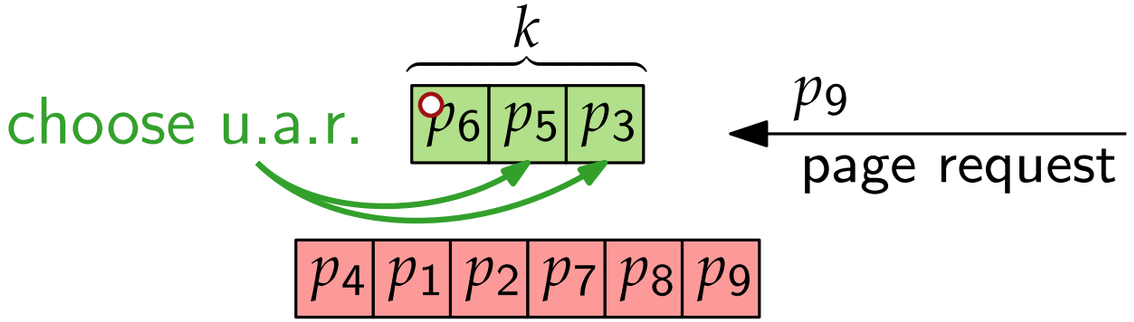

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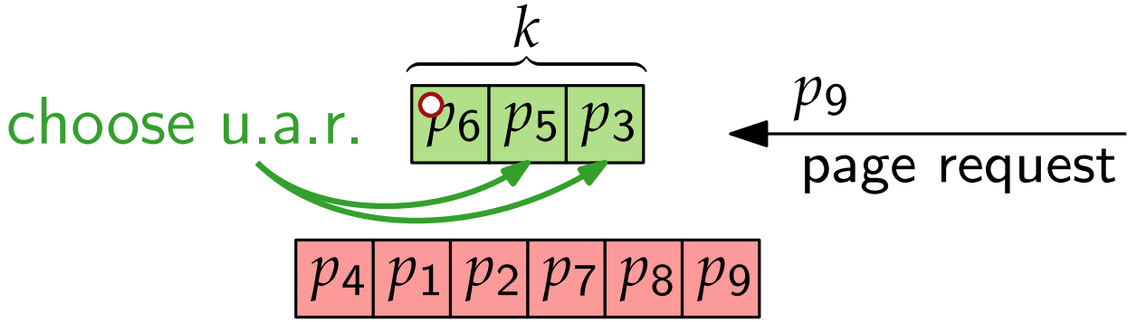
Theorem 3. MARKING is $2H_k$-competitive.

Paging – rand. strat.



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Remark.

$H_k = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{k}$ is the k-th harmonic number and for $k \geq 2$: $H_k < \ln(k) + 1$.

Paging – rand. strategy analysis

Theorem 3. MARKING is $2H_k$-competitive.

Proof.

Paging – rand. strategy analysis

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We consider phase P_i.

Paging – rand. strategy analysis

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  Over all phases, all $\frac{d_{\text{begin}}}{2}$ and $\frac{d_{\text{end}}}{2}$ cancel out, except the first $\frac{d_{\text{begin}}}{2}$ and the last $\frac{d_{\text{end}}}{2}$.
- Since the first $d_{\text{begin}} = 0$, MIN has at least $\frac{c}{2}$ faults per phase.

Paging – rand. strategy analysis

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Proof.

- For the clean pages, MARKING has c faults.

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Paging – rand. strategy analysis

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$s(j) = k + 1 - j$

Paging – rand. strategy analysis

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Paging – rand. strategy analysis

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Paging – rand. strategy analysis

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- So the competitive ratio of Marking is $\frac{c+c(H_k-1)}{c/2} = 2H_k$.

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Paging – rand. strategy analysis

Reminder.
No deterministic strategy is better than k-competitive.

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- $\sum_{j=1}^{s} E[F_j] \leq \sum_{j=1}^{s} \frac{c}{k+1-j} \leq \sum_{j=2}^{k} \frac{c}{j} = c \cdot (H_k - 1)$
- So the competitive ratio of Marking is $\frac{c+c(H_k-1)}{c/2} = 2H_k$. □

Paging – rand. strategy analysis

Reminder.
No deterministic strategy is better than k-competitive.

$\Rightarrow$ **Randomization helps!**

Theorem 3. MARKING is $2H_k$-competitive.

Proof.

- For the clean pages, MARKING has c faults.
- For the stale pages, there are $s = k - c \leq k - 1$ requests.
- For requests $j = 1, \ldots, s$ to stale pages, consider the expected number of faults $E[F_j]$.
- $c(j)$: # clean pages requested in this phase so far
  $s(j)$: # phase-initially stale pages having not been requested
- $E[F_j] = \frac{s(j)-c(j)}{s(j)} \cdot 0 + \frac{c(j)}{s(j)} \cdot 1 \leq \frac{c}{k+1-j}$ $\quad s(j) = k+1-j$
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- So the competitive ratio of Marking is $\frac{c+c(H_k-1)}{c/2} = 2H_k$. $\square$

Discussion

- Online Algorithms operate in a setting different from that of classical algorithms. However, this setting of incomplete information is very natural and occurs often in real-world applications. Can you think of further examples?

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Discussion

- Online Algorithms operate in a setting different from that of classical algorithms. However, this setting of incomplete information is very natural and occurs often in real-world applications. Can you think of further examples?
- We might also transform a classical problem with incomplete information into an online problem. E.g.: Matching problem for ride sharing.
- Randomization can help to improve our behavior on worst-case instances. You may also think of: we are less predictable for an adversary.

Literature

Main source:

- Sabine Storandt's lecture script "Randomized Algorithms" (2016–2017)

Original papers:

- [Belady'66] "A Study of Replacement Algorithms for Virtual-Storage Computer."
- [Sleator, Tarjan'85] "Amortized Efficiency of List Update and Paging Rules."
- [Fiat, Karp, Luby, McGeoch, Sleator, Young'91] "Competitive Paging Algorithms."