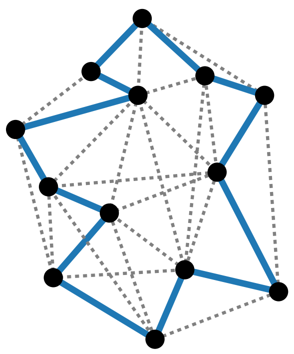
Julius-Maximilians-
UNIVERSITÄT
WÜRZBURG

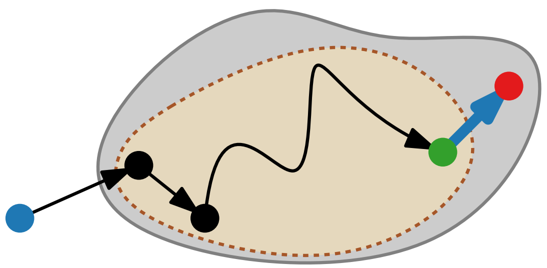
Advanced Algorithms

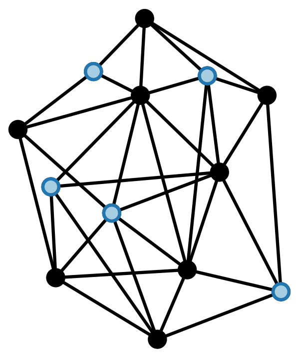
Exact algorithms for NP-hard problems

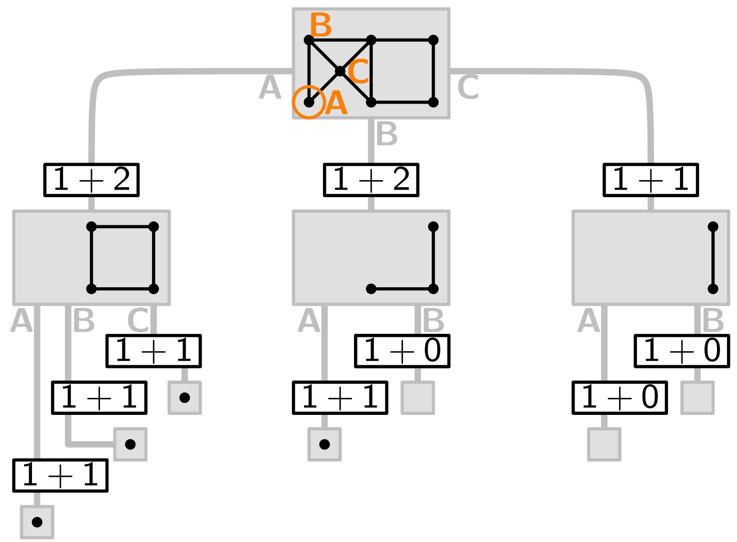
TSP and MIS

Jonathan Klawitter · WS20









Examples of NP-hard problems

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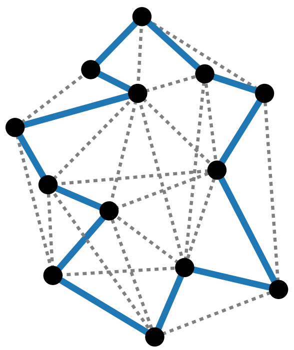
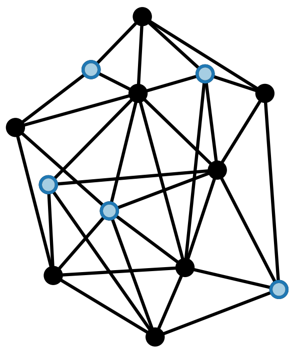
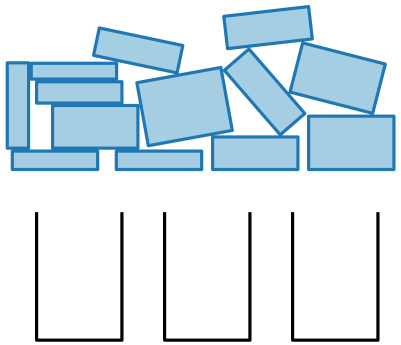


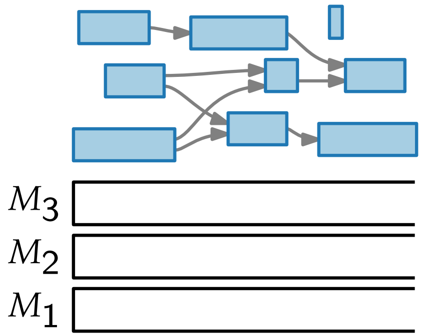
TSP



MIS



Bin Packing



Scheduling

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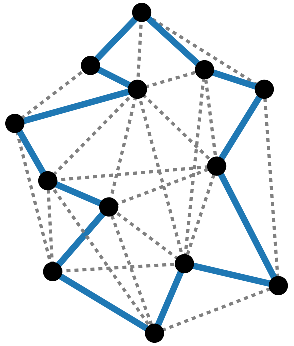

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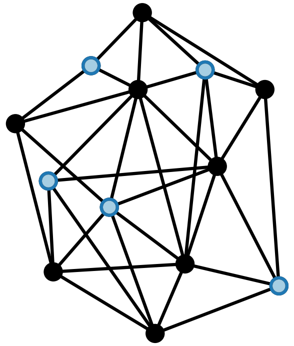

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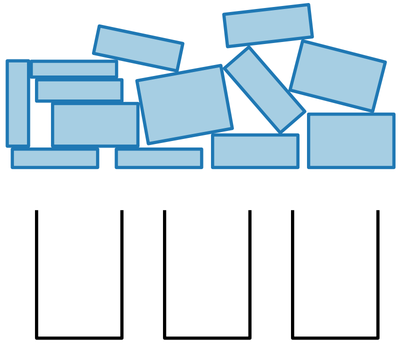

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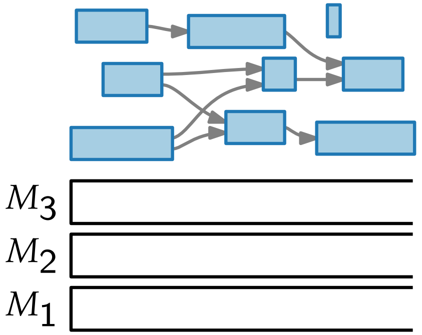
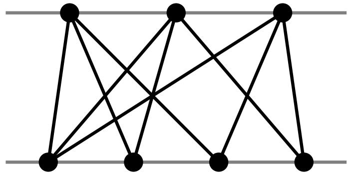


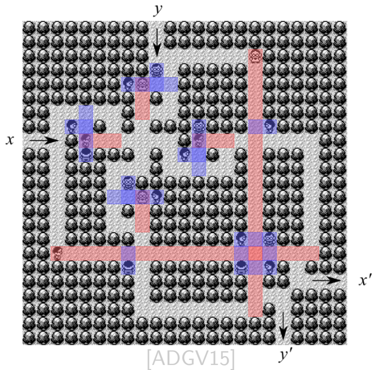
Scheduling

$(x_1 \vee x_2 \vee \neg x_4) \wedge$
$(\neg x_2 \vee x_3 \vee \neg x_4) \wedge$
$(x_3 \vee x_7 \vee \neg x_8) \wedge$
$\dots$

SAT



Graph Drawing



Games

...

Formal view on NP-hardness

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- or: There is a polynomial-time many-one reduction from an NP-hard problem L to H.
- If P $\neq$ NP, then NP-hard problems cannot be solved in polynomial time.

Misconceptions about NP-hardness

Common misconceptions [Mann '17]

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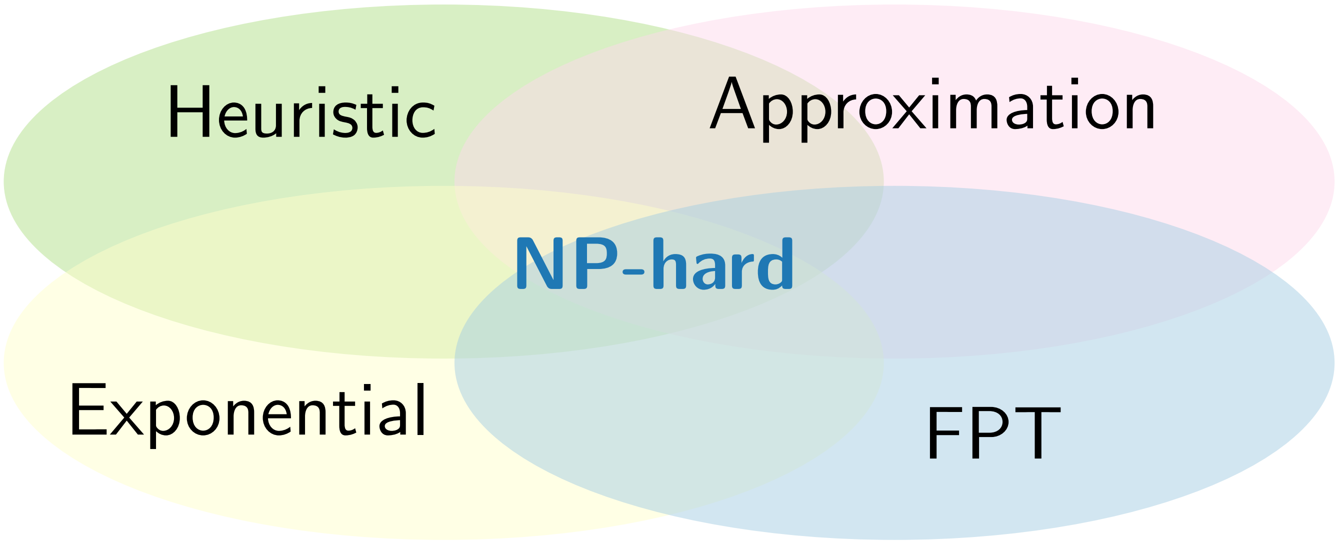
- Problems that are hard to solve in practice by an engineer are NP-hard.
- NP-hard problems cannot be solved optimally.
- NP-hard problems cannot be solved more efficiently than by exhaustive search.
- For solving NP-hard problems, the only practical possibility is the use of heuristics.

Dealing with NP-hard problems

What should we do?

- Sacrifice optimality for speed
  - Heuristics (Simulated Annealing, Tabu-Search)
  - Approximation Algorithms (Christofides-Algorithm)
- Optimal Solutions
  - Exact exponential-time algorithms
  - Fine-grained analysis – parameterized algorithms



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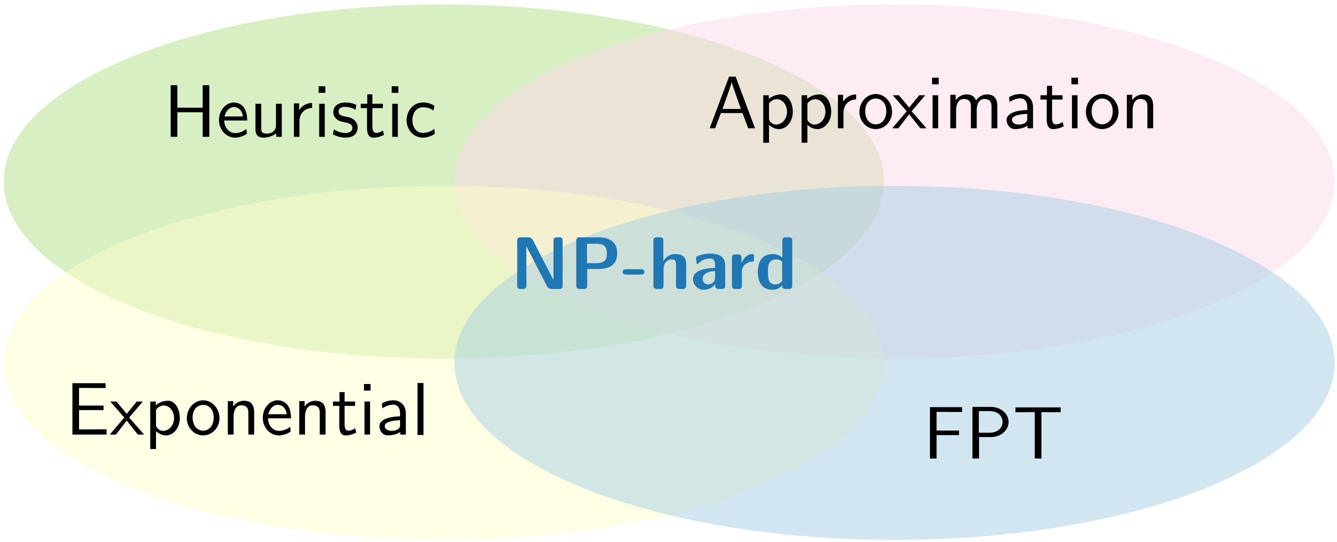
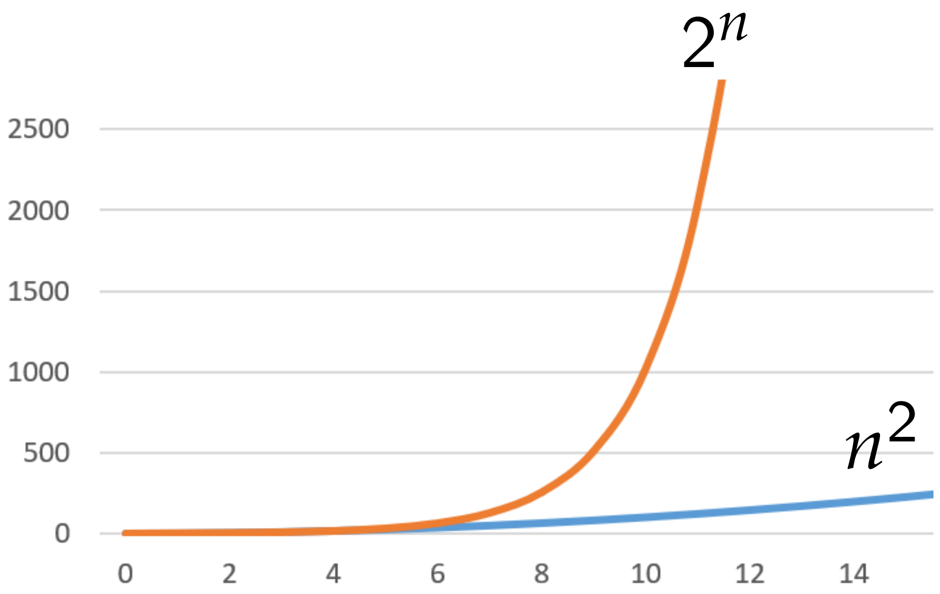


Motivation



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- polynomial-time vs.
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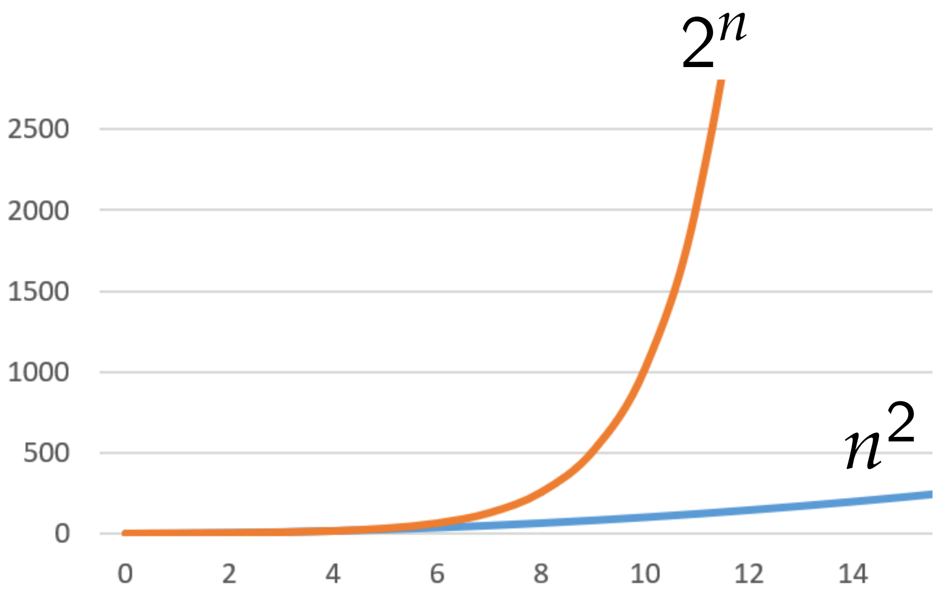
Exponential runningtime . . . should we just give up?



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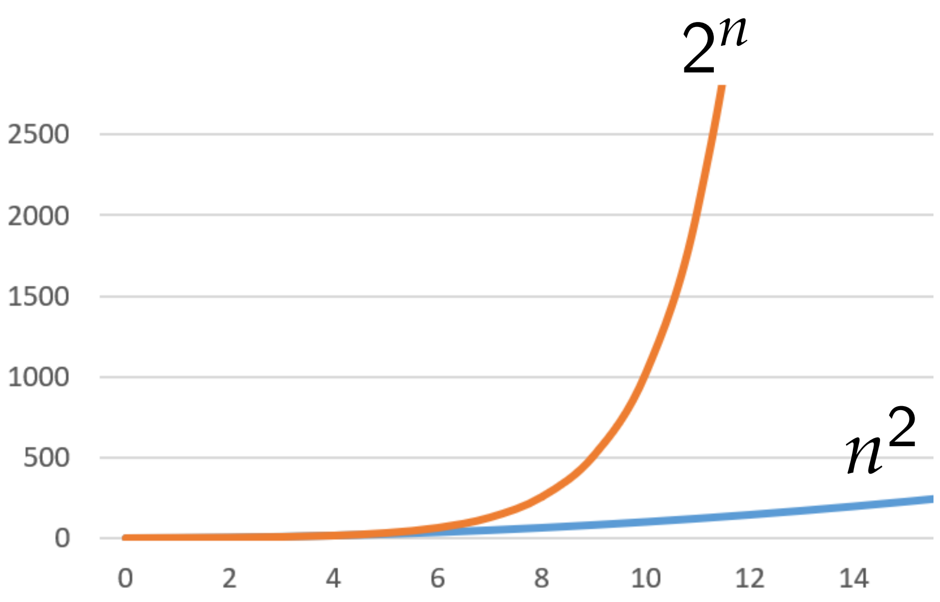

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Exponential runningtime . . . should we just give up?

- . . . can be *"fast"* for medium-sized instances:
  - $n^4 > 1.2^n$ for $n \leq 100$
  - TSP solvable exactly for $n \leq 2000$ and specialized instances with $n \leq 85900$
  - "hidden" constants in polynomial-time algorithms:
    $2^{100}n > 2^n$ for $n \leq 100$

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- Reducing the base of the runtime to $b < a$ results in a *multiplicative* increase:

$$b^{n_0'} = a^{n_0} \rightsquigarrow n_0' = n_0 \cdot \log_b a$$

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- TSP: Bellman-Held-Karp algorithm has running time $\mathcal{O}(2^n n^2)$ compared to a $\mathcal{O}(n!n)$-time brute-force search.
- MIS: algorithm by Tarjan & Trojanowski runs in $\mathcal{O}(2^{n/3})$ time compared to a trivial $\mathcal{O}(n2^n)$-time approach.
- $\textsc{Coloring}$: Lawler gaven an $\mathcal{O}(n(1+\sqrt[3]{3})^n)$ algorithm compared to $\mathcal{O}(n^{n+1})$-time brute-force.

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- SAT: No better algorithm than trivial brute-force search known.

$\mathcal{O}^*$-notation

$$\mathcal{O}(1.4^n \cdot n^2) \subsetneq \mathcal{O}(1.5^n \cdot n) \subsetneq \mathcal{O}(2^n)$$

- negligible polynomial factors
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- typical result

Approach	Runtime in $\mathcal{O}$-Notation	$\mathcal{O}^*$-Notation
Brute-Force	$\mathcal{O}(2^n)$	$\mathcal{O}^*(2^n)$
Algorithm A	$\mathcal{O}(1.5^n \cdot n)$	$\mathcal{O}^*(1.5^n)$
Algorithm B	$\mathcal{O}(1.4^n \cdot n^2)$	$\mathcal{O}^*(1.4^n)$

Traveling Salesperson Problem (TSP)

Input. Distinct cities $\{v_1, v_2, \ldots, v_n\}$ with distances $d(c_i, c_j) \in Q_{\geq 0}$;
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i.e. a Hamiltonian cycle $(v_{\pi(1)}, \ldots, v_{\pi(n)}, v_{\pi(1)})$ of G of minimum weight

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Brute-force.

- Try all permutations and pick the one with smallest weight.
- Runtime: $\Theta(n! \cdot n) = n \cdot 2^{\Theta(n \log n)}$

TSP – Dynamic programming

Bellman-Held-Karp algorithm

Idea.

- Reuse optimal substructures with dynamic programming.

Richard M. Karp

Richard E. Bellman

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  $\mathrm{OPT}[S, v] =$ length of a shortest s-v-path
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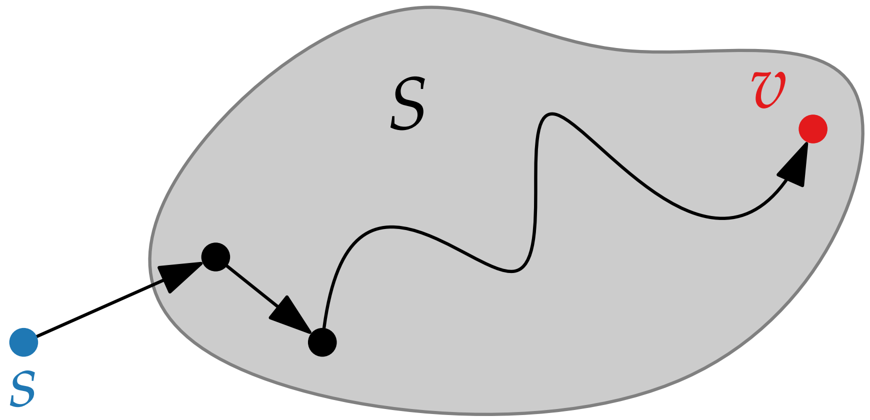

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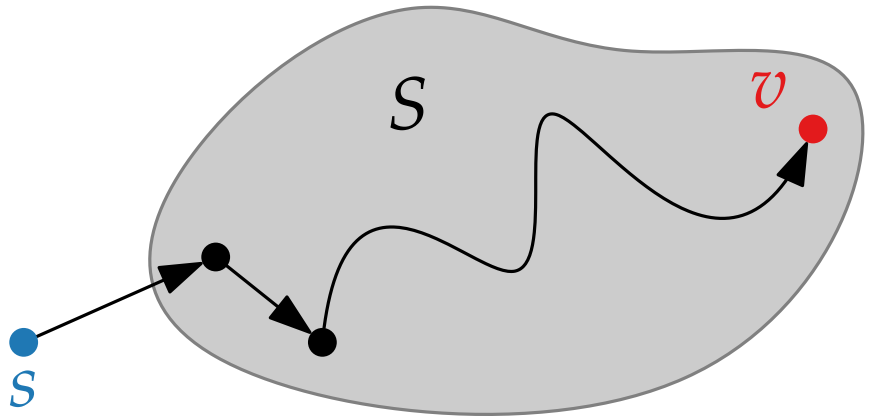

- Use $\mathrm{OPT}[S - v, u]$ to compute $\mathrm{OPT}[S, v]$.

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Details.

- The base case $S = \{v\}$ is easy: $\mathrm{OPT}[\{v\}, v] = d(s, v)$.

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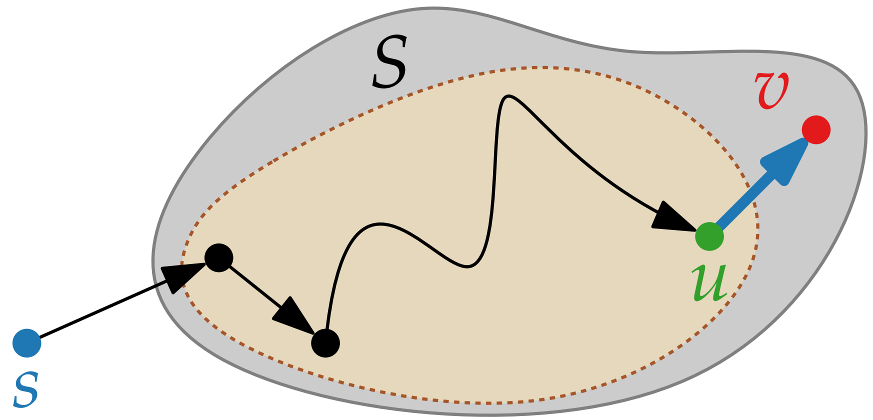
- When $|S| \geq 2$, compute $\text{OPT}[S, v]$ recursively:

$$\text{OPT}[S, v] = \min\{\text{OPT}[S - v, u] + d(u, v) \mid u \in S - v\}$$



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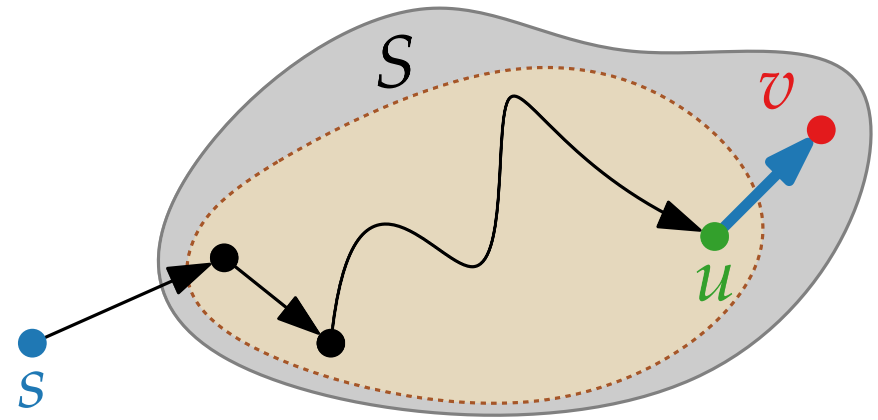

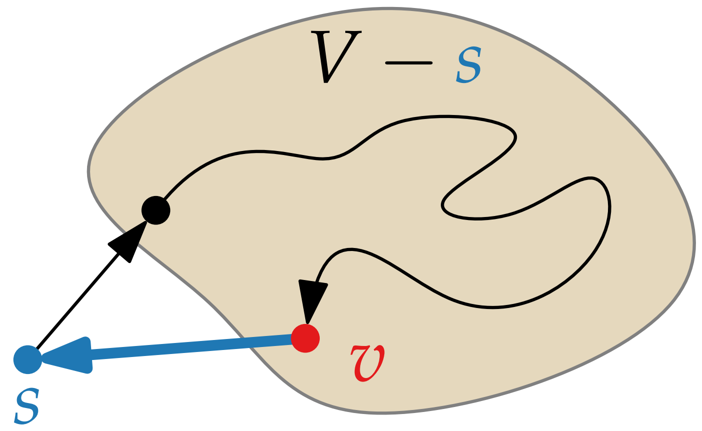

- After computing $\mathrm{OPT}[S, v]$ for each $S \subseteq V - s$ and each $v \in V - s$, the optimal solution is easily obtained as follows:

$$\mathrm{OPT} = \min\{\mathrm{OPT}[V - s, v]\} + d(v, s) \mid v \in V - s\}$$

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Pseudocode.

Algorithm Bellmann-Held-Karp(G, c)

foreach $v \in V - s$ **do**
- $\mathrm{OPT}[\{v\}, v] = c(s, v)$

for $j \leftarrow 2$ **to** $n - 1$ **do**
- **foreach** $S \subseteq V - s$ with $|S| = j$ **do**
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- Space usage in $\Theta(2^n \cdot n)$
- or actually better? What table values do we need to store?

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- DP algorithm that runs in $\mathcal{O}^*(2^n)$ time and $\mathcal{O}(2^n \cdot \overset{?}{n})$ space
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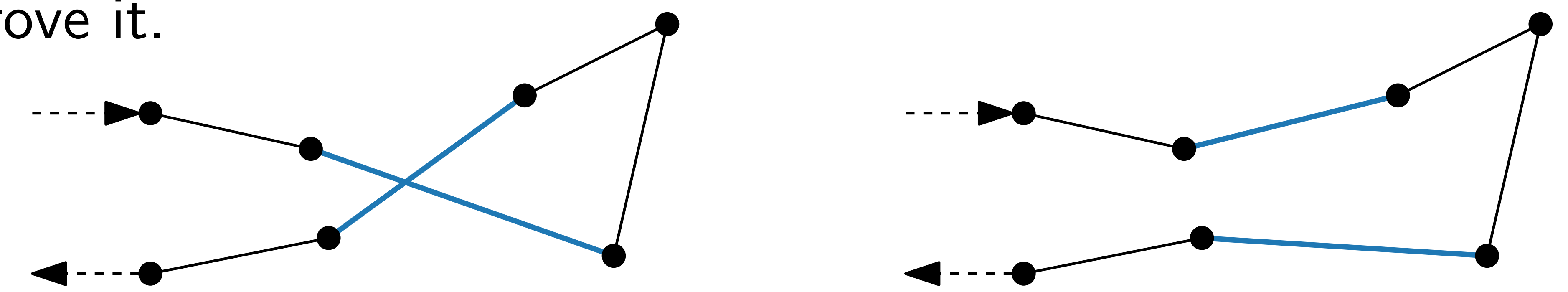
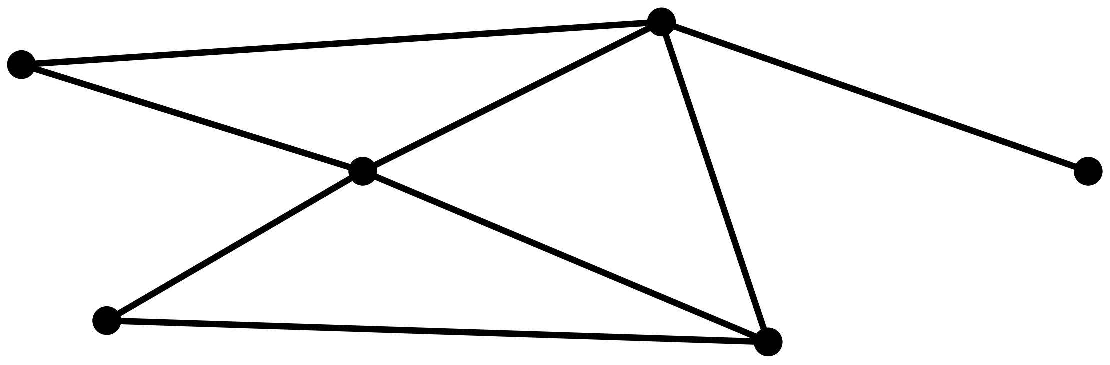
- Metric TSP can easily be 2-approximated. (Do you remember how?)
- Eucledian TSP considered in course Approximation Algorithms.
- In practice, one successful approach is to start with a greedily computed Hamiltonian cycle and then use 2-OPT and 3-OPT swaps to improve it.



Maximum Independent Set (MIS)

Input. Graph $G = (V, E)$ with n vertices.



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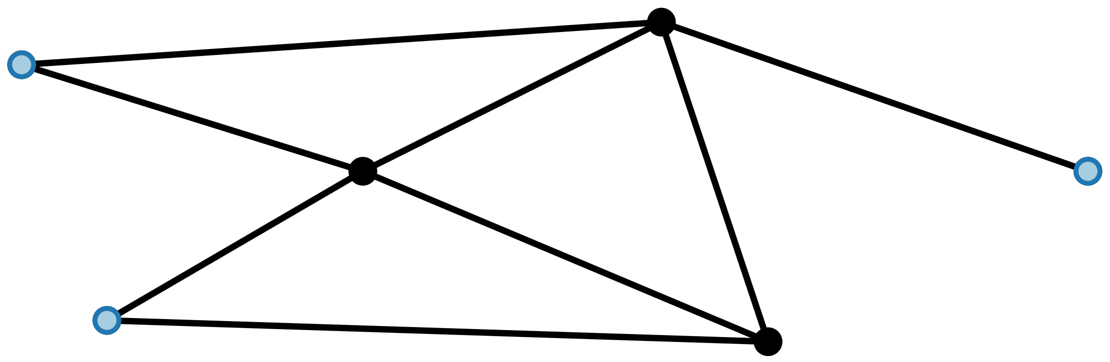

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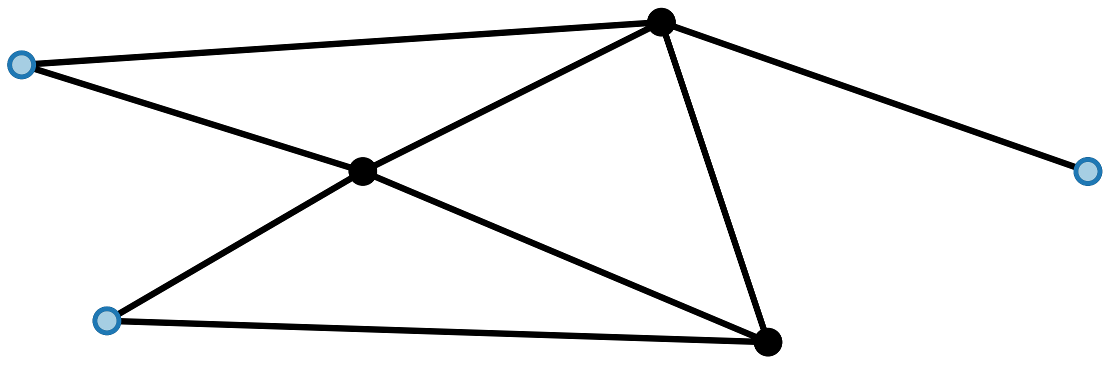

Brute-force.

- Try all subets of V.
- Runtime: $\mathcal{O}(2^n \cdot n)$

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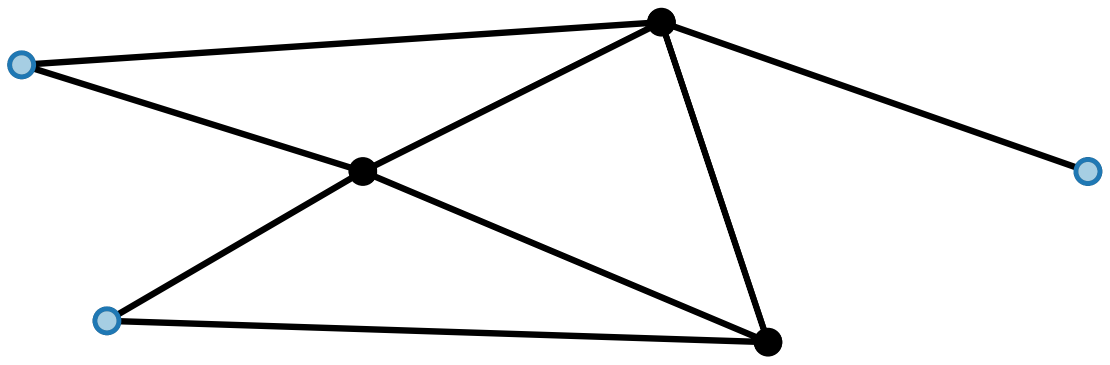


Brute-force.

- Try all subets of V.
- Runtime: $\mathcal{O}(2^n \cdot n)$

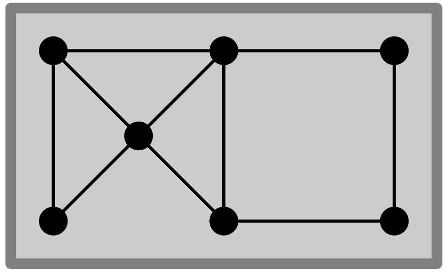
Naive MIS branching.

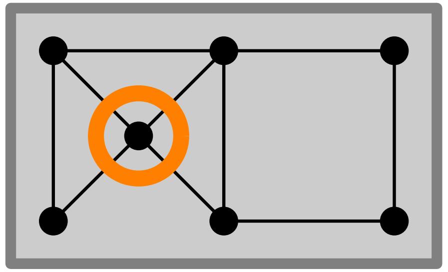
- Take a vertex v or don't take it.

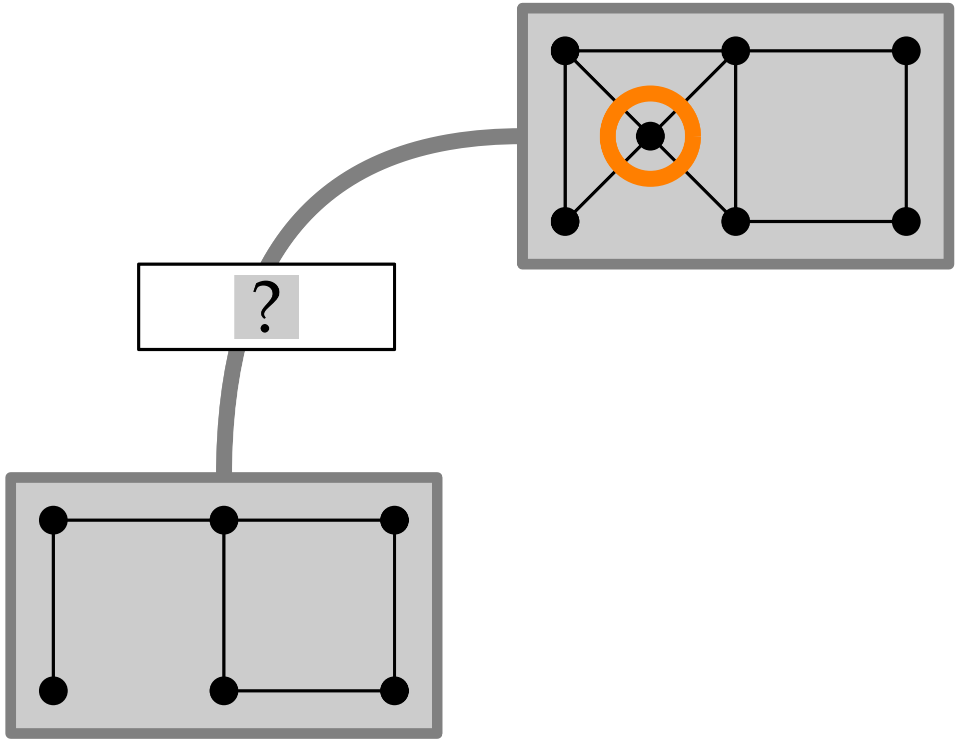
Algorithm NaiveMIS(G)

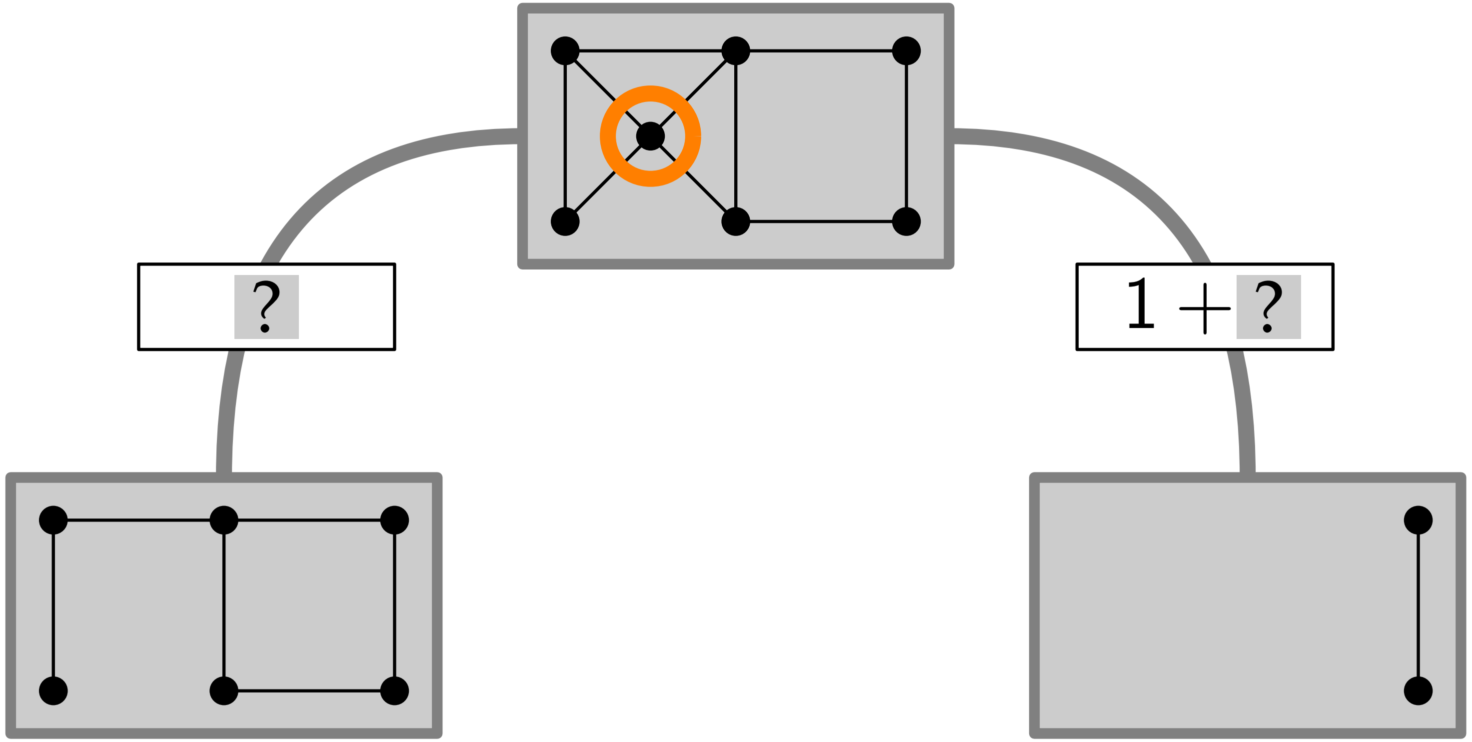
if $V = \emptyset$ **then**
  return 0

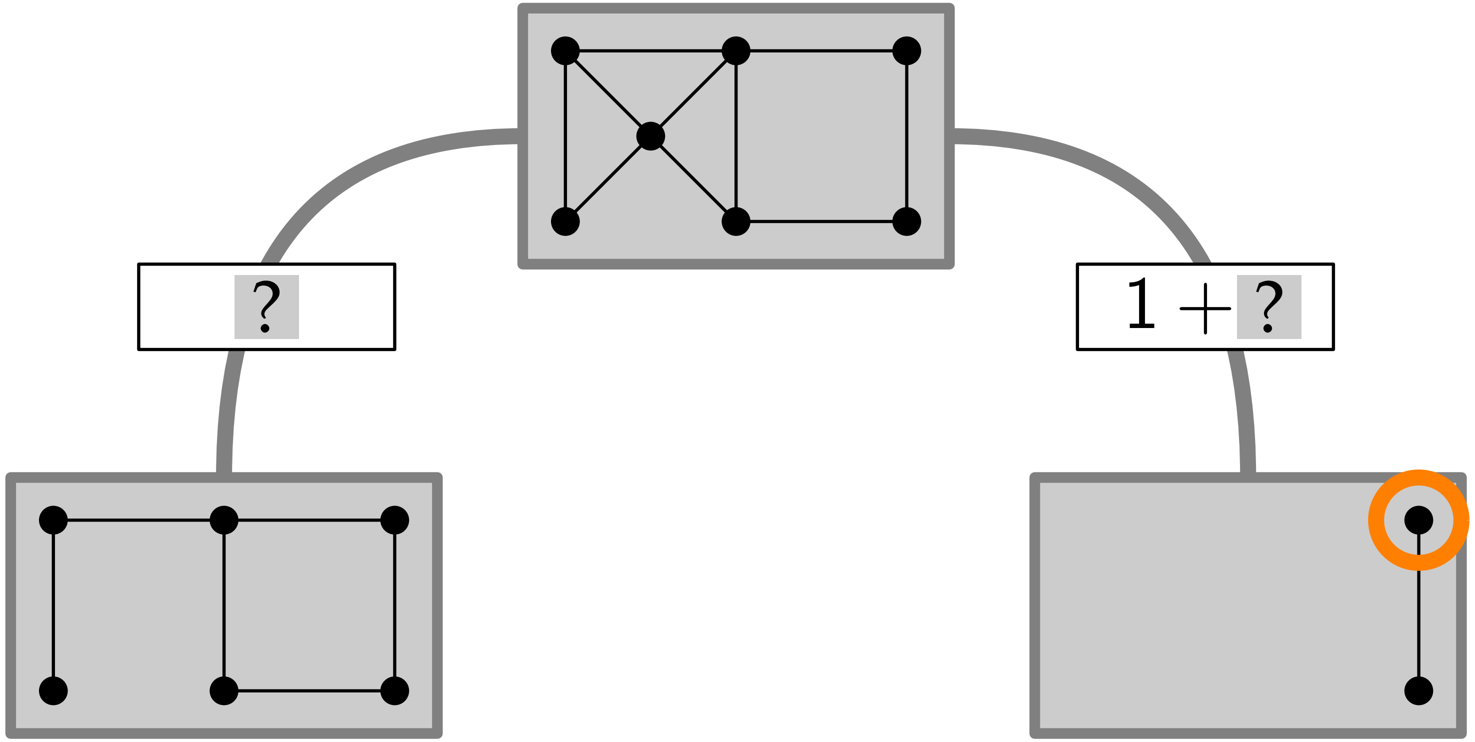
$v \leftarrow$ arbitrary vertex in $V(G)$
return max$\{1+$ NaiveMIS$(G - N(v) - \{v\})$,
 NaiveMIS$(G - \{v\})\}$

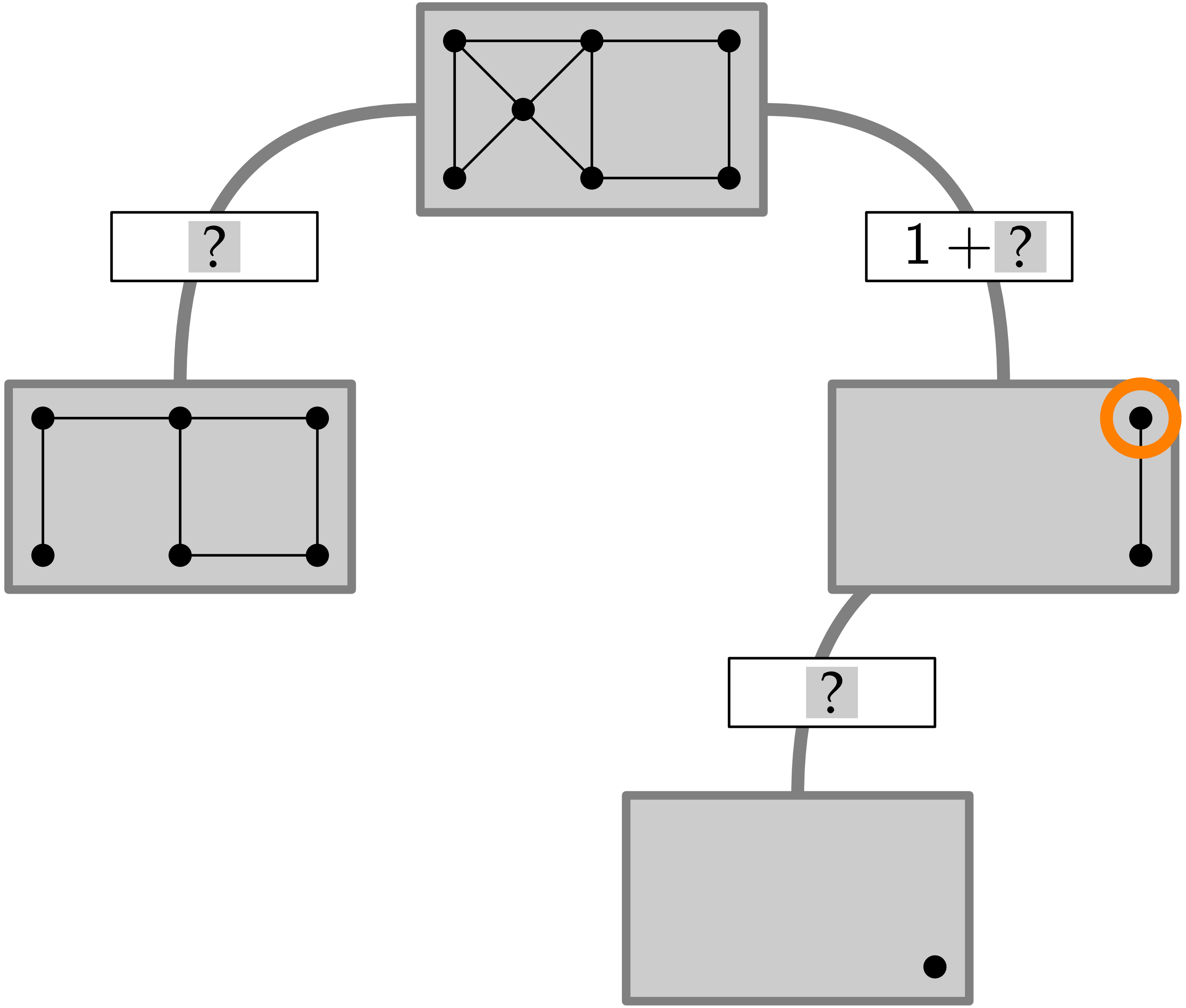


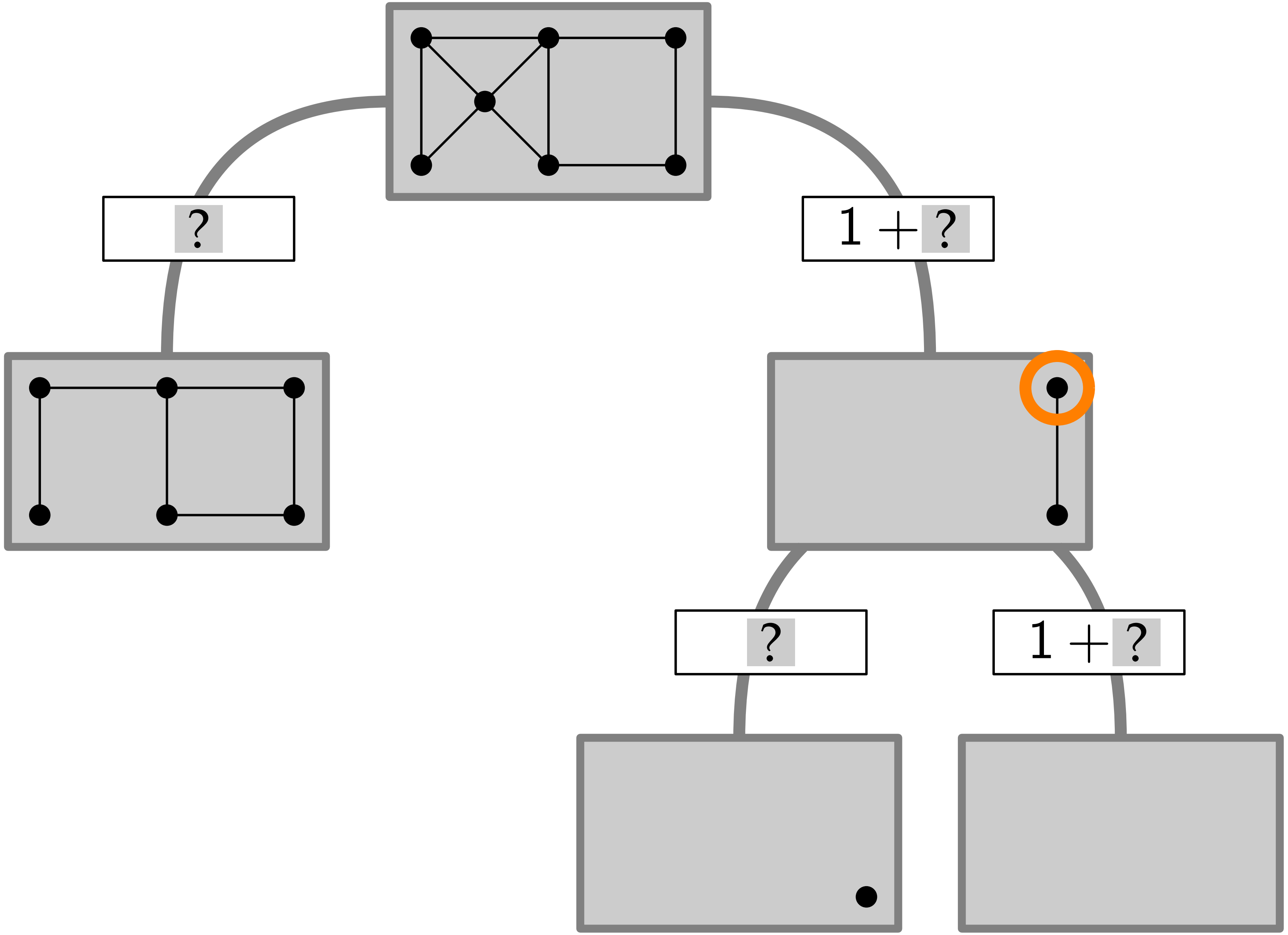


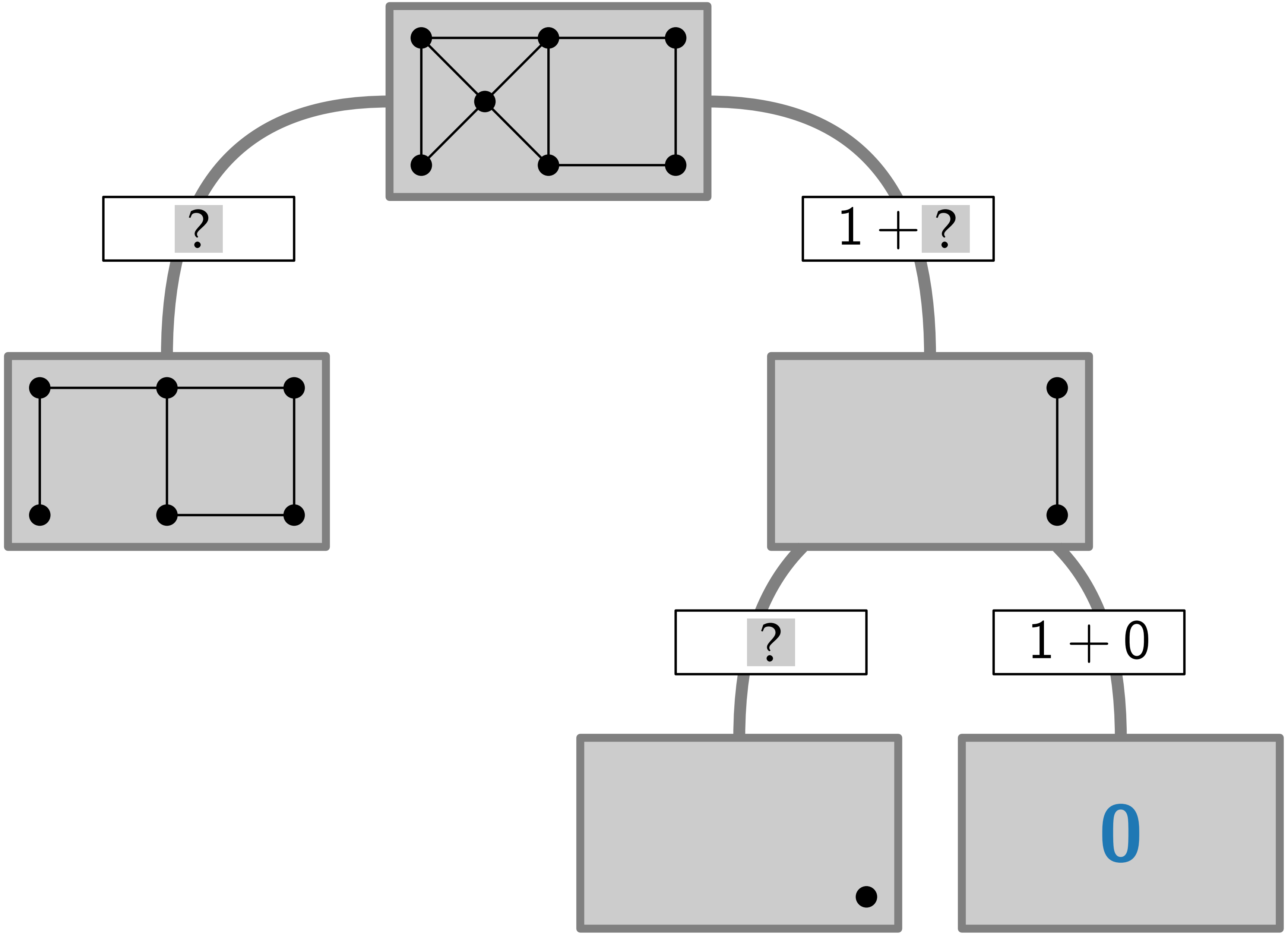


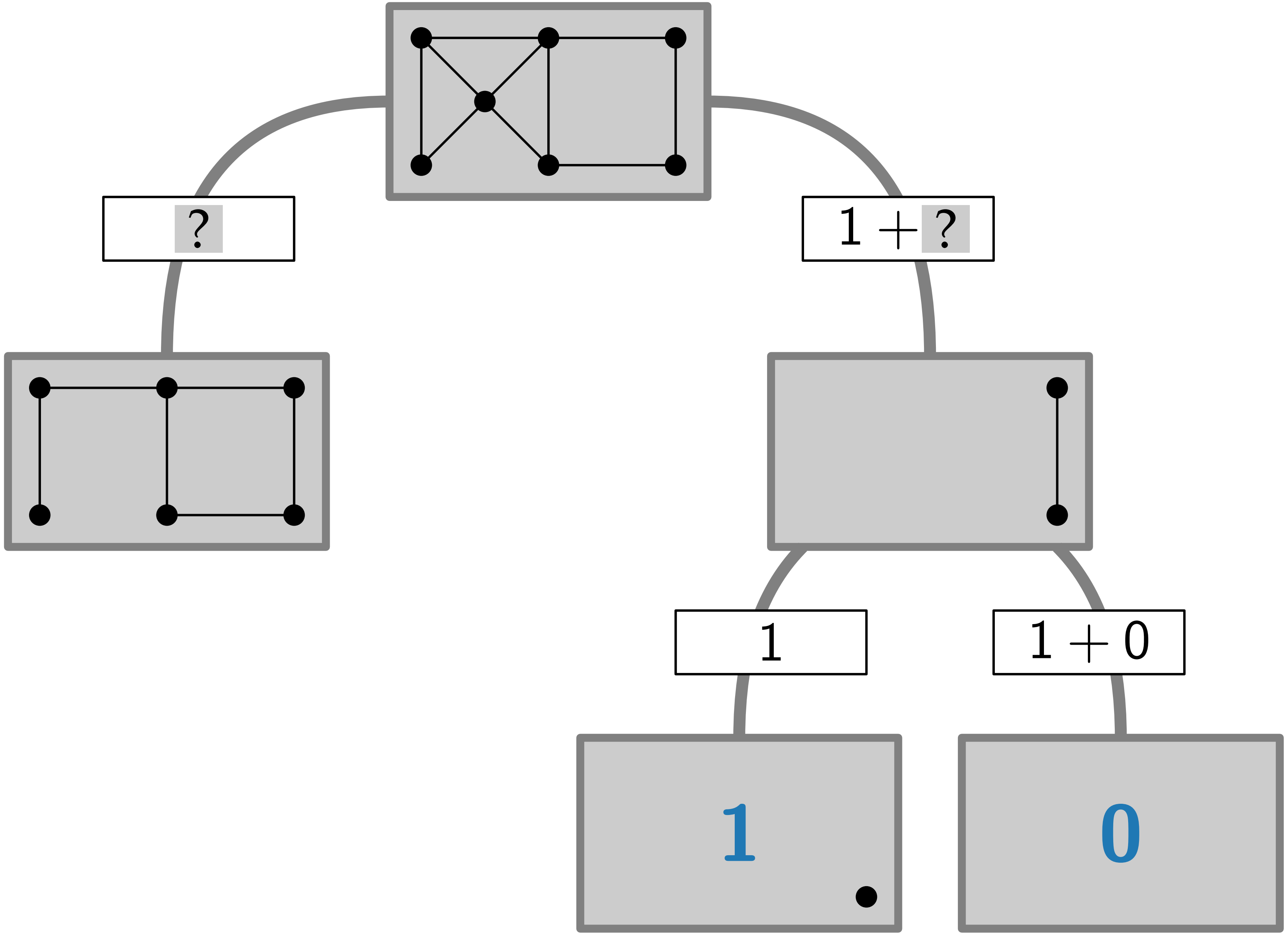

?
1 + ?

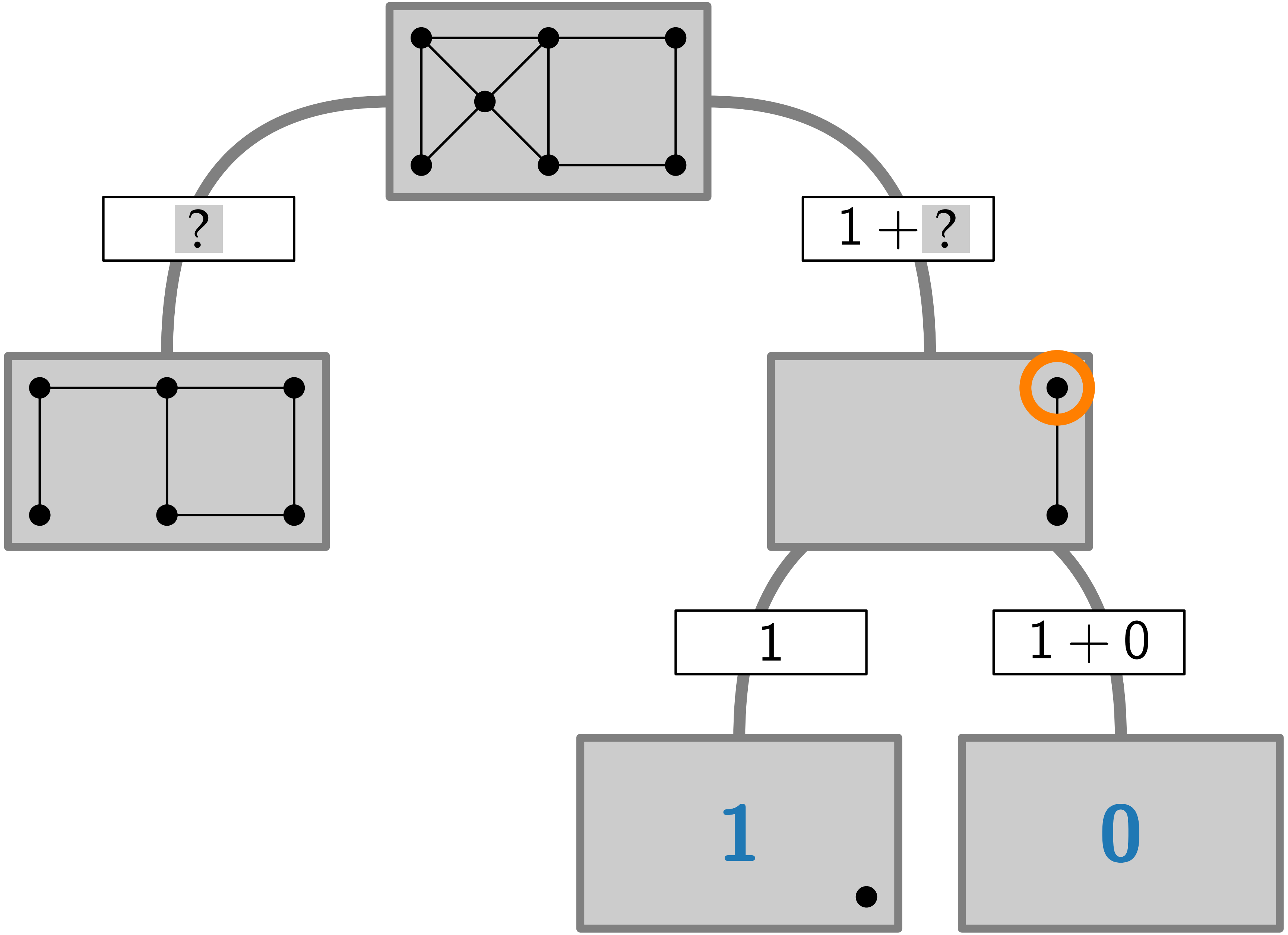

?
1 + ?

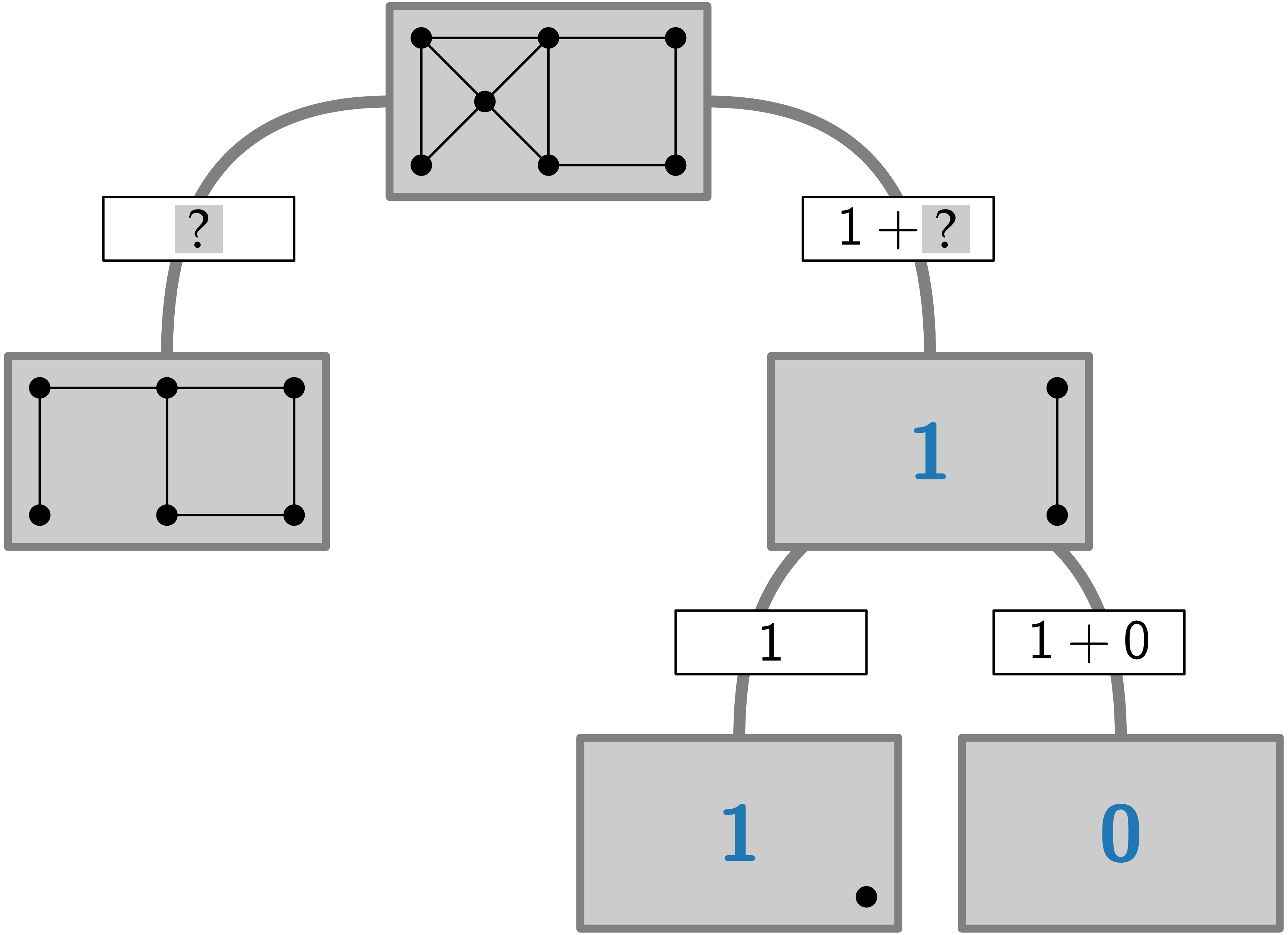

?
1 + ?
?


?
1 + ?
?
1 + ?

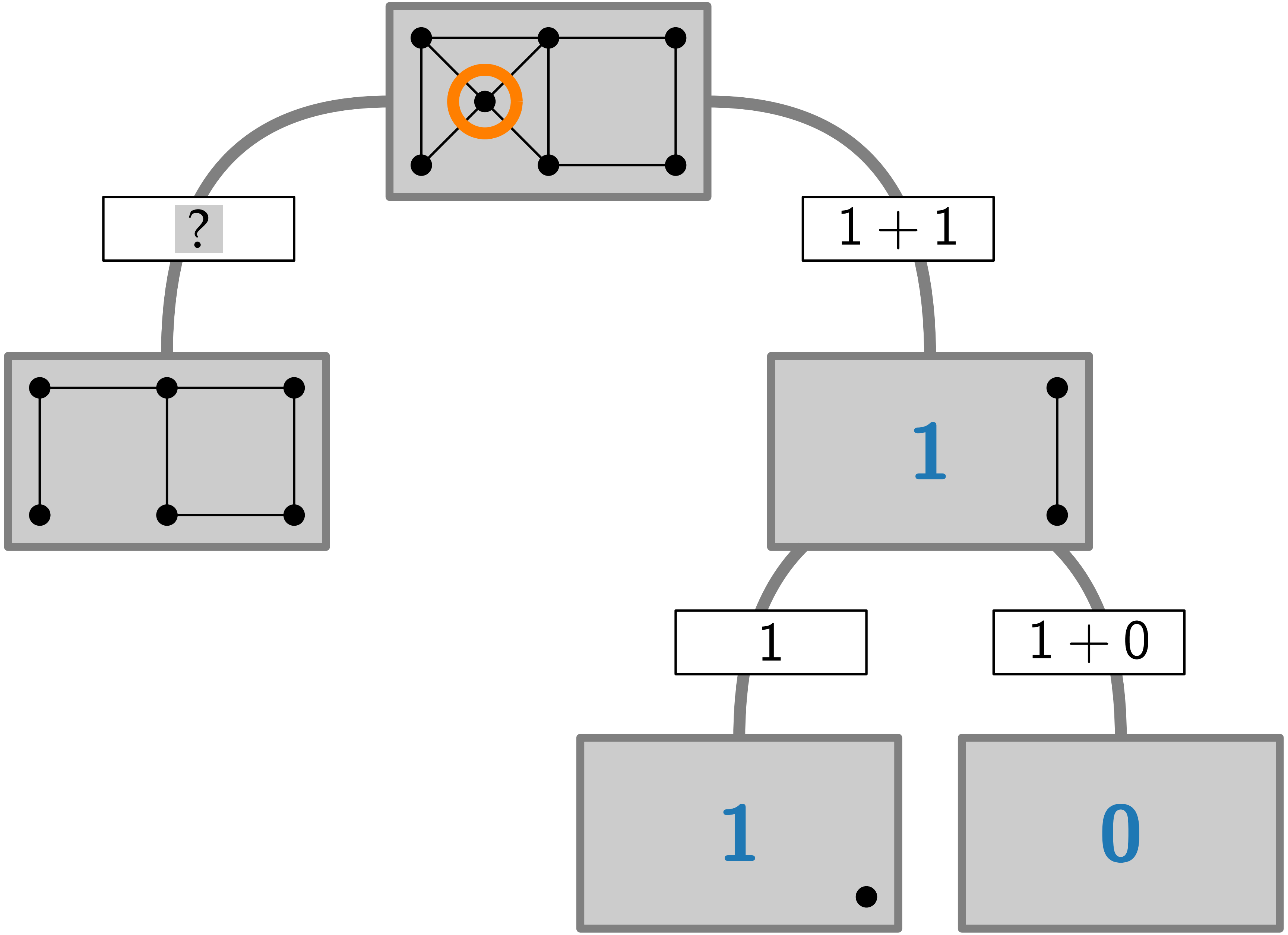

?
1 + ?
?
1 + 0
0

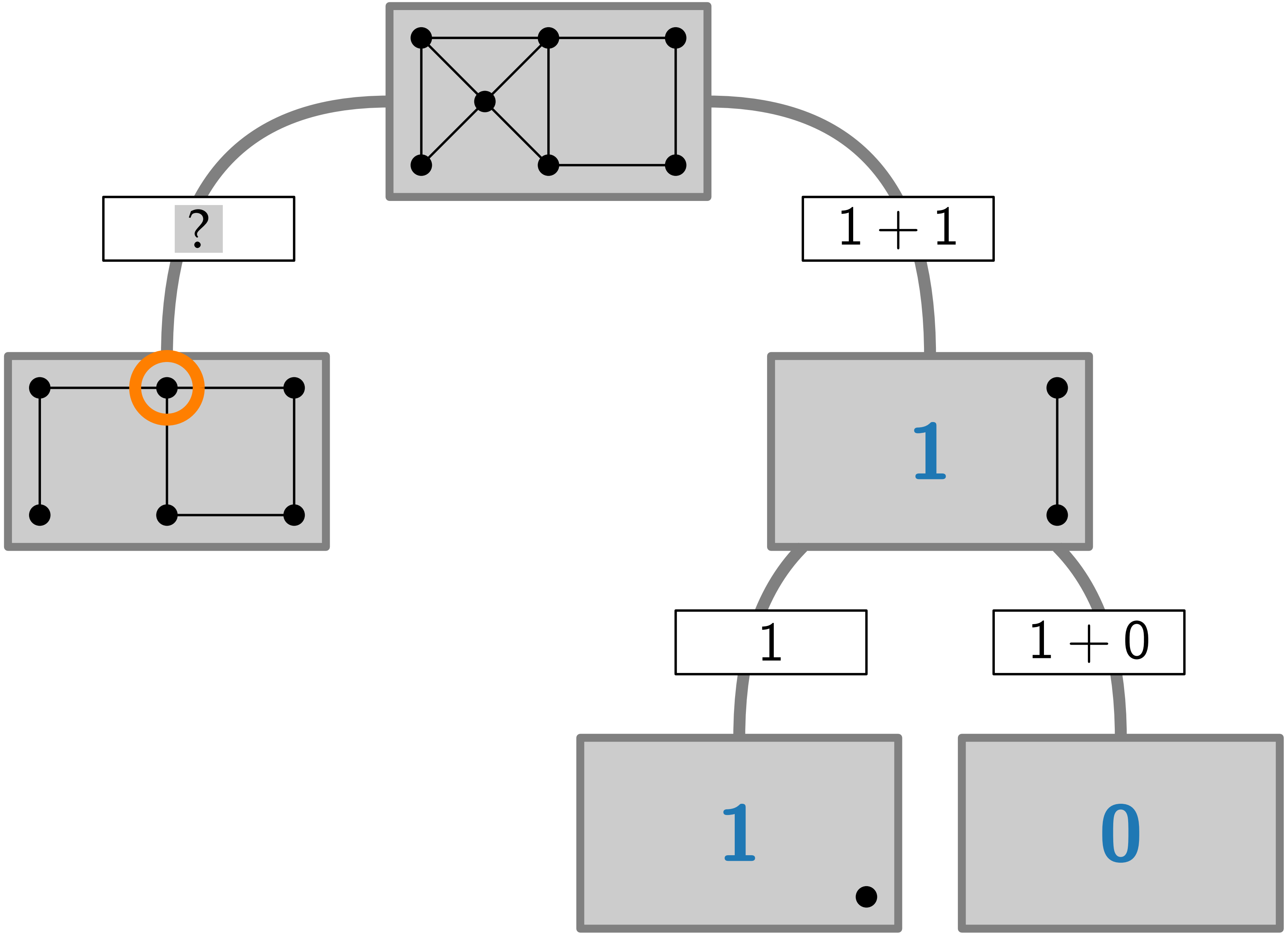

?
1 + ?
1
1 + 0
1
0

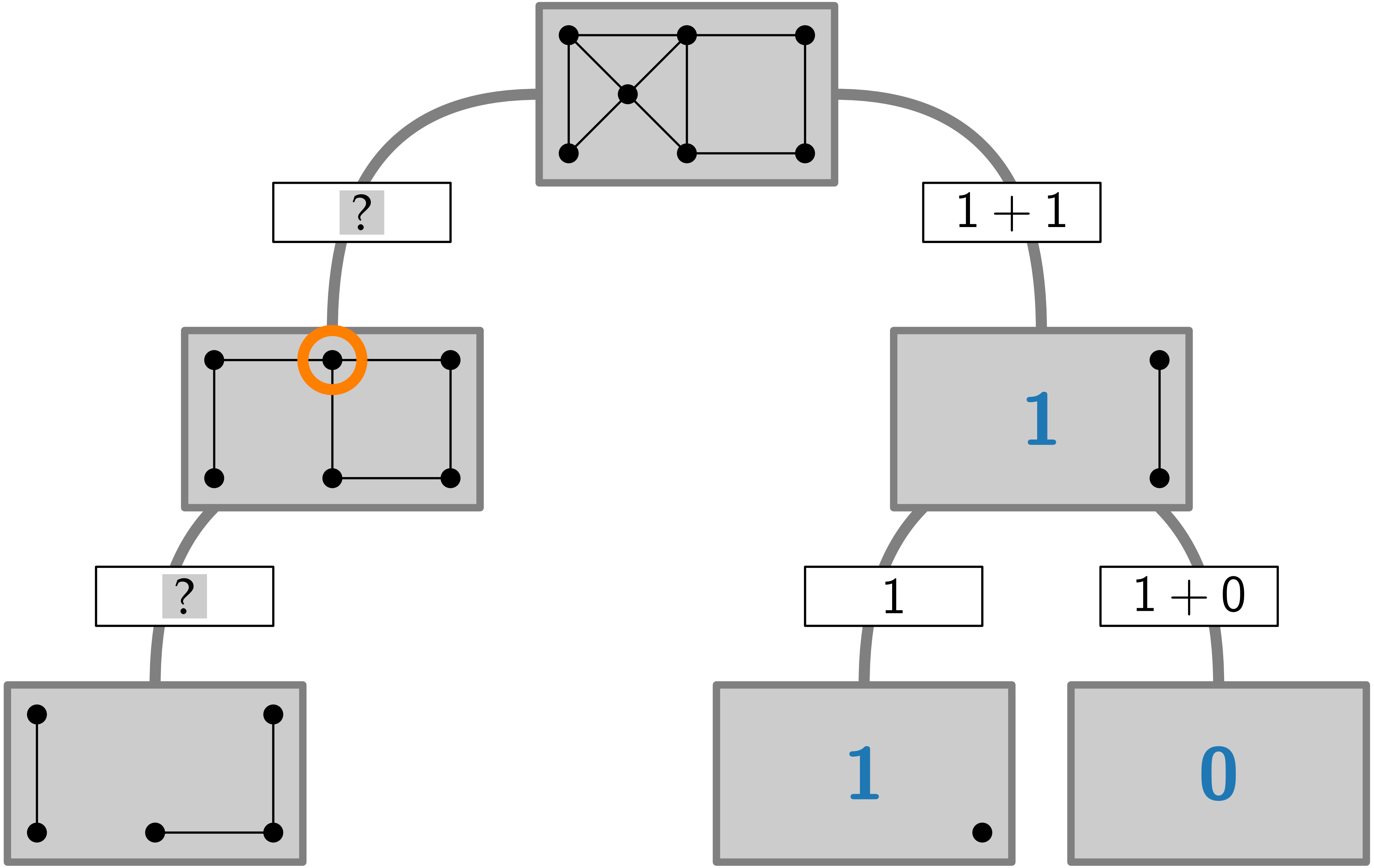

?
1 + ?
1
1 + 0
1
0

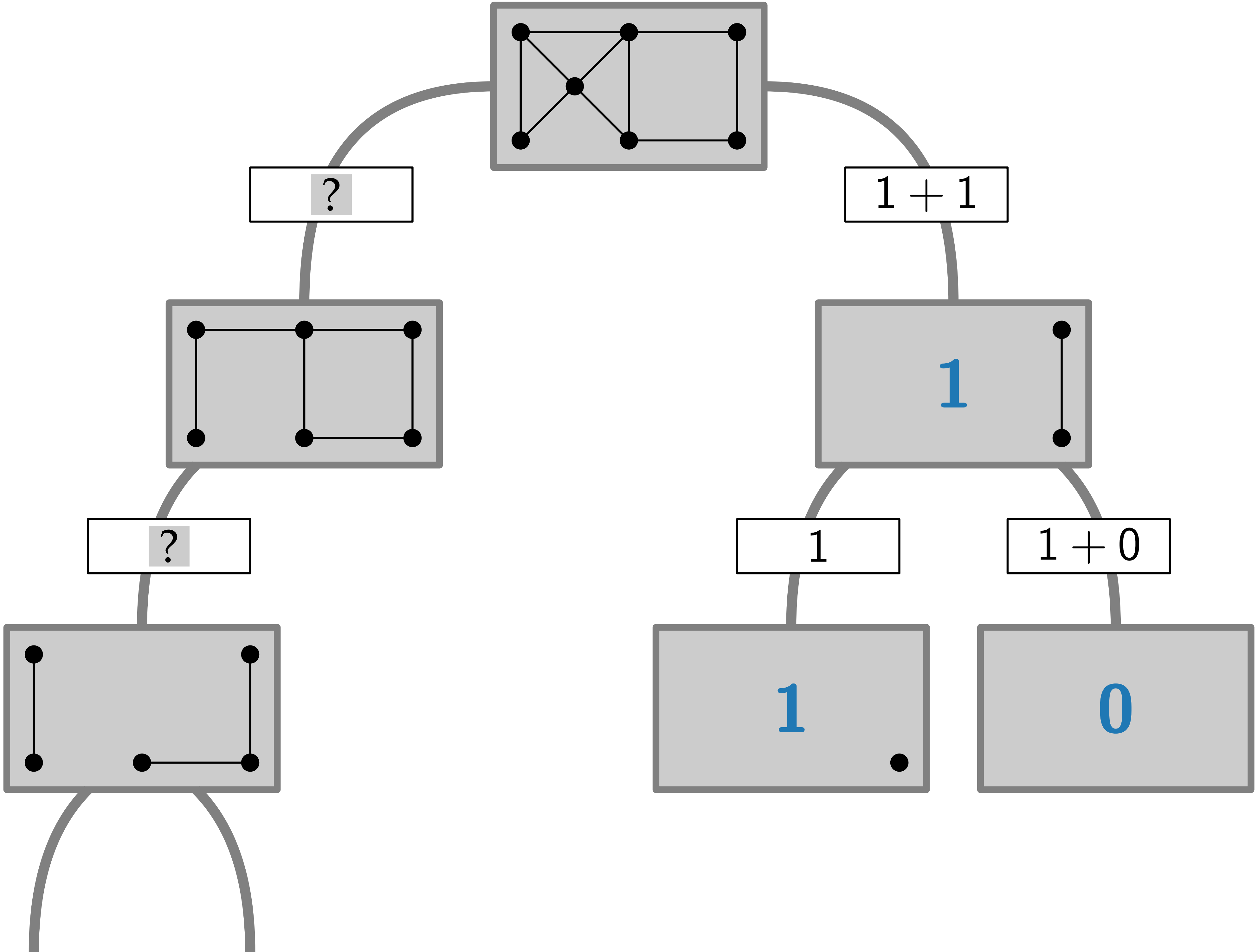


?
1 + ?
1
1
1 + 0
1
0

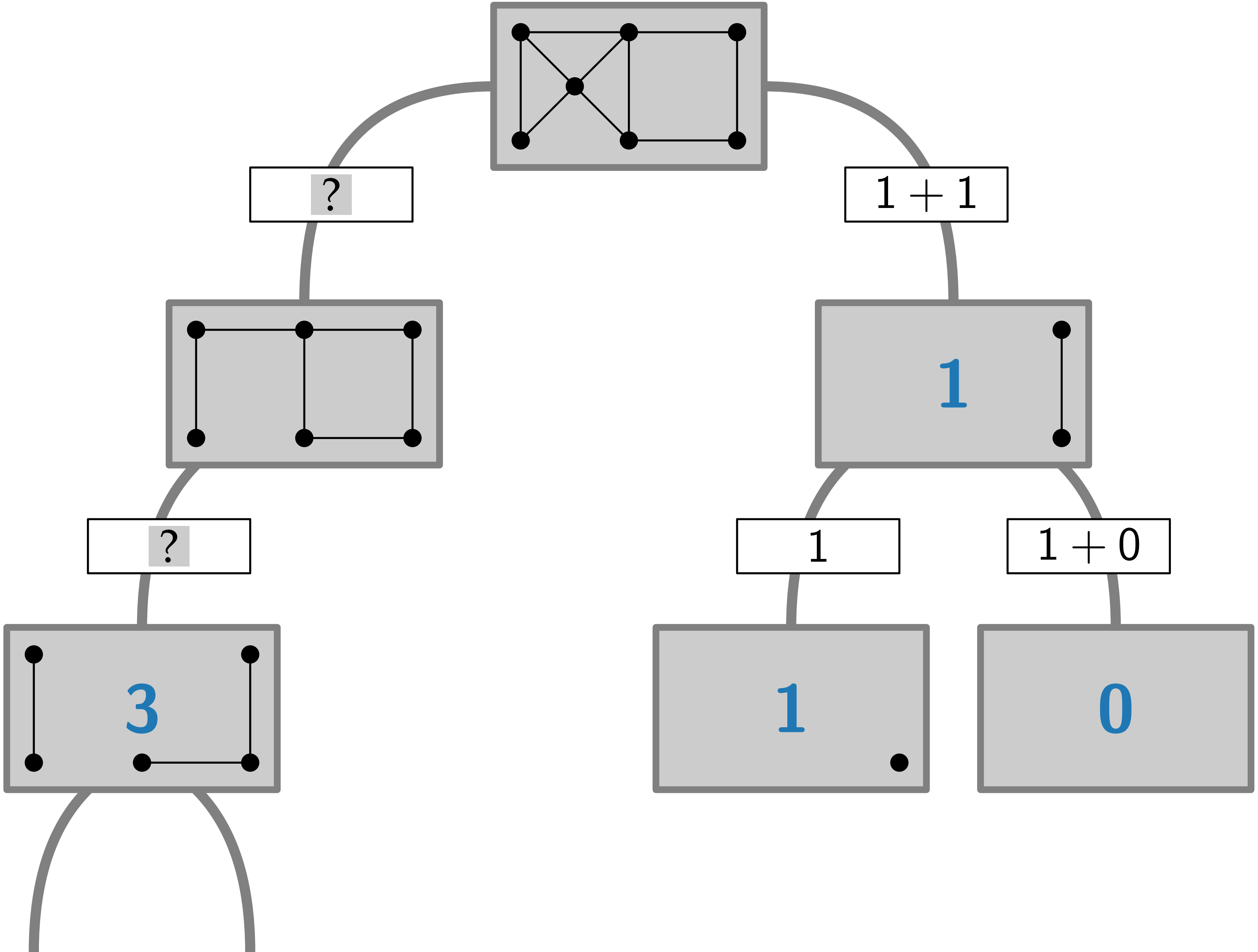

?
1 + 1
1
1
1 + 0
1
0

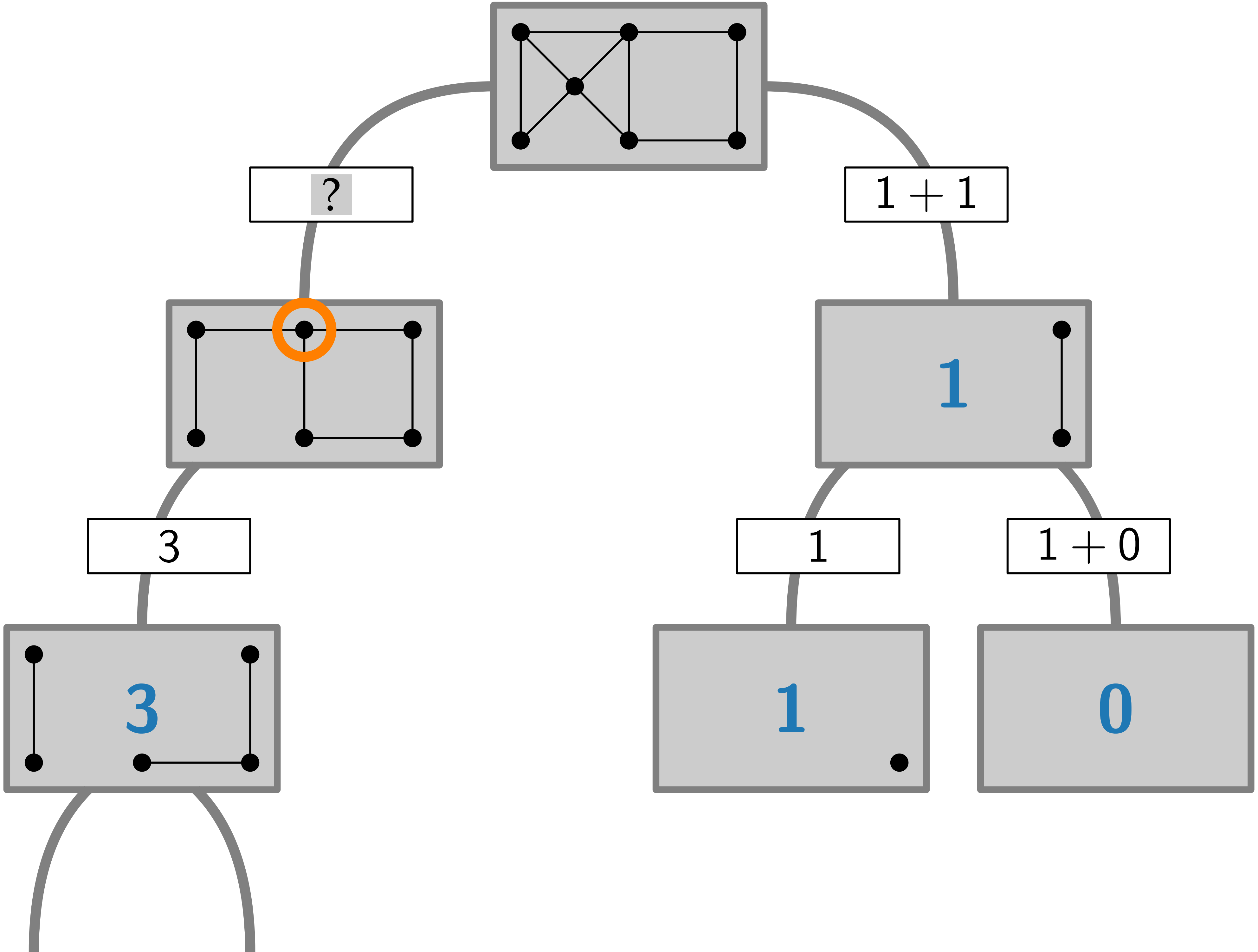

?
1 + 1
1
1
1 + 0
1
0


?
1 + 1
1
?
1
1 + 0
1
0

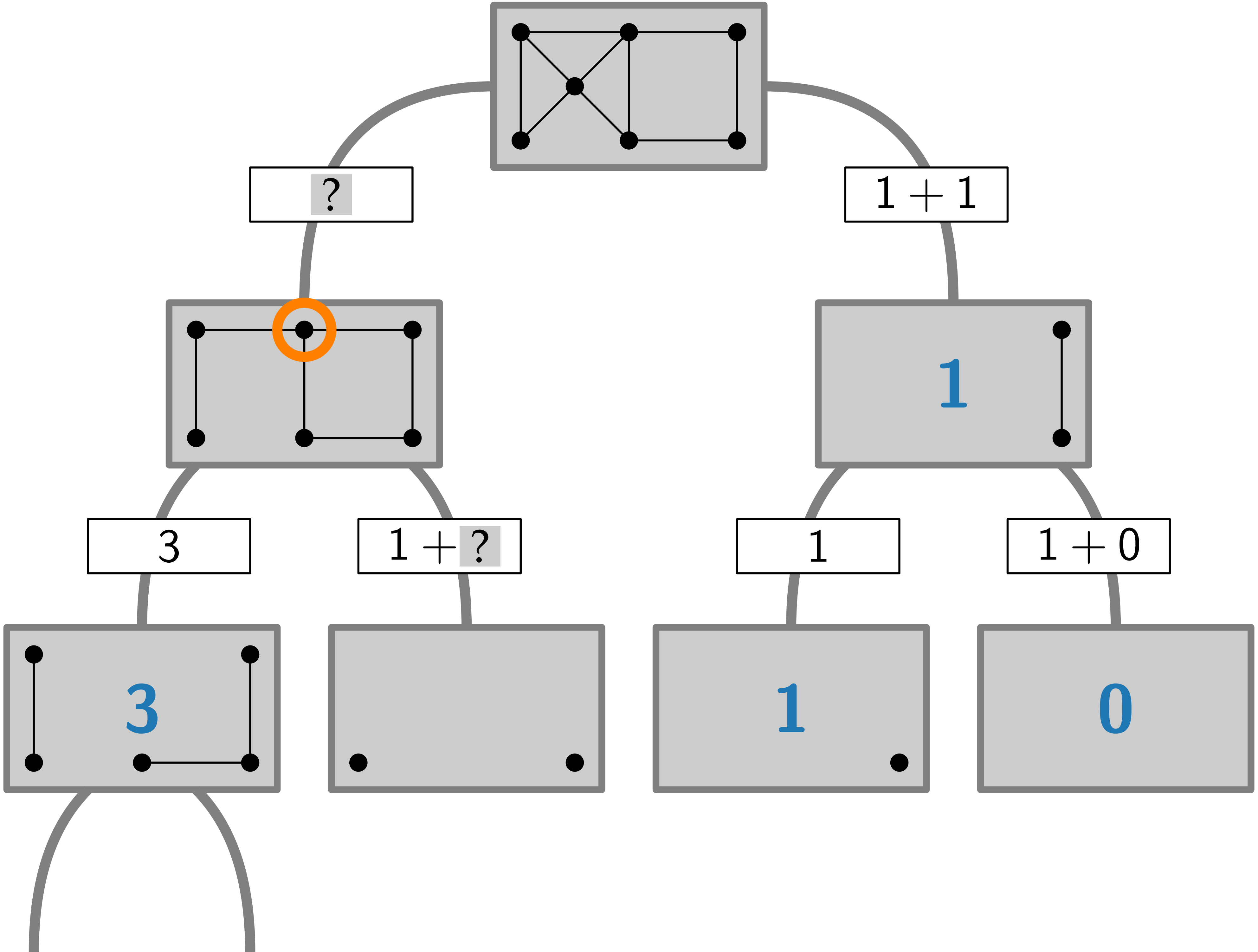


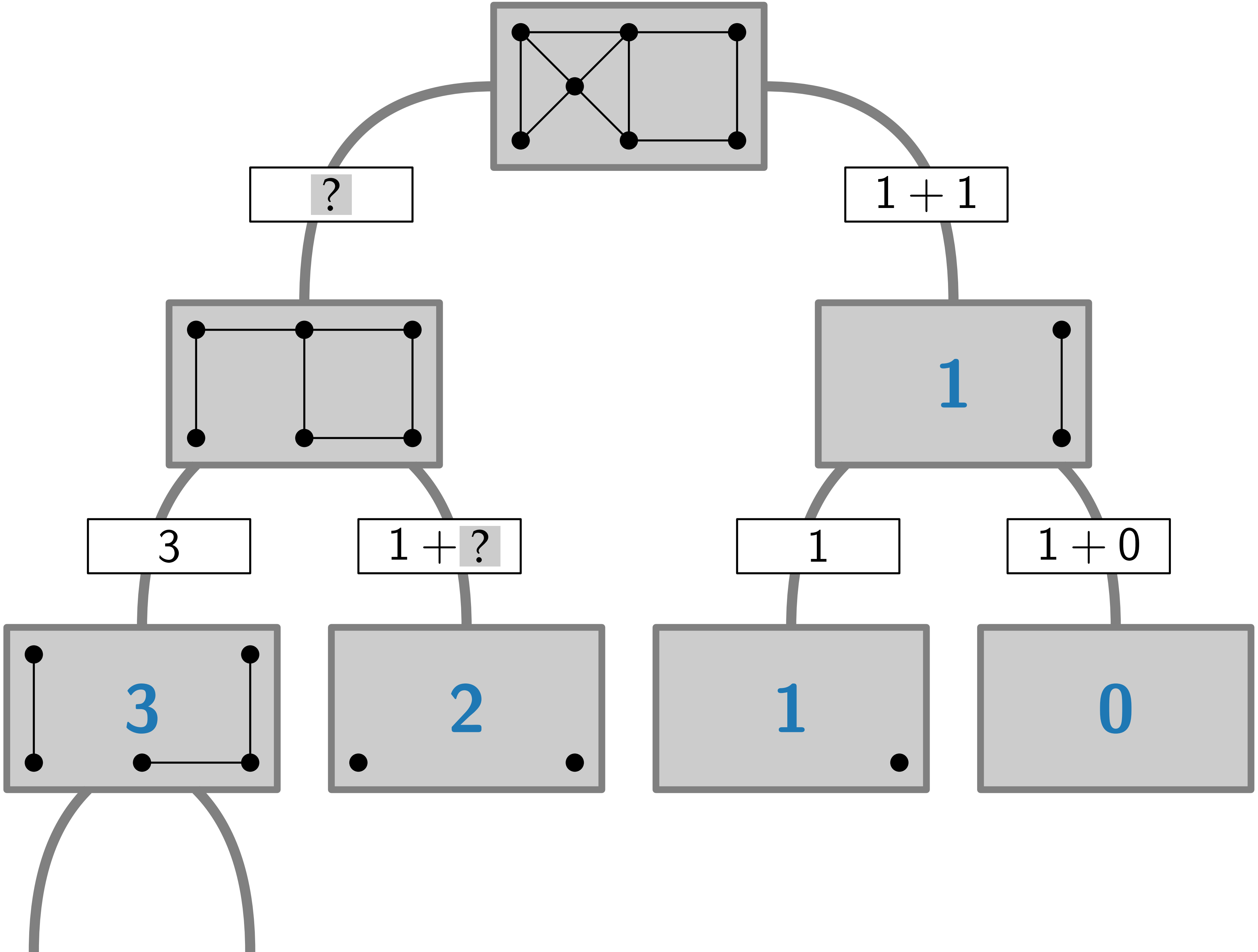
?
1 + 1
1
?
1
1 + 0
1
0

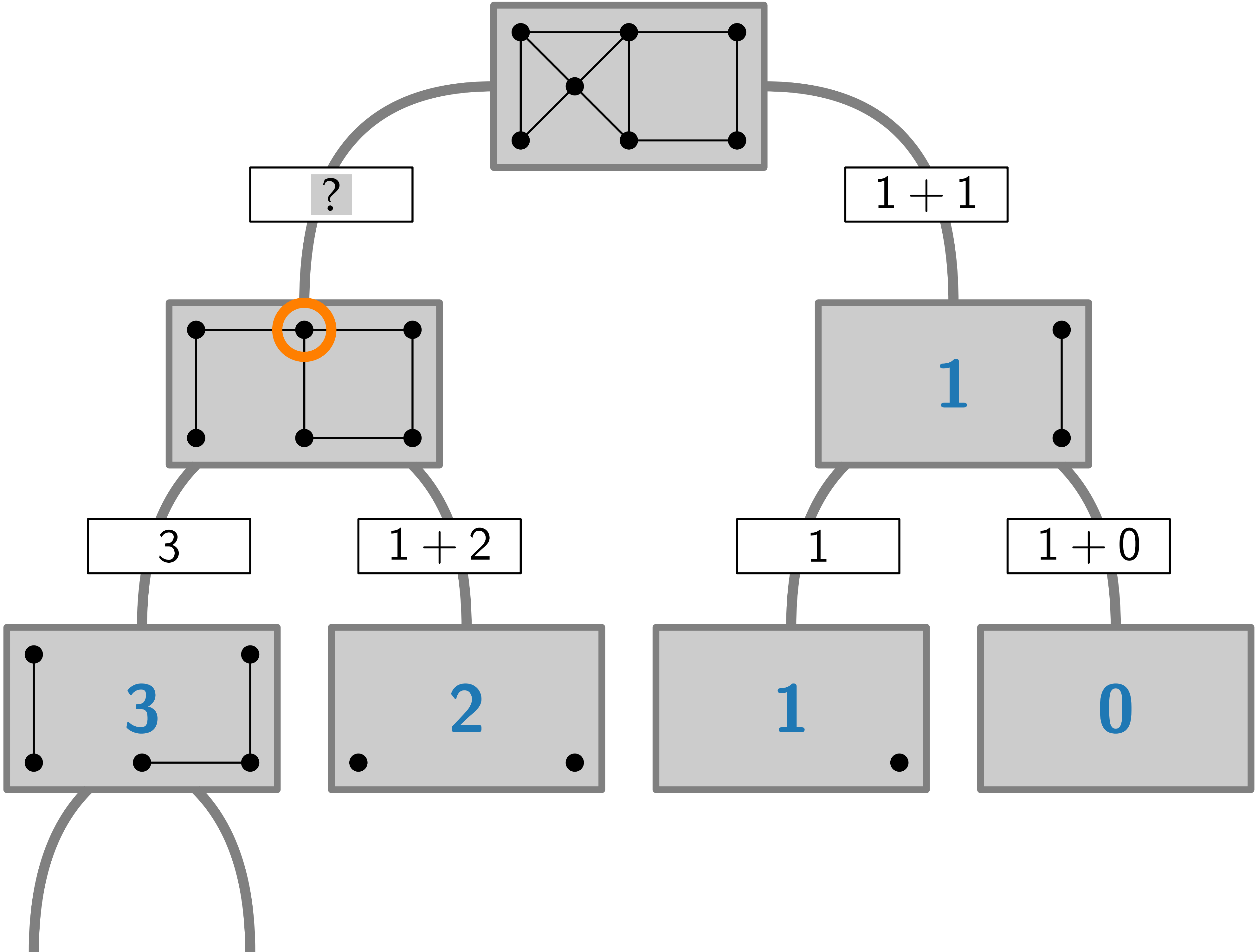

?
1 + 1
1
?
1
1 + 0
3
1
0

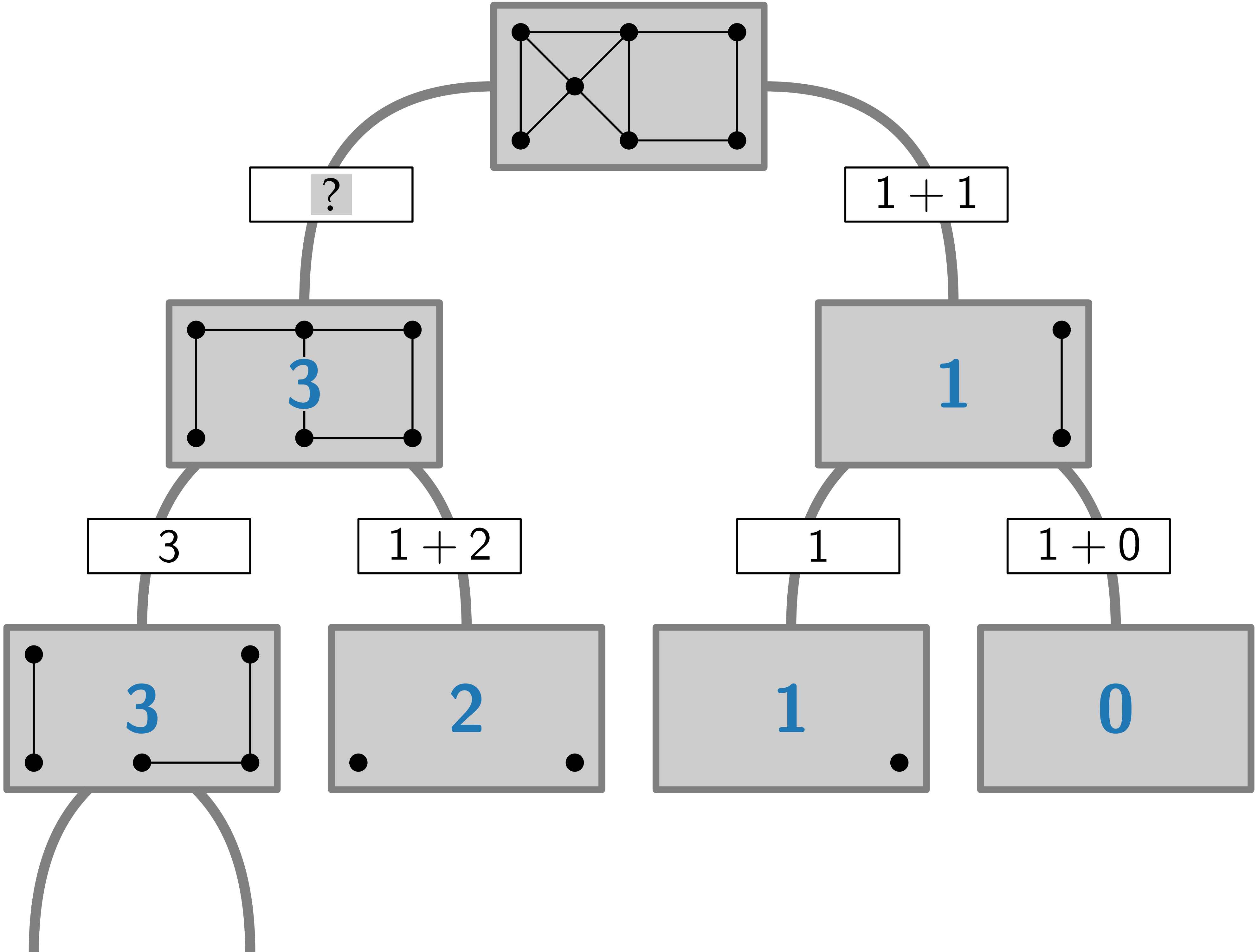


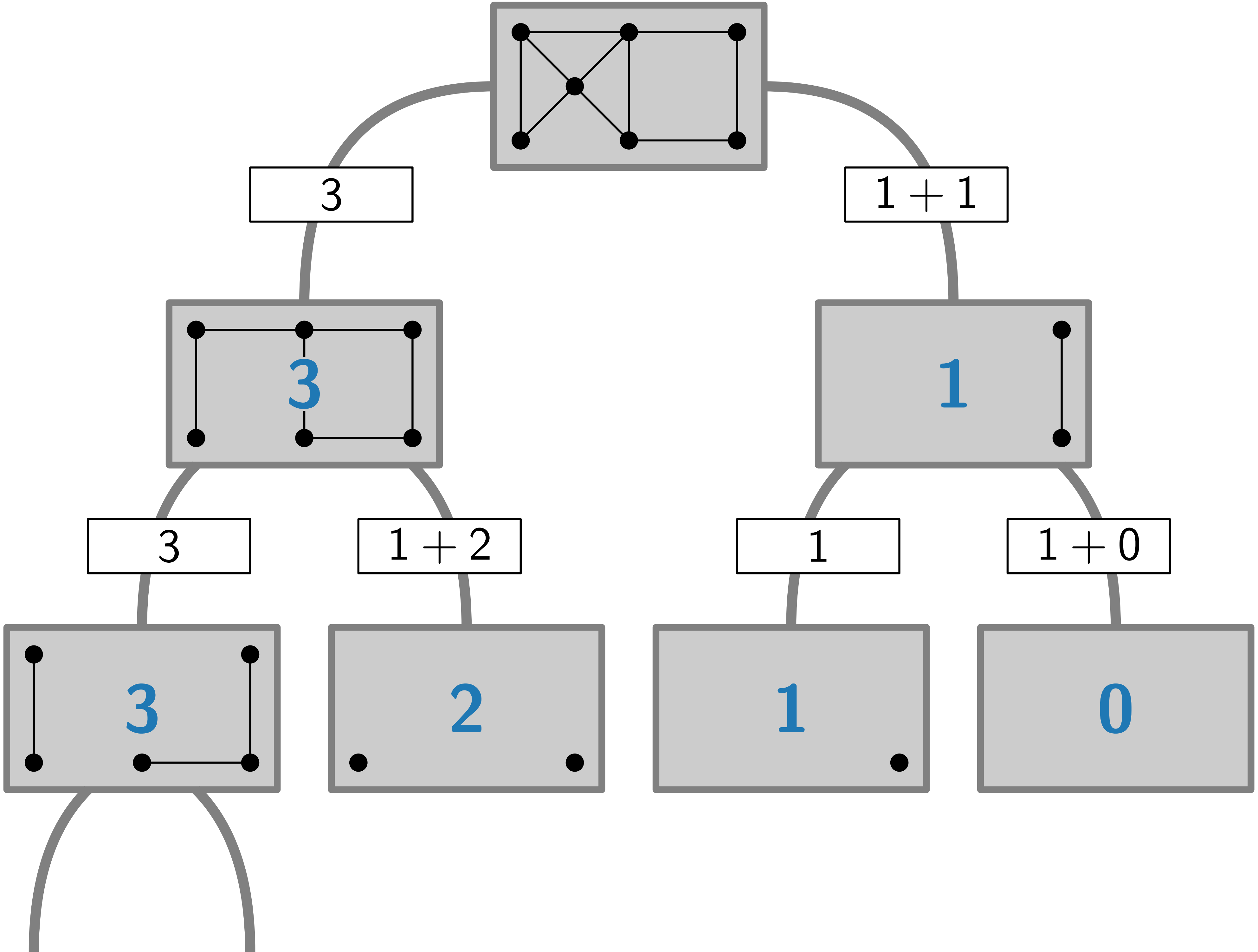
?
1 + 1
1
3
1
1 + 0
3
1
0

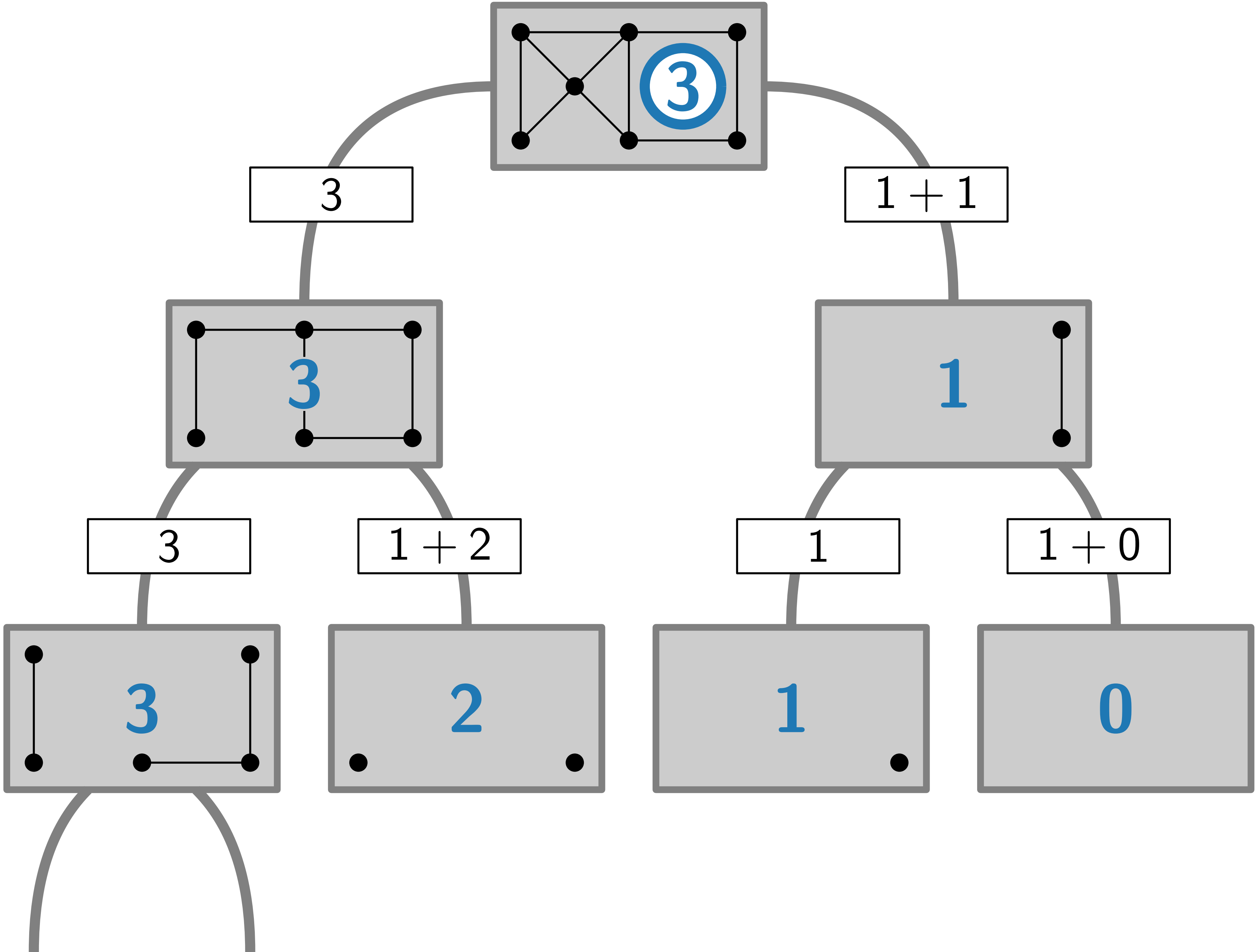

?
1 + 1
1
3
1 + ?
1
1 + 0
3
1
0


?
1 + 1
1
3
1 + ?
1
1 + 0
3
2
1
0


?
1 + 1
1
3
1 + 2
1
1 + 0
3
2
1
0


?
1 + 1
3
1
3
1 + 2
1
1 + 0
3
2
1
0


3
1 + 1
3
1
3
1 + 2
1
1 + 0
3
2
1
0

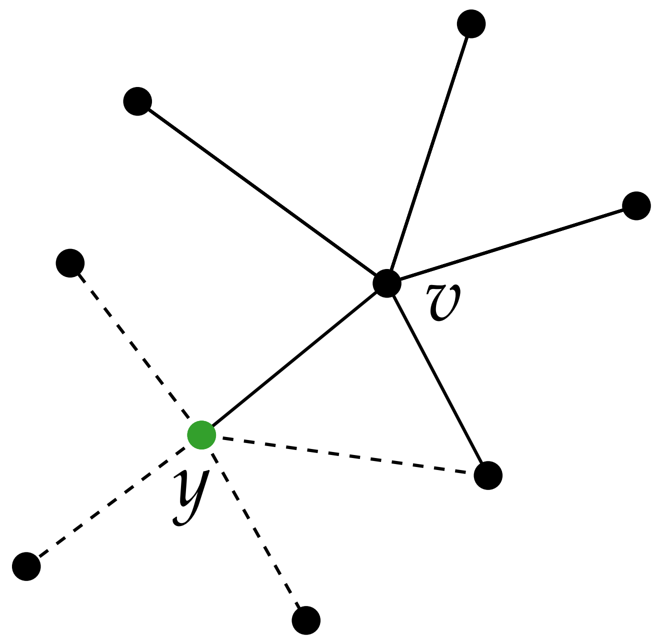

3
3
1 + 1
3
1
3
1 + 2
1
1 + 0
3
2
1
0

MIS – Smarter branching

Lemma.
Let U be a maximum independent set in G. Then for each $v \in V$:

1. $v \in U \Rightarrow N(v) \cap U = \emptyset$
2. $v \notin U \Rightarrow |N(v) \cap U| \geq 1$

Thus, $N[v] := N(v) \cup \{v\}$ contains some $y \in U$ and no other vertex of $N[y]$ is in U.



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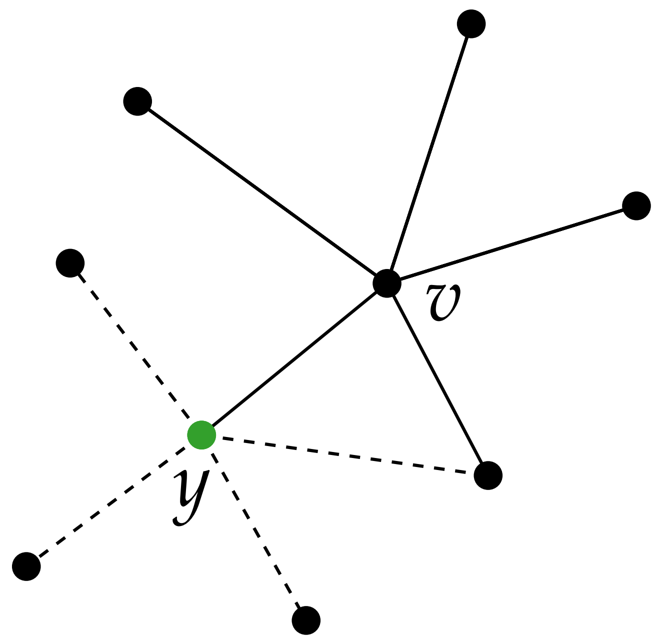

Smarter MIS branching.

- For some vertex v, branch on vertices in $N[v]$.

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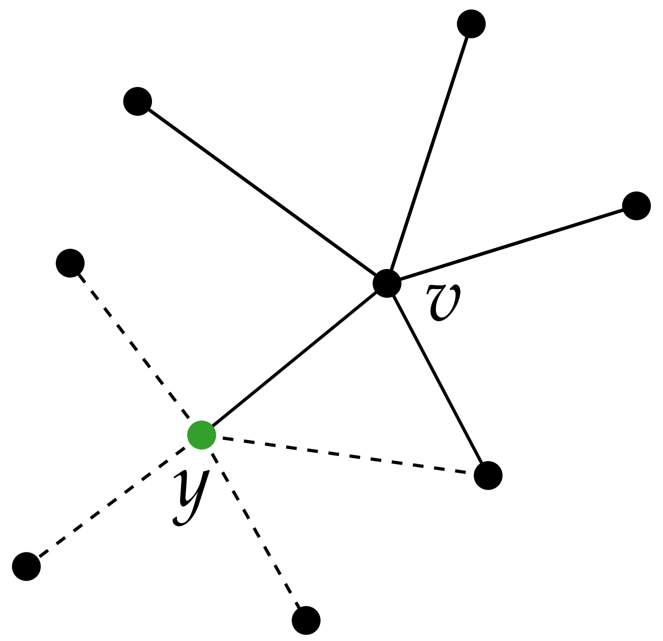

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Algorithm MIS(G)

  if $V = \emptyset$ **then**
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  return $1 + \max\{\text{MIS}(G - N[y]) \mid y \in N[v]\}$

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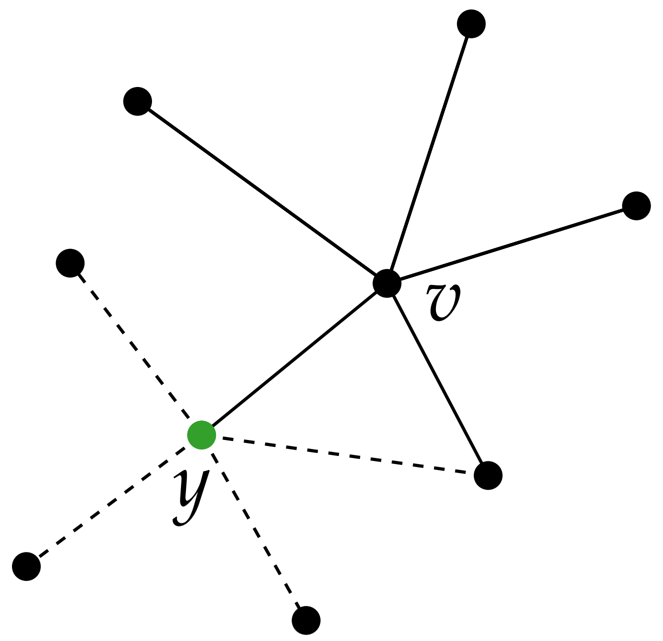

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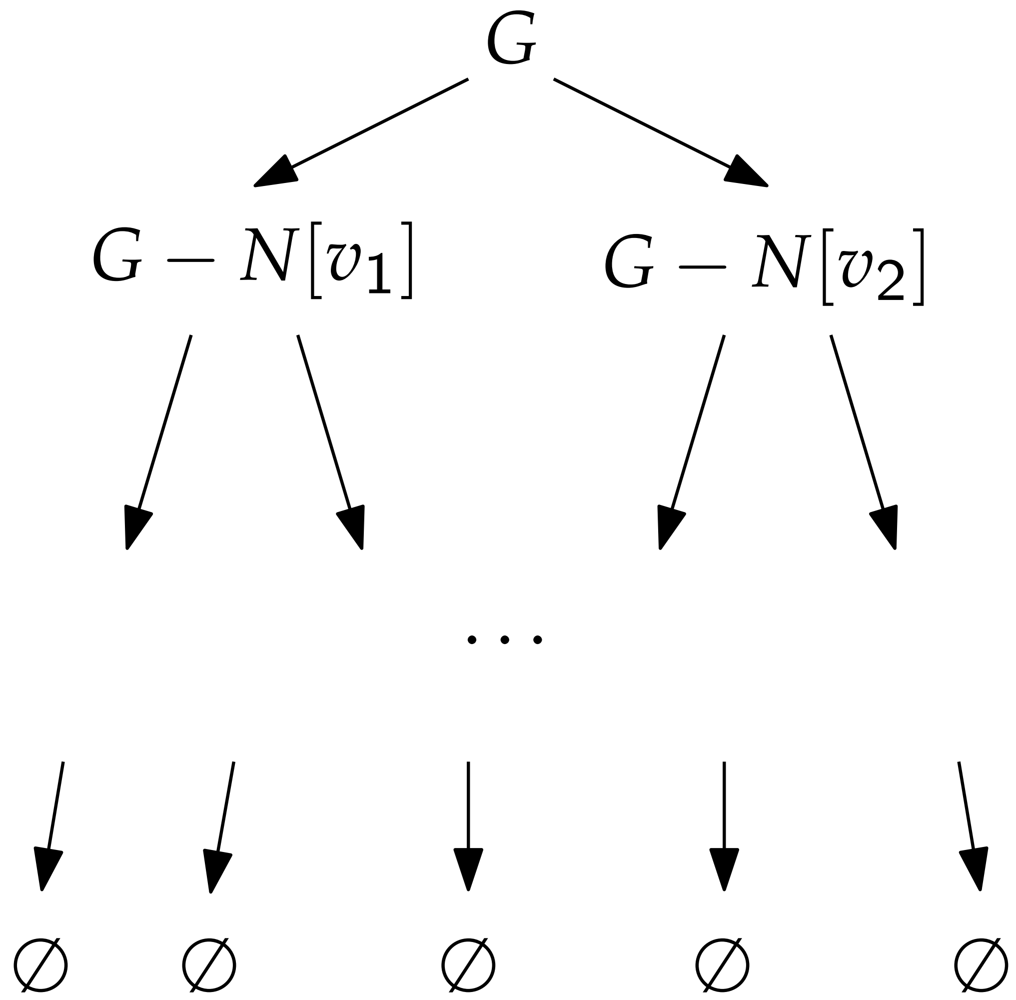
- Correctness follows from Lemma.
- We prove a runtime of $\mathcal{O}^*(3^{n/3}) = \mathcal{O}^*(1.4423^n)$.

MIS – Branching analysis

Execution corresponds to a **search tree** whose vertices are labeled with the input of the respective recursive call.



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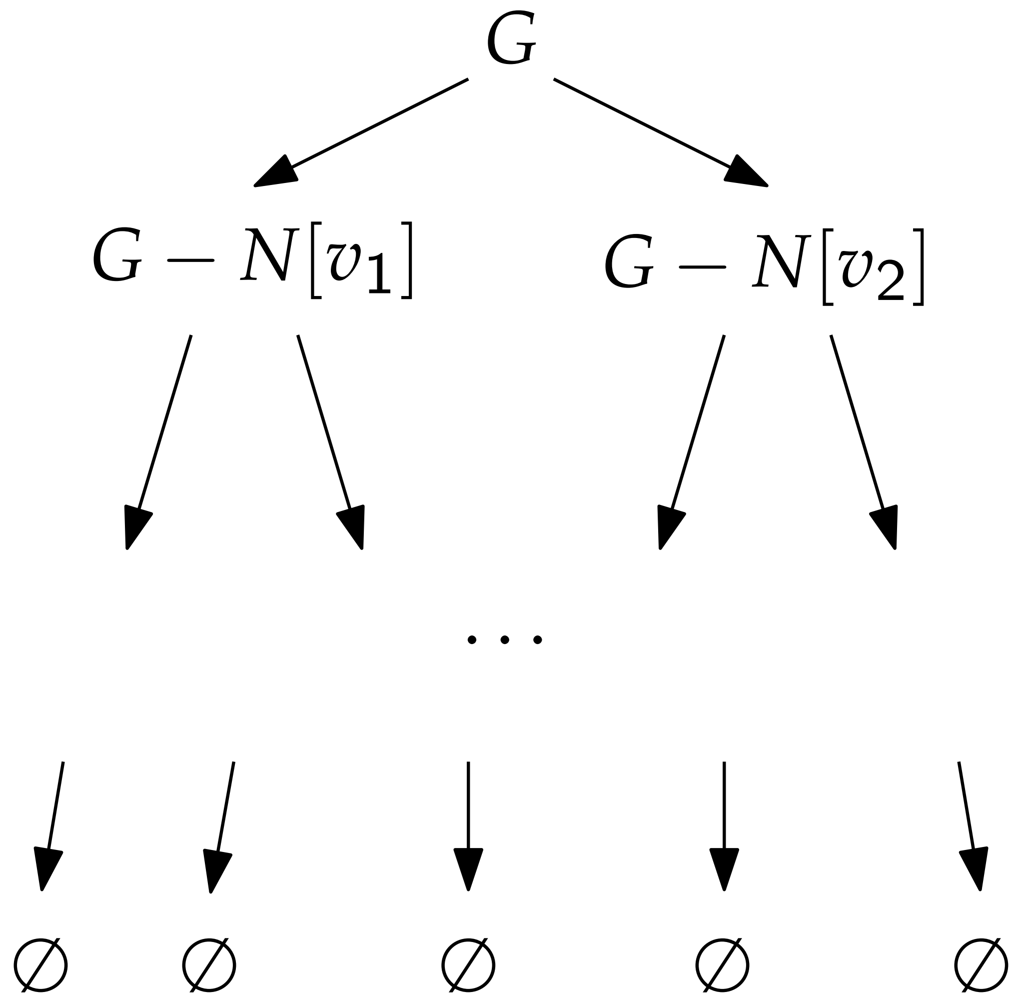
- Let $B(n)$ be the maximum number of leaves of a search tree for a graph with n vertices.



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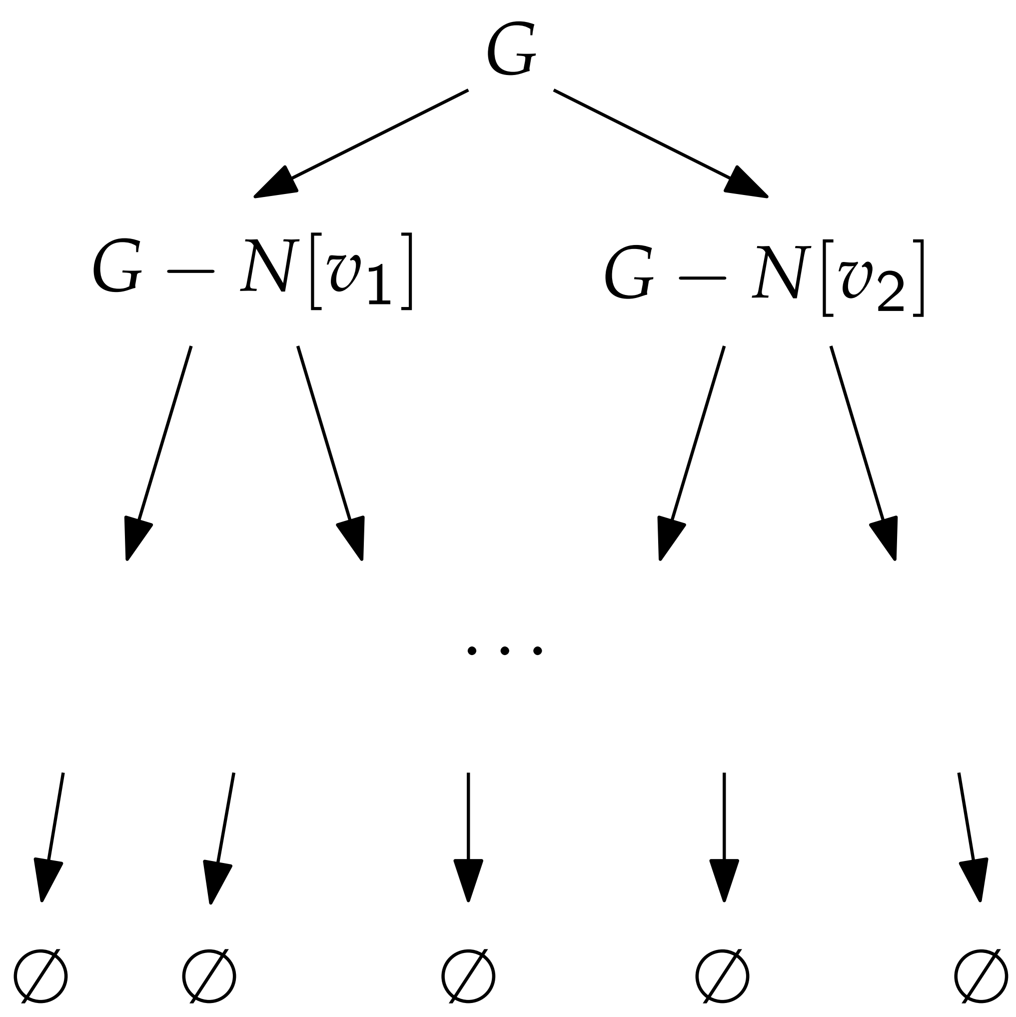

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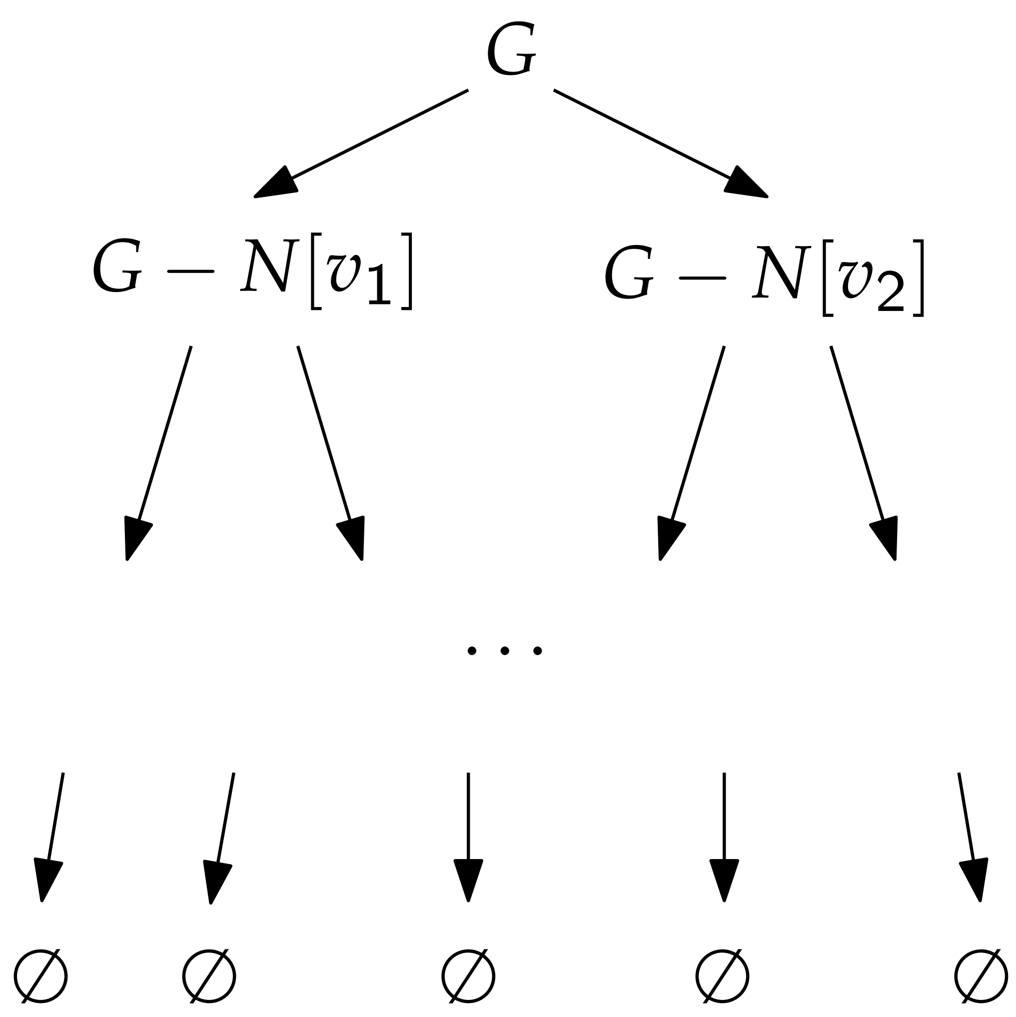

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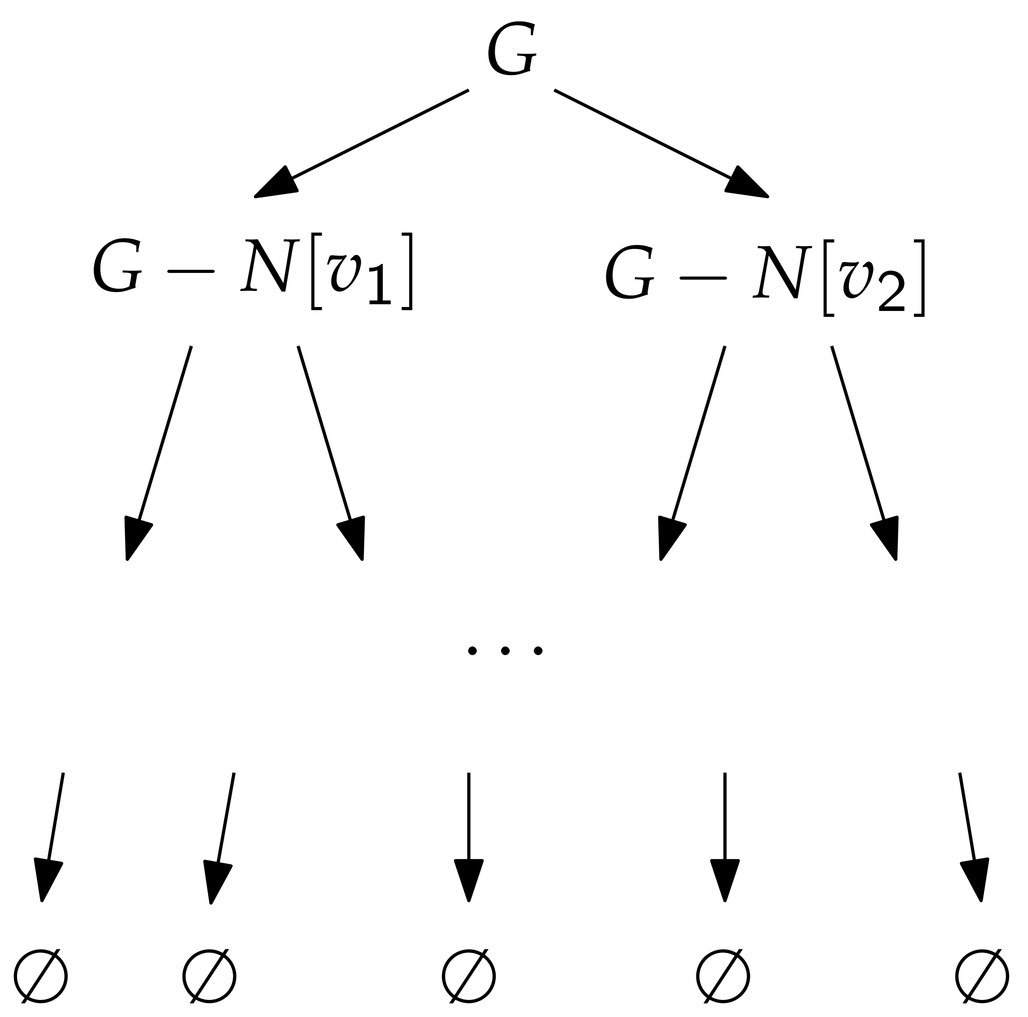
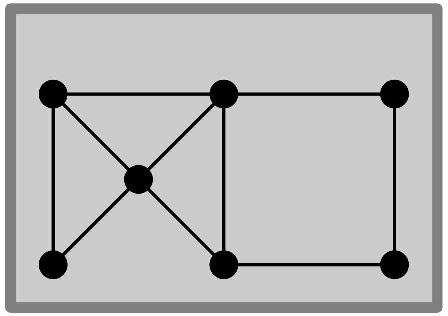
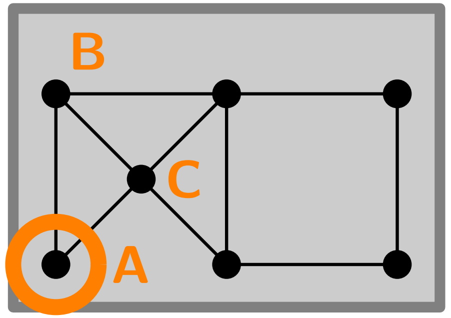
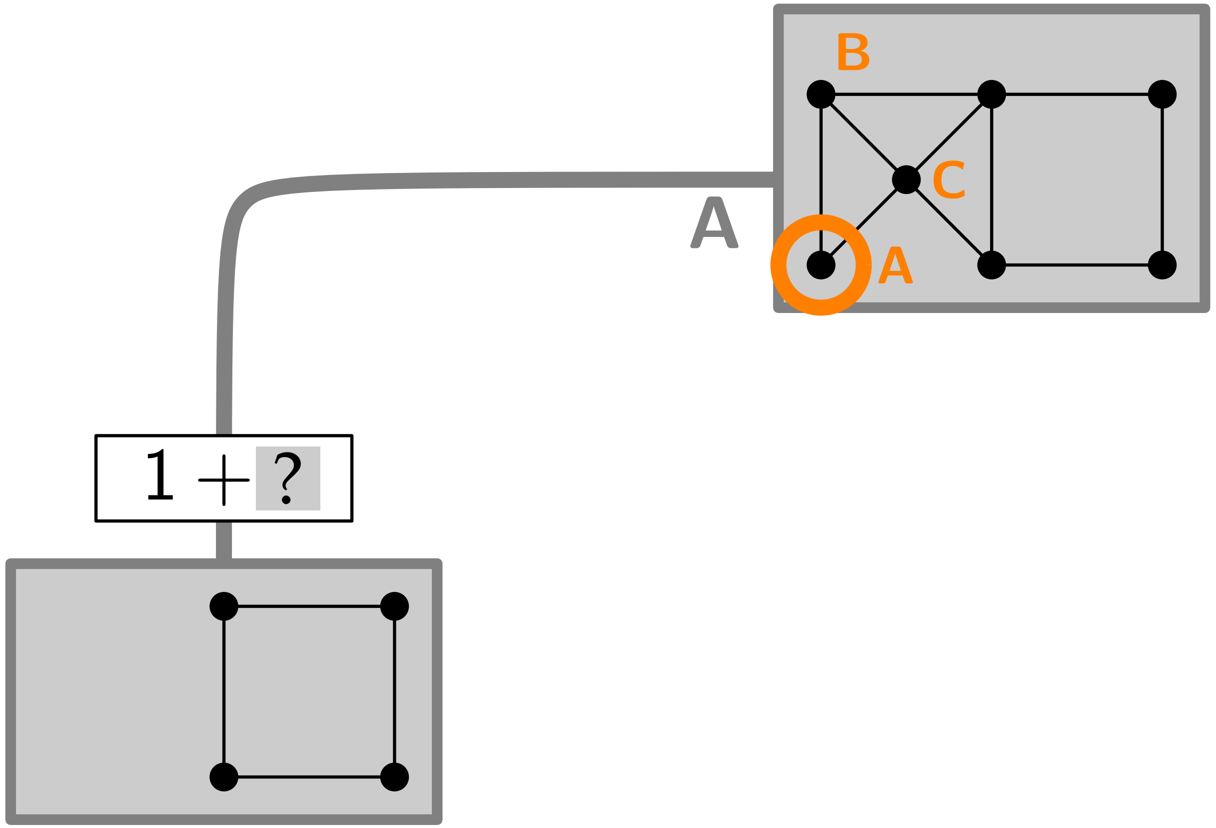
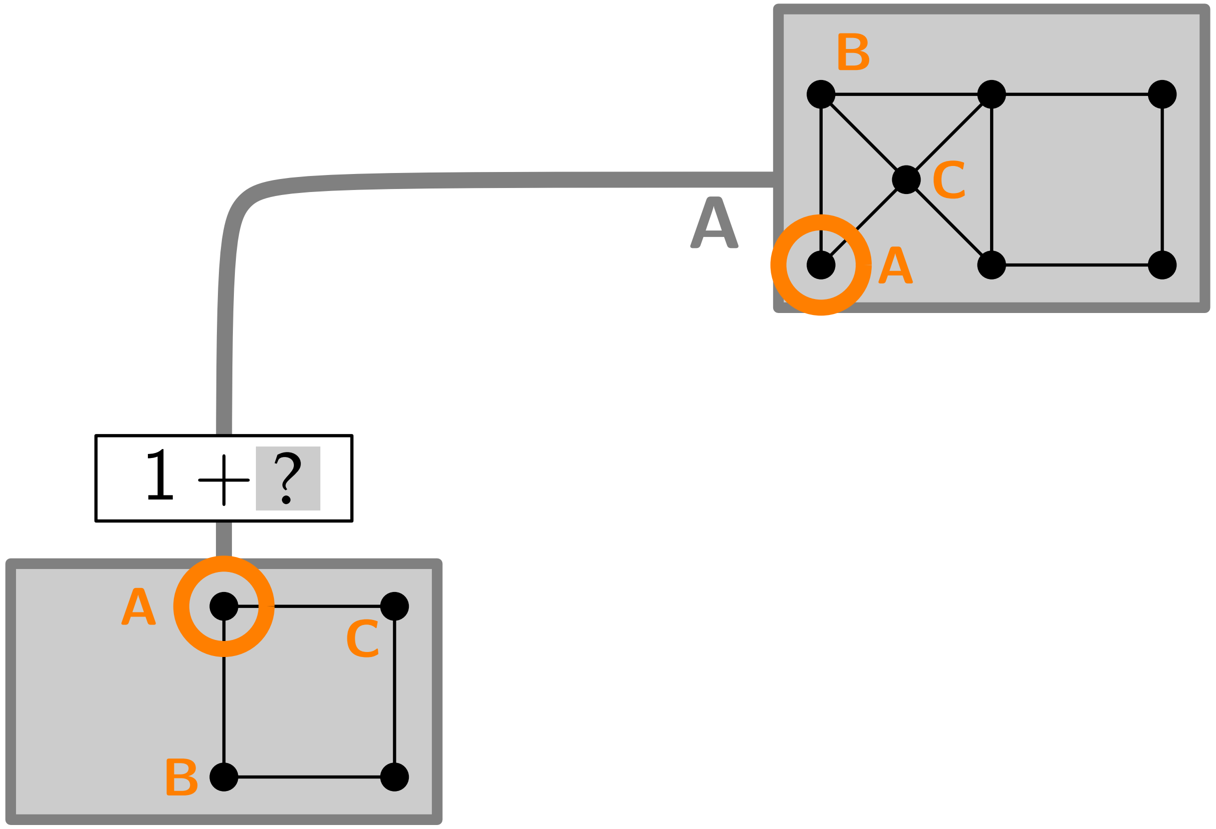
- Let's consider an example run.

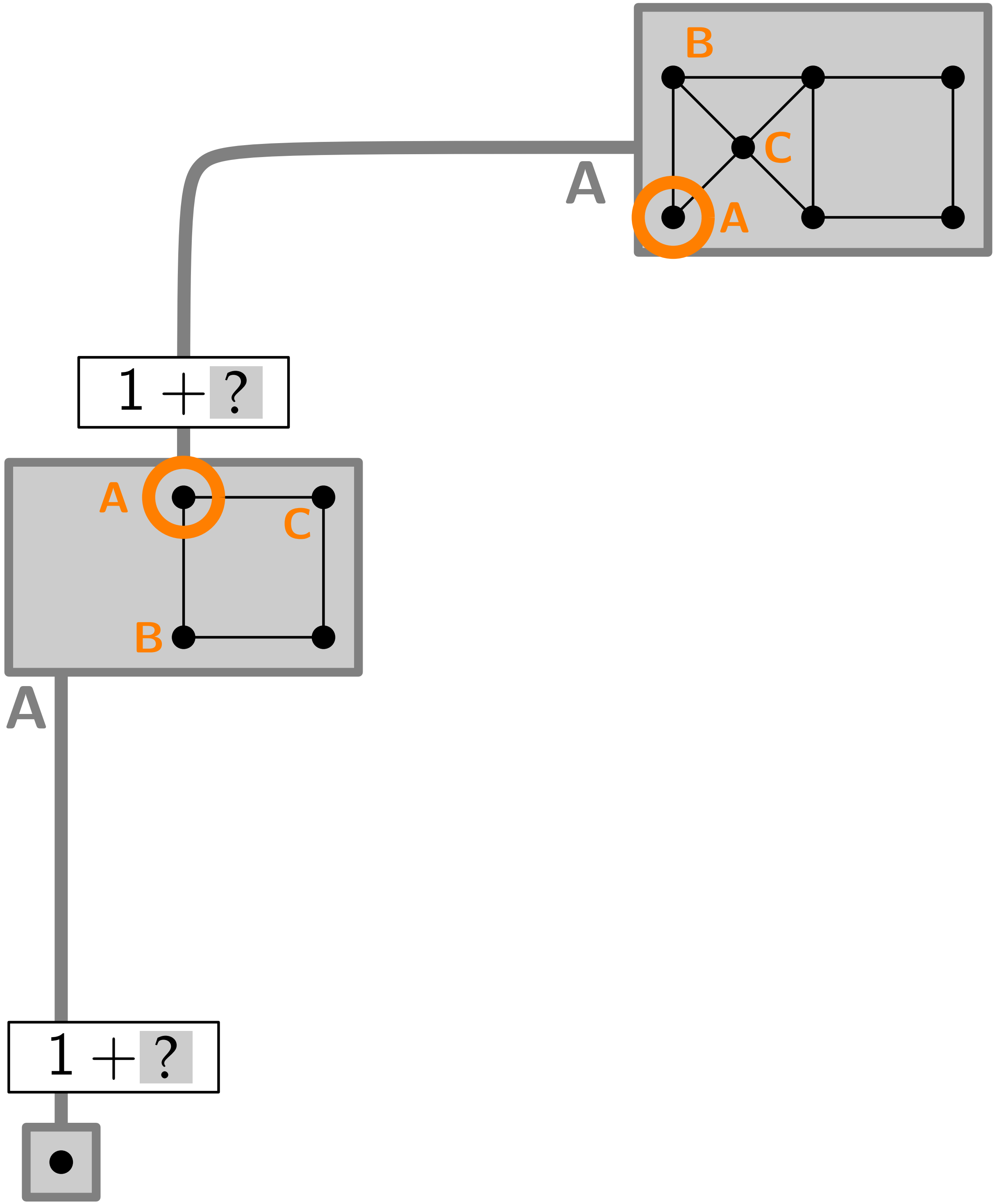


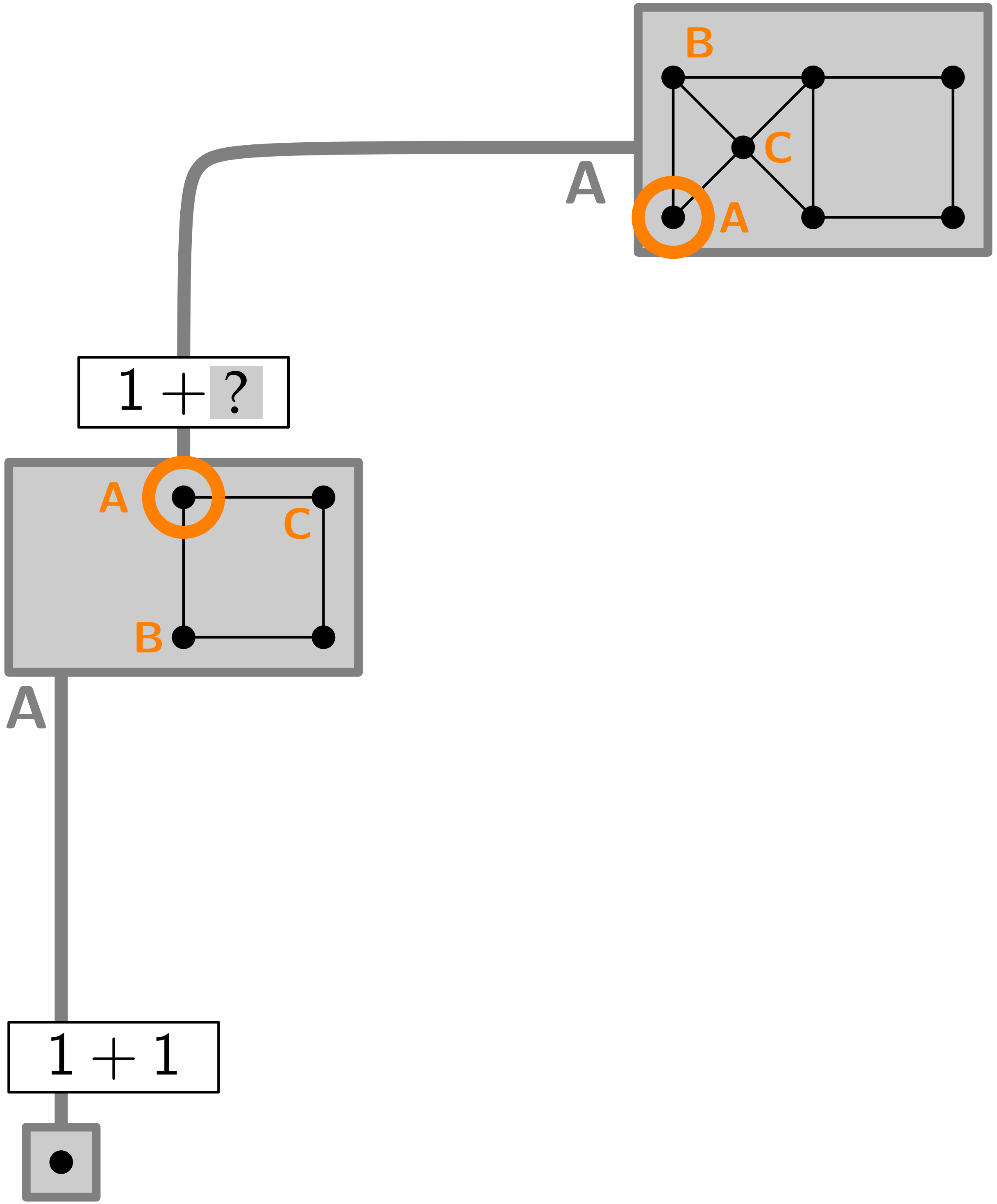


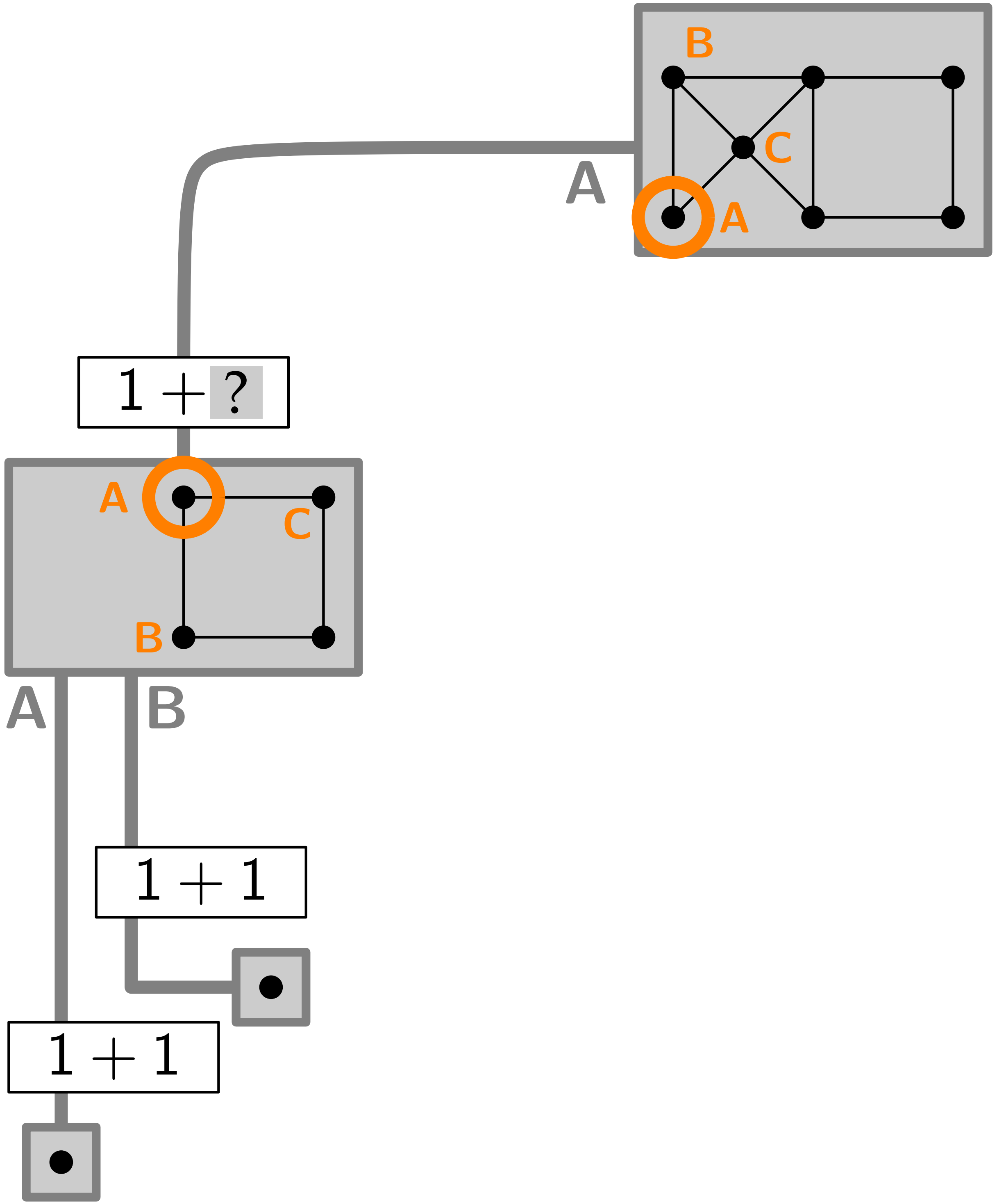

B
C
A

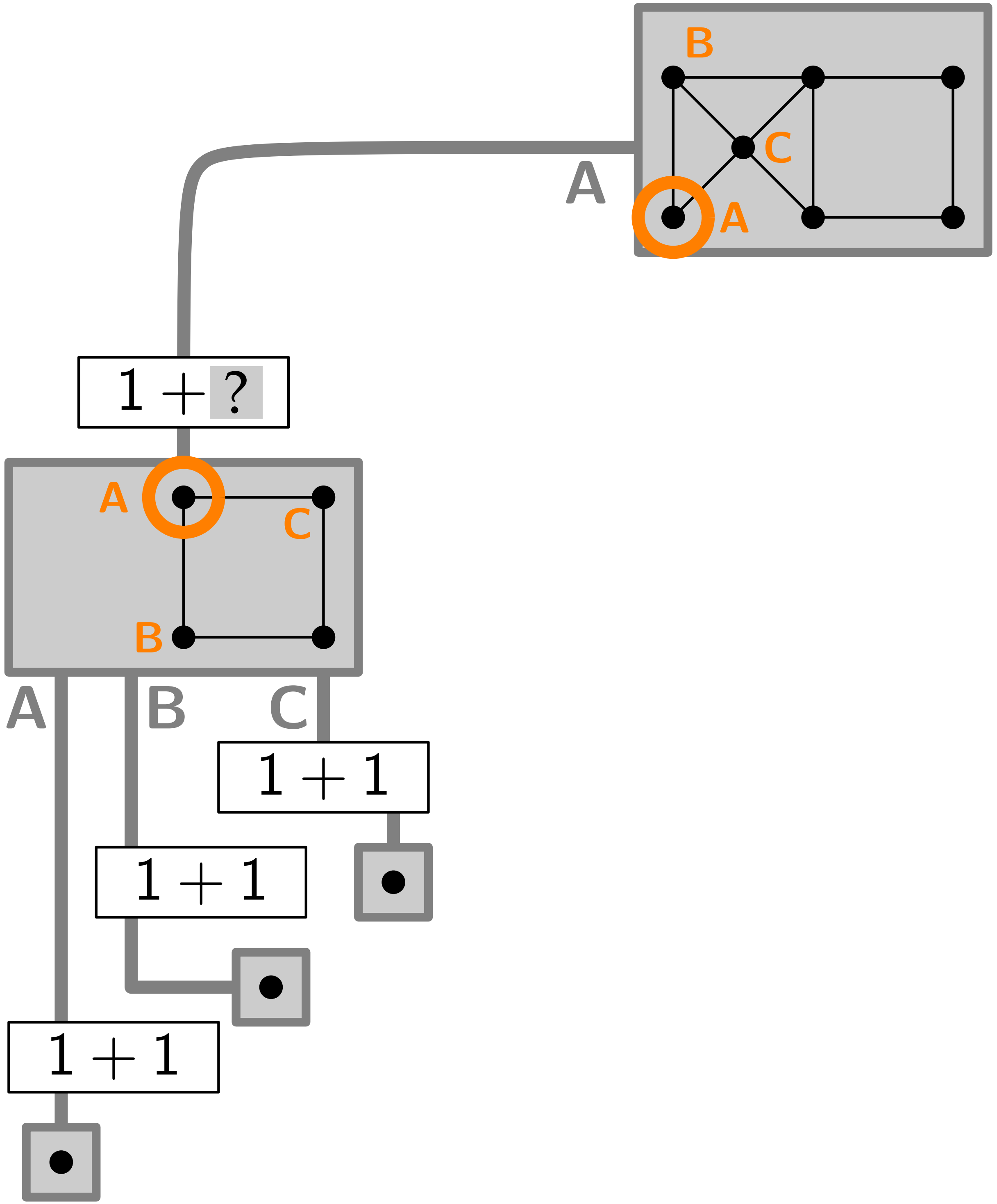

B
C
A
A
1 + ?


B
C
A
A
$1 + ?$
A
C
B

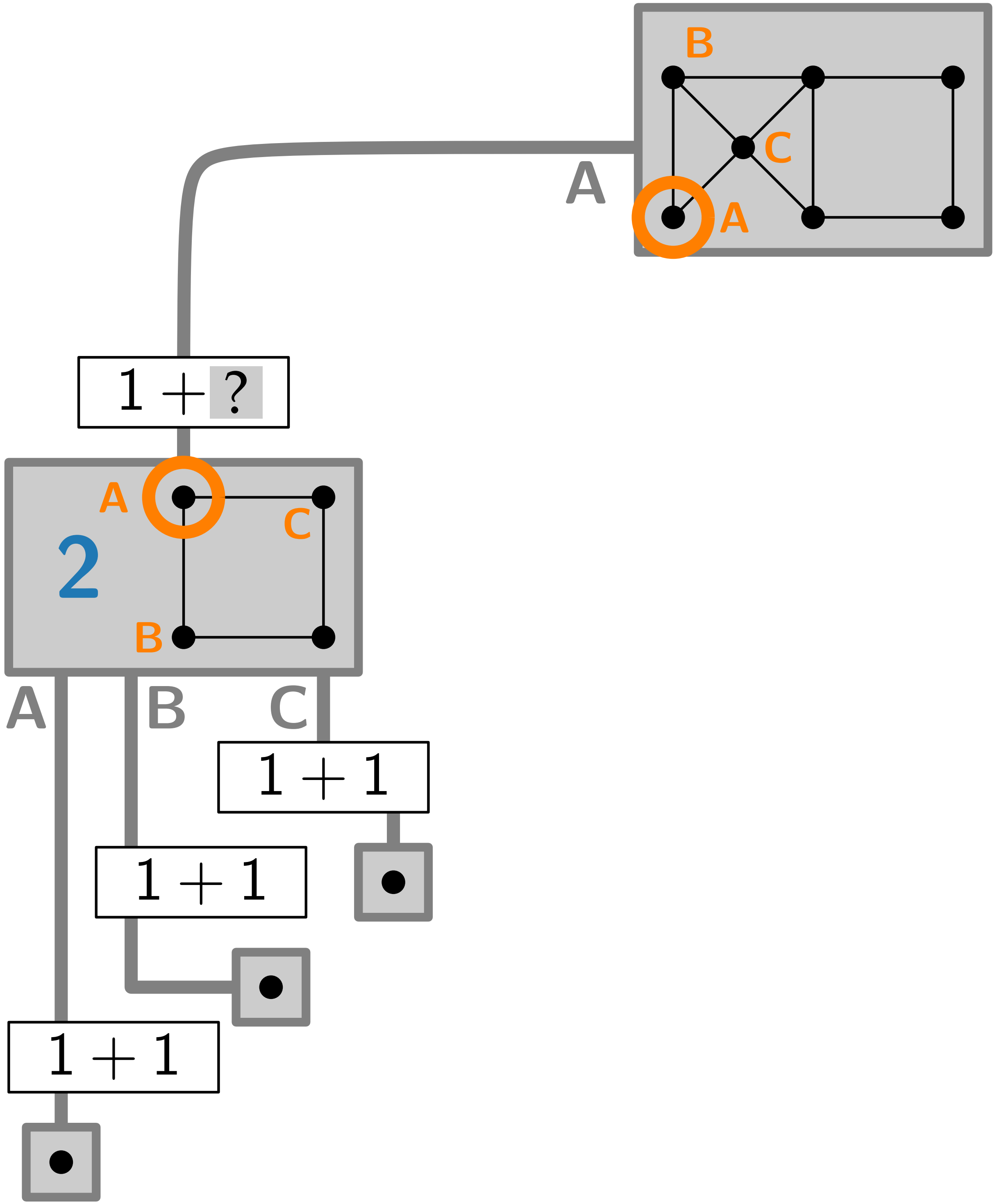

B
C
A
A
1 + ?
A
C
B
A
1 + ?

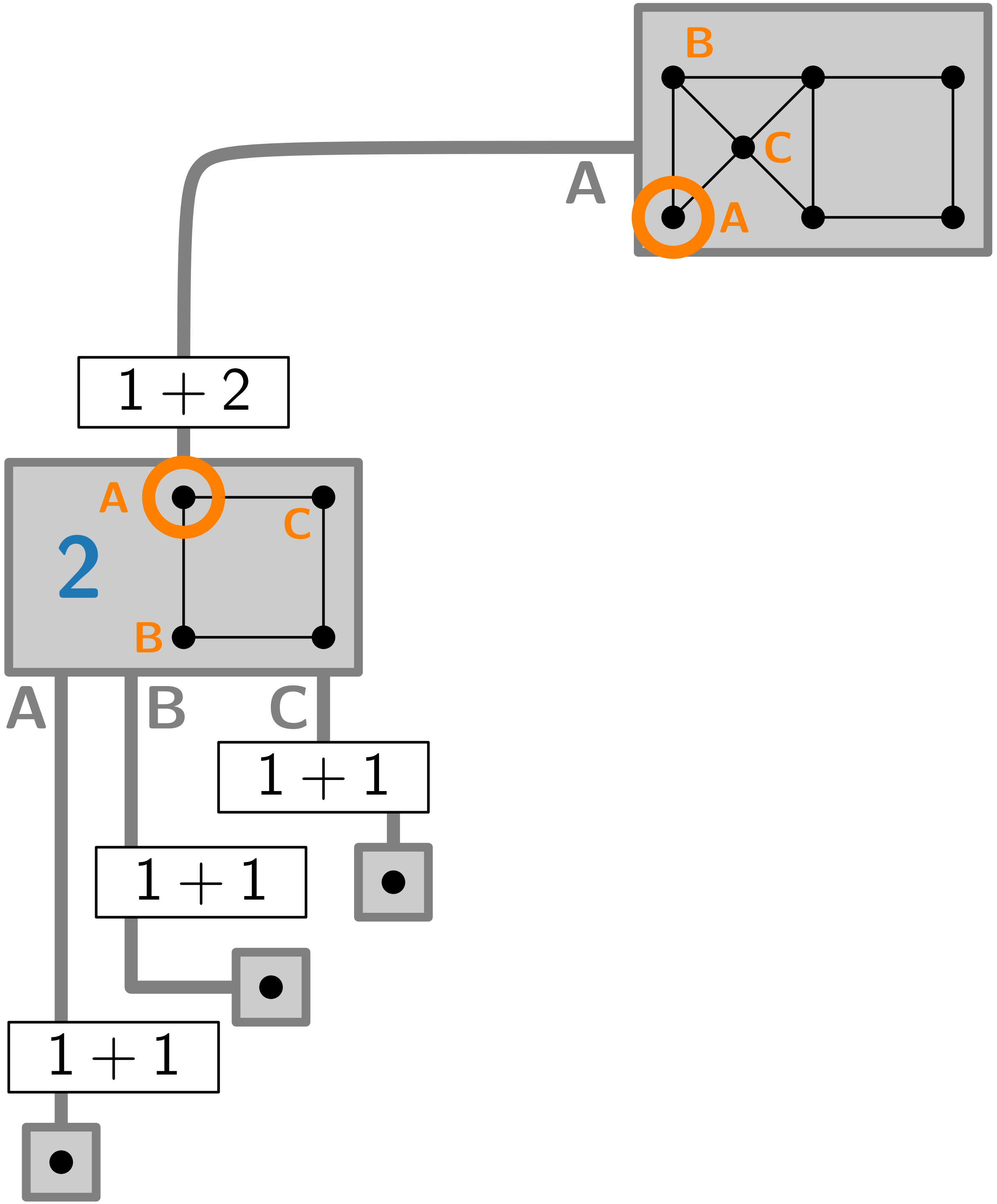

B
C
A
A
1 + ?
A
C
B
A
1 + 1

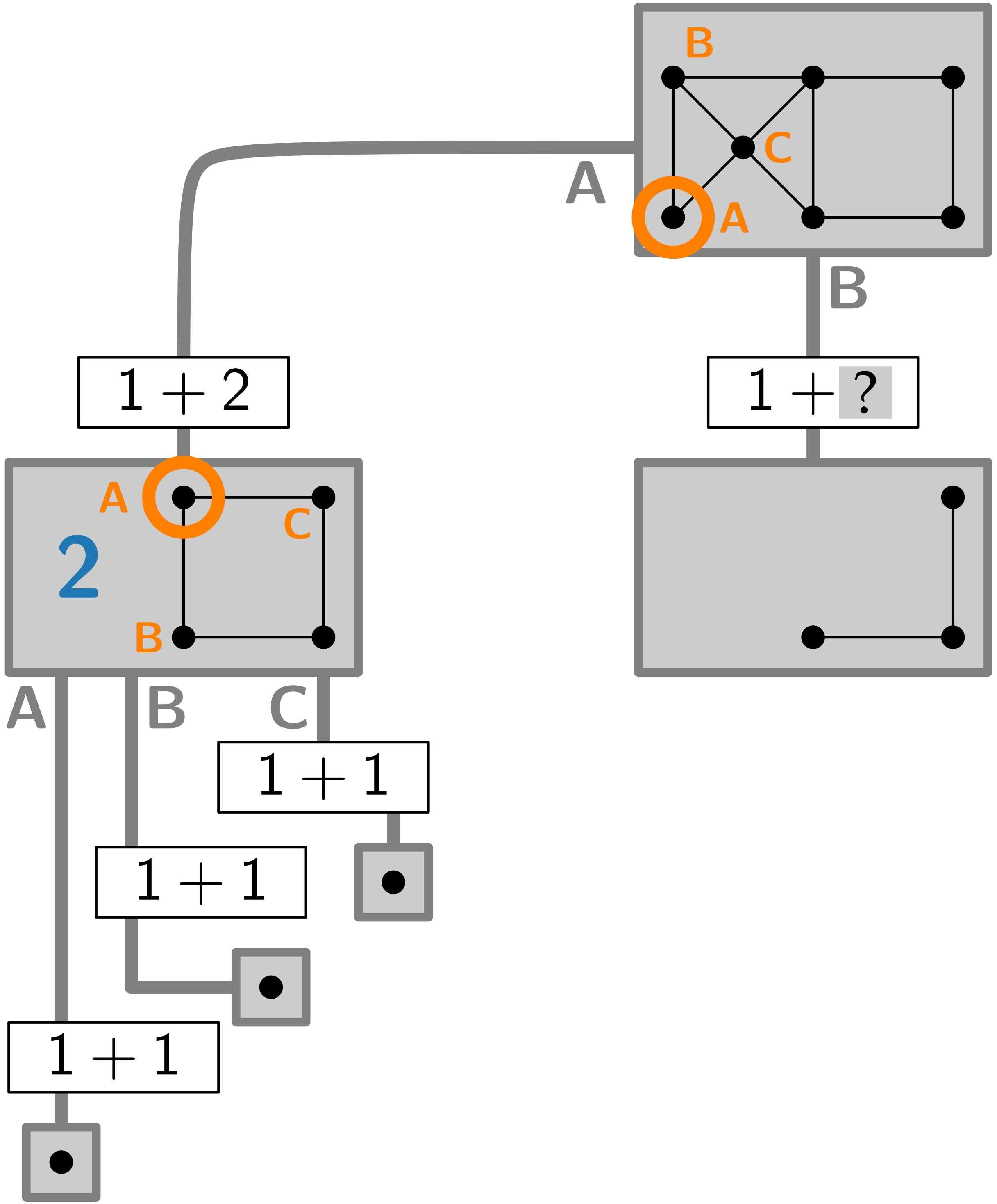

B
C
A
A
1 + ?
A
C
B
A
B
1 + 1
1 + 1

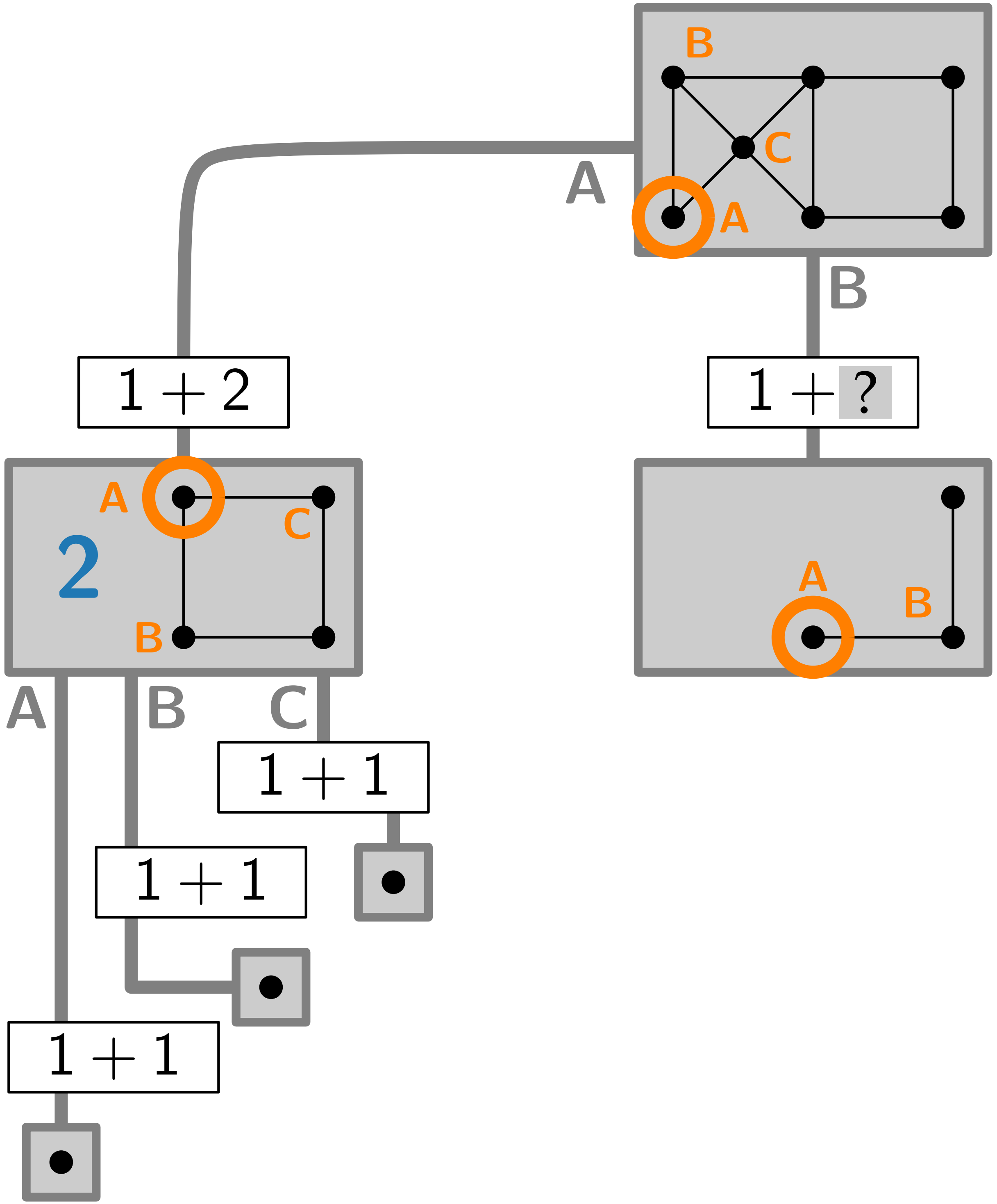


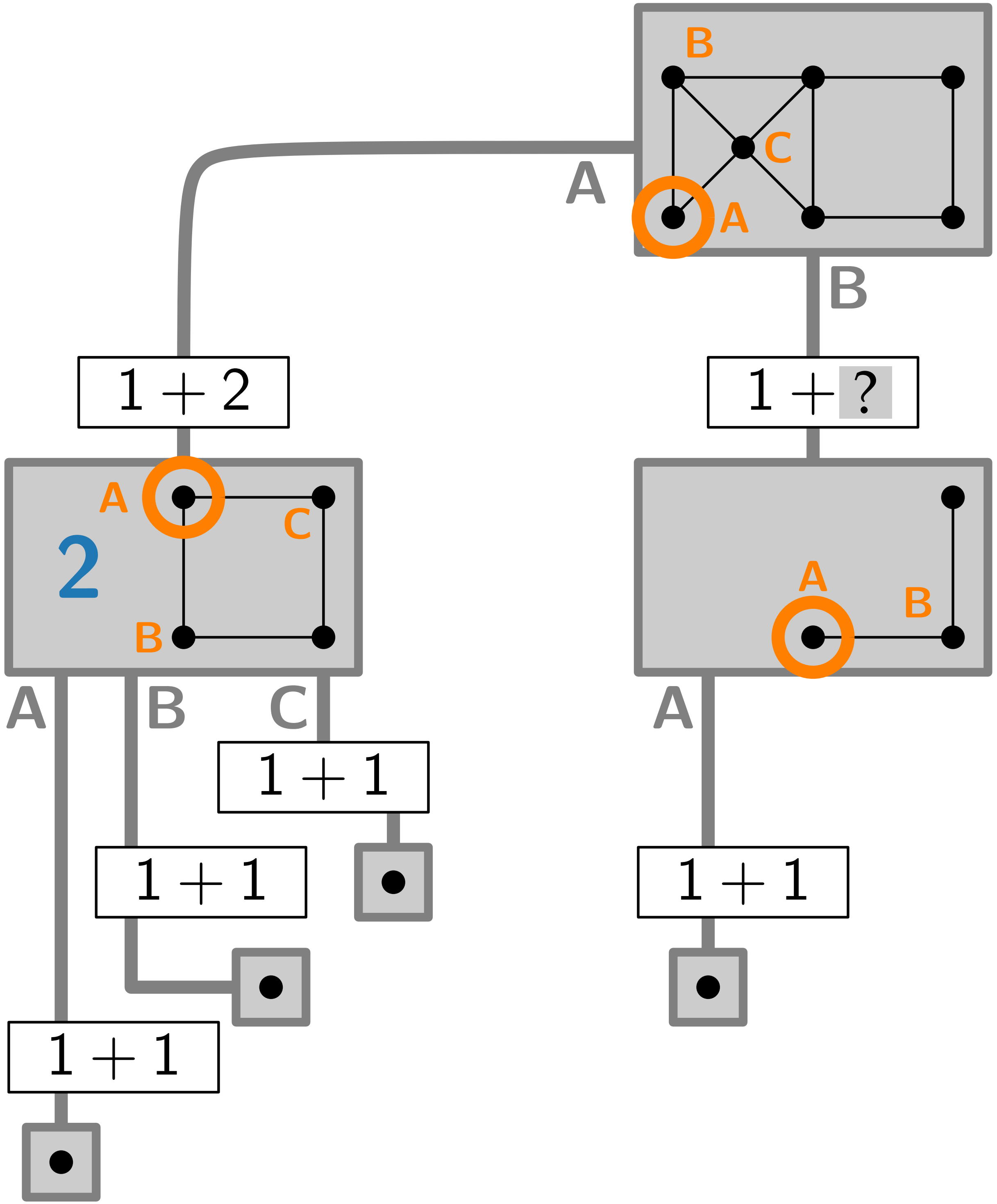
B
C
A
A
1 + ?
A
C
B
A
B
C
1 + 1
1 + 1
1 + 1

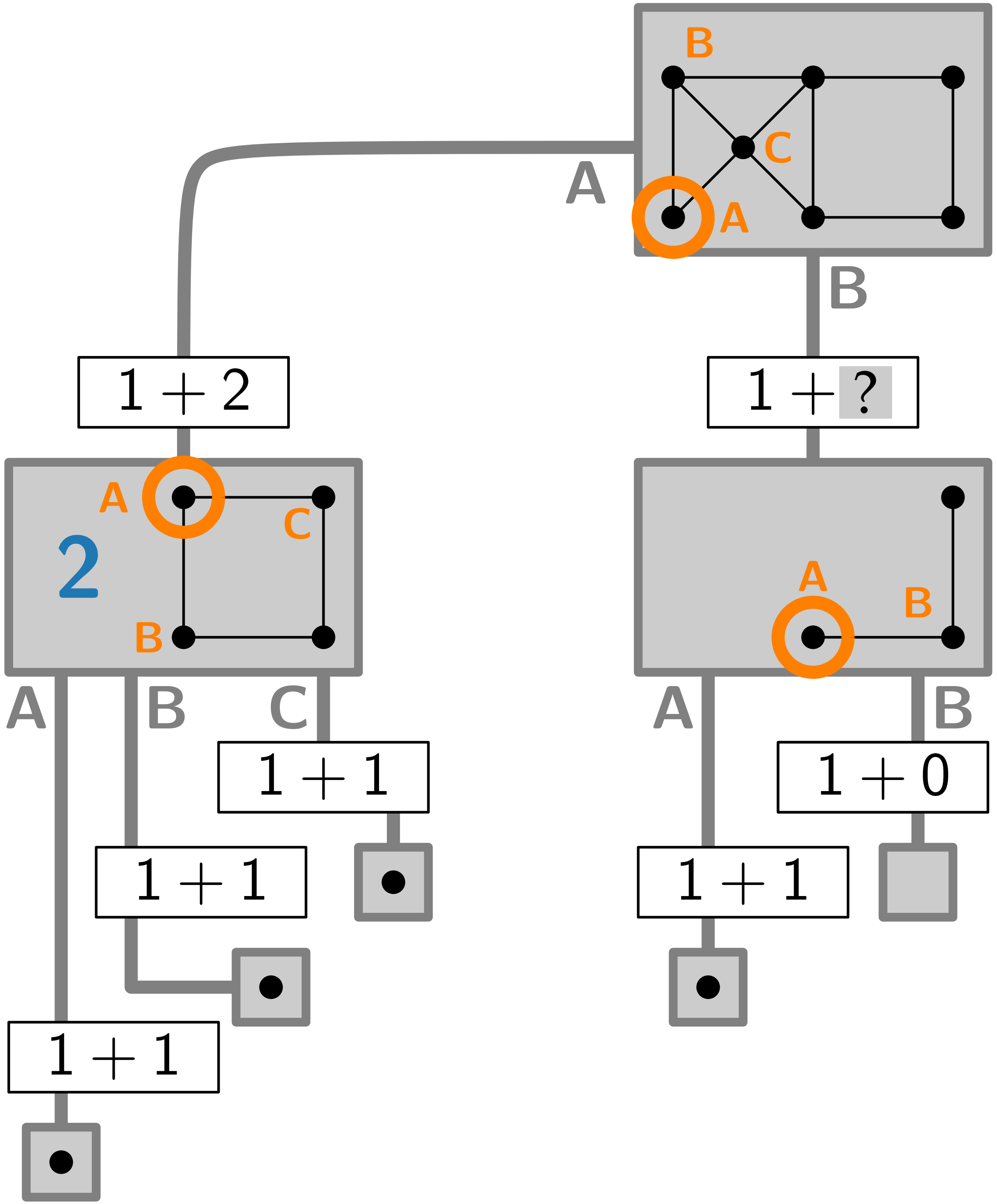

B
C
A
A
1 + ?
A
C
2
B
A
B
C
1 + 1
1 + 1
1 + 1

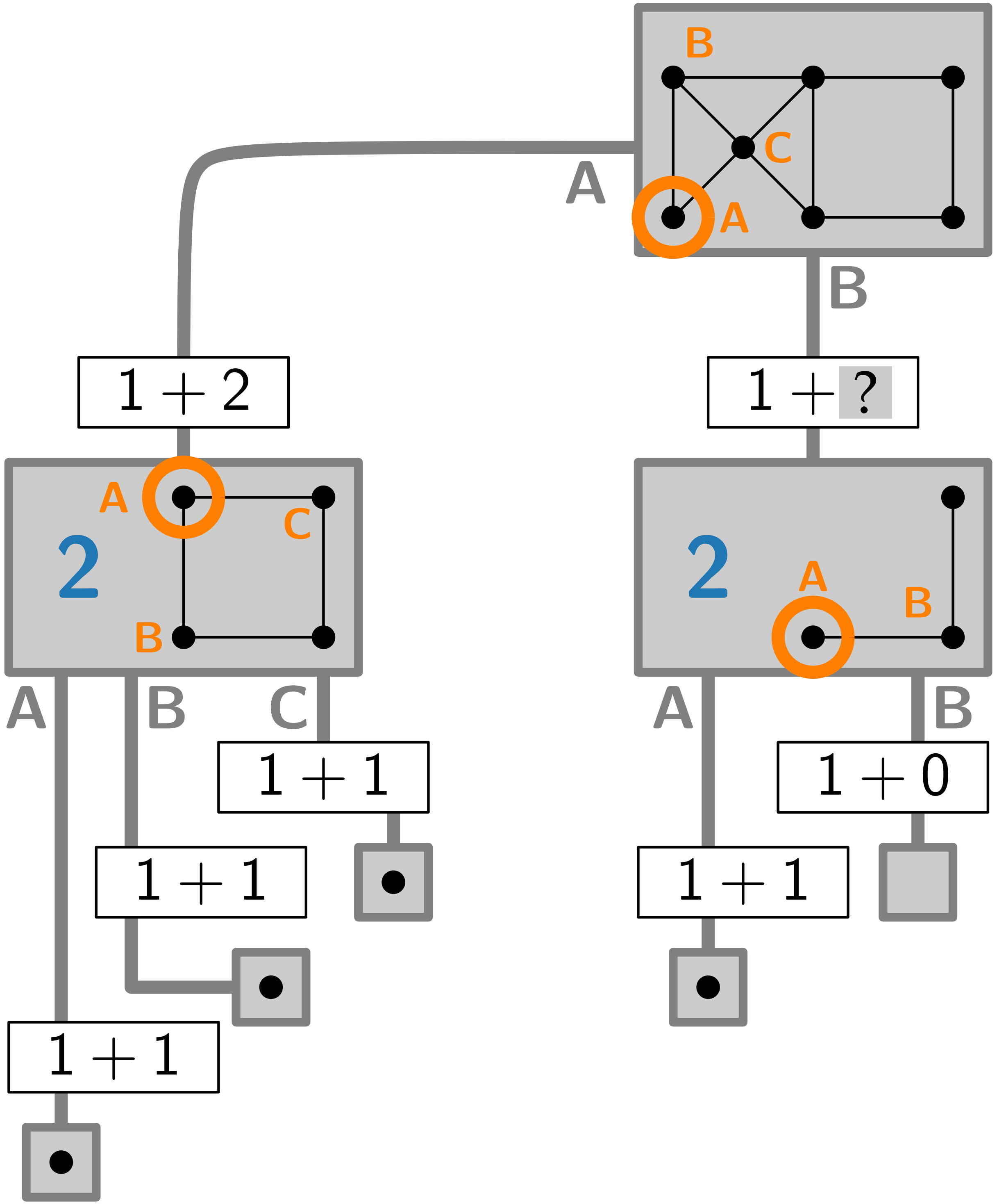

B
C
A
A
1 + 2
A
C
2
B
A
B
C
1 + 1
1 + 1
1 + 1


A
B
C
A
B
1 + 2
1 + ?
A
C
2
B
A
B
C
1 + 1
1 + 1
1 + 1


B
C
A
A
B
1 + 2
1 + ?
A
C
2
B
A
B
A
B
C
1 + 1
1 + 1
1 + 1


B
C
A
A
B
1 + 2
1 + ?
A
C
2
B
A
B
A
B
C
A
1 + 1
1 + 1
1 + 1
1 + 1


B
C
A
A
B
1 + 2
1 + ?
A
C
2
B
A
B
A
B
C
A
B
1 + 1
1 + 0
1 + 1
1 + 1
1 + 1


B
C
A
A
B
1 + 2
1 + ?
A
C
2
B
2
A
B
A
B
C
A
B
1 + 1
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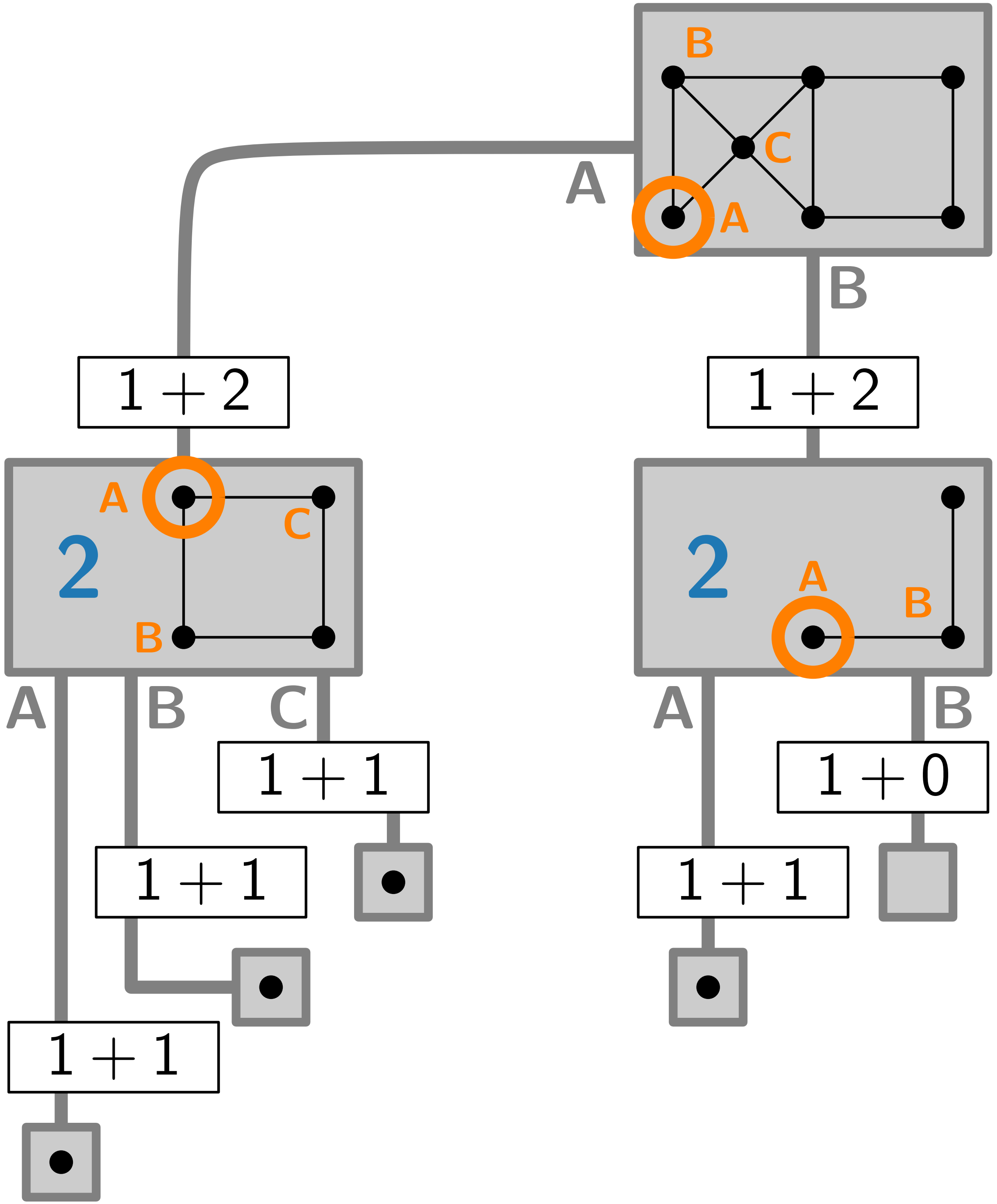
1 + 1
1 + 1



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C
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A
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1 + 2
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B
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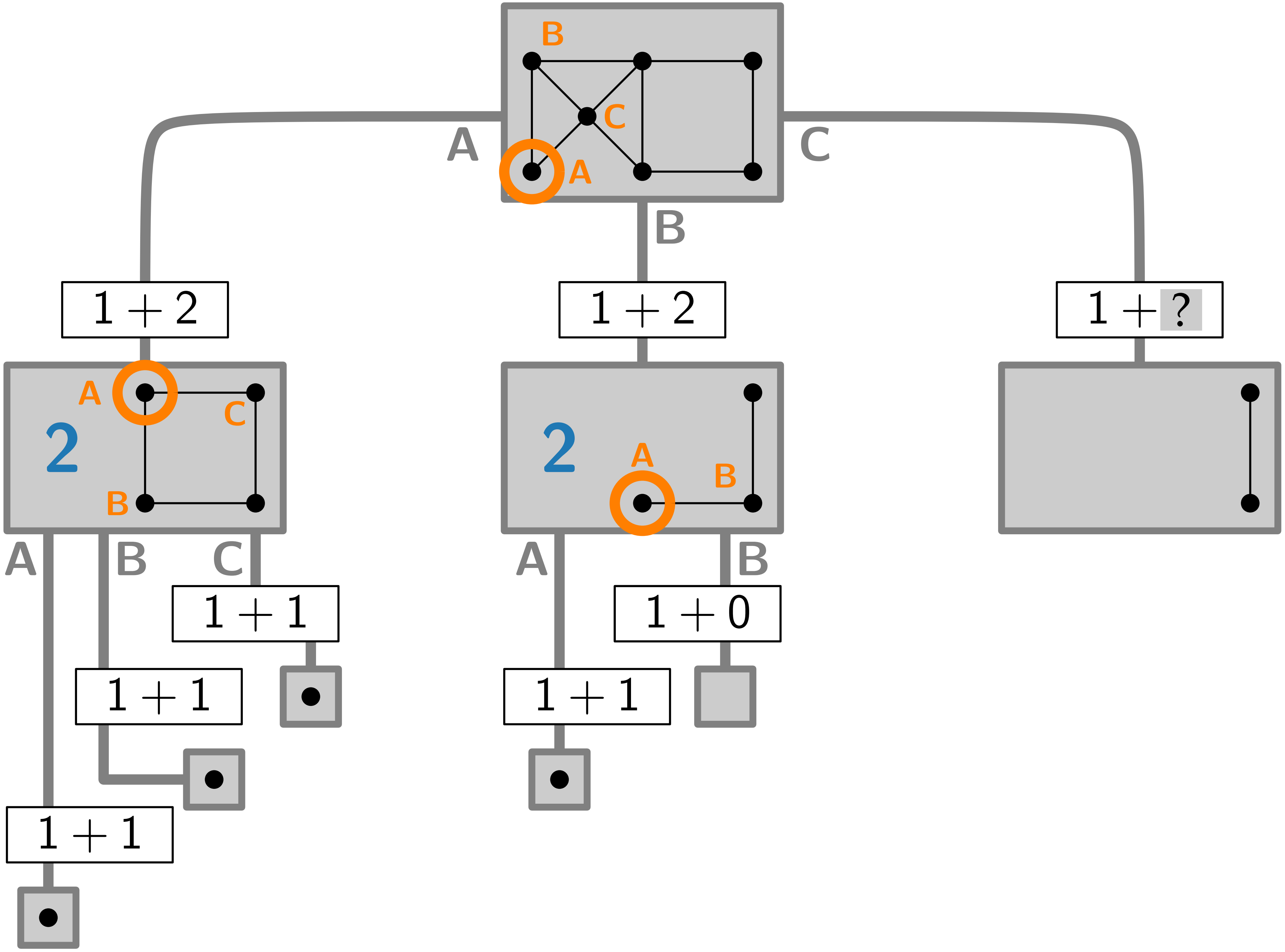
1 + 1
1 + 1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1 + 2
1 + 2
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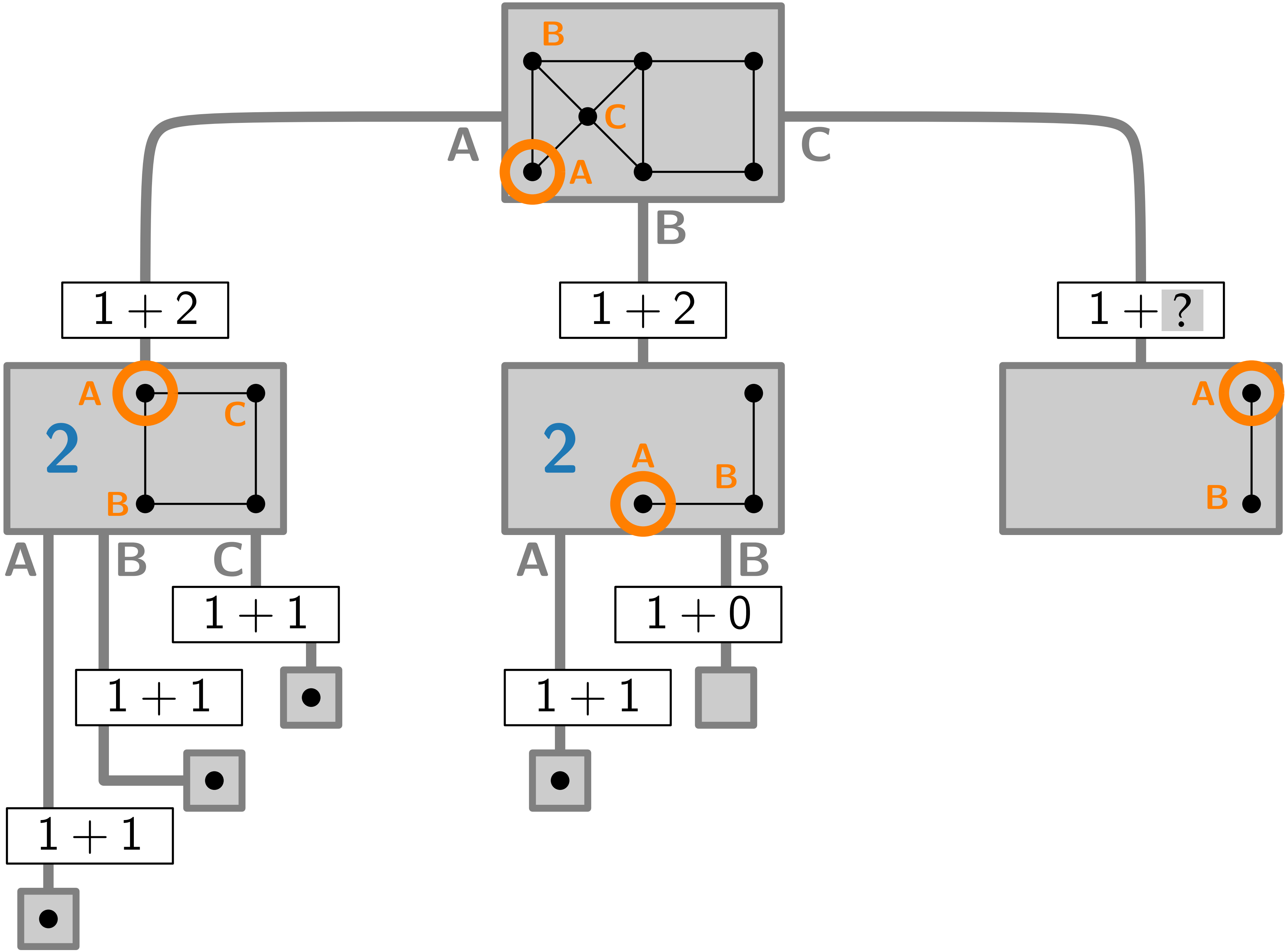

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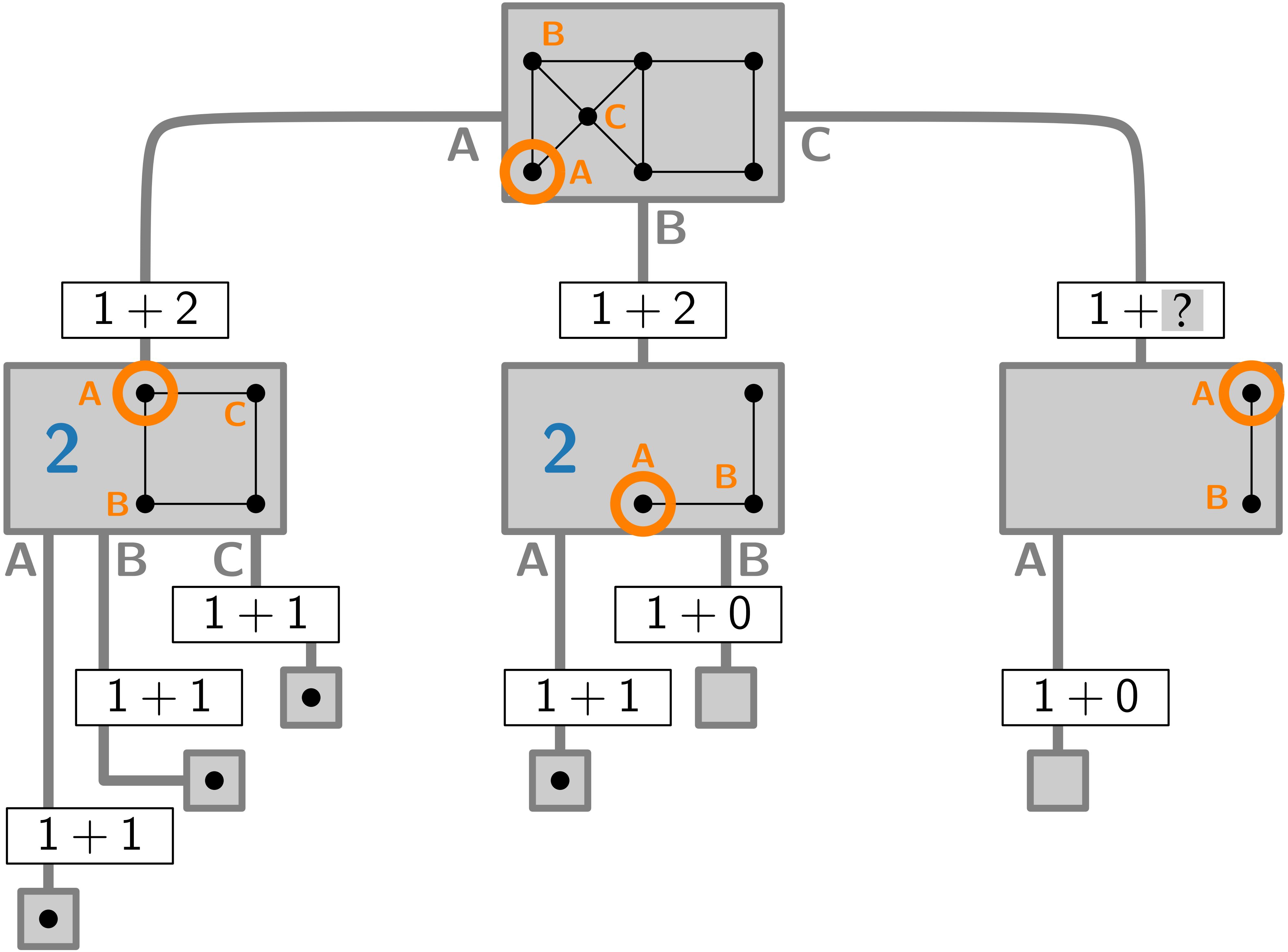
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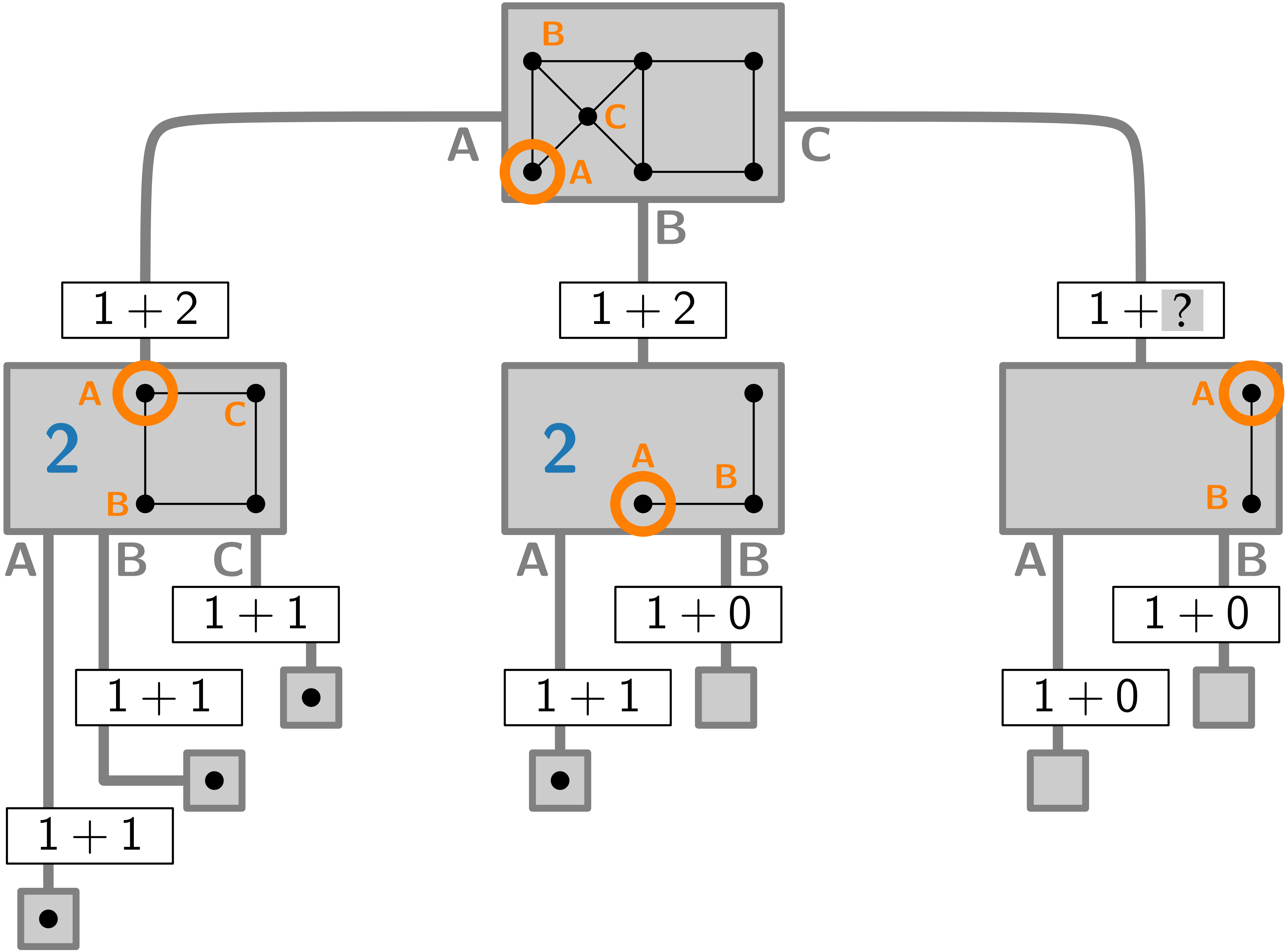
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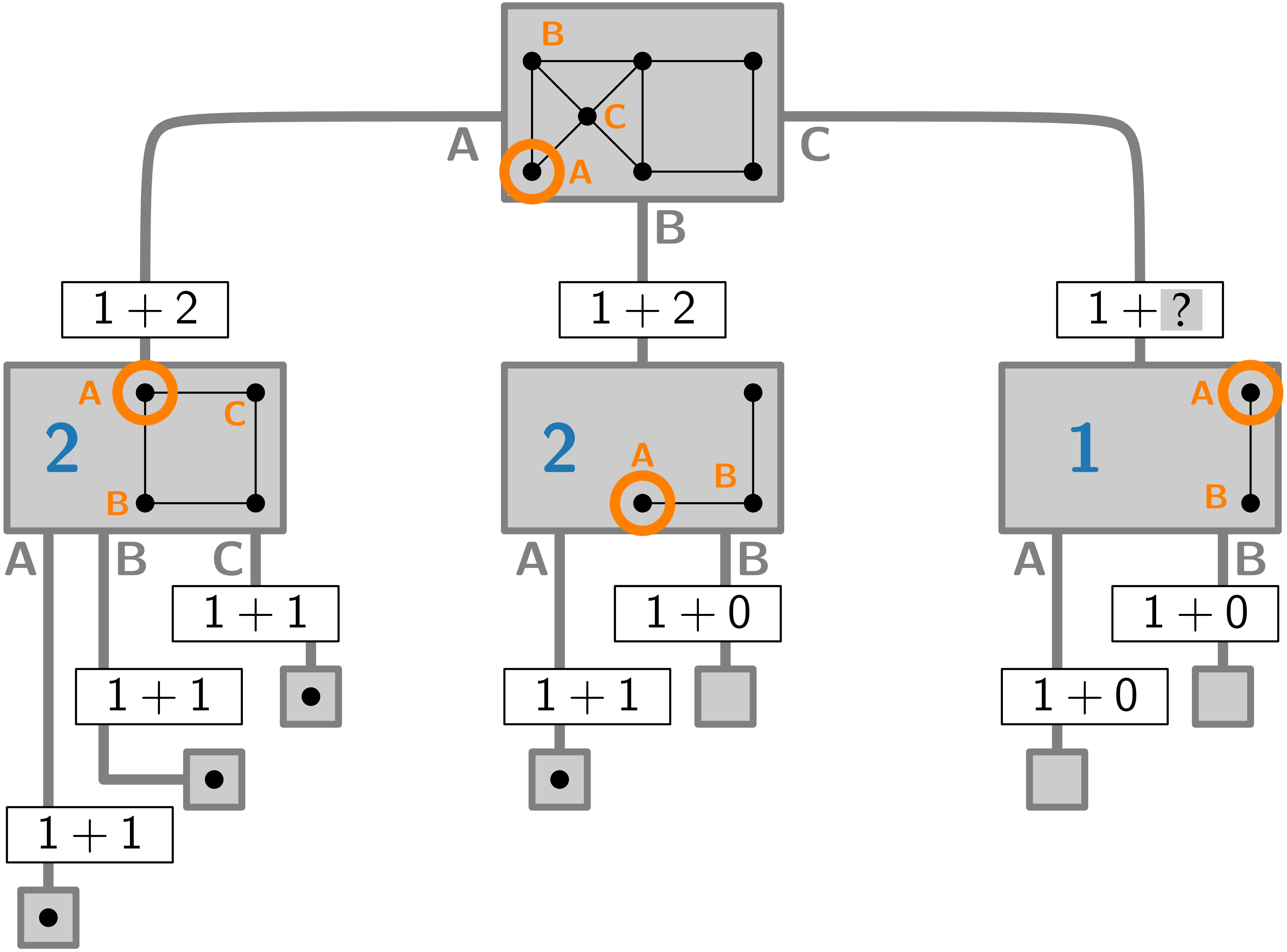
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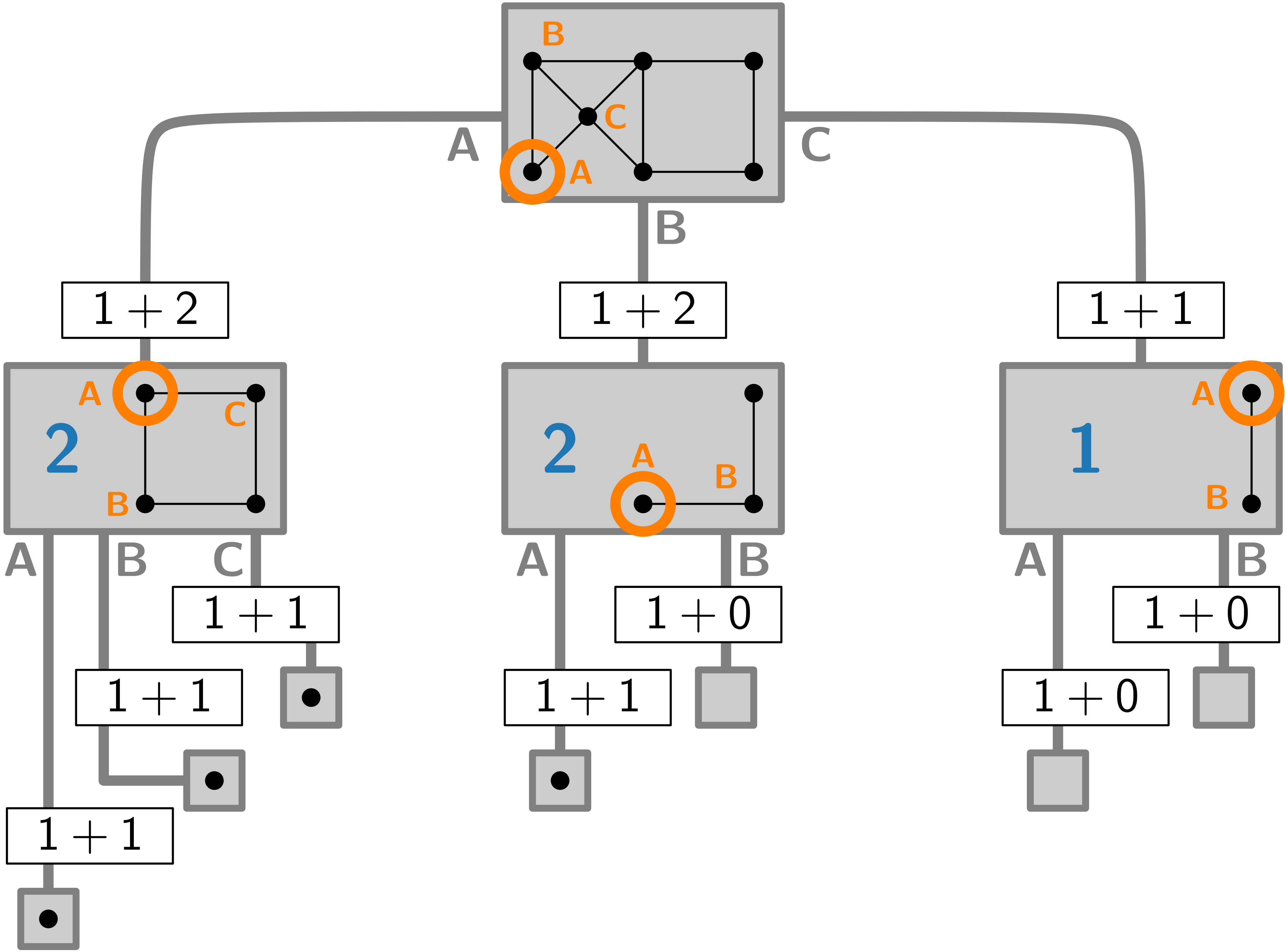
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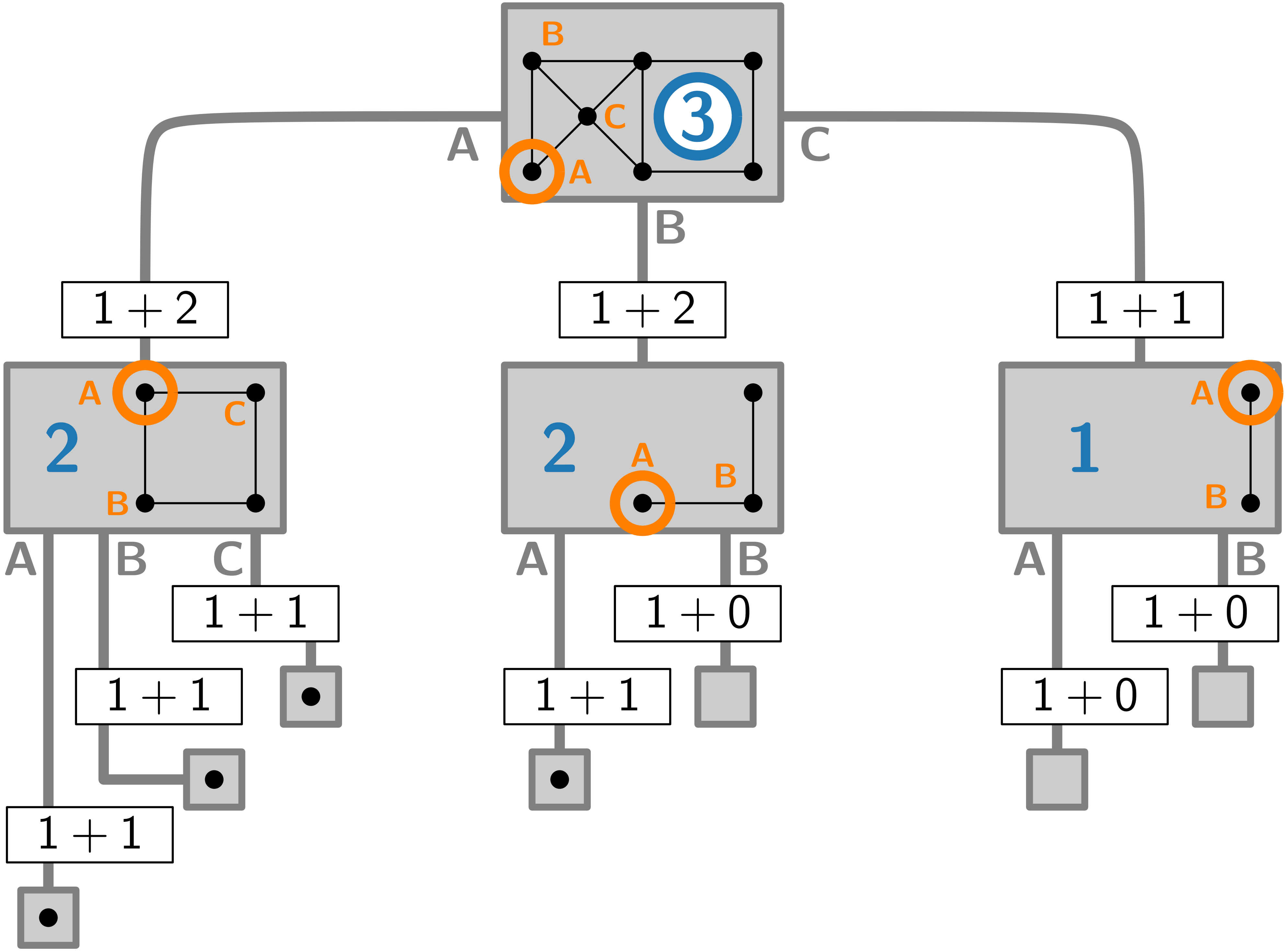
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A
B
A
B
C
A
B
A
B
1 + 1
1 + 0
1 + 0
1 + 1
1 + 1
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1 + 1

MIS – Runtime analysis

For a worst-case n-vertex graph G ($n \geq 1$):

$$B(n) \leq \sum_{y \in N[v]} B(n - (\deg(y) + 1))$$

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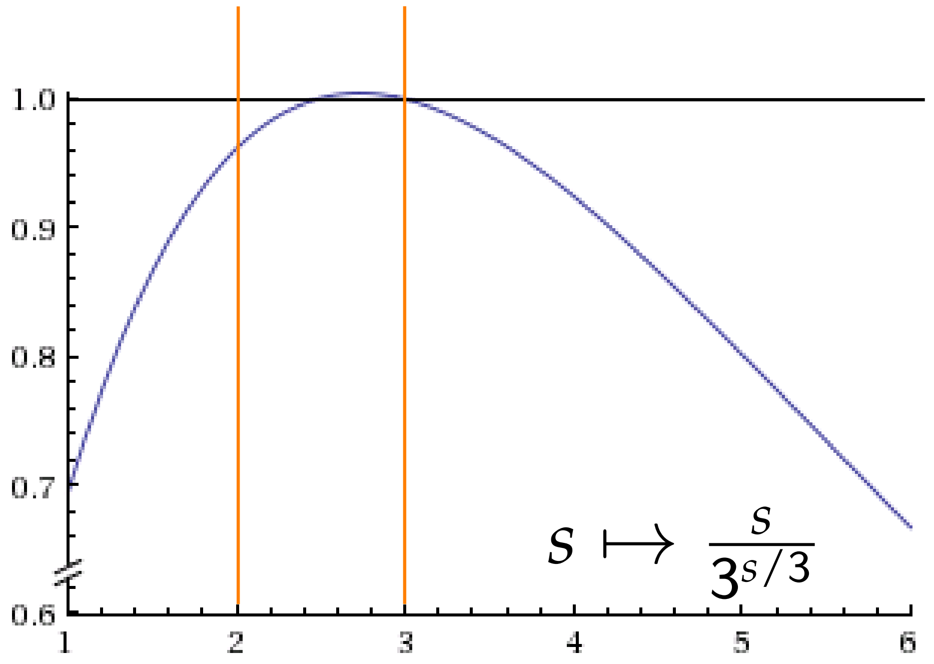

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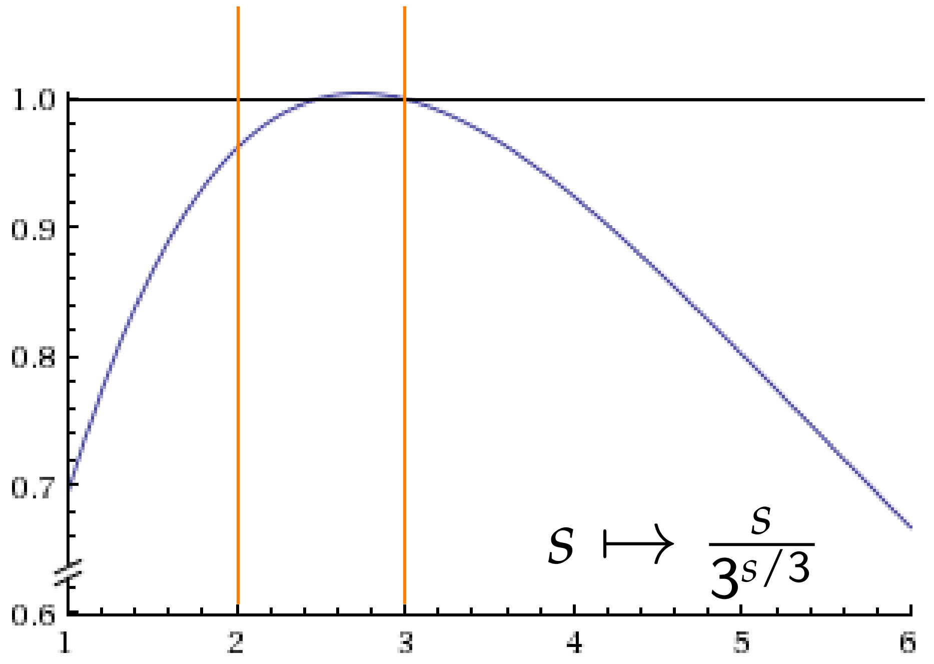
$$B(n) \in O^*(\sqrt[3]{3}^n) \subset O^*(1.44225^n)$$



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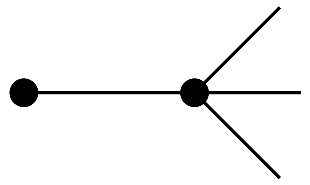

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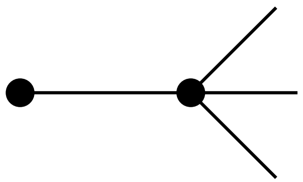

- Advanced case analysis in [Fomin, Kratsch Ch 2.3] leading to a $\mathcal{O}^*(1.2786^n)$-time algorithm.
- **Exercise**: Enumerating MISs
- **Exercise**: Edge-branching for MIS

Literature

Main source:

- [Fomin, Kratsch Ch1] “Exact Exponential Algorithms”

Referenced papers:

- [ADMV ’15] Classic Nintendo Games are (Computationally) Hard
- [Mann ’17] The Top Eight Misconceptions about NP-Hardness