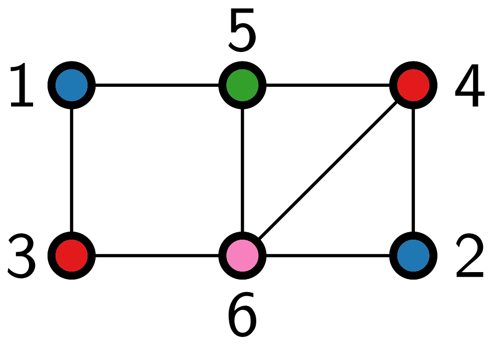
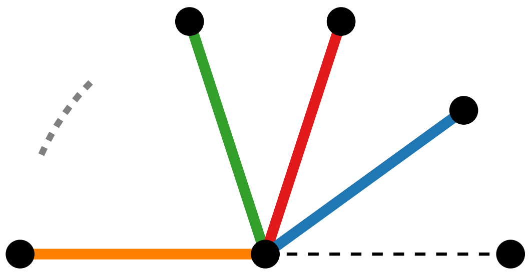
Advanced Algorithms

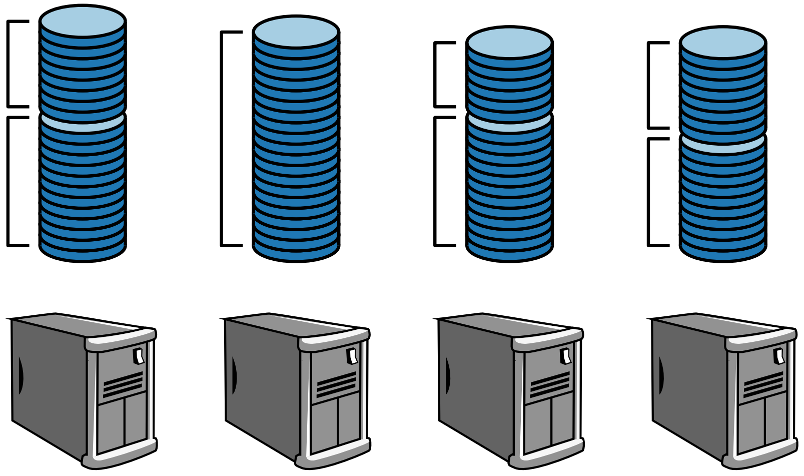
Approximation algorithms

Coloring and scheduling problems

Jonathan Klawitter · WS20



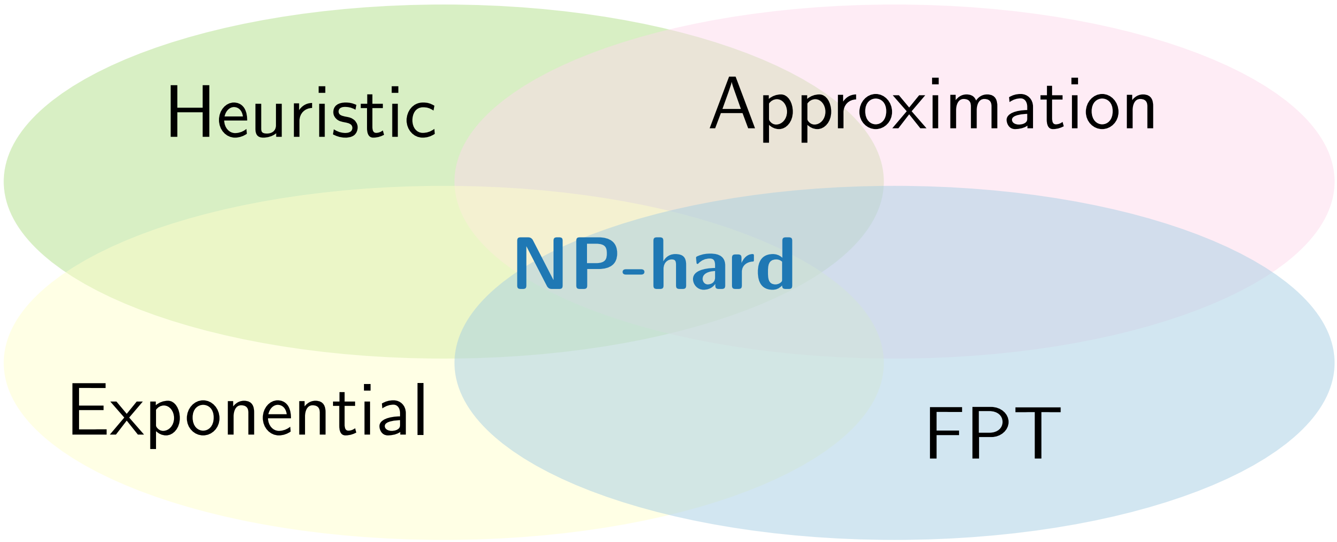




Dealing with NP-hard problems

What should we do?

- Sacrifice optimality for speed
  - Heuristics
  - Approximation Algorithms
- Optimal Solutions
  - Exact exponential-time algorithms
  - Fine-grained analysis – parameterized algorithms



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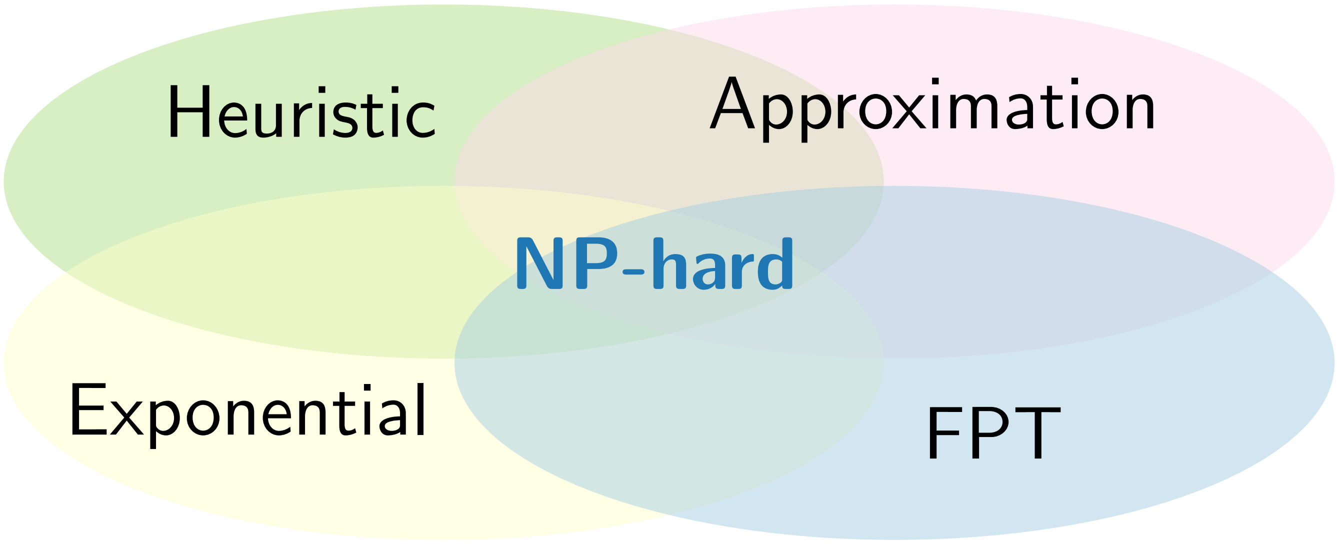

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- For NP-hard optimisation problems, we cannot compute the optimal solution of each instance efficiently (unless $\mathrm{P} = \mathrm{NP}$).
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Overview.

- Approximation algorithms that compute solutions with/that are
  - additive guarantee,
  - relative guarantee,
  - "arbitraility good".

Approximation with additive guarantee

Definition.
Let Π be an optimisation problem and let $\mathcal{A}$ be a polynomial-time algorithm that computes the value $\mathcal{A}(I)$ for an instance I of Π.
$\mathcal{A}$ is called an **approximation algorithm with additive guarantee** δ if

$$|\mathsf{OPT}(I) - \mathcal{A}(I)| \leq \delta(I)$$

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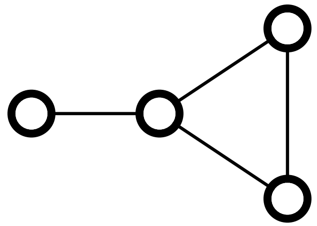
- Most problems do not admit an approximation algorithm with additive guarantee.

Minimum vertex coloring



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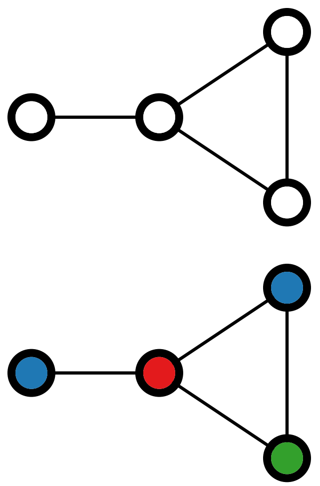

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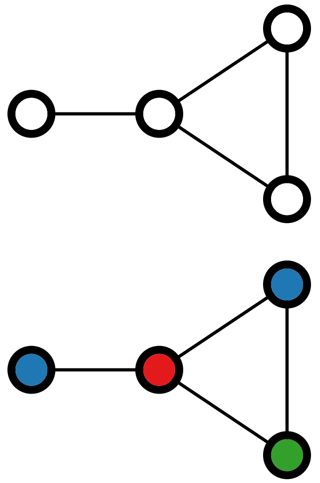

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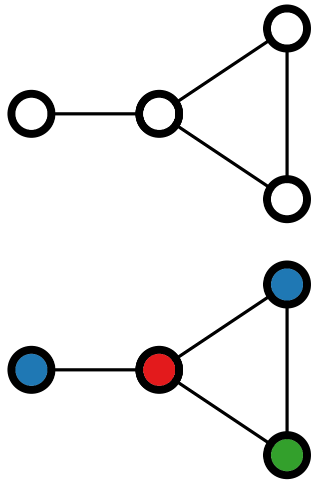

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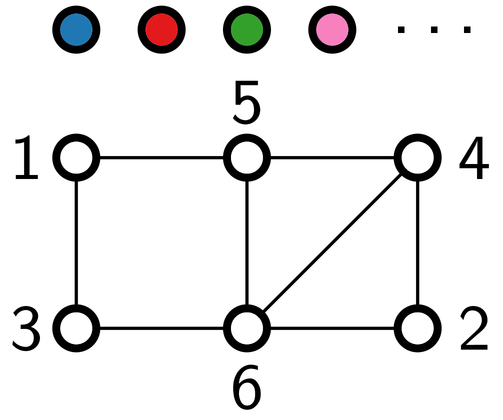

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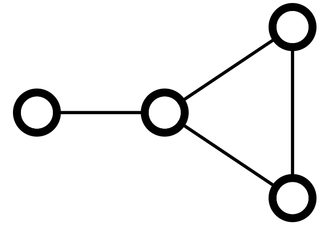

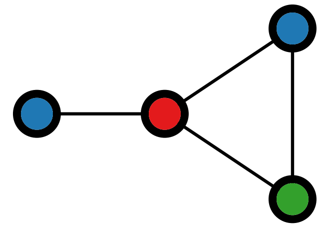

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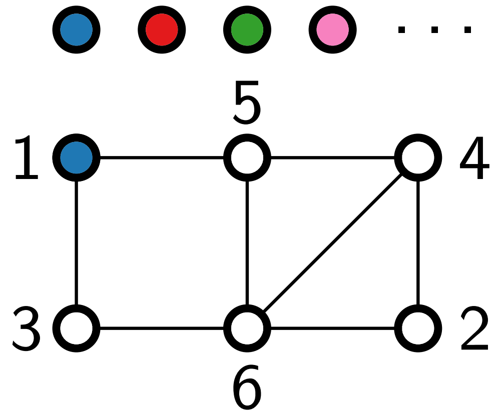

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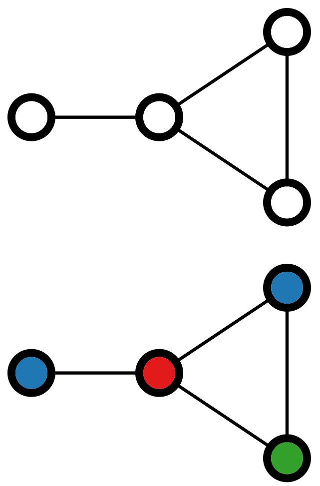

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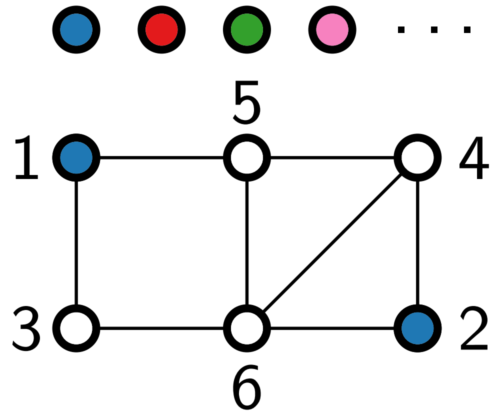

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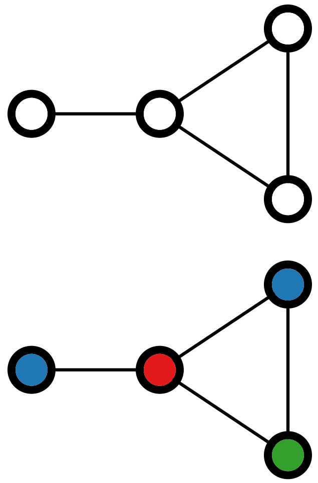

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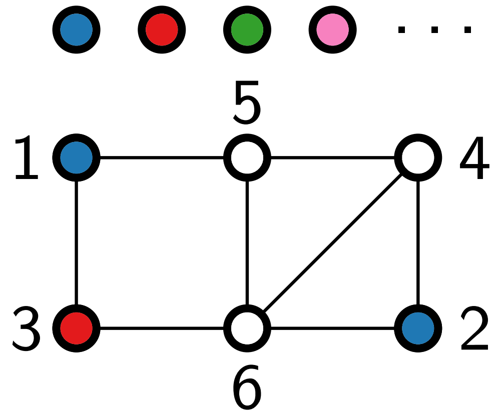

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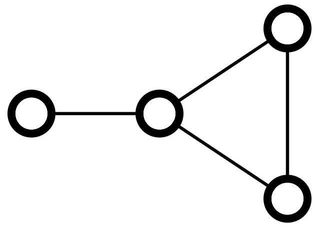

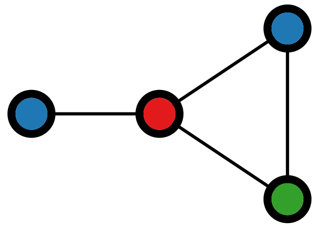

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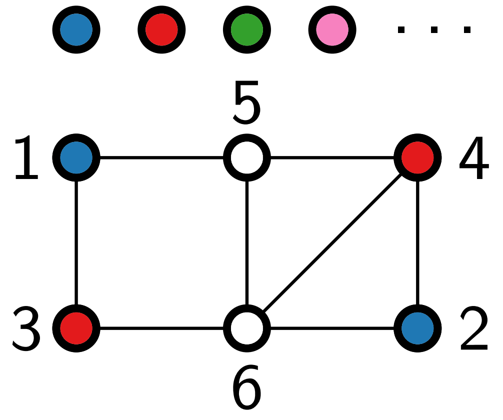

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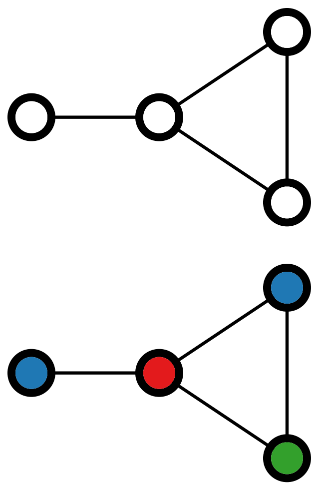

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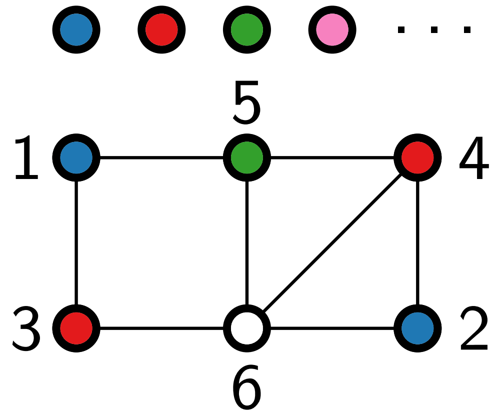

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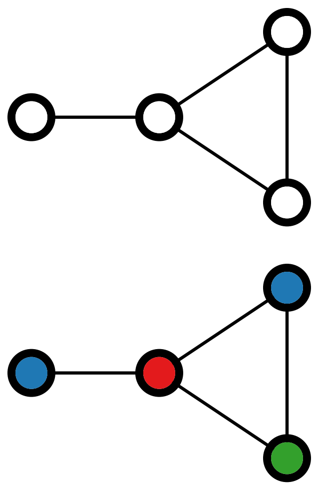

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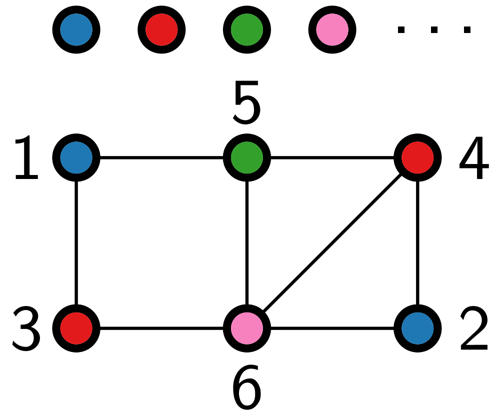

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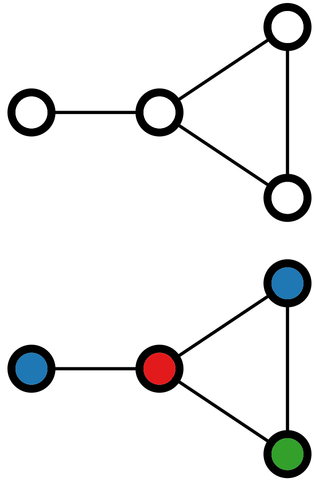

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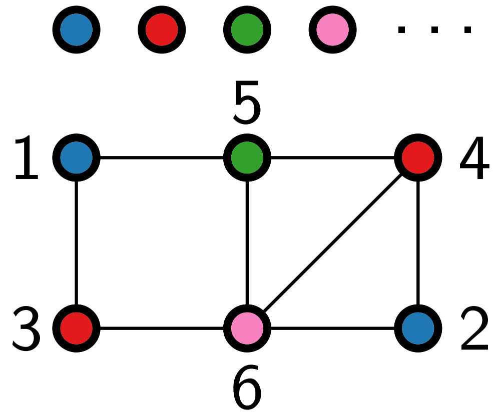

Theorem 1.
The algorithm `GreedyVertexColoring` computes a vertex coloring with at most $\Delta + 1$ colors in $\mathcal{O}(n + m)$ time.
Hence, it has an additive approximation gurantee of $\Delta - 1$.

Minimum edge coloring

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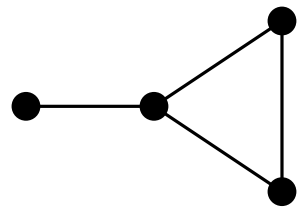

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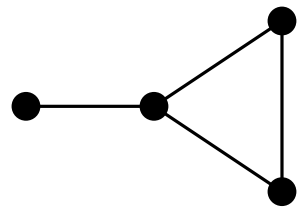

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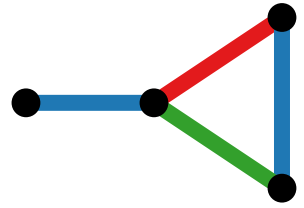

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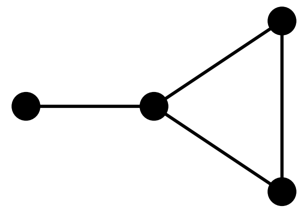

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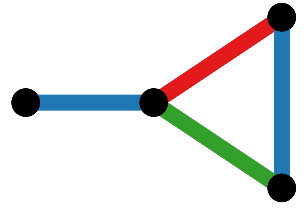

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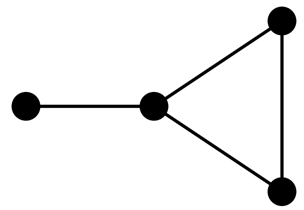

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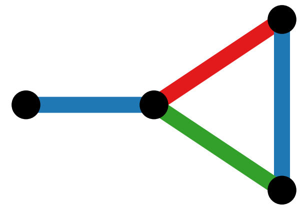

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- $\chi'(G)$ is lower bounded by

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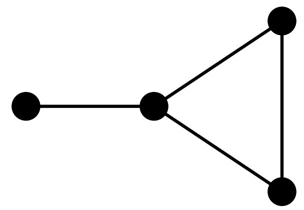

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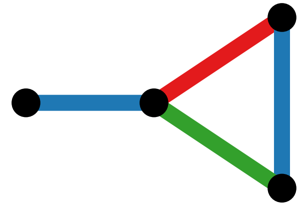

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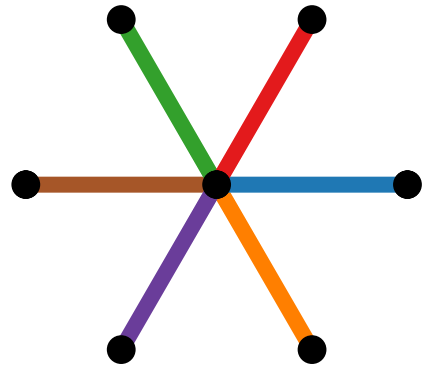

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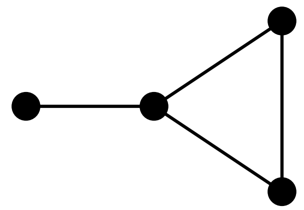

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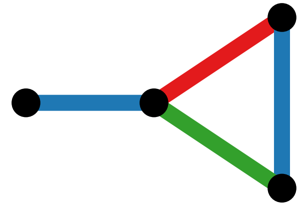

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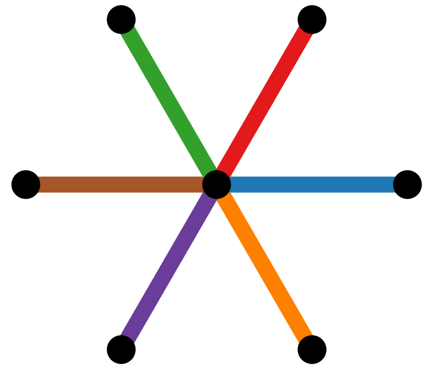
- We show that $\chi'(G) \leq \Delta + 1$.



Minimum edge coloring – upper bound

Vizing's Theorem.
For every graph $G = (V, E)$ with maximum degree Δ holds that $\Delta \leq \chi'(G) \leq \Delta + 1$.

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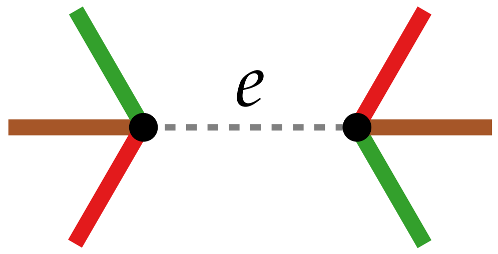

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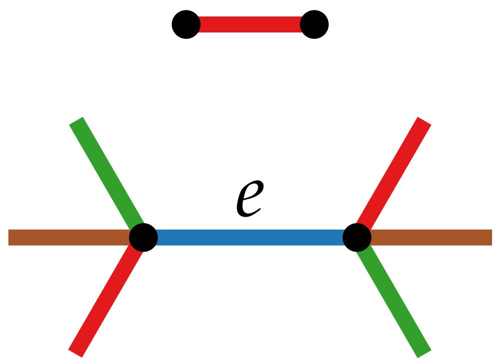

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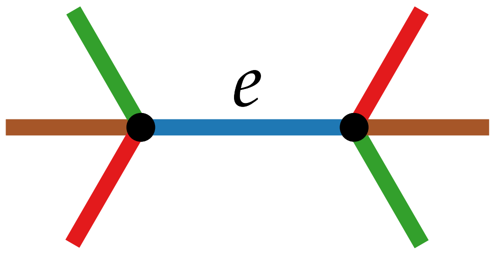

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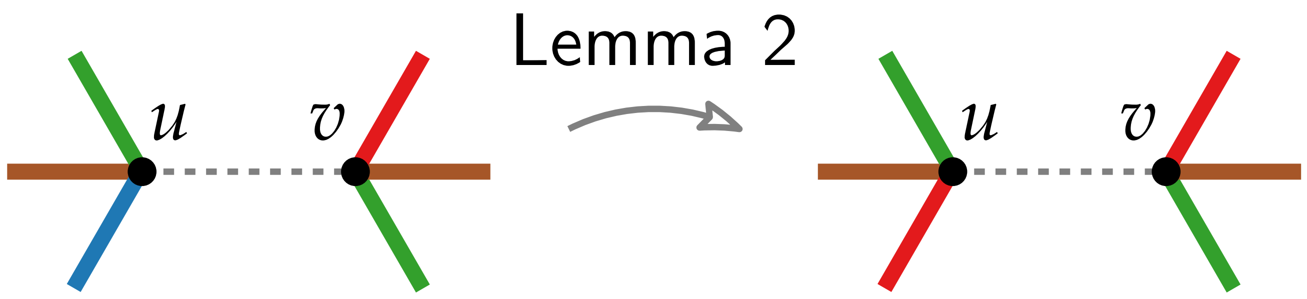
- If $\Delta(G) = \Delta(G - e)$, change coloring such that u and v (of $e = \{u, v\}$) miss the same color α.



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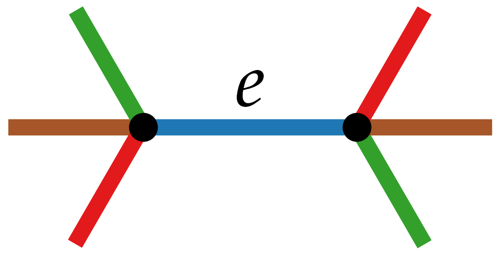

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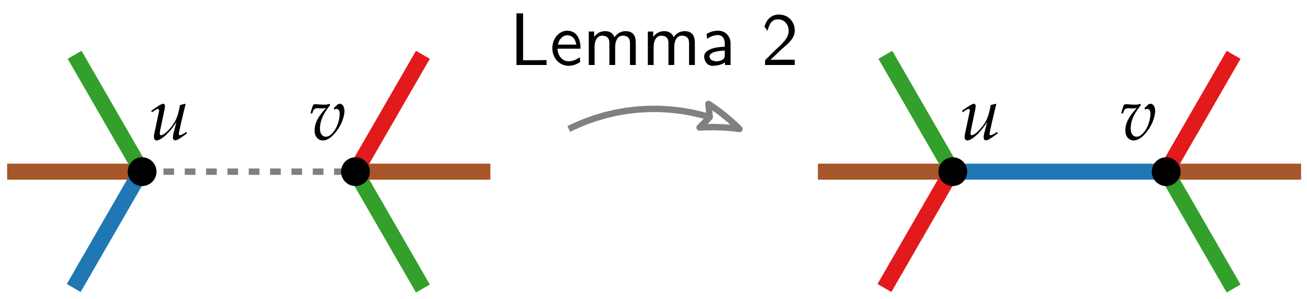

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Lemma 2.
Let G have a $(\Delta+1)$ edge coloring c, let u, v be non-adjacent, and $\deg(u), \deg(v) < \Delta$. Then c can be changed such that u and v miss the same color.

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Proof. Note, each vertex is **missing** a color.

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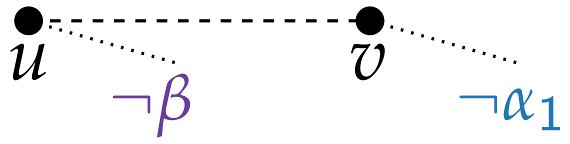

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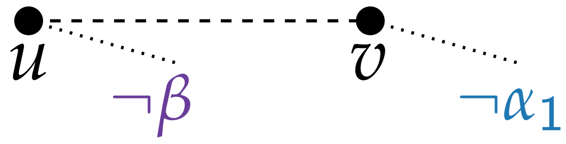

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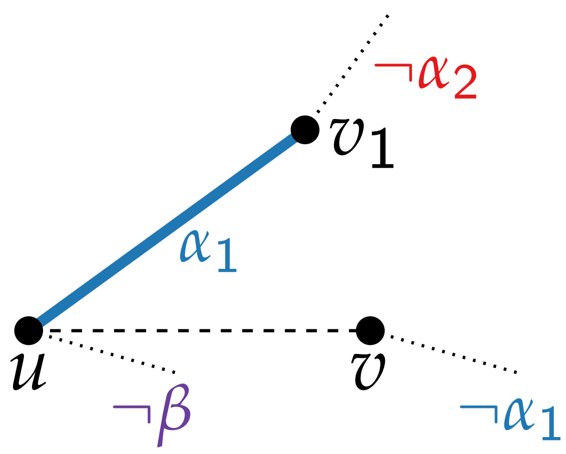

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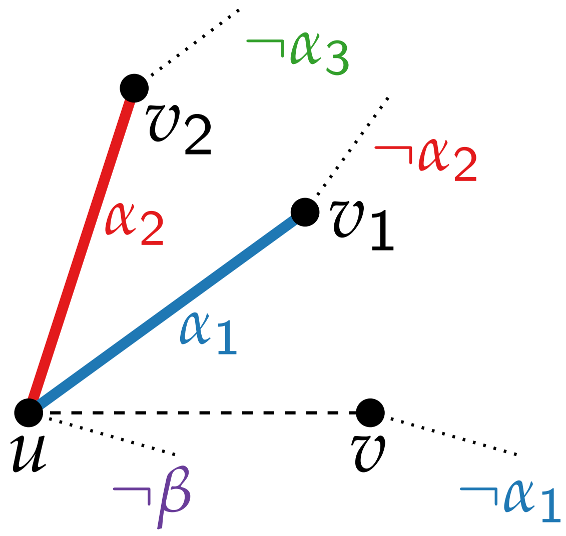

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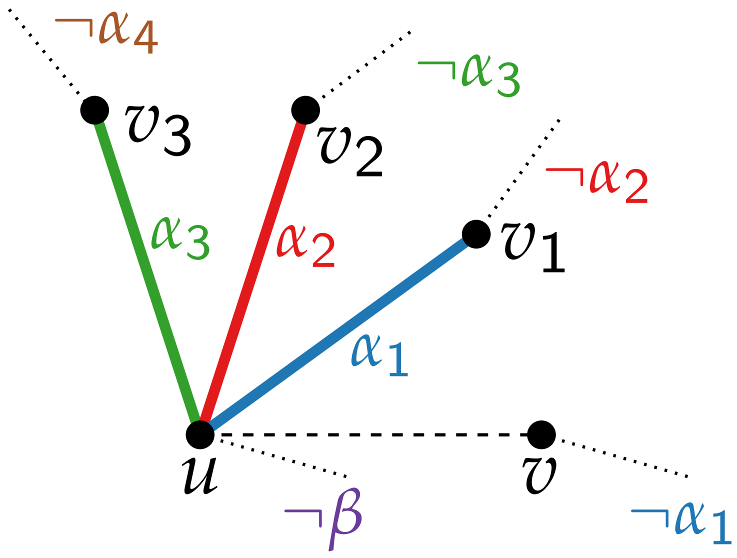

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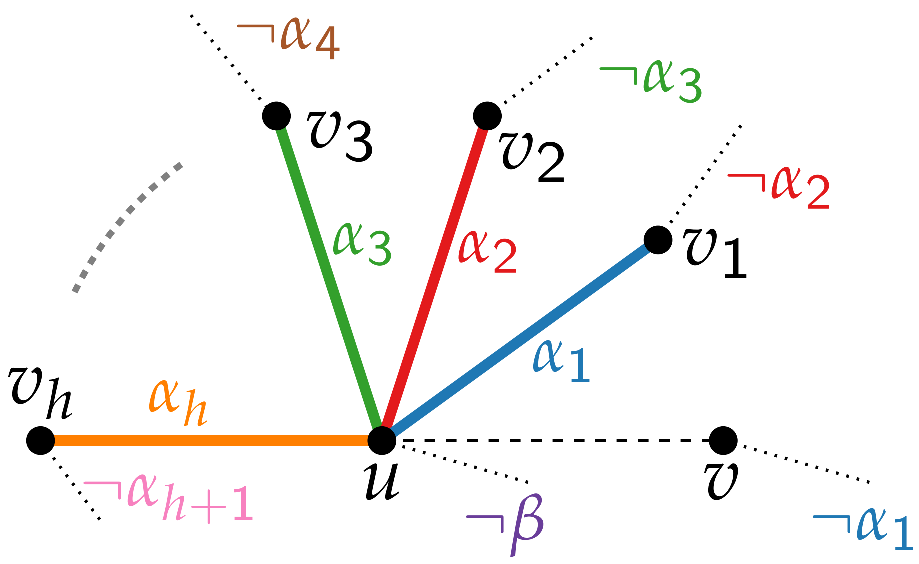

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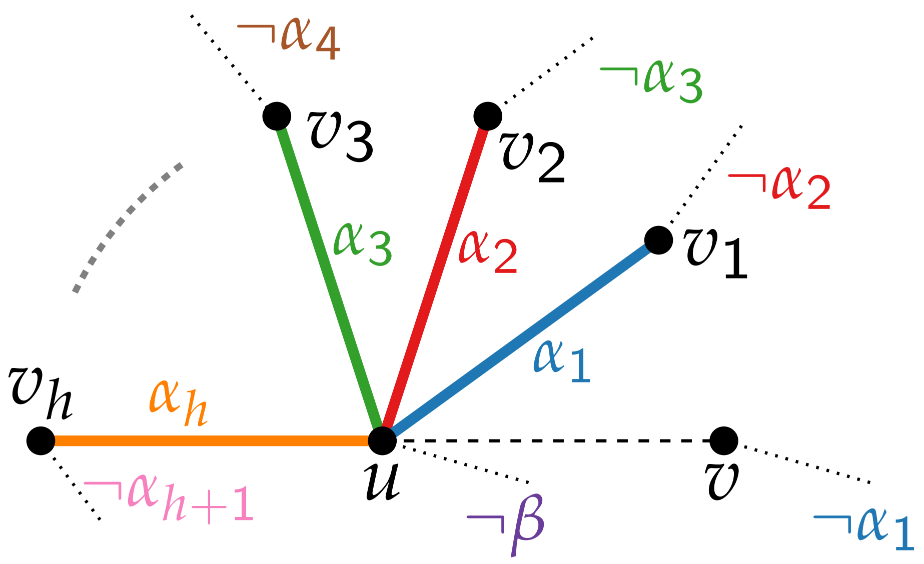

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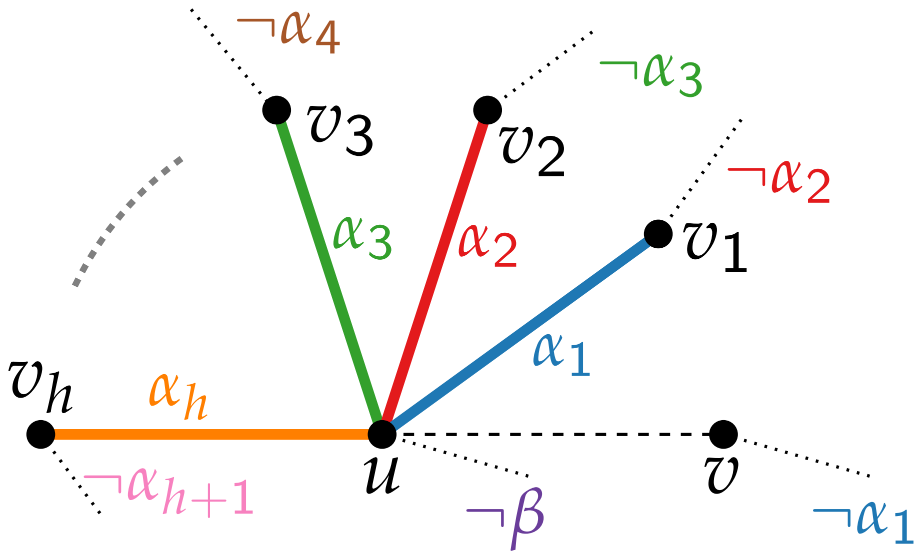

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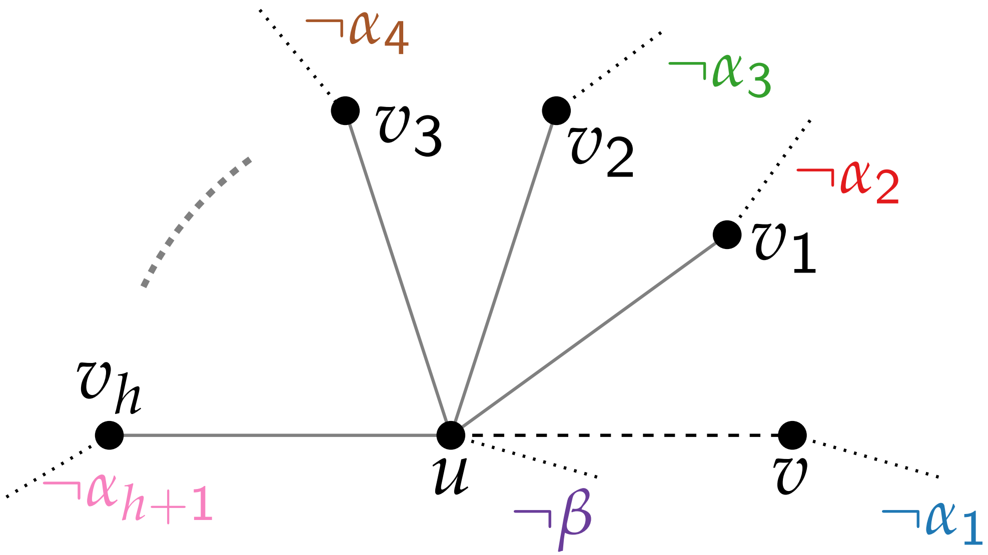

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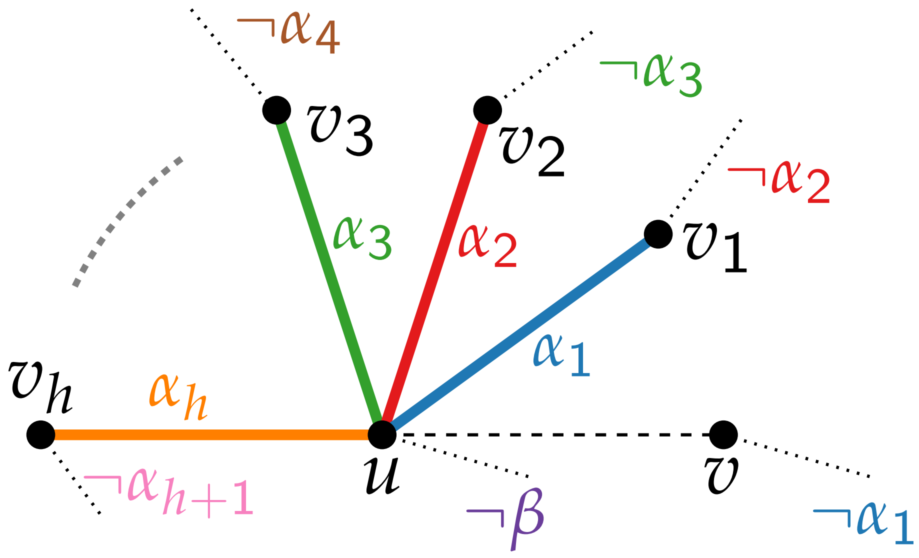

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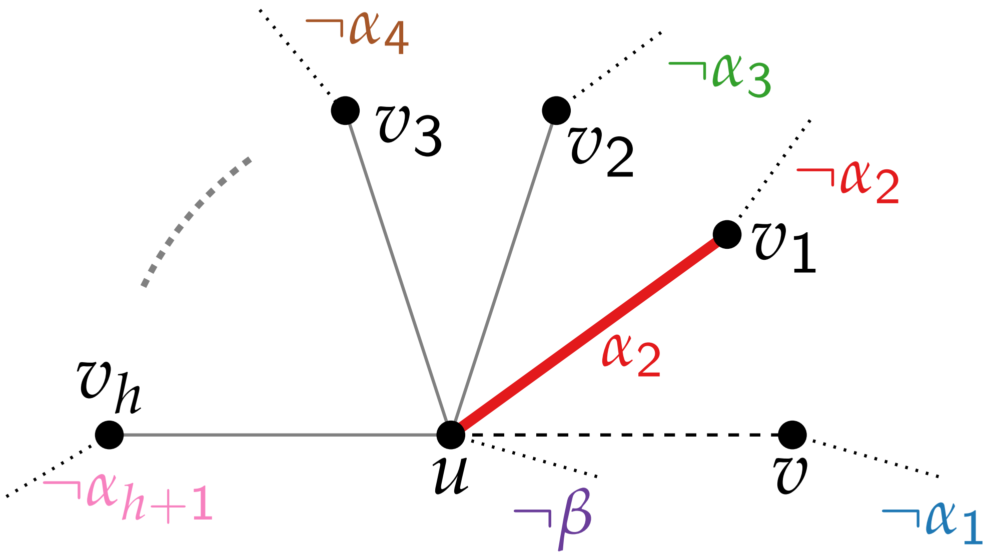

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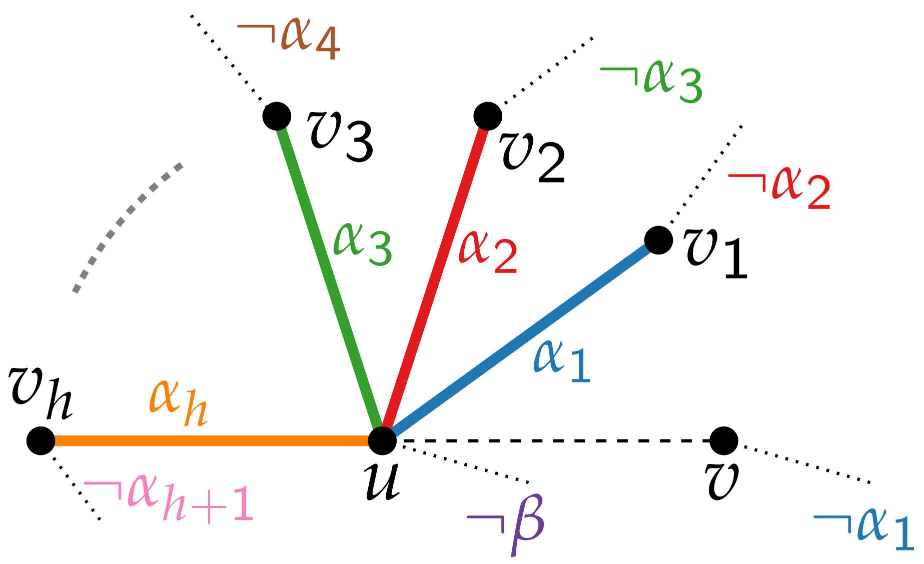

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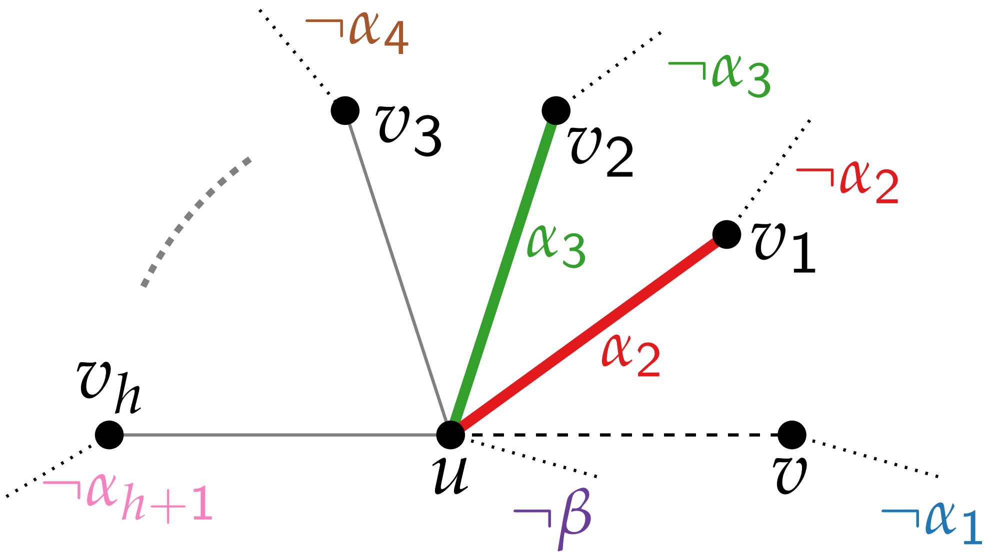

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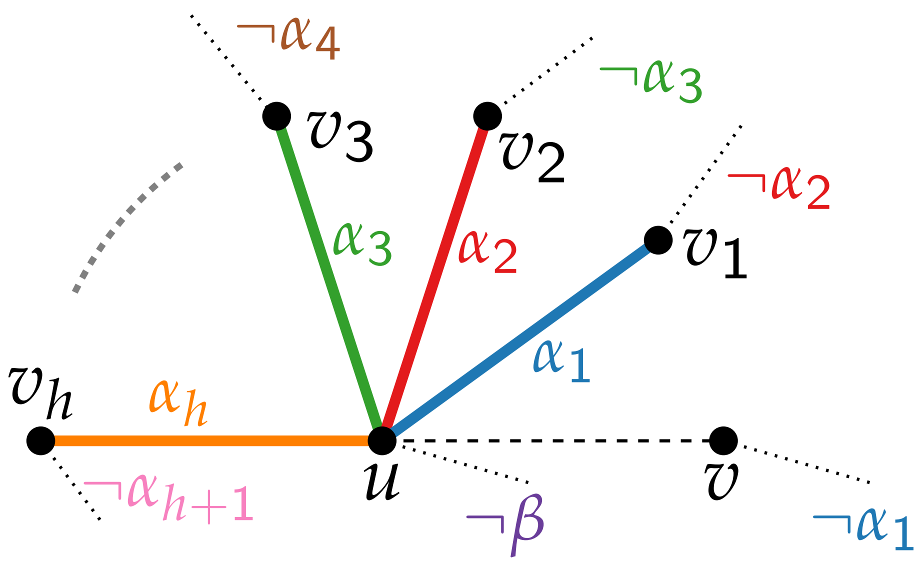

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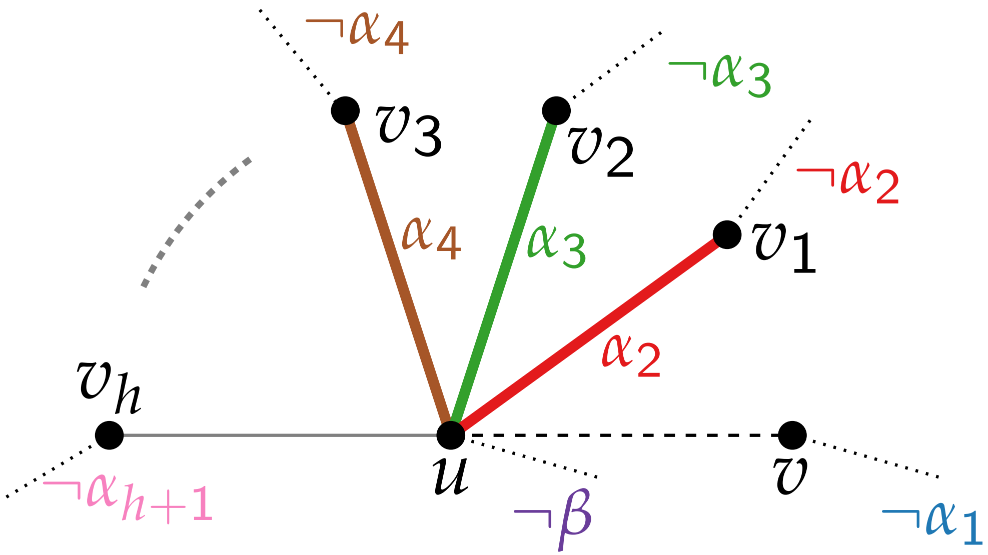

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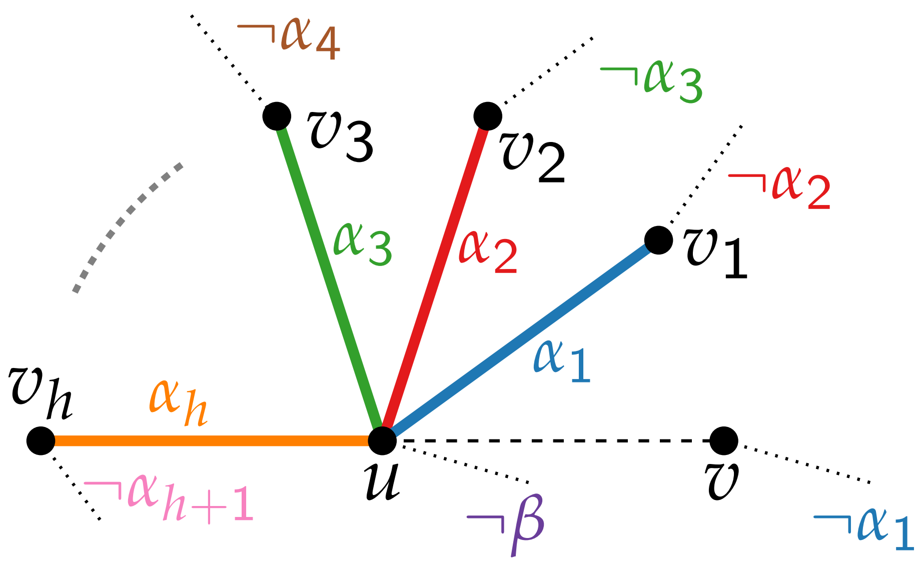

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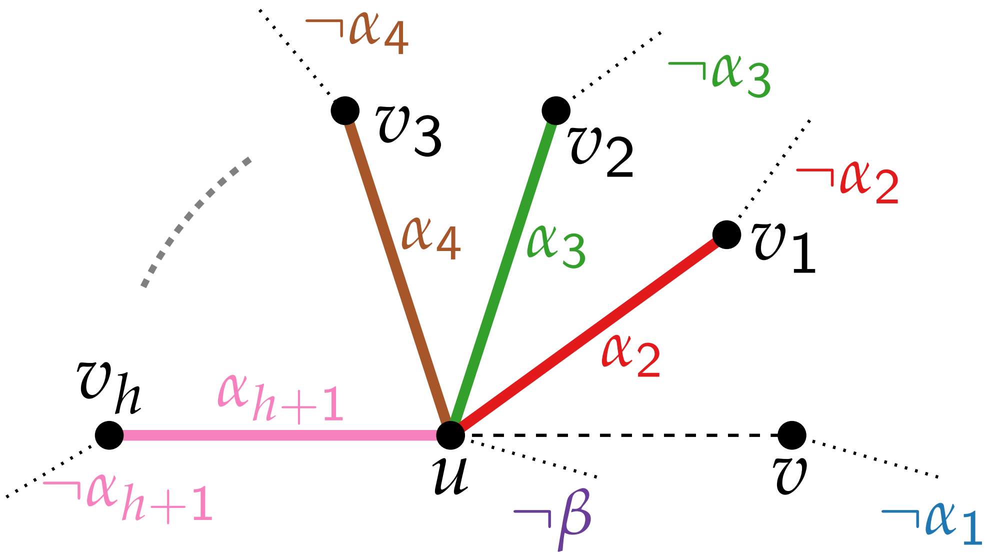

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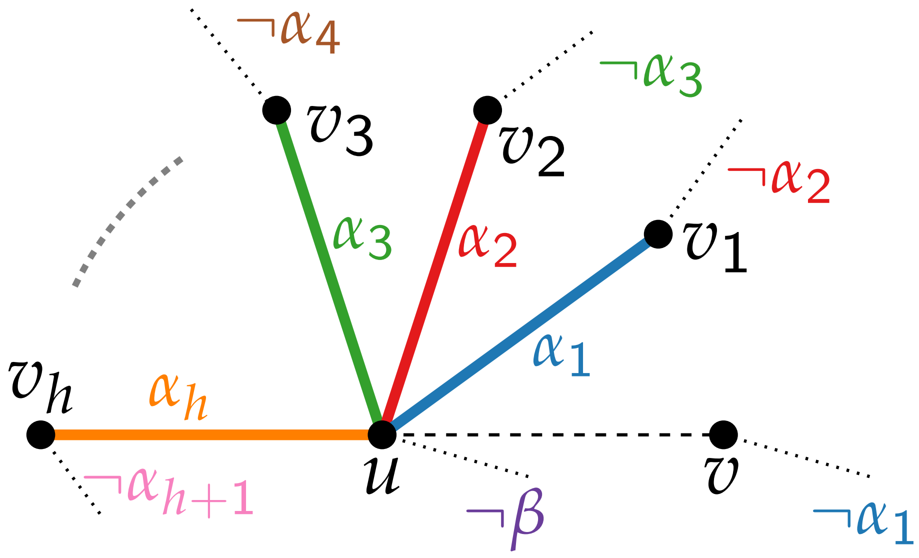

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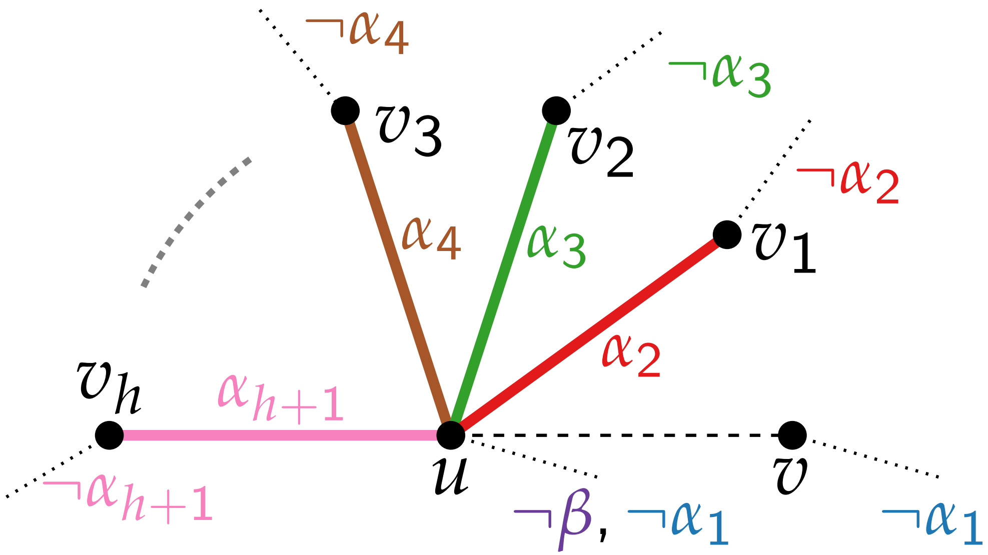

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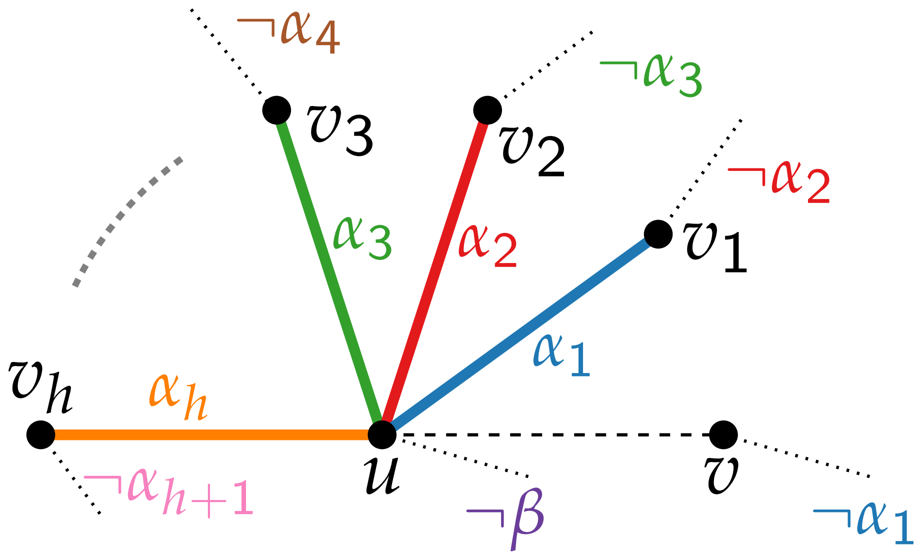

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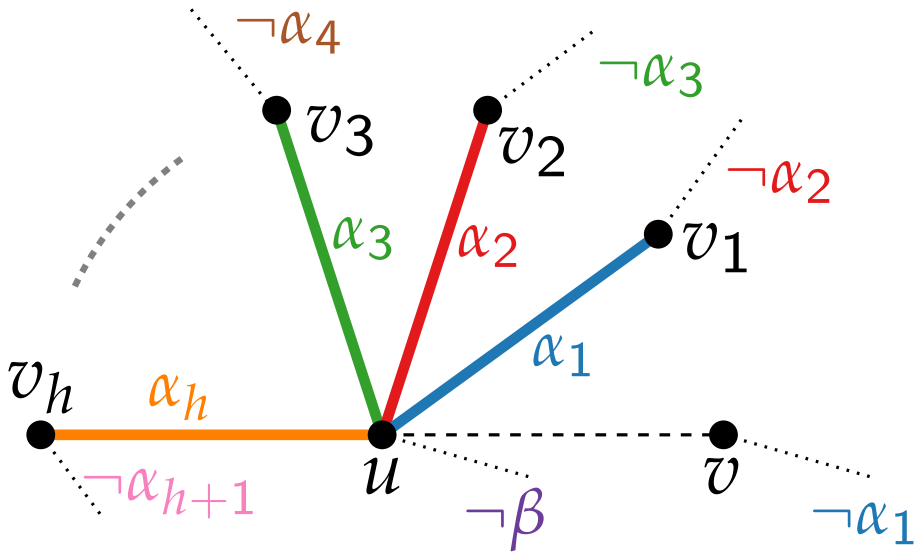

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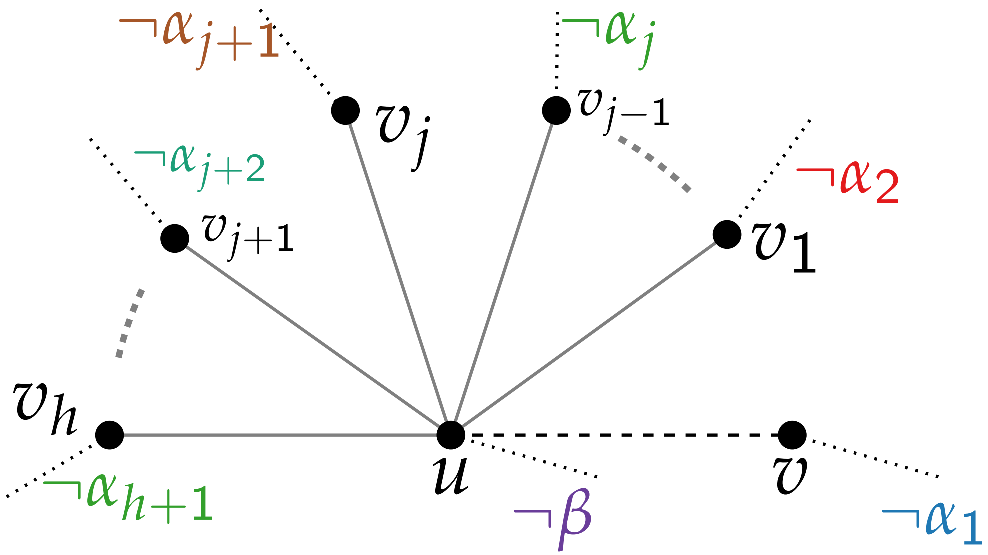

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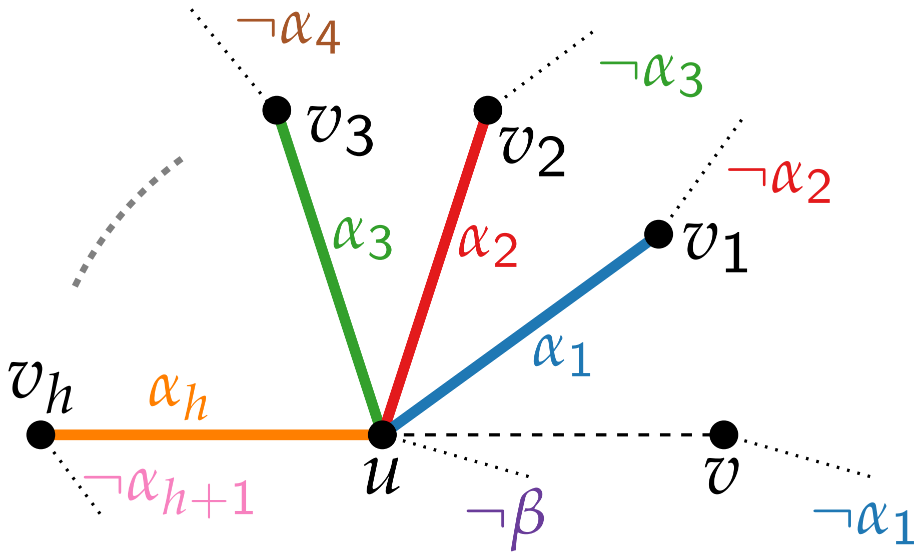

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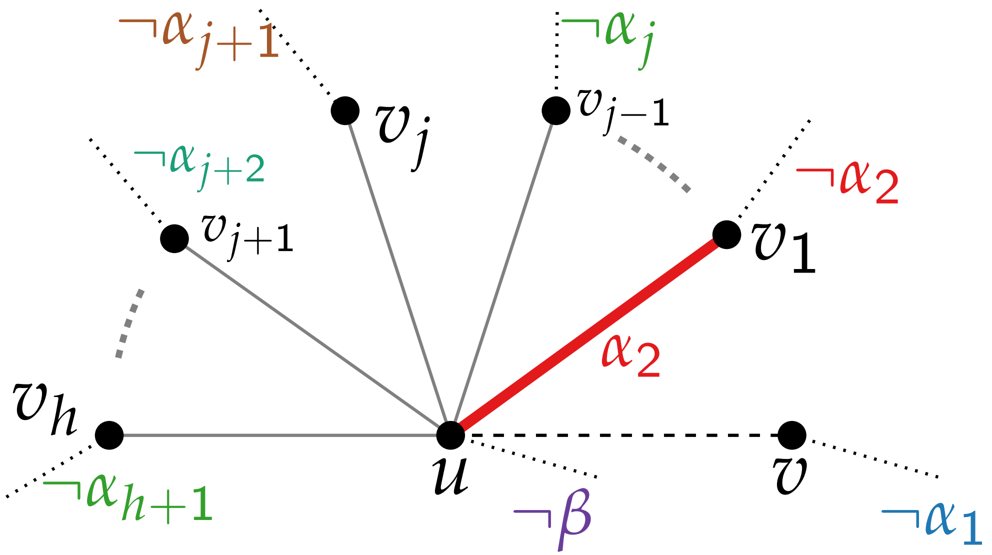

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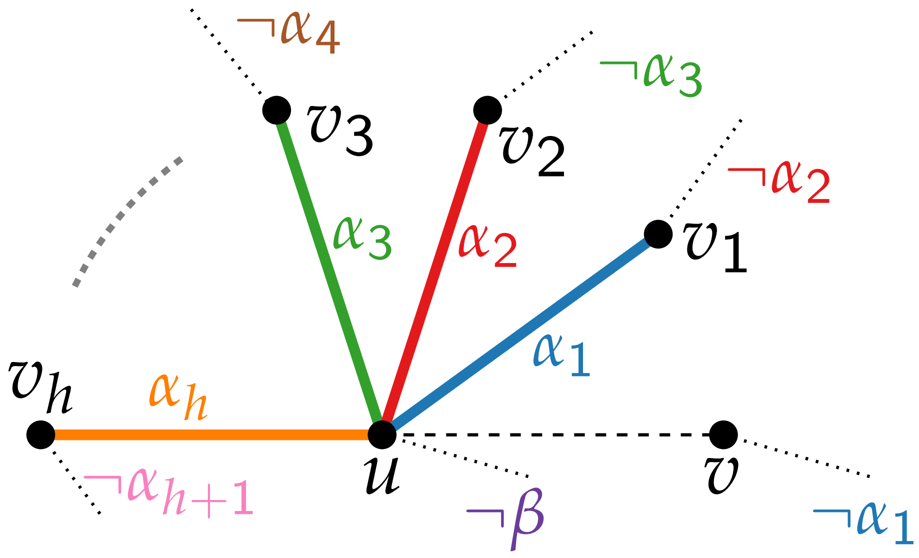

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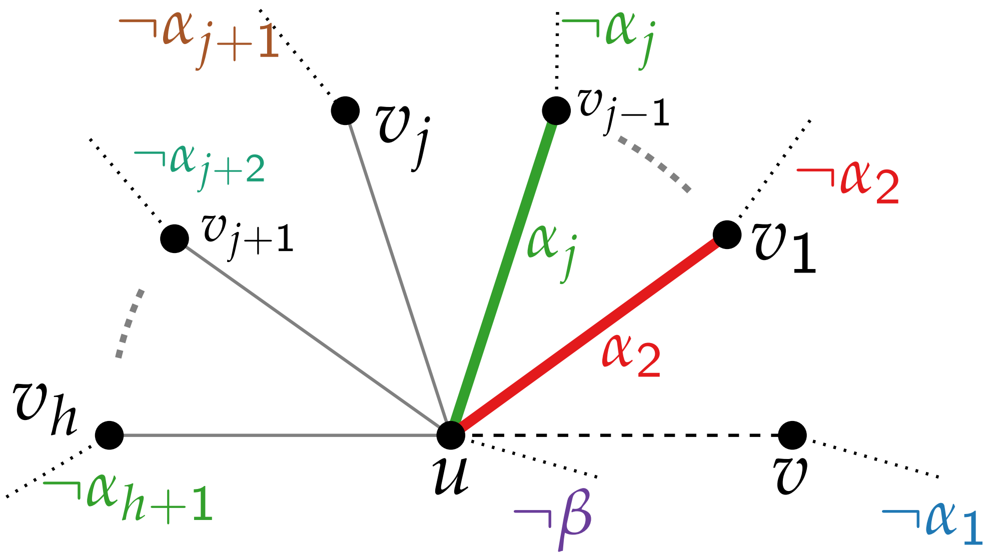

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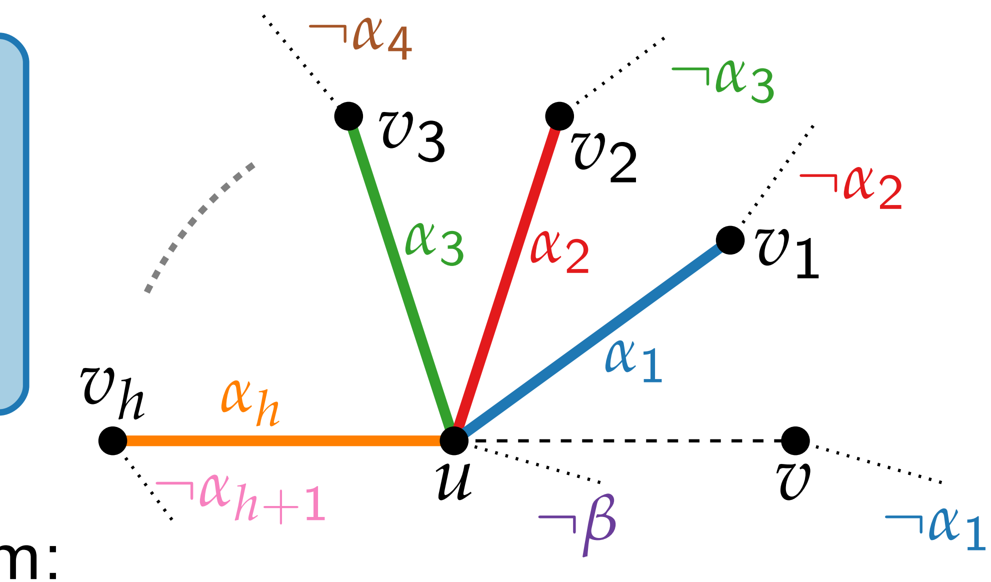

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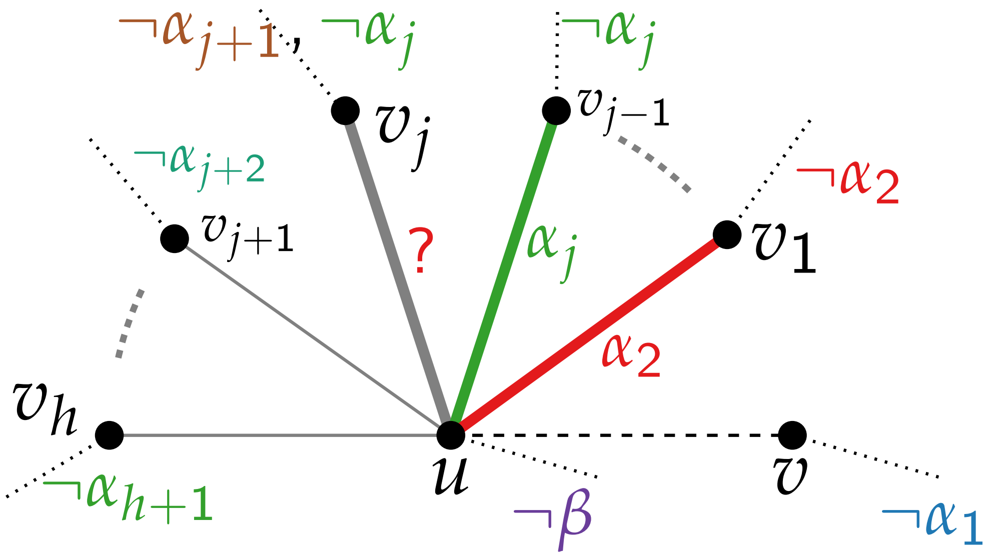

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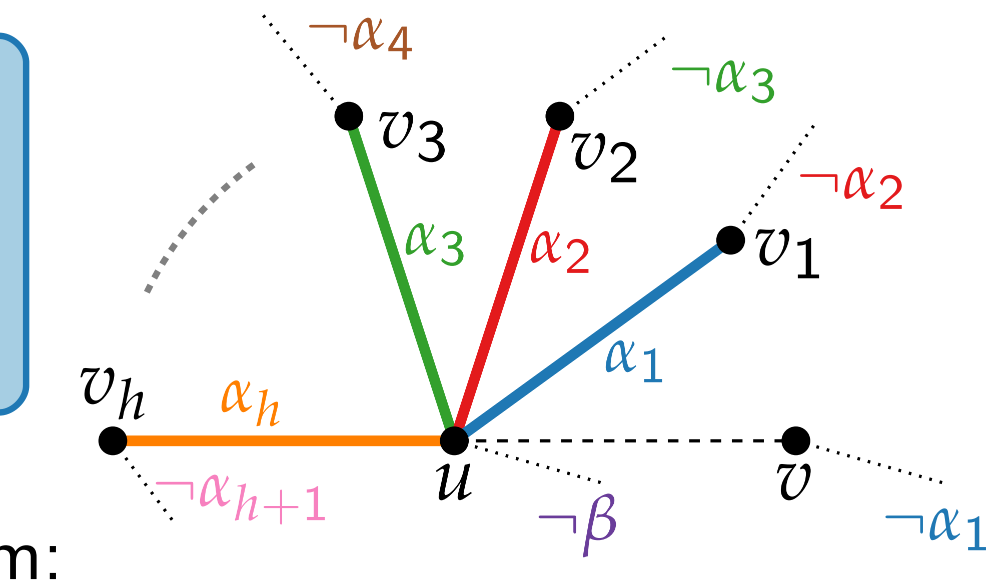

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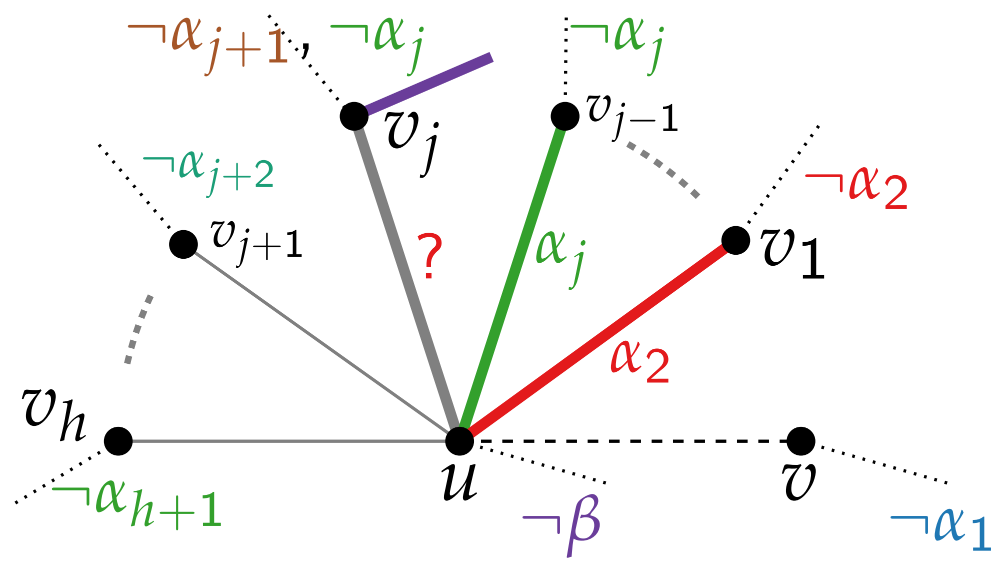

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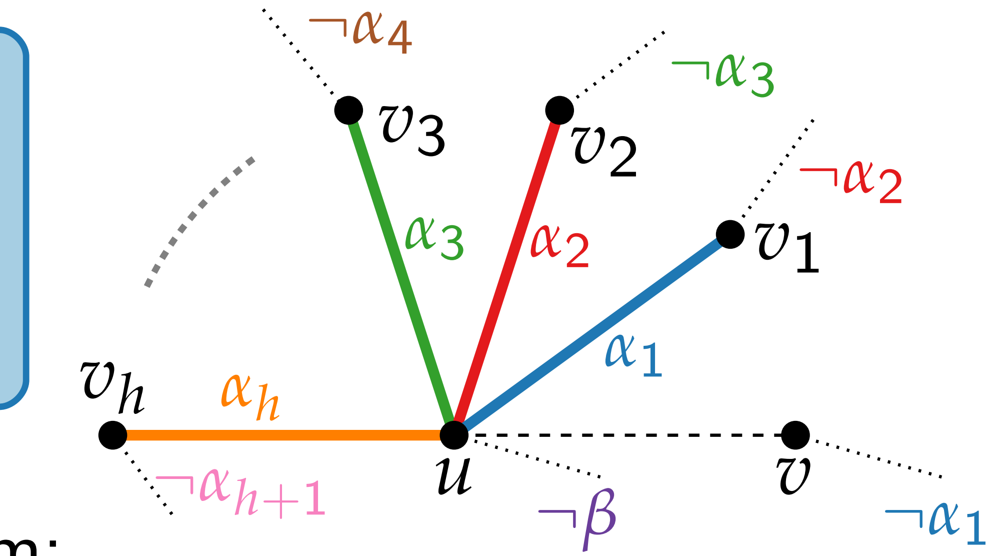

Proof. Note, each vertex is **missing** a color.
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$\textsc{VizingRecoloring}(G = (V, E), u, c, \alpha_1)$

- $i \leftarrow 1$
- **while** $\exists w \in N(u)\colon \; c(\{u, w\}) = \alpha_i \;\wedge\; w \notin \{v_1, \ldots, v_{i-1}\}$ **do**
  - $v_i \leftarrow w$
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  - $i{+}{+}$
- **return** $v_1, \ldots, v_i; \alpha_1, \ldots, \alpha_{i+1}$

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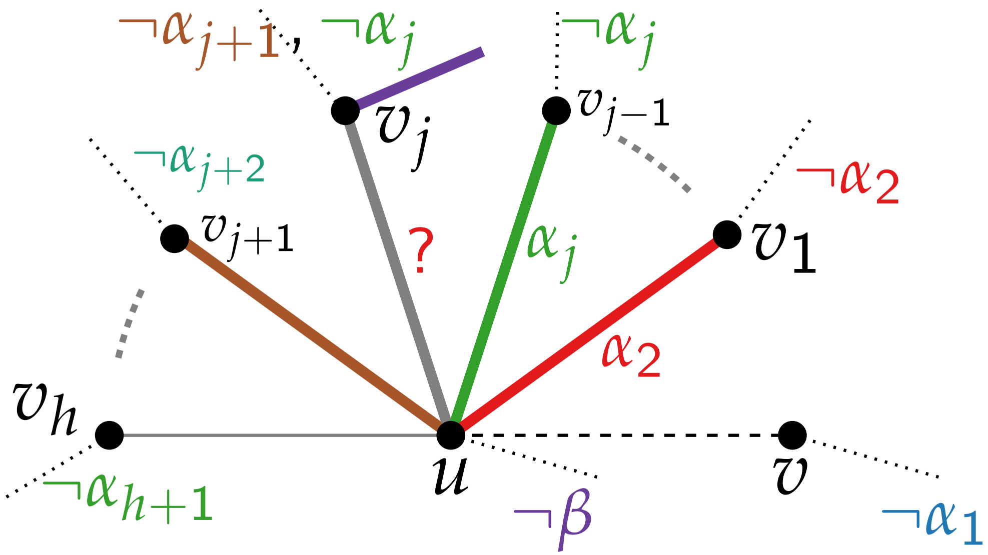

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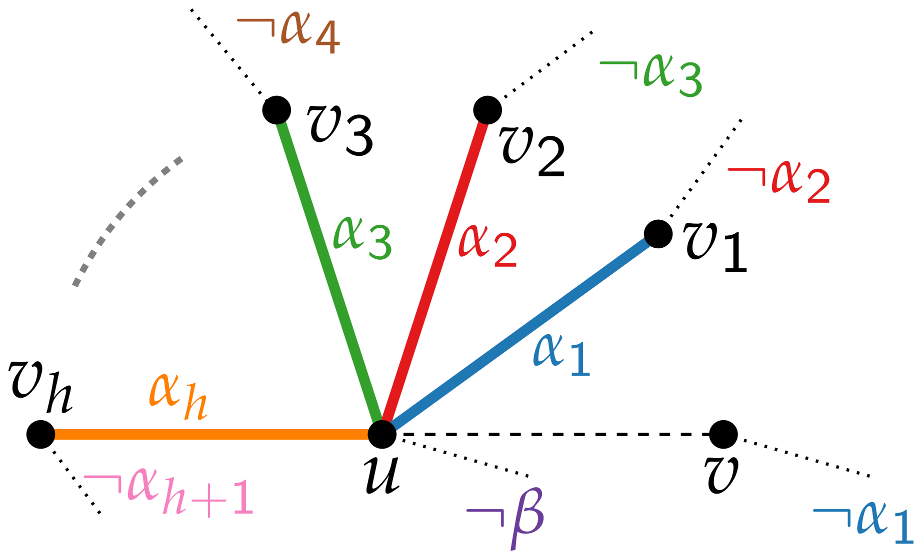

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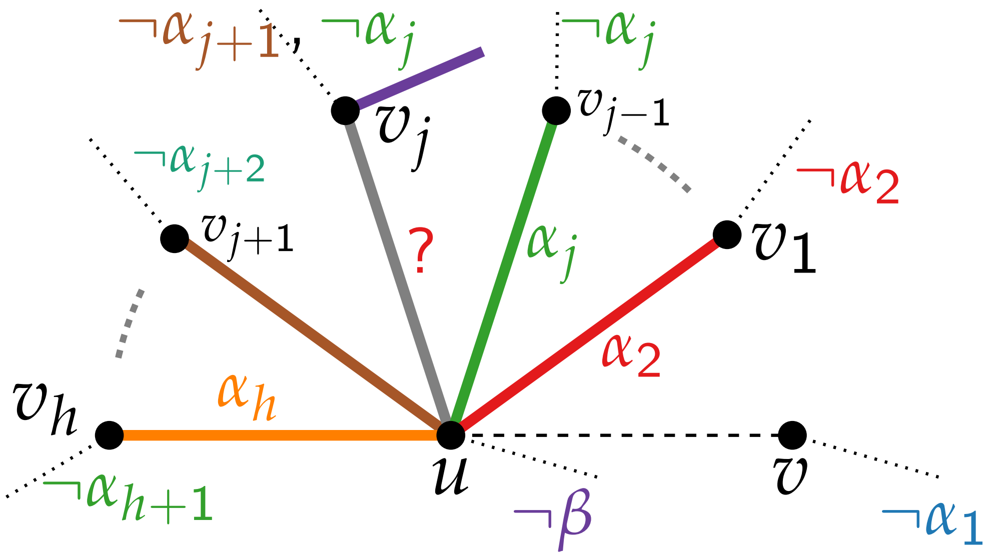

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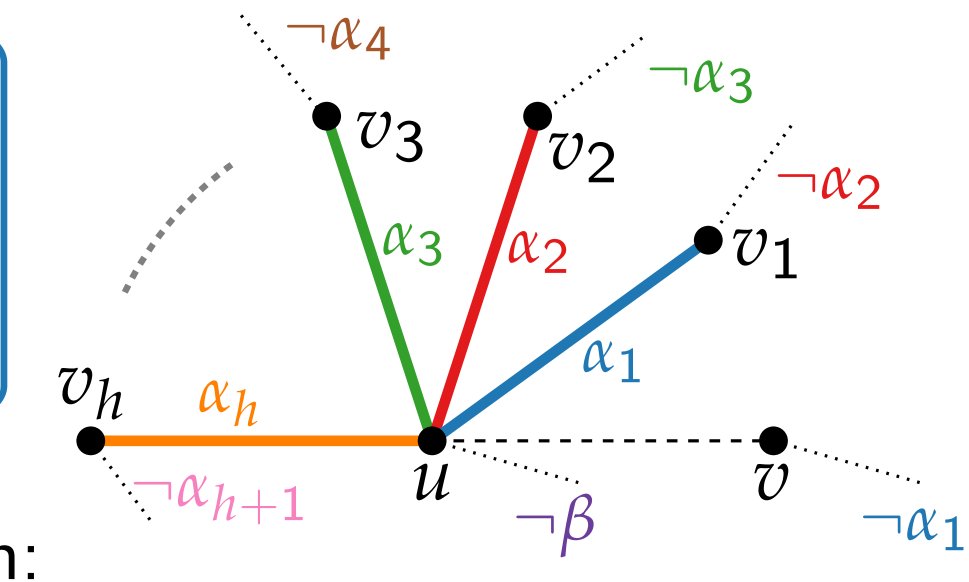

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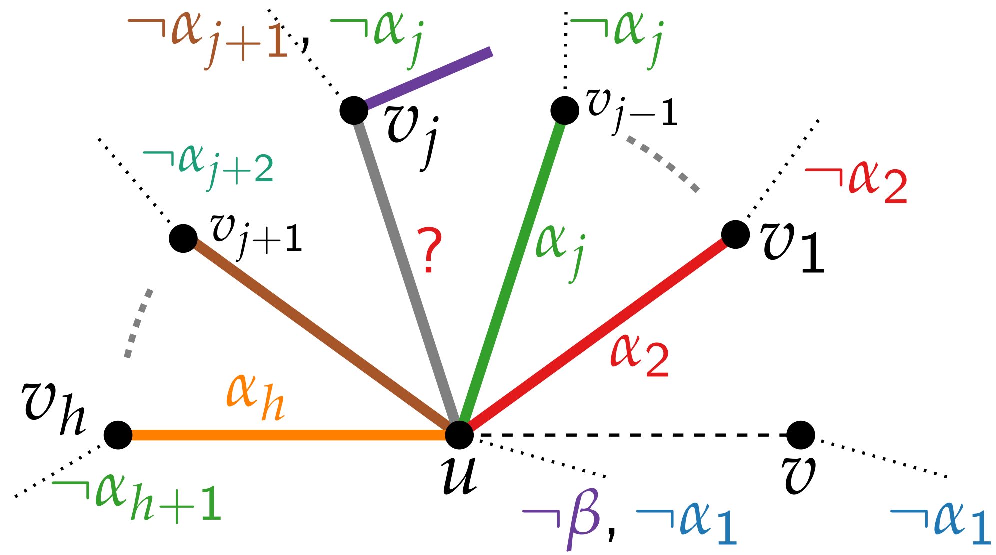

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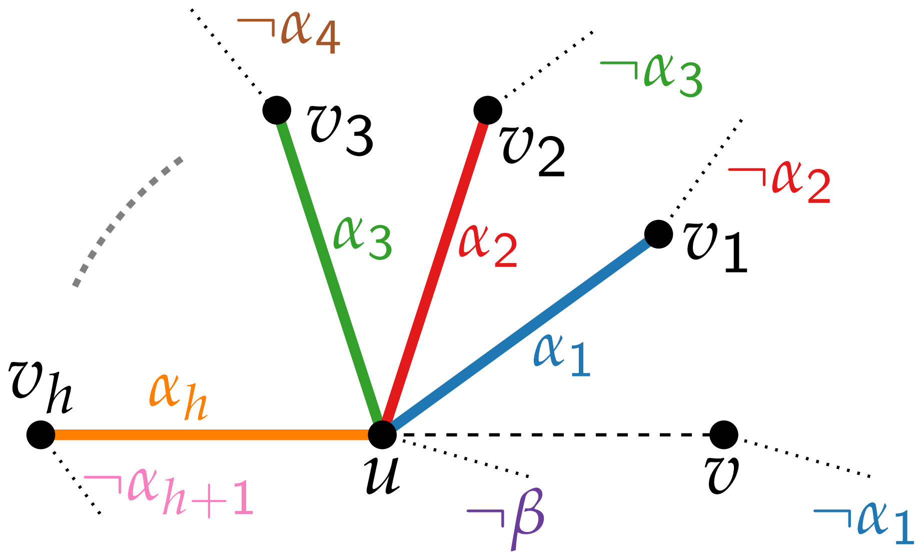

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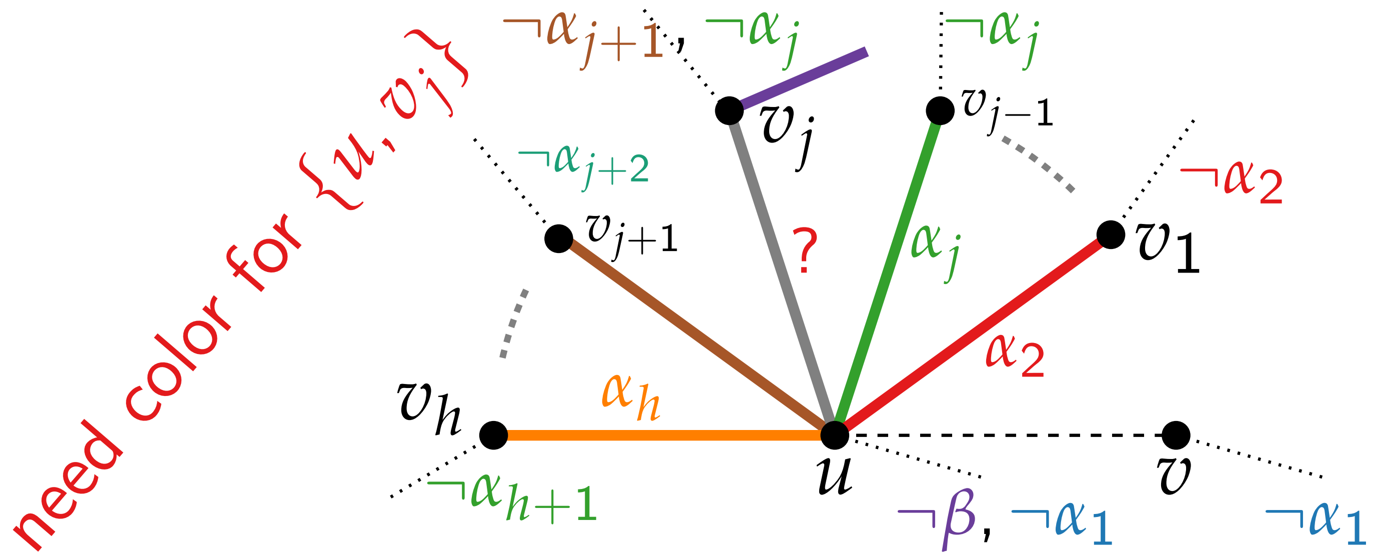

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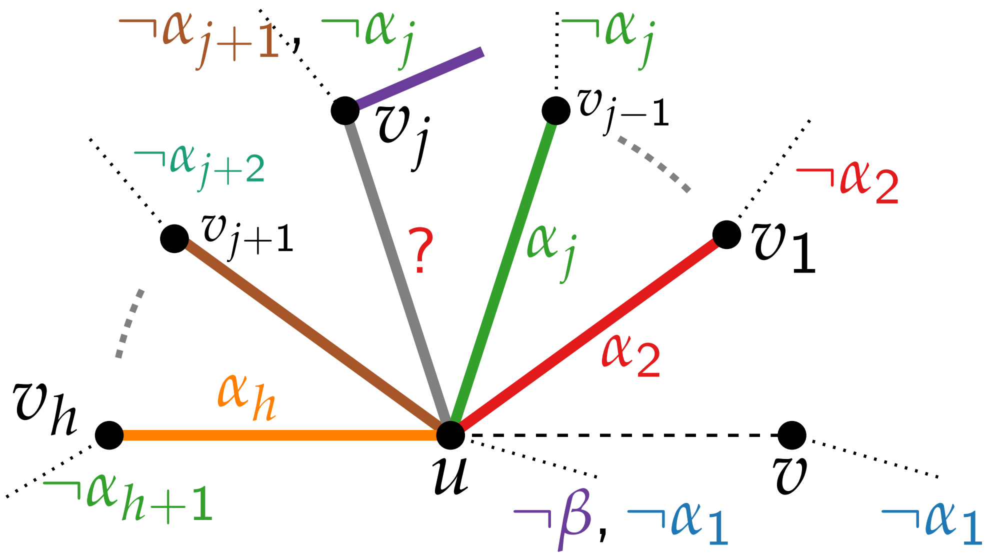

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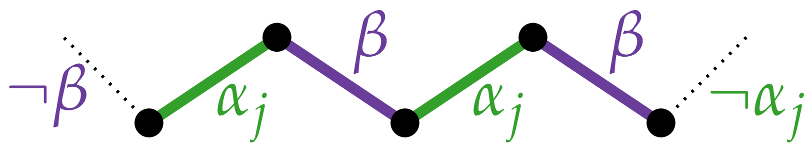

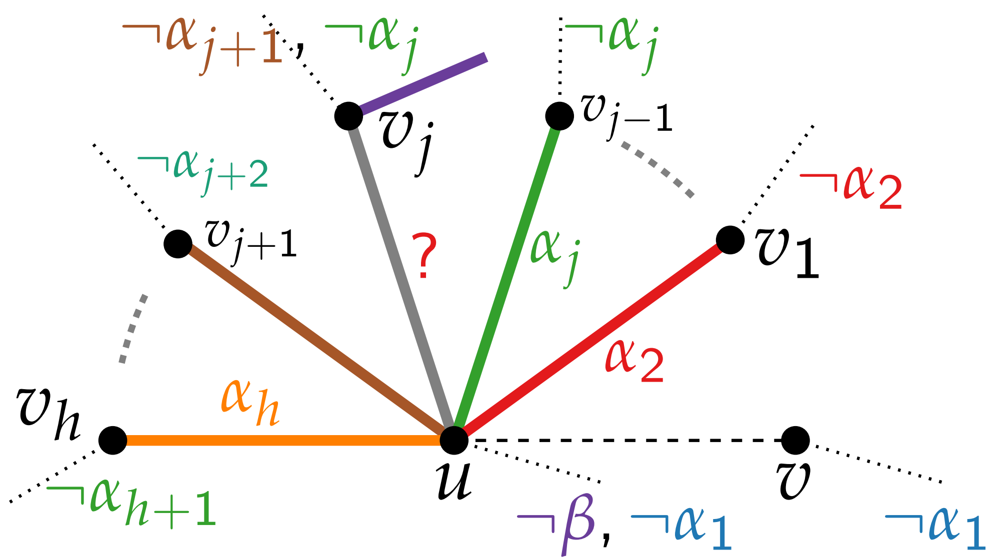

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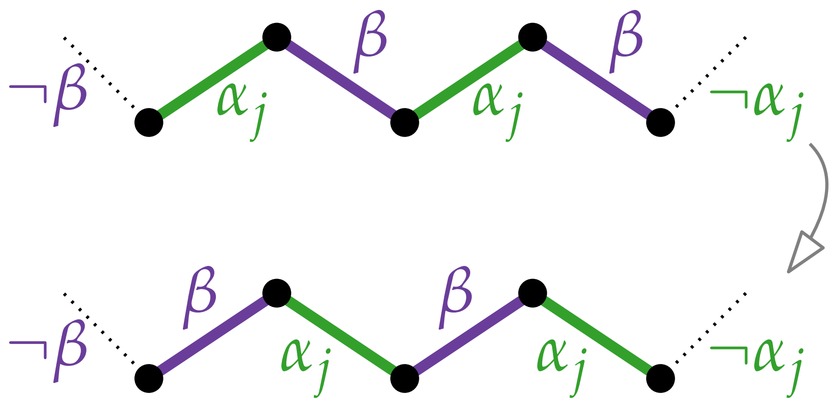

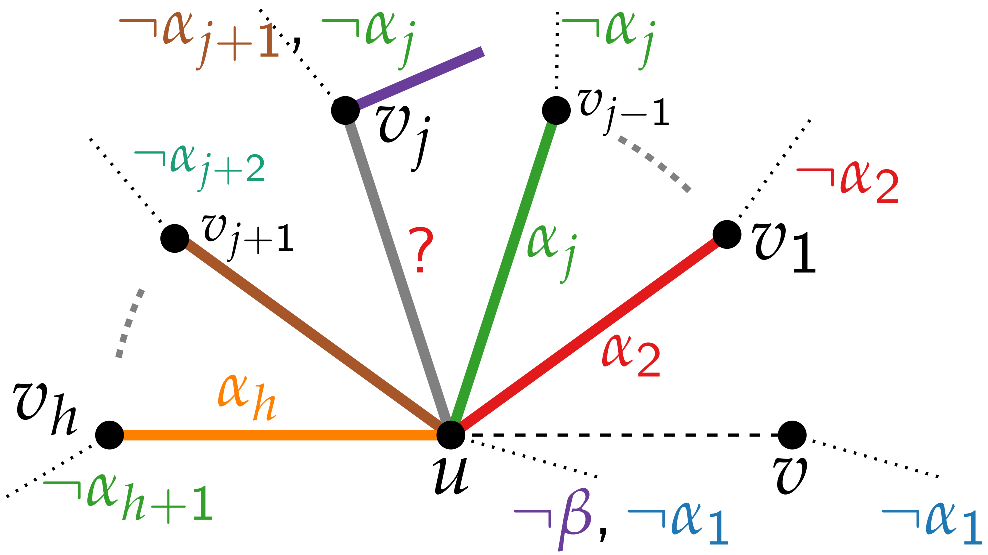

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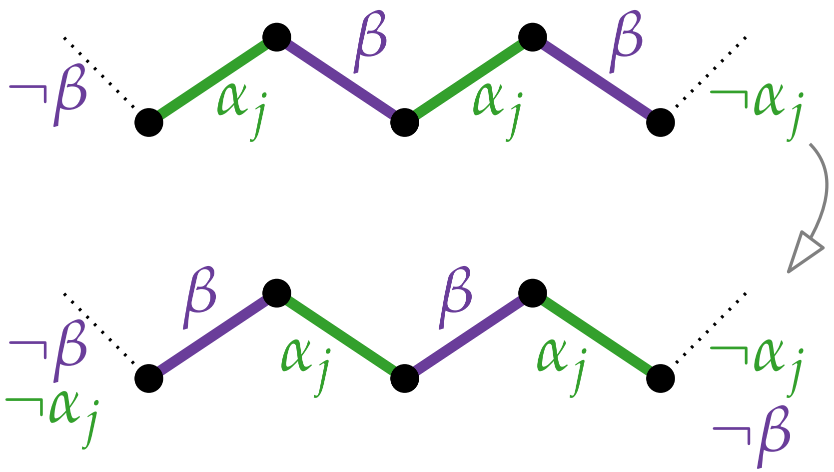

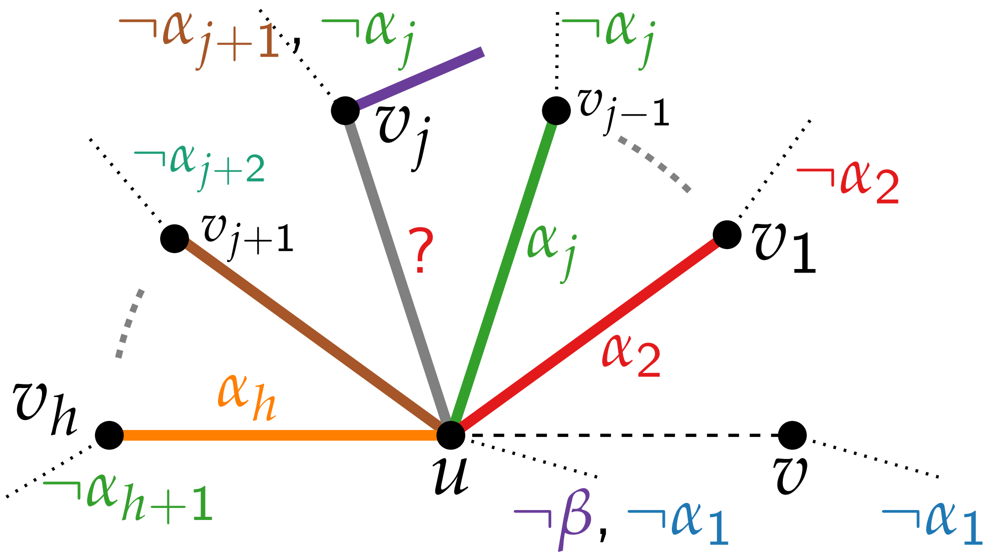

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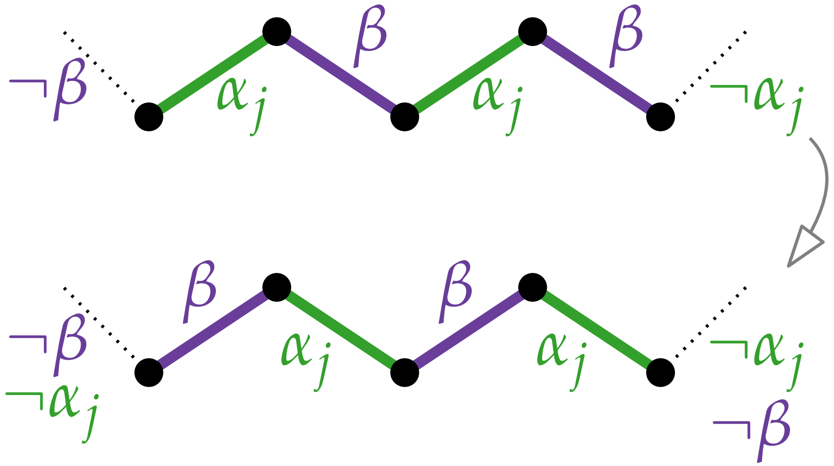

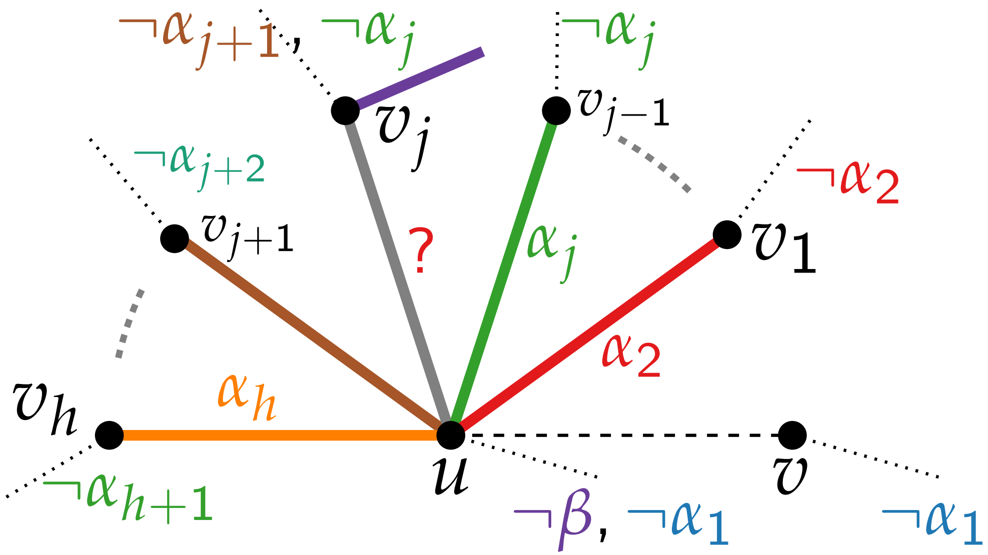

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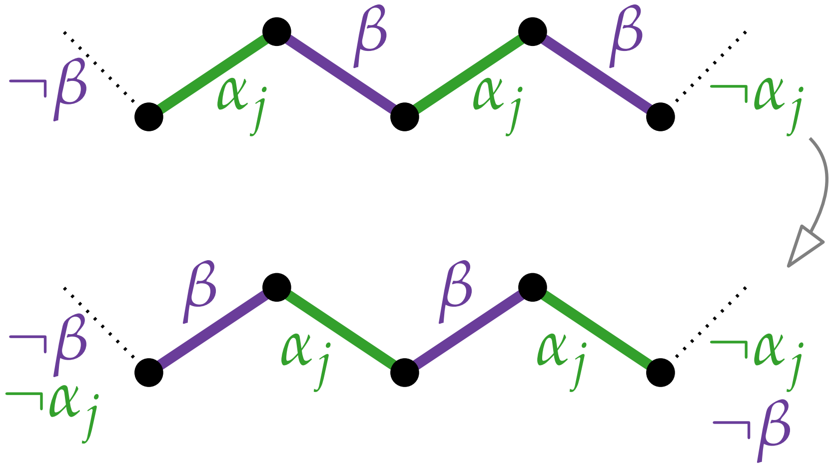

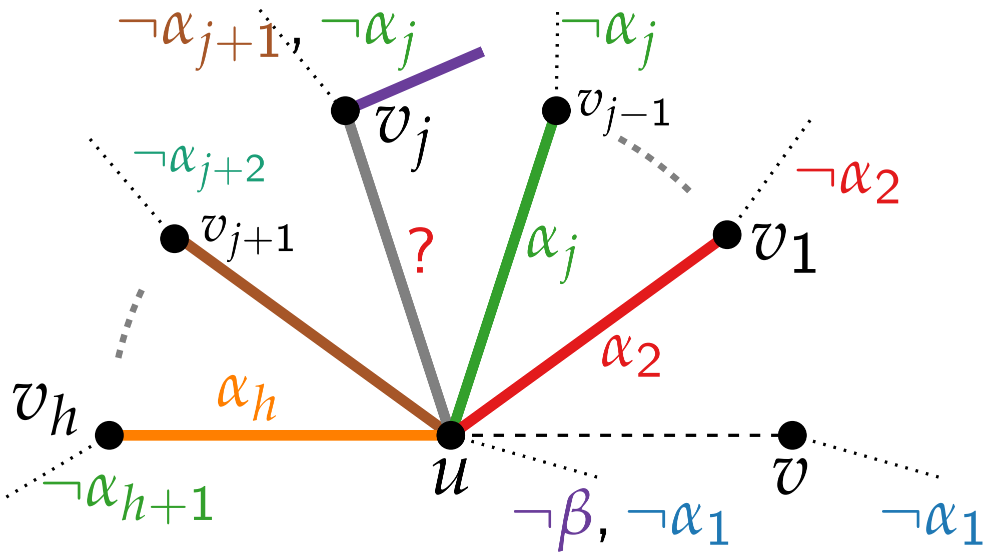

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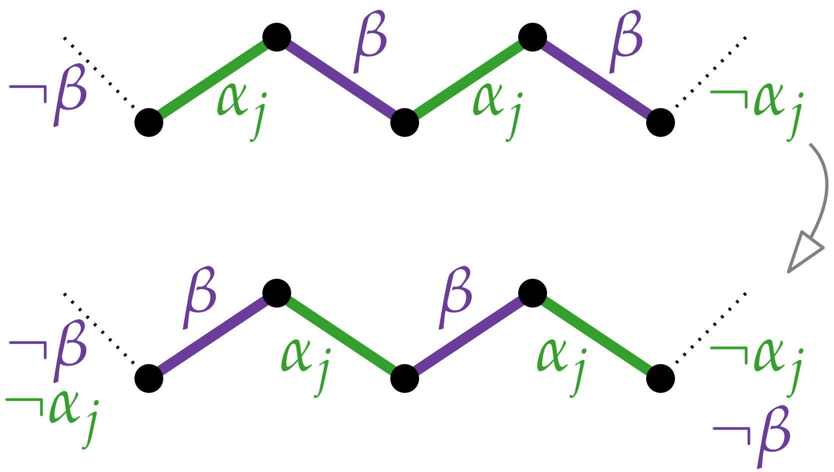

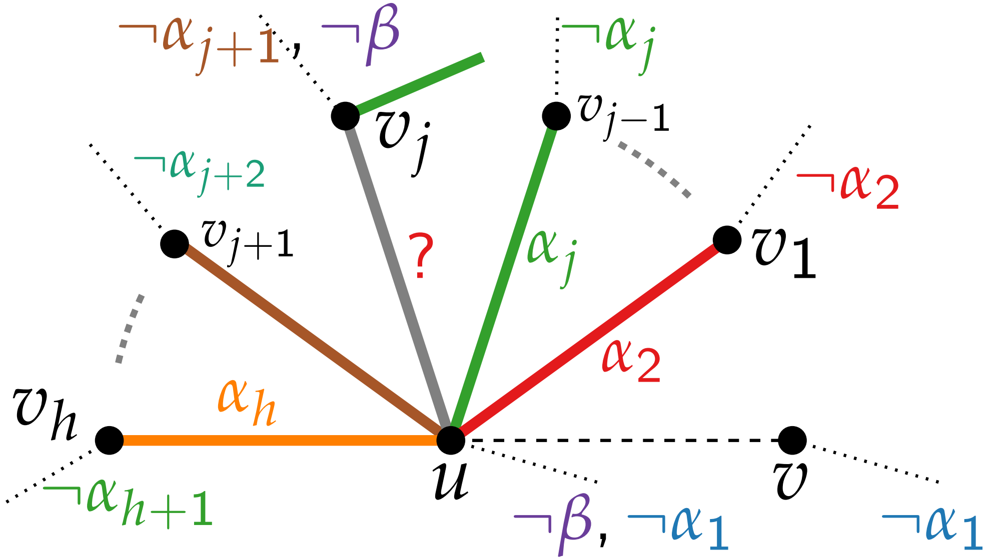

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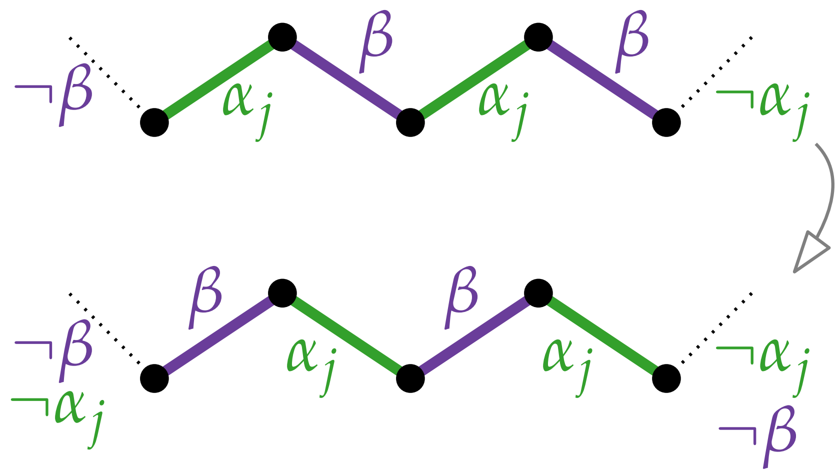
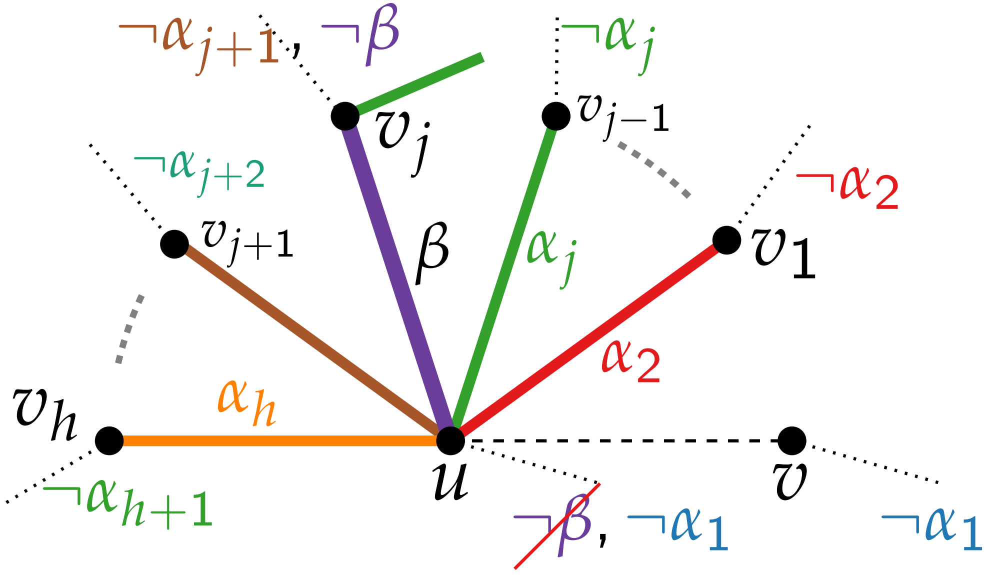
  - v_j now misses β
  - Color $\{u, v_j\}$ in β





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- If v_j and u are not in the same component:
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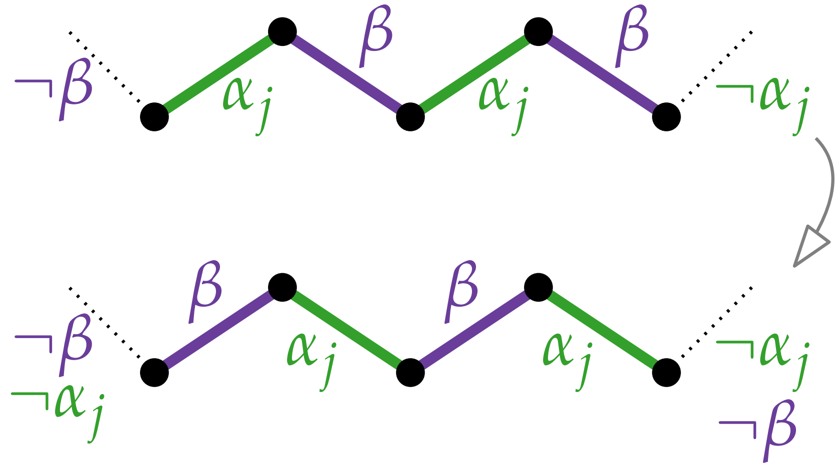
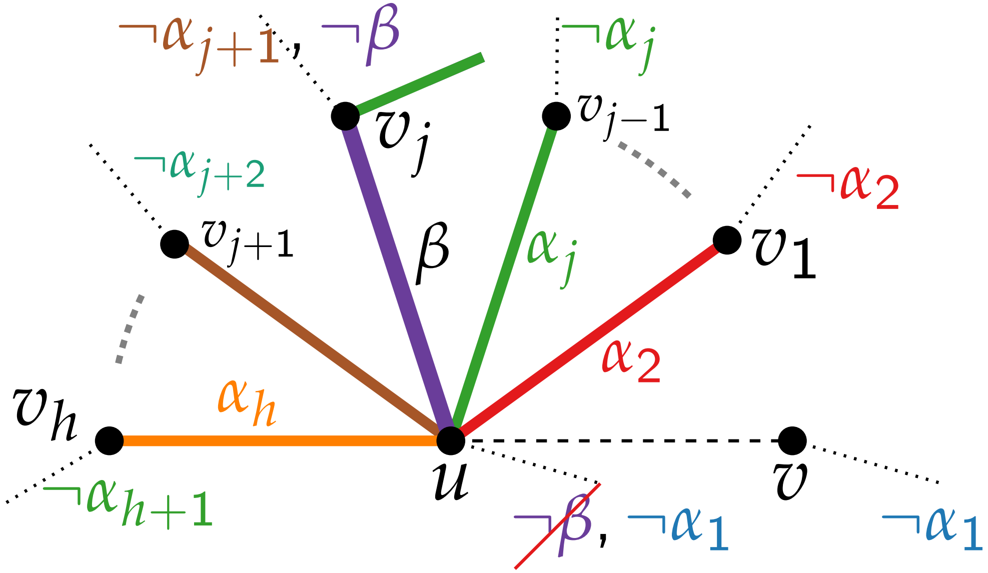
  - v_j now misses β
  - Color $\{u, v_j\}$ in β
- What if v_j and u are in the same component?





Minimum edge coloring - algorithm

```
VizingEdgeColoring(G = (V, E))
  if E = ∅ then
    return 0
  else
    {u, v} ← random edge of G
    G' ← G − e
    VizingEdgeColoring(G')
    if Δ(G') < Δ(G) then
      Color {u, v} with lowest free color
    else
      Recolor E with Lemma 2
      Color {u, v} with color now missing at u and v
```

Minimum edge coloring - algorithm

VizingEdgeColoring($G = (V, E)$)

if $E = \emptyset$ **then**
- **return** 0

else
- $\{u, v\} \leftarrow$ random edge of G
- $G' \leftarrow G - e$
- VizingEdgeColoring(G')
- **if** $\Delta(G') < \Delta(G)$ **then**
  - Color $\{u, v\}$ with lowest free color
- **else**
  - Recolor E with Lemma 2
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Theorem 4.
VizingEdgeColoring $\mathcal{A}$ is an approximation algorithm with additive approximation guarantee $\mathcal{A}(G) - \text{OPT}(G) \leq 1$.

Approximation with relative factor

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Let Π be an minimisation problem and $\alpha \in \mathbb{Q}^+$.
A **(factor) α-approximation algorithm** for Π is a polynomial-time algorithm $\mathcal{A}$, which computes for every instance I of Π a value $\mathcal{A}(I)$ such that

$$\frac{\mathcal{A}(I)}{\mathsf{OPT}(I)} \leq \alpha.$$

We call α the **approximation factor**.

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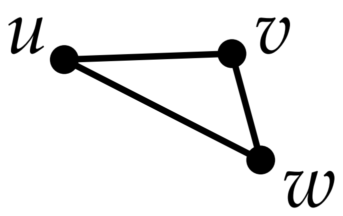
2-approximation for Metric TSP (from AGT)

Input. Complete graph $G = (V, E)$ and distance function $d : E \to \mathbb{R}_{\geq 0}$, which satisfies the triangle inequality, i.e. $\forall u, v, w \in V : d(u, w) \leq d(u, v) + d(v, w)$.



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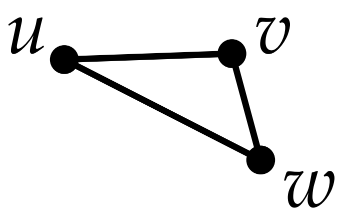

Output. Shortest Hamilton cycle.

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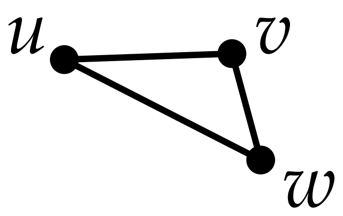

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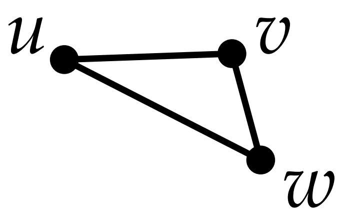

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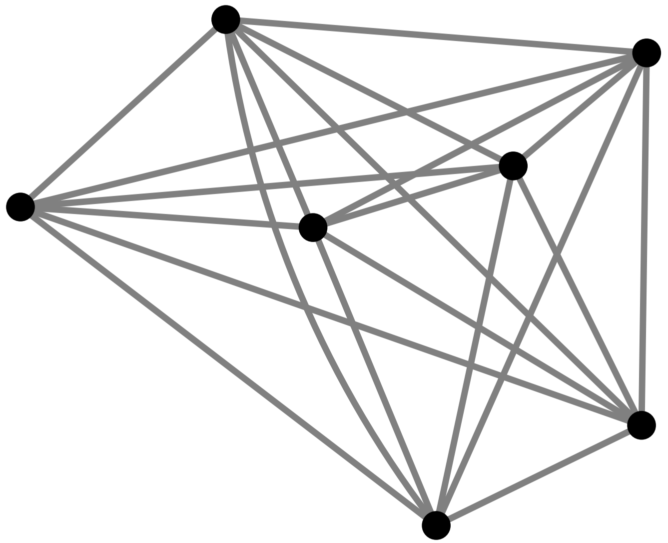

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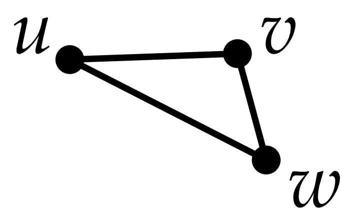

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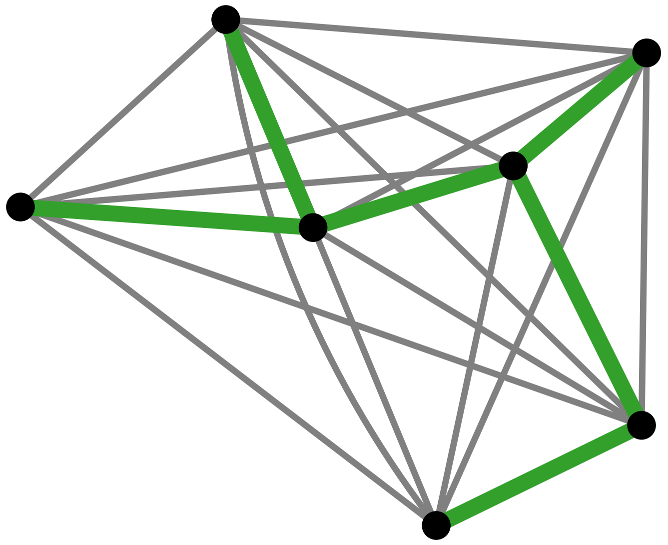
Algorithm.

- Compute MST.



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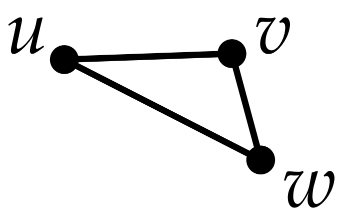

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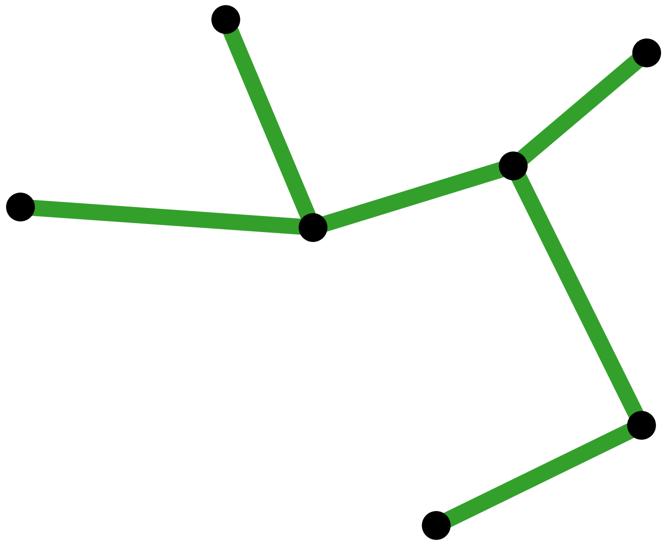

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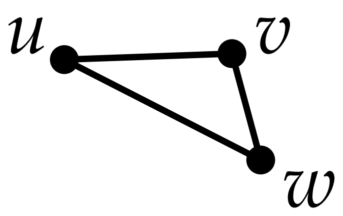

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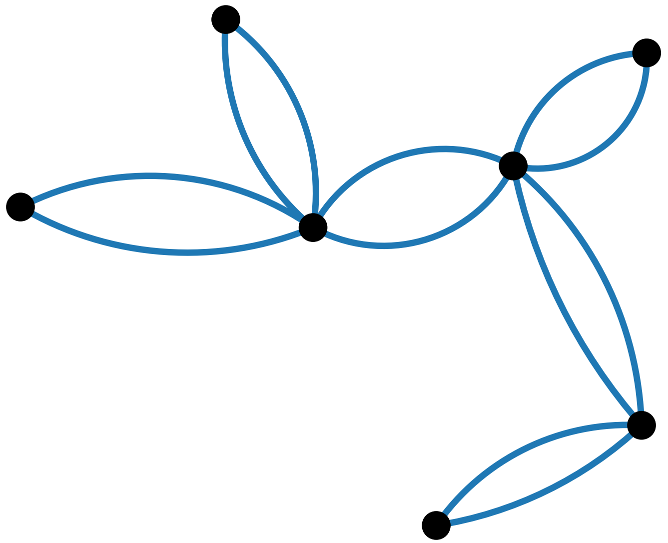
Algorithm.

- Compute MST.
- Double edges.



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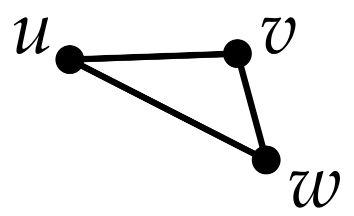

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Algorithm.

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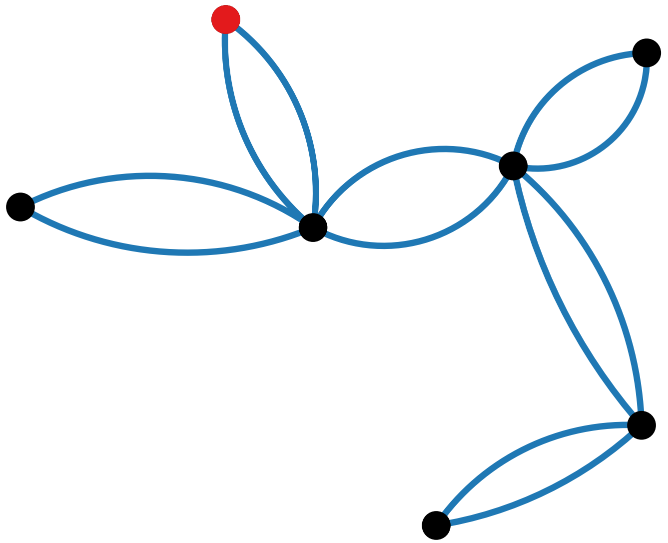
- Walk along tree,



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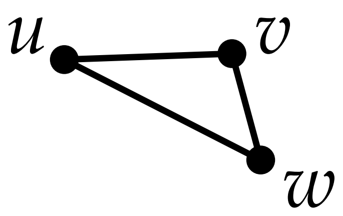

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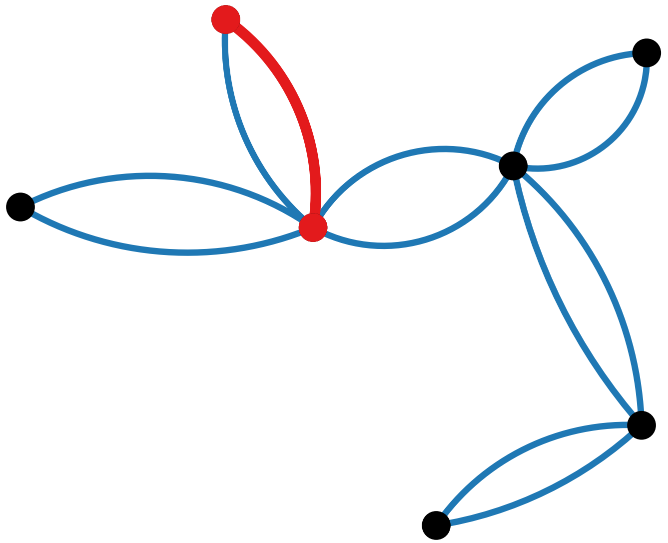

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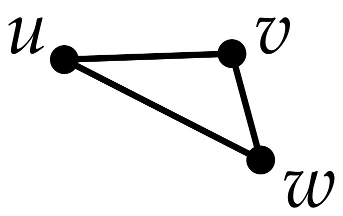
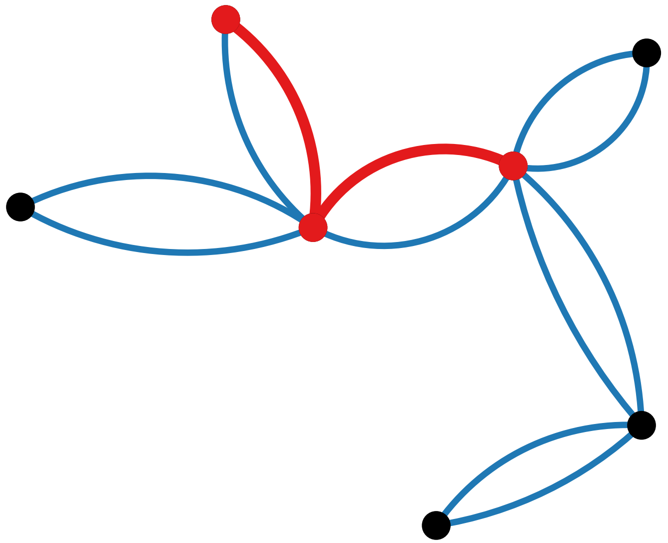


Output. Shortest Hamilton cycle.

Algorithm.

- Compute MST.
- Double edges.
- Walk along tree,



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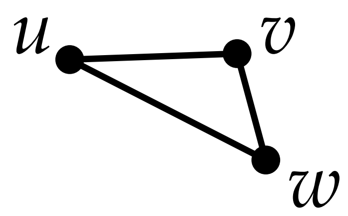

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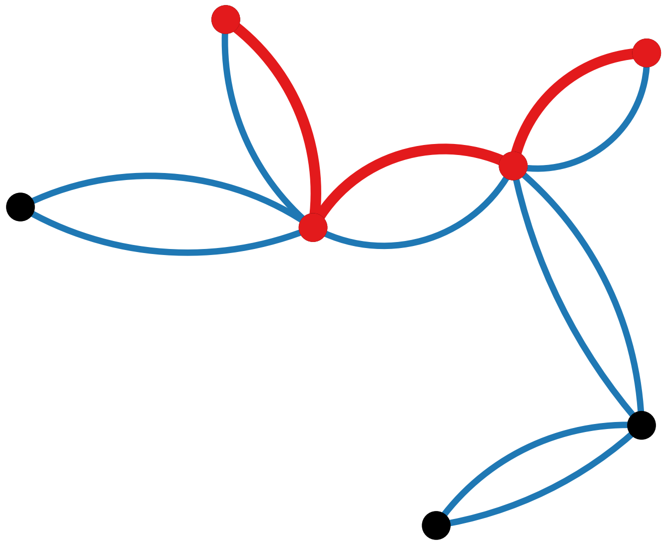

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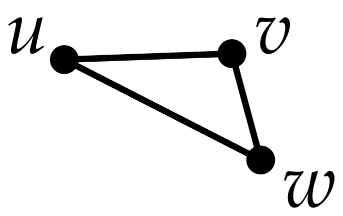

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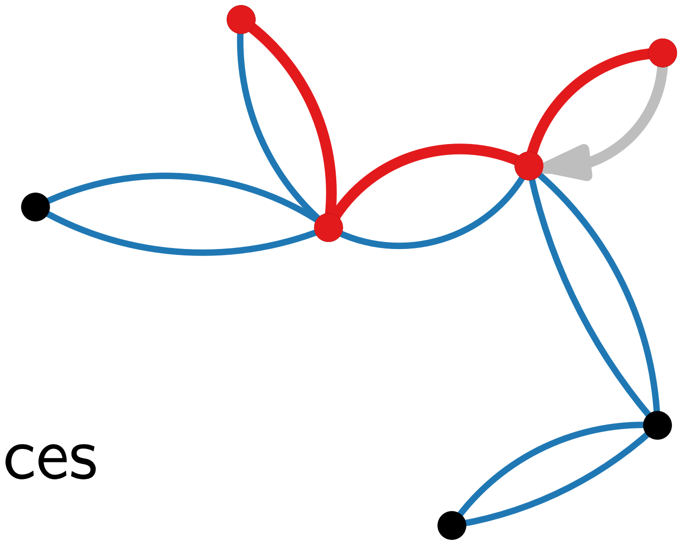

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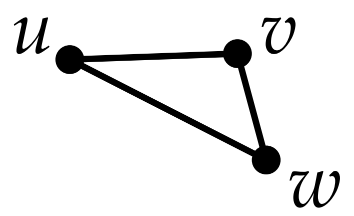

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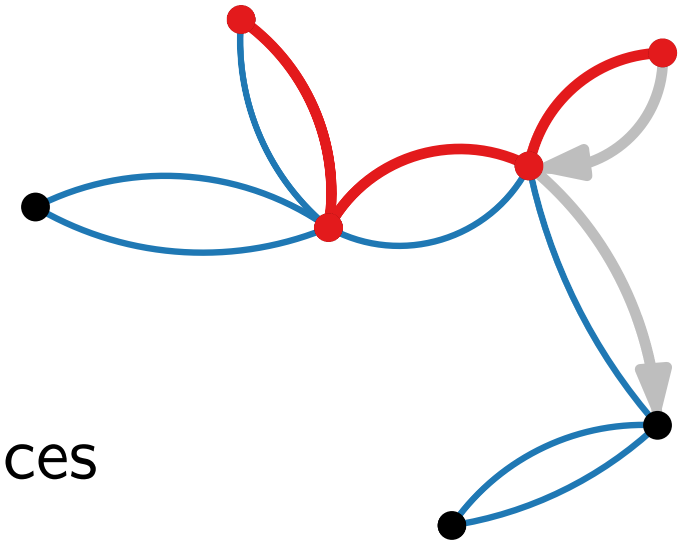

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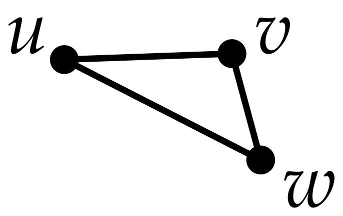

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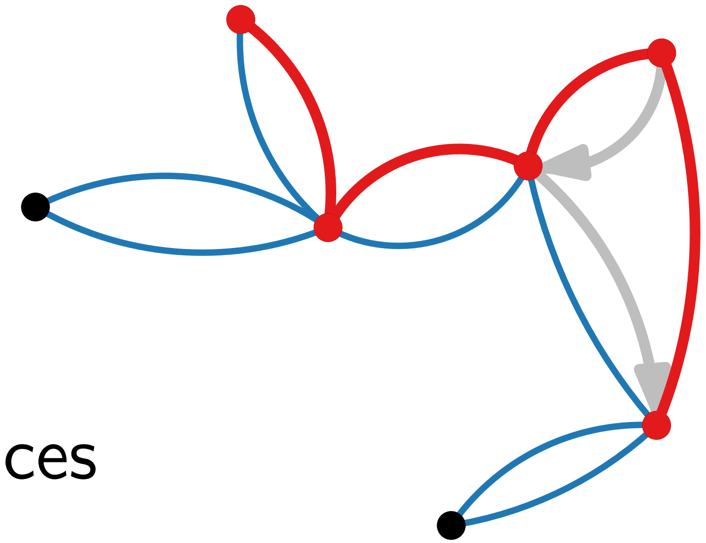

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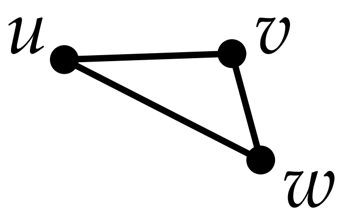

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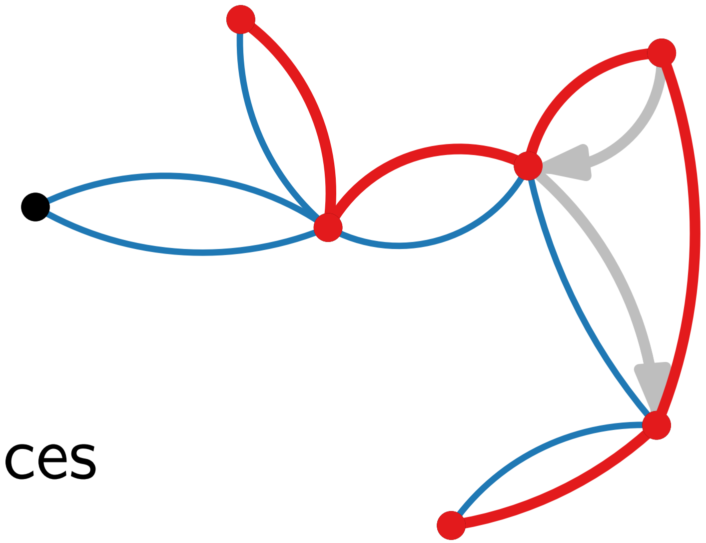

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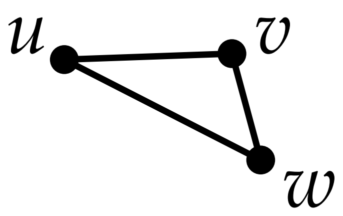

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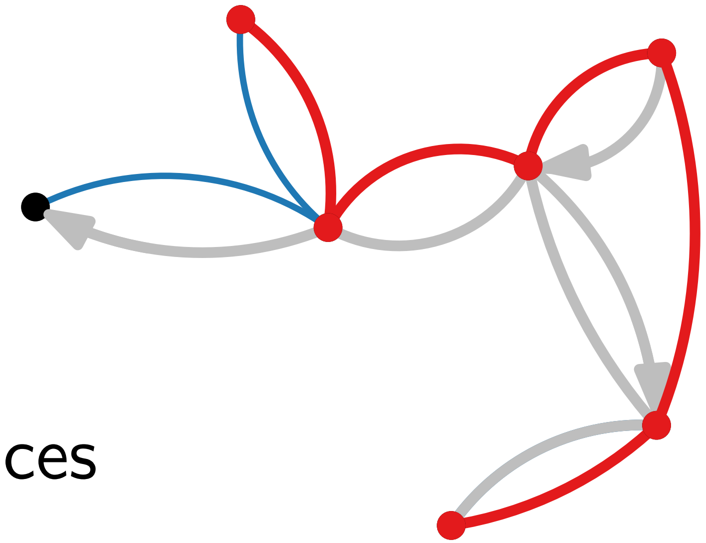

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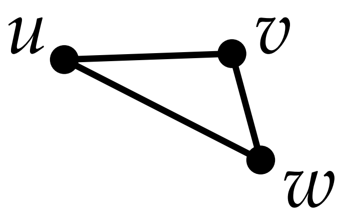

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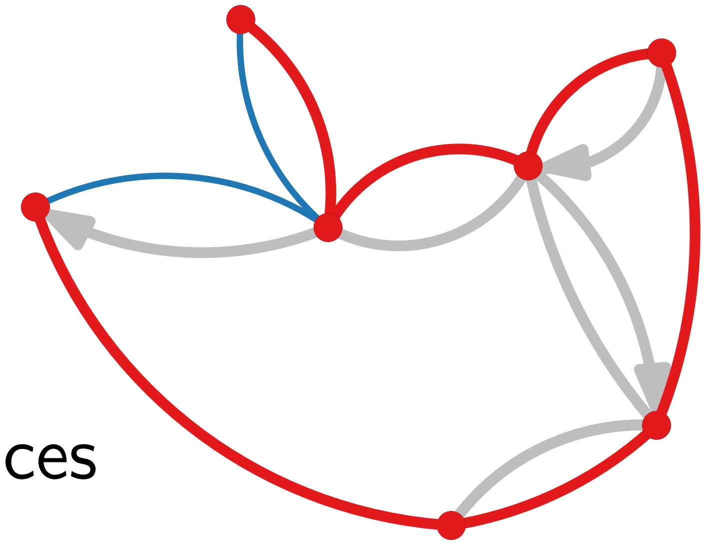

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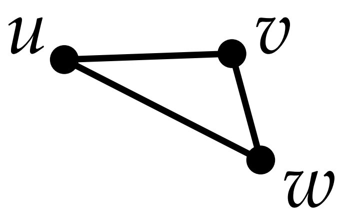

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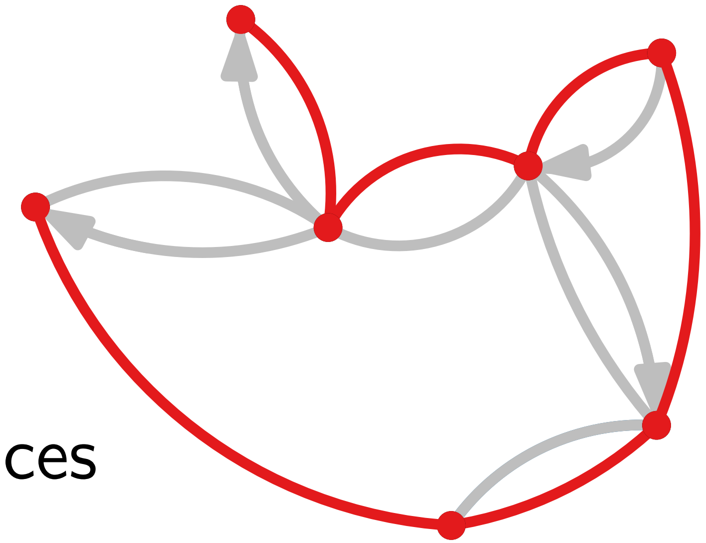

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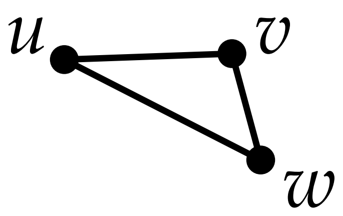

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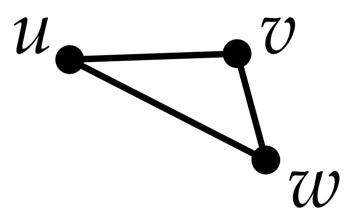

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Theorem 5.
The MST edge doubling algorithm is a 2-approximation algorithm for metric TSP.

Proof.
$d(\mathcal{A}) \leq d(\text{cycle}) = 2d(\text{MST}) \leq 2\text{OPT}$

Nearest addition algorithm for Metric TSP

NearestAdditionAlgorithm$(G = (V, E), d)$

- Find closest pair, say i and j
- Set tour T to go from i to j to i
- **for** $n - 2$ iterations **do**
  - Find pair $i \in T$ and $j \notin T$ with $\min d(i, j)$
  - Let k be vertex after i in T
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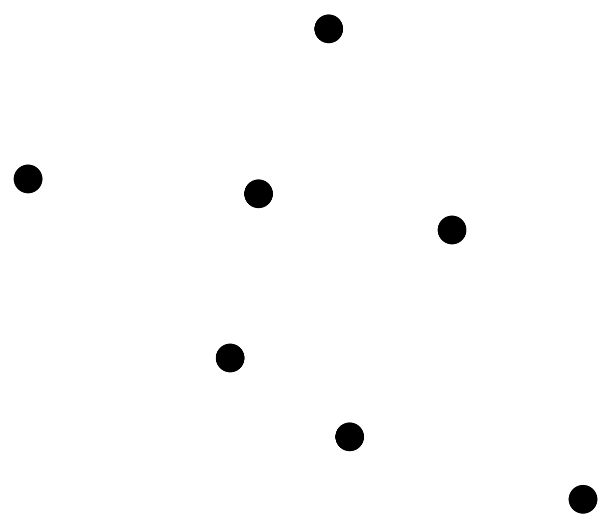

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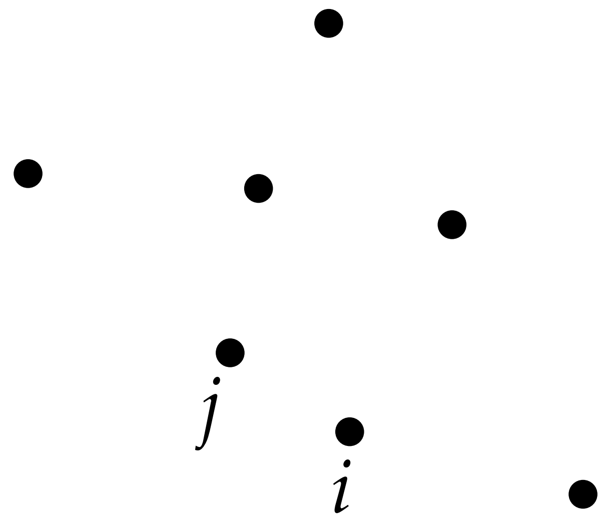

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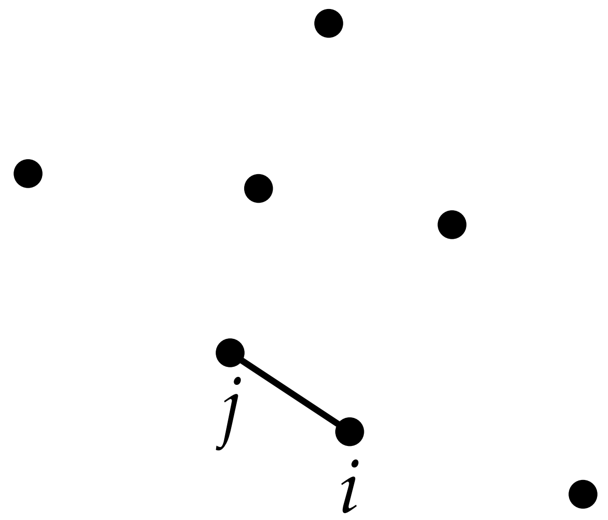

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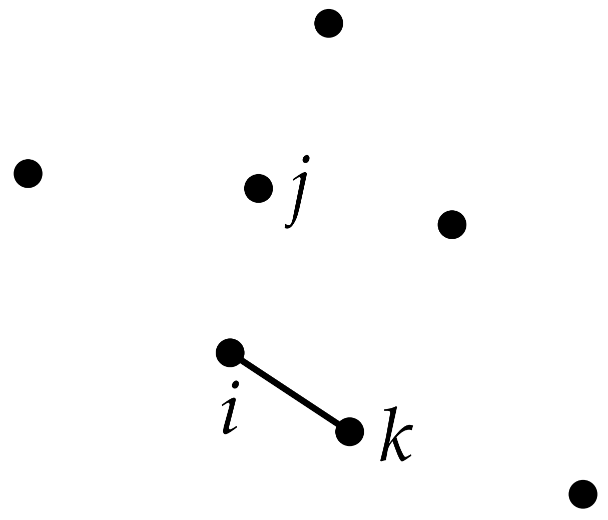

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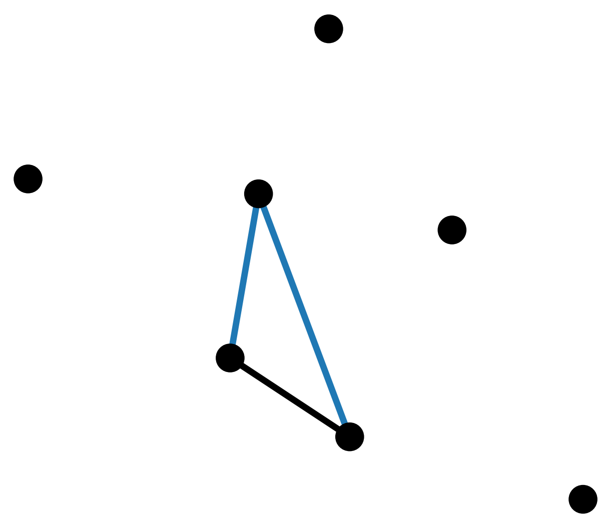

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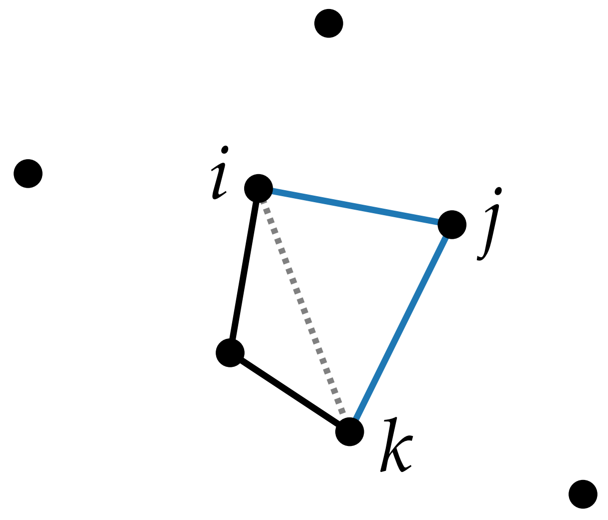

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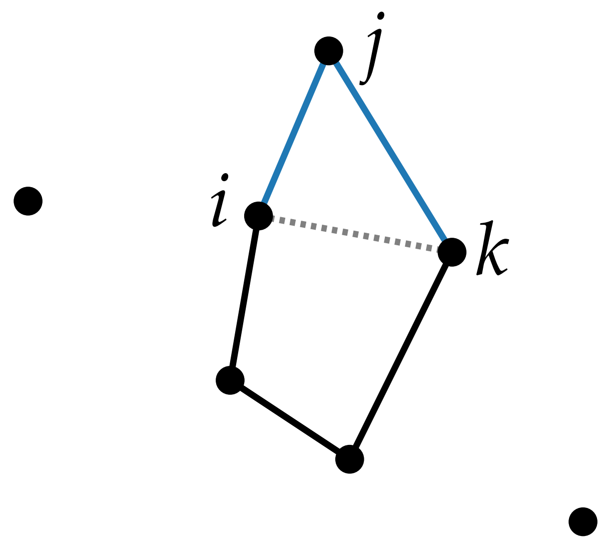

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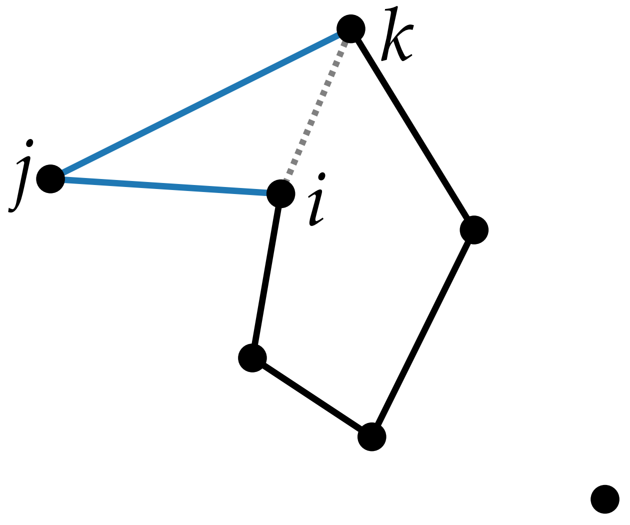

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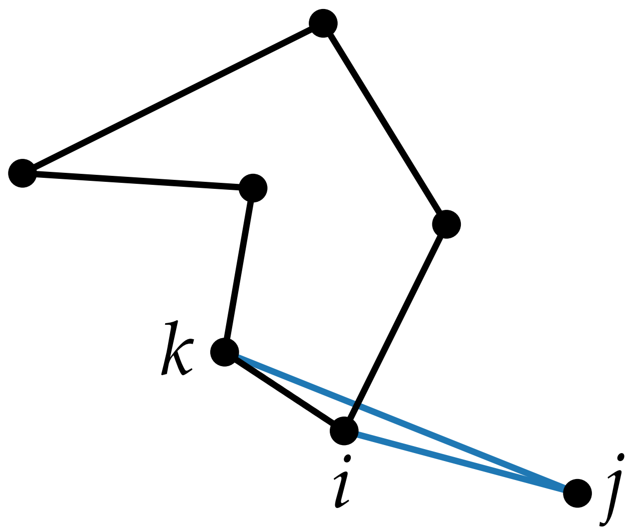

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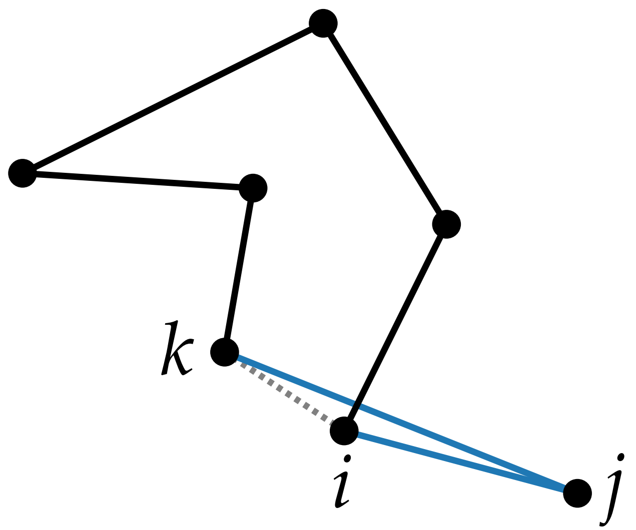

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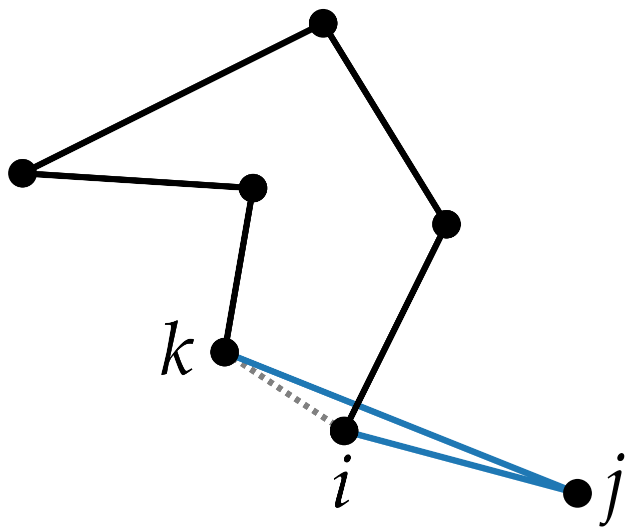

Theorem 6.
The NearestAdditionAlgorithm is a 2-approximation algorithm for metric TSP.

Nearest addition algorithm for Metric TSP

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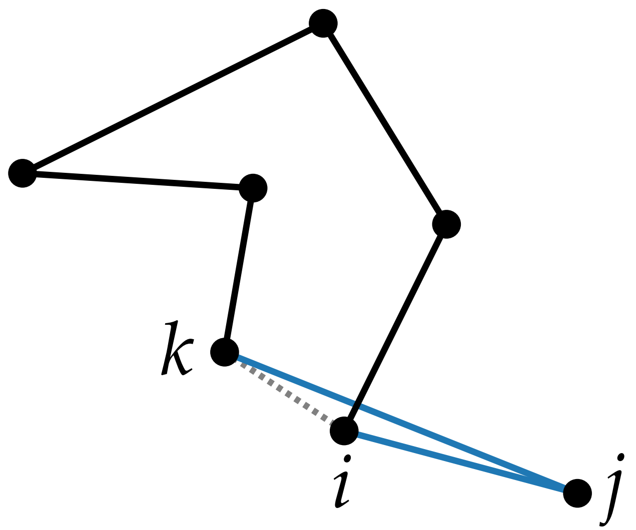

Theorem 6.
The NEARESTADDITIONALGORITHM is a 2-approximation algorithm for metric TSP.

Proof.
- Exercise.
- *Hints:* MST and Prim's algorithm.

Approximation schemes

- In some cases, we can get arbitrarily good approximations.

Approximation schemes

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Definition.
Let Π be a minimisation problem. An algorithm $\mathcal{A}$ is called an **polynomial-time approximation scheme (PTAS)**, if $\mathcal{A}$ computes for every input (I, ε) consisting of an instance I of Π and $\varepsilon > 0$ a value $\mathcal{A}(I)$, such that:

- $\mathcal{A}(I) \leq (1 + \varepsilon) \cdot \mathrm{OPT}$, and
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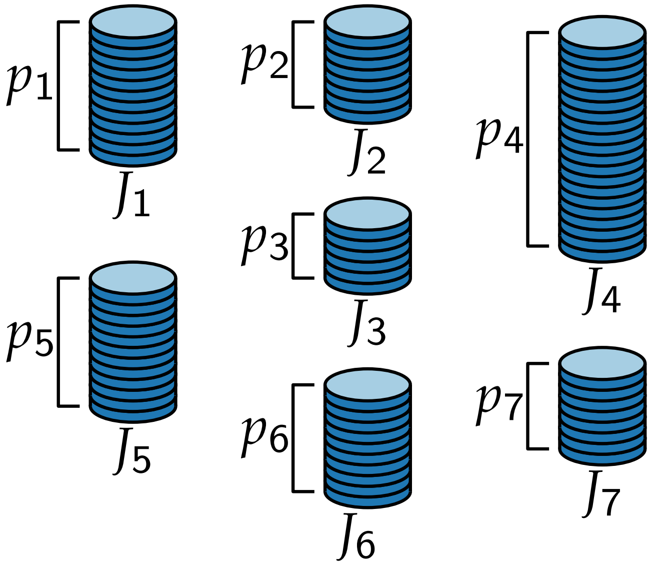
Examples.

- $\mathcal{O}\left(n^2 + n^{\frac{1}{\varepsilon}}\right) \Rightarrow$ PTAS but not FPTAS
- $\mathcal{O}\left(n^2 \cdot 3^{\frac{1}{\varepsilon}}\right) \Rightarrow$ PTAS but not FPTAS
- $\mathcal{O}\left(n^4 \cdot \left(\frac{1}{\varepsilon}\right)^2\right) \Rightarrow$ FPTAS

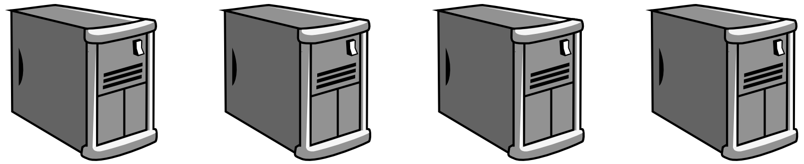
Multiprocessor Scheduling

Input.

- n jobs $J_1, \ldots, J_n$ with durations $p_1, \ldots, p_n$.



- m identical machines ($m < n$)



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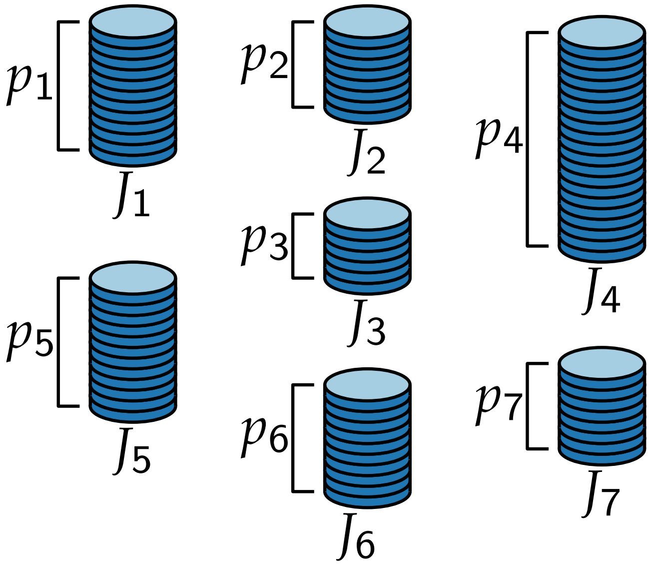

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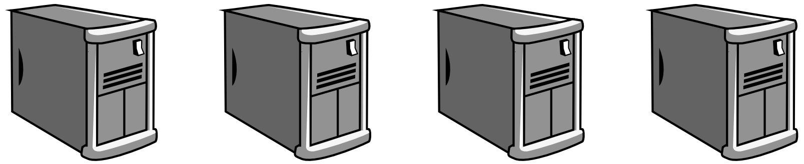

Output. Distribution of jobs to machines such that the time when all jobs have been processed is minimal.
This is called the **makespan** of the distribution.

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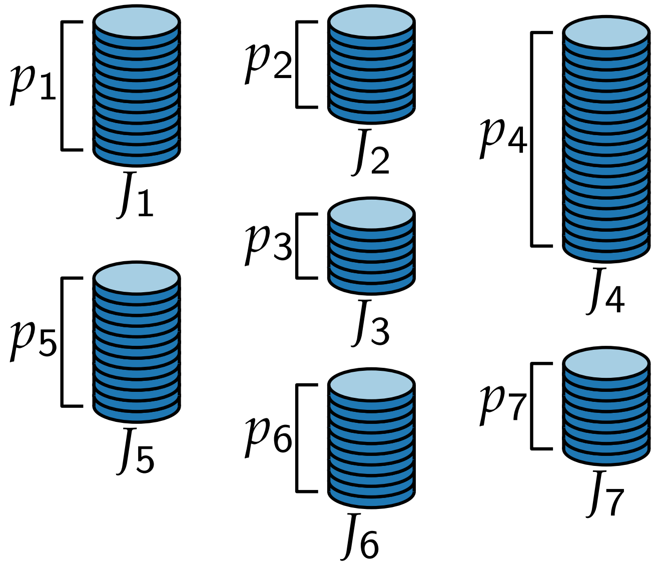

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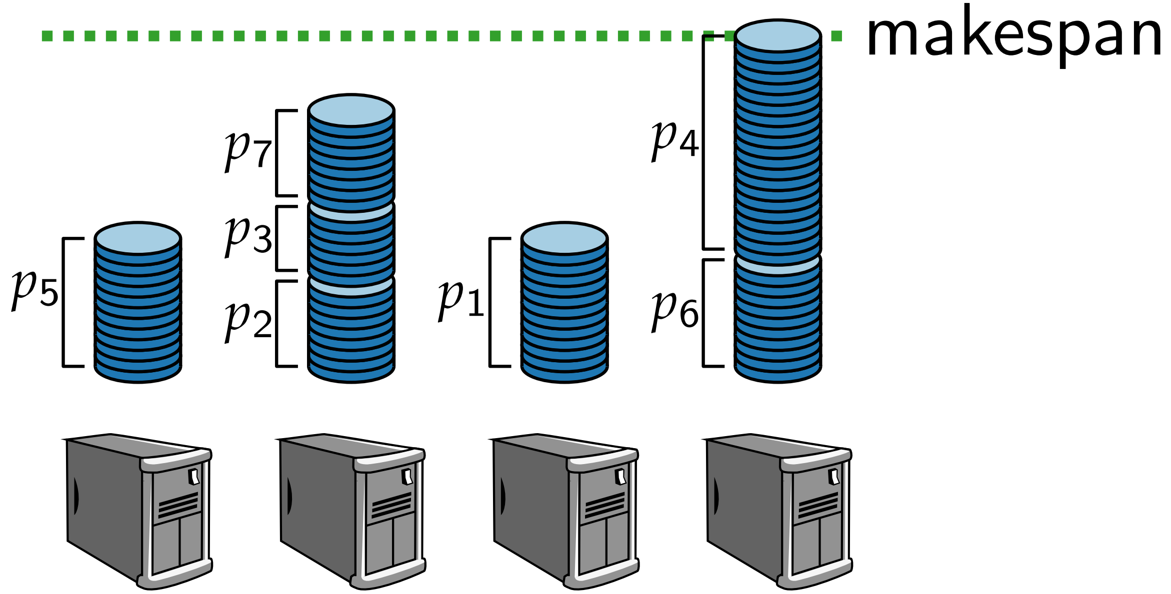

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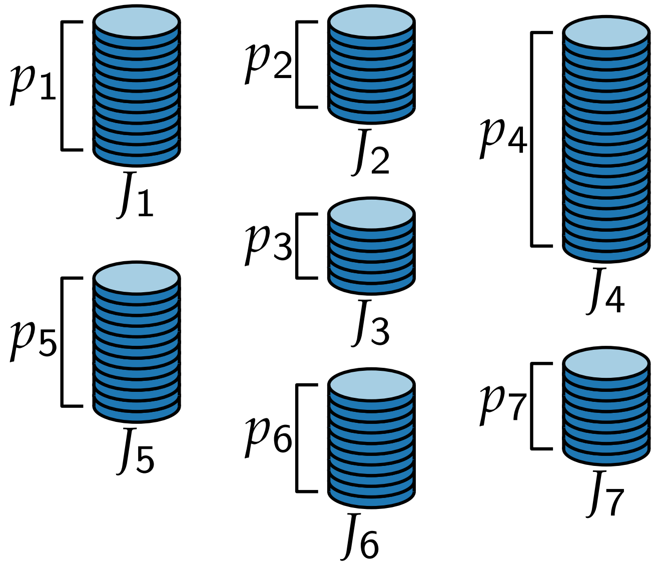

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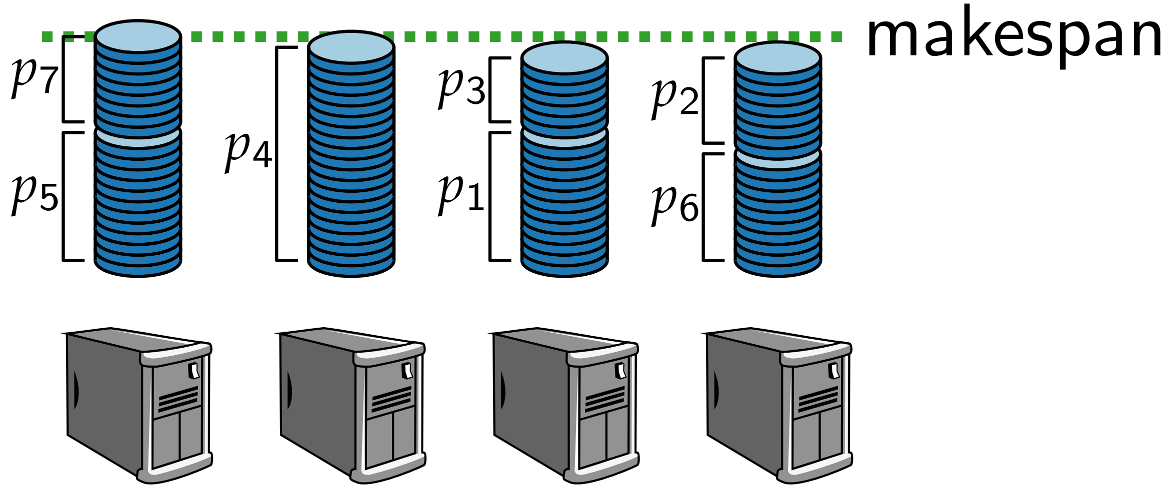
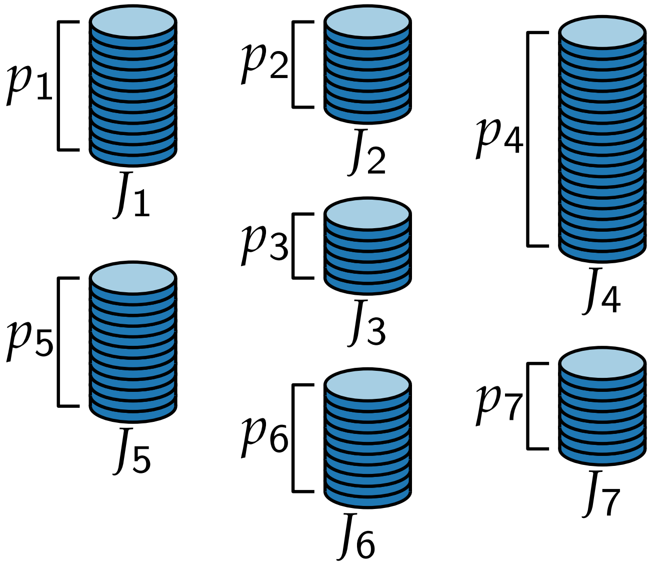


Output. Distribution of jobs to machines such that the time when all jobs have been processed is minimal.
This is called the **makespan** of the distribution.

Multiprocessor Scheduling

Input.

- n jobs $J_1, \dots, J_n$ with durations $p_1, \dots, p_n$.



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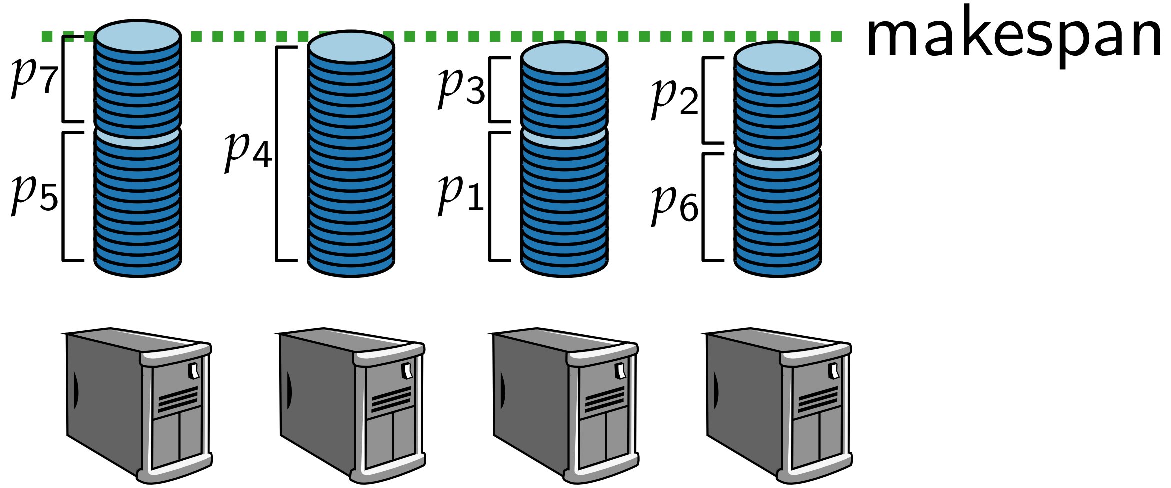


Output. Distribution of jobs to machines such that the time when all jobs have been processed is minimal.
This is called the **makespan** of the distribution.

- Multiprocess scheduling is NP-hard.

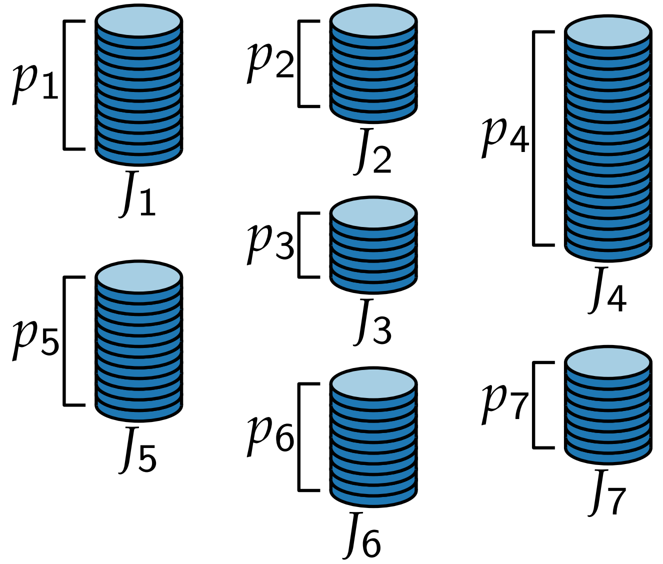
Multiprocessor Scheduling – List scheduling

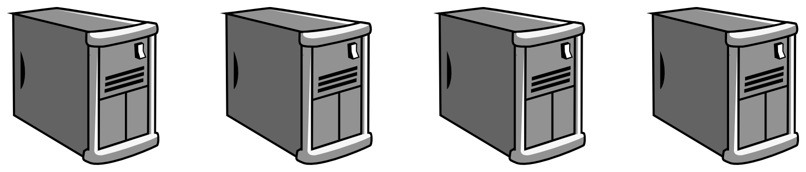
$\text{ListScheduling}(J_1, \ldots, J_n, m)$

Put the first m jobs on the m machines

Put next job on first free machine

Example.





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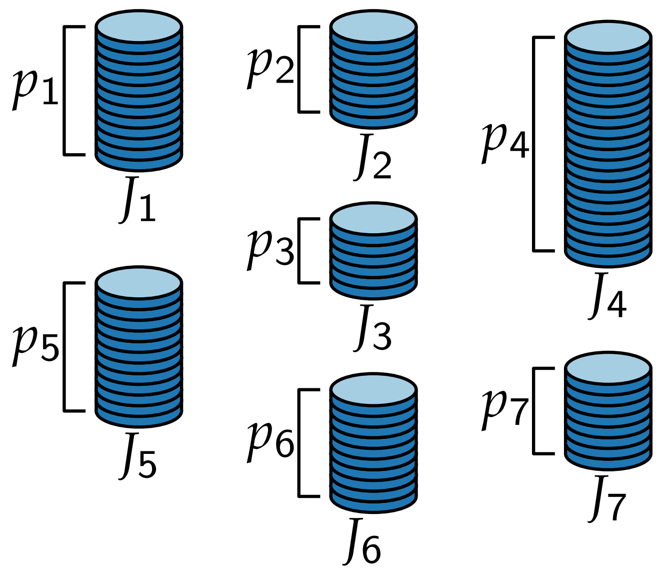

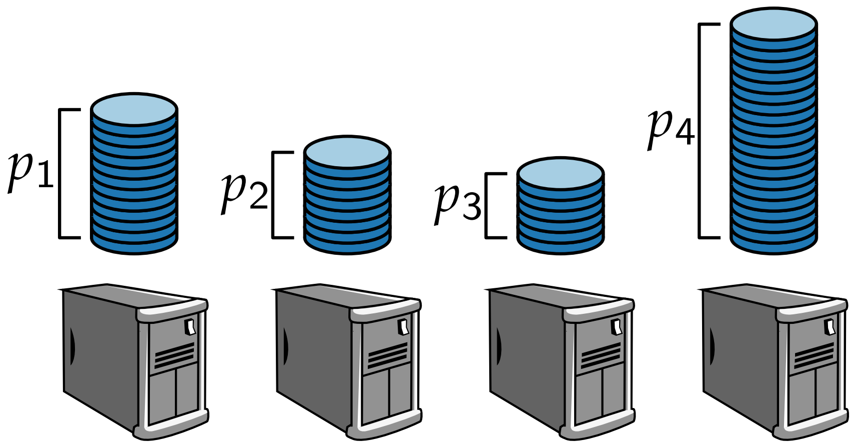

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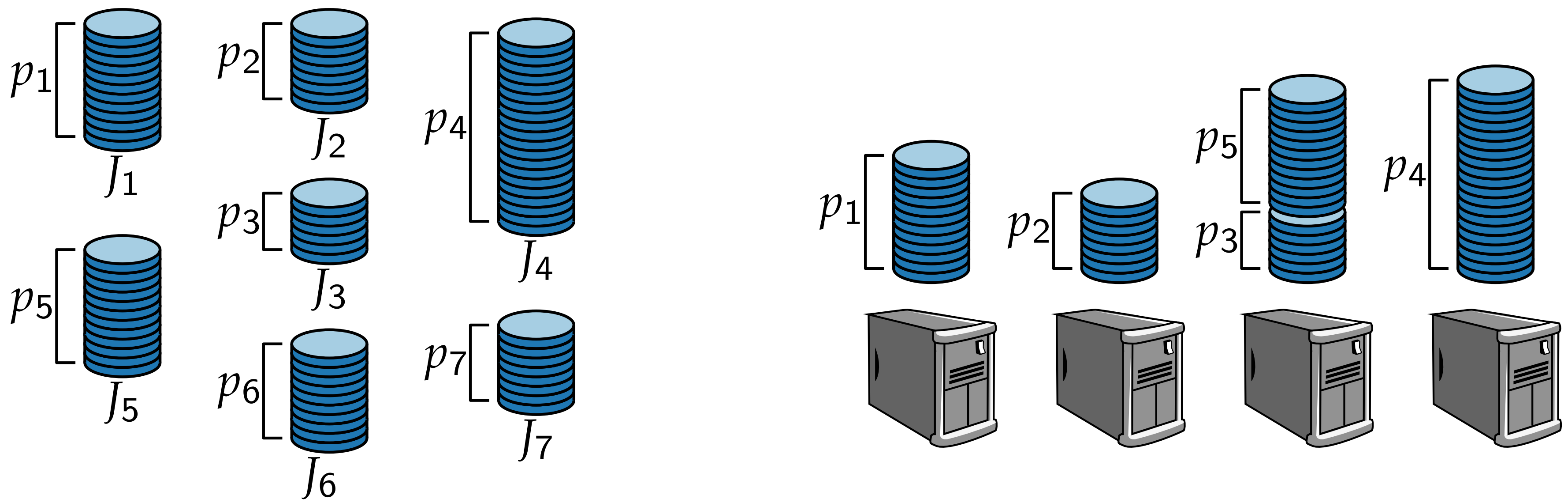

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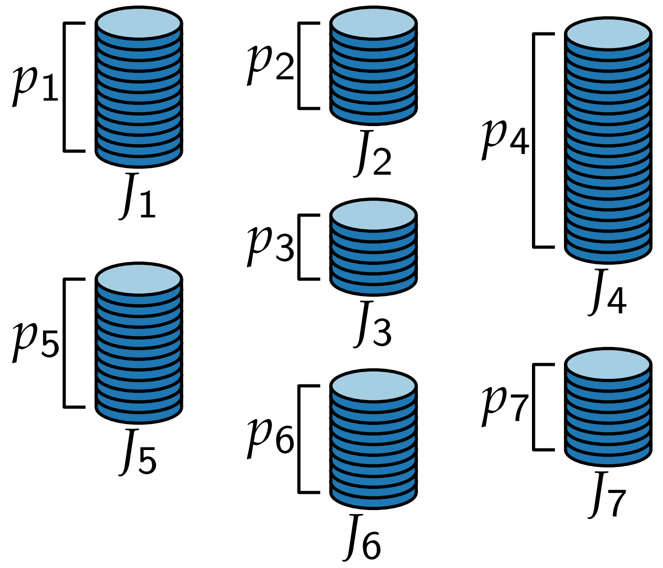

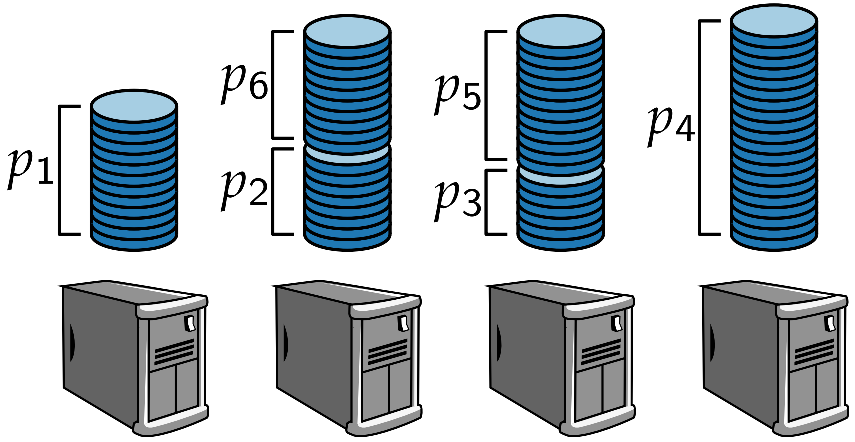

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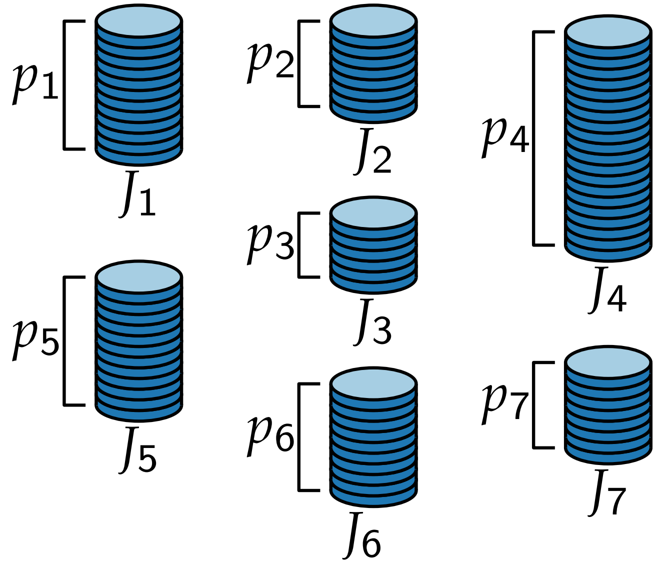

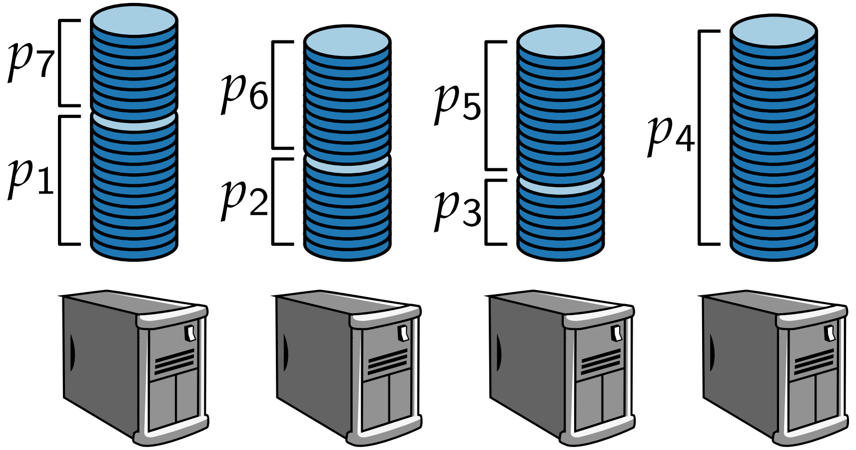

Multiprocessor Scheduling – List scheduling

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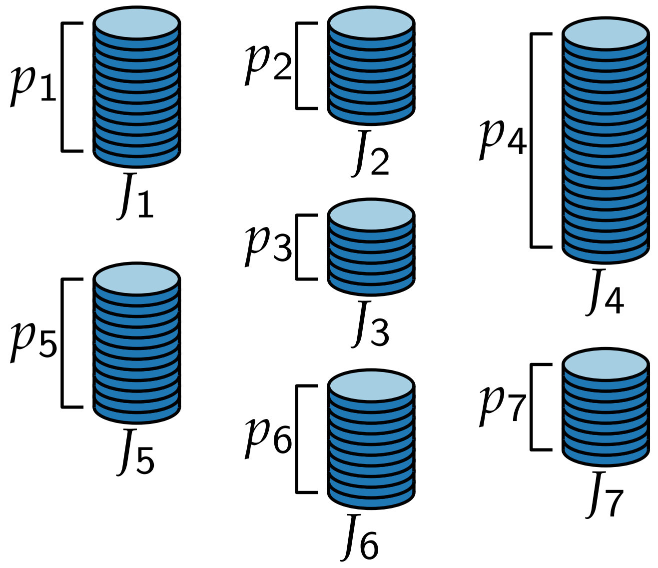
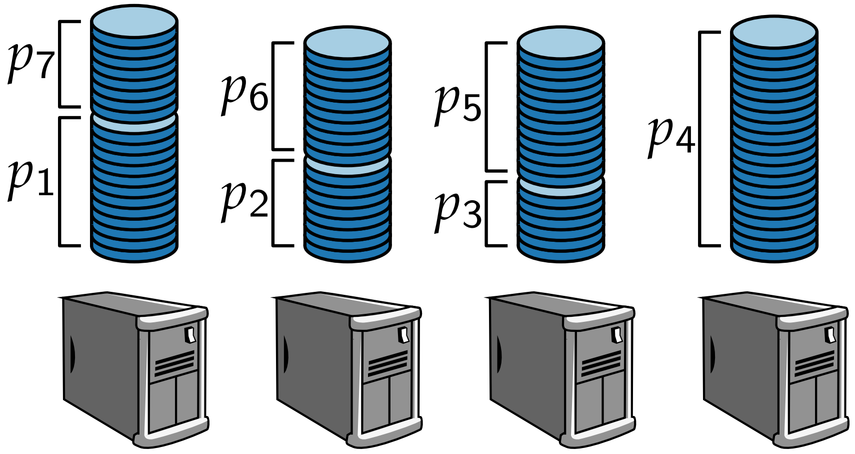




- ListScheduling runs in $\mathcal{O}(n)$ time.

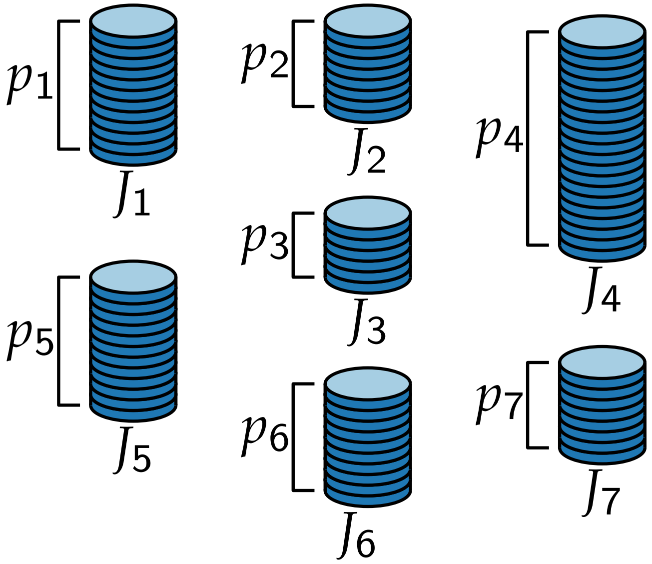
Multiprocessor Scheduling – List scheduling

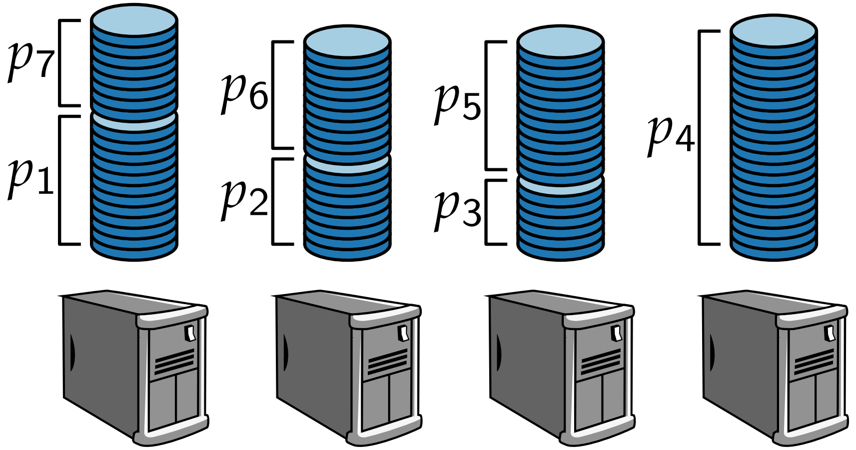
ListScheduling$(J_1, \ldots, J_n, m)$

Put the first m jobs on the m machines
Put next job on first free machine

Theorem 7.
ListScheduling is a $\left(2 - \frac{1}{m}\right)$-approximation algorithm.

Example.





- ListScheduling runs in $\mathcal{O}(n)$ time.

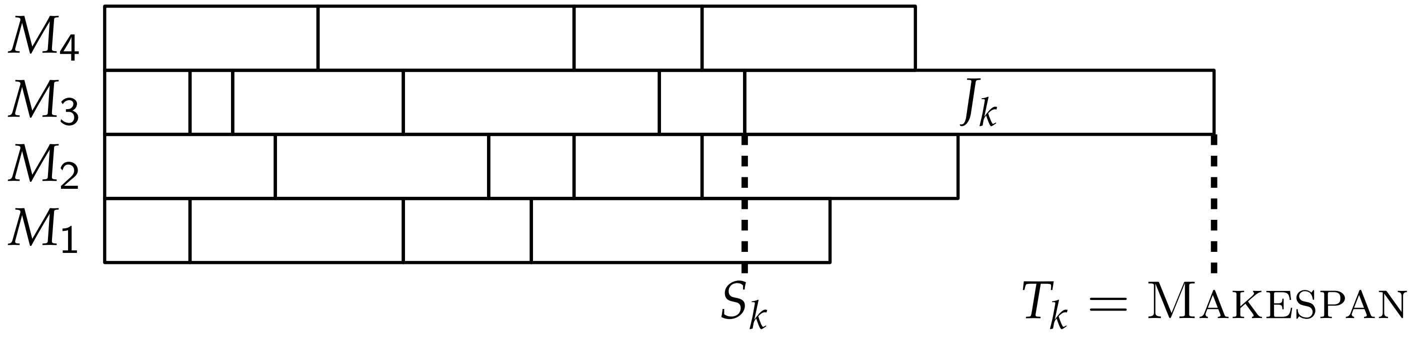
Multiprocessor Scheduling – List scheduling (proof)

ListScheduling($J_1, \dots, J_n, m$)

Put the first m jobs on the m machines
Put next job on first free machine

Theorem 7.
ListScheduling is a $(2 - \frac{1}{m})$-approximation algorithm.

Proof. Let J_k be the last job with start time S_k and finish time $T_k = \text{Makespan}$



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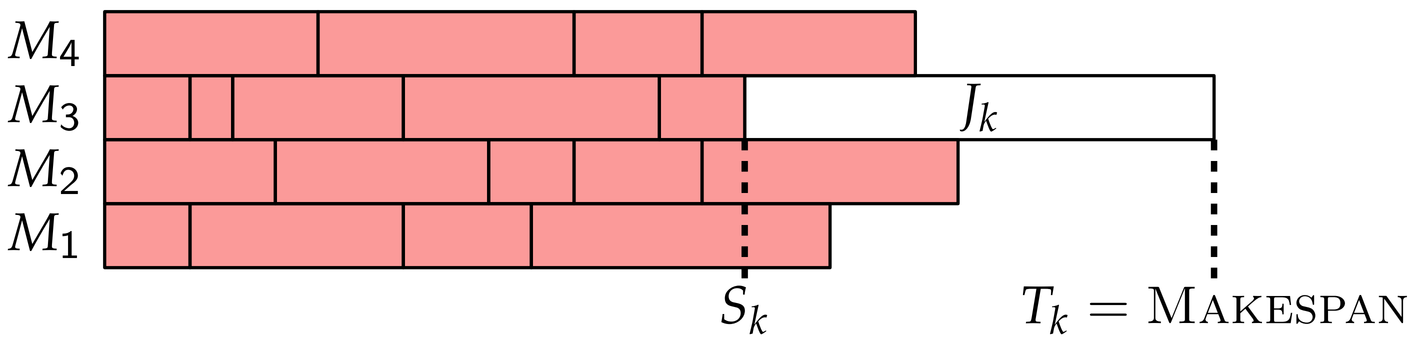
- No machine idles at time S_k.

$$S_k \leq \frac{1}{m} \sum_{i \neq k} p_i$$

weight of all jobs but J_k

evenly distributed on m machines



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- For an optimal Makespan T_{OPT}, we have:
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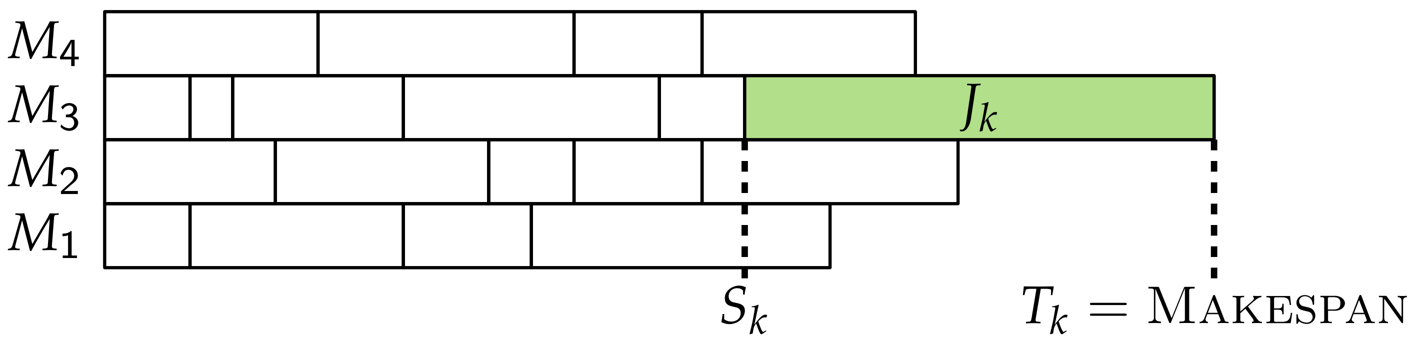

Multiprocessor Scheduling – List scheduling (proof)

$\textsc{ListScheduling}(J_1, \dots, J_n, m)$

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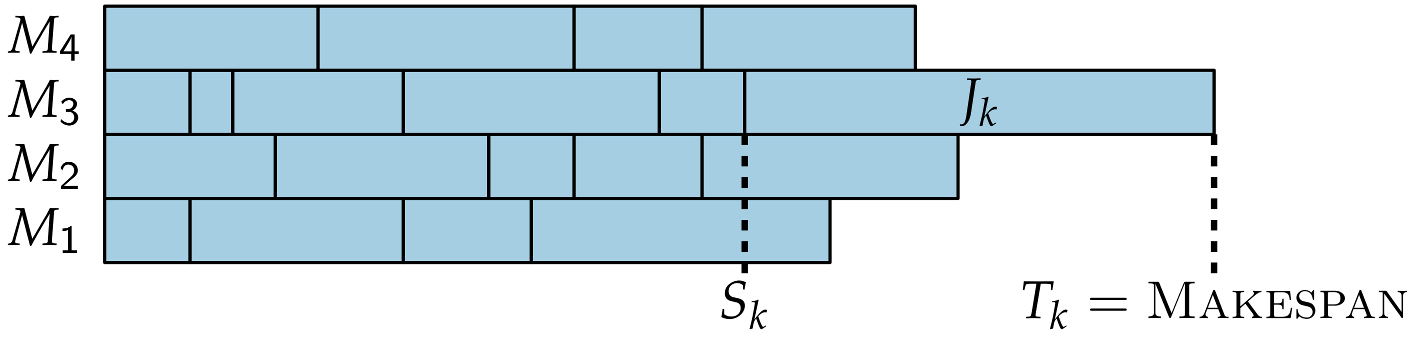
- $T_{\text{OPT}} \geq \frac{1}{m} \sum_{i=1}^{n} p_i$ weight of all jobs evenly distributed



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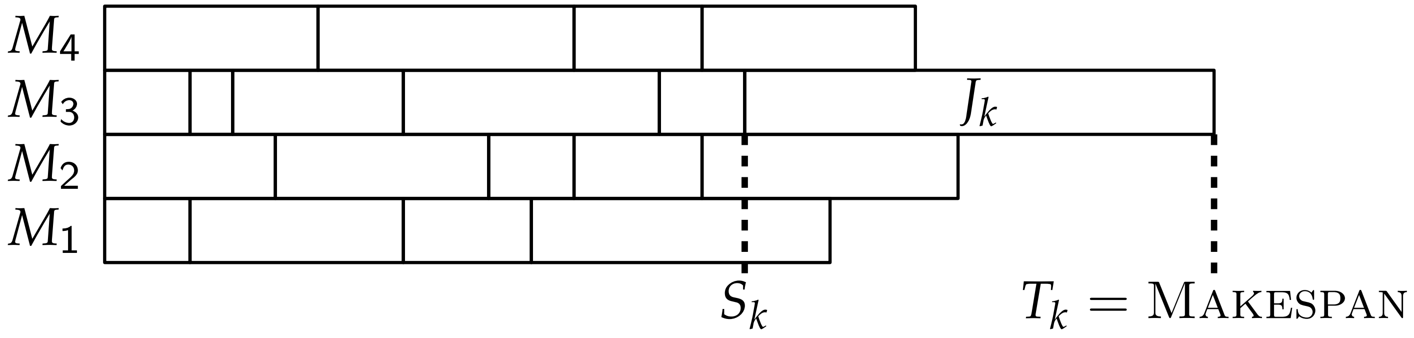

- Hence:

$$T_k = S_k + p_k$$

Multiprocessor Scheduling – List scheduling (proof)

$\textsc{ListScheduling}(J_1, \ldots, J_n, m)$

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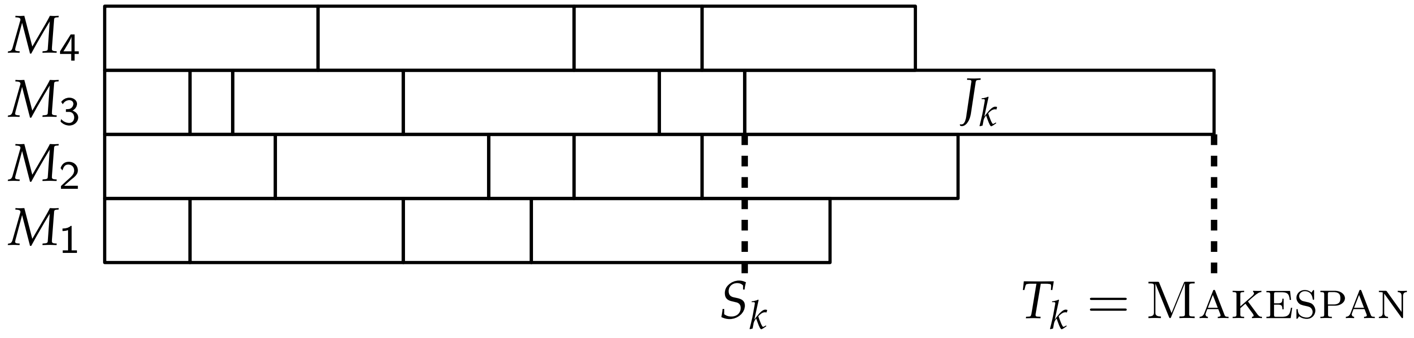
- Hence:

$$T_k = S_k + p_k \leq \frac{1}{m} \cdot \sum_{i \neq k} p_i + p_k$$



Multiprocessor Scheduling – List scheduling (proof)

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$$T_k = S_k + p_k$$
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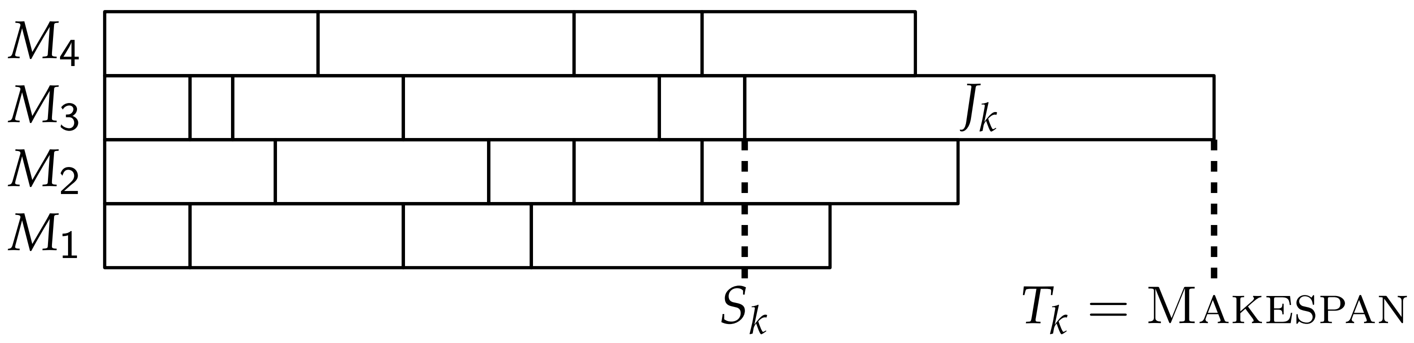
$$= \frac{1}{m} \cdot \sum_{i=1}^{n} p_i + \left(1 - \frac{1}{m}\right) \cdot p_k$$



Multiprocessor Scheduling – List scheduling (proof)

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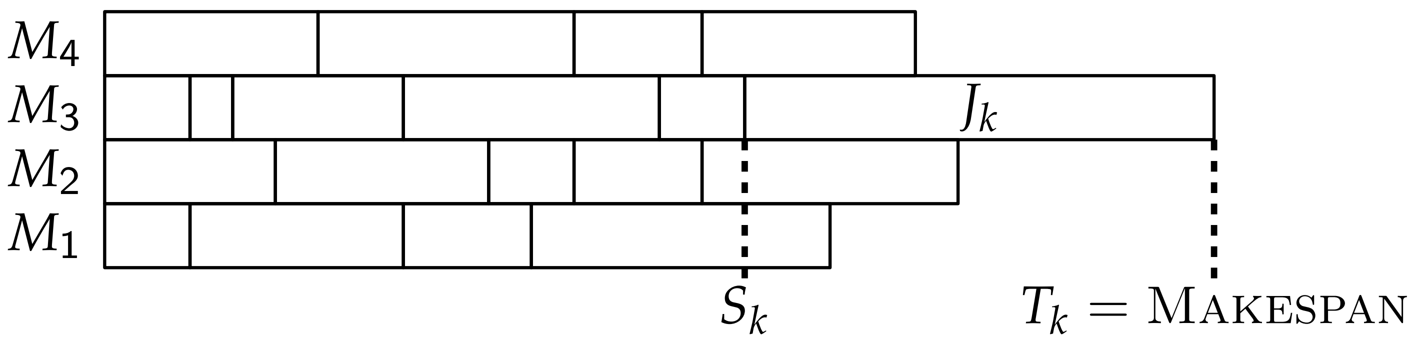

- Hence:

$$\begin{aligned} T_k &= S_k + p_k \\ &\leq \frac{1}{m} \cdot \sum_{i \neq k} p_i + p_k \\ &= \frac{1}{m} \cdot \sum_{i=1}^{n} p_i + \left(1 - \frac{1}{m}\right) \cdot p_k \\ &\leq T_{\text{OPT}} + \left(1 - \frac{1}{m}\right) \cdot T_{\text{OPT}} \end{aligned}$$

Multiprocessor Scheduling – List scheduling (proof)

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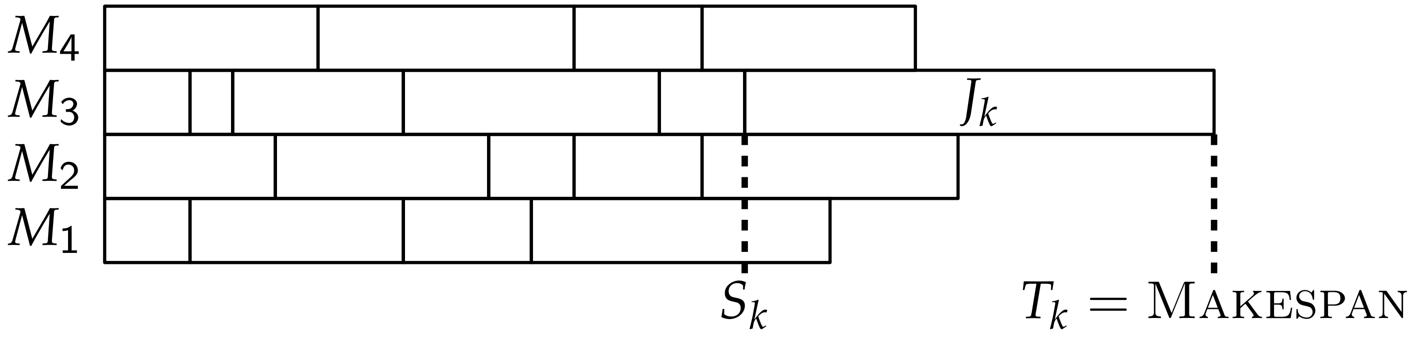

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Multiprocessor Scheduling – PTAS

For a constant ℓ ($1 \leq \ell \leq n$) define the algorithm $\mathcal{A}_\ell$ as follows.

$\mathcal{A}_\ell(J_1, \ldots, J_n, m)$

Sort jobs in descending order of runtime

Schedule the ℓ longest jobs $J_1, \ldots, J_\ell$ optimally

Use ListScheduling for the reamining jobs $J_{\ell+1}, \ldots, J_n$

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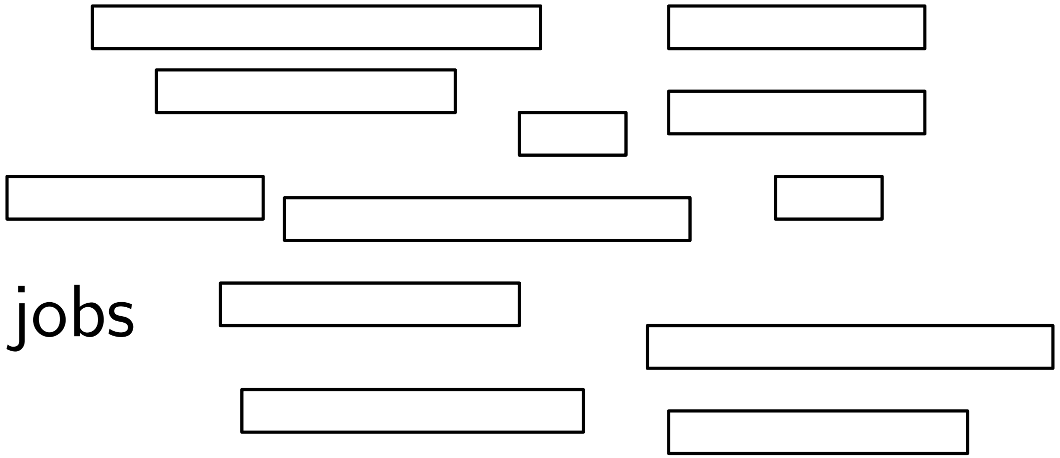
Example.

$\ell = 6$



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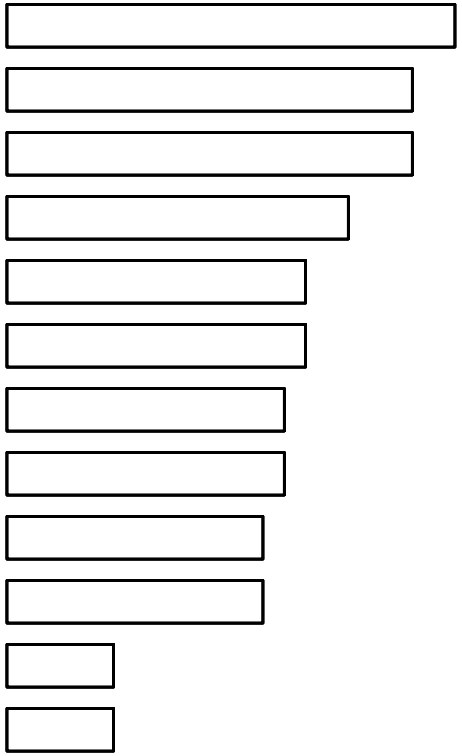

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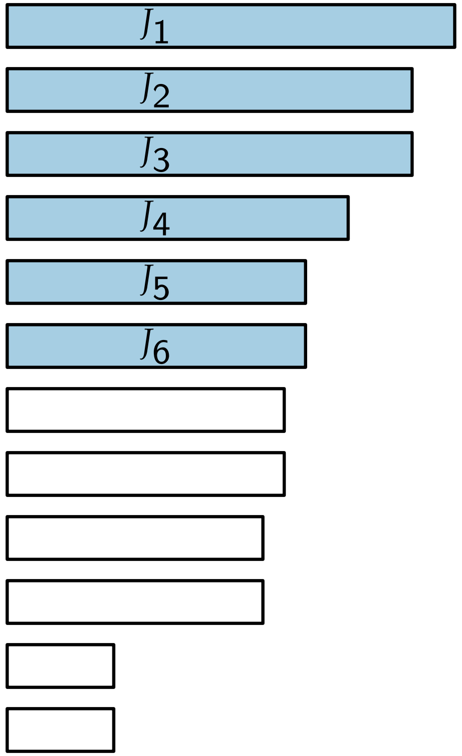

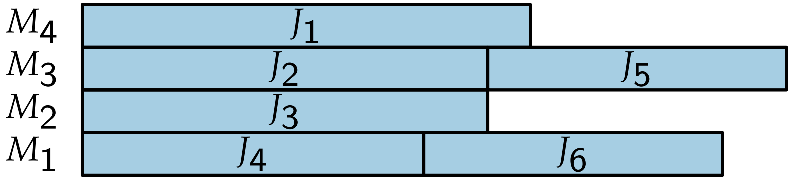

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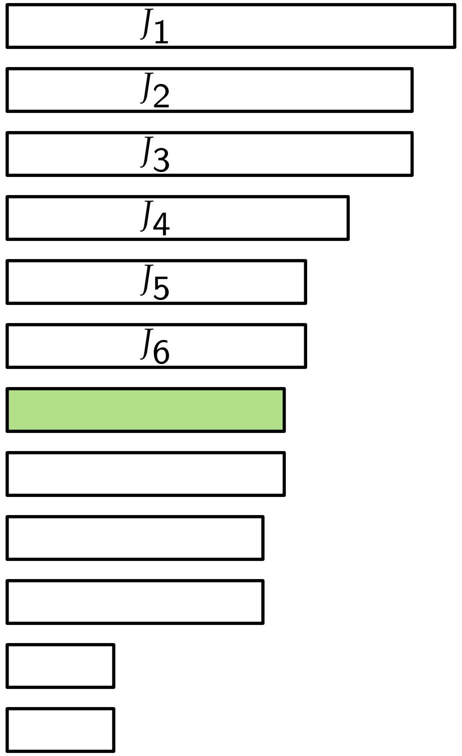

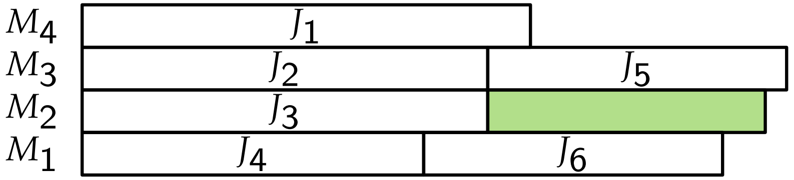

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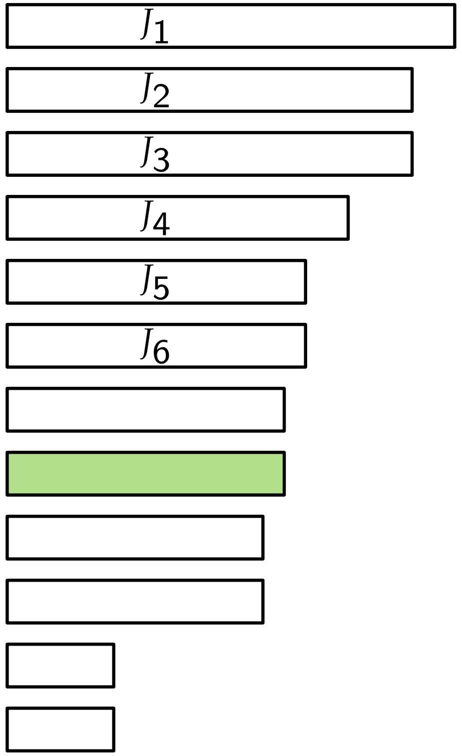

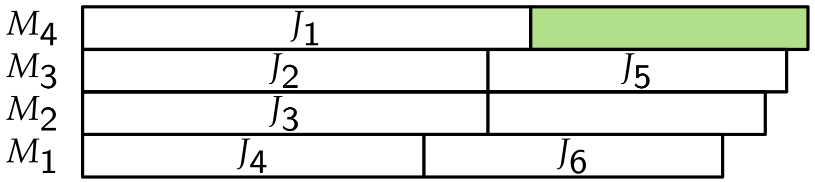

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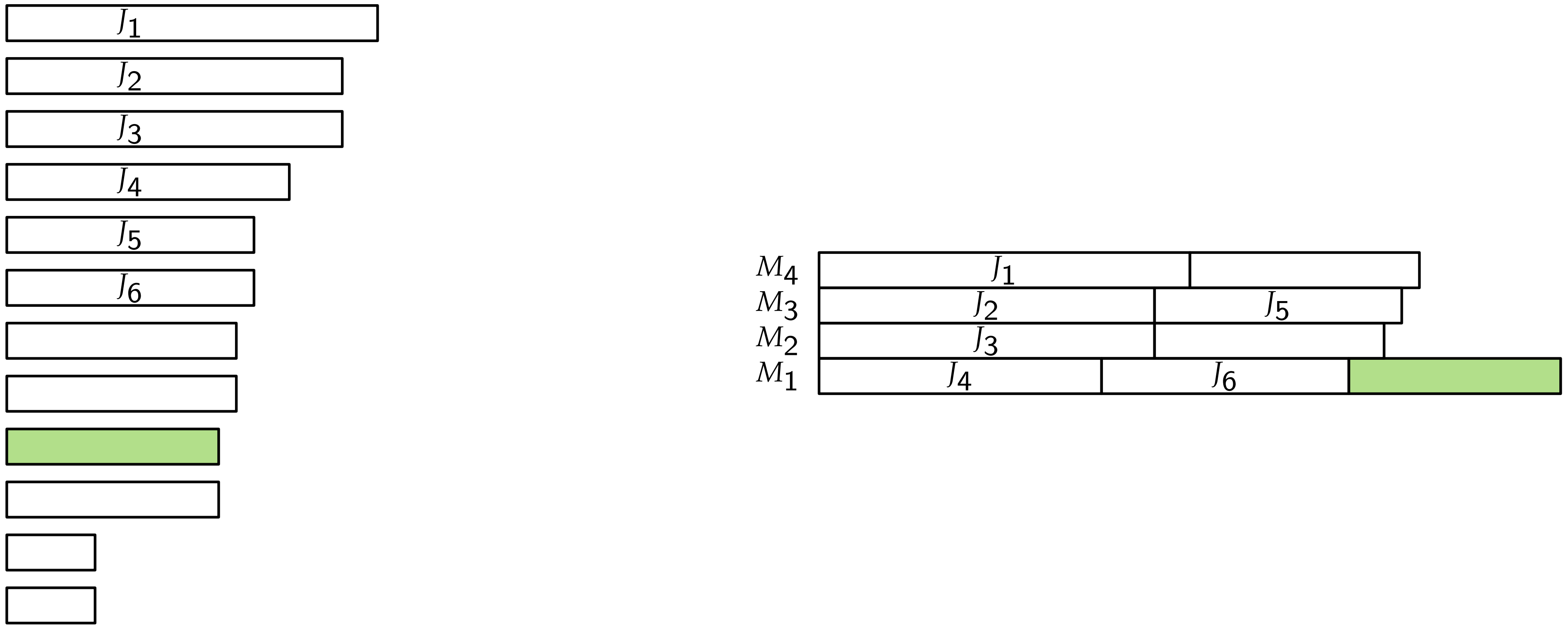

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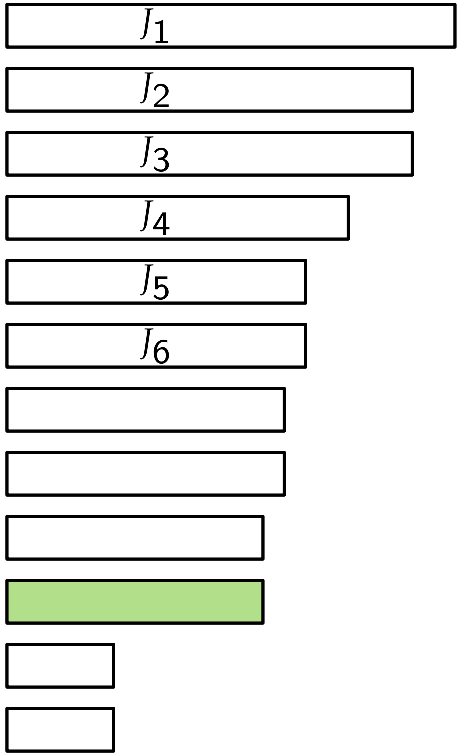
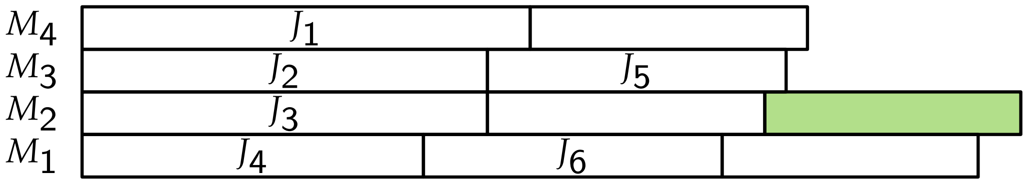
sorted jobs





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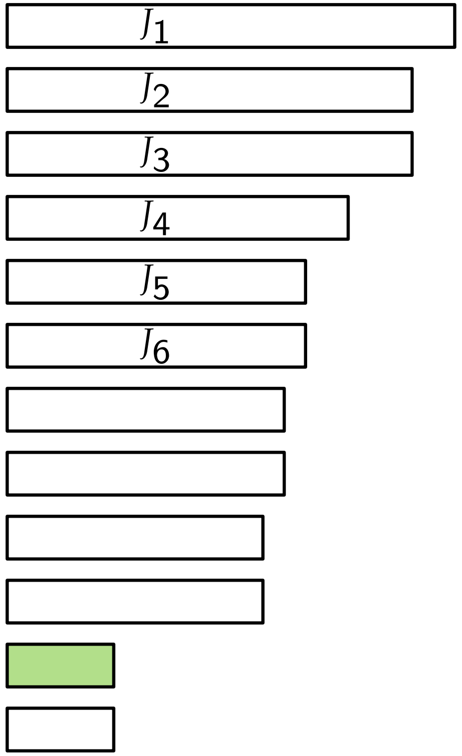

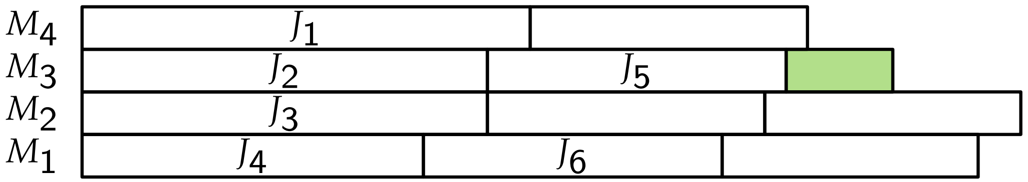

Multiprocessor Scheduling – PTAS

For a constant ℓ $(1 \leq \ell \leq n)$ define the algorithm $\mathcal{A}_\ell$ as follows.

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Example.

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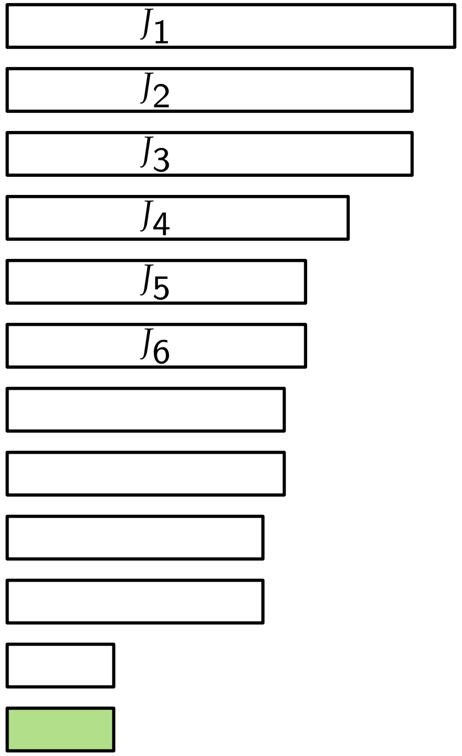

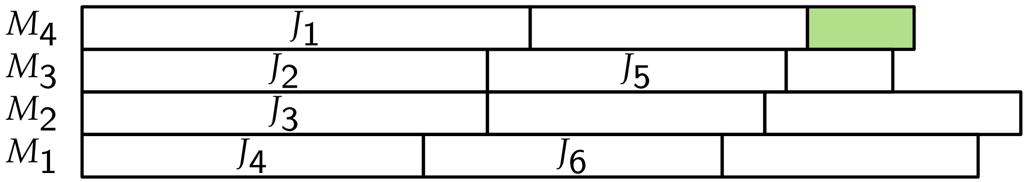

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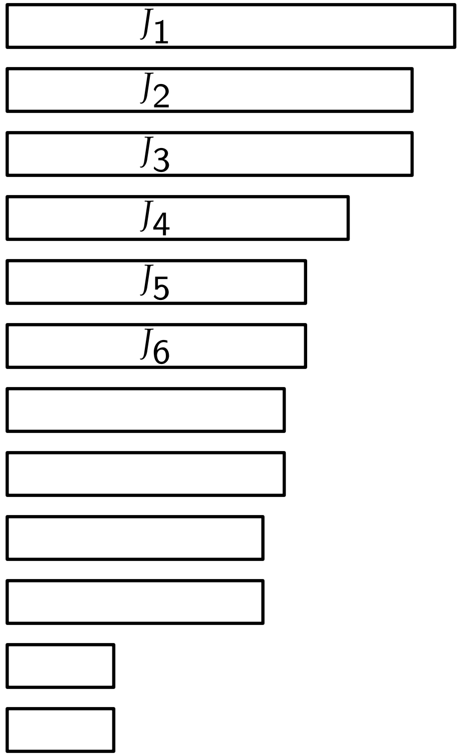

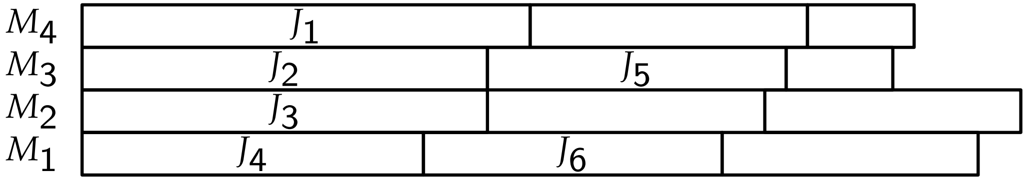

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For a constant ℓ $(1 \leq \ell \leq n)$ define the algorithm $\mathcal{A}_\ell$ as follows.

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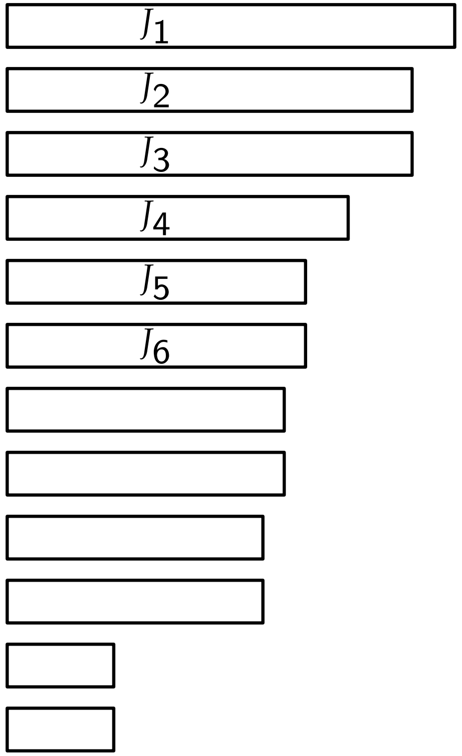

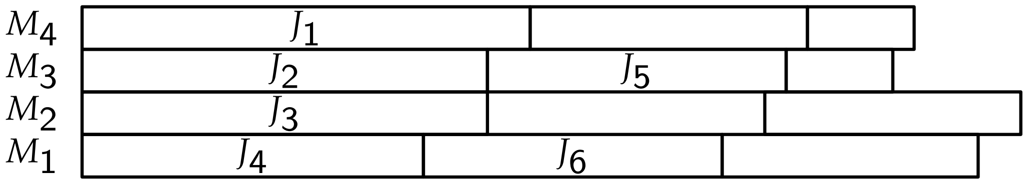

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Corollary 9.
For a constant number of machines, $\{\mathcal{A}_\varepsilon \mid \varepsilon > 0\}$ is a PTAS.

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For a constant number of machines, $\{\mathcal{A}_\varepsilon \mid \varepsilon > 0\}$ is a PTAS.

Multiprocessor Scheduling – PTAS (proof)

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Proof. Let J_k be the last job with start time S_k and finish time $T_k = \textsc{Makespan}$

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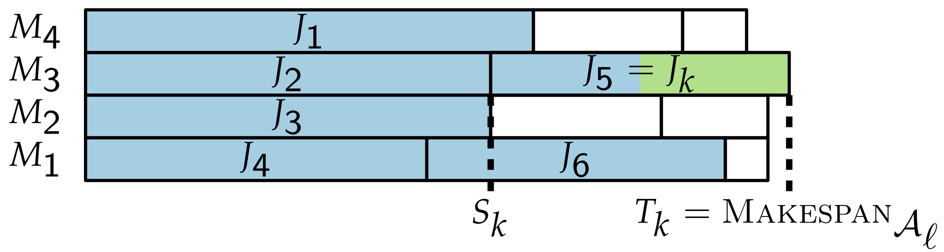

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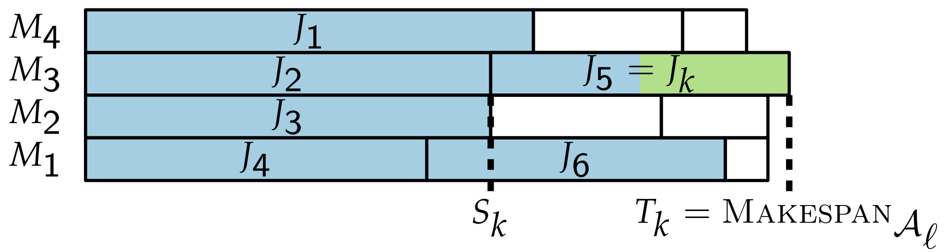

- Solution is optimal for $J_1, \ldots, J_k$
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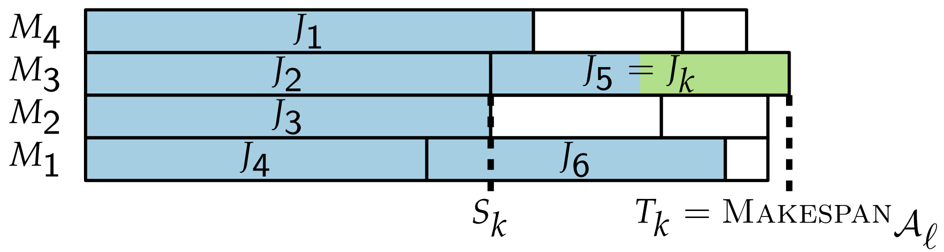

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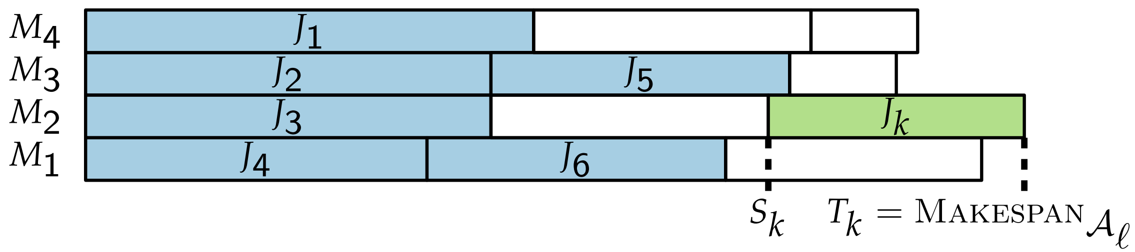

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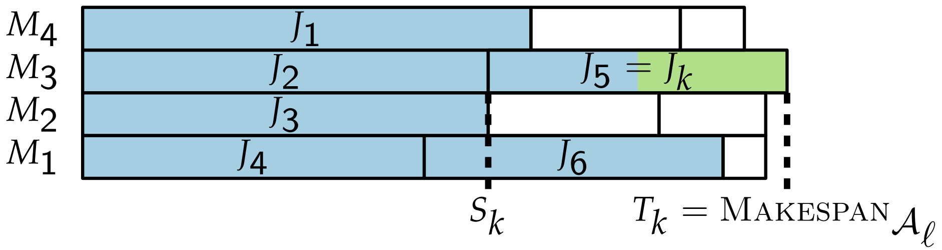

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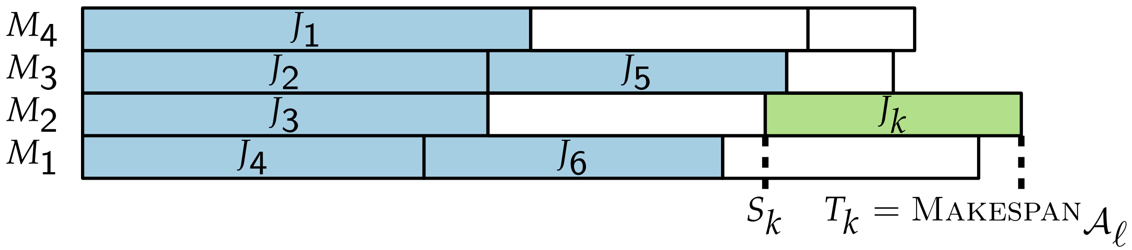
- Similar analysis to ListScheduling
- Use that there are $\ell + 1$ jobs that are at least as long as J_k (including J_k).



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- $S_k \leq \frac{1}{m} \sum_{i \neq k} p_i$
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$$T_k = S_k + p_k$$

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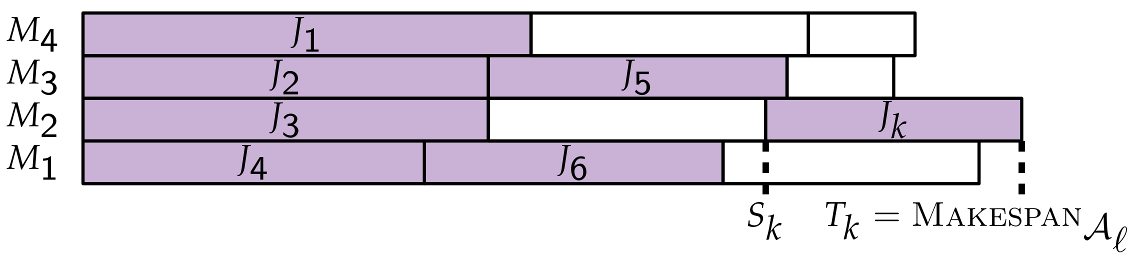

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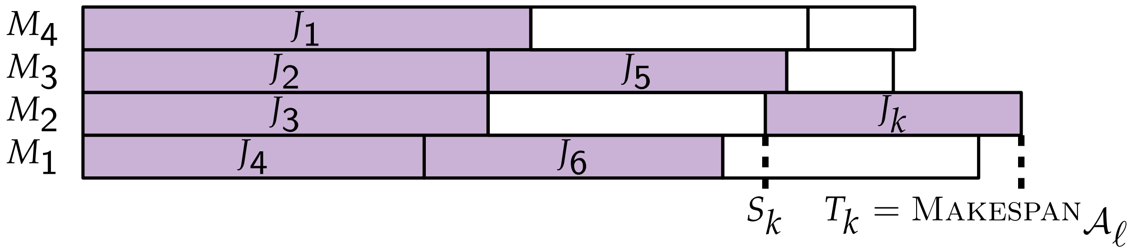

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- Min Vertex Coloring on planar graphs can be approximated with an additive approximation guarantee of 2.
- Christofides' approximation algorithm for Metric TSP has approximation factor 1.5.

Literature

Main references

- [Jansen, Margraf Ch3] “Approximative Algorithmen und Nichtapproximierbarkeit”
- [Williamson, Shmoys Ch3] “The Design of Approximation Algorithms”

Another book recommendation:

- [Vazirani] “Approximation Algorithms”

and don’t forget our lecture

- **Approximation Algorithms**.

For more precise definitions see

- https://go.uniwue.de/approxdef