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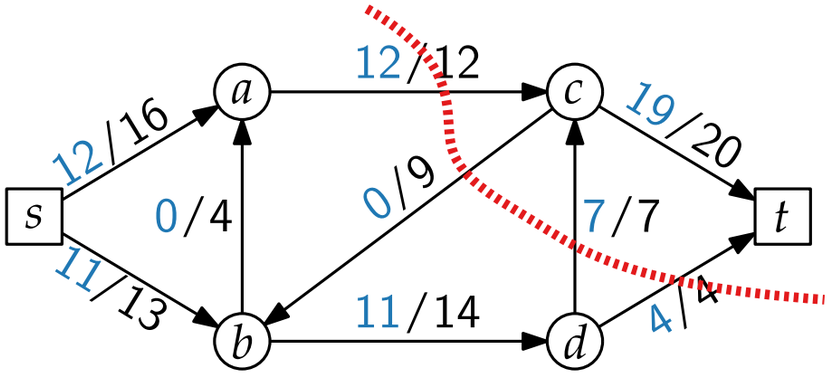
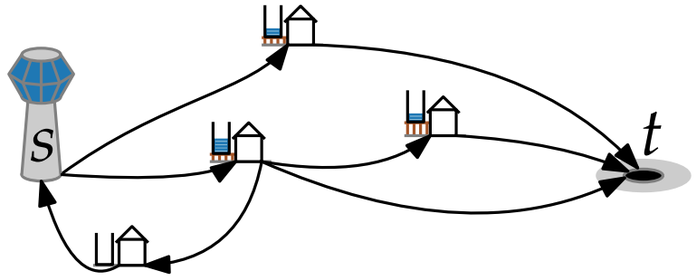
Advanced Algorithms

Maximum flow problem

Push-relabel algorithm

Jonathan Klawitter · WS20

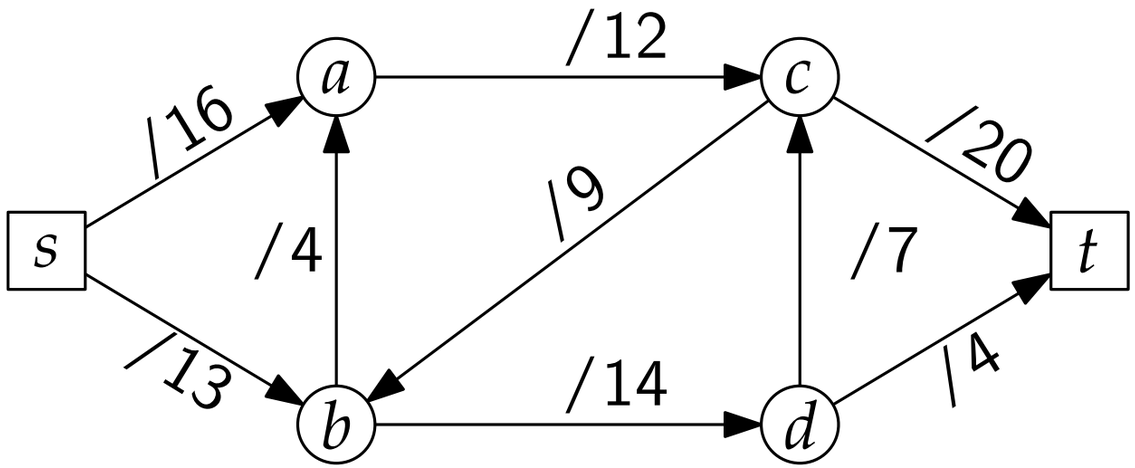




Flow network

A **flow network** $G = (V, E)$ is a digraph with

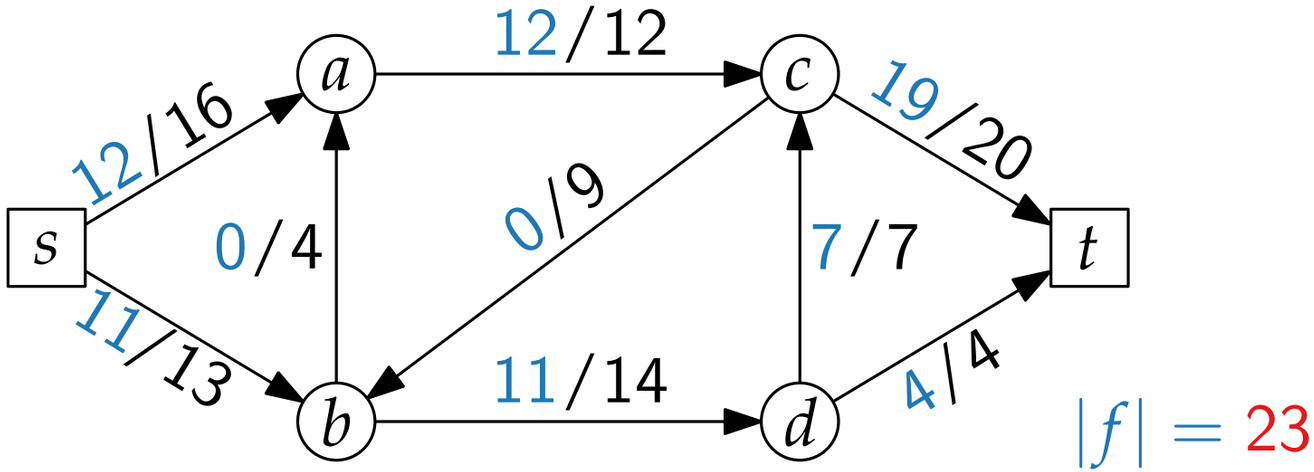
- unique **source** s and **sink** t,
- no antiparallel edges, and
- a **capacity** $c(u, v) \geq 0$ for every $(u, v) \in E$.



Flow

A **flow** in G is a real-value function $f : V \times V \to \mathbb{R}$ that satisfies

- **flow conservation**,
  $\sum_{v \in V} f(v, u) = \sum_{v \in V} f(u, v)$ for all $u \in V \setminus \{s, t\}$, and
- **capacity constraint**, $0 \le f(u, v) \le c(u, v)$.



$|f| = 23$

Maximum flow problem. Given a flow network G with source s and sink t, find a flow of maximum value.

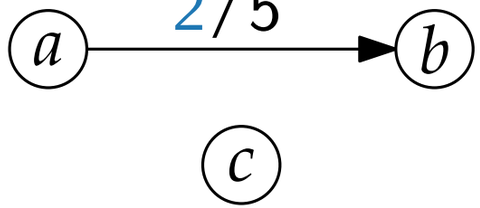
The **value** $|f|$ of a flow f is defined as

$$|f| = \sum_{v \in V} f(s, v) - \sum_{v \in V} f(v, s).$$

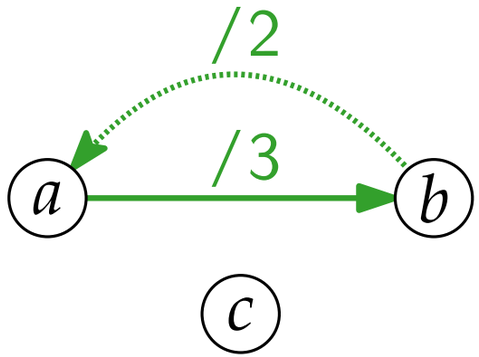
How much may flow change?

Given G and f, the **residual capacity** c_f for a pair $u, v \in V$ is

$$c_f(u,v) = \begin{cases} c(u,v) - f(u,v) & \text{if } (u,v) \in E \\ f(v,u) & \text{if } (v,u) \in E \\ 0 & \text{otherwise.} \end{cases}$$



$$c_f(a,b) = 3$$
$$c_f(b,a) = 2$$
$$c_f(a,c) = 0$$

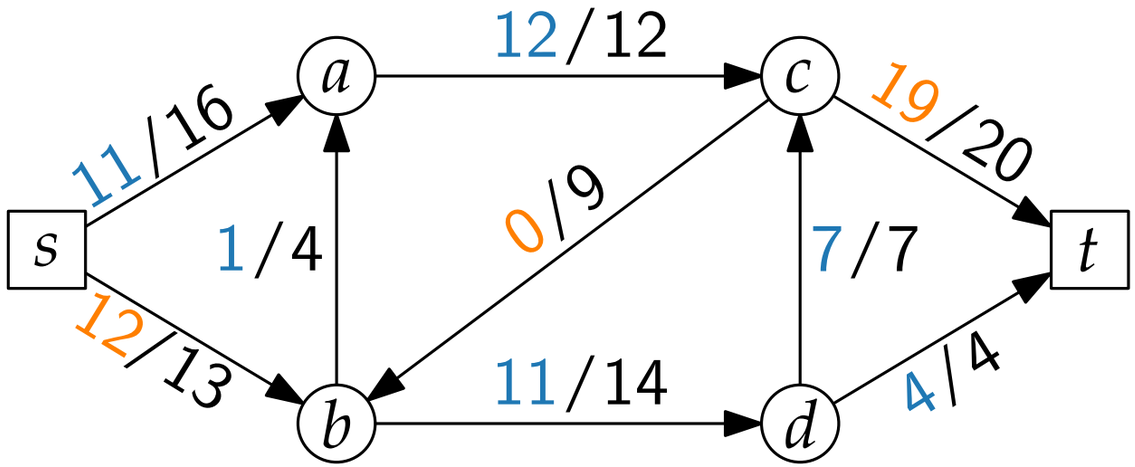


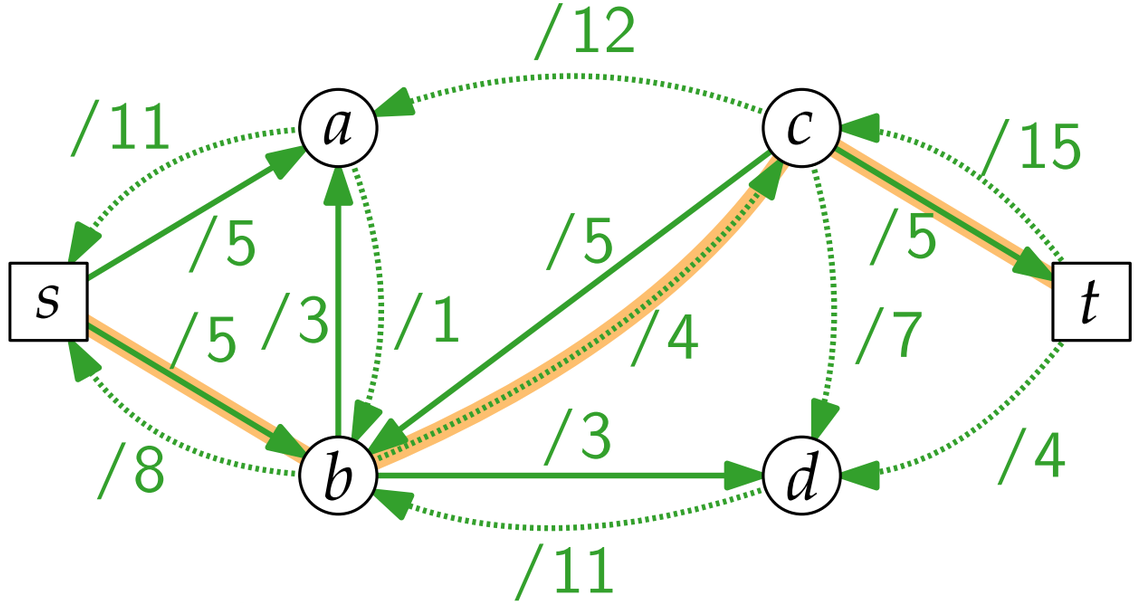
Residual network & augmenting path

The **residual network** $G_f = (V, E_f)$ for a flow network G with flow f has

- $E_f = \{(u, v) \in V \times V \mid c_f(u, v) > 0\}$.

An **augmenting path** is an st-path in G_f. $\Rightarrow$ use to increase f





flow/capacity

$$c_f(u, v) = \begin{cases} c(u, v) - f(u, v) & \text{if } (u, v) \in E \\ f(v, u) & \text{if } (v, u) \in E \\ 0 & \text{otherwise.} \end{cases}$$

$/c_f$ non-satured edges

reverse edges

Ford-Folkerson and Edmons-Karp algorithms

EdmondsKarp

~~FordFulkerson~~$(G = (V, E), c, s, t)$

foreach $uv \in E$ **do**
  $f_{uv} \leftarrow 0$
} initialising zero flow

while G_f contains shortest augmenting path p **do**
  $\Delta \leftarrow \min_{uv \in p} c_f(uv)$ } residucal capacity of p
  foreach $uv \in p$ **do**
    if $uv \in E$ **then**
      $f_{uv} \leftarrow f_{uv} + \Delta$
    else
      $f_{vu} \leftarrow f_{vu} - \Delta$
  } augmentation along p

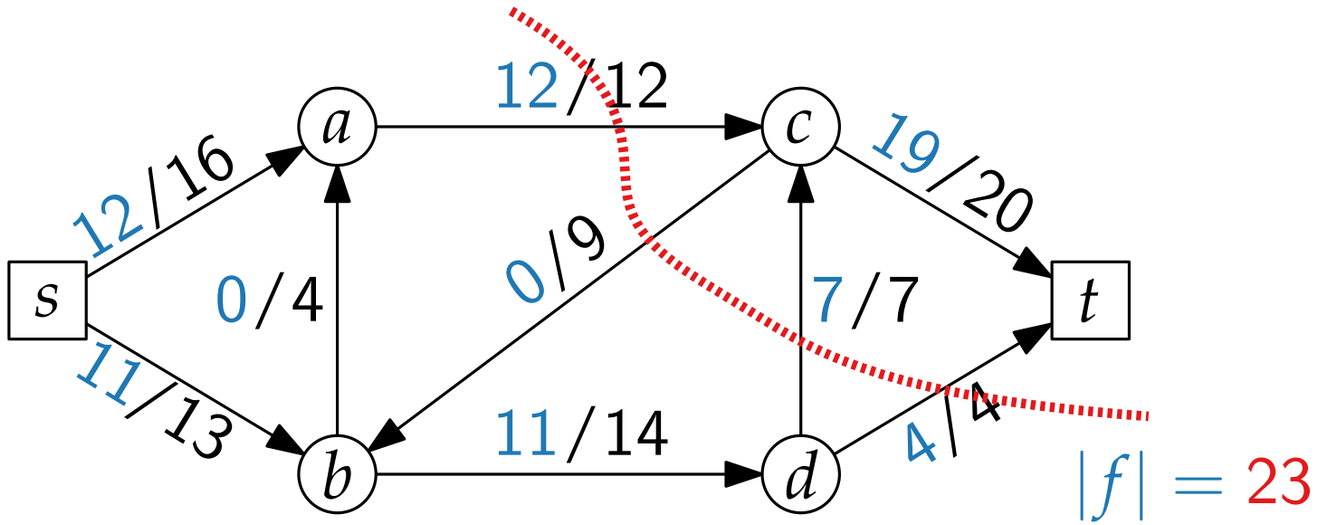
return f } return max flow

- Ford-Folkerson runs in $\mathcal{O}(|E||f^\star|)$ and Edmons-Karp in $\mathcal{O}(|V||E|^2)$ time.

Max-flow min-cut theorem

Theorem.
For a flow f in a flow network G, the following conditions are equivalent:

- f is a maximum flow in G.
- G_f contains no augmenting paths.
- $|f| = c(S, T)$ for some cut (S, T) of G.



Push-relabel idea

A New Approach to the Maximum-Flow Problem

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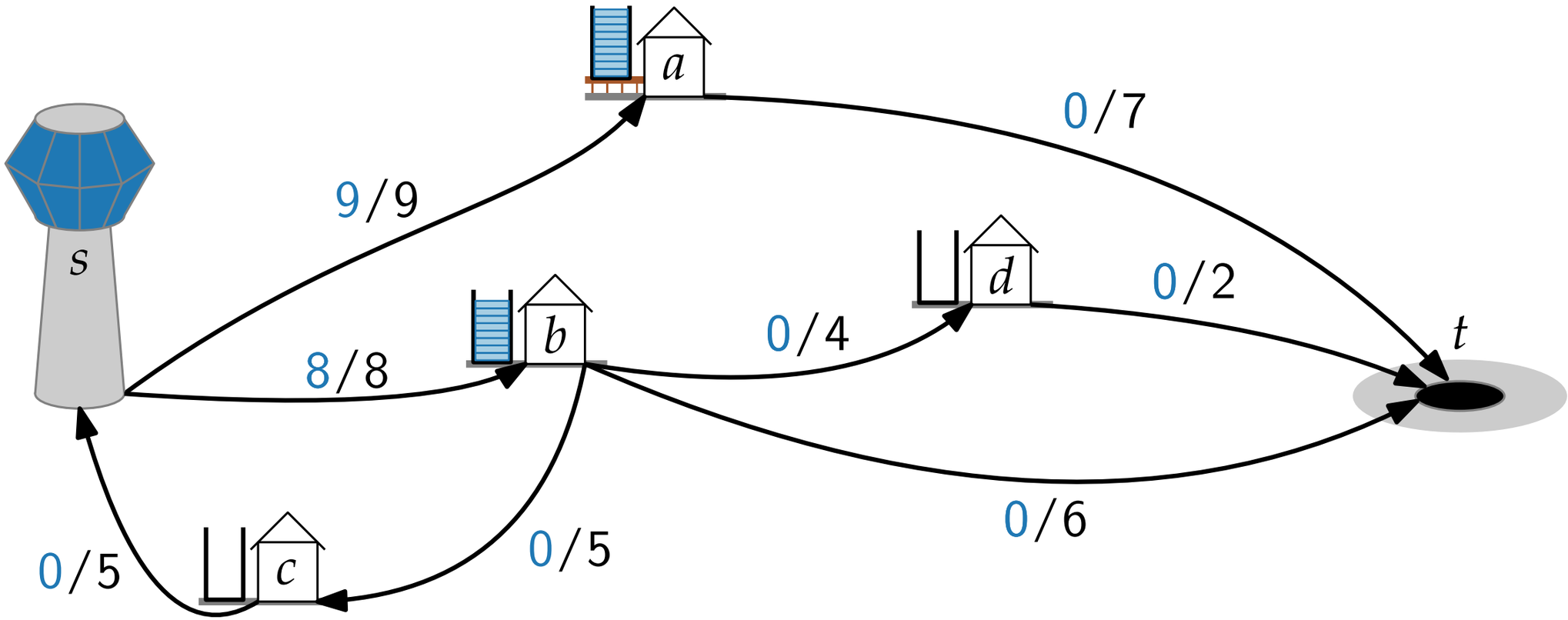
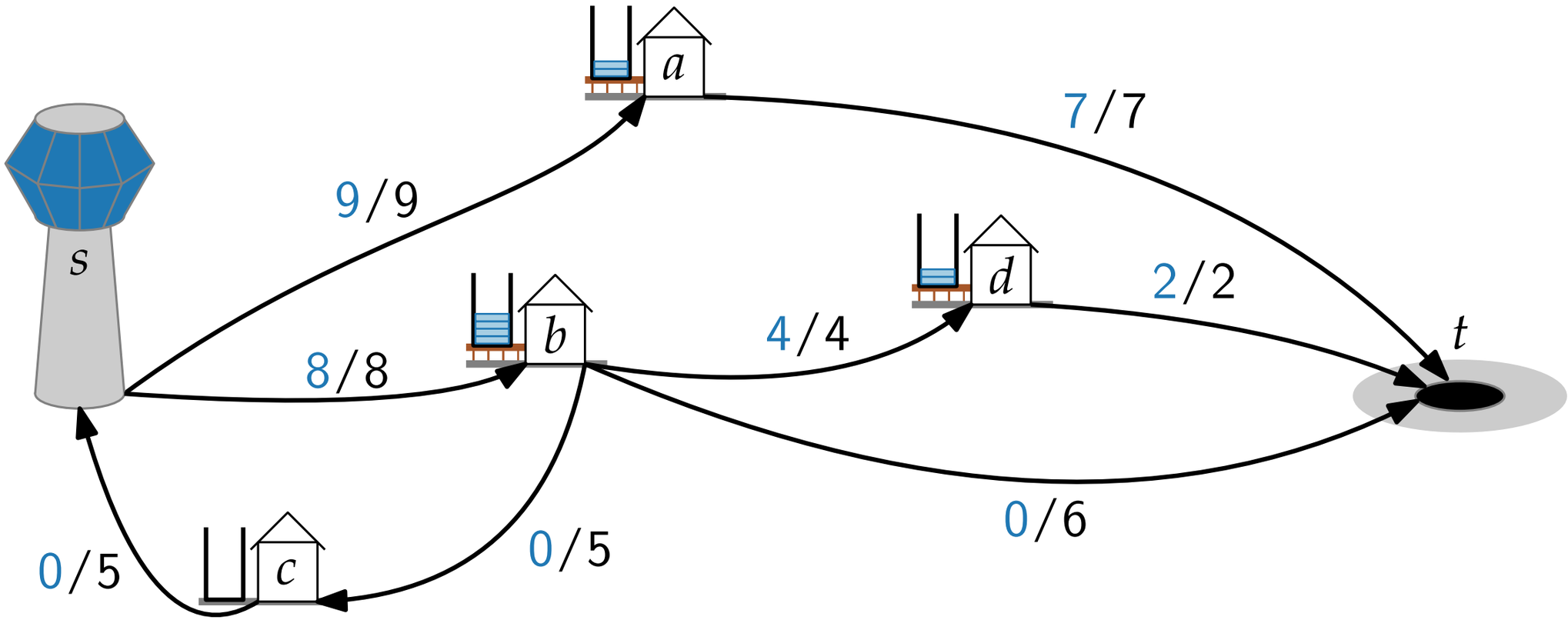
Abstract. All previously known efficient maximum-flow algorithms work by finding augmenting paths, either one path at a time (as in the original Ford and Fulkerson algorithm) or all shortest-length augmenting paths at once (using the layered network approach of Dinic). An alternative method based on the *preflow* concept of Karzanov is introduced. A preflow is like a flow, except that the total amount

for the next phase. Our algorithm abandons the idea of finding a flow in each phase and also abandons the idea of global phases. Instead, our algorithm maintains a preflow in the original network and pushes local flow excess toward the sink along what it estimates to be shortest paths in the residual graph. This pushing of flow changes the residual graph, and paths to the sink may become saturated. Excess that cannot be moved to the sink is returned to the source, also along estimated shortest paths. Only when the algorithm terminates does the preflow become a flow, and then it is a maximum flow.

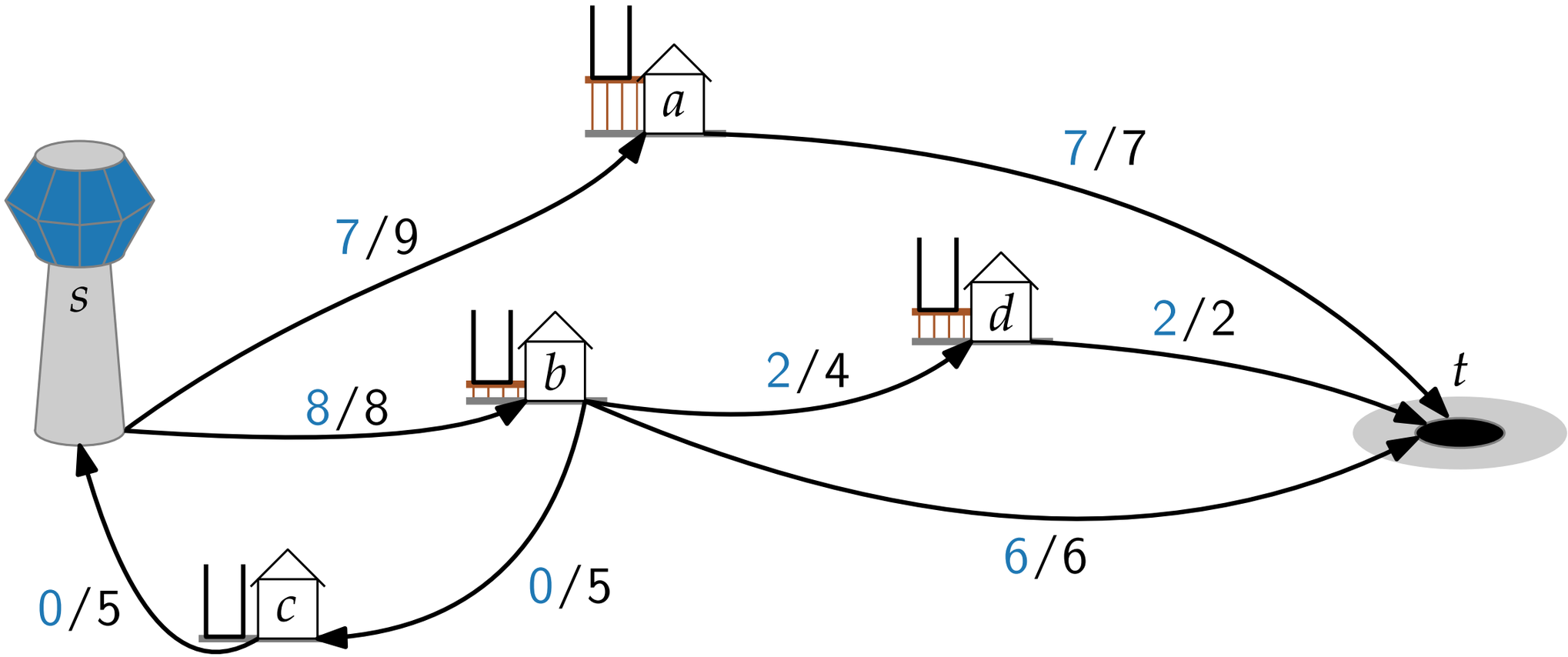
Push-relabel idea



Push-relabel idea



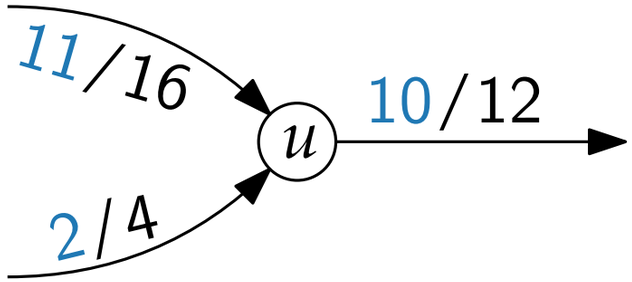
Push-relabel idea



Preflow, excess flow and height

A **preflow** in G is a real-value function $f\colon V \times V \to \mathbb{R}$ that satisfies the capacity constraint and, for all $u \in V \setminus \{s\}$,

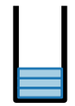
- $\sum_{v \in V} f(v,u) - \sum_{v \in V} f(u,v) \geq 0$.



The **excess flow** of a vertex u is

- $e(u) = \sum_{v \in V} f(v,u) - \sum_{v \in V} f(u,v)$.

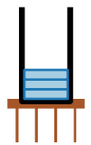
$e(u) = 3$



A vertex u is called **overflowing**, when $e(u) > 0$.

For a flow network G with preflow f, a **height function** is a function $h\colon V \to \mathbb{N}$ such that

- $h(s) = |V|$,
- $h(t) = 0$, and
- $h(u) \leq h(v) + 1$ for every residual edge $(u,v) \in E_f$.



PUSH operation

PUSH(u, v)

Condition: u is overflowing, $c_f(u, v) > 0$, and $h(u) = h(v) + 1$

Effect: Push $\min(e(u), c_f(u, v))$ overflow from u to v

$\Delta \leftarrow \min(e(u), c_f(u, v))$

if $(u, v) \in E$ **then**

  $f(u.v) \leftarrow f(u, v) + \Delta$

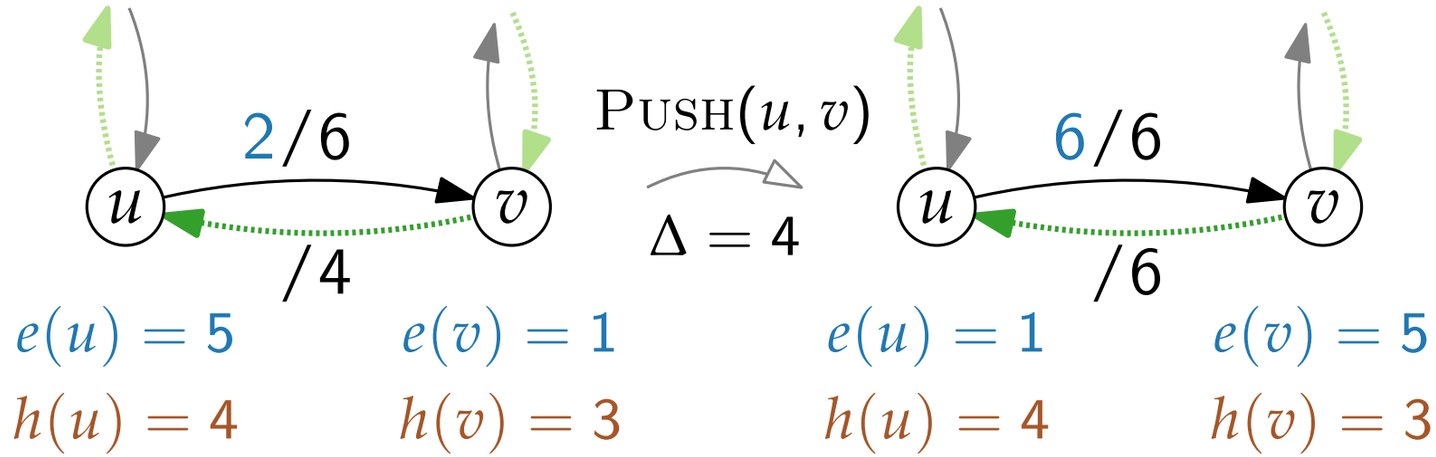
else

  $f(v.u) \leftarrow f(v, u) - \Delta$

$e(u) \leftarrow e(u) - \Delta$

$e(v) \leftarrow e(v) + \Delta$

Example.



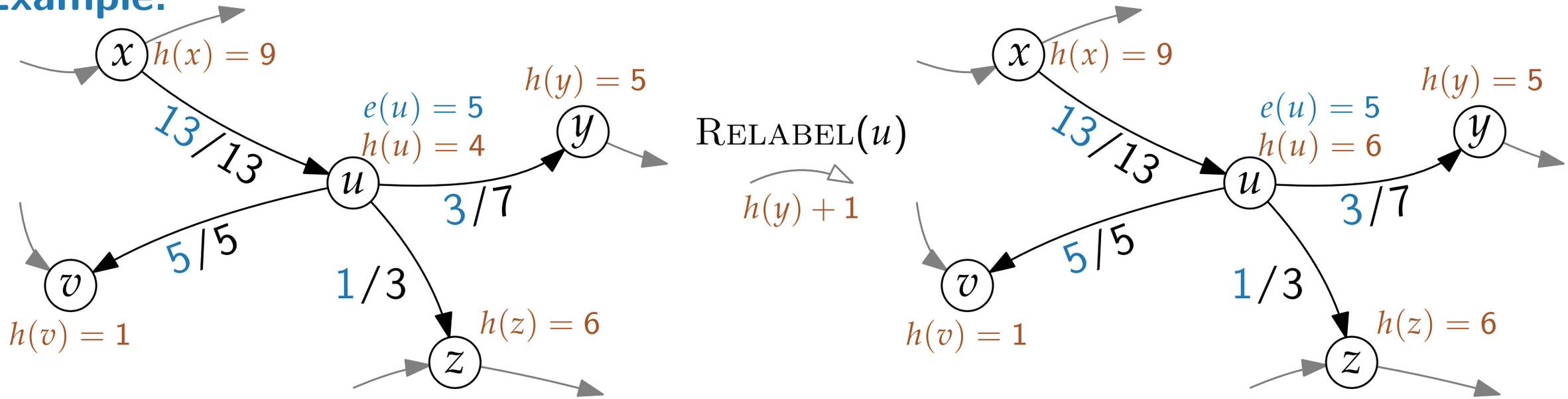
Relabel operation

Relabel(u)

Condition: u is overflowing and
$h(u) \leq h(v)$ for all $v \in V$ with $(u, v) \in E_f$

Effect: Increase the height of u
$h(u) \leftarrow 1 + \min\{h(v) : (u, v) \in E_f\}$

Example.



Push-Relabel algorithm

Push-Relabel(G)

  InitPreflow(G, s)
  while there exists an applicable
   Push or Relabel operation x **do**
  ⌊ apply x

InitPreflow(G, s)

  $h(v) \leftarrow 0, e(v) \leftarrow 0 \quad \forall v \in V$
  $h(s) \leftarrow |V|$
  $f(u, v) \leftarrow 0 \quad \forall (u, v) \in E$
  for each v adjacent to s **do**
  ⎢ $f(s, v) \leftarrow c(s, v)$
  ⌊ $e(v) \leftarrow c(s, v)$

- initialises heights
- pushes max flow over all outgoing edges of s

Correctness

Part 1.

If the algorithm terminates, the preflow is maximum flow.

- If an overflowing vertex exists, the algorithm can continue.
- The algorithm maintains f as a preflow and h as a height function.
- Sink t is not reachable from source s in G_f.

Part 2.

The algorithm terminates and the heights stay finite.

- Find upper bound on heights.
- Find upper bound for calls of Relabel.
- Find upper bound for calls of Push.

Continuation

Lemma 1.
If a vertex u is overflowing, either a push or a relabel operation applies to u.

Proof.

Assuming $h(u)$ is valid, we have

- $h(u) \leq h(v) + 1$ for all v with $(u, v) \in E_f$.

If no push operation valid for $(u, v) \in E_f$, then

- $h(u) \leq h(v)$ for all v with $(u, v) \in E_f$.

Therefore, $\textsc{Relabel}(u)$ is applicable.

Height function:

- $h(s) = |V|$,
- $h(t) = 0$, and
- $h(u) \leq h(v) + 1$ for every residual edge $(u, v) \in E_f$.

$\textsc{Push}(u, v)$

Condition: u is overflowing, $c_f(u, v) > 0$, and $h(u) = h(v) + 1$
$\Delta \leftarrow \min(e(u), c_f(u, v))$
if $(u, v) \in E$ **then**
  $f(u.v) \leftarrow f(u, v) + \Delta$
else
  $f(v.u) \leftarrow f(v, u) + \Delta$
$e(u) \leftarrow e(u) - \Delta$
$e(v) \leftarrow e(v) + \Delta$

$\textsc{Relabel}(u)$

Condition: u is overflowing, $h(u) \leq h(v)\ \forall v \in V$ with $(u, v) \in E_f$
$h(u) \leftarrow 1 + \min\{h(v) : (u, v) \in E_f\}$

Maintaining the preflow

Lemma 2.
The push-relabel algorithm maintains a preflow f.

Proof.

- InitPreflow initialises a preflow f. ✓
- Relabel(u) doesn't affect f. ✓
- Push(u, v) maintains f as a preflow. ✓

Height function:

- $h(s) = |V|$,
- $h(t) = 0$, and
- $h(u) \leq h(v) + 1$ for every residual edge $(u, v) \in E_f$.

Push(u, v)

Condition: u is overflowing, $c_f(u, v) > 0$, and $h(u) = h(v) + 1$
$\Delta \leftarrow \min(e(u), c_f(u, v))$
if $(u, v) \in E$ **then**
  $f(u.v) \leftarrow f(u, v) + \Delta$
else
  $f(v.u) \leftarrow f(v, u) + \Delta$
$e(u) \leftarrow e(u) - \Delta$
$e(v) \leftarrow e(v) + \Delta$

Relabel(u)

Condition: u is overflowing, $h(u) \leq h(v)\ \forall v \in V$ with $(u, v) \in E_f$
$h(u) \leftarrow 1 + \min\{h(v) : (u, v) \in E_f\}$

Maintaining the height function

Lemma 3.
The push-relabel algorithm maintains h as a height function.

Proof.

- InitPreflow initialises h as a height function. ✓
- Push(u, v) leaves h a height function. ✓
  - If (v, u) added to E_f, then $h(v) = h(u) - 1 < h(u) + 1$.
  - If (u, v) removed from E_f, then ✓.
- Relabel(u) leaves h a height function. ✓
  - $(u, v) \in E_f$, then $h(u) \leq h(v) + 1$
  - $(w, u) \in E_f$, then $h(w) < h(u) + 1$

Height function:

- $h(s) = |V|$,
- $h(t) = 0$, and
- $h(u) \leq h(v) + 1$ for every residual edge $(u, v) \in E_f$.

Push(u, v)

Condition: u is overflowing, $c_f(u, v) > 0$, and $h(u) = h(v) + 1$
$\Delta \leftarrow \min(e(u), c_f(u, v))$
if $(u, v) \in E$ **then**
  $f(u.v) \leftarrow f(u, v) + \Delta$
else
  $f(v.u) \leftarrow f(v, u) + \Delta$
$e(u) \leftarrow e(u) - \Delta$
$e(v) \leftarrow e(v) + \Delta$

Relabel(u)

Condition: u is overflowing, $h(u) \leq h(v)\ \forall v \in V$ with $(u, v) \in E_f$
$h(u) \leftarrow 1 + \min\{h(v) : (u, v) \in E_f\}$

Reachability of the sink

Lemma 4.
During the push-relabel algorithm, there is no path from s to t in G_f.

Height function:
- $h(s) = |V|$,
- $h(t) = 0$, and
- $h(u) \leq h(v) + 1$ for every residual edge $(u, v) \in E_f$.

Proof.

Suppose there is a path $s = v_0, v_1, \ldots, v_k = t$ in G_f.
Then

- $(v_i, v_{i+1}) \in E_f$ for $0 \leq i \leq k-1$, and
- $h(v_i) \leq h(v_{i+1}) + 1$ for $0 \leq i \leq k-1$.

$$\Rightarrow h(s) \leq h(t) + k = k$$

But then and since $k \leq |V| - 1$, follows $h(s) < |V|$. ✗

Partial correctness of the algorithm

Theorem 5.
If the push-relabel algorithm terminates, the computed preflow f is a maximum flow.

Proof.

- By Lemma 1, the algorithm stops, when there is no overflowing vertex.
- By Lemma 2, f is a preflow.

  $\Rightarrow$ f is a flow.
- By Lemma 3, h is a height function.
- So by Lemma 4, there is no st-path in G_f.

  $\Rightarrow$ By the Max-Flow Min-Cut Theorem, the flow f is a maximum flow.

Correctness

Part 1. ✓

If the algorithm terminates, the preflow is maximum flow.

- If an overflowing exists, the algorithm can continue.
- The algorithm maintains f as a preflow and h as a height function.
- Sink t is not reachable from source s in G_f.

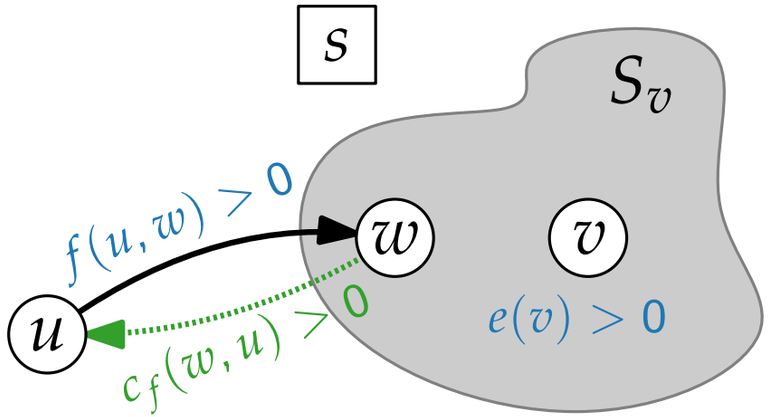
Part 2.

The algorithm terminates and the heights stay finite.

- Find upper bound on heights.
- Find upper bound for calls of Relabel.
- Find upper bound for calls of Push.

Reachability of source in residual graph

Lemma 6.
There is a path from every overflowing vertex v to s in G_f.



Proof.

- $S_v \leftarrow$ vertices reachable from v in G_f.
- Suppose $v \notin S_v$.
- Since f a preflow and $s \notin S_v$, we have $\sum_{w \in S_v} e(w) \geq 0$.
- Since $v \in S_v$, we even have $\sum_{w \in S_v} e(w) > 0$.
- There is edge (u, w) with $u \notin S_v, w \in S_v$ and $f(u, w) > 0$.
- But then $c_f(w, u) > 0$, meaning u is reachable from v. ✗

Upper bound on height and RELABEL operations

Lemma 7.
During the push-relable algorithm, we have $h(v) \leq 2|V| - 1$ for all $v \in V$.

Proof.

- Statement holds after initialisation.
- Let v be an overflowing vertex that is relabeled.
- By Lemma 6, there is a path $v = v_0, v_1, \ldots, v_k = s$ in G_f.
- Then $h(v_i) \leq h(v_{i+1}) + 1$ for $0 \leq i \leq k - 1$.
- Since $k \leq |V| - 1$, we have $h(v) \leq h(s) + k \leq 2|V| - 1$.

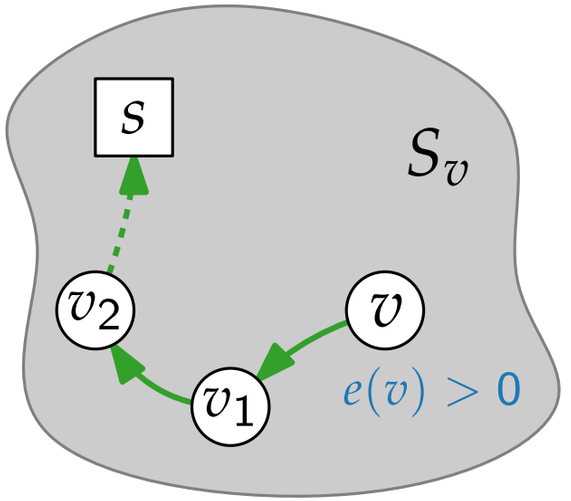
Corollary 8.
The push-relable algorithm executes at most $2|V|^2$ RELABEL operations.

Height function:

- $h(s) = |V|$,
- $h(t) = 0$, and
- $h(u) \leq h(v) + 1$ for every residual edge $(u, v) \in E_f$.

RELABEL(u)

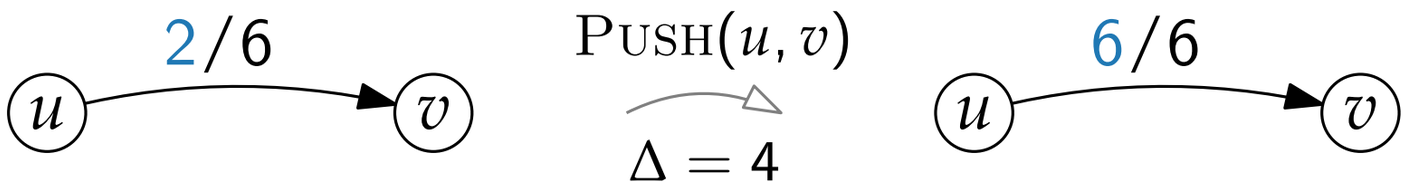
Condition: u is overflowing,
$h(u) \leq h(v)\ \forall v \in V$ with $(u, v) \in E_f$
$h(u) \leftarrow 1 + \min\{h(v) : (u, v) \in E_f\}$



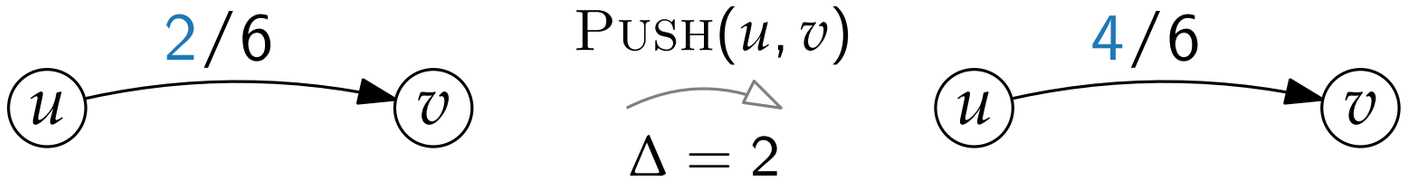
Saturating and unsaturating $\textsc{Push}$ operations

The operation $\textsc{Push}(u, v)$ is

- **saturating**, if afterwards $c_f(u, v) = 0$,



- and **unsaturating** otherwise.



Upper bound on saturating PUSH operations

Lemma 9.
The push-relable algorithm executes at most $2|V||E|$ saturating PUSH operations.

Proof.

- Consider saturating $\textsc{Push}(u, v)$
  - $h(u) = h(v) + 1$
- For another saturating $\textsc{Push}(u, v)$, first $\textsc{Push}(v, u)$ necessary
  - $h(v) = h(u) + 1$ necessary
- After another saturating $\textsc{Push}(u, v)$, both $h(u)$ and $h(v)$ have increased by at least two.
- But by Lemma 6, $h(u) \leq 2|V| - 1$ and $h(v) \leq 2|V| - 1$.
- There are at most $2|V| - 1$ sat. PUSH operations for edge (u, v).

$\textsc{Push}(u, v)$

Condition: u is overflowing, $c_f(u, v) > 0$, and $h(u) = h(v) + 1$
$\Delta \leftarrow \min(e(u), c_f(u, v))$
if $(u, v) \in E$ **then**
  $f(u.v) \leftarrow f(u, v) + \Delta$
else
  $f(v.u) \leftarrow f(v, u) + \Delta$
$e(u) \leftarrow e(u) - \Delta$
$e(v) \leftarrow e(v) + \Delta$

$\textsc{Push}(u, v)$
...
$\textsc{Push}(v, u)$
...
$\textsc{Push}(u, v)$
...

Upper bound on unsaturating PUSH operations

Lemma 10.
The push-relable algorithm executes at most $4|V|^2|E|$ unsaturating PUSH ops.

Proof.

- Consider $\mathcal{H} = \sum_{\substack{v \in V \setminus \{s,t\}, \\ v \text{ overflowing}}} h(v)$.
- After initialisation and at the end $\mathcal{H} = 0$.
- A saturating PUSH increases $\mathcal{H}$ by at most $2|V| - 1$.
- By Lemma 8, all saturating PUSH ops. increases $\mathcal{H}$ by at most $(2|V| - 1) \cdot 2|V||E|$.
- By Lemma 7, all RELABEL ops increases $\mathcal{H}$ by at most $(2|V| - 1) \cdot |V|$.
- An unsaturating $\text{PUSH}(u, v)$ decrease $\mathcal{H}$ by at least 1, since $h(u) - h(v) \geq 1$.

$\text{PUSH}(u, v)$

Condition: u is overflowing, $c_f(u, v) > 0$, and $h(u) = h(v) + 1$
$\Delta \leftarrow \min(e(u), c_f(u, v))$
if $(u, v) \in E$ **then**
  $f(u.v) \leftarrow f(u, v) + \Delta$
else
  $f(v.u) \leftarrow f(v, u) + \Delta$
$e(u) \leftarrow e(u) - \Delta$
$e(v) \leftarrow e(v) + \Delta$

Termination of the algorithm

Theorem 5.
If the push-relabel algorithm terminates, the computed preflow f is a maximum flow.

Theorem 11.
The push-relabel algorithm terminates after $\mathcal{O}(|V|^2|E|)$ valid $\textsc{Push}$ or $\textsc{Relabel}$ ops.

Proof.

- Follows by Corollary 8 and Lemma 9+10.

Implementation

The actual running time depends on the selection order of overflowing vertices:

- **FIFO implementation:**
  Pick overflowing vertex by *first-in-first-out* principle:
  $\mathcal{O}(|V|^3)$ running time.
  with dynamic trees: $\mathcal{O}(|V||E| \log \frac{|V|^2}{|E|})$
- **Highest label:**
  For $\textsc{Push}$ select highest overflowing vertex: $\mathcal{O}(|V|^2|E|^{\frac{1}{2}})$
- **Excess scaling:**
  For $\textsc{Push}(u, v)$ choose edge (u, v) such that u is overflowing, $e(u)$ is *sufficiently high* and $e(v)$ *sufficiently small*: $\mathcal{O}(|E| + |V|^2 \log C)$, where $C = \max_{(u,v)\in E} c(u, v)$

Discussion

- The push-relabel method offers an alternative framework to the Ford-Folkerson method to develope algorithms that solve the maximum flow problem.
- Push-relabel algorithms are regarded as benchmarks for maximum flow algorithms.
- In practice, heuristics are used to improve the performance of push-relabel algorithms. Any ideas?
- The algorithm can be extended to solve the minimum cost flow problem.

Literature

Main source:

- [CLRS Ch26] ← Cormen et al. “Introduction to Algorithms”

Original papers:

- [Goldberg, Tarjan ’88] A new approach to the maximum-flow problem

Links:

- MaxFlow Ford-Folkerson and Edmonds-Karp animations
  https://visualgo.net/en/maxflow